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SHARIPOV R. A.

FOUNDATIONS OF GEOMETRY FOR UNIVERSITY STUDENTS AND HIGH SCHOOL STUDENTS

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Referees: Prof. R. R. Gadylshin, Bashkir State Pedagogical University (БГПУ);

Prof. E. M. Bronshtein, Ufa State University for Aircraft and Technology (УГАТУ);

Mrs. N. V. Medvedeva, Honored teacher of the Republic Bashkortostan, High School No. 42, Ufa.

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PREFACE.

The elementary geometry is a part of geometry which we usually first meet in school. It describes the structure of our everyday material environment. In very ancient ages people learned to discriminate some primitive constituents in the large variety of forms they observe in the world surrounding them. These are a *point*, a *straight line* and a *segment* of it, a *plane*, a *circle*, a *cylinder*, a *ball*, and some others. People began to study their properties. Geometers of the Ancient Greece succeeded in it better than others. They noted that the properties of the simplest geometric forms are not a collection of facts, but they are bound to each other by many logical bonds. Some of these properties can be deduced from some others.

In 5-th century before Christ Euclid offered a list of simplest basic properties of geometric forms, which now are called *postulates* or *axioms* of Euclid. The elementary geometry or Euclidean geometry based on these axioms became the first axiomatic theory in mathematics.

The aim of this book is to explain the elementary geometry starting from the Euclid's axioms in their contemporary edition. It is addressed to University students as textbook for the course of foundations of geometry. It can also be recommended to High school students when they wish to know better what is their subject of elementary geometry from the professional point of view of mathematicians. Especially for the convenience of High school students in Chapter I of the book I give some preliminary material from the set theory.

In writing this book I used the books by I. Ya. Backelman [1] and by N. V. Efimov [2]. Proofs of some theorems are taken from the book by A. V. Pogorelov [3].

July, 1998;
January, 2007. R. A. Sharipov.

CHAPTER I

EUCLID'S GEOMETRY. ELEMENTS OF THE SET THEORY AND AXIOMATICS.

§ 1. Some basic concepts of the set theory.

The set theory makes a ground for constructing the modern mathematics in whole. This theory itself is based on two very simple concepts: the concept of a *set* and the concept of an *element* of a set. Saying a set one usually understand any collection of objects which for some reason should be grouped together. Individual objects composing a set are called its *elements*. A set A and its element a are in the relation of *belonging*: $a \in A$. This writing says that the element a *belongs* to the set A and the set A *comprises* its element a. The transposed writing $A \ni a$ means the same.

Let's mark a part of the elements in a set A. This marked part of elements can be treated as another set B. The fact that B is a part of A is denoted as $B \subset A$. If $B \subset A$, we say that B is a *subset* of A. One should clearly distinguish two writings:

$$a \in A, \qquad\qquad B \subset A.$$

The inclusion sign $\subset$ relates two sets, while the belonging sign $\in$ relates a set with its element.

When composing the set B above, we could mark all of the elements of A. Then we would get $B = A$. But even in this special case B can be treated as a part of A. This means that the writing $B \subset A$ does not exclude the possibility of coincidence $B = A$. If we wish to show that B is a subset of A different from A, we should write $B \subsetneq A$.

Another special case of $B \subset A$ arises when B contains no elements at all. Such a set is called the *empty* set. It is denoted by the special sign $\varnothing$. The empty set is treated as a subset of an arbitrary set A, i. e. $\varnothing \subset A$.

Let A and B be two arbitrary sets. Some of their elements could be common for them: $c \in A$ and $c \in B$. Such elements constitute a set C which is called the *intersection* of the sets A and B. This set is denoted $C = A \cap B$. If $A \cap B \neq \varnothing$, then we say that the sets A and B *do intersect*. Otherwise, if $A \cap B = \varnothing$, then we say that these sets *do not intersect*.

Again let A and B be two arbitrary sets. Let's gather into one set C all of the elements taken from A and B. The resulting set C in this case is called the *union* of the sets A and B. It is denoted $C = A \cup B$.

Elements composing the set $A \cup B$ are divided into three groups (into three subsets). These are

(1) elements that belong to the sets A and B simultaneously;
(2) elements that belong to the set A, but do not belong to the set B;
(3) elements that belong to the set B, but do not belong to the set A.

The first group of elements constitutes the intersection $A \cap B$. The second group of elements constitutes the set which is called the *difference* of the sets A and B. It is denoted $A \setminus B$. Now it is clear that the third group of elements constitutes the set being the difference $B \setminus A$. The sets $A \cap B$, $A \setminus B$, and $B \setminus A$ do not intersect with each other. Their union coincides with the union of A and B, i. e. we have the formula

$$A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A).$$

§ 2. Equivalence relations and breaking into equivalence classes.

Let M be some set. Let's consider ordered pairs of elements (a, b), where $a \in M$ and $b \in M$. Saying «ordered», we mean that a is the first element in the pair and b is the second element, the pair (b, a) being distinct from the pair (a, b). The set of all ordered pairs of the elements taken from M is called the *Cartesian square* of the set M. It is denoted $M \times M$.

Assume that some pairs in the set $M \times M$ are somehow marked. Then the marked pairs form a subset $R \subset M \times M$. If such a subset R is given, we say that a *binary relation* R in M is given. Indeed, each marked pair $(a, b) \in R$ can be understood as a sign indicating that its elements a and b are related in some way so that such a relation is absent among the elements of a non-marked pair. If a pair (a, b) is marked, this fact can be denoted in a special way, e. g. $a \overset{R}{\rightharpoondown} b$. The writing $a \overset{R}{\rightharpoondown} b$ is read as follows: the element a is in relation R to the element b.

The relation of equality and the relation of order among real numbers are well-known examples of binary relations. They are written as $a = b$, $a < b$, or $b > a$.

Definition 2.1. A binary relation R in a set M is called an *equivalence relation* and is denoted by the sign $\overset{R}{\sim}$ if the following conditions are fulfilled:

(1) *reflectivity:* $a \overset{R}{\sim} a$ for any $a \in M$;
(2) *symmetry:* $a \overset{R}{\sim} b$ implies $b \overset{R}{\sim} a$;
(3) *transitivity:* $a \overset{R}{\sim} b$ and $b \overset{R}{\sim} c$ imply $a \overset{R}{\sim} c$.

If a binary relation R is implicitly known from the context, the letter R in the writing $a \overset{R}{\sim} b$ can be omitted and the relation R among a and b is written as $a \sim b$.

Definition 2.2. Assume that in a set M an equivalence relation R is given. An *equivalence class* of an element $a \in M$ is the set of all elements $x \in M$ equivalent to a, i. e.

$$\mathrm{Cl}_R(a) = \{x \in M : x \overset{R}{\sim} a\}.$$

THEOREM 2.1. *If* $a \overset{R}{\sim} b$, *then* $\mathrm{Cl}_R(a) = \mathrm{Cl}_R(b)$. *If the elements a and b are not equivalent, then their classes do not intersect, i. e.* $\mathrm{Cl}_R(a) \cap \mathrm{Cl}_R(b) = \varnothing$.

Sometimes the classes determined by an equivalence relation R in M are considered as elements of another set. The set composed by all equivalence classes is called the *factorset*. It is denoted M/R. The passage from M to the factorset M/R is called the *factorization*.

The theorem 2.1 shows that if two equivalence classes are distinct, they have no common elements, while each element $a \in M$ belongs to at least one equivalence class. Therefore, each equivalence relation R defines the division of the set M into the union of non-intersecting equivalence classes:

$$M = \bigcup_{Q \in M/R} Q.$$

EXERCISE 2.1. *Using the properties of reflectivity, symmetry, and transitivity of the equivalence relation R, prove the above theorem 2.1.*

§ 3. Ordered sets.

DEFINITION 3.1. A binary relation P in a set M is called an *order relation* and is denoted by the symbol $\overset{P}{\prec}$ if the following conditions are fulfilled:

(1) *non-reflectivity:* $a \overset{P}{\prec} b$ implies $a \neq b$;
(2) *non-symmetry:* the condition $a \overset{P}{\prec} b$ excludes $b \overset{P}{\prec} a$;
(3) *transitivity:* $a \overset{P}{\prec} b$ and $b \overset{P}{\prec} c$ imply $a \overset{P}{\prec} c$.

The writing $a \overset{P}{\prec} b$ is read as «*a precedes b*» or as «b follows a». If a binary relation P is implicitly known from the context, the letter P in the writing $a \overset{P}{\prec} b$ can be omitted and the relation P among a and b is written as $a \prec b$.

If one of the mutually exclusive conditions $a \prec b$ or $b \prec a$ is fulfilled, we say that the elements a and b are *comparable.* A set M equipped with an order relation P is called a *linearly ordered* set if any two elements of M are comparable. Otherwise, if there are non-comparable pairs of elements in M, it is called a *partially ordered* set.

§ 4. Ternary relations.

Along with binary relations, sometimes ternary (or triple) relations are considered. A simple example is given by the addition operation for numbers. The equality $a + b = c$ means that the ordered triple of numbers (a, b, c) is distinguished as compared to other triples, for which such an equality is not fulfilled. One can easily formalize this example.

DEFINITION 4.1. We say that a *ternary relation* R in a set M is defined if some subset R in $M \times M \times M$ is fixed.

§ 5. Set theoretic terminology in geometry.

The primary set, which is studied in Euclidean geometry, is the *space.* Its elements are called *points.* The geometric space of Euclidean geometry is usually denoted by $\mathbb{E}$. Individual points of this space by tradition are denoted by capital letters of the Roman alphabet. Apart from the whole space and individual points, various other geometric forms are considered: *planes*, *straight lines*, *segments of straight lines*, *rays*, *polygons*, *polyhedra* etc. All of these geometric forms are subsets of the space, they consist of points.

The relations of belonging and inclusion denoted by the signs $\in$ and $\subset$ in geometry are expressed by various words corresponding to their visual meaning. Thus, for example, if a point A

belongs to a straight line m, then we say that A lies on the line m, while the line m passes through the point A. Similarly, if a straight line m is included into a plane α, we say that the line m lies on the plane α, while the plane α passes through the line m. Usually such a deliberate wording produces no difficulties for understanding and makes an explanation more vivid and visual.

§ 6. Euclid's axiomatics.

The geometric space $\mathbb{E}$ consists of points. All points of this space are equipollent, none of them is distinguished. If a separate point is taken, it has no geometric properties by itself. The properties of points reveal in their relation to other points. For example, if we take three points, they can be lying on a straight line and they can be not lying either. A triangle given by these points can be equilateral, or isosceles, or rectangular, or somewhat else. When composing a geometric form the points of the space $\mathbb{E}$ come into some definite relations with each other. The basic properties of such relations are formulated in Euclid's axioms. The total number of Euclid's axioms in their contemporary edition is equal to twenty. They are divided into five groups:

(1) axioms of *incidence* (8 axioms A1–A8);
(2) axioms of *order* (4 axioms A9–A12);
(3) axioms of *congruence* (5 axioms A13–A17);
(4) axioms of *continuity* (2 axioms A18 and A19);
(5) axiom of *parallels* (1 axiom A20).

In forthcoming chapters of this book we give a successive explanation of the above axioms and the geometry based on them.

§ 7. Sets and mappings.

Let X and Y be two sets. A *mapping* of the set X to the set Y is a rule that associates each element x of the set X with some definite element y in the set Y. The mappings, as well as the sets, are denoted by various letters (usually by small letters of the Roman alphabet). The writing $f\colon X \to Y$ means that f is a mapping of the set X to the set Y. If $x \in X$, then $f(x)$ denotes the result of applying the rule f to the element x. The element $y = f(x)$ of the set Y is called the *image* of the element x from X. An element $x \in X$ such that $y = f(x)$ is called a *preimage* of the element y from Y.

For the rule f to be treated as a mapping $f\colon X \to Y$ it should be unambiguous, i. e. each occasion of applying it to the same element $x \in X$ should yield the same result. In other words, $x_1 = x_2$ should imply $f(x_1) = f(x_2)$.

The simplest example of a mapping is an *identical mapping* of a set X to the same set X. It is denoted as $\mathrm{id}_X : X \to X$. The identical mapping id_X associates each element x of the set X with itself, i. e. $\mathrm{id}_X(x) = x$ for all $x \in X$.

Let $f\colon X \to Y$ and $g\colon Y \to Z$ are two mappings. In this case we can construct the third mapping. Let's define a rule h such that applying it to an element x of X consists in applying f to x and then applying g to $f(x)$. Ultimately this new rule yields $g(f(x))$, i. e. $h(x) = g(f(x))$. The newly constructed mapping $h\colon X \to Z$ is called the *composition* of the mappings g and f, it is denoted $h = g \circ f$. So we have

$$g \circ f(x) = g(f(x)) \tag{7.1}$$

for all x of X. Thus the relationship (7.1) is a short form for the definition of the composition $g \circ f$. The operation of composition can also be understood as a multiplication, where the multiplicands are two mappings.

THEOREM 7.1. *If three mappings $f\colon Z \to W$, $g\colon Y \to Z$ and $h\colon X \to Y$ are given, then the relationship*

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{7.2}$$

is valid. It expresses the associativity of the composition.

PROOF. We have mappings both in the left and in the right sides of (7.2), i. e. (7.2) is an equality of mappings. Two mappings in our case are two rules that associate the elements of X with some elements in W. The statements of these rules can be quite different, not similar to each other. However, these rules are treated to be equal if the results of applying them to an element x do coincide for all $x \in X$. For this reason the proof (7.2) reduces to verifying the equality

$$(f \circ g) \circ h(x) = f \circ (g \circ h)(x) \tag{7.3}$$

for all x of X. Let's do it by means of direct calculations on the base of the formula (7.1) defining the composition of mappings:

$$(f \circ g) \circ h(x) = f \circ g(h(x)) = f(g(h(x))),$$

$$f \circ (g \circ h)(x) = f(g \circ h(x)) = f(g(h(x))).$$

As a result of these rather simple calculations both left and right sides of (7.3) are reduced to the same expression $f(g(h(x)))$. The equality (7.3) and, hence, the equality (7.2) are proved. □

Definition 7.1. Let $f\colon X \to Y$ be a mapping of a set X to a set Y and let A be some non-empty subset in X. Then the set $B \subset Y$ composed by the images of all elements of the set A is called the *image* of the set A. It is denoted $B = f(A)$.

According to this definition, the image of a non-empty set is not empty. For the empty set we set $f(\varnothing) = \varnothing$. The image of the set X under the mapping $f\colon X \to Y$ is sometimes denoted by $\operatorname{Im} f$, i.e. $\operatorname{Im} f = f(X)$. The set X is called the *domain* of the mapping f, the set Y is called its *domain of values*, and the set $\operatorname{Im} f$ is called the *image* of the mapping f. The domain of values and the image of a mapping f often do not coincide.

Definition 7.2. Let $f\colon X \to Y$ be a mapping of a set X to a set Y and let y be some element of the set Y. The set composed by all those elements of X which are taken to the element y by the mapping f is called the *total preimage* of the element y. This set is denoted by $f^{-1}(y)$.

Definition 7.3. Let $f\colon X \to Y$ be a mapping of a set X to a set Y and let B be a non-empty subset of the set Y. The set composed by all those elements x of X whose images $f(x)$ are in B is called the *total preimage* of the set B. This set is denoted by $f^{-1}(B)$.

According to the definition 7.3 the total preimage of the set Y coincides with X, i. e. $f^{-1}(Y) = X$. For the empty set we set $f^{-1}(\varnothing) = \varnothing$ by definition. However, even for a non-empty set B its preimage $f^{-1}(B)$ can be empty.

Definition 7.4. A mapping $f\colon X \to Y$ is called *injective* if for any $y \in Y$ the total preimage $f^{-1}(y)$ contains not more than one element.

Definition 7.5. A mapping $f\colon X \to Y$ is called *surjective*, if for any $y \in Y$ the total preimage $f^{-1}(y)$ is not empty.

Definition 7.6. A mapping f is called *bijective*, or a *one-to-one mapping* if it is injective and surjective simultaneously.

Theorem 7.2. *A mapping $f\colon X \to Y$ is injective if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.*

Theorem 7.3. *A mapping $f\colon X \to Y$ is surjective if and only if $\operatorname{Im} f = Y$.*

Exercise 7.1. *Prove the theorems 7.2 and 7.3, which are often used for checking injectivity and surjectivity of mappings instead of the initial definitions 7.4 and 7.5.*

Assume that a mapping $f : X \to Y$ is bijective. Then for any element $y \in Y$ the total preimage $f^{-1}(y)$ is not empty, but it contains not more then one element. Hence, it contains exactly one element. For this reason we can define the mapping $h\colon Y \to X$ which associates each element y of Y with that very unique element of the set $f^{-1}(y)$. Such mapping h is called the *inverse mapping* for f. It is denoted $h = f^{-1}$.

Theorem 7.4. *The mapping $h\colon Y \to X$ inverse to a bijective mapping $f\colon X \to Y$ is bijective and $h = f^{-1}$ implies $h^{-1} = f$.*

Theorem 7.5. *A mapping $f\colon X \to Y$ and its inverse mapping $f^{-1}\colon Y \to X$ are related to each other as follows:*

$$f \circ f^{-1} = \mathrm{id}_Y, \qquad\qquad f^{-1} \circ f = \mathrm{id}_X\,.$$

Theorem 7.6. *The composition of two injective mappings is an injective mapping.*

Theorem 7.7. *The composition of two surjective mappings is a surjective mapping.*

Theorem 7.8. *The composition of two bijective mappings is a bijective mapping.*

Exercise 7.2. *Prove the theorems 7.4, 7.5, 7.6, 7.7, and 7.8.*

§ 8. Restriction and extension of mappings.

Let X' be a subset in a set X and assume that two mappings $f : X \to Y$ and $h : X' \to Y'$ are given. If $h(x) = f(x)$ for all $x \in X'$, then we say that h is a *restriction* of the mapping f to the subset X'. To the contrary, the mapping f is called an *extension* or a *continuation* of the mapping h from the set X' to the bigger set X.

If a mapping $f : X \to Y$ is given, one can easily construct its restriction to an arbitrary subset $X' \subset X$. It is sufficient to forbid applying f to the elements not belonging to X'. As a result we get the mapping $f : X' \to Y$ being a restriction of the initial one.

To extent a mapping $f : X' \to Y'$ from X' to a bigger set X is usually more complicated. For this purpose one should define the values $f(x)$ for those elements x of X, which do not belong to X'. This can be done in many ways. However, in a typical case the mapping $f : X' \to Y'$ possesses some properties that should be preserved in extending it to X. This makes the problem of extending sapid, but substantially reduces the arbitrariness in choosing possible extensions.

CHAPTER II

AXIOMS OF INCIDENCE AND AXIOMS OF ORDER.

§ 1. Axioms of incidence.

AXIOM A1. *Each straight line contains at least two points.*

AXIOM A2. *For each two distinct points A and B there is a straight line passing through them and this line is unique.*

AXIOM A3. *In the space there are at least three points which do not lie on one straight line.*

The axioms A1 and A2 show that each straight line can be fixed by fixing two points on it. This fact is used for denoting straight lines: saying the line AB, we understand the line passing through the points A and B. Certainly, the line AB can coincide with the line CD for some other two points C and D. This occasion is not excluded.

The axiom A3 shows that in the space there is at least one triangle. However, it is not yet that very triangle in usual sense because the axioms A1, A2, and A3, taken separately, do not define a segment. On the base of these three axioms one cannot distinguish the interior of a triangle from its exterior.

AXIOM A4. *For any three points A, B, and C not lying on one straight line there is some plane passing through them. Such a plane is unique.*

Axiom A5. *Each plane contains at least one point.*

Axiom A6. *If some two distinct points A and B of a straight line a lie on a plane α, then the whole line a lies on the plane α.*

Axiom A7. *If two planes do intersect, their intersection contains at least two points.*

Axiom A8. *In the space there are at least four points not lying on one plane.*

The incidence axioms A1–A8 are yet too few in order to derive complicated and sapid propositions from them. However, some simple and visually evident facts can be proved on the base of these axioms.

Theorem 1.1. *If two distinct straight lines do intersect, their intersection consists exactly of one point.*

Proof. Let $a \neq b$ be two distinct straight lines with non-empty intersection and let A be a point of $a \cap b$. Provided the proposition of the theorem is not valid, one could find another point B in the intersection $a \cap b$. Thus we would have two straight lines a and b passing through the points A and B. This fact would contradict the axiom A2. $\square$

Theorem 1.2. *If two distinct planes do intersect, then their intersection is a straight line.*

Proof. Let A be a common point of two distinct planes $\alpha \neq \beta$. Let's apply the axiom A7. According to this axiom, there is at least one more commom point of these two planes α and β. We denote it by B and consider the straight line AB.

The points A and B lie on the plane α. Let's apply the axiom A6 to them. This axiom says that the line AB in whole lies on the plain α.

Let's repeat these arguments for the plane β. As a result we find that the line AB in whole lies on the plain β. Thus, the straight line AB is a part of the intersection $\alpha \cap \beta$. It is a common line for these two planes.

The rest is to prove that the intersection of planes $\alpha \cap \beta$ contains no points other than those lying on the line AB. If such a point C would exist, then we would have three points A, B, and C not lying on one straight line, and we would have two distinct planes α and β passing through these three points. But it contradicts to the axiom A4. The contradiction obtained shows that the intersection $\alpha \cap \beta$ coincides with the line AB. $\square$

Theorem 1.3. *For a straight line and a point not lying on this straight line there is a plane passing through this line and through this point. Such a plane is unique.*

Proof. Let C be a point not lying on a line a. Let's apply the axiom A1 to the line a. According to this axiom, we can find two points A and B on the line a. Then the points A, B, and C appear to be three points not lying on one straight line. Due to the axiom A4 there is exactly one plane α passing through the points A, B, and C. Let's apply the axiom A6 to the line a and the plane α. From this axiom we derive that the line a is contained within the plane α. Thus, the plane α a required plane passing through the point C and the line a.

Now let's show that the plane α passing through the point C and the line a is unique. Indeed, each plane of this sort should pass through the above three points A, B, and C not lying in one straight line. Due to the axiom A4 it is unique. $\square$

Theorem 1.4. *A straight line a not lying on a plane α has not more than one common point with that plane.*

If the intersection $a \cap \alpha$ is empty, then a straight line a is said to be *parallel* to a plane α. Let's consider the case, where this intersection is non-empty. Assume that A is a point from the intersection $a \cap \alpha$. If this intersection contains more than one point, then there is another point $B \in a \cap \alpha$. The points A and B of the line a both lie on the plane α. Applying the axiom A6, we get that the whole line a should lie on the plane α. However, this contradicts the initial premise of the theorem: $a \not\subset \alpha$. The contradiction obtained shows that the intersection $a \cap \alpha$ consists of exactly one point A. The theorem is proved.

THEOREM 1.5. *For a pair of intersecting, but not coinciding straight lines there is exactly one plane containing both of them.*

PROOF. Let a and b be a pair of intersecting, but not coinciding straight lines. According to the theorem 1.1 their intersection consists of one point, we denote this point by A. Then we apply the axiom A1 to the line b. According to this axiom, there is another point B on b distinct from A. The point B does not lie on a because it lies on the other line b and does not belong to the intersection $a \cap b$.

Now let's apply the theorem 1.3 to the aline a and the point B. According to this theorem, there is exactly one plane α passing through the line a and the point B. The points A and B of the line b lie on the plane α. Therefore we can apply the axiom A6, which says that that the line b in whole should lie on the plane α. Thus, the plane α contains both lines a and b.

The rest is to prove that the plane α is unique. If it is not unique and if β is another plane containing both lines a and b, then β passes through the line a and the point B not lying on a. According to the theorem 1.3, such a plane is unique. Therefore the plane β should coincide with the plane α. $\square$

LEMMA 1.1. *For any plane α there is a point in the space not lying on this plane.*

PROOF. Let α be some arbitrary plane. Using the axiom A8, we find four points A, B, C, and D, not lying on one plane. It is clear that at least one of these four points does not lie on the plane α. Otherwise they would be lying on one plane α in spite of their choice. $\square$

LEMMA 1.2. *For any straight line a there is a point in the space not lying on this line.*

PROOF. Let a be some arbitrary straight line. Let's apply the axiom A3 and find three points A, B, and C not lying on one straight line. It is clear that at least one of these three points does not lie on the line a. Otherwise the points A, B, and C would be lying on one line a in spite of their choice. $\square$

THEOREM 1.6. *On each plane there are at least three points not lying in one straight line.*

PROOF. Assume that some arbitrary plane α is given. Applying the axiom A5, we choose the first point A on this plane. Then we use the lemma 1.1. Because of this lemma we can choose a point X outside the plane α. Therefore, the straight line AX intersects with the plane α at the point A, but it does not lie on that plane. Now we can apply the theorem 1.4. This theorem says that the point A is the unique common point of the line AX and the plane α.

Now let's apply the lemma 1.2 to the line AX. According to this lemma, there is a point Z not lying on the line AX. The points A, X, Z do not lie on one straight line. Therefore, according to the axiom A4, they fix a unique plane $\beta = AXZ$ passing through these three points.

The planes α and β do intersect and have the common point A. Let's apply the axiom A7 and conclude that, apart from the point A, there is at least one other common point of the planes α and β. We denote it B. As a result we have found that there are two distinct points A and B on the plane α.

Now we apply the lemma 1.1 to the plane β. Due to this lemma we can find a point Y not lying on the plane β. The line AX lies in the plane β, while the point Y is outside of this plane. Therefore $Y \notin AX$. Hence the three points A, X, and Y do not lie on one straight line. According to the axiom A4 they determine a unique plane $\gamma = AXY$ passing through them.

The planes α and γ do not coincide since there is the point X belonging to γ and not belonging to α. These planes do intersect since they have the common point A. Hence, we can apply the theorem 1.2. It says that the intersection of the planes α and γ is a straight line a containing their common point A.

By construction the straight line a lies on the plane α. It intersects the line AX at the unique point A. This fact follows from the theorem 1.1 and from $X \notin a$. Let's prove that the straight line a does not contain the point B. Remember that A is the unique point of intersection of the line AX and the plane α. Therefore $B \notin AX$ and the three points A, X, and B do not lie on one straight line. If we admit that $B \in a$, then both planes β and γ pass through the three points A, X, and B. Die to the axiom A4 they should coincide: $\gamma = \beta$. However, by construction the plane γ contains the point Y not belonging to the plane β, i. e. $\gamma \neq \beta$. This contradiction shows that $B \notin a$.

As a result of the above considerations within the plane α we have constructed a straight line a passing through the point A, and we have constructed a point B not lying on that line. Applying the axiom A1 to the line a, we find another point $C \in a$ distinct from A. The three points A, B, and C is a required triple of points of the plane α not lying on one straight line. □

EXERCISE 1.1. *Draw figures illustrating the proofs of the above six theorems 1.1–1.6 and two lemmas 1.1 and 1.2.*

Let's some set consisting of four elements. For example, this can be the set of four initial positive integers $\{1, 2, 3, 4\}$. Let's call this set the *space*, while the numbers 1, 2, 3, and 4 are its points. The subsets

$$\{1, 2\},\ \{1, 3\},\ \{1, 4\},\ \{2, 3\},\ \{2, 4\},\ \{3, 4\}$$

are called the *straight lines* in this space. For the planes we choose the following four subsets:

$$\{1, 2, 3\},\ \{1, 2, 4\},\ \{1, 3, 4\},\ \{2, 3, 4\}.$$

The above sets constitute a *finite model* of geometry with the axioms of incidence.

EXERCISE 1.2. *Prove that the above finite model of geometry satisfies all eight axioms of incidence A1–A8.*

§ 2. Axioms of order.

Axioms of order constitute the second group of Euclid's axioms. Mostly, they describe the inner structure of separate straight lines. For any three distinct points A, B, and C lying on one straight line one of them lies between two others. If B lies between A and C, we write this fact as

$$(A \blacktriangleright B \blacktriangleleft C). \tag{2.1}$$

The axioms of order determine the properties of the ternary relation of the points of a fixed straight line written as (2.1).

Axiom A9. *If a point B lies between the points A and C, then it lies between C and A.*

Using the notation (2.1), this axiom can be written as follows:

$$(A \blacktriangleright B \blacktriangleleft C) \implies (C \blacktriangleright B \blacktriangleleft A). \tag{2.2}$$

The axiom A9 and the formula (2.2) mean the symmetry of the ternary relation, which is sometimes called the «betweenness» relation, under the exchange of its first and third arguments.

Let A and B be two arbitrary distinct points. According to the axiom A2 they fix the straight line AB. An *open interval* (or simply an *interval*) is the set of all points of the line AB lying between the points A and B:

$$(AB) = \{X \in AB : \quad (A \blacktriangleright X \blacktriangleleft B)\}.$$

The axiom A9 means that the interval (AB) coincides with the interval (BA). The points A and B are called the *ending points* of the interval (AB). Joining the ending points to an open interval, we get a *closed interval* or a *segment*:

$$[AB] = \{A\} \cup \{B\} \cup (AB).$$

According to the axiom A9 the segment $[AB]$ coincides with the segment $[BA]$.

The interval (AB) is called the *interior* of the segment $[AB]$, while A and B are its *ending points*. The points of the straight line AB not belonging to the segment $[AB]$ constitute the *exterior* of the segment $[AB]$. Along with the open and closed intervals, sometimes one defines semi-open intervals:

$$[AB) = \{A\} \cup (AB), \qquad [BA) = \{B\} \cup (AB).$$

Axiom A10. *For any two points A and B on the straight line AB there is a point C such that B lies between A and C.*

Axiom A11. *For any three distinct points A, B, and C lying on one straight line only one of them can lie between two others.*

The axiom A11 means that not more than one of the following three conditions can be fulfilled:

$$(A \blacktriangleright B \blacktriangleleft C), \qquad (B \blacktriangleright C \blacktriangleleft A), \qquad (C \blacktriangleright A \blacktriangleleft B). \tag{2.3}$$

Generally speaking, the axiom A11 does not exclude the case where none of the above conditions (2.3) is fulfilled.

Theorem 2.1. *The exterior of any segment $[AB]$ is not empty.*

Proof. Let's apply the axiom A10 to the points A and B. It yields the existence of a point C lying on the straight line AB such that the condition $(A \blacktriangleright B \blacktriangleleft C)$ is fulfilled. Due to the axiom A11 this condition excludes the other condition $(B \blacktriangleright C \blacktriangleleft A)$. Hence, the point C is not an inner point of the segment $[AB]$. It is not an ending point either since it does coincide neither with A, nor with B. Hence, C is a point on the line AB external for the segment $[AB]$. $\square$

Actually, the axioms A9, A10 and A11 can yield more. From them one can derive the existence of at least two points in the exterior of any segment $[AB]$.

THEOREM 2.2. *For any segment $[AB]$ there are two points C_1 and C_2 on the line AB such that the conditions $(A \blacktriangleright B \blacktriangleleft C_1)$ and $(B \blacktriangleright A \blacktriangleleft C_2)$ are fulfilled.*

PROOF. Note that the points A and B enter the statement of the axiom A10 in an asymmetric way. For the beginning we apply the axiom A10 in its standard form. It yields the existence of a point C_1 on the line AB such that the condition $(A \blacktriangleright B \blacktriangleleft C_1)$ is fulfilled. Then we exchange A and B and apply the axiom A10 once more. Now it yields the existence of a point C_2 on the line AB such that $(B \blacktriangleright A \blacktriangleleft C_2)$.

The points C_1 and C_2 both are in the exterior of the segment $[AB]$. However, they cannot coincide. Indeed, if $C = C_1 = C_2$, then, using the axiom A9, we would derive that the conditions $(A \blacktriangleright B \blacktriangleleft C)$ and $(C \blacktriangleright A \blacktriangleleft B)$ are fulfilled simultaneously. But this opportunity is prohibited by the axiom A11. $\square$

AXIOM A12. *Let A, B and C be three points of a plane α not lying on one straight line and let a be a straight line on the same plane α passing through none of these three points. If the line a intersects the segment $[AB]$ at its interior point, then it necessarily passes through an interior point of at least one of the segments $[AC]$ or $[BC]$.*

The axiom A12 is known as Pasch's axiom. It is important for all of the further constructions in this section.

THEOREM 2.3. *The interior of any segment $[AB]$ is not empty.*

PROOF. Let's apply the lemma 1.2 and find a point C not lying on the line AB (see Fig. 2.1 below). Then we apply the axiom A10 to the points A and C. It yields the existence of a point D lying on the line AC and such that C is in the interior of the segment $[AD]$. The next step is to draw the line DB and apply the theorem 2.1 to the segment $[DB]$. As a result we find a point E on the line DB lying outside the segment $[DB]$.

The line CE crosses the line AD at the unique point C, which is in the interior of the segment $[AD]$. It crosses the line DB at the point E, which is distinct from B. Hence, the line CE contains none of the points A, D, and B. We denote the line CE by a and apply Pasch's axiom A12 to the points A, D, and B. According to this axiom, the line CE should cross the segment $[AB]$ or the segment $[DB]$ somewhere at an interior point. In our case it cannot cross the segment $[DB]$. Indeed, the line CE crosses the line DB at the unique point E, which is in the exterior of the segment $[DB]$ by construction. Therefore, the line CE crosses the segment $[AB]$ at some its interior point F. This means that the interior of the segment $[AB]$ is not empty. The theorem 2.3 is proved. □

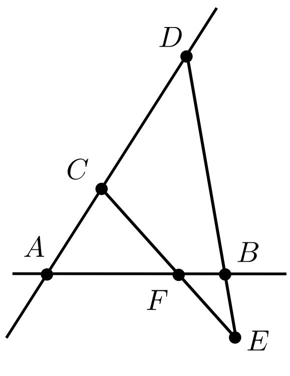


Fig. 2.1

Theorem 2.4. *For any three points A, B, and C lying on one straight line exactly one of them lies between two others.*

Proof. Let A, B and C be three arbitrary points lying on one straight line. Assume that the A does not lie between B and C and assume also that C does not lie between A and B. Under these assumptions let's prove that B lies between A and C. For the beginning we apply the lemma 1.2 and find a point D not lying on the line AC (see Fig. 2.2). Then we apply the axiom A10 to the points A and D and find a point E on the line AD such that D is an interior point of the segment $[AE]$. Let's draw the lines DC, EC, and EB. At the intersection of the lines DC and EB we get the point F. Through the points A and F

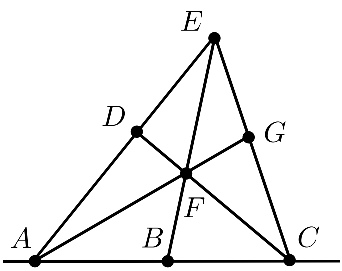


Fig. 2.2

we draw the line AF. At the intersection of the line AF with the line EC we get the point G.

Let's consider the triangle ABE. The line DC intersects its side $[AE]$ at the interior point D. According to Pasch's axiom A12 this line should intersect one of the segments $[AB]$ or $[EB]$ at some interior point. In our case the segment $[AB]$ is excluded since the lines DC and AB intersect at the point C. This point, according to the assumption made in the beginning of the proof, does not lie on the segment $[AB]$ between the points A and B. Hence, the point F obtained as the intersection of the lines DC and EB is an interior point of the segment $[EB]$.

Now let's consider the triangle EBC. The line AF crosses its side $[EB]$ at the interior point F. According to Pasch's axiom A12 this line should intersect one of the segments $[BC]$ or $[EC]$ at some interior point. The segment $[BC]$ is excluded since the lines AF and BC intersect at the point A, while this point, according to the assumption made in the beginning of the proof, does not lie between the points B and C. The rest is the segment $[EC]$, which should intersect the line AF at its interior point G.

The nest step is to consider the triangle EDC and the line AG which intersect its side EC at the interior point G. Applying Pasch's axiom A12 in this case, we get that the line AG intersect one of the segments $[DC]$ or $[DE]$ at some interior point. The segment $[DE]$ is excluded. Indeed, the line AG crosses the line DE at the point A, while the point E is chosen so that the condition $(A \blacktriangleright D \blacktriangleleft E)$ is fulfilled. Due to the axiom A11 this condition excludes the condition $(E \blacktriangleright A \blacktriangleleft D)$, i. e. the point A cannot be an interior point of the segment $[DE]$. Hence, F is an interior point of the segment $[DC]$.

In the last step we consider the triangle ADC and the line EB which intersects its side $[DC]$ at the interior point F. Let's apply Pasch's axiom A12 in this case. From this axiom we derive that the line EB should intersect one of the segments $[AD]$ or $[AC]$ at some interior point. The segment $[AD]$ is excluded. Indeed, the line EB crosses the line AD at the point E. The condition $(A \blacktriangleright D \blacktriangleleft E)$ for the point E excludes the condition $(D \blacktriangleright E \blacktriangleleft A)$ and, hence, the point E does not lie in the interior of the segment $[AD]$. The rest is the segment $[AC]$. The point B at the intersection of the lines EB and AC should be an interior point of the segment $[AC]$. The theorem 2.4 is proved. □

The theorem 2.4 proved just above strengthens the axiom A11. Now for any three points A, B, and C lying on one straight line one of the conditions (2.3) is necessarily fulfilled, thus excluding other two conditions (2.3). Pasch's axiom A12 also can be strengthened.

THEOREM 2.5. *Let A, B, and C be three points of a plane α not lying on one straight line and let a be a straight line on the plane α passing through neither of these three points. Then if a crosses the segment $[AB]$ at some interior point, it passes through an interior point of exactly one of the segments $[AC]$ or $[BC]$.*

PROOF. Assume that the proposition of the theorem is not valid. Then the line a crosses each of the three segments $[AB]$, $[BC]$, and $[CA]$ at their interior points. Let's denote these points by P, Q, and R. The points P, Q, and R lie on three distinct straight lines AB, BC, and CA intersecting each other at three points A, B, and C. According to the statement of the theorem, none of the points A, B, and C lies on the line a, hence, none of the points P, Q, and R can coincide with another one.

Let's prove that the point R does not lie between P and Q. If we admit that the point lies between P and Q, then we can apply Pasch's axiom A12 to the line AR and to the triangle PQB. It says that the line AR should intersect one of the segments $[PB]$ or $[QB]$ at some interior point. However, we know that the line AR crosses the line PB at the point A, and it crosses the line QB at the point C. If A is in the interior of the segment $[PB]$, this contradicts the fact that P lies between A and B. Similarly, if C is in the interior of the segment $[QB]$, then Q cannot lie between B and C.

The above contradiction proves that the point R cannot lie between P and Q. Similarly, one can prove that Q does not lie between R and P, while P does not lie between Q and R. Thus, none of the points P, Q, and R on the line a lies between two others. This contradicts the previous theorem 2.4. Therefore the initial assumption that the line a crosses both segments $[AC]$ and $[BC]$ at their interior points is invalid. The theorem 2.5 is proved. $\square$

§ 3. Segments on a straight line.

Lemma 3.1. *Let A, B, C, and D be a group of four points. Assume that the point B lies between A and C, while the point C lies between B and D. Then both points B and C lie between the points A and D.*

Proof. From $(A \blacktriangleright B \blacktriangleleft C)$ it follows that the point A lies on the line BC, while from $(B \blacktriangleright C \blacktriangleleft D)$ it follows that D also lies on the line BC. Thus, under the assumptions of the lemma 3.1 all of the four points A, B, C, and D lie on one straight line.

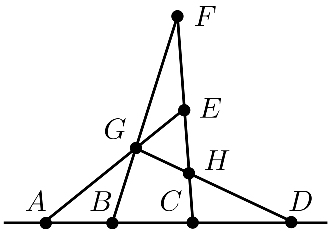


Fig. 3.1

Using the lemma 1.2 we find a point E not lying on the line AD (see Fig. 3.1). Then we apply the axiom A10 to the points C and E. As a result on the line CE we find a point F such that the point E lies in the interior of the segment $[CF]$. Let's draw the lines AE and FB, then consider the triangle FBC. The line AE crosses the line FC at the point E which is an interior point for the segment $[FC]$. The intersection of the lines AE and BC coincides with the point A which is outside the segment $[BC]$. Therefore, according to Pasch's axiom A12, the line AE should cross the side $[FB]$ of the triangle FBC at some interior point G.

Now let's consider the triangle AEC. The line FB crosses the line AC at the point B lying in the interior of the segment $[AC]$. The same line FB crosses the line EC at the point F outside the segment $[EC]$. Hence, according to Pasch's axiom A12, the point G obtained as the intersection of the lines FB and AE should be an interior point of the segment $[AE]$.

In the next step we consider again the triangle FBC and draw the line GD. This line crosses its side $[FB]$ at the interior point G and it has no common points with the side $[BC]$ since the point D lies outside the segment $[BC]$. Hence, due to Pasch's axiom A12 we conclude that the line GD crosses the segment $[FC]$ at some interior point H.

Now let's consider the triangle GBD and the line FC. We use the fact that the point C is in the interior of the segment $[BD]$ and that $F \notin [GB]$. Then from Pasch's axiom A12 we derive that H is an interior point of the segment $[GD]$.

In the last step we consider the triangle AGD and the line FC. The line FC crosses the side $[GD]$ of this triangle at the interior point H and it does not intersect the side $[AG]$ since the point E lies outside the segment $[AG]$. Now from Pasch's axiom A12 we find that C is an interior point for the segment $[AD]$. This is one of the propositions of the lemma, which we had to prove.

The second proposition of the lemma does not require a separate proof. In order to prove that the point B lies in the interior of the segment $[AD]$ it is sufficient to exchange the notations of the points A with D and B with C. Thereafter the rest is to use the first proposition, which is already proved, and then return to the initial notations. $\square$

LEMMA 3.2. *Let A, B, C, and D be a group of four points. If the point C lies in the interior of the segment $[AD]$ and if the point B lies in the interior of the segment $[AC]$, then B is in the interior of $[AD]$ and C is in the interior of $[BD]$.*

PROOF. It is easy to see that if C lies in the interior of the segment $[AD]$ and if B lies in the interior of the segment $[AC]$, then all of the four points A, B, C, and D lie on one straight line AD. (see Fig. 3.2) Applying the lemma 1.2, we choose a point E not lying on the line AD. Then we apply the axiom A10 and find a point F on the line BE such that the point E lies between F and B. Let's draw the line FC and consider the triangle ABE. The line FC does not intersect the sides $[AB]$ and $[BE]$ of this triangle. Indeed, the lines AB and FC intersect at the point C outside the segment $[AB]$. The point F is the intersection of the lines BE and FC, it lies outside the segment $[BE]$. If the line FC would intersect the segment $[AE]$ at an interior point, then due to Pasch's axiom A12 it would intersect on of the segments $[AB]$ or $[BE]$. However, it is not so. Therefore the line FC has no common points with the segment $[AE]$.

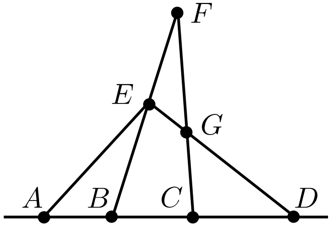


Fig. 3.2

Now let's draw the line DE that intersects the line FC at some point G. Then we consider the triangle AED. The line FC intersects the side $[AD]$ of this triangle at the interior point C, but it has no common points with the side $[AE]$. Applying Pasch's axiom A12, we find that G is an interior point of the segment $[ED]$.

In the next step we consider the triangle BED. The line FC crosses its side $[ED]$ at the interior point G. The intersection of the line FC with the line BE is the point F. It is outside the segment $[BE]$. Therefore, due to Pasch's axiom A12 the point C, which is the intersection of the lines FC and BD, should be an interior point of the segment $[BD]$. Thus we have proved the first proposition of the lemma 3.2 saying that C lies between the points B and D.

Note that now we can apply the previous lemma 3.1. Indeed, the point B lies between A and C, while the point C lies between

B and D. From the lemma 3.1 we derive that B lies in the interior of the segment $[AD]$. Thus we have proved the second proposition of the lemma 3.2. $\square$

THEOREM 3.1. *If a point B lies between two other points A and C, then the segments $[AB]$ and $[BC]$ are subsets of the segment $[AC]$.*

PROOF. Let's prove the proposition of the theorem concerning the segment $[AB]$. Remember that a segment consists of two ending points and of all points lying between these ending points. For the point A we have $A \in [AC]$. The point B lies between the points A and C. Therefore B is an interior point of the segment $[AC]$, i. e. $B \in [AC]$.

Let X be an arbitrary interior point of the segment $[AB]$. Then X lies between A and B. While the point B lies between A and C. In this case the lemma 3.2 is applicable. It yields $X \in [AC]$. Thus, we have proved that $[AB] \subset [AC]$.

In order to prove the second proposition $[BC] \subset [AC]$ it is sufficient to exchange the notations of the points A and C and then apply the first proposition $[AB] \subset [AC]$, which is already proved, and afterward return to the initial notations. $\square$

THEOREM 3.2. *If a point B lies between two other points A and C, then the segment $[AC]$ is the union of the segments $[AB]$ and $[BC]$.*

PROOF. According to the previous theorem, the segments$[AB]$ and $[BC]$ are subsets of the segment $[AC]$. Therefore, we have

$$[AB] \cup [BC] \subset [AC]. \tag{3.1}$$

Let's prove the opposite inclusion $[AC] \subset [AB] \cup [BC]$. The ending points of the segment $[AC]$ and the point B belong to the union $[AB] \cup [BC]$. Therefore, we consider some arbitrary interior point of the segment $[AC]$, different from the point B. Let's denote it X.

If $X \notin [AB]$, then due to the theorem 2.4 exactly one of the following two conditions is fulfilled: $A \in [BX]$ or $B \in [AX]$. First of these condition combined with $X \in [AC]$ allows us to apply the lemma 3.1. From this lemma we derive $A \in [BC]$ and $X \in [BC]$. But $A \in [BC]$ contradicts the fact that B is an interior point of the segment $[AC]$. Hence, we should study the second condition $B \in [AX]$. When combined with $X \in [AC]$, it allows us to apply the lemma 3.2. From the lemma 3.2 we derive $B \in [AC]$ and $X \in [BC]$.

Thus, for an arbitrary interior point $X \neq B$ of the segment $[AC]$ we have shown that $X \notin [AB]$ implies $X \in [BC]$. Hence, the required inclusion $[AC] \subset [AB] \cup [BC]$ is proved. When combined with (3.1) it yields the equality $[AB] \cup [BC] = [AC]$. The proof of the theorem 3.2 is complete. □

THEOREM 3.3. *If a point B lies between two other points A and C, then the intersection of the segments $[AB]$ and $[BC]$ consists of exactly one point B.*

PROOF. The point B is an ending point for both segments $[AB]$ and $[BC]$. Therefore, this point belongs to the intersection $[AB] \cap [BC]$. The ending points A and C do not belong to the intersection $[AB] \cap [BC]$ since $A \notin [BC]$ and $C \notin [AB]$. Hence, each point X of the intersection $[AB] \cap [BC]$ distinct from B should be an interior point of the segments $[AB]$ and $[BC]$.

Let X be an interior point of the segment $[AB]$. Then from the conditions $B \in [AC]$ and $X \in [AB]$, applying the lemma 3.2, we derive $X \in [AC]$ and $B \in [XC]$. Due to the theorem 2.4 the condition $B \in [XC]$ excludes the condition $X \in [BC]$. Thus, the segment $[AB]$ cannot have common interior points with the segment $[BC]$. □

§ 4. Directions. Vectors on a straight line.

Let's consider a set of n on some straight line a. We enumerate these points denoting them $A_1, \ldots, A_n$. Let's call $A_1, \ldots, A_n$ a *monotonic sequence of points* on a line if $n \geq 3$ and if each point

A_i lies between the points A_{i-1} and A_{i+1} for all $i = 2, \ldots, n-1$. The points of a monotonic sequence $A_1, \ldots, A_n$ determine the family of $n - 1$ segments

$$[A_1, A_2],\ [A_2, A_3],\ \ldots,\ [A_{n-1}, A_n]. \tag{4.1}$$

Adjacent segments in (4.1) have non-empty intersections consisting of one point: $[A_i, A_{i+1}] \cap [A_{i+1}, A_{i+2}] = \{A_{i+1}\}$. This fact follows from the theorem 3.3. Applying the theorem 3.2, we get

$$[A_i, A_{i+1}] \cup [A_{i+1}, A_{i+2}] = [A_i, A_{i+2}].$$

Applying this theorem several times, we find

$$[A_i, A_{i+m}] = \bigcup_{q=1}^{m} [A_{i+q-1}, A_{i+q}]. \tag{4.2}$$

From (4.2) one can conclude that $A_i \in [A_{i-q}, A_{i+k}]$. In other words, the point A_i lies between the points A_{i-q} and A_{i+k}. Moreover, the following relationship is valid:

$$[A_i, A_{i+1}] \cap [A_j, A_{j+1}] = \varnothing \quad \text{for } j \geqslant i + 2. \tag{4.3}$$

In order to prove (4.3) we use the fact that for $j \geqslant i+2$ the point A_j lies between the points A_{i+1} and A_{j+1}. Therefore, from the theorem 3.1, we derive $[A_j, A_{j+1}] \subset [A_{i+1}, A_{j+1}]$. And, applying the theorem 3.3, we get

$$[A_i, A_{i+1}] \cap [A_j, A_{j+1}] \subset [A_i, A_{i+1}] \cap [A_{i+1}, A_{j+1}] = \{A_{i+1}\}.$$

But the point A_{i+1} does not belong to the segment $[A_j, A_{j+1}]$. Therefore, the intersection of the segment $[A_i, A_{i+1}]$ and the segment $[A_j, A_{j+1}]$ is empty, which is in concordance with the formula (4.3).

THEOREM 4.1. *Let $A_1, \ldots, A_n$ be a monotonic sequence of points on a straight line and let B be some point of this line coinciding with none of the points $A_1, \ldots, A_n$. Then one can join the point B to the points $A_1, \ldots, A_n$ and enumerate the resulting set of points so that the monotonic sequence of points $A_1, \ldots, A_{n+1}$ will be formed.*

PROOF. Let's consider the three points A_1, A_n, and B. According to the theorem 2.4, exactly one of the following three conditions is fulfilled:

$$(A_1 \blacktriangleright A_n \blacktriangleleft B), \quad (A_1 \blacktriangleright B \blacktriangleleft A_n), \quad (B \blacktriangleright A_1 \blacktriangleleft A_n). \tag{4.4}$$

If the first condition is valid, we denote $B = A_{n+1}$ and immediately get the required monotonic sequence $A_1, \ldots, A_{n+1}$.

If the second condition (4.4) is valid, then the point B lies in the interior of the segment $[A_1, A_n]$ and does not coincide with $A_1, \ldots, A_n$. But from the relationship (4.2) we get

$$[A_1, A_n] = \bigcup_{i=1}^{n-1} [A_i, A_{i+1}], \tag{4.5}$$

the segments in the right hand side of this equality intersecting only by their ending points. Hence, the point B is an interior point for exactly one of the segments in the right hand side of (4.5). Assume that $B \in [A_q, A_{q+1}]$. We advance by one the numbers of the points $A_{q+1}, \ldots, A_n$:

$$A_{q+1} \to A_{q+2}, \ \ldots, \ A_n \to A_{n+1}.$$

Then assign $B = A_{q+1}$ and get the required monotonic sequence $A_1, \ldots, A_{n+1}$.

In the case where the third condition (4.4) is fulfilled we need to advance the numbers in the whole sequence $A_1, \ldots, A_n$:

$$A_1 \to A_2, \ \ldots, \ A_n \to A_{n+1}.$$

Then we assign $B = A_1$ and as a result we obtain the required monotonic sequence of points $A_1, \ldots, A_{n+1}$ including the point B and all of the initial points $A_1, \ldots, A_n$. $\square$

THEOREM 4.2. *Any set of n points, where $n \geqslant 3$, lying on one straight line can be enumerated so that a monotonic sequence of points $A_1, \ldots, A_n$ will be produced.*

PROOF. Let's choose some three point from the given set of n points on a straight line. According to the theorem 2.4, exactly one of the chosen three points lies between two others. We denote it A_2, while two other points are denoted A_1 and A_3. As a result we get the monotonic sequence of three points A_1, A_2, A_3. The rest is to add step by step the other points of the given set, relying on the theorem 4.1 in each step. $\square$

For any set of $n \geqslant 3$ points on a straight line there are exactly two ways of numbering these points converting them into a monotonic sequence of points. If one of these two numberings $A_1, \ldots, A_n$ is given, the other numbering $B_1, \ldots, B_n$ is obtained from the first one as follows:

$$B_1 = A_n,\ B_2 = A_{n-1},\ \ldots,\ B_n = A_1. \tag{4.6}$$

DEFINITION 4.1. A segment $[AB]$ of a straight line is called a *directed segment* or a *vector*, if one of its ending points is somehow distinguished with respect to the other.

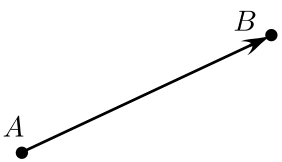


Fig. 4.1

The distinguished ending point of a vector is usually marked by an arrow in drawing. This point is called the true *ending point*, while the other (not distinguished) ending point is called the *starting point* of a vector. An arrow is also used for denoting vectors in writing: $\overrightarrow{AB}$. Note that $[AB]$ and $[BA]$ are two equivalent notations for the same segment. However, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are two different vectors.

Each vector defines a direction on a straight line. It is important to be able to compare two directions given by two vectors.

Definition 4.2. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lying on one straight line are called *codirected* if there is a monotonic sequence of points $A_1, \ldots, A_n$ including the points $A = A_i$, $B = A_k$, $C = A_j$, and $D = A_q$ such that $\operatorname{sign}(k - i) = \operatorname{sign}(q - j)$.

Note that adding new points to a monotonic sequence of points $A_1, \ldots, A_n$ as described in the theorem 4.1 does not change the signs of $(k - i)$ and $(q - j)$. Renumbering the points $A_1, \ldots, A_n$ as described in (4.6) changes these signs to opposite ones:

$$\operatorname{sign}(k - i) \to -\operatorname{sign}(k - i), \qquad \operatorname{sign}(q - j) \to -\operatorname{sign}(q - j).$$

Thus, we see that the equality $\operatorname{sign}(k-i) = \operatorname{sign}(q-j)$ being valid or not does not depend on a particular choice of the monotonic sequence of points that includes the starting and ending points of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Therefore, in order to verify if two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are codirected or not it is sufficient to enumerate the set of starting and ending points of these vectors as described in the theorem 4.2.

The codirectedness is a binary relation in the set of vectors lying on one line. This relation possesses the following properties:

(1) $\overrightarrow{AB} \upuparrows \overrightarrow{AB}$ for any vector $\overrightarrow{AB}$;
(2) $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ implies $\overrightarrow{CD} \upuparrows \overrightarrow{AB}$;
(3) $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{CD} \upuparrows \overrightarrow{EF}$ imply $\overrightarrow{AB} \upuparrows \overrightarrow{EF}$;
(4) if a vector $\overrightarrow{AB}$ is not codirected with $\overrightarrow{CD}$, while $\overrightarrow{CD}$ is not codirected with $\overrightarrow{EF}$, then $\overrightarrow{AB} \upuparrows \overrightarrow{EF}$.

The properties (1)–(4) are easily proved if one considers some monotonic sequence of points $A_1, \ldots, A_n$, including all of the points A, B, C, D, E, and F. The first three of these properties mean that the codirectedness relation is reflective, symmetric, and transitive. The fourth property shows that if we factorize the vectors on a straight line with respect to this relation, we get only two equivalence classes, each corresponding one of two possible directions on this line.

Assume that some vector $\overrightarrow{MN}$ on a straight line a is fixed. Let's agree to call *positive* the direction given by this vector. Then the opposite vector $\overrightarrow{NM}$ fixes the *negative* direction. With these prerequisites, for any two points X and Y on the line a we say that the point X *precedes* the point Y if if the vector $\overrightarrow{XY}$ is in positive direction, i. e. if $\overrightarrow{XY} \upuparrows \overrightarrow{MN}$. The relation of *precedence* is denoted as $X \prec Y$. It possesses the following properties, which are easy to verify:

(1) $A \prec B$ implies $A \neq B$;
(2) $A \prec B$ excludes $B \prec A$;
(3) $A \prec B$ and $B \prec C$ imply $A \prec C$;
(4) for any two points A and B exactly one of the two conditions $A \prec B$ or $B \prec A$ is fulfilled.

The properties (1)–(4) show that the precedence relation turns a line with a distinguished vector $\overrightarrow{MN}$ into a linearly ordered set.

Theorem 4.3. *On a straight line withe a fixed direction on it a point B lies between A and C if and only if one of the two conditions $A \prec B \prec C$ or $C \prec B \prec A$ is fulfilled.*

Exercise 4.1. *Prove that the only way of renumbering the points of a monotonic sequence $A_1, \ldots, A_n$ preserving the property of being monotonic is given by the formula* (4.6).

Exercise 4.2. *Verify the properties* (1)–(4) *for the relation of codirectedness of vectors.*

Exercise 4.3. *Verify the properties* (1)–(4) *for the relation of precedence of points on a straight line with a fixed direction.*

Exercise 4.4. *Prove the theorem 4.3.*

§ 5. Partitioning a straight line and a plain.

Let's consider some point O on a straight line a. According to the axiom A1, on the line a there is at least one point other than O. Let's denote it E. The vector $\overrightarrow{OE}$ fixes one of two possible directions on a and defines the precedence relation for the points of a. Let's consider two infinite intervals:

$$\begin{aligned}(O,+\infty)&=\{X\in a:\quad O\prec X\},\\(-\infty,O)&=\{X\in a:\quad X\prec O\}.\end{aligned}$$

Using the properties (1)–(4) of the binary relation of precedence, one can show that the intervals $(O,+\infty)$ and $(-\infty,O)$ do not intersect, while the whole line a is divided into three subsets:

$$a=(-\infty,O)\cup\{O\}\cup(O,+\infty).\tag{5.1}$$

Joining the point O to each of the infinite intervals $(-\infty,O)$ and $(O,+\infty)$, we get two sets which are called *half-lines* or *rays*:

$$[O,-\infty)=(-\infty,O)\cup\{O\},\qquad[O,+\infty)=\{O\}\cup(O,+\infty).$$

Thus, each point O on a line a determines the division of this line into two rays with one common point O.

Now let's consider a line a lying on a plane α. The theorem 1.6 is applicable to the plane α. It says that on any plane there are at least three points not lying on one line. Hence the set $\alpha\setminus a$ is not empty. Let's define an equivalence relation on $\alpha\setminus a$ by setting $A\sim B$ if $A=B$ or if the segment $[AB]$ has no common points with the line a. The reflexivity ans symmetry of such binary relation are obvious. The rest is to verify its transitivity.

Let $A\sim B$ and $B\sim C$. If $A=B$ or if $B=C$, then $A\sim C$ is a trivial consequence of one of the relations $A\sim B$ or $B\sim C$. The coincidence $A=C$ implies $A\sim C$ by itself. Therefore, we can assume that A, B, and C are three distinct points. Under this assumption let's consider two cases:

(1) where the points A, B, and C lie on one straight line;
(2) where A, B, and C do not lie on one straight line.

In the first case if we assume that the points A and C are not equivalent, then the lines AC and a intersect at some point O interior for the segment $[AC]$. Let's define a positive direction on the line AC by means of the vector $\overrightarrow{OA}$. Then $C \prec O \prec A$. The point B does not lie on the line a, therefore, $B \neq O$. Hence, B belongs to one of the intervals $(-\infty, O)$ or $(O, +\infty)$. If $B \in (-\infty, O)$, then $B \prec O \prec A$, which contradicts the condition $A \sim B$. If $B \in (O, +\infty)$, then $C \prec O \prec B$, which contradicts the condition $B \sim C$. In both cases the assumption of non-equivalence of A and C leads to a contradiction. Therefore, the required condition $A \sim C$ is fulfilled.

In the second case, assuming that A and C are not equivalent, we find that the line a passing through none of the points A, B, and C intersects the segment $[AC]$ at some interior point O. Then due to Pasch's axiom A12 it should intersect one of the segments $[AB]$ or $[BC]$ at an interior point. This contradicts to the fact that both conditions $A \sim B$ and $B \sim C$ are fulfilled simultaneously. The contradiction obtained proves that $A \sim C$.

The above equivalence relation determines the division of the set $\alpha \setminus a$ into classes. As appears, the number of such classes is equal to two. Taking into account the axiom A1, let's choose some point O lying on the line a. Then we choose and fix some point A lying on the plane α, but not lying on the line a. Let's draw the line AO and apply the axiom A10 to the points A and O on this line. As a result we find a point B on the line AO such that the point O lies in the interior of the segment $[AB]$.

The points A and B belong to the set $\alpha \setminus a$. They are not equivalent since the segment $[AB]$ intersects the line a at the point O. Hence, the equivalence classes $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are distinct. Let's prove that an arbitrary point X of the set $\alpha \setminus a$ belongs to one of these classes. Let's study two cases:

(1) where the point X lies on the line AO;
(2) where the point X does not lie on the line AO.

Naturally, we can assume that the point X differs from A and

B. In the first case the vector $\overrightarrow{OA}$ fixes one of the two possible directions on the line AO and determines a precedence relation on this line. Due to the division (5.1) the point X is in one of the infinite intervals $(-\infty, O)$ or $(O, +\infty)$.

If $X \in (-\infty, O)$, then $X \prec O$ and $B \prec O$. Hence, applying the theorem 4.3 to the points X, B, and O, we conclude that the point O cannot lie in the interior of the segment $[BX]$. Therefore, we have $X \in \mathrm{Cl}(B)$.

If $X \in (O, +\infty)$, then $O \prec X$ and $O \prec A$. Hence, applying the theorem 4.3 again, we get $X \in \mathrm{Cl}(A)$.

In the case, where the point X does not lie on the line OA, we can consider the triangle ABX lying on the plane α. The line a lies on the same plane and does not passes through the points A, B, and X. This line intersects the side $[AB]$ in the interior point O. Let's apply the theorem 2.5, which strengthens Pasch's axiom. According to this theorem, the line a intersects exactly one of two remaining sides of the triangle ABX — the side $[AX]$ or the side $[BX]$. If a intersects $[AX]$, then a does not intersect $[BX]$ and $X \in \mathrm{Cl}(B)$. Otherwise, if a intersects $[BX]$, then a does not intersect $[AX]$ and we have $X \in \mathrm{Cl}(A)$.

Let's denote $a_+ = \mathrm{Cl}(A)$ and $a_- = \mathrm{Cl}(B)$. The above considerations show that the line a lying on the plane α determines the division of this plane into three subsets:

$$\alpha = a_- \cup a \cup a_+. \tag{5.2}$$

The division (5.2) is analogous to the division (5.1). The subsets a_- and a_+ are called *open half-planes*. Extending the analogy with (5.1), we define *closed half-planes*:

$$\overline{a_-} = a_- \cup a, \qquad\qquad \overline{a_+} = a_+ \cup a.$$

Let's consider two non-coinciding straight lines a and b intersecting at a point O. According to the theorem 1.5, such lines fix a unique plane α containing both of them. Each of the lines

a and b determines a division of the plane alpha α into two half-planes. The intersection of two closed half-planes is called an *angle*. Let's choose a point A other than O on the line a. In a similar way, on the line b we choose a point B, different from O. The point B does not lie on the line a, hence, it belongs only to one of the closed half-planes — to $\overline{a_+}$ or to $\overline{a_-}$. For the sake of certainty assume that $\overline{a_+}$ is that very half-plane which contains the point B, while by $\overline{b_+}$ we denote the half-plane containing the point A. The angle produced as the intersection of the closed half-planes $\overline{a_+}$ and $\overline{b_+}$ is usually denoted as follows:

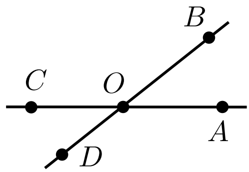


Fig. 5.1

$$\angle AOB = \overline{a_+} \cap \overline{b_+}. \tag{5.3}$$

Applying the axiom A10, now we choose a point C on the line a such that the point O lies between A and C. A point D on the line b is chosen in a similar way. The lines a and b define four angles at a time on the plane α:

$$\begin{aligned} \angle AOB &= \overline{a_+} \cap \overline{b_+}, & \angle BOC &= \overline{a_+} \cap \overline{b_-}, \\ \angle COD &= \overline{a_-} \cap \overline{b_-}, & \angle DOA &= \overline{a_-} \cap \overline{b_+}. \end{aligned}$$

The points A and B marking the half-planes $\overline{a_+}$ and $\overline{b_+}$ play equal roles in defining the angle $\angle AOB$. Therefore $\angle AOB$ and $\angle BOA$ are different notations for the same angle.

Let's consider the angle $\angle AOB$ from (5.3). The intersection of the open half-planes $a_+ \cap b_+$ is called the *interior* of the angle $\angle AOB$. The point O determines the division of the lines a and b into four closed half-lines (four rays). We denote them as follows:

$$[OA\rangle, \qquad [OB\rangle, \qquad [OC\rangle, \qquad [OD\rangle.$$

The previous notations $[O, +\infty)$ and $[O, -\infty)$ for rays are convenient only if we consider rays lying on one fixed line.

THEOREM 5.1. *Any angle $\angle AOB$ is the union of its interior and two rays $[OA\rangle$ and $[OB\rangle$.*

The rays $[OA\rangle$ and $[OB\rangle$ are called the *sides* of the angle $\angle AOB$, while the point O is called its *vertex*. The angles $\angle AOB$ and $\angle COB$ on Fig. 5.1 have the common vertex O and the common side $[OB\rangle$, while the other sides of these angles $[OA\rangle$ and $[OC\rangle$ lie on one line a and intersect at the unique point O. Such angles are called *adjacent angles*.

The union of two adjacent angles is a half-plane. Indeed, let's consider the union of the angles $\angle AOB$ and $\angle COB$:

$$\begin{gathered}\angle AOB \cup \angle COB = (\overline{a_+} \cap \overline{b_+}) \cup (\overline{a_+} \cap \overline{b_-}) = \\ = \overline{a_+} \cap (\overline{b_+} \cup \overline{b_-}) = \overline{a_+} \cap \alpha = \overline{a_+}.\end{gathered} \tag{5.4}$$

A closed half-plane $\overline{a_+}$ with a marked point O on the line a can be treated as an angle. Such an angle is called a *straight angle*. One should be careful when using the notation $\angle AOC$ for a straight angle since this notation fits for $\overline{a_+}$ and for $\overline{a_-}$ either.

Let's consider the angles $\angle AOB$ and $\angle COD$ on Fig. 5.1. The sides $[OC\rangle$ and $[OD\rangle$ of the second angle complement the sides $[OA\rangle$ and $[OB\rangle$ of the first one up to the lines a and b. Such angles are called *vertical angles*.

THEOREM 5.2. *Any three points A, B, and O not lying on one straight line determine exactly one angle $\angle AOB$ with the vertex at the point O.*

Let's consider three points A, B, and C not lying on one straight line. We denote by a the line BC, by b the line AC, and by c the line AB. The lines a, b, and c lie on the plane α determined by the points A, B, and C according to the axiom A4. Let's denote by a_+ the half-plane on the plane α determined by the line a and possessing the point A. In a similar way, let $B \in b_+$ and $C \in c_+$. The *triangle ABC* is the set of points of the plane α obtained as the intersection of three closed half-planes $\overline{a_+}$, $\overline{b_+}$, and $\overline{c_+}$. Before now, saying a triangle ABC, we could understand the collection of three segments $[AB]$, $[BC]$, and $[AC]$ connecting some three points A, B, and C not lying on one straight line. Now a triangle ABC is equipped with the interior. The *interior* of a triangle ABC is the intersection of three open half-planes $a_+ \cap b_+ \cap c_+$.

THEOREM 5.3. *A triangle ABC is the union of its interior and its three sides $[AB]$, $[BC]$, and $[AC]$.*

EXERCISE 5.1. *Prove the theorems 5.1, 5.2, and 5.3 by proving the the following lemma before it .*

LEMMA 5.1. *For any points A and B the intersection of the rays $[AB\rangle$ and $[BA\rangle$ is the segment $[AB]$.*

EXERCISE 5.2. *Verify the calculations (5.4) using some set-theoretic considerations.*

EXERCISE 5.3. *Let A, B, and C be three arbitrary points not lying on one straight line. Prove that the interior of the triangle ABC is not empty.*

§ 6. Partitioning the space.

Let α be a plane. According to the lemma 1.1, there is a point not lying on the plane α. Therefore, the set $\mathbb{E} \setminus \alpha$ is not empty. We define an equivalence relation in $\mathbb{E} \setminus \alpha$ by setting $A \sim B$ if $A = B$ or if the segment $[AB]$ does not intersect the plane α. The reflexivity and symmetry of this binary relation are obvious.

Let's verify its transitivity. Assume that $A \sim B$ and $B \sim C$. If $A = B$ or $B = C$, then $A \sim C$ is a trivial consequence of one of the conditions $A \sim B$ or $B \sim C$. The coincidence $A = C$ implies $A \sim C$ by itself. Therefore we should consider a general case, where A, B, and C are three distinct points. Under this assumption we study two cases:

(1) where the points A, B, and C lie on one straight line;
(2) where A, B, and C do not lie on one straight line.

In the first case if we assume that A and C are not equivalent, then the line AC intersects the plane α at some point O lying in the interior of the segment $[AC]$. According to the theorem 1.4, the point O is the unique common point of the line AC and the plane α. Let's define a positive direction on the line AC by means of the vector $\overrightarrow{OA}$. Then $C \prec O \prec A$. The point B does not lie on the plane α, therefore, $B \neq O$. Hence, B belongs to one of the intervals $(-\infty, O)$ or $(O, +\infty)$ determined by the division (5.1). If $B \in (-\infty, O)$, then $B \prec O \prec A$, which contradicts the condition $A \sim B$. If $B \in (O, +\infty)$, then $C \prec O \prec B$, which contradicts the condition $B \sim C$. Thus, the assumption that A and C are not equivalent leads to a contradiction. Therefore the required condition $A \sim C$ is fulfilled.

In the second case one can draw a plane β through the points A, B, and C. This fact follows from the axiom A4. Here the assumption that A and C are not equivalent means that the line AC intersects the plane α at some interior point O of the segment $[AC]$. But $[AC] \subset \beta$, therefore the non-coinciding planes α and β have the common point O. Let's apply the theorem 1.2 and denote by a the line obtained as the intersection $\alpha \cap \beta$. The line a lies on the plane β and passes through none of the points A, B, and C. It intersects the segment $[AC]$ at the interior point O. Then due to Pasch's axiom A12 it should intersect one of the segments $[AB]$ or $[BC]$ at some interior point. But this contradicts to the fact that the conditions $A \sim B$ and $B \sim C$ are fulfilled simultaneously. The contradiction obtained proves that the points A and C are equivalent.

The above equivalence relation divides the set $\mathbb{E} \setminus \alpha$ into equivalence classes. Here, as in the case of partitioning a plane, the number of equivalence classes is equal to two. They are denoted α_+ and α_- and are called *open half-spaces*. Thus, each plane α defines the division of the space

$$\mathbb{E} = \alpha_- \cup \alpha \cup \alpha_+ \tag{6.1}$$

analogous to the divisions (5.1) and (5.2) for a line and for a plane respectively. Relying on (6.1) we define *closed half-spaces*

$$\overline{\alpha_-} = \alpha_- \cup \alpha, \qquad\qquad \overline{\alpha_+} = \alpha_+ \cup \alpha.$$

LEMMA 6.1. *If A, B, C, and D are four points not lying on one plane, then neither three of them can lie on one straight line.*

Let A, B, C, and D are some four points not lying on one plane. The existence of at least one of such quadruples of points is granted by the axiom A8. Due to the lemma 6.1 and the axiom A4 each three of these four points determine some plane. Let's denote these planes as follows:

$$\alpha = BCD, \qquad \beta = ACD, \qquad \gamma = ABD, \qquad \delta = ABC.$$

Each of the four planes α, β, γ, and δ determine two half-spaces. Let's choose the notations for these half-spaces so that the following conditions are fulfilled:

$$A \in \alpha_+, \qquad B \in \beta_+, \qquad C \in \gamma_+, \qquad D \in \delta_+.$$

The intersection of closed half-spaces $\overline{\alpha_+}$, $\overline{\beta_+}$, $\overline{\gamma_+}$, and $\overline{\delta_+}$ is called a *tetrahedron*. The points A, B, C, and D are called *vertices* of the tetrahedron $ABCD$, the segments $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, and $[DC]$ are called its *edges*, while the triangles ABC, ABD, ACD, and BCD are called the *faces* of the tetrahedron $ABCD$. The intersection of the open half-spaces α_+, β_+, γ_+, and δ_+ is called the *interior* of the tetrahedron $ABCD$.

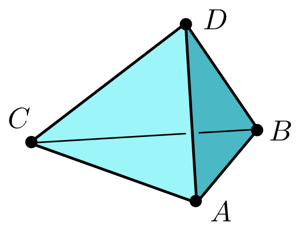


Fig. 6.1

A tetrahedron is called a *three-dimensional simplex*. A triangle is a *two-dimensional simplex*, while a segment is a *one-dimensional simplex*. A point is a *zero-dimensional simplex*. Such a terminology is popular in algebraic topology (see [4]).

EXERCISE 6.1. *Prove that the number of classes into which the set* $\mathbb{E} \setminus \alpha$ *is broken by the above equivalence relation is two.*

EXERCISE 6.2. *Prove the lemma 6.1. For this purpose use the axioms of incidence and the results of § 1.*

LEMMA 6.2. *Let* $\angle AOB$ *be the angle determined by three points* A*,* O*, and* B *not lying on one straight line. Then a ray coming out from the point* O *lies within the angle* $\angle AOB$ *if and only if it intersects the segment* $[AB]$.

EXERCISE 6.3. *Prove the lemma 6.2. For this purpose complete the ray* $[OA\rangle$ *up to a whole straight line and upon choosing a point* C *on the line* OA *not belonging to the ray* $[OA\rangle$ *do draw the triangle* ABC*.*

EXERCISE 6.4. *Let* A*,* B*,* C*, and* D *be four points not lying on one plane. Show that the interior of the tetrahedron* $ABCD$ *is not empty.*

EXERCISE 6.5. *Prove that each tetrahedron* $ABCD$ *is the union of its interior and the triangles* ABC*,* ABD*,* ACD*, and* BCD*.*

CHAPTER III

AXIOMS OF CONGRUENCE.

§ 1. Binary relations of congruence.

The axioms of congruence form the third group of Euclid's axioms. In formulating these axioms it is assumed that in the set of all straight line segments a binary relation is defined which is called the *congruence*. A similar binary relation is assumed to be given in the set of all angles. It is also called the *congruence*, though the congruence of segments and the congruence of angles are certainly two different binary relations. For denoting the congruence of segments and the congruence of angles usually the same sign $\cong$ is used.

A straight line segment is given by two points. An angle can be given by three points. Therefore, the congruence of segments can be treated as a tetrary (or quadruple) relation in the set of points, while the congruence of angles can be treated as a hexary (or sextuple) relation in the set of points. Such a treatment would be more consistent from the formal point of view. But it is less visual and, hence, is less convenient.

§ 2. Congruence of segments.

Axiom A13. *Any straight line segment $[AB]$ is congruent to itself and for any ray beginning at an arbitrary point C there is a unique point D on this ray such that $[AB] \cong [CD]$.*

Axiom A14. *The binary relation of congruence for segments is transitive, i. e.* $[AB] \cong [CD]$ *and* $[CD] \cong [EF]$ *imply* $[AB] \cong [EF]$.

The reflexivity of the congruence of segments is stated explicitly in the axiom A13, while the transitivity of this relation forms the content of the axiom A14. Let's prove its symmetry.

Lemma 2.1. *The binary relation of congruence for segments is symmetric, i. e.* $[AB] \cong [CD]$ *implies* $[CD] \cong [AB]$.

Proof. Assume that $[AB] \cong [CD]$. Let's apply the axiom A13 to the segment $[CD]$ and to the ray $[AB\rangle$ beginning at the point A. From this axiom we get that there is a point E on the ray $[AB\rangle$ such that $[CD] \cong [AE]$. Since $[AB] \cong [CD]$ and $[CD] \cong [AE]$, due to the axiom A14 we derive $[AB] \cong [AE]$.

Now we apply the axiom A13 to the segment $[AB]$ and to the ray $[AB\rangle$. It says that the point E on the ray $[AB\rangle$ such that $[AB] \cong [AE]$ is unique. But $[AB] \cong [AB]$. Therefore, the point E coincides with B. Hence, $[CD] \cong [AB]$. Lemma is proved. $\square$

Thus, due to the axioms A13 and A14 and due to the lemma 2.1 proved just above the relation of congruence is an equivalence relation in the set of all straight line segments.

Axiom A15. *Let* B *be a point lying between* A *and* C *on a straight line* AC*, while* L *be a point lying between* K *and* M *on a straight line* KM*. Then the following propositions are valid:*

(1) $[AB] \cong [KL]$ *and* $[BC] \cong [LM]$ *imply* $[AC] \cong [KM]$;
(2) $[AB] \cong [KL]$ *and* $[AC] \cong [KM]$ *imply* $[BC] \cong [LM]$.

Note that the propositions (1) and (2) under the assumptions of the axiom A15 can be complemented with one more proposition of the same sort:

(3) $[AC] \cong [KM]$ and $[BC] \cong [LM]$ imply $[AB] \cong [KL]$.

The proposition (3) is obtained from the proposition (2) by reformulating it upon exchanging the notations of points: A with C and K with M.

Under the assumptions of the axiom A15 the point B breaks the segment $[AC]$ into two segments $[AB]$ and $[BC]$, whose intersection consists of the unique point B and whose union coincides with the whole segment $[AC]$ (see theorems 3.2 and 3.3 from Chapter II). In this situation we say that the segment $[AC]$ is *composed* of the segments $[AB]$ and $[BC]$ or, in other words, $[AC]$ is the *sum* of $[AB]$ and $[BC]$. Therefore, the first proposition of the axiom A15 can be shorten to the following one: a segment composed of segments congruent to $[AB]$ and $[BC]$ is congruent to their sum $[AC]$. If we call $[BC]$ the *difference* of the segments $[AC]$ and $[AB]$, then the second proposition of the axiom A15 can be stated as follows: the difference of segments congruent to $[AC]$ and $[AB]$ is congruent to their difference $[BC]$.

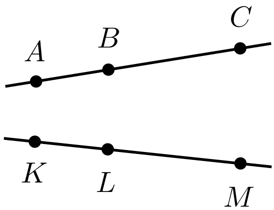


Fig. 2.1

THEOREM 2.1. *Let the segment $[KM]$ be congruent to the segment $[AC]$ and let B be an arbitrary point of the line AC distinct from A and C. Then on the line KM there is a unique point L such that $[KL] \cong [AB]$ and $[LM] \cong [BC]$.*

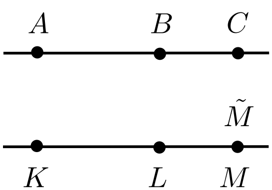


Fig. 2.2

PROOF. Let's consider three possible locations of the point B relative to the points A and C:

$$
\begin{aligned}
&(A \blacktriangleright B \blacktriangleleft C),\\
&(C \blacktriangleright A \blacktriangleleft B),\\
&(B \blacktriangleright C \blacktriangleleft A).
\end{aligned}
\tag{2.1}
$$

According to theorem 2.4 from Chapter II exactly one of the conditions (2.1) is necessarily fulfilled. If it is the first condition, we apply the axiom A13 to the segment $[AB]$ and to the ray $[KM\rangle$. As a result we find a unique point L on the ray $[KM\rangle$ such that $[KL] \cong [AB]$ (see Fig. 2.2). Then we consider the ray coming out from the point L in the direction opposite to the ray $[LK\rangle$. On this ray we find a point $\tilde{M}$ such that $[L\tilde{M}] \cong [BC]$. Then the following conditions are fulfilled:

$$(A \blacktriangleright B \blacktriangleleft C), \qquad\qquad (K \blacktriangleright L \blacktriangleleft \tilde{M}). \tag{2.2}$$

Due to (2.2) we can apply the item (1) of the axiom A15 to the points A, B, C, K, L, and $\tilde{M}$. It yields $[K\tilde{M}] \cong [AC]$. According to the premise of the theorem, $[KM] \cong [AC]$. Moreover, by construction both points M and $\tilde{M}$ lie on the same ray $[KL\rangle$ beginning at the point K. Hence, due to the axiom A13 we derive the coincidence of points $M = \tilde{M}$. Then

$$[KL] \cong [AB], \qquad\qquad [LM] \cong [BC]. \tag{2.3}$$

This means that L is a required point on the line KM. Let's prove its uniqueness. The second condition in (2.3) admits the existence of exactly two points L and $\tilde{L}$ satisfying this condition. The first of them lies on the ray $[MK\rangle$, it is the point L. The second point $\tilde{L}$ lies on the opposite ray coming out from the point M. If we assume that the point L is not fixed uniquely by the conditions (2.3), then the second point $\tilde{L}$ should also satisfy both conditions (2.3) simultaneously. Under this assumption on the ray coming out from the point $\tilde{L}$ in the direction opposite to the ray $[\tilde{L}K\rangle$ we choose a point $\tilde{M}$ such that $[\tilde{L}\tilde{M}] \cong [BC]$. From $[K\tilde{L}] \cong [AB]$ and $[\tilde{L}\tilde{M}] \cong [BC]$ due to the item (1) of the axiom A15 we derive $[K\tilde{M}] \cong [AC]$. But the point M, according to the premise if the theorem, satisfies the same condition $[KM] \cong [AC]$. This fact contradicts the axiom A13. The coincidence $M = \tilde{M}$ is excluded since M and $\tilde{M}$ by construction lie on different sides with respect to the point $\tilde{L}$.

Fig. 2.3

The contradiction obtained proves the uniqueness of the point L in the case where the first condition (2.1) is fulfilled.

Now let's study the second case of mutual disposition of the points A, B, and C in (2.1). Let's apply the axiom A13 to the ray coming out from the point K in the direction opposite to the ray $[KM\rangle$. As a result we get a point L on the line KM such that $[KL] \cong [AB]$. Upon combining $[KL] \cong [AB]$ and $[KM] \cong [AC]$ we apply the item (1) of the axiom A15. This yields $[LM] \cong [BC]$. Thus, L is a required point on the line KM. The rest is to prove the uniqueness of the point L.

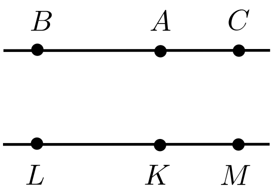


Fig. 2.4

The condition $[LM] \cong [BC]$ in (2.3) admits the existence of exactly two points L on the line KM satisfying this condition. The first of them is the above point L. The second one is the point $\tilde{L}$ that lies on the ray coming out from the point M in the direction opposite to the ray $[MK\rangle$. The assumption of non-uniqueness of the point L means that both points L and $\tilde{L}$ satisfy both conditions (2.3). Under this assumption we apply the axiom A13 to the ray $[KL\rangle$ and mark a point $\tilde{M}$ on this ray such that $[K\tilde{M}] \cong [AC]$. Now from $[K\tilde{L}] \cong [AB]$ and $[K\tilde{M}] \cong [AC]$, applying the item (1) of the axiom A15, we derive $[\tilde{L}\tilde{M}] \cong [BC]$. Then for two points M and $\tilde{M}$ on the ray $[\tilde{L}K\rangle$ we have $[\tilde{L}\tilde{M}] \cong [BC]$ and $[\tilde{L}M] \cong [BC]$, which contradicts the axiom A13. This contradiction proves the uniqueness of L for the second disposition of points in (2.1).

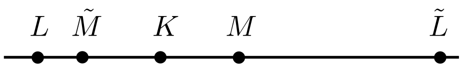


Fig. 2.5

The third disposition of the points A, B, and C in (2.1) does not require a special treatment. This disposition is reduced to the second one by simultaneous exchanging the notations of the points: A with C and K with M. $\square$

Assume that a segment $[AC]$ on a straight line a is congruent to a segment $[KM]$ on another straight line b. Using the theorem 2.1 proved above, one can define a mapping $f: a \to b$ by setting $f(A) = K$, $f(C) = M$, and determining $f(X)$ by means of the conditions $[AX] \cong [Kf(X)]$ and $[CX] \cong [Mf(X)]$ for all other points

The mapping $h: b \to a$ is constructed in a similar way. For it we set $h(K) = A$, $h(M) = C$, and we set $[KZ] \cong [Ah(Z)]$ and $[MZ] \cong [Ch(Z)]$ for all other points $Z \in b$. Due to the uniqueness of the point L in the theorem 2.1 the mappings f and h appear to be inverse to each other. In particular, this means that both of them are bijective.

Note, that following the proof of the theorem 2.1, one can establish the following fact characterizing f:

$$\begin{aligned}
&(A \blacktriangleright X \blacktriangleleft C) \text{ implies } (K \blacktriangleright f(X) \blacktriangleleft M),\\
&(C \blacktriangleright A \blacktriangleleft X) \text{ implies } (M \blacktriangleright K \blacktriangleleft f(X)),\\
&(X \blacktriangleright C \blacktriangleleft A) \text{ implies } (f(X) \blacktriangleright M \blacktriangleleft K).
\end{aligned} \tag{2.4}$$

In other words, the mutual disposition of the points K, M, and $f(X)$ on the line b mimics the disposition if the initial points A, C, and X on the line a.

Theorem 2.2. *Assume that a segment $[KM]$ of a straight line b is congruent to a segment $[AC]$ of a line a. Let's denote $f: a \to b$ the mapping given by the relationships $f(A) = K$, $f(C) = M$, and by the conditions $[AX] \cong [Kf(X)]$ and $[CX] \cong [Mf(X)]$ for all $X \in a$ distinct from A and C. Under these assumptions, if we introduce distinguished directions on the lines a and b by virtue of the vectors $\overrightarrow{AC}$ and $\overrightarrow{KM}$, then for any two points X and Y on the line a the following conditions are fulfilled:*

(1) *$X \prec Y$ implies $f(X) \prec f(Y)$;*
(2) *the segment $[XY]$ is congruent to the segment $[f(X)f(Y)]$.*

Proof. Let's study various cases of mutual disposition of the points X, Y, A, and C on the line a. If the point X or the point Y coincides with one of the points A or C, in this case the proposition of the theorem easily follows from (2.4) and from the way in which the mapping f is defined. For this reason, without loss of generality we can assume that X, Y, A, and C are four distinct points on the line a.

The points A and C break the line a into three parts: the ray $(-\infty, A]$, the segment $[AC]$, and the ray $[C, +\infty)$. If X and Y are in different parts of this partitioning, then the interval (XY) contains at least one of the points A or C. In this case the item (1) of the theorem can be derived from (2.4). The item (2) of the theorem 2.2 after that is proved by applying the first item from the axiom A15.

Now let's consider the case where the points X and Y lie on the segment $[AC]$. Then $X \prec Y$ implies $A \prec X \prec Y \prec C$. Applying the axiom A13, we choose a point $\tilde{Y}$ on the ray $[f(X)M\rangle$ such that $[f(X)\tilde{Y}] \cong [XY]$. Then, using the same axiom A13, we draw the segment $[\tilde{Y}\tilde{C}] \cong [YC]$ on the ray coming out from the point $\tilde{Y}$ in the direction opposite to the ray $[\tilde{Y}K\rangle$. Applying the item (1) of the axiom A15 to the segments $[Kf(X)]$ and $[f(X)\tilde{Y}]$, we derive $[K\tilde{Y}] \cong [AY]$. Then we apply the same item of the axiom A15 to the segments $[K\tilde{Y}]$ and $[\tilde{Y}\tilde{C}]$. It yields $[K\tilde{C}] \cong [AC]$. Hence, $\tilde{C} = M$, which implies $[\tilde{Y}M] \cong [YC]$. From this relationship we derive the coincidence $\tilde{Y} = f(Y)$.

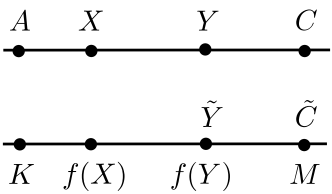


Fig. 2.6

The coincidences $\tilde{C} = M$ and $\tilde{Y} = f(Y)$ due to the above construction yield $K \prec f(X) \prec f(Y) \prec M$. This proves the first proposition of the theorem $f(X) \prec f(Y)$. The second proposition $[f(X)f(Y)] \cong [XY]$ follows from $[f(X)\tilde{Y}] \cong [XY]$ and from the coincidence $\tilde{Y} = f(Y)$.

Now let's consider the case, where the points X and Y lie on the ray $[C, +\infty)$. Here from $X \prec Y$ we derive $A \prec C \prec X \prec Y$.

Let's apply the axiom A13 to the ray coming out from the point $f(X)$ in the direction opposite to the ray $[f(X)M]$. We mark on this ray a point $\tilde{Y}$ such that $[f(X)\tilde{Y}] \cong [XY]$. Then we apply the first item of the axiom A15 to the segments $[Mf(X)]$ and $[f(X)\tilde{Y}]$. This yields $[M\tilde{Y}] \cong [CY]$, which in turn leads to the coincidence of points $\tilde{Y}$ and $f(Y)$. Now the relationships $f(X) \prec f(Y)$ and $[f(X)f(Y)] \cong [XY]$ are fulfilled by construction of the point $\tilde{Y}$.

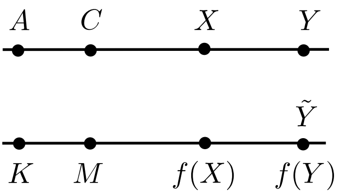


Fig. 2.7

The last case, where the points X and Y lie on the ray $(-\infty, A]$, does not require a special consideration. It is reduced to the second case if we exchange the notations of points A with C and K with M and if we change the distinguished directions on a and b for the opposite ones. □

§ 3. Congruent translation of straight lines.

Definition 3.1. A mapping $f : a \to b$ is called a *congruent translation* of a straight line a to a straight line b if for any two points X and Y on the line a the condition of congruence $[f(X)f(Y)] \cong [XY]$ is fulfilled.

Let f and h be two congruent translations of a straight line a to a straight line b. If at some two points A and B on te line a these mappings coincide

$$f(A) = h(A), \qquad\qquad f(B) = h(B),$$

then they coincide at all points $X \in a$, i.e. $f = h$. This fact is easily derived from the theorem 2.1. This theorem together with the theorem 2.2 show that congruent translations of lines do exist. Indeed, in order to define such a mapping $f : a \to b$ it is sufficient to choose two points A and B on the line a and construct the segment $[KM]$ congruent to $[AB]$ on the line b.

Assume that on a straight line a some point O is marked and one of two possible directions is distinguished. Then a can be broken into two rays $[O,-\infty)$ and $[O,+\infty)$. Assume that oh another straight line b some point Q is marked and some direction is distinguished. Then this line is also broken into two rays $[Q,-\infty)$ and $[Q,+\infty)$. Now we choose a point E_+ on the ray $[O,+\infty)$ and, using the axiom A13, on the line b we construct two segments $[QF_+]$ and $[QF_-]$ congruent to the segment $[OE_+]$. The segment $[QF_+]$ lies on the ray $[Q,+\infty)$, while the segment $[QF_-]$ lies in the opposite ray $[Q,-\infty)$. Because of the presence of two segments congruent to $[OE_+]$ we can define two mappings of congruent translation of the line a to the line b:

$$\begin{aligned} &f^+(O)=Q, && f^+(E_+)=F_+,\\ &f^-(O)=Q, && f^-(E_+)=F_-. \end{aligned} \tag{3.1}$$

Theorem 3.1. *For any point O on a straight line a with a distinguished direction and for any point Q on another straight line b with a distinguished direction there are exactly two mappings $f\colon a\to b$ performing congruent translation of the line a to the line b. The first of them f^+_{OQ} preserves the precedence of points, i. e. $X\prec Y$ implies $f^+_{OQ}(X)\prec f^+_{OQ}(Y)$. The second mapping inverts the precedence of points, i. e. $X\prec Y$ implies $f^-_{OQ}(Y)\prec f^-_{OQ}(X)$.*

Exercise 3.1. *Prove the theorem 3.1 by showing that the mappings f^+_{OQ} and f^-_{OQ} do not depend on a particular choice of the point $E_+\in[O,+\infty)$ in formulas (3.1) defining them. Moreover, show that these mappings remain unchanged if one changes simultaneously the distinguished directions on the lines a and b for opposite ones.*

Assume that the line b coincides with the line a. We choose two points O and Q and fix one of two possible directions on this line. In this case the mapping f^+_{OQ} is called the *congruent translation by the vector $\overrightarrow{OQ}$*. It is denoted $f^+_{OQ}=p_{OQ}$. The coincidence $O = Q$ makes a special case. In this special case the points O and Q do not define a vector (understood as an arrowhead segment), while the the mapping of congruent translation p_{OO} appears to be the identical mapping: $p_{OO} = \mathrm{id}$. The mapping f^{-}_{OQ} is not identical even in the case of coinciding points O and Q. For $O = Q$ the mapping f^{-}_{OO} is called the *inversion with respect to the point* O. It is denoted as $f^{-}_{OO} = i_O$.

THEOREM 3.2. *The mappings of congruent translation by vectors and the mappings of inversion satisfy the relationships*

$$\begin{aligned} &p_{BC} \circ p_{AB} = p_{AC}, && p_{AB} \circ i_C = i_C \circ p_{BA},\\ &i_C \circ i_C = \mathrm{id}, && i_A \circ i_B = p_{BC}, \text{ where } C = i_A(B). \end{aligned}$$

THEOREM 3.3. *Assume that on each of two straight lines a and b with distinguished directions two points are fixed: $O, \tilde{O} \in a$ and $Q, \tilde{Q} \in b$. Then we have*

$$f^{+}_{\tilde{O}\tilde{Q}} \circ = p_{Q\tilde{Q}} \circ f^{+}_{OQ} \circ p_{\tilde{O}O}, \qquad f^{-}_{\tilde{O}\tilde{Q}} = p_{Q\tilde{Q}} \circ f^{-}_{OQ} \circ p_{\tilde{O}O}.$$

A remark. By means of the small circle in theorems 3.2 and 3.3 we denote the operation of composing two mappings: $f \circ h(x) = f(h(x))$ (see § 7 in Chapter I).

EXERCISE 3.2. *Relying on the theorems 2.1, 2.2, and 3.1, prove the properties of mappings of congruent translations of lines stated in theorems 3.2 and 3.3.*

The domain of any of the above mappings of congruent translation is some straight line. At this moment we have no tools for extending this domain. The only exception is the inversion i_O. Assume that some point O in the space is fixed. For any point X different from O there is a unique line $a = OX$ passing through O and X. On this line the inversion mapping i_O is defined. Let's set $i(X) = i_O(X)$. For the point O itself we set $i(O) = O$. As a result we get the mapping $i\colon \mathbb{E} \to \mathbb{E}$ which is called the *inversion* or the *central symmetry* with the center at the point O.

§ 4. Slipping vectors. Addition of vectors on a straight line.

Definition 4.1. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lying on one straight line are called *equal* if they are codirected and if the segment $[AB]$ is congruent to the segment $[CD]$.

The equality of points, straight lines, planes, segments, and many other geometric forms is understood as pure coincidence. The equality of vectors, according to the definition 4.1, is of different nature.

Exercise 4.1. *Verify that the relation of equality of vectors is a binary relation of equivalence.*

Vectors understood as arrowhead segments are sometimes called *geometric vectors*. They have strictly fixed positions in the space. In contrast to geometric vectors, a *slipping vector* on a straight line is a class of mutually equivalent vectors in the sense of the definition 4.1. A slipping vector has many representatives lying on a given line. They are called *geometric realizations* of this slipping vector.

Theorem 4.1. *For any four points A, B, C, and D lying on one straight line $\overrightarrow{AB} = \overrightarrow{CD}$ implies $\overrightarrow{AC} = \overrightarrow{BD}$ and, conversely, $\overrightarrow{AC} = \overrightarrow{BD}$ implies $\overrightarrow{AB} = \overrightarrow{CD}$.*

Proof. Let's consider the first proposition of the theorem. Assume that $\overrightarrow{AB} = \overrightarrow{CD}$. We choose the direction of the vector $\overrightarrow{AB}$ for the positive direction on the line where both vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lie. Then the following relationships are fulfilled:

$$A \prec B, \qquad\qquad C \prec D. \tag{4.1}$$

From (4.1) we derive the complete list of possible mutual dispositions of the points A, B, C, and D:

$$A \prec B \prec C \prec D, \qquad C \prec D \prec A \prec B, \tag{4.2}$$

$$A \prec C \prec B \prec D, \qquad C \prec A \prec D \prec B, \tag{4.3}$$

$$A \prec C \prec D \prec B, \qquad C \prec A \prec B \prec D. \tag{4.4}$$

All of the above cases in (4.2), (4.3), and (4.4) are grouped into pairs. We can study only one case in each pair since the other case is obtained by transposition of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$, which does not change the proposition of the theorem in whole.

Let's show that the case (4.4) is impossible. It is not compatible with the condition $[AB] \cong [CD]$ which follows from the equality $\overrightarrow{AB} = \overrightarrow{CD}$. Applying the axiom A13, on the right of B we choose a point $\tilde{D}$ such that $[B\tilde{D}] \cong [AC]$. Combining this condition with the condition $[CD] \cong [AB]$, from the axiom A15 we derive the congruence of the segments $[AD] \cong [A\tilde{D}]$. This relationship contradicts the axiom A13 since both points D and $\tilde{D}$ lie on the same ray coming out from the point A. The contradiction obtained excludes the case (4.4) from our further consideration.

Fig. 4.1

Note that the first proposition of the theorem for the case (4.3) is equivalent to the second proposition for the case (4.2). Therefore, it is sufficient to consider only the case $A \prec B \prec C \prec D$ and prove both propositions of the theorem for this case. The codirectedness conditions $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{AC} \upuparrows \overrightarrow{BD}$ follow from the disposition of points $A \prec B \prec C \prec D$. From $[AB] \cong [CD]$ and from the obvious relationship $[BC] \cong [BC]$, applying the first item of the axiom A15, we derive $[AC] \cong [BD]$. Conversely, from $[AC] \cong [BD]$ and

Fig. 4.2

$[BC] \cong [BC]$, upon applying the item (2) of the axiom A15, we get $[AB] \cong [CD]$. The theorem is proved. $\square$

THEOREM 4.2. *The equality $p_{AB} = p_{CD}$ is valid if and only if $\overrightarrow{AB} = \overrightarrow{CD}$ in the sense of the definition 4.1.*

PROOF. Assume that p_{AB} and p_{CD} are the mappings of congruent translation on a straight line a and assume that $p_{AB} = p_{CD} = p$. Let's define the positive direction on the line a by means of the vector $\overrightarrow{AC}$ and thus define a relation of precedence for the points of this line a. Then $A \prec C$. Let's apply the mapping p to the points A and C and use the theorem 2.2:

$$p(A) \prec p(C), \qquad\qquad [p(A)p(C)] \cong [AC].$$

But $p(A) = p_{AB}(A) = B$ and $p(C) = p_{CD}(C) = D$. Hence, $A \prec C$ and $B \prec D$, which means that the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are codirected. Moreover, $[BD] \cong [AC]$, therefore, $\overrightarrow{AC} = \overrightarrow{BD}$. Applying the theorem 4.1, we derive the required equality of vectors $\overrightarrow{AB} = \overrightarrow{CD}$.

Now, conversely, assume that $\overrightarrow{AB} = \overrightarrow{CD}$. From this equality, applying the theorem 4.1, we derive $\overrightarrow{AC} = \overrightarrow{BD}$. Hence, we have $[AC] \cong [BD]$, while from $A \prec C$ it follows that $B \prec D$. Let's apply the mapping p_{AB} to the point C and denote $\tilde{D} = p_{AB}(C)$. Then, according to the theorem 2.2, we get $[AC] \cong [B\tilde{D}]$, while $A \prec C$ implies $B \prec \tilde{D}$. From $[AC] \cong [B\tilde{D}]$ and $[AC] \cong [BD]$ we derive $[BD] \cong [B\tilde{D}]$ and from $B \prec D$ and $B \prec \tilde{D}$ we conclude that the points D and $\tilde{D}$ lie on the same ray coming out from the point B. Hence, $D = \tilde{D}$, which follows from the axiom A13. Now $D = p_{AB}(C)$. This fact yields the coincidence $p_{AB} = p_{CD}$. $\square$

THEOREM 4.3. *Any two mappings of congruent translation on the same straight line do commute: $p_{AB} \circ p_{CD} = p_{CD} \circ p_{AB}$.*

PROOF. Let's choose some arbitrary point E on the line a on which the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lie. Then denote $F = p_{AB}(E)$,

$G = p_{CD}(F)$, and $H = p_{CD}(E)$. As a result we get

$$p_{AB} = p_{EF}, \qquad\qquad p_{CD} = p_{FG} = p_{EH}. \tag{4.5}$$

To the last equality $p_{FG} = p_{EH}$ in (4.5) the previous theorem 4.2 is applicable. It yields $\overrightarrow{FG} = \overrightarrow{EH}$. We apply the theorem 4.1 to this equality. It yields $\overrightarrow{EF} = \overrightarrow{HG}$. Hence, $p_{EF} = p_{HG}$. By means of direct calculations we derive

$$\begin{aligned} &p_{CD} \circ p_{AB} = p_{FG} \circ p_{EF} = p_{EG},\\ &p_{AB} \circ p_{CD} = p_{HG} \circ p_{EH} = p_{EG}. \end{aligned} \tag{4.6}$$

Here in (4.6) we used the theorem 3.2. The rest to compare the right hand sides of the formulas (4.6), which immediately yields the required result $p_{AB} \circ p_{CD} = p_{CD} \circ p_{AB}$. □

Some fixed mapping of congruent translation on a straight line can be given by various pairs of points. However, the theorem 4.2 show that all such pairs of correspond to geometric vectors equal to each other in the sense of the definition 4.1. Therefore, passing from geometric vectors to slipping vectors, we get the one-to-one correspondence of the set of congruent translations and the set of slipping vectors on a line: $p = p_a$.

The identical mapping id is also a mapping of congruent translation: $\mathrm{id} = p_{AA}$. But a single point A does not define an arrowhead segment. Especially for to describe this situation the concept of *zero vector* is introduced. The zero vector $\mathbf{0}$ is a formal object complementing the set of slipping vectors on a line so that $p_{\mathbf{0}} = \mathrm{id}$. Any one-point set on a line treated as a «degenerate» arrowhead segment $\overrightarrow{AA}$ can be taken for a geometric realization of the zero vector.

The set of mappings of congruent translation is naturally equipped with the operation of composition. According to the theorem 3.2, the composition of two congruent translations is a congruent translation. Let's set by definition

$$p_{\mathbf{a}} \circ p_{\mathbf{b}} = p_{\mathbf{a}+\mathbf{b}}. \tag{4.7}$$

The formula (4.7) is the definition of the addition operation for slipping vectors on a straight line.

THEOREM 4.4. *The addition operation of slipping vectors on a straight line possesses the following properties:*

(1) *it is commutative, i. e.* $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$*;*
(2) *it is associative, i. e.* $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$*;*
(3) *there is a vector* $\mathbf{0}$*, such that* $\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}$ *for an arbitrary vector* $\mathbf{a}$*;*
(4) *for any vector* $\mathbf{a}$ *there is an opposite vector* $\mathbf{a}'$ *such that* $\mathbf{a} + \mathbf{a}' = \mathbf{a}' + \mathbf{a} = \mathbf{0}$.

The first property follows from the theorem 4.3. The associativity follows from the formula (4.7) since the property of associativity is peculiar to the composition of any mappings. The third property follows from the formula (4.7) and from the definition of the zero vector $p_{\mathbf{0}} = \mathrm{id}$. The rest is to prove the fourth property. Let $\overrightarrow{AB}$ be a geometric realization for a slipping vector $\mathbf{a}$. Let's denote by $\mathbf{a}'$ the slipping vector whose geometric realization is the vector $\overrightarrow{BA}$. Then $p_{\mathbf{a}+\mathbf{a}'} = p_{\mathbf{a}} \circ p_{\mathbf{a}'} = p_{AB} \circ p_{BA} = p_{BB} = \mathrm{id} = p_{\mathbf{0}}$. This means that, $\mathbf{a} + \mathbf{a}' = \mathbf{0}$.

The addition is an algebraic operation on the set of slipping vectors. Sets equipped with various algebraic operations are studied in course of general algebra (see, for instance, [5]). Let's recall the definition of a group — it is a set with one algebraic operation, which is usually called the *group multiplication.*

DEFINITION 4.2. A set G is called a *group* if for any two elements a and b of this set a third element of this set $a \cdot b$, which is called the *product* of a and b, is assigned so that the following three conditions are fulfilled:

(1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ i.e. the group multiplication is associative;
(2) there is an element $e \in G$ such that $e \cdot a = a \cdot e = a$ for any element $a \in G$;

(3) for any element $a \in G$ there is an element a' such that $a \cdot a' = a' \cdot a = e$.

The element e is called the *unity* of the group G, while the element a' is called the *inverse element* for an element $a \in G$.

DEFINITION 4.3. A group G is called *commutative* or *Abelian* if the group multiplication in it is commutative, i. e. $a \cdot b = b \cdot a$.

Comparing the definitions 4.2 and 4.3 with the properties of the addition of vectors in the theorem 4.4 shows that the set of slipping vectors is an Abelian group with respect to the addition. The multiplication sign in it is replaced by the plus sign, while the zero vector plays the role of the unity.

§ 5. Congruence of angles.

Let h and k be two rays coming out from one point and not lying on one straight line. Let's choose a point A on the ray h and a point B on the ray k. Assume that the points A and B are distinct from the point O. Let's apply the theorem 5.2 from Chapter II to the points A, B, and O and construct the angle $\angle AOB$. It is clear that this angle does not depend on a particular choice of the points A and B on the rays h and k. It is determined by the rays h and k themselves. For this reason we shall denote such an angle as $\angle hOk$ or even as $\angle hk$.

AXIOM A16. *Any angle $\angle hk$ is congruent to itself and for any half-plane a_+ with a ray m lying on the boundary line a there is a unique ray n within the half-plane a_+ such that $\angle hk \cong \angle mn$.*

AXIOM A17. *Let A, B, and C be three points not lying on one straight line and let $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ be other three points also not lying on one straight line. If the conditions*

$$[AB] \cong [\tilde{A}\tilde{B}], \qquad [AC] \cong [\tilde{A}\tilde{C}], \qquad \angle BAC \cong \angle \tilde{B}\tilde{A}\tilde{C}$$

are fulfilled, then the other two conditions $\angle ABC \cong \angle \tilde{A}\tilde{B}\tilde{C}$ and $\angle ACB \cong \angle \tilde{A}\tilde{C}\tilde{B}$ are also fulfilled.

The axiom A16 for angles is analogous to the axiom A13 for segments. The axiom A17 can be formulated as follows: if an angle and two sides forming this angle in some triangle are congruent to an angle and to the corresponding sides of some other triangle, then the remaining two angles of the first triangle are congruent to the corresponding angles of the second triangle. Let's define the concept of congruence for triangles.

DEFINITION 5.1. Two triangles are called *congruent* if their vertices are in one-to-one correspondence so that the angles at the vertices and the sides of one triangle are congruent to the corresponding angles and sides of the other triangle.

For example, the triangle ABC is congruent to the other triangle FGH if the following six conditions are fulfilled:

$$
\begin{array}{lll}
[AB] \cong [FG], & [BC] \cong [GH], & [CA] \cong [HF], \\
\angle ABC \cong \angle FGH, & \angle BCA \cong \angle GHF, & \angle CAB \cong \angle HFG.
\end{array}
$$

THEOREM 5.1. *If for triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ the conditions $[AB] \cong [\tilde{A}\tilde{B}]$, $[AC] \cong [\tilde{A}\tilde{C}]$, and $\angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}$ are fulfilled, then the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

PROOF. The congruence of three corresponding angles and the congruence of two corresponding sides in the triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ easily follow from the axiom A17. The rest is to prove the congruence $[BC] \cong [\tilde{B}\tilde{C}]$. Using the axiom A13, on the ray $[\tilde{B}\tilde{C}\rangle$ we find a point C' such that $[BC] \cong [\tilde{B}C']$ and consider the triangle $\tilde{A}\tilde{B}C'$. For this triangle the conditions

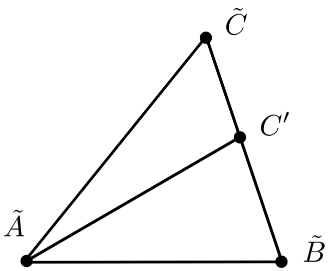


Fig. 5.1

$$
\begin{gathered}
[AB] \cong [\tilde{A}\tilde{B}], \\
[BC] \cong [\tilde{B}C'], \\
\angle ABC \cong \angle\tilde{A}\tilde{B}C',
\end{gathered}
$$

are fulfilled. Due to these conditions one can apply the axiom A17 to the angle $\angle\tilde{A}\tilde{B}C'$ and to the sides $[\tilde{A}\tilde{B}]$ and $[\tilde{B}C']$ of the triangle $\tilde{A}\tilde{B}C'$. It yields $\angle BAC \cong \angle\tilde{B}\tilde{A}C'$. Moreover, in the statement of the theorem we have $\angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}$. Therefore, if we assume that $\tilde{C} \neq C'$, we would have two rays $[\tilde{A}\tilde{C}\rangle$ and $[\tilde{A}C'\rangle$ on one half-plane bounded by the line $\tilde{A}\tilde{B}$ which form with the ray $[\tilde{A}\tilde{B}\rangle$ two angles congruent to the angle $\angle BAC$. But this would contradict the axiom A16, hence, $\tilde{C} = C'$. As a result we get the required congruence of segments $[BC] \cong [\tilde{B}\tilde{C}]$. Thus, the theorem is proved. $\square$

The theorem 5.1 is known as the congruence criterion for triangles by two sides and the angle between them. The next theorem is called the congruence criterion for triangles by a side and two angles adjoint to this side.

THEOREM 5.2. *If for triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ the conditions $[AB] \cong [\tilde{A}\tilde{B}]$, $\angle ABC \cong \angle\tilde{A}\tilde{B}\tilde{C}$, and $\angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}$ are fulfilled, then the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

PROOF. Applying the axiom A13, on the ray $[\tilde{B}\tilde{C}\rangle$ we choose a point C' such that $[BC] \cong [\tilde{B}C']$. Then to the triangles ABC and $\tilde{A}\tilde{B}C'$ the previous theorem 5.1 is applicable. It means that the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}C'$, hence, $\angle BAC \cong \angle\tilde{B}\tilde{A}C'$. From the statement of the theorem we get $\angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}$. Now, if we assume that $C' \neq \tilde{C}$, then on the half-plane bounded by the line $\tilde{A}\tilde{B}$ we would have two rays $[\tilde{A}\tilde{C}\rangle$ and $[\tilde{A}C'\rangle$ which form with the ray $[\tilde{A}\tilde{B}\rangle$ two angles congruent to the angle $\angle BAC$. This would be a contradiction to the axiom A16. Hence, $C' = \tilde{C}$ and the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$. $\square$

THEOREM 5.3. *Let h, k, and l be three distinct rays coming out from one point O and lying on one plane. Let h', k', and l' be other three distinct rays coming out from one point O' and lying* *on one plane. If the ray l is inside the angle $\angle hk$ and if the ray l' is inside the angle $\angle h'k'$, then*

(1) $\angle hl \cong \angle h'l'$ *and* $\angle lk \cong \angle l'k'$ *imply* $\angle hk \cong \angle h'k'$;
(2) $\angle hk \cong \angle h'k'$ *and* $\angle hl \cong \angle h'l'$ *imply* $\angle lk \cong \angle l'k'$;
(3) $\angle hk \cong \angle h'k'$ *and* $\angle lk \cong \angle l'k'$ *imply* $\angle hl \cong \angle h'l'$.

Proof. Let's choose a point A on the ray h and a point C on the ray k. We connect them by means of the segment $[AC]$ and then we apply the lemma 6.2 from Chapter II to the ray l passing within the angle $\angle AOC$. Let's denote by B the point at which the ray l crosses the segment $[AC]$. The point B is an interior point of the segment $[AC]$ since the ray l does coincide neither with the ray h nor with the ray k.

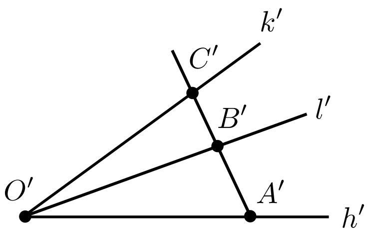


Fig. 5.2

The condition $\angle hl \cong \angle h'l'$ is common for the first and the second items of the theorem. Let's use it in the following constructions. Applying the axiom A13, on the ray h' we choose a point A' such that $[OA] \cong [O'A']$. Then in the same way on the ray l' we choose a point B' such that $[OB] \cong [O'B']$. Due to the condition $\angle hl \cong \angle h'l'$ we can apply the theorem 5.1. As a result we find that the triangle AOB is congruent to the triangle $A'O'B'$. Hence, we have

$$[AB] \cong [A'B'], \qquad\qquad \angle OAB \cong \angle O'A'B'. \tag{5.1}$$

The point B' divides the line $A'B'$ into two rays, the ray $[B'A'\rangle$ being one of them. On the ray opposite to $[B'A'\rangle$ we choose a point C' such that $[BC] \cong [B'C']$. Applying the first item of the axiom A15 to the segments $[A'B']$ and $[B'C']$, we get $[AC] \cong [A'C']$. From this relationship and from (5.1) due to the theorem 5.1 we conclude that the triangle AOC is congruent to the triangle $A'O'C'$. Hence, the following relationships are valid:

$$\begin{aligned}
&[OC] \cong [O'C'],\\
&\angle ACO \cong \angle A'C'O',\\
&\angle AOC \cong \angle A'O'C'.
\end{aligned} \tag{5.2}$$

Let's combine the first two relationships (5.2) with $[BC] \cong [B'C']$. Applying the theorem 5.1 to this combination, we derive that the triangle BOC is congruent to the triangle $B'O'C'$. From this congruence, in particular, we get

$$\angle BOC \cong \angle B'O'C'. \tag{5.3}$$

On order to prove the first item of the theorem 5.3 we consider the angles $\angle l'k'$ and $\angle B'O'C'$. They lie on one half-plane bounded by the line $O'B'$ and have the common side $l' = [O'B'\rangle$. For this angles from the statement of the theorem and from the formula (5.3) we extract the relationships

$$\angle lk \cong \angle l'k', \qquad\qquad \angle lk \cong \angle B'O'C'.$$

Applying the axiom A16 now yields the coincidence of rays $[O'C'\rangle = k'$. Being combined with the last relationship (5.2), this coincidence immediately yields the required result $\angle hk \cong h'k'$.

In order to prove the second item of the theorem 5.3 we derive the coincidence of rays $[O'C'\rangle = k'$ by considering the angles $\angle h'k'$ and $\angle A'O'C'$, which lie on one half-plane bounded by the line $O'A'$ and have the ray h' as their common side. From the statement of the theorem and from (5.2) for these angles we get

$$\angle hk \cong \angle h'k', \qquad\qquad \angle hk \cong \angle A'O'C'.$$

Applying the axiom A16 to the above relationships, we derive $[O'C'\rangle = k'$. Now from $[O'C'\rangle = k'$ and (5.3) we get the required result $\angle lk \cong l'k'$.

The third item of the theorem 5.3 does not require a separate proof. It is reduced to the second item upon exchanging the notations of the rays: h with k and h' with k'. $\square$

DEFINITION 5.2. A triangle ABC is called *isosceles* if some two sides of it are congruent. For example, $[AB] \cong [AC]$. The side $[BC]$ in this case is called the *base* of the isosceles triangle ABC, while the congruent sides $[AB]$ and $[AC]$ are called the *lateral sides* of this isosceles triangle.

THEOREM 5.4. *The angles at the base of an isosceles triangle are congruent to each other.*

PROOF. Let ABC be an isosceles triangle with lateral sides $[AB]$ and $[AC]$. Let's introduce the duplicate notations for the vertices of this triangle: $\tilde{A} = A$, $\tilde{B} = C$, and $\tilde{C} = B$. Then from $[AB] \cong [AC]$ due to the axiom A16 we get

$$[AB] \cong [\tilde{A}\tilde{B}], \qquad [AC] \cong [\tilde{A}\tilde{C}], \qquad \angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}.$$

In this situation the axiom A17 is applicable. Applying this axiom we get $\angle ABC \cong \angle\tilde{A}\tilde{B}\tilde{C}$ and $\angle ACB \cong \angle\tilde{A}\tilde{C}\tilde{B}$. When taking into account the above duplicate notations for the vertices it means that $\angle ABC \cong \angle ACB$ and $\angle ACB \cong \angle ABC$. The required congruence of angles at the base of the isosceles triangle ABC is proved. $\square$

LEMMA 5.1. *Let ABC and ADC be two triangles with the common side $[AC]$ lying on two different half-planes of the same plane separated by the line AC. In this case if $[AB] \cong [AD]$ and if $[CB] \cong [CD]$, then $\angle ABC \cong \angle ADC$ and $\angle ADC \cong \angle ABC$.*

PROOF. The points B and D lie on different sides of the line AC, hence, the segment $[BD]$ intersects this line at some interior point S. There are the following five cases for the disposition of the point S relative to the points A and C:

(1) the point A lies between S and C;

(2) the point S coincides with the point A;
(3) the point S lies between A and C;
(4) the point S coincides with the point C;
(5) the point C lies between A and S.

The first three cases of mutual disposition of points are shown on Fig. 5.3, Fig. 5.4 and Fig. 5.5. The fourth case is reduced to the second one and the fifth case is reduced to the first one when exchanging the notations of the points A and C. These two cases do not require a separate consideration.

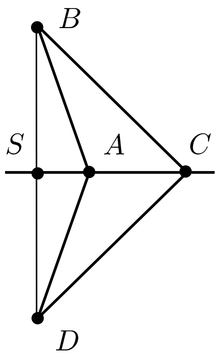


Fig. 5.3

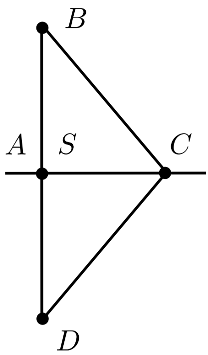


Fig. 5.4

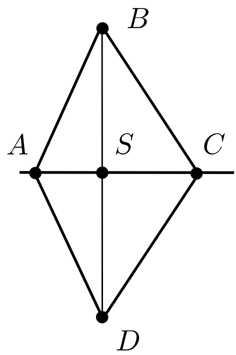


Fig. 5.5

Let's consider the first case. From $[AB] \cong [AD]$ and from $[CB] \cong [CD]$ we conclude that the triangles BCD and BAD on Fig. 5.3 are isosceles. Therefore, we have $\angle SBC \cong \angle SDC$ and $\angle SBA \cong \angle SDA$. The ray $[BA\rangle$ lies inside the angle $\angle SBC$, while the ray $[DA\rangle$ is inside the angle $\angle SDC$. Hence, we can apply the second item of the theorem 5.3 and derive the required relationship $\angle ABC \cong \angle ADC$. The second relationship $\angle ADC \cong \angle ABC$ is derived similarly.

In the second case the required relationships are derived immediately since due to the congruence $[CB] \cong [CD]$ the triangle BCD on Fig. 5.4 is isosceles.

And finally, let's consider the third case. From $[AB] \cong [AD]$ and $[CB] \cong [CD]$ we conclude that the triangles BCD and BAD on Fig. 5.5 are isosceles, which means that $\angle SBC \cong \angle SDC$ and $\angle SBA \cong \angle SDA$. Due to these relationships we can apply the first item of the theorem 5.3. It yields $\angle ABC \cong \angle ADC$. The second relationship $\angle ADC \cong \angle ABC$ is derived similarly. □

LEMMA 5.2. *For any triangle ABC in the plane of this triangle there is exactly one point D different from B and not lying on the line AC such that $[AB] \cong [AD]$ and $[CB] \cong [CD]$.*

PROOF. For the beginning we prove the existence of the point D. Let's denote by h the ray $[AC\rangle$ lying on the line AC. This line divides the plane ABC into two half-planes. The triangle ABC lies on one of them. Applying the axiom A16, in the other half-plane we draw a ray k coming out from the point A and such that $\angle CAB \cong hk$. Applying the axiom A13, on the ray k we choose a point D satisfying the condition $[AB] \cong [AD]$. Then $\angle CAB \cong \angle CAD$ and due to the theorem 5.1 the triangle ABC appears to be congruent to the triangle ADC. Hence, $[CB] \cong [CD]$. Thus, a required point D is constructed. The points B and D lie on different half-planes outside the boundary line AC. Therefore $D \neq B$.

Let's prove the uniqueness of the point D. Assume that $\tilde{D}$ is another point satisfying the conditions of the lemma. According to the statement of the lemma, the point $\tilde{D}$ does not lie on the line AC, therefore it lies on one of the open half-planes determined my this line.

Let's consider that half-plane which contains the point D. Applying the lemma 5.1, we get $\angle ABC \cong \angle A\tilde{D}C$. Combining this congruence with $[AB] \cong [A\tilde{D}]$ and $[CB] \cong [C\tilde{D}]$, we conclude that the triangle ABC is congruent to the triangle $A\tilde{D}C$. Hence, we have the following congruence of angles:

$$\angle CAB \cong \angle CA\tilde{D}, \qquad\qquad \angle ACB \cong \angle AC\tilde{D}. \tag{5.4}$$

The similar relationships are fulfilled for the point D too:

$$\angle CAB \cong \angle CAD, \qquad \angle ACB \cong \angle ACD. \tag{5.5}$$

They are derived from $[AB] \cong [AD]$ and $[CB] \cong [CD]$ by means of the lemma 5.1. Comparing (5.4) with (5.5) and applying the axiom A13, we prove the coincidence of the rays $[AD\rangle = [A\tilde{D}\rangle$ and $[CD\rangle = [C\tilde{D}\rangle$. Two non-coinciding straight lines AD and CD can have at most one common point. Therefore $\tilde{D} = D$.

If we assume that the point $\tilde{D}$ lies on the same half-plane as the point B, we get $\tilde{D} = B$. This fact is derived with the use of the lemma 5.1 on the base of considerations quite similar to the above ones. But the coincidence $\tilde{D} = B$ is excluded by the provisions of the lemma. Therefore, the point D constructed above is unique. □

Theorem 5.5. *If for triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ the conditions $[AB] \cong [\tilde{A}\tilde{B}]$, $[AC] \cong [\tilde{A}\tilde{C}]$, and $[BC] \cong [\tilde{B}\tilde{C}]$ are fulfilled, then the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

Proof. Let's denote by $\tilde{h}$ the ray $[\tilde{A}\tilde{C}\rangle$. The line $\tilde{A}\tilde{C}$ divides the plane of the second triangle $\tilde{A}\tilde{B}\tilde{C}$ into two half-planes. The triangle $\tilde{A}\tilde{B}\tilde{C}$ itself lies on one of them. Applying the axiom A16 to both half-planes, we draw two rays $\tilde{k}$ and $\tilde{k}'$ coming out from the point $\tilde{A}$ and such that $\angle CAB \cong \angle\tilde{h}\tilde{k}$ and $\angle CAB \cong \angle\tilde{h}\tilde{k}'$. The ray $\tilde{k}'$ is chosen to be lying in the same half-plane as the triangle $\tilde{A}\tilde{B}\tilde{C}$. The ray $\tilde{k}$ lies in the other half-plane. Let's apply the axiom A13 to the rays $\tilde{k}$ and $\tilde{k}'$ and choose two points $\tilde{D}$ and $\tilde{B}'$ on them such that $[AB] \cong [\tilde{A}\tilde{D}]$ and $[AB] \cong [\tilde{A}\tilde{B}']$. By construction the triangle ABC appears to be congruent to the triangles $\tilde{A}\tilde{B}'\tilde{C}$ and $\tilde{A}\tilde{D}\tilde{C}$. This fact follows

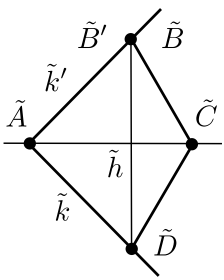


Fig. 5.6

from the theorem 5.1 if we take into account the congruence $[AC] \cong [\tilde{A}\tilde{C}]$. Hence, for the points $\tilde{D}$ and $\tilde{B}'$ we get

$$
\begin{aligned}
&[AB] \cong [\tilde{A}\tilde{B}'], && [CB] \cong [\tilde{C}\tilde{B}'],\\
&[AB] \cong [\tilde{A}\tilde{D}], && [CB] \cong [\tilde{C}\tilde{D}].
\end{aligned}
$$

According to the statement of the theorem, exactly the same conditions are fulfilled for the point $\tilde{B}$ lying on the same half-plane as the point $\tilde{B}'$:

$$
[AB] \cong [\tilde{A}\tilde{B}], \qquad [CB] \cong [\tilde{C}\tilde{B}].
$$

Hence, the lemma 5.2 yields $\tilde{B} = \tilde{B}'$. This means that the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$. $\square$

The theorem 5.5 proved just above is known as the congruence criterion for triangles by three sides.

THEOREM 5.6. *The congruence of angles is a reflexive, symmetric, and transitive binary relation. Therefore, it is the an equivalence relation.*

EXERCISE 5.1. *The reflexivity of the congruence of angles is explicitly stated in the axiom A16. Prove the symmetry and transitivity of this binary relation with the use of the theorem 5.5.*

§ 6. A right angle and orthogonality.

THEOREM 6.1. *The congruence of two angles imply the congruence of their adjacent angles.*

PROOF. Assume that an angle $\angle hk$ with the vertex at a point O is congruent to an angle $\angle\tilde{h}\tilde{k}$ with the vertex at a point $\tilde{O}$. Let's complement the ray h with the ray l up to a whole straight line. Similarly, we complement the ray $\tilde{h}$ with the ray $\tilde{l}$ up to a whole straight line. As a result we get two angles $\angle kl$ and $\angle\tilde{k}\tilde{l}$.

Their congruence should be proved. For this purpose we choose some points A, B, and C on the rays h, k, and l respectively. Then, applying the axiom A13, we mark three points $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ on the rays $\tilde{h}$, $\tilde{k}$ and $\tilde{l}$ so that the conditions

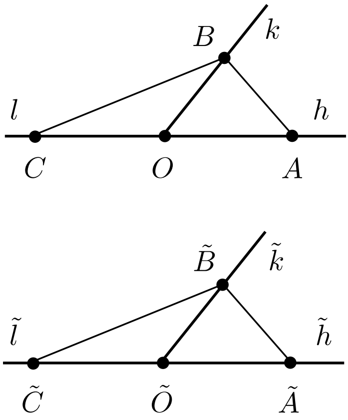


Fig. 6.1

$$
\begin{aligned}
&[OA] \cong [\tilde{O}\tilde{A}],\\
&[OB] \cong [\tilde{O}\tilde{B}],\\
&[OC] \cong [\tilde{O}\tilde{C}]
\end{aligned}
\tag{6.1}
$$

are fulfilled. From the first two conditions (6.1) complemented with $\angle hk \cong \angle\tilde{h}\tilde{k}$ we derive that the triangles AOB and $\tilde{A}\tilde{O}\tilde{B}$ are congruent. This yields $[AB] \cong [\tilde{A}\tilde{B}]$ and $\angle BAO \cong \angle\tilde{B}\tilde{A}\tilde{O}$. Moreover, from the first and the last conditions (6.1), applying the axiom A15, we derive $[AC] \cong [\tilde{A}\tilde{C}]$. Combining the three conditions obtained and applying the theorem 5.1, we derive that the triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ are congruent. This congruence yields $[BC] \cong [\tilde{B}\tilde{C}]$. Now we can apply the theorem 5.5 to the triangles BOC and $\tilde{B}\tilde{O}\tilde{C}$. This theorem yields the congruence of these triangles, which, in turn, yields the required relationship $\angle BOC \cong \angle\tilde{B}\tilde{O}\tilde{C}$. The theorem is proved. □

Theorem 6.2. *Vertical angles are congruent to each other.*

The proof of this theorem is obvious. Two vertical angles always have a common adjacent angle (see Fig. 5.1 in Chapter II). Therefore, due to the previous theorem 6.1 and the axiom A16 these angles are congruent.

Definition 6.1. An angle is called a *right angle*, if it is congruent to its adjacent angle.

LEMMA 6.1. *Right angles do exist.*

PROOF. Let $\angle hk$ be an arbitrary angle formed by two rays coming out from a point A. If this angle appears to be a right angle, then the proof is over. Assume that this angle is not a right angle. Let's complete the ray h up to a whole straight line. This line divides the plane of the angle $\angle hk$ into two half-planes. The angle $\angle hk$ belongs to one of these half-planes. Applying the axiom A16, on the other half-plane we draw a ray $\tilde{k}$ coming out from the point A and such that $\angle hk \cong \angle h\tilde{k}$. Then we choose a point B on the ray k and, applying the axiom A13, on the ray $\tilde{k}$ we find a point $\tilde{B}$ satisfying the condition $[AB] \cong [A\tilde{B}]$.

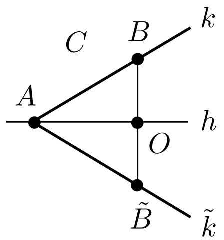


Fig. 6.2

The points B and $\tilde{B}$ lie on different sides of the line containing the ray h. Therefore, the segment $[B\tilde{B}]$ intersects this line at some its interior point $O \neq A$ (the equality $O = A$ leads to the case, where $\angle hk$ and $\angle h\tilde{k}$ both are right angles). The angles $\angle AOB$ and $\angle AO\tilde{B}$ are adjacent angles. They are congruent since the triangles AOB and $AO\tilde{B}$ are congruent, this fact follows from the theorem 5.1 due to $[AB] \cong [A\tilde{B}]$ and $\angle OAB \cong \angle OA\tilde{B}$. Hence, the angles $\angle AOB$ and $\angle AO\tilde{B}$ are right angles. □

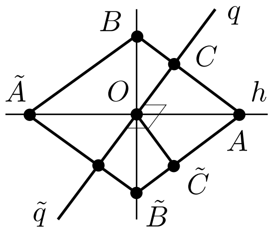


Fig. 6.3

LEMMA 6.2. *All right angles are congruent to each other.*

PROOF. In order to prove the lemma it is sufficient to show that all right angles are congruent to one of them. As such a reference model we choose the right angle $\angle AOB$ constructed in the proof of the lemma 6.1. Let's complement Fig. 6.2 with one more point $\tilde{A}$ lying on the line OA. Let's determine this point

$\tilde{A}$ by means of the condition $[O\tilde{A}] \cong [OA]$. Then we draw the segments $[\tilde{A}B]$ and $[\tilde{A}\tilde{B}]$. As a result we get four right angles with the common vertex at the point O.

Let $\angle h'q'$ be some arbitrary right angle. The line $A\tilde{A}$ divides the plane of Fig. 6.3 into two half-planes. In that half-plane which contains the point B we draw a ray q coming out from the point O so that the angle $\angle hq$ is congruent to the angle $\angle h'q'$. Then the angle $\angle hq$ is also a right angle.

Let's prove that the ray q coincides with the ray $[OB\rangle$. If it is not so, the ray q lies within one of the angles $\angle AOB$ or $\angle\tilde{A}OB$. For the sake of certainty assume that it lies within the angle $\angle AOB$ (the second case is reduced to this one by exchanging the notations of the points A and $\tilde{A}$). Applying the lemma 6.2 from Chapter II to the ray q, we find that it intersects the segment $[AB]$ at some interior point C. Let's complement q up to the whole line with the ray $\tilde{q}$ lying inside the angle $\tilde{A}O\tilde{B}$. Then we perform the congruent translation f of the line AB to the line $A\tilde{B}$ such that $f(A) = A$ and $f(B) = \tilde{B}$. Let's denote $\tilde{C} = f(C)$. Then $\tilde{C}$ is an interior point of the segment $[A\tilde{B}]$ and $[A\tilde{C}] \cong [AC]$. This relationship complemented with $\angle OAB \cong \angle OA\tilde{B}$ yields the congruence of the triangles AOC and $AO\tilde{C}$. Hence, the angle $\angle AO\tilde{C}$ is congruent to the angle $\angle AOC = \angle hk$. The angle $\angle h\tilde{q}$ is an adjacent angle for $\angle hq$. Therefore, $\angle h\tilde{q} \cong \angle hq$ since $\angle hq$ is a right angle. From $\angle AO\tilde{C} \cong \angle hq$ and $\angle h\tilde{q} \cong \angle hq$ we derive the coincidence of the rays $[O\tilde{C}\rangle$ and $\tilde{q}$. But such a coincidence is forbidden since $[O\tilde{C}\rangle$ lies within the angle $\angle AO\tilde{B}$, while $\tilde{q}$ is within its adjacent angle $\angle\tilde{A}O\tilde{B}$.

The contradiction obtained just above proves that $C = B$ and $q = [OB\rangle$. Hence, an arbitrary right angle $\angle h'q'$ is congruent to the reference right angle $\angle AOB$. The lemma is proved. $\square$

Definition 6.2. Two intersecting straight lines are called *perpendicular* to each other if all four angles formed by them at the intersection point are right angles.

The perpendicularity of lines a and b is denoted as $a \perp b$.

Actually, for two lines to be perpendicular it is sufficient that one of the four angles formed by them at the intersection point is a right angle. Then other three angles are also right angles due to the theorems 6.1 and 6.1.

THEOREM 6.3. *Let a be some straight line lying on a plane α. Then for any point $A \in a$ there is exactly one straight line b passing trough this point, lying on the plane α, and being perpendicular to the line a.*

THEOREM 6.4. *Through any point of a plane one can draw at most two straight lines lying on this plane and being perpendicular to each other.*

The theorems 6.3 and 6.4 are easily derived from the lemmas 6.1 and 6.2. The theorem 6.4 means the two-dimensionality of a plane.

Let a be some straight line and assume that B is some point outside this line. A segment $[BA]$ connecting the point B with a point $A \in a$ is called a *perpendicular* dropped from B onto the line a if the line AB is perpendicular to the line$a$.

THEOREM 6.5. *For any line a and for any point $B \notin a$ there is exactly one perpendicular dropped from B onto the line a.*

PROOF. For the beginning let's prove the existence of a perpendicular dropped from the point B onto the line a. For this purpose we draw the plane α passing through this line and through the point B. Let's choose some point O on the line a. If $OB \perp a$, the required perpendicular is found. Otherwise, we consider two rays produced by the point O on the line a. Let's denote one of them through h and let $k = [OB\rangle$. The angle $\angle hk$ lies on one of the half-planes produced by the line a on the plane α. On the other half-plane we draw a ray

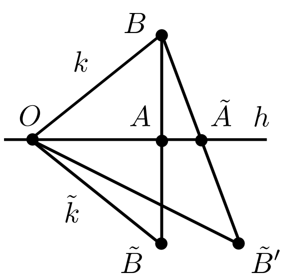


Fig. 6.4

coming out from the point O and such that $\angle h\tilde{k} \cong \angle hk$. On this ray we choose a point $\tilde{B}$ satisfying the condition $[O\tilde{B}] \cong [OB]$. Then the segment $[B\tilde{B}]$ intersects the line a at some interior point A, while the triangles OAB and $OA\tilde{B}$ are congruent. Hence, the angle $\angle OAB$, which is congruent to its adjacent angle $\angle OA\tilde{B}$, is a right angle. Thus, $[BA]$ is a required perpendicular.

Now let's prove the uniqueness of the perpendicular dropped from the point B onto the line a. Assume that it is not unique and consider two perpendiculars $[BA]$ and $[B\tilde{A}]$. As above, we draw the plane α passing through the point B and the line a. Let's choose a point O on the line a outside the segment $[A\tilde{A}]$ and denote by h the ray $[OA\rangle = [O\tilde{A}\rangle$. On the lines AB and $\tilde{A}B$ we choose the points $\tilde{B}$ and $\tilde{B}'$ distinct from B and satisfying the conditions $[AB] \cong [A\tilde{B}]$ and $[\tilde{A}B] \cong [\tilde{A}\tilde{B}']$. Now note that the right angle $\angle OAB$ is congruent to its adjacent angle $\angle OA\tilde{B}$, while the right angle $\angle O\tilde{A}B$ is congruent to the angle $\angle O\tilde{A}\tilde{B}'$. Hence, we derive that $\triangle OAB \cong \triangle OA\tilde{B}$ and that $\triangle O\tilde{A}B \cong \triangle O\tilde{A}\tilde{B}'$. As a result we have

$$\begin{aligned} &\angle AO\tilde{B} \cong \angle AOB, &\qquad &\angle AO\tilde{B}' \cong \angle AOB,\\ &[O\tilde{B}] \cong [OB], & &[O\tilde{B}'] \cong [OB]. \end{aligned}$$

The first pair of the above relationships yields the coincidence of the rays $[O\tilde{B}\rangle = [O\tilde{B}'\rangle$. Then from the second pair we derive $\tilde{B} = \tilde{B}'$. Hence, $A = \tilde{A}$ and $[BA] = [B\tilde{A}]$. □

Theorem 6.6. *A triangle cannot have two right angles.*

Exercise 6.1. *Prove the theorems 6.3 and 6.4 on the base of the lemmas 6.1 and 6.2.*

Exercise 6.2. *Derive the theorem 6.6 from the theorem 6.5.*

§ 7. Bisection of segments and angles.

Definition 7.1. A point O is called a *center* of a segment $[AB]$ if it lies in the interior of this segment and if $[AO] \cong [OB]$.

Definition 7.2. Let $\angle hk$ be an angle with the vertex at a poit O. A ray l coming out from the point O is called a *bisector* of the angle $\angle hk$ if it lies within this angle and if $\angle hl \cong \angle lk$.

Definition 7.3. A segment $[AO]$ connecting a vertex A of a triangle ABC with a point O on the line BC is called a *height* of this triangle if the lines AO and BC are perpendicular.

Definition 7.4. A segment $[AO]$ connecting a vertex A of a triangle ABC with a center of the side $[BC]$ is called a *median* of this triangle.

Theorem 7.1. *In an isosceles triangle ABC a median AO drawn from a vertex A to a center of the base $[BC]$ is a height and a bisector of the angle $\angle BAC$ simultaneously.*

The proof of the theorem 7.1 is sufficiently simple provided a of the triangle ABC is already drawn. The problem of existing a median is not considered in this theorem at all.

Theorem 7.2. *For any segment $[AB]$ there is a point O being its center.*

Proof. Assume that a straight line segment $[AB]$ is given. Let's consider some plane α containing the line AB. Applying the theorem 6.3, on the plane α we draw two lines being perpendicular to the line AB and passing through the points A and B respectively. These two lines do not intersect each other since if they intersect at some point M, the triangle ABM would have two right angles. Let's choose some point C on the line passing through the point A. Then on the line passing through the point B we choose a point D such that C and D are on different half-planes separated by the line AB and such that $[BD] \cong [AC]$. Then the segment $[CD]$ intersects the line AB at some interior point O.

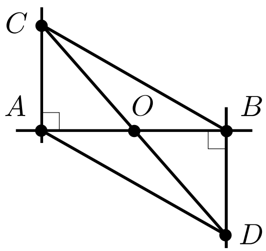


Fig. 6.5

Let's consider the triangle BOD and the line AC. The line AC cannot intersect the side $[BD]$ of this triangle since the lines AC and BD have no common points at all. The line AC intersects the line OD at the point C outside the segment $[OD]$. Hence the line AC cannot intersect the third side $[OB]$ of the triangle BOD at an interior point (see Pasch's axiom A12). This fact excludes the disposition $(O \blacktriangleright A \blacktriangleleft B)$ of the points O, A, and B on the line AB. The second disposition $(A \blacktriangleright B \blacktriangleleft O)$ is excluded by considering the line BD and the triangle AOC. The rest is the third disposition $(A \blacktriangleright O \blacktriangleleft B)$. In this disposition O is an interior point of the segment $[AB]$.

Now let's consider the triangles ABC and BAD. They are congruent due to the theorem 5.1 since $\angle BAC \cong \angle ABD$ and $[AC] \cong [BD]$. From this congruence of the triangles ABC and BAD we derive $[AD] \cong [BC]$ and $\angle ABC \cong \angle BAD$. Applying the item (1) of the theorem 5.3, we get $\angle CBD \cong \angle DAC$, which means that $\triangle CBD \cong \triangle DAC$. Hence, we have

$$\angle DCB \cong \angle CDA, \qquad\qquad \angle CDB \cong \angle DCA.$$

From these two relationships, applying the theorem 5.2, we derive that the triangle AOC is congruent to the triangle BOD, while the triangle COB is congruent to the triangle DOA. These congruences of triangles yield the relationships

$$[AO] \cong [OB], \qquad\qquad [CO] \cong [OD].$$

They mean that the point O is a required center of the segment $[AB]$, and simultaneously, it is a center of the segment $[CD]$. $\square$

The theorem 7.2 complemented with the theorem 7.1 yields an algorithm for bisecting angles. Indeed, assume that an angle $\angle hk$ with the vertex at a point O is given. On the sides of this angle we choose two points A and B such that $[OA] \cong [OB]$. Then the triangle AOB is isosceles. Bisecting its base $[AB]$ by a point C, we construct its median being a bisector of the angle $\angle AOB$ at the same time. In other words we have the following theorem.

THEOREM 7.3. *For any angle $\angle hk$ there is a ray l being a bisector of this angle.*

THEOREM 7.4. *The center of any segment is unique.*

THEOREM 7.5. *Each angle has exactly one bisector.*

EXERCISE 7.1. *Using the theorem 2.1, prove the theorem 7.4 for a segment $[AB]$. In other words, show that on the line AB there is exactly one point O satisfying the condition $[AO] \cong [OB]$. Then from the theorem 7.4 derive the theorem 7.5.*

§ 8. Intersection of two straight lines by a third one.

Let's consider three straight lines a, b, and c lying on one plane. Assume that the line c intersects the lines a and b at the points A and B respectively. At the intersection point A we have four angles. Other four angles arise at the point B. The angles $\angle CAB$ and $\angle ABD$ have the sides $[AB\rangle$ and $[BA\rangle$ being two oppositely-directed rays lying on one line c and intersecting along the segment $[AB]$. The angles $\angle CAB$ and $\angle ABD$ lie on different half-planes separated by the line c. Such angles are called *inner crosswise lying* angles.

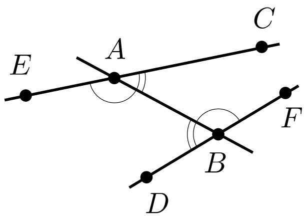


Fig. 8.1

Apart from $\angle CAB$ and $\angle ABD$, by the intersection of the lines a and b with the third line c another pair of inner crosswise lying angles arise. These are the angles $\angle EAB$ and $\angle ABF$. The angle $\angle EAB$ is an adjacent angle for the angle $\angle CAB$, while $\angle ABF$ is an adjacent angle for $\angle ABD$.

DEFINITION 8.1. Two straight lines a and b are called *parallel* if they coincide $a = b$ or if they lie on one plane and do not intersect each other.

The relation of *parallelism* of two straight lines is reflexive and symmetric by definition. It is denoted as $a \parallel b$. In order to prove the transitivity of this relation one should use the axiom A20, which is not yet considered.

THEOREM 8.1. *Assume that a and b are two straight lines lying on one plane and intersecting with a third straight line c at the points A and B. If the inner crosswise lying angles at the points A and B are congruent, then the lines a and b are parallel.*

PROOF. Note that it is sufficient to require the congruence of one pair of inner crosswise lying angles, e. g. $\angle EAB \cong \angle ABF$. Then the relationship $\angle CAB \cong \angle ABD$ follows from the above relationship due to the theorem 6.1.

Let's prove the theorem by contradiction. Assume that the angles $\angle EAB$ and $\angle ABF$ are congruent, but the lines a and b are not parallel. Then they intersect. For the sake of certainty we can take the points C and F to be coinciding with the intersection point of the lines a and b. We choose the point E to be satisfying the condition $[AE] \cong [BF]$. Such a choice is possible due to the axiom A13. Let's connect the point E with the point B by means of the segment $[EB]$ and consider the triangles EAB and ABF. From the statement of the theorem and from the above additional drawings we derive

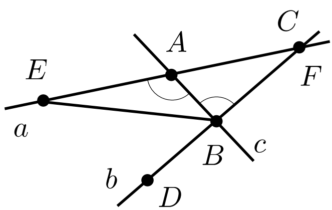


Fig. 8.2

$$[AE] \cong [BF], \qquad [AB] \cong [BA], \qquad \angle EAB \cong \angle ABF.$$

Applying the theorem 5.1 to these relationships, we get the congruence $\triangle EAB \cong \triangle ABF$. This congruence immediately yields $\angle EBA \cong \angle BAC$. But $\angle BAC \cong \angle ABD$, which, as we already mentioned above, follows from $\angle EAB \cong \angle ABF$. Therefore, $\angle EBA \cong \angle DBA$. Hence, due to the axiom A16 we get the coincidence of the rays $[BE\rangle$ and $[BD\rangle$. This means that the point E should lie on the line b, which is impossible since the lines a and b do not coincide. The contradiction obtained shows that the lines a and b cannot intersect, i. e. they are parallel. $\square$

THEOREM 8.2. *Any two perpendiculars to one straight line lying on one plane are parallel.*

EXERCISE 8.1. *Derive the theorem 8.2 from the theorem 8.1.*

CHAPTER IV

CONGRUENT TRANSLATIONS AND MOTIONS.

§ 1. Orthogonality of a straight line and a plane.

Definition 1.1. Assume that a straight line a intersects a plane α at a point O. The line a is said to be *perpendicular* to the plane α if it is perpendicular to all straight lines lying on the plane α and passing through the point O.

Theorem 1.1. *A line a intersecting a plane α at a point O is perpendicular to this plane if and only if it is perpendicular to some two distinct straight lines lying on the plane α and passing through the point O.*

Proof. The necessity of the condition formulated in the theorem is obvious: if the line a is perpendicular to the plane α, then it is perpendicular to all lines on this plane passing through the point O, including those two of them mentioned in the theorem.

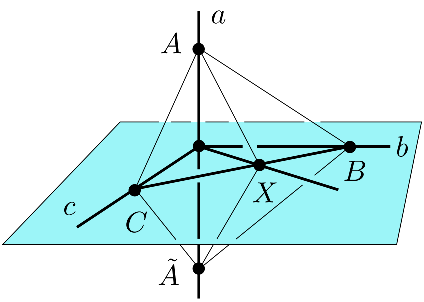


Fig. 1.1

Let's prove the sufficiency. Assume that the line a is perpendicular to the lines b and c lying on the plane α and intersecting at the point O. Let's consider some arbitrary straight line x lying on the plane α and passing through the point O. The point O divides the line x into two rays. Let's consider one of these rays $[O,+\infty)$. It lies inside one the four angles formed by the lines a and b at the intersection point O. On the sides of this angle we mark two points B and C, then we draw the segment $[BC]$. According to the lemma 6.2 from Chapter II, the ray $[O,+\infty)$ intersects the segment $[BC]$ at some interior point X.

Let's choose some arbitrary point A different from O on the line a and then, applying the axiom A13, on the ray opposite to $[OA\rangle$ we mark a point $\tilde{A}$ such that $[O\tilde{A}]\cong[OA]$. Since $a\perp b$, we conclude that the angles $\angle AOB$ and $\angle\tilde{A}OB$ are right angles. They are congruent to each other $\angle AOB\cong\angle\tilde{A}OB$. Moreover, $[OB]\cong[OB]$. From these three relationships

$$[OA]\cong[O\tilde{A}],\qquad [OB]\cong[OB],\qquad \angle AOB\cong\angle\tilde{A}OB$$

we derive that the triangles AOB and $\tilde{A}OB$ are congruent. Hence, $[AB]\cong[\tilde{A}B]$. In a similar way from $a\perp c$ we derive the congruence $[AC]\cong[\tilde{A}C]$. Let's complement these two relationships with one more:

$$[AB]\cong[\tilde{A}B],\qquad [AC]\cong[\tilde{A}C],\qquad [BC]\cong[BC].$$

Now from these three relationships we derive that the triangles ABC and $\tilde{A}BC$ are congruent, which yields $\angle ABC\cong\angle\tilde{A}BC$.

Let's consider the triangles ABX and $\tilde{A}BX$. For their sides and angles the following relationships are fulfilled:

$$[AB]\cong[\tilde{A}B],\qquad [BX]\cong[BX],\qquad \angle ABX\cong\angle\tilde{A}BX.$$

These relationships yield the congruence of the triangles ABX and $\tilde{A}BX$. Hence, for the segments $[AX]$ and $[\tilde{A}X]$ we have $[AX]\cong[\tilde{A}X]$. In other words, the triangle $AX\tilde{A}$ is isosceles, while the segment $[OX]$ is a median of it since O is the center of the segment $[A\tilde{A}]$ by our choice of the points A and $\tilde{A}$. The rest is to apply the theorem 7.1 from Chapter III. According to this theorem the median $[OX]$ in the isosceles triangle $\tilde{A}XA$ is its height at the same time. Therefore, we get the required relationship $a \perp x$. $\square$

THEOREM 1.2. *For any line a and a point O on this line there is exactly one plane α passing through the point O and being perpendicular to the line a.*

PROOF. For the beginning we prove the existence of the required plane α, then we prove its uniqueness. Let's choose a point B outside the line a and draw the plane β passing through the line a and the point B. On the plane β we apply the theorem 6.3 from Chapter III to the line a and the point $O \in a$. This yields a line $b \in \beta$ passing through the point O and being perpendicular to the line a.

In the next step we choose a point C not lying on the plane β. Through the line a and the point C we draw a plane γ. It is clear that the planes β and γ are distinct,the line a being the intersection of these planes. Applying the theorem 6.3 from Chapter III to the line $a \in \gamma$, and the point O, we get a line c on the plane γ passing through the point O and being perpendicular to the line a.

The lines b and c belong to the different planes β and γ and pass through the point O. They are two distinct straight lines intersecting at the point O. There is a plane α passing through such two lines (see theorem 1.5 in Chapter II). But $a \perp b$ and $a \perp c$ by construction. Therefore, according to the above theorem 1.1, we have $a \perp \alpha$. A required plane α is constructed.

Now let's prove the uniqueness of the constructed plane α. Assume that there is another plane $\tilde{\alpha} \perp a$ passing through the point O. The planes α and $\tilde{\alpha}$ are distinct but have the common point O. Therefore, the intersect along some line b which passes trough the point O. Since $\alpha \perp a$, we have $b \perp a$.

The lines a and b are perpendicular to each other. They intersect at the point O. There is a plane β passing through such two lines a and b. Let's choose a point C not lying on the plane β and let's draw a plane γ through the point C and the line a. Note that such a plane γ is different from α and from $\tilde{\alpha}$. It follows from the fact that the plane γ intersects with the line b at the single point O, while the planes α and $\tilde{\alpha}$ contain the line b in whole. The plane γ intersecting with the planes α and $\tilde{\alpha}$ yields two lines c and $\tilde{c}$. The lines c and $\tilde{c}$ lie on the plane γ and pass through the point O. From $\alpha \perp a$ and $\tilde{\alpha} \perp a$ we derive $c \perp a$ and $\tilde{c} \perp a$ for them. If $c \neq \tilde{c}$, then on the plane γ we would have two perpendiculars to the line a passing through the point $O \in a$. This would contradict the theorem 6.3 from Chapter III. Hence, $c = \tilde{c}$. From $c = \tilde{c}$ we easily derive $\alpha = \tilde{\alpha}$. The uniqueness of the plane α is proved. $\square$

Theorem 1.3. *For any plane α and a point $O \in \alpha$ there is exactly one straight line a passing through the point O and being perpendicular to the plane α.*

Proof. Let's begin with proving the existence of a required line. On the plane α we choose some arbitrary point B different from O and draw the line OB. Let's denote this line through b. Then we apply the above theorem 1.2 to the line b and the point O. As a result we get a plane γ passing through the point O and being perpendicular to the line b. Intersecting with α, the plane γ produces a line c passing through the point O. To γ, c, and O we apply the theorem 6.3 from Chapter III. As a result we get a line $a \in \gamma$ passing through the point O and being perpendicular to the line c. From $a \subset \gamma$ and $\gamma \perp b$ we derive $a \perp b$. Thus, for the constructed line a we have

$$a \perp b, \qquad\qquad a \perp c.$$

This means that the line a passes through the point O and perpendicular to the lines b and c lying on the plane α and intersecting at the point O. According to the theorem 1.1, the line a is perpendicular to the plane α. The existence of a required line a is proved.

The rest is to prove the uniqueness of the constructed line a. Assume that there is another line $\tilde{a}$ passing through the point O and being perpendicular to the plane α. If $a \neq \tilde{a}$, then the pair of lines a and $\tilde{a}$ intersecting at the point O defines a plane β. Intersecting with α, the plane β produces a line b passing through the point O. From $b \subset \alpha$, from $a \perp \alpha$, and from $\tilde{a} \perp \alpha$ we conclude that on the plane β there are two perpendiculars a and $\tilde{a}$ to the line b passing through the point $O \in b$. This fact contradicts the theorem 6.3 from Chapter III. The contradiction obtained proves the coincidence $a = \tilde{a}$ and, thus, it proves the uniqueness of the required line a. $\square$

Theorem 1.4. *Assume that a line b intersects a line a at a point O. The line b is perpendicular to the line a if and only if it lies in a plane α passing through the point O and being perpendicular to the line a.*

Theorem 1.5. *A plane α passing through a point O on a line a and being perpendicular to this line is the union of all straight lines passing through the point O and being perpendicular to a.*

Exercise 1.1. *Draw figures illustrating the proofs of the theorems 1.2 and 1.3.*

Exercise 1.2. *Prove the theorems 1.4 and 1.5.*

§ 2. A perpendicular bisector of a segment and the plane of perpendicular bisectors.

Definition 2.1. A straight line a passing through the center of a segment $[AB]$ and being perpendicular to it is called a *perpendicular bisector* of this segment.

Let O be the center of a segment $[AB]$ and let M be some point on its perpendicular bisector a distinct from the point O. Then from $[AO] \cong [BO]$ and $[OM] \cong [OM]$ it follows that the triangles AOM and BOM are congruent to each other. Hence, we get $[AM] \cong [BM]$.

Conversely, assume that $[AM] \cong [BM]$. Then the triangle AMB is isosceles. Its median $[MO]$ is its height at the same time (see theorem 7.1 in Chapter III). Hence, the line AM is a perpendicular bisector for the segment $[AB]$. The conclusion is that a point M satisfies the relationship $[AM] \cong [BM]$ if and only if it lies on some perpendicular bisector of the segment $[AB]$.

Let's consider the set of all perpendicular bisectors of the given segment $[AB]$. This is the set of all straight lines being perpendicular to the line AB and passing through the point O. According to the theorem 1.5, such a set is a plane passing through the point O and being perpendicular to the segment $[AB]$. This plane is called the *plane of perpendicular bisectors* of the segment $[AB]$. Now we can formulate the following theorem.

Theorem 2.1. *For any two points A and B a point M satisfies the condition $[AM] \cong [BM]$ if and only if it lies on the plane of perpendicular bisectors of the segment $[AB]$.*

§ 3. Orthogonality of two planes.

Lemma 3.1. *Assume that two planes α and β have a common point O. Under this assumption if the plane β contains the perpendicular to the plane α passing through the point O, then the plane α contains the perpendicular to the plane β passing through the same point O.*

Proof. Let a be the perpendicular to the plane α passing through the point O and assume that b is the perpendicular to the plane β also passing through the point O. Let's denote by c the intersection of the planes α and β. According to the statement of the lemma, $a \subset \beta$. Hence, $b \perp a$ and $b \perp c$. Moreover, $a \perp c$, since the line c lies on the plane α, while a is perpendicular to the plane α.

Two lines b and c intersecting at the point O define some plane γ. From $a \perp b$ and $a \perp c$ due to the theorem 1.1 we get

$a \perp \gamma$. But the plane passing through the point O and being perpendicular to the line a is unique. Therefore, the plane γ should coincide with the plane α. Hence, we get the required inclusion $b \subset \alpha$. $\square$

Definition 3.1. Assume that two planes α and β have a common point O. The plane α is said to be *perpendicular to the plane β at the point O* if it contains the perpendicular to the plane β passing through the point O.

The above lemma 3.1 shows that the relation of orthogonality of planes at a point is symmetric, i. e. $\alpha \perp \beta$ implies $\beta \perp \alpha$.

Theorem 3.1. *Assume that two planes α and β intersect along a line c. In this case if the plane β is perpendicular to the plane α at some point $A \in c$, then the plane β is perpendicular to the plane α at any other point B of the line c.*

Proof. Let's denote by a and b the perpendiculars to the planes α and β respectively passing through the point A. Since α is perpendicular to β at the point A, we have $a \subset \beta$ and $b \subset \alpha$. Let's apply the theorem 6.3 from Chapter III to the line AB and the point B on the plane β. It yields a straight line lying on the plane β, passing through the point B, and perpendicular to the line AB. Let's show that this line is a perpendicular to the plane α. For this purpose we mark some point C on it, then we choose a segment $[AD]$ on the line b such that $[AD] \cong [BC]$. For the rectangular triangles ABC and BAD we have the relationships

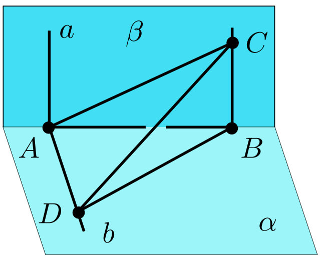


Fig. 3.1

$$[AB] \cong [BA], \qquad [BC] \cong [AD], \qquad \angle ABC \cong \angle BAD.$$

From these relationships we derive the congruence of these triangles ABC and BAD. Hence, $[AC] \cong [BD]$.

Now let's consider the triangles CAD and DBC. These triangles appear to be congruent due to the following congruence relationships for their sides:

$$[AD] \cong [BC], \qquad [AC] \cong [BD], \qquad [CD] \cong [DC].$$

Since the triangles CAD and DBC are congruent, we have the congruence of the angles $\angle CAD \cong \angle DBC$. But b is a perpendicular to the plane β, whic means that the angle $\angle CAD$ is a right angle. Hence, the angle $\angle DBC$ is also aright angle. This yields $BC \perp BD$ and $BC \perp AB$. In other words, the line BC is perpendicular to the pair of lines AB and BD lying on the plane α. Therefore, it is a perpendicular to the plane α passing through the point B. The line BC belongs to the plane β, hence, we obtain the required orthogonality of the plane α and the plane β at the point B. $\square$

The theorem 3.1 shows that the orthogonality of planes is their global property: if it takes place at some point, it is present at all other points of the intersection of two planes.

Theorem 3.2. *For any two perpendiculars to a given plane α there is a plane β containing both of them. This plane β is perpendicular to the plane α.*

The theorem 3.2 does not require a separate proof. The required plane β was constructed in proving the theorem 3.1.

Theorem 3.3. *Any two perpendiculars to a given plane α are parallel to each other.*

The theorem 3.3 is easily derived from the theorem 3.2 and from the theorem 8.2 from Chapter III.

Definition 3.2. Assume that B is some point not lying on a plane α. A segment $[BA]$ connecting the point B with some point $A \in \alpha$ is called a *perpendicular* dropped from the point B onto the plane α if the line AB is perpendicular to the plane α. The point $A \in \alpha$ is called the *foot* of the perpendicular or the *orthogonal projection* of the point B onto the plane α.

THEOREM 3.4. *From any point $B \notin \alpha$ one can drop exactly one perpendicular onto the plane α.*

PROOF. For the beginning let's prove the existence of a perpendicular dropped from the point B onto the plane α. Let's choose some point $O \in \alpha$. If the line OB is perpendicular to the plane α, then the segment $[BO]$ is a required perpendicular.

Let's conside the case where $[BO]$ is not a perpendicular to the plane α. Using the theorem 1.3, we draw the line a being perpendicular to the plane α and passing through the point O. The segment $[BO]$ does not lie on the line a, hence, $B \notin a$. Mow we draw a plane β passing through the line a and the point B. Let's denote by c the line being the intersection of the planes α and β and then apply the theorem 6.5 from Chapter III to the point B and the line c. As a result we get a point A on the line c such that the line AB is perpendicular to the line c.

Let's show that the segment $[BA]$ is a required perpendicular dropped from the point B onto the plane α. For this purpose note that the plane β contains the perpendicular to the plane α drawn at the point O. According to the theorem 3.1 it contains a perpendicular to α drawn at the point A either. Let's denote this perpendicular by $\tilde{a}$. From $\tilde{a} \subset \beta$ and $\tilde{a} \perp c$, applying the theorem 6.5 from Chapter III, we derive the coincidence $\tilde{a} = AB$.

Let's show that the perpendicular $[BA]$ constructed above is unique. If we assume that another perpendicular $[BA']$ does exist, then in the triangle ABA' we would have two right angles, which contradicts the theorem 6.6 from Chapter III. The uniqueness of the perpendicular $[BA]$ can be derived from the theorem 3.3 too. $\square$

EXERCISE 3.1. *Prove the theorems 3.2 and 3.3.*

§ 4. A dihedral angle.

Let α and β be two planes intersecting along some straight line a. Each of these two planes α and β divides the space $\mathbb{E}$ into two half-spaces. We describe this fact as follows:

$$\mathbb{E} = \alpha_- \cup \alpha \cup \alpha_+, \qquad\qquad \mathbb{E} = \beta_- \cup \beta \cup \beta_+.$$

DEFINITION 4.1. A *dihedral angle* is the intersection of two closed half-spaces determined by two intersecting planes. The intersection of the corresponding open half-spaces is called the *interior* of a dihedral angle. The straight line produced as the intersection of two planes confining a dihedral angle is called an *edge* of this dihedral angle.

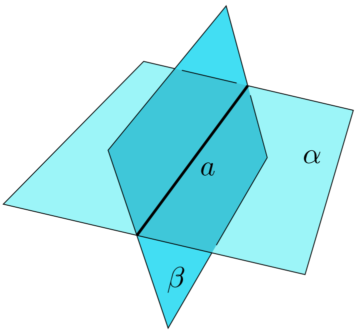


Fig. 4.1

By the intersection of two planes α and β four dihedral angles arise at a time. These are

$$\begin{aligned} &\overline{\alpha_+} \cap \overline{\beta_+}, &\qquad &\overline{\alpha_+} \cap \overline{\beta_-},\\ &\overline{\alpha_-} \cap \overline{\beta_-}, &\qquad &\overline{\alpha_-} \cap \overline{\beta_-}. \end{aligned}$$

Note that each dihedral angle is the union of its interior and two closed half-planes cut by the edge a on the planes α and β. These two half-planes ate called the *sides* of a dihedral angle.

Let's consider some dihedral angle with the sides on two planes α and β (see Fig. 4.2 below). Let's choose some arbitrary point O on its edge a and draw the plane γ passing through the point O and being perpendicular to the line a. The plane γ intersecting with the sides of the dihedral angle yields two rays h and k lying on the planes α and β and being perpendicular to the edge of the dihedral angle. They form the angle $\angle hk$ which is called the *plane angle* of the dihedral angle at the point O. The plane angle of a dihedral angle depends on a point O on its edge. However, the following theorem shows that all plane angles of a dihedral angle are equipollent.

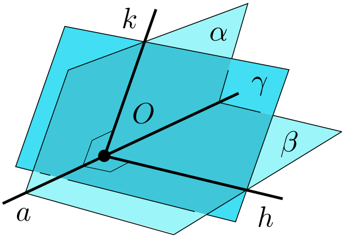


Fig. 4.2

THEOREM 4.1. *All plane angles of a given dihedral angle are congruent to each other.*

Before proving this theorem we consider some additional constructions on a plane. Let c be some straight line lying on a plane α and assume that A and B are two points of this line. On the plane α we draw two lines passing through these points and being perpendicular to the line c. According the theorem 8.2 from Chapter III, such two lines are parallel. On these lines we choose the rays lying in one half-plane with respect to the line c. We denote these rays through a and b. On the ray a coming out from the point A we mark some point C. Then on the ray b we choose a point D so that the condition $[BD] \cong [AC]$ is fulfilled. Such a choice is enabled by the axiom A13. Moreover, due to this axiom such a point D is unique. Let's connect the points C and D with the points A and B by means of the segments $[AD]$ and $[BC]$.

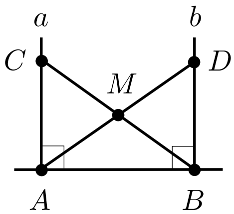


Fig. 4.3

Since the lines AC and BD are parallel, the segment $[BD]$ does not intersect the line AC. Hence, the points B and D lie on one half-plane with respect to the line AC. Moreover, the points C and D by construction lie on one half-plane with respect to the line AB. Hence, the point D and the ray $[AD\rangle$ lie inside the angle $\angle BAC$. Applying the lemma 6.2 from Chapter II, we find that the ray $[AD\rangle$ intersects the segment $[BC]$ at some its interior point M.

The similar considerations can be applied to the ray $[BC\rangle$ and the angle $\angle ABD$. They prove that the point M is an interior point of the segment $[AD]$.

Let's consider the rectangular triangles ABD and BAC. For these triangles the following conditions are fulfilled:

$$[AB] \cong [BA], \qquad [AC] \cong [BD], \qquad \angle BAC \cong \angle ABD.$$

From these relationships we derive that the triangles ABD and BAC are congruent. Hence, $\angle DAB \cong \angle CBA$. We apply this congruence to the triangle AMB. According to the theorem 5.2 from Chapter III, it is congruent to itself under exchanging its vertices A and B. This yields $[AM] \cong [BM]$, i. e. the triangle AMB is isosceles.

Apart from $\angle DAB \cong \angle CBA$, the congruence of the triangles ABD and BAC yields $[AD] \cong [BC]$. Combining this relationship with the relationship $[AM] \cong [BM]$, we get $[CM] \cong [DM]$. This result follows from the axiom A15. It means that the triangle CMD is also isosceles. In Euclidean geometry the isosceles triangles AMB and CMD are congruent. However, in order to prove this congruence one should use the axiom A20, which is not yet considered.

Proof of the theorem 4.1. Let's consider a dihedral angle and mark two arbitrary points A and B on its edge (see Fig. 4.4 below). Then we draw two planes passing through these points and being perpendicular to the edge of our dihedral angle. These planes cut out four rays on the sides of the dihedral angle, which determine two plane angles with the vertices at the points A and B. On the sides of one of these plane angles we choose the points C and E. On the sides of the other plane angle we mark two points D and F such that the conditions

$$[AC] \cong [BD], \qquad\qquad [AE] \cong [BF]$$

are fulfilled. Now let's draw the segments $[AD]$, $[BC]$, $[AF]$, and

$[BE]$. Then on each side of the dihedral angle we get the pattern shown on Fig. 4.3. This pattern is already studied in details. We are going to apply the above results.

At the intersection of the segments $[AD]$ and $[BC]$ we have the point M and, similarly, the point N is at the intersection of the segments $[AF]$ and $[BE]$. As we know, the triangles AMB and ANB are isosceles. This fact yields the relationships

$$[AM] \cong [BM],$$
$$[AN] \cong [BN].$$

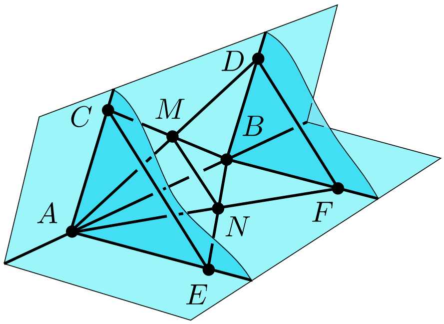


Fig. 4.4

Let's complement these relationships with

$$[MN] \cong [MN].$$

Then, applying the theorem 5.5 from Chapter III, we derive the congruence of the triangles AMN and BMN. Hence, we have $\angle MAN \cong \angle MBN$ or, equivalently, $\angle DAF \cong \angle CBE$.

Now let's use the relationships $[AD] \cong [BC]$ and $[AF] \cong [BE]$, which arise in considering the rectangular triangles CAB, ABD, EAB, and ABF. Complementing them with $\angle DAF \cong \angle CBE$ and applying the theorem 5.1 from Chapter III, we derive the congruence of the triangles DAF and CBE. Now we have

$$[CE] \cong [DF], \qquad [AC] \cong [BD], \qquad [AE] \cong [BF],$$

which implies the congruence of the triangles CAE and DBF. Hence, $\angle CAE \cong \angle DBF$, which means that two plane angles of our dihedral angle are congruent. $\square$

§ 5. Congruent translations of a plane and the space.

The concept of congruent translation for straight lines was introduced in § 3 of Chapter III (see definition 3.1). It is easily generalized for the case of planes and for the whole space.

Definition 5.1. A mapping $f\colon \alpha \to \beta$ is called a *congruent translation* of a plane α to a plane β if for any two points X and Y on the plane α the condition $[f(X)f(Y)] \cong [XY]$ is fulfilled.

Definition 5.2. A mapping $f\colon \mathbb{E} \to \mathbb{E}$ is called a *congruent translation* of the space if for any two points X and Y the condition congruence $[f(X)f(Y)] \cong [XY]$ is fulfilled.

Theorem 5.1. *Let f be a congruent translation of a plane to another plane or a congruent translation of the space in whole. In both cases the following propositions are valid:*

(1) *for any three points X, Y, and Z from the domain of the mapping f if they lie on one straight line, then their images $f(X)$, $f(Y)$, and $f(Z)$ also lie on one straight line so that $(X \blacktriangleright Y \blacktriangleleft Z)$ implies $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$;*
(2) *for any three points X, Y, and Z from the domain of f if they do not lie on one straight line, then their images $f(X)$, $f(Y)$, and $f(Z)$ also do not lie on one straight line and the triangle XYZ is congruent to the triangle $f(X)f(Y)f(Z)$.*

Before proving this theorem we shall formulate and prove the following auxiliary lemma.

Lemma 5.1. *Assume that A, B, and C are three points not lying on one straight line. Then for any point $\tilde{B}$ on the line AC the conditions $[AB] \cong [A\tilde{B}]$ and $[BC] \cong [\tilde{B}C]$ cannot be fulfilled simultaneously.*

Proof. The proof is by contradiction. Assume that $\tilde{B} \neq B$ is a point for which both conditions $[AB] \cong [A\tilde{B}]$ and $[BC] \cong [\tilde{B}C]$ are fulfilled. Applying the theorem 2.1, we find that the points A and C lie on the plane of perpendicular bisectors of the segment

$[B\tilde{B}]$. The intersection of this plane with the plane of the triangle ABC is some particular perpendicular bisector of the segment $[B\tilde{B}]$ containing both points A and C. If O is the center of the segment, it belongs to any perpendicular bisector, in particular, we have $O \in AC$. If we assume that $\tilde{B} \in AC$, then from $\tilde{B} \in AC$ and from $O \in AC$ we would conclude that the lines $\tilde{B}O$ and AC do coincide. Hence, $B \in AC$, which contradicts the premise of the lemma: A, B, and C are three points not lying on one straight line. The contradiction obtained shows that $\tilde{B} \notin AC$. The proof of the lemma is complete. $\square$

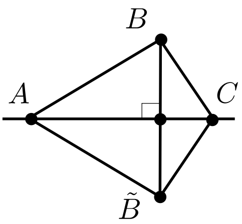


Fig. 5.1

Proof of the theorem 5.1. Let's begin with proving the item (1) of the theorem. Assume that the points X, Y, Z taken from the domain of the mapping f lie on one straight line. If the main proposition in the item (1) is not valid, then $f(X)$, $f(Y)$, and $f(Z)$ does not lie on one straight line at least for one particular choice of the points X, Y, Z lying on one straight line. Then they define a triangle $f(X)f(Y)f(Z)$ for whose sides the relationships $[XY] \cong [f(X)f(Y)]$, $[YZ] \cong [f(Y)f(Z)]$, and $[XZ] \cong [f(X)f(Z)]$ are fulfilled. Relying upon the last relationship $[XZ] \cong [f(X)f(Z)]$, we apply the theorem 2.1 from Chapter III to the points X, Y, and Z. Due to this theorem we can find a point $\tilde{Y}$ on the line $[f(X)f(Z)]$ such that $[XY] \cong [f(X)\tilde{Y}]$ and $[YZ] \cong [\tilde{Y}f(Z)]$. Comparing these relationships with $[XY] \cong [f(X)f(Y)]$ and $[YZ] \cong [f(Y)f(Z)]$, we note that we got exactly in a situation forbidden by the above lemma 5.1. Hence, our assumption that the points $f(X)$, $f(Y)$, and $f(Z)$ do not lie on one straight line is wrong. Thus, we have

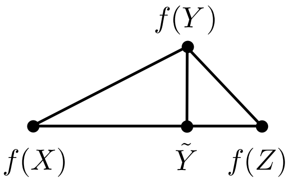


Fig. 5.2

proved the main proposition of the item (1) in the theorem 5.1. Now the relationship $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$ is derived from $(X \blacktriangleright Y \blacktriangleleft Z)$ by means of the theorem 2.2 from Chapter III.

Let's prove the item (2) of the theorem. Now the points X, Y, and Z do not lie on one straight line. Assume that their images $f(X)$, $f(Y)$, and $f(Z)$ appear to be lying on one straight line. Let's use the relationship $[XZ] \cong [f(X)f(Z)]$ and apply the theorem 2.1 from Chapter III. Due to this theorem we can find a point $\tilde{Y}$ lying on the segment $[XZ]$ and such that $[X\tilde{Y}] \cong [f(X)f(Y)]$ and $[\tilde{Y}Z] \cong [f(Y)f(Z)]$. Now, if we recall the relationship $[XY] \cong [f(X)f(Y)]$ and $[YZ] \cong [f(Y)f(Z)]$, we see that again we are in a situation forbidden by the lemma 5.1. Hence, our preliminary assumption that the points $f(X)$, $f(Y)$, and $f(Z)$ lie on one straight line is not valid. The main proposition of the item (2) in the theorem 5.1 is proved. The rest is to prove the congruence of the triangles XYZ and $f(X)f(Y)f(Z)$. It follows from $[XY] \cong [f(X)f(Y)]$, $[YZ] \cong [f(Y)f(Z)]$, and $[XZ] \cong [f(X)f(Z)]$. $\square$

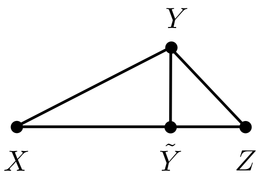


Fig. 5.3

As a corollary of the theorem 5.1 and the theorem 2.2 from Chapter III we find that each congruent translations maps a straight line onto a straight line and a ray onto a ray. Under such a mapping each angle is mapped onto a congruent angle. In particular, this means that congruent translations preserve orthogonality of lines.

Let $f\colon \mathbb{E} \to \mathbb{E}$ be a congruent translation of the space. Using the theorem 1.5, we conclude that such a mapping takes a plane onto a plane preserving the orthogonality of planes and preserving the orthogonality of a plane and a straight line. Hence, the restriction of a congruent translation of the space to some plane appears to be a congruent translation of planes, while the restriction of a congruent translation of a plane to some straight line appears to be a congruent translation of straight lines.

Let's consider a congruent translation f mapping a plane α onto a plane β. Let a be some straight line on the plane α. Then, as we noted above, the points of the line a are mapped to the points of some straight line b lying on the plane β. Let X and Y be two points of the plane α lying in different half-planes with respect to the line a. Then the segment $[XY]$ crosses the line a at some its interior point O. Due to the theorem 5.1 the point $f(O)$ is the intersection point of the lines $f(X)f(Y)$ and b. From $(X \blacktriangleright O \blacktriangleleft Y)$ we derive $(f(X) \blacktriangleright f(O) \blacktriangleleft f(Y))$. This means that the points $f(X)$ and $f(Y)$ lie on different sides of the line b on the plane β.

Now let's consider two points X and Y lying on the same half-plane with respect to the line a on the plane α. Let Z be a point lying on the other half-plane. Then due to above considerations the points $f(X)$ and $f(Z)$ lie on different half-planes with respect to the line b on the plane β. The points $f(Y)$ and $f(Z)$ are also on different half planes. Therefore, the points $f(X)$ and $f(Y)$ lie on the same side of the line b on the plane β. Thus, we have the following result.

THEOREM 5.2. *Any congruent translation of planes and any congruent translation of the whole space are «half-planes preserving maps», i. e. they take a half-plane onto a half-plane.*

THEOREM 5.3. *Each congruent translation of the whole space takes a half-space onto a half-space.*

The proof of the theorem 5.3 is analogous to the proof of the theorem 5.2. We do not give it here.

Let $f : \alpha \to \beta$ be a congruent translation of planes. Let's consider some arbitrary straight line a on the plane α. The points of this line are mapped into some definite line b on the plane β. The line a divides the plane α into two half-planes a_+ and a_-. Let's denote by b_+ that half-plane on the plane β to which the points of a_+ are mapped. Let X be some point of the half-plane a_+. Within the plane α we can drop the perpendicular onto the line a. Let's denote by X_0 the foot of this perpendicular. Any congruent translation preserves the orthogonality, therefore, the point $f(X_0)$ is the foot of the perpendicular dropped from $f(X)$ onto the line b within the plane β. Being more precise, the following three conditions are fulfilled for the point $f(X)$:

$$[f(X)f(X_0)] \perp b, \qquad [f(X)f(X_0)] \cong [XX_0], \qquad f(X) \in b_+.$$

Note that these conditions fix uniquely the point $f(X)$ on the plane β provided the point $f(X_0)$ on the line b is given. In a

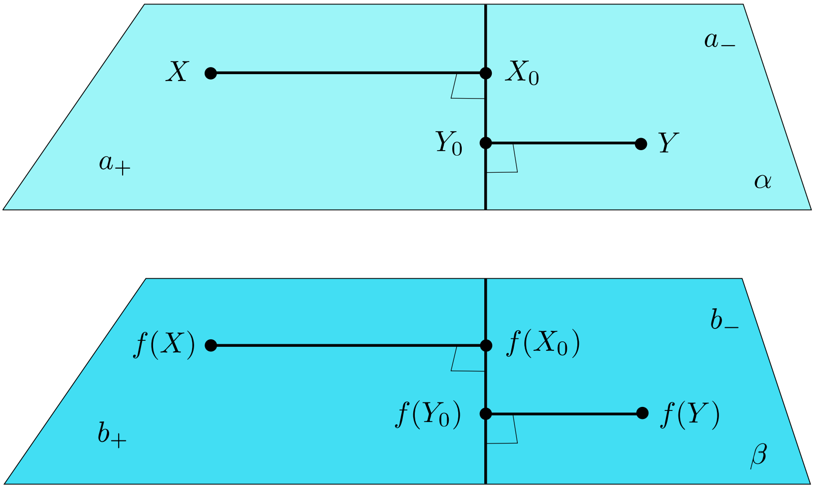


Fig. 5.4

similar way, for an arbitrary point Y from the half-plane a_- and for the foot Y_0 of the perpendicular dropped from Y onto the line a the following conditions are fulfilled:

$$[f(Y)f(Y_0)] \perp b, \qquad [f(Y)f(Y_0)] \cong [YY_0], \qquad f(Y) \in b_-.$$

These conditions fix uniquely the point $f(Y)$ on the plane β upon fixing the point $f(Y_0)$ on the line b. These observations lead to the following theorem.

THEOREM 5.4. *Let a be some straight line on a plane α dividing this plane into two half-planes a_+ and a_-. Let b be some straight line on a plane β dividing this plane into two half-planes b_+ and b_-. Then any congruent translation of lines $f : a \to b$ has a unique extension up to a congruent translation of planes $f : \alpha \to \beta$ such that the half-plane a_+ is mapped onto b_+ and the half-plane a_- is mapped onto b_-.*

PROOF. Let $X \in a_+$. We determine $f(X)$ by means of the following construction. From the point X we drop the perpendicular onto the line a and denote by X_0 the foot of this perpendicular. Applying the congruent translation of lines $f : a \to b$ to the point X_0, we get the point $f(X_0)$ on the line b. Then on the plane β we draw the line passing through the point $f(X_0)$ and being perpendicular to the line b. On this line we choose the ray coming out from the point $f(X_0)$ and lying on the closed half-plane $\overline{b_+}$. Then the point $f(X)$ on this ray is determined by the condition $[XX_0] \cong [f(X)f(X_0)]$. For a point $Y \in a_-$ the procedure of constructing the point $f(Y)$ differs only by the choice of the ray $[f(Y_0)f(Y)\rangle$ lying not on $\overline{b_+}$, but on $\overline{b_-}$ (see Fig. 5.4 above).

The above construction yields a mapping $f : \alpha \to \beta$. For the points $X \in a$ this mapping coincide with the initial mapping $f : a \to b$. Let's prove that this mapping is a congruent translation of planes. For this purpose we need to prove that $[XY] \cong [f(X)f(Y)]$ for any two points X and Y on the plane α. Let's consider the following four cases:

(1) both point X and Y lie on the line a;
(2) only one of the points X or Y lies on the line a;
(3) the points X and Y do not belong to the line a and are on the same side with respect to this line;
(4) the points X and Y are on different sides with respect to a.

In the first case the relationship $[XY] \cong [f(X)f(Y)]$ follows from the fact that the initial mapping $f\colon a \to b$ is a congruent translation of lines.

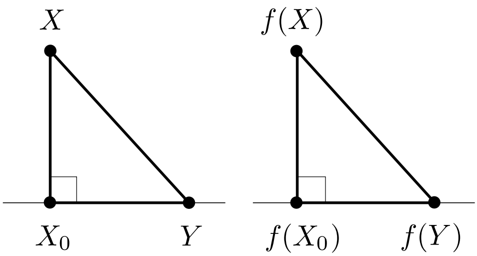


Fig. 5.5

In the second case we assume that $Y \in a$ and $X \notin a$ for the sake of certainty. Then on the planes α and β we have two rectangular triangles (see Fig. 5.5). In these triangles for their sides $[X_0Y]$ and $[f(X_0)f(Y)]$ we have the relationship $[X_0Y] \cong [f(X_0)f(Y)]$ following from the fact that $f\colon a \to b$ is a congruent translation of lines. Moreover, by construction of the point $f(X)$ we have $[XX_0] \cong [f(X)f(X_0)]$. Combining this relationship with the congruence of the right angles $\angle XX_0Y \cong \angle f(X)f(X_0)f(Y)$, we get the congruence of the triangles XX_0Y and $f(X)f(X_0)f(Y)$. This congruence yields the required relationship $[XY] \cong [f(X)f(Y)]$.

Let's consider the third case. The congruence of the rectangular triangles X_0Y_0Y and $f(X_0)f(Y_0)f(Y)$ on this case is proved just like above in the previous case. From this congruence we derive that the angle $\angle YX_0Y_0$ and the angle $\angle f(Y)f(X_0)f(Y_0)$ are congruent. We also derive the congruence of the segments $[X_0Y]$ and $[f(X_0)f(Y)]$. Then we take into account the congruence of the right angles $\angle f(X)f(X_0)f(Y_0)$ and $\angle XX_0Y_0$ and apply the theorem 5.3 from Chapter III. As a result we obtain the con-

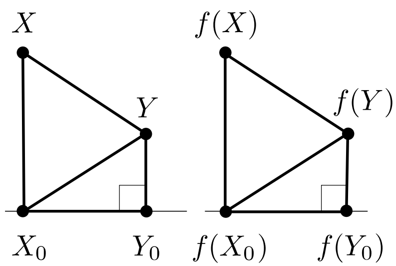


Fig. 5.6

gruence of the angle $\angle f(X)f(X_0)f(Y)$ and the angle $\angle XX_0Y$. From the relationship $[XX_0] \cong [f(X)f(X_0)]$ and from the relationship $[X_0Y] \cong [f(X_0)f(Y)]$ now we derive that the triangles XX_0Y and $f(X)f(X_0)f(Y)$ are congruent. This congruence yields the required relationship $[XY] \cong [f(X)f(Y)]$ for X and Y.

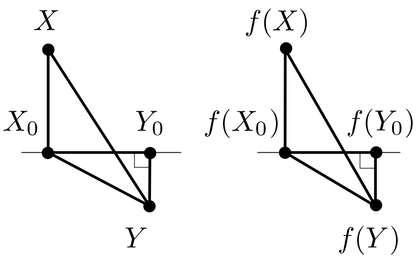


Fig. 5.7

The proof of the congruence $[XY] \cong [f(X)f(Y)]$ in the fourth case almost literally the same as the proof of this fact in the third case. However, we have quite different picture in this case since the points X and Y are on different sides of the line a. By construction their images $f(X)$ and $f(Y)$ are also on different sides of the corresponding line b.

In the cases (2), (3), and (4) there are three degenerate subcases, where $X_0 = Y_0$. We leave the proof of the relationship $[XY] \cong [f(X)f(Y)]$ in these subcases to the reader as an exercise.

Thus we have constructed a mapping $f\colon \alpha \to \beta$ extending the initial mapping $f\colon a \to b$ and have proved that it is a congruent translation. The uniqueness of such an extension follows from the considerations preceding the statement of the theorem 5.4. $\square$

THEOREM 5.5. *Let α be some plane dividing the space into two half-spaces α_+ and α_-. Let β be another plane dividing the space into two half-spaces β_+ and β_-. Then any congruent translation of planes $f : \alpha \to \beta$ has a unique extension up to a congruent translation of the whole space $f\colon \mathbb{E} \to \mathbb{E}$ such that α_+ is mapped onto β_+ and α_- is mapped onto β_-.*

The construction of the required mapping $f\colon \mathbb{E} \to \mathbb{E}$ is analogous to that we used above in proving the theorem 5.4. Other details of the proof for the theorem 5.5 are also very similar to those for the theorem 5.4.

EXERCISE 5.1. *Prove the theorems 5.3 and 5.5.*

EXERCISE 5.2. *Prove that the inversion $i_O : \mathbb{E} \to \mathbb{E}$ with respect to a point O is a congruent translation of the whole space.*

§ 6. Mirror reflection in a plane and in a straight line.

Let's apply the theorems 5.4 and 5.5 in order to construct some particular mappings of congruent translation. Let's consider a plane α and denote by α_+ and α_- the half-planes produced in the space $\mathbb{E}$ by this plane. Let's set $\beta = \alpha$, $\beta_+ = \alpha_-$, and $\beta_- = \alpha_+$. For the initial mapping we take the identical mapping of the plane α onto itself. Since $\beta = \alpha$, we can treat it as $f : \alpha \to \beta$. Applying the theorem 5.5 to $f : \alpha \to \beta$, we get the mapping of congruent translation $z_\alpha : \mathbb{E} \to \mathbb{E}$. This mapping exchanges the half-spaces α_+ and α_-, leaving stable the points of the plane α. Such a mapping z_α is called a *mirror reflection in a plane* α.

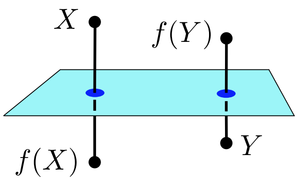


Fig. 6.1

Let a be some straight line lying on the plane α. It divides this plane into two half-planes a_+ and a_-. Let's denote $b = a$, $b_+ = a_-$, and $b_- = a_+$. For the initial mapping $f : a \to b$ we take the identical mapping of the line a onto itself. Then, applying the theorem 5.4, we get a mapping of congruent translation of plane $z_a : \alpha \to \alpha$ that exchanges the half-planes a_+ and a_-. It is called the *mirror reflection of a plane* α *in a line* a.

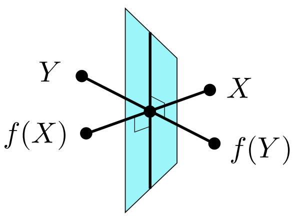


Fig. 6.2

The mapping z_a defined just above can be extended up to a mapping of the whole space. For this purpose we denote again $\beta = \alpha$, $\beta_+ = \alpha_-$, and $\beta_- = \alpha_+$. For the initial mapping $f : \alpha \to \beta$ mapping now we take the mirror reflection of the plane α in the line a. Applying the theorem 5.5, we define a mapping $z_a\colon \mathbb{E} \to \mathbb{E}$, which is called the *mirror reflection of the space* $\mathbb{E}$ *in a line* a. The plane α plays an auxiliary role in defining this mapping $z_a\colon \mathbb{E} \to \mathbb{E}$. There is the following theorem that yields a different way for constructing the mapping $z_a\colon \mathbb{E} \to \mathbb{E}$.

THEOREM 6.1. *For any point $X \notin a$ the segment connecting X with its mirror symmetric point $z_a(X)$ intersects the line a at the point X_0 being its center and this segment is perpendicular to the line a.*

EXERCISE 6.1. *Verify that the theorem 6.1 fixes uniquely the point $z_a(X)$ provided the point X is given.*

EXERCISE 6.2. *Prove the theorem 6.1 and show that the mirror reflections in a line z_a and in a plane z_α satisfy the identities $z_\alpha \circ z_\alpha = \mathrm{id}$ and $z_a \circ z_a = \mathrm{id}$.*

EXERCISE 6.3. *Let a straight line a be a perpendicular to a plane α passing through some point $O \in \alpha$. In this case prove that $z_a \circ z_\alpha = z_\alpha \circ z_a = i_O$.*

§ 7. Rotation of a plane about a point.

Assume that in a plane α an angle $\angle hk$ with the vertex at a point O is given. Let's extend the ray h up to the whole line a, and extend the ray k up to the whole line b. The angle $\angle hk$ is the intersection of two half-planes determined by the lines a and b. For the sake of certainty let's set $\angle hk = \overline{a_+} \cap \overline{b_-}$. According to the result of § 3 in Chapter III, there is a unique mapping of congruent translation of lines $f^+_{OO}\colon a \to b$ taking the point O to itself and mapping the ray to the ray k. Applying the theorem 5.4, we extend this mapping up to a congruent translation of planes $\theta_{hk}\colon \alpha \to \alpha$ mapping a_+ to b_+ and a_- to b_-. Such a mapping is

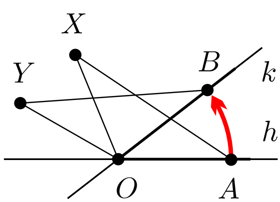


Fig. 7.1

called the *rotation of a plane α about a point O by the angle $\angle hk$ from the ray h toward the ray k.*

Exchanging the rays h and k we do not change the angle $\angle hk$. However, in this case we get the rotation in the opposite direction from the ray k toward the ray h. The mapping θ_{kh} is inverse for θ_{hk}, i. e. we have $\theta_{hk} \circ \theta_{kh} = \theta_{kh} \circ \theta_{hk} = \mathrm{id}_\alpha$.

Assume that in a plane α some rotation angle $\angle hk$ is given. Let's study the procedure of constructing the point $Y = \theta_{hk}(X)$ for some arbitrary point $X \in \alpha$. For the sake of certainty e assume that $X \in a_+$. On the rays h and k we choose two points A and B so that the relationships $[OA] \cong [OX]$ and $[OB] \cong [OX]$ are fulfilled. Then $B = \theta_{hk}(A)$. Let's consider the isosceles triangle AOX. It lies on the closed half-plane $\overline{a_+}$, which is taken to $\overline{b_+}$ under the rotation θ_{hk}. Let's construct the triangle BOY congruent to AOX in the half-plane $\overline{b_+}$. For this purpose in $\overline{b_+}$ we choose a ray forming with the ray k an angle congruent to the angle $\angle XOA$. Afterwards, on this ray we mark a point Y such that $[OY] \cong [OX]$. Now from $O = \theta_{hk}(O)$, from $B = \theta_{hk}(A)$, and from the congruence of the triangles AOX and BOY we derive $Y = \theta_{hk}(X)$. For the case, where $X \in a_-$, the procedure of constructing the point $Y = \theta_{hk}(X)$ is analogous to the above one. The only difference is that the triangle BOY is chosen in the other half-plane $\overline{b_-}$.

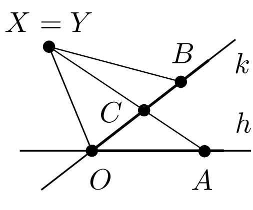


Fig. 7.2

THEOREM 7.1. *The rotation mapping $\theta_{hk} \colon \alpha \to \alpha$ has exactly one stable point. This is the point $O = \theta_{hk}(O)$, about which the rotation is performed.*

PROOF. The mapping θ_{hk} is an extension of the mapping $f^+_{OO} \colon a \to b$ due to the theorem 5.4. The point O is a stable point for $f^+_{OO} : a \to b$, hence it is a stable point for θ_{hk} either. Let's show that the mapping $\theta_{hk} \colon \alpha \to \alpha$ has no other stable points.

Assume that it is not so. If X is another stable point, then the line OX consists of the stable points of the mapping θ_{hk} (this fact follows from the theorem 2.1 in Chapter III). The line OX is distinct from a (since a is mapped to b), therefore, we can assume that the stable point $X \neq O$ is initially chosen on the half-plane a_{+}. Let's apply to X the above procedure of constructing the point $Y = \theta_{hk}(X)$. The condition $X = Y$ leads to the situation shown on Fig. 7.2. From $Y \in b_{+}$ and $A \in b_{-}$ we conclude that the points X and A are on different sides of the line b. Therefore the segment $[AX]$ crosses the line OB at some its interior point C and A is an external point with respect to the segment $[XC]$. Hence, from $X \in a_{+}$ it follows that $C \in a_{+}$. But B lies on the half-plane a_{+}. Hence, the points B and C lie on one side with respect to the line OA and the point O is outside the segment $[BC]$. From this fact it follows that the points B and C lie on one side with respect to the line OX. Now, applying the congruence of angles $\angle OYB \cong \angle OXA$, we derive that the angles $[XA\rangle$ and $[XB\rangle$ do coincide. Then from $[AX] \cong [BY]$ we conclude that $A = B$. But this contradicts the fact that the rays $h \neq k$ form the angle $\angle hk$. The contradiction obtained shows that the mapping θ_{hk} has no stable points other than the point O. $\square$

Theorem 7.2. *Let h and k be two rays coming out from the point O and lying on a plane α. There are exactly two mappings of congruent translation $f\colon \alpha \to \alpha$ with the stable point O that take the ray h to the ray k. The first of them $f = \theta_{hk}$ is the rotation by the angle $\angle hk$ about the point O and the second one $f = z_m$ is the mirror reflection of the plane α in the line m containing the bisector of the angle $\angle hk$.*

Proof. It is easy to see that both mappings θ_{hk} and z_m take the ray h to the ray k. Let a be the line containing the ray h and let b be the line containing the ray k. The restrictions of θ_{hk} and z_m to the line a coincide with the mapping of congruent translation of lines $f^{+}_{OO}\colon a \to b$. But due to the theorem 5.4 we have exactly two extensions of this mapping up to a congruent translation of planes $f\colon \alpha \to \alpha$. The first extension takes a_{+} to

b_+ and a_- to b_-, it is the rotation $f = \theta_{hk}$. The second one takes a_+ to b_- and a_- to b_+, it is the mirror reflection $f = z_m$. $\square$

A remark. In theorem 7.2 there are two special dispositions of the rays h and k where they lie on one straight line n. If $h = k$, then we set $\theta_{hk} = \theta_{hh} = \mathrm{id}$ by definition. In this case the line m coincides with n. If the rays h and k are opposite to each other, they define a straight angle, the bisector of this angle is the perpendicular to the line n passing through the point O. The rotation θ_{hk} in this case is set to be coinciding with the inversion of the plane α with respect to the point O. With these additional provisions the theorem 7.2 remains valid both special dispositions of the rays h and k.

THEOREM 7.3. *For any ray l coming out from a point O and lying on a plane α the equality $\theta_{hk}(l) = q$ implies $\theta_{hk} = \theta_{lq}$.*

PROOF. Indeed, $\theta_{hk}(l) = q$ means that the mapping of congruent translation θ_{hk} takes the ray l to the ray q. It has the unique stable point O. Therefore θ_{hk} coincides with θ_{lq}. $\square$

THEOREM 7.4. *The mappings of rotation of a plane α about a point O and the mappings of mirror reflections in straight lines passing through this point possess the following properties:*

$$\begin{aligned}
&\theta_{kl} \circ \theta_{hk} = \theta_{hl}; && \theta_{hk} \circ z_m = z_m \circ \theta_{kh};\\
&z_m \circ z_m = \mathrm{id}; && z_m \circ z_n = \theta_{hk}, \quad \text{where } \ h \subset n,\ k = z_m(h).
\end{aligned}$$

PROOF. Let h, k, and l be three rays coming out from a point O and lying on a plane α. Assume that none two of them lie on one straight line. Then they define three angles $\angle hk$, $\angle kl$, $\angle hl$ and three rotations θ_{hk}, θ_{kl}, θ_{hl}. Let's denote by f the composition $f = \theta_{kl} \circ \theta_{hk}$. It is easy to see that the mapping f takes the ray h to the ray l. It has the stable point O. Due to the theorem 7.2 we have two options: $f = \theta_{hl}$ or $f = z_m$.

Let's prove that f has the unique stable point O. If we assume that there is another stable point X, then the ray $q = [OX\rangle$

consists of stable points for the mapping f. In this case we would have $\theta_{kl} \circ \theta_{hk}(q) = q$ or $\theta_{hk}(q) = \theta_{lk}(q)$. We denote $\theta_{hk}(q) = p$ and apply the theorem 7.3. As a result we get $\theta_{hk} = \theta_{qp}$ and $\theta_{lk} = \theta_{qp}$. Hence, $\theta_{kh} = \theta_{kl}$, which leads to the coincidence of the rays $h = l$. But this contradicts to the initial assumption that none of the rays h, k, and l lies on one straight line. This contradiction excludes the option $f = z_m$ and proves the first relationship $\theta_{kl} \circ \theta_{hk} = \theta_{hl}$.

Let m be some straight line lying on the plane α and passing through the point O. The point O divides the line m into two rays. Let's denote one of them by l and denote $q = \theta_{hk}(l)$. Then $\theta_{hk} = \theta_{lq}$, which follows from the theorem 7.3. Let's denote $p = \theta_{ql}(l)$. By construction the rays p and q lie on different sides of the line m. From $p = \theta_{ql}(l)$ and $l = \theta_{ql}(q)$ we get these rays form two congruent angles $\angle lq$ and $\angle lp$ with the ray l lying on the line m. Hence, $p = z_m(q)$ and $q = z_m(p)$. This yields

$$\theta_{hk} \circ z_m(l) = \theta_{hk}(l) = \theta_{lq}(l) = q,$$
$$z_m \circ \theta_{kh}(l) = z_m \circ \theta_{ql}(l) = z_m(p) = q.$$

Two mappings $f = \theta_{hk} \circ z_m$ and $g = z_m \circ \theta_{kh}$ take the ray l to the ray q. The equality $f = \theta_{lq}$ is excluded since $f = \theta_{hk} \circ z_m = \theta_{lq}$ implies $z_m = \mathrm{id}$. The equality $g = \theta_{lq}$ is also excluded since $g = z_m \circ \theta_{kh} = \theta_{lq}$ would lead to $z_m = \theta_{lq} \circ \theta_{pl} = \theta_{pq}$. Hence, due to the theorem 7.2 the mappings f and g coincide with the mirror reflection in the line containing the bisector of the angle $\angle lq$. Therefore, $f = g$. This fact proves the second relationship $\theta_{hk} \circ z_m = z_m \circ \theta_{kh}$ in the theorem 7.4.

The third relationship $z_m \circ z_m = \mathrm{id}$ follows immediately from the definition of the mirror reflection $z_m \colon \alpha \to \alpha$ of a plane in a line (see § 6 above).

Let $m \neq n$ be two straight lines lying on the plane α and intersecting each other at the point O. The point O divides the line n into two rays. We denote one of them by h and set $k = z_m(h)$. Then for the mapping $f = z_m \circ z_n$ we have $f(h) = k$.

Then the theorem 7.2 provides two options: $f = \theta_{hk}$ or $f = z_u$, where u is the line containing the bisector of the angle $\angle hk$.

Let's show that the mapping f has the unique stable point O. If we assume that there is another stable point X, then from $z_m \circ z_n(X) = X$ we derive $z_m(X) = z_n(X)$. Let's denote $Y = z_m(X) = z_n(X)$. The coincidence $X = Y$ is excluded since the point O is the unique common stable point for the mappings z_m and z_n. Then from $Y = z_m(X)$ we get that m with the perpendicular bisector of the segment $[XY]$ lying on the plane α. Due to $Y = z_n(X)$ the line n also coincides with this perpendicular bisector, which contradicts the initial assumption $m \neq n$. Hence, f nas no stable points other than O. Therefore, $f \neq z_u$. In this case we have $f = \theta_{hk}$, which completes the proof of the fourth relationship and completes the proof of the theorem in whole. $\square$

Exercise 7.1. *Compare the theorem 7.4 with the theorem 3.2 in Chapter III.*

Exercise 7.2. *Prove that the theorem 7.3 and the theorem 7.4 remain valid in special cases where some two straight lines do coincide or some rays appear to be lying on one straight line.*

§ 8. The total rotation group and the group of pure rotations of a plane.

Let $f : \alpha \to \alpha$ be some mapping of the congruent with a stable point O. Such a mapping is sometimes called a *generalized rotation of a plane α about a point O*. Let's choose some arbitrary ray $h \subset \alpha$ coming out from the point O. The mapping f takes it to the ray $k = f(h)$ coming out from the same point. Now, applying the theorem 7.2, we conclude that any generalized rotation f of a plane α about a point O is some rotation θ_{hk} about this point or a reflection z_m in a line passing through the point O. As a simple consequence of this fact we get that such a mapping f is bijective and has the inverse mapping f^{-1}, which is also a generalized rotation about the point O. The set of all generalized rotations of a plane α about a point O is a group with respect to the composition (see definition 4.2 in Chapter III). The unity of this *total rotation group* is the identical mapping id, which can be interpreted as a special case of the rotation: $\mathrm{id} = \theta_{hh}$ (see the remark to the theorem 7.2).

The set of pure (non-generalized) rotations of a plane α about a point O is also a group. This fact follows from the theorem 7.3 and from the first relationship in the theorem 7.4.

THEOREM 8.1. *Let h, k, l, and q be four rays coming out from one point and lying on one plane α. The equality $\theta_{hk} = \theta_{lq}$ takes place if and only if the bisectors of the angles $\angle lk$ and $\angle hq$ lie on one straight line.*

PROOF. Let $\theta_{hk} = \theta_{lq}$. We denote by m the bisector of the angle formed by the rays l and k. Then $z_m(l) = k$ and $z_m(k) = l$. Let's consider the mapping $f = z_m \circ \theta_{kh} \circ z_m$ and calculate $f(l)$:

$$f(l) = z_m(\theta_{kh}(z_m(l))) = z_m(\theta_{kh}(k)) = z_m(h).$$

On the other hand, applying the second and the third relationships from the theorem 7.4, for the mapping f we get

$$f = (z_m \circ \theta_{kh}) \circ z_m = (\theta_{hk} \circ z_m) \circ z_m = \theta_{hk} \circ (z_m \circ z_m) = \theta_{hk}.$$

Therefore, $z_m(h) = f(l) = \theta_{hk}(l) = \theta_{lq}(l) = q$. This means that the ray q is produced from the ray l with the use of the mirror reflection in the line m. Hence, the bisector of the angle $\angle hq$ lies on the same line m as the bisector of the angle $\angle lk$. The necessity of the proposition stated in the theorem is proved.

Let's prove its sufficiency. Assume that the bisectors of the angles $\angle lk$ and $\angle hq$ lie on one straight line. We denote this line by m. Then $z_m(h) = q$ and $z_m(l) = k$. Let's consider the mapping $f = z_m \circ \theta_{kh} \circ z_m = \theta_{hk}$ again and calculate $\theta_{hk}(l)$:

$$\theta_{hk}(l) = f(l) = z_m(\theta_{kh}(z_m(l))) = z_m(\theta_{kh}(k)) = z_m(h) = q.$$

The relationship obtained $\theta_{hk}(l) = q$ and the theorem 7.3 yield the required result $\theta_{hk} = \theta_{lq}$. $\square$

THEOREM 8.2. *The equality $\theta_{hk} = \theta_{lq}$ implies $\theta_{hl} = \theta_{kq}$ and, conversely, $\theta_{hl} = \theta_{kq}$ implies $\theta_{hk} = \theta_{lq}$.*

THEOREM 8.3. *Any two rotations of a plane α about a point $O \in \alpha$ commute: $\theta_{hk} \circ \theta_{lq} = \theta_{lq} \circ \theta_{hk}$.*

The theorem 8.2 is easily derived from the theorem 8.1. It is analogous to the theorem 4.1 in Chapter III. The theorem 8.3 is analogous to the theorem 4.3 in Chapter III. Ti means that the group of pure rotations of a plane α about a point $O \in \alpha$ is commutative (Abelian) (see definition 4.3 in Chapter III). The following theorem is derived from the theorem 8.1.

THEOREM 8.4. *The equality $\theta_{hk} = \theta_{lq}$ implies the congruence of angles $\angle hk \cong \angle lq$.*

EXERCISE 8.1. *Using the theorem 8.1, prove the theorem 8.2 and the theorem 8.4.*

EXERCISE 8.2. *Using the analogy of the rotations θ_{hk} on a plane and the congruent translations p_{AB} on a straight line, prove the theorem 8.3.*

§ 9. Rotation of the space about a straight line.

Let $\theta_{hk} : \alpha \to \alpha$ be the rotation of a plane α by the angle $\angle hk$ about a point O. The plane α divides the space $\mathbb{E}$ into two half-spaces α_+ and α_-. Let's set $\beta = \alpha$, $\alpha_+ = \beta_+$, $\alpha_- = \beta_-$. Applying the theorem 5.5, we get the mapping $\theta_{hk} : \mathbb{E} \to \mathbb{E}$. Let's draw the straight line c passing through the point O and being perpendicular to the plane α. This line is called the *axis of the rotation*, while the mapping $\theta_{hk} : \mathbb{E} \to \mathbb{E}$ itself is called the *rotation by the angle $\angle hk$ about the axis c.*

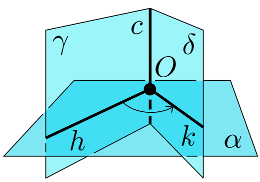


Fig. 9.1

The ray h and the axis of the rotation c have the common point O. Let's draw the plane γ through the ray h and the line c. Similarly, we draw the plane δ passing through the ray k and the line c. Both planes γ and δ are perpendicular to the plane α. They intersect along the line c and define a dihedral angle with the edge c, for which the angle $\angle hk$ lying on the plane α is a plane angle.

Each dihedral angle defines a rotation of the space about an axis. Indeed, assume that a dihedral angle with the edge c is given. It is the intersection of two closed half-spaces: $\overline{\gamma_+} \cap \overline{\delta_-}$. Denote $c = d$ and denote by $c_+ \subset \gamma$ and $d_+ \subset \delta$ the sides of this angle. Then for the mapping $f\colon c \to d$ we take the identical mapping $\mathrm{id}\colon c \to c$. Applying the theorem 5.4, we extend it up to a mapping $f\colon \gamma \to \delta$ taking the half-plane c_+ to the half-plane d_+ and taking the half-plane c_- to the half-plane d_-. Then we apply the theorem 5.5 and extend this mapping up to a mapping $f\colon \mathbb{E} \to \mathbb{E}$ taking γ_+ to δ_+ and taking γ_- to δ_-.

Exercise 9.1. *Show that the above construction of the mapping $f : \mathbb{E} \to \mathbb{E}$ on the base of a dihedral angle yields the same result as the previous construction extending the rotation of a plane about a point up to a rotation of the space about an axis.*

The rotations of the space about a fixed axis inherit all of the properties of the rotations of a plane about a fixed point.

Theorem 9.1. *Let $\theta_{hk} : \mathbb{E} \to \mathbb{E}$ be a rotation of the space about some axis c different from the identical mapping ($h \neq k$). Then the set of stable point of the mapping θ_{hk} coincides with the rotation axis c.*

Theorem 9.2. *Let h and k be two half-planes with common boundary c. There are exactly two mappings of congruent translations $f\colon \mathbb{E} \to \mathbb{E}$ which preserves stable the points of the line c and takes the half-plane h to the half-plane k. The first of them $f = \theta_{hk}$ is the rotation about the axis c, while the second one $f = z_\gamma$ is the mirror reflection in the plane γ that contains the bisector of the dihedral angle formed by the half-planes h and k.*

A remark. The *bisector of a dihedral angle* is a half-plane bounded by the edge of this angle and containing the bisector of any one of its plane angles. The bisector divides a dihedral angle into two dihedral angles whose plane angles are congruent.

Like in theorem 7.2, in theorem 9.2 there are special cases where the half-planes h and k lie on one plane. If the half-planes h and k do coincide, the rotation θ_{hk} is the identical mapping: $\theta_{hk} = \mathrm{id}$. If h and k are two complementary half-planes on one plane, then the rotation θ_{hk} coincides with the mirror reflection in the line c separating these half-planes: $\theta_{hk} = z_c$.

Theorem 9.3. *The rotations of the space about a fixed axis and the mirror reflections in planes passing through this axis possess the following properties:*

$$\begin{aligned}
&\theta_{kl} \circ \theta_{hk} = \theta_{hl}, &&\theta_{hk} \circ z_\alpha = z_\alpha \circ \theta_{kh},\\
&z_\alpha \circ z_\alpha = \mathrm{id}, &&z_\alpha \circ z_\beta = \theta_{hk}, \quad \text{where } \ h \subset \beta,\ k = z_\alpha(h).
\end{aligned}$$

Theorem 9.4. *The equality $\theta_{hk} = \theta_{lq}$ implies $\theta_{hl} = \theta_{kq}$ and, conversely, the equality $\theta_{hl} = \theta_{kq}$ implies $\theta_{hk} = \theta_{lq}$.*

Theorem 9.5. *Any two rotations about the same axis commute: $\theta_{hk} \circ \theta_{lq} = \theta_{lq} \circ \theta_{hk}$.*

Exercise 9.2. *Compare the theorems 9.1, 9.2, 9.3, 9.4, and 9.5 with the corresponding theorems for the rotations of a plane. Suggest your scheme for proving them.*

Theorem 9.6. *Let θ_{hk} and θ_{lq} be two rotations whose axes c_1 and c_2 do not coincide but intersect at some point O. Then the composition $\theta_{hk} \circ \theta_{lq}$ is some rotation θ_{rp} about a third axis c_3 passing through the point O.*

Proof. Let's draw the plane β passing through the intersecting lines c_1 and c_2. The line c_1 divides this plane into two half-planes. Let's denote one of these half-planes through h and denote $k = \theta_{hk}(h)$. The half-planes h and k define a dihedral angle with the edge c_1. Let's denote by α the plane containing its bisector. Then the mirror reflection z_α takes the half-plane h to the half-plane k. Therefore we can apply the fourth relationship from the theorem 9.3 in order to expand the rotation θ_{hk} into the composition of two mirror reflections: $\theta_{hk} = z_\alpha \circ z_\beta$. We use this expansion in the following calculations:

$$\theta_{hk} \circ \theta_{lq} = z_\alpha \circ z_\beta \circ \theta_{lq} = z_\alpha \circ (z_\beta \circ \theta_{lq}) = z_\alpha \circ (\theta_{ql} \circ z_\beta).$$

By construction the plane β contains the axis c_2 of the second rotation, therefore, we used the second relationship from the theorem 9.3 in the form $z_\beta \circ \theta_{lq} = \theta_{ql} \circ z_\beta$. Now, applying the above trick to the rotation θ_{ql}, we get the expansion $\theta_{ql} = z_\gamma \circ z_\beta$. For the initial composition $\theta_{hk} \circ \theta_{lq}$ it yields

$$\theta_{hk} \circ \theta_{lq} = z_\alpha \circ (z_\gamma \circ z_\beta \circ z_\beta) = z_\alpha \circ z_\gamma \circ (z_\beta \circ z_\beta) = z_\alpha \circ z_\gamma.$$

The plane α contains the line c_1, but it does not contain c_2. Similarly, the plane γ contains c_2, but it does not contain the line c_1. Therefore, these planes do not coincide, but they have a common point O. Let's denote by c_3 the line arising as their intersection. It is clear that $O \in c_3$. Now due to the fourth relationship from the theorem 9.3 for the composition $z_\alpha \circ z_\beta$ we derive $z_\alpha \circ z_\beta = \theta_{rp}$, where θ_{rp} is some rotation about the axis c_3. Hence, $\theta_{hk} \circ \theta_{lq} = \theta_{rp}$. The theorem is proved. $\square$

Let's combine the first relationship from the theorem 9.3 and the theorem 9.6. As a result we get the following result.

THEOREM 9.7. *The composition of two rotations of the space whose axes have a common point O is a rotation about an axis passing through the point O.*

§ 10. The theorem on the decomposition of rotations.

Above we considered several types of the mappings of congruent translation of the space. They are rotations about axes, mirror reflections in planes or in a lines, and inversions with respect to various points. Their common property is that they have stable points.

Definition 10.1. A mapping of congruent translation of the space $f\colon \mathbb{E} \to \mathbb{E}$ having a stable point O, i. e. such that $f(O) = O$, is called a *generalized rotation of the space about the point O.*

Let f and g be two mappings of congruent translation of the space. Then their composition $f \circ g$ is obviously a mapping of congruent translation of the space. If f and g are generalized rotations about a point O, then $f \circ g$ is also a generalized rotation about this point.

Theorem 10.1. *Any generalized rotation $f : \mathbb{E} \to \mathbb{E}$ of the space about a point O is either a rotation about some axis passing through the point O, i. e. $f = \theta_{hk}$, or the composition of such a rotations and a mirror reflection in some plane containing the point O, i. e. $f = z_\alpha \circ \theta_{hk}$.*

The term *rotation* in this theorem is treated so that the identical mapping and any reflection in a line are assumed to be rotations (see the remark to the theorem 9.2). In order to prove the theorem 10.1 we need two auxiliary lemmas.

Lemma 10.1. *Let M be some point lying on the plane of a triangle ABC. If for a point X in the space the relationships $[XA] \cong [MA]$, $[XB] \cong [MB]$, and $[XC] \cong [MC]$ are fulfilled, then the point X coincides with M.*

Proof. Assume that $X \neq M$. Then from the relationships $[XA] \cong [MA]$, $[XB] \cong [MB]$, $[XC] \cong [MC]$ due to the theorem 2.1 we derive that the points A, B, and C lie on the plane of perpendicular bisectors of the segment $[MX]$. Let's denote this plane through α. Then due to the theorem 6.1 we have $X = z_\alpha(M)$, where z_α is the mirror reflection in the plane α. But, according to the premise of the theorem, the point M lies on the plane α. Therefore, $z_\alpha(M) = M$. Hence, $X = M$ despite the assumption that $X \neq M$. Thus, the lemma is proved. $\square$

Lemma 10.2. *Let M be some point not lying on the plane of a triangle ABC. In the space there is exactly one point X different from M and such that the relationships $[XA] \cong [MA]$, $[XB] \cong [MB]$, $[XC] \cong [MC]$ are fulfilled. It is the mirror image of the point M in the plane of the triangle ABC.*

Proof. Let's denote through α the plane of the triangle ABC and let $X = z_\alpha(M)$. The relationships $[XA] \cong [MA]$, $[XB] \cong [MB]$, and $[XC] \cong [MC]$ for X follow from the fact that the mirror reflection z_α is a congruent translation of the space. The existence of the required point X is proved.

Let's prove the uniqueness of the point $X \neq M$ satisfying the relationships $[XA] \cong [MA]$, $[XB] \cong [MB]$, and $[XC] \cong [MC]$. From these relationships and from the theorem 2.1 we derive that the points A, B, and C lie on the plane of perpendicular bisectors for the segment $[MX]$. Hence, due to the theorem 6.1 we get $X = z_\alpha(M)$. This formula fixes the point X uniquely. □

Proof of the theorem 10.1. Assume that the mapping f being a generalized rotation about the point O is different from the identical mapping. Let's show that either this mapping or its composition with the mirror reflection in some plane has a stable point Z different from O.

Let's study various pairs of points X and $f(X)$, where $X \neq O$. If $X = f(X)$, then the required stable point is found. If $X \neq f(X)$, we construct the plane of perpendicular bisectors for the segment $[Xf(X)]$. From $[OX] \cong [f(O)f(X)]$, since O is a stable point, we derive $[OX] \cong [Of(X)]$. Then, according to the theorem 2.1 the point O lies on the plane of perpendicular bisectors for the segment $[Xf(X)]$. In other words, all planes of perpendicular bisectors for the segments of the form $[Xf(X)]$ have the common point O. There is a case where all such planes do coincide. In this case we denote by α the common plane of perpendicular bisectors for all segments of the form $[Xf(X)]$. Then f is the mirror reflection in the plane α. Thus we have $f = z_\alpha \circ \mathrm{id}$, where the identical mapping is treated as a special case of the rotation about an axis.

Now let's consider the case where there are two points X and Y for which the plane of perpendicular bisectors of the segments $[Xf(X)]$ and $[Yf(Y)]$ do not coincide. These planes have the common point O, hence, they intersect along some straight line a. Let's choose some point $Z \neq O$ on this line. Then

$$[ZX] \cong [Zf(X)], \qquad [ZY] \cong [Zf(Y)], \qquad [ZO] \cong [Zf(O)].$$

The first two relationship follow from the theorem 2.1, while the last one is the trivial consequence of $f(O) = O$. From the fact that f is a congruent translation of the space we derive

$$[ZX] \cong [f(Z)f(X)],\ \ [ZY] \cong [f(Z)f(Y)],\ \ [ZO] \cong [f(Z)f(O)].$$

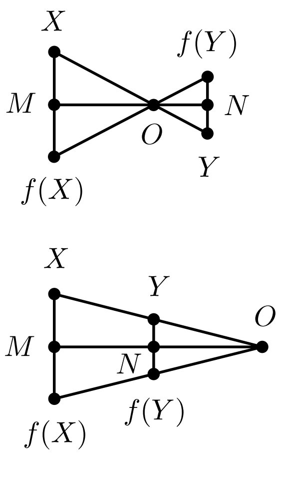


Fig. 10.1

Let's denote $\tilde{Z} = f(Z)$ and, comparing the above sets of relationships, we find

$$\begin{aligned}
&[Zf(X)] \cong [\tilde{Z}f(X)],\\
&[Zf(Y)] \cong [\tilde{Z}f(Y)],\\
&[Zf(O)] \cong [\tilde{Z}f(O)].
\end{aligned}\tag{10.1}$$

In order to apply one of the lemmas 10.1 or 10.2, let's prove that the points X, Y, and O do not lie on one straight line. If we assume that these points lie on one straight line, then the points $f(X)$, $f(Y)$, and $f(O)$ also lie on one straight line (see theorem 5.1) and we have the situation shown on one of the figures 10.1. All of the five points X, Y, $f(X)$, $f(Y)$, and O lie on one plane. From $[OX] \cong [Of(X)]$ and $[OY] \cong [Of(Y)]$ we conclude that the triangles $XOf(X)$ and $YOf(Y)$ are isosceles. Their medians $[OM]$

and $[ON]$ lie on one straight line because they are bisectors of one angle or bisectors of of two vertical angles. They are the height of the corresponding triangles at the same time. For this reason the line MN is the common perpendicular bisector of the segments $[X f(X)]$ and $[Y f(Y)]$ that lie on one plane. Through the point O we draw the perpendicular to the plane of the triangles $XOf(X)$ and $YOf(Y)$. Then through this perpendicular and through the line MN one can draw the plane β which (due to the theorem 3.1) is the common plane of perpendicular bisectors for the segments $[X f(X)]$ and $[Y f(Y)]$. But we consider the case where the planes of perpendicular bisectors for the segments $[X f(X)]$ and $[Y f(Y)]$ do not coincide. Due to this contradiction, our initial assumption that the points X, Y, and O lie on one straight line is invalid.

Now, having proved that the points X, Y, and O do not lie on one line, we return to the relationships (10.1). Due to the theorem 5.1 the points $f(X)$, $f(Y)$, and $f(O)$ also do not lie on one straight line. Let's denote by β the plane of the triangle $f(X)f(Y)f(O)$. If $Z \in \beta$, then relying upon the relationships (10.1), we apply the lemma 10.1. It yields $Z = \tilde{Z} = f(Z)$, i. e. $Z \neq O$ is a required stable point of the mapping f. If $Z \notin \beta$, then the lemma 10.2 is applicable. In this case $Z = \tilde{Z}$ or the points Z and $\tilde{Z}$ are mirror symmetric with respect to the plane β. If $Z = \tilde{Z}$, then Z again is a stable point for f. Otherwise, if $Z \neq \tilde{Z}$, then Z is a stable point for the mapping $z_\beta \circ f$, which is also a generalized rotation about the point O.

Let's denote $g = f$ in the first case and denote $g = z_\beta \circ f$ in the second case. From $g(O) = O$ and $g(Z) = Z$ due to the theorem 2.1 from Chapter III we conclude that all points of the line OZ are stable under the action of the mapping g. Let h be some arbitrary half-plane having the line OZ as its boundary. Let's denote $k = g(h)$ and apply the theorem 9.2. Due to this theorem g is the rotation about the axis OZ taking the half-plane h to the half-plane k, or g is the mirror reflection in the plane δ containing the bisector of the dihedral angle formed by the half-planes h and k. Hence, for the initial mapping f we have the following four possible expansions:

$$f = \theta_{hk}, \qquad f = z_\beta \circ \theta_{hk}, \qquad f = z_\delta, \qquad f = z_\beta \circ z_\delta.$$

If the first or the second case takes places, the proof is over. In the third case we can write $f = z_\delta \circ \mathrm{id}$, therefore, in this case the proof is also completed. The rest is the fourth case. The planes β and δ have the common point O. Hence, either $\beta = \delta$ or these planes intersect along some line b. If $\delta = \beta$, then $f = z_\beta \circ z_\beta = \mathrm{id}$. Otherwise, if $\delta \neq \beta$, we apply the fourth relationship from the theorem 9.3. For the mapping f it yields $f = z_\beta \circ z_\beta = \theta_{lq}$, where θ_{lq} is the rotation about the line $b = \beta \cap \delta$. Thus, the theorem 10.1 is proved. $\square$

§ 11. The total rotation group and the group of pure rotations of the space.

The theorem 10.1 proved in previous section has a very important consequence. It means that any mapping of generalized rotation about a point is bijective. Indeed, a rotation about an axis and a mirror reflection both are bijective mappings, while the composition of two bijective mappings is also a bijective mapping. From this fact we derive that the set of all generalized rotations of the space about a fixed point O is a group with respect to the composition. This group is called the *total rotation group of the space* about a fixed point O.

According to the theorem 9.6 the set of rotations about various axes passing through a fixed point O is also a group with respect to the composition. This group is called the *group of pure rotations of the space* about a fixed point O.

The theorem 10.1 determines the division of generalized rotations into *even* and *odd* ones. Pure rotations belong to even rotations, while mirror reflections and their compositions with pure rotations are odd rotations. The same generalized rotation cannot be even and odd simultaneously. Indeed, $f = \theta_{hk}$ and $f = z_\beta \circ \theta_{lq}$ would imply $z_\beta = \theta_{hk} \circ \theta_{ql} = \theta_{rp}$, which is impossible.

EXERCISE 11.1. *Show that the composition of two even rotations and the composition of two odd rotations both are even rotations, while the composition of an even rotation and an odd rotation is an odd rotation.*

§ 12. Orthogonal projection onto a straight line.

Let a be some straight line in the space. Choosing some arbitrary point X not lying on the line a, we drop the perpendicular $[XY]$ from the point X onto the line a. The foot of this perpendicular (the point Y) is called the *orthogonal projection* of the point X onto the line a. According to the theorem 6.5 from Chapter III, once a point X is given, its projection Y is fixed uniquely. Hence, we can define a mapping $\pi_a : \mathbb{E} \to a$. For a point $Y \in a$ we set $\pi_a(Y) = Y$. The mapping π_a is called the *orthogonal projection* onto the line a.

The orthogonal projection is not a congruent translation. Moreover, two different points $X_1 \neq X_2$ can be taken to one point $\pi_a(X_1) = \pi_a(X_2)$ under this mapping. Let α be some plane perpendicular to the line a. According to the definition 1.1, such a plane intersects the line a at some point A. Comparing this definition with the construction of the mapping π_a, we see that all points of the plane α are taken to the point A by the projection π_a.

Let α and β be two different planes perpendicular to the line a and intersecting the line a at the points A and B. Such planes have no common points. Indeed, the existence of a point $M \in \alpha \cap \beta$ would mean that $\pi_a(M) = A$ and $\pi_a(M) = B$. But the result of orthogonal projection of the point M onto the line a is defined uniquely. Therefore, $\alpha \cap \beta = \varnothing$.

THEOREM 12.1. *Let π_a be the orthogonal projection onto a straight line a. If the projections of some two points A and B do coincide, then the whole line AB is projected onto one point $C = \pi_a(A) = \pi_a(B)$ of the line a.*

PROOF. The relationship $C = \pi_a(A)$ means that $C = A$ or C

is the foot of the perpendicular dropped from the point A onto the line a. In both cases the point A lies on the plane α, passing through the point C and being perpendicular to the line a (see theorems 1.2 and 1.4). Similar considerations yield $B \in \alpha$. Hence the whole line AB lies on the plane α which is projected onto the single point $C \in a$. $\square$

THEOREM 12.2. *Let π_a be the orthogonal projection onto a straight line a and assume that b is some straight line which is not projected onto a single point of the line a. Then for the points of the line b the following propositions are valid:*

(1) *$A \neq B$ implies $\pi_a(A) \neq \pi_a(B)$;*
(2) *$(A \blacktriangleright B \blacktriangleleft C)$ implies $(\pi_a(A) \blacktriangleright \pi_a(B) \blacktriangleleft \pi_a(C))$.*

PROOF. The first item of this theorem is an immediate consequence of the previous theorem 12.1.

Let's consider the second item of the theorem. Assume that $(A \blacktriangleright B \blacktriangleleft C)$. Let's use the following notations for the projections of the points A, B, C:

$$\tilde{A} = \pi_a(A),$$

$$\tilde{B} = \pi_a(B),$$

$$\tilde{C} = \pi_a(C).$$

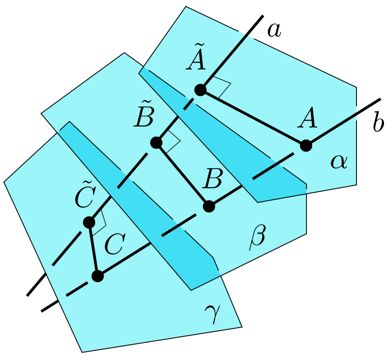


Fig. 12.1

Let's draw the plane β passing through the point $\tilde{B} \in a$ and being perpendicular to the line a. Such a plane does exist and it is unique (see theorem 1.2). The points B and $\tilde{B}$ lie on this plane. From $\tilde{A} \neq \tilde{B}$ and from $\tilde{C} \neq \tilde{B}$ we conclude that the four points A, C, $\tilde{A}$, and $\tilde{C}$ do not lie on the plane β. The plane β divides the set of points not lying on this plane into two open half-spaces β_+ and β_-.

The half-spaces β_+ and β_- arise as the equivalence classes, where two points X and Y are treated to be equivalent if $X = Y$ or if the segment $[XY]$ does not intersect the plane β (see § 6 in Chapter II).

In our case the points A and $\tilde{A}$ lie on the plane α passing through the point $\tilde{A} \in a$ and being perpendicular to the line a. The plane α does not intersect the plane β. Therefore either $A = \tilde{A}$ or, if $A \neq \tilde{A}$, the segment $[A\tilde{A}]$ lies on the plane α and does not intersect the plane β. Hence, $A \sim \tilde{A}$. In a similar way we get $C \sim \tilde{C}$. But $(A \blacktriangleright B \blacktriangleleft C)$ implies that the segment $[AC]$ intersects the plane β at the point B. Hence the points A and C are not equivalent. Then from $A \sim \tilde{A}$ and $C \sim \tilde{C}$ we get that the points $\tilde{A}$ and $\tilde{C}$ are also not equivalent. Therefore the segment $[\tilde{A}\tilde{C}]$ intersects the plane β at the point $\tilde{B}$. Hence, we immediately derive the required relationship $(\tilde{A} \blacktriangleright \tilde{B} \blacktriangleleft \tilde{C})$. $\square$

As a corollary of the theorem 12.2 we get that if the mapping π_a does not project a line b onto one point, then any ray lying on the line b is projected onto a ray ands any segment of the line b is projected onto a segment.

§ 13. Orthogonal projection onto a plane.

Let α be some plane and let X be a point outside this plane. Let's drop the perpendicular from the point X onto the plane α and denote by $\pi_\alpha(X)$ the foot of this perpendicular. For the points Y lying on the plane α we set $\pi_\alpha(Y) = Y$. Due to the theorem 3.4 this construction defines a mapping $\pi_\alpha\colon \mathbb{E} \to \alpha$ which is called the *orthogonal projection* onto the plane α.

The orthogonal projection onto a plane π_α is not a congruent translation. It can take some different points to one point. Let Y be a point on the plane α and let a be the perpendicular to the plane α passing through the point Y (see theorem 1.3). Then all points of the line a and only these points are projected onto the point $Y \in \alpha$.

THEOREM 13.1. *Let π_α be the orthogonal projection on some plane α. If for some two points A and B their projections do coincide, then the whole straight line AB is projected onto one point $C = \pi_a(A) = \pi_a(B)$ on the plane α. In this case the line AB coincides with the perpendicular to the plane α passing through the point C.*

PROOF. The relationship $C = \pi_\alpha(A)$ means that $C = A$ or C is the foot of the perpendicular dropped from the point A onto the plane α. In any of these two cases the point A lies on the perpendicular a to the plane α passing through the point C (see theorem 1.3). In a similar way we prove that $B \in a$. Hence, the line AB coincides with the line a which is projected onto one point $C \in \alpha$. $\square$

THEOREM 13.2. *Let π_α be the orthogonal projection onto a plane α and assume that b is some straight line which is not projected onto one point. Then the line b is projected onto some line a lying on the plane α and for the points of b the following propositions are valid:*

(1) *$A \neq B$ implies $\pi_a(A) \neq \pi_a(B)$;*
(2) *$(A \blacktriangleright B \blacktriangleleft C)$ implies $(\pi_a(A) \blacktriangleright \pi_a(B) \blacktriangleleft \pi_a(C))$.*

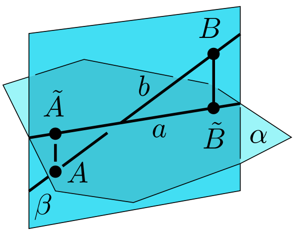


Fig. 13.1

PROOF. The line b is not projected onto one point. Therefore, there are at least two points A and B on this line whose projections are different. Let $\tilde{A} = \pi_\alpha(A)$ and $\tilde{B} = \pi_\alpha(B)$ and denote by a the line $\tilde{A}\tilde{B}$. Then we draw the perpendiculars m and n to the plane α passing through the points $\tilde{A}$ and $\tilde{B}$ respectively. According to the theorem 3.2 there is a plane β containing both these perpendiculars. This plane is perpendicular to the plane α. It intersects the plane alpha along the line a since $\tilde{A}$ and $\tilde{B}$ belong to $\alpha \cap \beta$.

The points A and B lie on the perpendiculars m and n, hence, they belong to the plane β. This yields $b \subset \beta$. Let X be some arbitrary point of the plane β not lying on the line a. Within the plane β we drop the perpendicular $[X\tilde{X}]$ from the point X onto the line a. From $\alpha \perp \beta$ due to the definitions 3.1 and 1.1 and due to the theorem 6.5 from Chapter III we derive that the segment $[X\tilde{X}]$ is the perpendicular dropped from the point X onto the plane α. Hence, the whole plane β is projected onto the line a and the restriction of the mapping $\pi_\alpha\colon \mathbb{E} \to \alpha$ to the plane β coincides with the restriction of the mapping $\pi_a\colon \mathbb{E} \to a$ to the same plane. This observation reduces the theorem 13.2 to the theorem 12.2 proved in previous section. $\square$

As an auxiliary result in proving the theorem 13.2 we have shown that the plane β perpendicular to the plane α is projected onto the line $a = \alpha \cap \beta$ under the projection π_α. This result can be strengthened.

Theorem 13.3. *A plane β is projected onto a line $a \subset \alpha$ by the mapping $\pi_\alpha\colon \mathbb{E} \to \alpha$ if an only if $\beta \perp \alpha$.*

The following theorem is well-known. It is called the theorem *on three perpendiculars.*

Theorem 13.4. *A straight line b intersecting a plane α at a point O is perpendicular to a line c lying on this plane and passing through the point O if and only if its projection $a = \pi_\alpha(b)$ is perpendicular to c.*

Proof. If the line b lies on the plane α, then $b = a$. In this case proposition of the theorem is trivial.

Assume that b intersects the plane α at a point O, but does not lie on this plane. Let's choose some point $B \neq O$ on the line b and denote $A = \pi_\alpha(B) \in a$. The line AB is the perpendicular to the plane α passing through the point A. We draw the other perpendicular to the plane α through the point O and we denote it by d. According to the theorem 3.2, there is a plane β containing both perpendiculars to the plane α. This plane is perpendicular to α. The points O and B lie on the plane β, hence $b \subset \beta$. Similarly, from $O \in \beta$ and $A \in \beta$ we find that $a \subset \beta$.

Since $d \perp \alpha$, the line d is perpendicular to any line lying on the plane α and passing through the point O. In particular, $d \perp c$. Now if $b \perp c$, then from $c \perp b$ and $c \perp d$ due to the theorem 1.1 we derive $c \perp \beta$. Hence, $c \perp a$.

Conversely, if $c \perp a$, then, complementing this condition with the condition $c \perp d$, we again get the orthogonality $c \perp \beta$. Hence, $c \perp b$. The proof is over. $\square$

THEOREM 13.5. *Let π_α be the orthogonal projection onto a plane α and let β be some plane which is not projected onto a line under the projection π_α. Then for the points of the plane β the following propositions are valid:*

(1) *$A \neq B$ implies $\pi_a(A) \neq \pi_a(B)$;*
(2) *if the points A, B, and C do not lie on one straight line, then their projections also do not lie on one straight line;*
(3) *if the points A, B, and C lie on one straight line, then their projections also lie on one straight line and $(A \blacktriangleright B \blacktriangleleft C)$ implies $(\pi_\alpha(A) \blacktriangleright \pi_\alpha(B) \blacktriangleleft \pi_\alpha(C))$.*
(4) *if b is some straight line on the plane β and if the points A and C lie on different sides of the line b, then their projections $\pi_\alpha(A)$ and $\pi_\alpha(C)$ lie on different sides of the line $a = \pi_\alpha(b)$ on the plane α.*

PROOF. If we assume that the projections of two distinct points A and B do coincide, then the line AB connecting them is perpendicular to the plane α. The plane β comprising the perpendicular to the plane α is perpendicular to α (see definition 3.1). Then it is projected onto a line despite to the premise of the theorem. This contradiction proves the item (1).

Let A, B, and C be three points of the plane β not lying on one straight line. We denote by $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ their projections:

$$\tilde{A} = \pi_\alpha(A), \qquad \tilde{B} = \pi_\alpha(B), \qquad \tilde{C} = \pi_\alpha(C).$$

Die to the item (1), which is already proved, the coincidences $\tilde{A} = \tilde{B}$, $\tilde{B} = \tilde{C}$, and $\tilde{A} = \tilde{C}$ are impossible. Let's draw two perpendiculars to the plane α trough the points $\tilde{A}$ and $\tilde{B}$. According to the theorem 3.2, there is a plane γ containing both these perpendiculars. This plane γ is perpendicular to α. It contains the line $\tilde{A}\tilde{B}$.

Now if we assume that the points $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ lie on one straight line, then the point $\tilde{C}$ belong to the line $\tilde{A}\tilde{B}$ which lies on the plane γ. From $\gamma \perp \alpha$ we derive that the perpendicular to the plane α passing through the point $\tilde{C}$ lies on the plane γ. Hence, all of the three points A, B, and C lie on the plane γ. This fact implies $\beta = \gamma$ and $\beta \perp \alpha$. But this is not possible since β is not projected onto a line according to the premise of the theorem. The contradiction obtained proves the second item of the theorem 13.5.

Let's proceed to the third item. In this case the points A, B, and C lie on one straight line. We denote this line by b. Due to the item (1) of the theorem, which is already proved, the line b is not projected onto one point. Therefore, the third item of the theorem follows from the theorem 13.2.

Assume that the points A and C lie on the plane β on different sides of the line b. Then the segment $[AC]$ crosses the line b at some its interior point B, i. e. $(A \blacktriangleright B \blacktriangleleft C)$. Passing to the projections $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, due to the item (3) we conclude that the segment $[\tilde{A}\tilde{C}]$ lying on the plane α intersects the line $a = \pi_\alpha(b)$ at some its interior point $\tilde{B}$. Thus, the fourth item and the theorem 13.5 in whole are proved. $\square$

The theorem 13.5 has an important corollary. If the planes α and β are not perpendicular, then the mapping $\pi_\alpha \colon \mathbb{E} \to \alpha$ takes each line b lying on the plane β to some line a on the plane α so that the half-planes bounded by the line b are projected onto the half-planes on α bounded by the line a.

Exercise 13.1. *Prove the theorem 13.3.*

§ 14. Translation by a vector along a straight line.

Let a be some line in a plane α. It divides the plane α into two half-planes a_+ and a_-. Let $\beta = \alpha$ and $b = a$. Assume also that $b_+ = a_+$ and $b_- = a_-$, while for the mapping $f\colon a \to a$ we take $p_{\mathbf{c}}$, where $\mathbf{c}$ is some slipping vector on the line a. The mapping $p_{\mathbf{c}}$, which is called a congruent translation by a vector, was defined above in § 3 of Chapter III. Applying the theorem 5.4, we can extend $p_{\mathbf{c}}$ up to the mapping $p_{a\mathbf{c}}\colon \alpha \to \alpha$, which is called the *translation of the plane α by the vector $\mathbf{c}$ along the line a.*

The plane α divides the space into two half-spaces α_+ and α_-. Let's denote $\beta = \alpha$, $\beta_+ = \alpha_+$, and $\beta_- = \alpha_-$. For the mapping $f : \alpha \to \alpha$ we choose $p_{a\mathbf{c}}$, then we apply the theorem 5.5. As a result we get the mapping $p_{a\mathbf{c}} : \mathbb{E} \to \mathbb{E}$, which is called the *translation of the space by the vector $\mathbf{c}$ along the line a.*

The plane α containing the line a is an auxiliary object in constructing the mapping $p_{a\mathbf{c}} : \mathbb{E} \to \mathbb{E}$. This fact is explained by the following theorem.

THEOREM 14.1. *The restriction of the mapping $p_{a\mathbf{c}} : \mathbb{E} \to \mathbb{E}$ to any plane β containing the line a coincides with the translation of the plane β by the vector $\mathbf{c}$ along the line a.*

PROOF. Let's consider the mapping $p_{a\mathbf{c}} : \mathbb{E} \to \mathbb{E}$ constructed with the use of the auxiliary plane α containing the line a. Let β be some other plane containing the line a and let X be some point of the plane β not lying on the plane α. For the beginning we study the case, where the plane β is not perpendicular to the plane α. Let's denote by α_+ the half-space bounded by the plane α, and containing the point X. By b_+ we denote the half-plane on

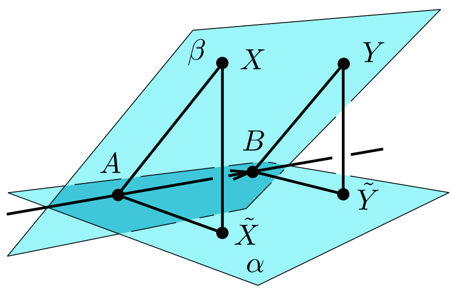


Fig. 14.1

the plane β bounded by the line a and containing the point X. Then $b_+ = \beta \cap \alpha_+$. Let $Y = p_{a\mathbf{c}}(X)$. Following the general scheme of constructing the extensions for the mappings of congruent translation (see § 5), in order to fix the point Y we drop the perpendicular from the point X onto the plane α. As a result we get the point $\tilde{X} = \pi_\alpha(X)$ and the segment $[X\tilde{X}]$. We apply the mapping $p_{a\mathbf{c}} : \alpha \to \alpha$ to the point $\tilde{X}$ and get the point $\tilde{Y} = p_{a\mathbf{c}}(\tilde{X})$. Let's draw the perpendicular to the plane α through the point $\tilde{Y}$. On this perpendicular we mark a point Y such that $Y \in a_+$ and $[Y\tilde{Y}] \cong [X\tilde{X}]$. These conditions fix the point $Y = p_{a\mathbf{c}}(X)$ uniquely.

Remember that the mapping $p_{a\mathbf{c}}\colon \alpha \to \alpha$ itself is an extension of the mapping $p_{\mathbf{c}}\colon a \to a$ from the line a to the plane α. Due to the theorem 13.5 under the orthogonal projection $\pi_\alpha\colon \mathbb{E} \to \alpha$ the half-plane b_+ is mapped to some half-plane with the boundary a. We denote it a_+. From $\tilde{X} = \pi_\alpha(X)$ we get $\tilde{X} \in a_+$. In order to fix the point $\tilde{Y} = p_{a\mathbf{c}}(\tilde{X})$ we apply the procedure of extending the mapping $p_{\mathbf{c}} : a \to a$ (see theorem 5.4, its proof, and comments preceding this theorem in § 5). On the plane α we drop the perpendicular from the point $\tilde{X}$ onto the line a. We denote by A the foot of such perpendicular. Let $B = p_{\mathbf{c}}(A)$. Then the vector $\overrightarrow{AB}$ is a geometric realization for the slipping vector $\mathbf{c}$ on the line a (see § 3 and § 4 in Chapter III). Having fixed the point B, on the plane α we draw the perpendicular to the line a through this point. Then on this perpendicular we choose a point $\tilde{Y}$ such that $\tilde{Y} \in a_+$ and $[B\tilde{Y}] \cong [A\tilde{X}]$. These two conditions fix the point $\tilde{Y} = p_{a\mathbf{c}}(\tilde{X})$ uniquely. By construction this point appears to be the orthogonal projection of the point Y onto the plane α, i. e. $\tilde{Y} = \pi_\alpha(Y)$.

Now let's consider the planes of the triangles $AX\tilde{X}$ and $BY\tilde{Y}$, which are congruent due to $[X\tilde{X}] \cong [Y\tilde{Y}]$, $[A\tilde{X}] \cong [B\tilde{Y}]$ and since the angles $\angle A\tilde{X}X$ and $\angle B\tilde{Y}Y$ both are right angles. Let's denote the first of these two planes by γ and the second one by δ. The plane γ contains the line $X\tilde{X}$, which is perpendicular to the plane

α. Hence, $\gamma \perp \alpha$ (see definition 3.1). Due to $\tilde{X} = \pi_\alpha(X)$ the line $A\tilde{X}$ is the projection of the line AX. Therefore we can apply the theorem on three perpendiculars.(see theorem 13.4 above). Due to this theorem since the lines $A\tilde{X}$ and a are perpendicular, we derive $a \perp AX$. Now from $a \perp A\tilde{X}$ and $a \perp AX$ we get $a \perp \gamma$. Thus, the half-planes b_+ and a_+ define a dihedral angle with the edge $a = b$, while the plane γ cut the angle $\angle XA\tilde{X}$ being a plane angle for this dihedral angle.

Let's repeat the above considerations with respect to the plane δ. This yields $a \perp \delta$, i. e. the plane δ cut the other plane angle of the dihedral angle formed by half-planes b_+ and a_+. From the congruence of the triangles $AX\tilde{X}$ and $BY\tilde{Y}$ we derive the congruence of the angles $\angle YB\tilde{Y}$ and $\angle XA\tilde{X}$. Now, applying the theorem 4.1 and taking into account the fact that the point Y lies on the half-plane α_+, we conclude that the angle $\angle YB\tilde{Y}$ coincides with the plane angle of the dihedral angle formed by the half-planes b_+ and a_+. Hence, $Y \in b_+ \subset \beta$. Thus, the segments $[AX]$ and $[BY]$ lie on the plane β on one side of the line $b = a$. They both are perpendicular to the line a and congruent to each other since the triangles $AX\tilde{X}$ and $BY\tilde{Y}$ are congruent. We write the obtained results for X and Y as follows:

(1) the points X and Y lie on the plane β on one side of the line a;
(2) the segments $[AX] \cong [BY]$ are perpendicular to a;
(3) the vector $\overrightarrow{AB}$ is a geometric realization of the slipping vector $\mathbf{c}$.

The conditions listed above mean exactly that the point Y is produced from the point X by applying the mapping $p_{a\mathbf{c}} \colon \beta \to \beta$ that translate the plane β by the vector $\mathbf{c}$ along the line a.

The case where $\beta \perp \alpha$ appears to be much simpler than the case we have already considered. Here the points X and Y are projected to the points A and B on the line a, i. e. $A = \pi_\alpha(X)$ and $B = \pi_\alpha(Y)$. By construction of the mapping $p_{a\mathbf{c}} : \mathbb{E} \to \mathbb{E}$ this means that the conditions (2) and (3) are fulfilled. The relationship $Y \in \beta$ follows from $\beta \perp \alpha$ and $[YB] \perp \alpha$ due to the definition 3.1. The fact that the points X and Y lie on one side of the line a follows from the fact that they are on one side of the plane α by construction of the mapping $p_{a\mathbf{c}}\colon \mathbb{E} \to \mathbb{E}$. Hence the condition (1) in the case $\beta \perp \alpha$ is also fulfilled and the point Y is the result of applying the translation $p_{a\mathbf{c}}\colon \beta \to \beta$ to the point X. The theorem is proved. $\square$

The mappings of translation along some line a inherit the properties of the translations on this line considered in § 3 and in § 4 of Chapter III. They are commutative and the equality $\mathbf{d} = \mathbf{c} + \mathbf{e}$ for the vectors $\mathbf{d}$, $\mathbf{c}$, and $\mathbf{e}$ implies

$$p_{a\mathbf{d}} = p_{a\mathbf{c}} \circ p_{a\mathbf{e}}.$$

The translation by the zero vector appears to be the identical mapping: $p_{a\mathbf{0}} = \mathrm{id}$. The translations along nonzero vectors constitute a separate class of the mappings of congruent translation. For a vector $\mathbf{c} \neq \mathbf{0}$ the mapping $p_{a\mathbf{c}}$ cannot be reduced to a generalized rotation about some point. This fact is derived from the following theorem.

Theorem 14.2. *For $\mathbf{c} \neq \mathbf{0}$ the mapping $p_{a\mathbf{c}}\colon \mathbb{E} \to \mathbb{E}$ has no stable points at all.*

Let $\mathbf{c}$ be some nonzero slipping vector on the line a. Let's choose a point A not lying on this line. Then let's draw the plane α passing through the point A and the line a. Denote $B = p_{a\mathbf{c}}(A)$. According to the theorem 14.1, the point B belongs to the plane α. Let's denote by b the line AB and consider the vector $\overrightarrow{BA}$. We denote by $\mathbf{d}$ the slipping vector on the line b corresponding to the geometric vector $\overrightarrow{BA}$. It is obvious that $A = p_{b\mathbf{d}}(B)$, because of which the point A appears to be a stable point for the composition $f = p_{b\mathbf{d}} \circ p_{a\mathbf{c}}$. According to the theorem 10.1 the mapping f is either a rotation of the space about some axis or the composition of such a rotation with the mirror reflection in some plane. As we shall see, in our case only the firs option is possible. The mapping $f = p_{b\mathbf{d}} \circ p_{a\mathbf{c}}$ is a rotation about the axis passing through the point A and being perpendicular to the plane α. This proposition can be strengthened a little bit.

THEOREM 14.3. *Let $p_{a\mathbf{c}}$ and $p_{b\mathbf{d}}$ be two mappings of translation by two vectors $\mathbf{c}$ and $\mathbf{d}$ along two lines a and b. If their composition $f = p_{b\mathbf{d}} \circ p_{a\mathbf{c}}$ has a stable point O, then the lines a and b lie on some plane α, while the mapping f itself is a rotation about an axis passing through the point O and being perpendicular to the plane α.*

THEOREM 14.4. *Let $p_{a\mathbf{m}}$, $p_{b\mathbf{n}}$, and $p_{c\mathbf{k}}$ be three mappings of translation by the vectors $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{k}$ along the lines a, b, and c. If their composition $f = p_{c\mathbf{k}} \circ p_{b\mathbf{n}} \circ p_{a\mathbf{m}}$ has a stable point O, then this composition is a rotation about some axis passing through the point O.*

We do not prove the theorems 14.3 and 14.4 here. Their proofs are left to the reader as exercises. The matter is that upon formulating the axiom A20 the statements of these theorems simplify substantially. The proofs of the corresponding simplified theorems based on the axiom A20 are given in Chapter VI

EXERCISE 14.1. *Prove the theorem 14.2 on the base of the theorem 14.1.*

EXERCISE 14.2. *Prove the theorems 14.3 and 14.4*

The theorems 14.3 and 14.4 differ only in the number of translations in the composition. As appears, this number could be arbitrary. Using the theorems 9.6, 14.2, and 14.4, by induction one can prove the following fact.

THEOREM 14.5. *Let $f = f_1 \circ \ldots \circ f_n$ be the composition of n translations by n vectors along n lines. If such a composition has a stable point O, then it is a rotation about some axis passing through the point O.*

§ 15. Motions and congruence of complicated geometric forms.

THEOREM 15.1. *Each mapping of congruent translation of the space $f:\mathbb{E}\to\mathbb{E}$ is the composition $f=g\circ p_{a\mathbf{c}}$, where g is a generalized rotation about some point O, while $p_{a\mathbf{c}}$ is a translation by some vector $\mathbf{c}$ along some straight line a.*

PROOF. If the mapping f has a stable point O, then the mapping f itself is a generalized rotation about this point. In this case we choose the zero vector $\mathbf{c}=\mathbf{0}$ on an arbitrary line a and assign $g=f$. From $p_{a\mathbf{c}}=\mathrm{id}$ we get the required expansion $f=g\circ p_{a\mathbf{c}}$ for the initial mapping f.

Assume that f has no stable points. Let's choose some arbitrary point A and denote $O=f(A)$. Then $O\neq A$. Let's draw the line AO and denote it by a. Then we consider the vector $\mathbf{c}=\overrightarrow{AO}$. The translation $p_{a\mathbf{c}}:\mathbb{E}\to\mathbb{E}$ also maps the point A to the point O. This mapping is bijective and the translation by the opposite vector $\overrightarrow{OA}=-\mathbf{c}$ along the line a is the inverse mapping for it. Let's denote $g=f\circ p_{a\mathbf{c}}^{-1}$. It is easy to verify that the point O is a stable point for the mapping g, i. e. g is a generalized rotation about the point O. From $g=f\circ p_{a\mathbf{c}}^{-1}$ one easily derives the required expansion $f=g\circ p_{a\mathbf{c}}$ for f. $\square$

Using the theorems 10.1 and 15.1, one finds that an arbitrary congruent translation of the space f admits an expansion of one of the following two sorts: $f=\theta_{hk}\circ p_{a\mathbf{c}}$ or $f=z_\beta\circ\theta_{hk}\circ p_{a\mathbf{c}}$. Due to the existence of such expansions the mapping f is bijective since mirror reflections, rotations about axes, and translation along lines all are bijective mappings.

Let f and g be two mappings of congruent translation. Since f is bijective, we calculate its inverse mapping f^{-1} and construct the composition $g'=f\circ g\circ f^{-1}$. Such a procedure is called the *conjugation* of the mapping g by means of the mapping f. The mapping g' obtained as a result of conjugation is called a conjugate mapping for g.

Theorem 15.2. *Let f and g are two mappings of congruent translation and let $g' = f \circ g \circ f^{-1}$ be obtained through the conjugation of the mapping g by means of f. In this case*

(1) *if $g = z_\beta$ is a mirror reflection in a plane β, then g' is a mirror reflection in the plane $f(\beta)$;*
(2) *if $g = \theta_{hk}$ is a rotation about an axis a by an angle $\angle hk$, then g' is the rotation about the axis $f(a)$ by an angle congruent to $\angle hk$;*
(3) *if $g = p_{a\mathbf{c}}$ is a translation by some vector $\mathbf{c}$ along some line a, then g' is the translation by the vector $f(\mathbf{c})$ along the line $f(a)$.*

Exercise 15.1. *Prove the theorem 15.2. For this purpose use the fact that if g maps a point A to a point B, then its conjugate mapping $g' = f \circ g \circ f^{-1}$ maps the point $f(A)$ to the point $f(B)$.*

Definition 15.1. A congruent translation of the space f, which expands into the composition of a rotation about some axis and a translation along some line ($f = \theta_{hk} \circ p_{a\mathbf{c}}$), is said to be *even*. If there is a mirror reflection in a plane in the expansion of f (i. e. $f = z_\beta \circ \theta_{hk} \circ p_{a\mathbf{c}}$), then f is said to be *odd*.

A mapping of congruent translation cannot be even and odd at the same time. For the mappings which are generalized rotations about a fixed point this fact was proved in § 11. In general case it should be proved separately. Let's prove it by contradiction. Assume that f is a mapping of congruent translation possessing some expansions of two sorts:

$$f = g_A \circ p_{a\mathbf{c}}, \qquad\qquad f = \tilde{g}_B \circ p_{b\mathbf{d}}. \tag{15.1}$$

Here g_A is an even rotation about some fixed point A, while $\tilde{g}_B$ is an odd rotation about some fixed point B. From the relationships (15.1) we derive $g_A \circ p_{a\mathbf{c}} \circ p_{b\mathbf{e}} = \tilde{g}_B$, where $\mathbf{e}$ is the vector opposite to the vector $\mathbf{d}$.

For the beginning let's study the case $A = B$. Here we have $g_A \circ p_{a\mathbf{c}} \circ p_{b\mathbf{e}} = \tilde{g}_A$. If this relationship is fulfilled, the composition of translations $p_{a\mathbf{c}} \circ p_{b\mathbf{e}}$ has the stable point A. Applying the theorem 14.3, we get $p_{a\mathbf{c}} \circ p_{b\mathbf{e}} = \theta_{hk}$, where θ_{hk} is the rotation about some axis passing through the point A. The composition of g_A with such a rotation does not change the parity, therefore the equality $g_A \circ \theta_{hk} = \tilde{g}_A$ contradicts the fact that $\tilde{g}_A$ is odd. This contradiction proves that the relationships (15.1) cannot be fulfilled simultaneously in the case $A = B$.

Now assume that $A \neq B$. In this case we consider the translation along the line AB by the vector $\mathbf{s} = \overrightarrow{AB}$. Let's denote it $p_{r\mathbf{s}}$. Then $B = p_{r\mathbf{s}}(A)$. Using this equality, we set $g_B = p_{r\mathbf{s}} \circ g_A \circ p_{r\mathbf{s}}^{-1}$. The mapping g_B produced from g_A by conjugation is a generalized rotation about a fixed point B, having the same parity as the mapping g_A, i. e. it is even. This fact follows from the theorem 15.2. From $g_B = p_{r\mathbf{s}} \circ g_A \circ p_{r\mathbf{s}}^{-1}$ for the initial mapping g_A we derive

$$g_A = p_{r\mathbf{s}}^{-1} \circ g_B \circ p_{r\mathbf{s}} = g_B \circ (g_B^{-1} \circ p_{r\mathbf{s}}^{-1} \circ g_B) \circ p_{r\mathbf{s}}.$$

The mapping $g_B^{-1} \circ p_{r\mathbf{s}}^{-1} \circ g_B$ is produced from $p_{r\mathbf{s}}^{-1}$ through conjugation by means of g_B^{-1}. According to the theorem 15.2 it is a translation by a vector $\mathbf{v}$ along some line u. Then for g_A we have $g_A = g_B \circ p_{u\mathbf{v}} \circ p_{r\mathbf{s}}$. Let's substitute this formula into the relationship $g_A \circ p_{a\mathbf{c}} \circ p_{b\mathbf{e}} = \tilde{g}_B$, which follows from (15.1). Then

$$p_{u\mathbf{v}} \circ p_{r\mathbf{s}} \circ p_{a\mathbf{c}} \circ p_{b\mathbf{e}} = g_B^{-1} \circ \tilde{g}_B. \tag{15.2}$$

Due to the relationship (15.2) the composition of four translations in the left hand side of this relationship has the stable point B. Applying the theorem 14.5 with $n = 4$, we get that this composition is a rotation about some axis passing through the point B. Hence, $\tilde{g}_B = g_B \circ \theta_{hk}$, which contradicts the initial assumption that $\tilde{g}_B$ is odd. Thus, in the case $A \neq B$ the relationships (15.1) cannot be fulfilled simultaneously either.

Definition 15.2. A congruent translation of the space $f : \mathbb{E} \to \mathbb{E}$ is called a *motion*, if it is even.

The congruence of segments and angles are basic concepts, they enter the statements of the axioms. The congruence of triangles is a derived concept. Passing from triangles to more complicated geometric forms, we could formulate the definitions of congruence for each particular form. However, the concept of congruent translations enable us to do it at once.

Definition 15.3. Two geometric forms Φ_1 and Φ_2 are called *congruent*, if there is a congruent translation $f\colon \mathbb{E} \to \mathbb{E}$ performing one-to-one correspondence for the points of these forms.

Since the congruent translations are divided into even and odd ones, we can sharpen our concept of congruence for arbitrary geometric forms.

Definition 15.4. Two geometric forms Φ_1 and Φ_2 are called *strictly congruent* if there is a motion performing one-to-one correspondence for the points of these forms.

Definition 15.5. Two geometric forms Φ_1 and Φ_2 are called *mirror congruent* or being *mirror images of each other*, if there is an odd congruent translation $f\colon \mathbb{E} \to \mathbb{E}$ performing one-to-one correspondence for the points of these forms.

Exercise 15.2. *Show that for triangles the definitions 15.3, 15.4, and 15.5 are equivalent to the definition 5.1 from Chapter III.*

CHAPTER V

AXIOMS OF CONTINUITY.

§ 1. Comparison of straight line segments.

Definition 1.1. Let $[AB]$ and $[CD]$ be two straight line segments. We say that the segment $[AB]$ is *smaller* than the segment $[CD]$, and write it as $[AB] < [CD]$, if in the interior of the segment $[CD]$ there is a point E such that $[AB] \cong [CE]$.

Theorem 1.1. *The binary relation of comparison for straight line segments introduced in the definition 1.1 possesses the following five properties:*

1. *the condition $[AB] < [CD]$ excludes $[AB] \cong [CD]$;*
2. *the condition $[AB] < [CD]$ excludes $[CD] < [AB]$;*
3. *$[AB] < [CD]$ and $[CD] < [EF]$ imply $[AB] < [EF]$;*
4. *if $[\tilde{A}\tilde{B}] \cong [AB]$ and $[\tilde{C}\tilde{D}] \cong [CD]$, then $[AB] < [CD]$ implies $[\tilde{A}\tilde{B}] < [\tilde{C}\tilde{D}]$;*
5. *for arbitrary two non-congruent segments $[AB]$ and $[CD]$ one of the two conditions $[AB] < [CD]$ or $[CD] < [AB]$ is always fulfilled.*

Proof. The first item of the theorem is a direct consequence of the axiom A13. Indeed, if the conditions $[AB] < [CD]$ and $[AB] \cong [CD]$ are fulfilled simultaneously, then it would mean that on the ray $[CD\rangle$ there are two points D and E such that $[AB] \cong [CD]$ and $[AB] \cong [CE]$, which contradicts the axiom A13.

Let's prove the second item of the theorem by contradiction. Assume that both conditions $[AB] < [CD]$ and $[CD] < [AB]$ are fulfilled simultaneously. Then in the interior of the segment $[CD]$ there is a point E and in the interior of the segment $[AB]$ there is a point F such that the following relationships are fulfilled:

$$[AB] \cong [CE], \qquad\qquad [CD] \cong [AF].$$

Using the relationship $[CD] \cong [AF]$, we define the mapping of congruent translation f such that it maps the line CD to the line AB and such that $f(C) = A$ and $f(D) = F$ (see theorems 2.1, 2.2, and § 3 of Chapter III). Let's denote $\tilde{E} = f(E)$. Then from $(C \blacktriangleright E \blacktriangleleft D)$ we get $(A \blacktriangleright \tilde{E} \blacktriangleleft F)$. The segments $[A\tilde{E}]$ and $[CE]$ are congruent. As a result on the ray $[AB\rangle$ we get two points $\tilde{E} \neq B$ such that $[A\tilde{E}] \cong [CE]$ and $[AB] \cong [CE]$, which contradicts the axiom A13. The contradiction obtained shows that the conditions $[AB] < [CD]$ and $[CD] < [AB]$ cannot be fulfilled simultaneously.

Now let's consider the third item of the theorem. The relationships $[AB] < [CD]$ and $[CD] < [EF]$ mean that in the interiors of the segments $[CD]$ and $[EF]$ there are two points M and N such that the following conditions are fulfilled:

$$[AB] \cong [CM], \qquad\qquad [CD] \cong [EN].$$

Using the second of these two conditions, we construct a congruent translation f mapping the line CD to the line EF and such that $f(C) = E$ and $f(D) = N$. By means of the mapping f we define the point $K = f(M)$ in the interior of the segment $[CN]$. For this point $[EK] \cong [CM]$. From $[EK] \cong [CM]$ and from $[AB] \cong [CM]$ we derive $[AB] \cong [EK]$. From $(E \blacktriangleright K \blacktriangleleft N)$ and $(E \blacktriangleright N \blacktriangleleft F)$ we get $(E \blacktriangleright K \blacktriangleleft F)$ (see lemma 3.2 in Chapter II). In other words, the point K lies in the interior of the segment $[EF]$ and $[AB] \cong [EF]$. The required relationship $[AB] < [EF]$ is proved.

Let's prove the fourth item of the theorem. From $[AB] < [CD]$ we derive the existence of a point E in the interior of the segment

$[CD]$ such that $[AB] \cong [CE]$. Using $[CD] \cong [\tilde{C}\tilde{D}]$, we construct a congruent translation f of the line CD to the line $\tilde{C}\tilde{D}$ such that $f(C) = \tilde{C}$ and $f(D) = \tilde{D}$. Applying f to the point E, we define the point $\tilde{E} = f(E)$ lying in the interior of the segment $[\tilde{C}\tilde{D}]$ such that $[CE] \cong [\tilde{C}\tilde{E}]$. Combining this relationship with $[\tilde{A}\tilde{B}] \cong [AB]$ and with $[AB] \cong [CE]$, we get $[\tilde{A}\tilde{B}] \cong [\tilde{C}\tilde{E}]$. The relationship $[\tilde{A}\tilde{B}] < [\tilde{C}\tilde{D}]$ is proved.

In order to prove the fifth item of the theorem, applying the axiom A13, we mark a point E on the ray $[CD\rangle$ such that the segment $[AB]$ is congruent to the segment $[CE]$. The coincidence $D = E$ is impossible since it would mean the congruence of the segments $[AB]$ and $[CD]$. Therefore the point E lies in the interior of the segment $[CD]$ or outside this segment. In the first case $[AB] < [CD]$. In the second case $[CD] < [CE]$. Complementing this relationship with $[AB] \cong [CE]$ and applying the fourth item of the theorem, which is already proved, we get $[CD] < [AB]$. The theorem is proved. $\square$

The properties (1)–(3) of the comparison for segments are very similar to the corresponding properties of the order relation (see § 3 in Chapter I). The only difference is that instead of «$[AB] < [CD]$ excludes $[AB] = [CD]$» the theorem 1.1 here says «$[AB] < [CD]$ excludes $[AB] \cong [CD]$». This means the comparison of segments is an order relation not in the set of segments, but in the factorset consisting of classes of congruent segments. The next two properties (4) and (5) mean that this factorset is linearly ordered.

The comparison relation $[AB] < [CD]$ is sometimes written as $[CD] > [AB]$. In this case we say that the segment $[CD]$ is *bigger* than the segment $[AB]$. The writings $[AB] \leqslant [CD]$ and $[CD] \geqslant [AB]$ mean that one or the two conditions $[AB] < [CD]$ or $[AB] \cong [CD]$ is fulfilled.

THEOREM 1.2. *If two points C and D lie in the interior of the segment $[AB]$, then $[CD] < [AB]$.*

PROOF. Let's introduce the order relation on the line AB

by setting $A \prec B$ (see § 4 in Chapter II). If C and D are in the interior of the segment $[AB]$, then one of the following two conditions is fulfilled:

$$A \prec C \prec D \prec B, \qquad\qquad A \prec D \prec C \prec B.$$

Let's consider the case where the first condition is fulfilled. The case where the second condition is fulfilled is reduced to this case by exchanging the notations of the points C and D. From the relationship $A \prec C \prec D \prec B$ we derive $(A \blacktriangleright C \blacktriangleleft D)$ and $(A \blacktriangleright D \blacktriangleleft B)$. From the first of these two relationships we get $[CD] < [AD]$, while from the second one we derive $[AD] < [AB]$. Now, applying the third item of the theorem 1.1, we obtain the required relationship $[CD] < [AB]$. The theorem is proved. $\square$

§ 2. Comparison of angles.

Definition 2.1. Let $\angle hk$ and $\angle lq$ be two arbitrary angles. We say that the $\angle hk$ is *smaller* than the angle $\angle lq$ and write it as $\angle hk < \angle lq$ if there is a ray m coming out from the vertex of the angle $\angle lq$ and lying inside it so that $\angle hk \cong \angle lm$.

Theorem 2.1. *The binary relation of comparison for angles possesses the following five properties:*

(1) *the condition* $\angle hk < \angle lq$ *excludes* $\angle hk \cong \angle lq$*;*
(2) *the condition* $\angle hk < \angle lq$ *excludes* $\angle lq < \angle hk$*;*
(3) $\angle hk < \angle lq$ *and* $\angle lq < \angle mn$ *imply* $\angle hk < \angle mn$*;*
(4) *if* $\angle\tilde{h}\tilde{k} \cong \angle hk$ *and if* $\angle\tilde{l}\tilde{q} \cong \angle lq$*, then the relationship* $\angle hk < \angle lq$ *implies* $\angle\tilde{h}\tilde{k} < \angle\tilde{l}\tilde{q}$*;*
(5) *for any two non-congruent angles* $\angle hk$ *and* $\angle lq$ *one of the two conditions* $\angle hk < \angle lq$ *or* $\angle lq < \angle hk$ *is fulfilled .*

Theorem 2.2. *If the rays* l *and* q *coming out from the vertex of the* $\angle hk$ *lie inside this angle, then* $\angle lq < \angle hk$*.*

Exercise 2.1. *By analogy to theorems 1.1 and 1.2 prove the theorem 2.1 and the theorem 2.2.*

In the case of angles we have two reference angles, they are the right angle and the straight angle. Each angle can be enclosed into some straight angle, therefore, any angle is smaller than any straight angle. Comparing angles with the right angle we divide those different from a right angle into two sets. An angle smaller than a right angle is called an *acute angle*. An angle bigger than a right angle is called an *obtuse angle*. According to the theorem 2.1 any acute angle is smaller than any obtuse angle.

Definition 2.2. An angle adjacent to an internal angle of a triangle is called an *external angle* of this triangle.

Theorem 2.3. *In a triangle any internal angle is smaller than any external angle not adjacent with it.*

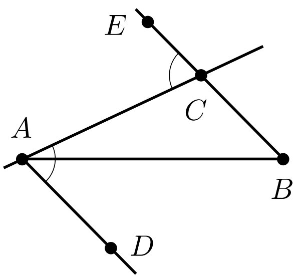


Fig. 2.1

Proof. Let ABC be a triangle . We extend its side $[BC]$ by drawing the ray $[CE\rangle$. The angle $\angle ACE$ adjacent to the angle $\angle ACB$ is an external angle of the triangle ABC. We compare it with the internal angle $\angle CAB$ not adjacent to $\angle ACE$. The line AC divides the plane of the triangle ABC into two half-planes, the points E and B lying on different sides of this line. Applying the axiom A16, in the half plane containing the point B we draw the ray $[AD\rangle$ so that the angle $\angle CAD$ is congruent to the angle $\angle ACE$. If we assume that $\angle CAD \cong \angle CAB$ or $\angle CAD < \angle CAB$, then the ray $[AD\rangle$ coincides with the ray $[AB\rangle$ or it lies inside the angle $\angle CAB$. In both cases the ray $[AD\rangle$ should intersect the segment $[BC]$ (see lemma 6.2 in Chapter II). But in our case the angles $\angle CAD$ and $\angle ACE$ are inner crosswise lying angles at the intersections of the line AC with two lines AD and BC. They are congruent

$\angle CAD \cong \angle ACE$, therefore, due to the theorem 8.1 from Chapter III the lines AD and BC cannot intersect. The contradiction obtained proves the required relationship $\angle CAB < \angle ACE$. □

THEOREM 2.4. *In a triangle a bigger side is opposite to a bigger angle and, conversely, a bigger angle is opposite to a bigger side.*

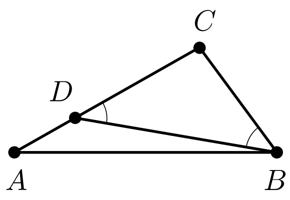


Fig. 2.2

PROOF. Let ABC be a triangle. For the beginning we prove that the relationship $[AC] > [BC]$ implies the relationship $\angle ABC > \angle BAC$. From $[AC] > [BC]$ we conclude that in the interior of the segment $[AC]$ there is a point D such that the segment $[DC]$ is congruent to the segment $[BC]$. Let's draw the segment $[BD]$ and consider the triangle BCD. Due to $[DC] \cong [BC]$ it is isosceles. Hence, $\angle BDC \cong \angle DBC$.

By construction the ray $[BD\rangle$ is inside the angle ABC. Therefore $\angle ABC > \angle DBC$. On the other hand, the angle $\angle BDC$ is an external angle of the triangle ABD. Applying the theorem 2.3, we get $\angle BDC > \angle BAC$. As a result we have three relationships

$$\angle ABC > \angle DBC, \quad \angle BDC \cong \angle DBC, \quad \angle BDC > \angle BAC.$$

Applying the theorem 2.1, from these relationships we derive $\angle ABC > \angle BAC$. The first proposition of the theorem is proved.

We prove the converse proposition by contradiction. Assume that the relationship $\angle ABC > \angle BAC$ is fulfilled, but the relationship $[AC] > [BC]$ is not fulfilled. Then $[AC] \cong [BC]$ or $[AC] < [BC]$. If $[AC] \cong [BC]$, then the triangle ABC is isosceles and $\angle ABC \cong \angle BAC$, which contradicts $\angle ABC > \angle BAC$. If $[AC] < [BC]$, then due to the first proposition of the theorem, which is already proved, we have $\angle ABC < \angle BAC$, which contradicts $\angle ABC > \angle BAC$. Hence, $\angle ABC > \angle BAC$ implies $[AC] > [BC]$. The theorem is proved. □

Assume that a point B lies in the interior of a segment $[AC]$

between the points A and C. The segment $[AC]$ in this case is composed of two segments $[AB]$ and $[BC]$ (see theorems 3.1 and 3.2 in Chapter II). It is sometimes called the *sum* of the segments $[AB]$ and $[BC]$. The segment $[BC]$ is called the *difference* of the segments $[AC]$ and $[AB]$.

Having two arbitrary segments $[MN]$ and $[PQ]$, we can draw two segments congruent to them on one line so that they lie on different sides of some point B on this line:

$$[MN] \cong [AB], \qquad\qquad [PQ] \cong [BC].$$

Then the segment $[AC]$ is the sum of the segments $[MN]$ and $[PQ]$. Such a segment is not unique, however, all segments representing the sum of the segments $[MN]$ and $[PQ]$ are congruent to each other.

Let $[MN] < [PQ]$. Let's draw a segment $[AC]$ congruent to $[PQ]$ on some line. Then $[MN] < [AC]$, therefore, in the interior of the segment $[AC]$ there is a point B such that $[AB] \cong [MN]$. For such a point B the segment $[BC]$ is the difference of the segments $[PQ]$ and $[MN]$.

Theorem 2.5. *In an arbitrary triangle the sum of any two sides is bigger than the third side.*

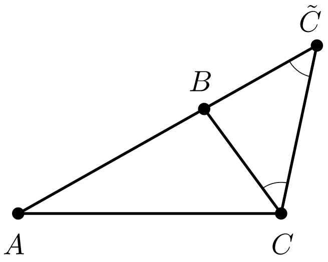


Fig. 2.3

Proof. Let ABC be some triangle. The point B divides the line AB into two rays. Let's choose the ray opposite to the ray $[BA\rangle$ and mark the segment $[B\tilde{C}]$ congruent to the segment $[BC]$ on this ray. The ray $[CB\rangle$ crosses the segment $[A\tilde{C}]$, therefore it lies inside the angle $\angle AC\tilde{C}$ (see lemma 6.2 in Chapter II). This yields the formula $\angle AC\tilde{C} > \angle BC\tilde{C}$. On the other hand, in the isosceles triangle $CB\tilde{C}$ the angles $\angle BC\tilde{C}$ and $\angle B\tilde{C}C$ are congruent. Hence,

$\angle AC\tilde{C} > \angle A\tilde{C}C$. Applying the theorem 2.4 to the triangle $AC\tilde{C}$, we get $[A\tilde{C}] > [AC]$. By construction the segment $[A\tilde{C}]$ is the sum of the segments $[AB]$ and $[BC]$. It is bigger that the segment $[AC]$. The theorem is proved. □

Theorem 2.6. *In any triangle at least two angles are acute.*

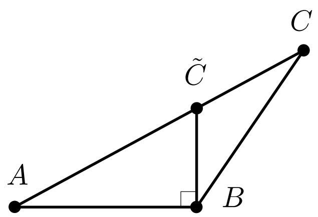


Fig. 2.4

Proof. Let ABC be a triangle, the angle $\angle ABC$ of which is obtuse. Let's draw the perpendicular to the line AB through the point B. Since an obtuse angle is bigger than a right angle, this perpendicular crosses the side AC at some interior point $\tilde{C}$. As a result we get the triangle $AB\tilde{C}$ in which the obtuse angle is replaced by the right angle, while the angle at the vertex A remains unchanged. If this angle was initially obtuse, repeating the procedure, we can replace id by a right angle either. Thus, we conclude: if a triangle has two obtuse angle or if it has an obtuse angle and a right angle, it can be transformed to a triangle with two right angles. However, we know that a triangle with two right angles is impossible (see theorem 8.2 in Chapter III). □

Due to the theorem 2.6 all triangles are divided into three sorts: *acute-angular triangles*, *right-angular triangles*, and *obtuse-angular triangles*. In an acute-angular triangle all angles are acute; a right-angular triangle has one right angle and two acute angles; an obtuse-angular triangle has one obtuse angle, while two other angles are acute.

In a right-angular triangle the side opposite to the right angle is called the *hypotenuse*. Other two sides are called *legs*. Due to the theorems 2.4 and 2.6 in a right-angular triangle the hypotenuse is bigger than any one of two legs.

§ 3. Axioms of real numbers.

The comparison of segments and angles introduced in definitions 1.1 and 2.1 yields very rough concept of what are the sizes of objects being compared. More precise knowledge of a size require the concept of length for segments and a quantitative measure for angles. In order to introduce these concepts we need to use some facts from the theory of real numbers.

Real numbers constitute the set $\mathbb{R}$, where two algebraic operations are defined — the *addition* and the *multiplication*. Basic properties of real numbers are formulated in seventeen axioms R1–R17. The whole theory of real numbers is deduced from these axioms.

Axiom R1. *The addition of real numbers is commutative, i. e.* $a + b = b + a$ *for all* a *and* b *in* $\mathbb{R}$.

Axiom R2. *The addition of real numbers is associative, i. e.* $(a + b) + c = a + (b + c)$ *for all* a, b, *and* c *in* $\mathbb{R}$.

Axiom R3. *There is a number* 0 *in* $\mathbb{R}$, *which is called zero, such that* $a + 0 = a$ *for all* $a \in \mathbb{R}$.

One can prove that zero is unique. Indeed, if we assume that there is another number $\tilde{0}$ with the same property, then $\tilde{0}+0 = \tilde{0}$ and $0 + \tilde{0} = 0$. From the axiom R1 we derive that they do coincide: $\tilde{0} = \tilde{0} + 0 = 0 + \tilde{0} = 0$.

Axiom R4. *For any number* $a \in \mathbb{R}$ *there is an opposite number* $a' \in \mathbb{R}$ *such that* $a + a' = 0$.

For any number $a \in \mathbb{R}$ its opposite number a' is unique. If we assume that there is another opposite number $\tilde{a}'$ for a, then from the axioms R1, R2, and R3 we derive $a' = \tilde{a}'$:

$$a' = a' + 0 = a' + (a + \tilde{a}') = (a' + a) + \tilde{a}' = 0 + \tilde{a}' = \tilde{a}'.$$

Exercise 3.1. *Let* a' *be the number opposite to the number*

$a \in \mathbb{R}$. *Prove that the number opposite to* a' *coincides with the number* a, *i. e.* $(a')' = a$.

EXERCISE 3.2. *Prove that the number opposite to zero coincides with itself, i. e.* $0' = 0$.

The concept of an opposite number is the base for introducing the *subtraction*: $b - a = b + a'$. Moreover, the opposite number is usually denoted $a' = -a$ for the sake of uniformity of notations.

EXERCISE 3.3. *Prove the following relationships:*

$$\begin{aligned} &a - b = -(b - a), && (a - b) + c = a + (c - b),\\ &(a + b) - c = a + (b - c), && (a - b) - c = a - (b + c). \end{aligned}$$

AXIOM R5. *The multiplication of real numbers is commutative, i. e.* $a \cdot b = b \cdot a$ *for all* a *and* b *in* $\mathbb{R}$.

AXIOM R6. *The multiplication of real numbers is associative, i. e.* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all* a, b, *and* c *in* $\mathbb{R}$.

AXIOM R7. *There is a number* 1, *which is called one, such that* $a \cdot 1 = a$ *for all* $a \in \mathbb{R}$.

AXIOM R8. *For any nonzero number* $a \neq 0$ *there is an inverse number* $a^* \in \mathbb{R}$ *such that* $a \cdot a^* = 1$.

EXERCISE 3.4. *Prove that the number one is unique. Also prove that for any number* $a \neq 0$ *its inverse number* a^* *is unique.*

The concept of an inverse number is the base for introducing the *division*: $b : a = b \cdot a^*$. There are several ways for denoting the operation of division for real numbers:

$$b : a = b/a = \frac{b}{a} = b \cdot a^{-1}.$$

The last form writing the quotient b/a is due to the notation $a^* = a^{-1}$ for an inverse number.

AXIOM R9. *The multiplication and the addition of real numbers are related trough the distributivity law:* $(a+b)\cdot c = a\cdot c+b\cdot c$.

EXERCISE 3.5. *Prove the distributivity law relating the multiplication and the subtraction:* $(a-b)\cdot c = a\cdot c-b\cdot c$. *Also derive the following rules for operating with fractions:*

$$\frac{b}{a}+\frac{d}{c}=\frac{b\cdot c+a\cdot d}{a\cdot c}, \qquad\qquad \frac{b}{a}\cdot\frac{d}{c}=\frac{b\cdot d}{a\cdot c}.$$

The set of real numbers $\mathbb{R}$ is equipped with a binary relation of order with respect to which it is a linearly ordered set (see § 3 in Chapter I).

AXIOM R10. *For any two numbers a and b in $\mathbb{R}$ at least one of the three condition $a<b$, $a=b$, or $b<a$ is fulfilled.*

AXIOM R11. *The condition $a<b$ excludes $a=b$.*

AXIOM R12. *The condition $a<b$ excludes $b<a$.*

AXIOM R13. *The conditions $a<b$ and $b<c$ imply $a<c$.*

The following two axioms bind the relation of order with algebraic operations of addition and multiplication for real numbers.

AXIOM R14. *The condition $a<b$ implies $a+c<b+c$ for any number $c\in\mathbb{R}$.*

AXIOM R15. *The conditions $a<b$ and $0<c$ imply $a\cdot c<b\cdot c$.*

EXERCISE 3.6. *Prove that the relation of order in the set of real numbers possesses the following properties:*

(1) $a>b$ *implies* $-a<-b$;
(2) $a<b$ *and* $c<0$ *imply* $a\cdot c>b\cdot c$;
(3) $1>0$;
(4) $a>b>0$ *implies* $b^{-1}>a^{-1}>0$.

Positive integers are obtained by successive adding the unity: $2 = 1 + 1$, $3 = 1 + 1 + 1 = 2 + 1$ etc. They constitute the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. Complementing $\mathbb{N}$ with the number zero and with the numbers opposite to positive integers, we get the set of all integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Fractions n/m, where $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, constitute the set of rational numbers $\mathbb{Q}$. We consider also the set fractions of the special form

$$r = \frac{n}{2^m}, \quad \text{where} \quad n \in \mathbb{Z} \quad \text{and} \quad m \in \mathbb{N}.$$

Such fractions constitute the set of *binary-rational numbers*. THe set of binary-rational numbers is closed with respect to the addition, subtraction, and multiplication, but it is not closed with respect to the division.

Axiom R16. *For any real number $\xi \in \mathbb{R}$ there is a positive integer $n \in \mathbb{N}$ such that $n > \xi$.*

The axiom R16 is known as the Archimedes axiom. It means that each real number has an upper estimate in the set of positive integers. This estimate can be strengthened.

Theorem 3.1. *For any real number ξ there is a positive integer $n \in \mathbb{N}$ such that $-n < \xi < n$.*

Proof. If $\xi = 0$, we can take $n = 1$. If $\xi > 0$, then the number n given by the Archimedes axiom R16 provides the estimate $-n < \xi < n$. If $\xi < 0$, we apply the Archimedes axiom to the number $-\xi$. The resulting number n in this case provides the estimate $-n < \xi < n$. $\square$

Theorem 3.2. *For any real number $\xi > 0$ there is a positive integer m such that $2^{-m} < \xi$.*

Proof. Let's consider the number ξ^{-1} and apply the Archimedes axiom R16 to it. As a result we get a positive integer m such that $\xi^{-1} < m$. Let's use the inequality $m < 2^m$ which is fulfilled for all positive integers $m \in \mathbb{N}$. It is easily proved by induction on m. Now from $\xi^{-1} < 2^m$ and $\xi > 0$ we derive $\xi > 2^{-m}$. The theorem is proved. $\square$

Axiom R17. *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two monotonic sequences of real numbers such that*

$$a_1 \leqslant \ldots \leqslant a_n \leqslant a_{n+1} \leqslant \ldots \leqslant b_{n+1} \leqslant b_n \leqslant \ldots \leqslant b_1.$$

Then there is a real number $\xi \in \mathbb{R}$ separating these sequences, i. e. such that $a_n \leqslant \xi \leqslant b_n$ for all $n \in \mathbb{N}$.

The axiom R17 is known as Cantor's axiom. It expresses the property of completeness of real numbers. Cantor's axiom R17 lies in the base of proving many well-known facts of mathematical analysis (see details in [6]).

§ 4. Binary rational approximations of real numbers.

Let p, q, and ξ be three real numbers. We say that the real numbers p and q *approximate* the number real ξ if

$$p \leqslant \xi \leqslant q.$$

The number p is called the *lower estimate*, while q is the *upper estimate*. The difference $q - p$ is called the *accuracy* of the approximation.

Let's study the problem of approximating real numbers with binary rational numbers. Let's denote by $I(m, k)$ the following semi-open intervals within the set of real numbers:

$$I(m,k) = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right) = \left\{x \in \mathbb{R} : \quad \frac{k}{2^m} \leqslant x < \frac{k+1}{2^m}\right\}.$$

The integer $k \in \mathbb{Z}$ is called the *number* of such interval, while the number $m \in \mathbb{N}$ is its *order*.

THEOREM 4.1. *For any fixed $m \in \mathbb{N}$ each real number ξ is trapped exactly in one interval $I(m, k)$.*

PROOF. The intervals $I(m, k_1)$ and $I(m, k_2)$ with different numbers $k_1 \neq k_2$ do not intersect each other. Therefore, once the number ξ is in some interval, it cannot be in any other.

Applying the theorem 3.1 to the number ξ, we find that there is an integer n, such that ξ is in some interval $(-n, n)$ enclosed between the numbers $-n$ and n. But for this interval we have

$$(-n, n) \subset \bigcup_{k=-2^m n}^{2^m n} I(m, k).$$

Therefore, the number ξ is certainly contained in one of the intervals $I(m, k)$ whose union contains the interval $(-n, n)$. □

The theorem 4.1 proved just above means that for any real number ξ and for any positive integer m there is a unique integer k_m defining two binary-rational numbers a_m and b_m such that

$$\frac{k_m}{2^m} = a_m \leqslant \xi < b_m = \frac{k_m + 1}{2^m}. \tag{4.1}$$

The numbers a_m and b_m are called the *binary-rational approximations* of the order m for the number ξ.

LEMMA 4.1. *The sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_m\}_{m\in\mathbb{N}}$ of binary-rational approximations of a given real number ξ are monotonic: $a_{m+1} \geqslant a_m$ and $b_{m+1} \leqslant b_m$. The sequence $\{a_m\}_{m\in\mathbb{N}}$ stabilizes if and only if the number ξ itself is a binary-rational number.*

PROOF. When advancing the number m by one the interval $I(m, k)$ divides into two mutually non-intersecting intervals of the next order: $I(m, k) = I(m + 1, 2k) \cup I(m + 1, 2k + 1)$. Therefore, when passing from m to $m + 1$ we obtain $k_{m+1} = 2k_m$ or $k_{m+1} = 2k_m + 1$. For a_{m+1} and b_{m+1} this yields

$$a_{m+1} = a_m, \qquad\qquad b_{m+1} = b_m - \frac{1}{2^{m+1}},$$

$$a_{m+1} = a_m + \frac{1}{2^{m+1}}, \qquad\qquad b_{m+1} = b_m.$$

In any of these two cases the monotony conditions $a_{m+1} \geqslant a_m$ and $b_{m+1} \leqslant b_m$ are fulfilled.

Let's consider the case where the number ξ is a binary-rational. Then ξ can be written as a fraction

$$\xi = \frac{q}{2^n} = \frac{q \cdot 2^{m-n}}{2^m}. \tag{4.2}$$

From (4.2) for k_m, where $m \geqslant n$, we get $k_m = q \cdot 2^{m-n}$. For the numbers a_m and b_m, where $m \geqslant n$, this yields

$$a_m = \xi, \qquad\qquad b_m = \xi + \frac{1}{2^m}.$$

This means that the sequence $\{a_m\}_{m\in\mathbb{N}}$ stabilizes at the value $a_m = \xi$ for $m \geqslant n$.

Assume, conversely, that the sequence $\{a_m\}_{m\in\mathbb{N}}$ stabilizes for $m \geqslant n$. Let $a = a_n$ be the value at which this sequence stabilizes. Then from (4.1) we derive

$$a \leqslant \xi < a + \frac{1}{2^m}. \tag{4.3}$$

The inequalities (4.3) are easily transformed to

$$\frac{1}{2^m} > \xi - a \geqslant 0.$$

If $\xi \neq a$, then $\xi - a \neq 0$ and we get a contradiction with the theorem 3.2. Hence, the number ξ coincides with the binary-rational number a. The lemma is proved. $\square$

EXERCISE 4.1. *Show that the second sequence $\{b_m\}_{m\in\mathbb{N}}$ cannot stabilize for any choice of the number ξ.*

Let $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_m\}_{m\in\mathbb{N}}$ be the sequences of binary-rational approximations of the number ξ. Due to the lemma 4.1 we can apply Cantor's axiom R17. Let $\tilde{\xi}$ is a number whose existence is declared in this axiom. Then we have two inequalities

$$a_m \leqslant \xi < b_m, \qquad\qquad a_m \leqslant \tilde{\xi} \leqslant b_m.$$

From these inequalities we derive the estimate for the modulus of the difference of two numbers ξ and $\tilde{\xi}$:

$$|\xi - \tilde{\xi}| \leqslant b_m - a_m = \frac{1}{2^m}.$$

If we assume that $\xi \neq \tilde{\xi}$, then we immediately get a contradiction to the theorem 3.2. Therefore, $\xi = \tilde{\xi}$. Thus, we have an important conclusion: any real number ξ is uniquely fixed by two sequences of its binary-rational approximations (4.1).

Theorem 4.2. *Let $\xi < \eta$ be two real numbers and let a_m, b_m, $\tilde{a}_m$, and $\tilde{b}_m$ be the sequences of their binary-rational approximations. Then there is an integer number n such that $b_m < \tilde{a}_m$ for all integers $m > n$.*

Proof. Let's consider the relationships (4.1). From these relationships for the numbers a_m and b_m we derive the estimates

$$\begin{aligned} b_m &= a_m + \frac{1}{2^m} \leqslant \xi + \frac{1}{2^m},\\ a_m &= b_m - \frac{1}{2^m} > \xi - \frac{1}{2^m}. \end{aligned} \tag{4.4}$$

The estimates analogous to (4.4) can be written for the numbers $\tilde{a}_m$ and $\tilde{b}_m$ either. Let's write such estimates for b_m and $\tilde{a}_m$:

$$\tilde{a}_m > \eta - \frac{1}{2^m}, \qquad\qquad b_m \leqslant \xi + \frac{1}{2^m}.$$

Subtracting the second inequality from the first one, we get

$$\tilde{a}_m - b_m > (\eta - \xi) - \frac{1}{2^{m-1}}. \tag{4.5}$$

Relying on the theorem 3.2, we choose a number n such that

$$\eta - \xi > \frac{1}{2^{n-1}}. \tag{4.6}$$

From (4.5) and from (4.6) for the difference $\tilde{a}_m - b_m$ we get the lower estimate

$$\tilde{a}_m - b_m > \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}.$$

For $m > n$ it yields the required inequality $b_m < \tilde{a}_m$. The theorem is proved. $\square$

§ 5. The Archimedes axiom and Cantor's axiom in geometry.

Assume that a segment $[OE]$ on some line a is given. Let's define the positive direction on this line by means of the vector $\mathbf{c} = \overrightarrow{OE}$. Let's denote $A(0) = O$, $A(1) = E = p_{\mathbf{c}}(A(0))$ and construct the sequence of points $A(2) = p_{\mathbf{c}}(A(1))$, $A(3) = p_{\mathbf{c}}(A(2))$ etc by applying the congruent translation by the vector $\mathbf{c}$ repeatedly. Let's enumerate with the negative integers the points obtained by several subsequent translations of the point O by the opposite vector $-\mathbf{c}$: $A(-1) = p_{-\mathbf{c}}(A(0))$, $A(-2) = p_{-\mathbf{c}}(A(-1))$ etc. This sequence of points enumerated with integer numbers appears to be monotonic (see definition in § 4 of Chapter II):

$$\ldots \prec A(-2) \prec A(-1) \prec A(0) \prec A(1) \prec A(2) \prec \ldots .$$

The segments connecting the neighboring points in this sequence are congruent to each other:

$$[A(-n)A(-n+1)] \cong \ldots \cong [OE] \cong \ldots \cong [A(n-1)A(n)]$$

Such a monotonic sequence of points is called *equidistant*. The segment $[A(0)A(2)]$ is obtained by adding two segments — the segment $[A(0)A(1)]$ and the segment $[A(1)A(2)]$, each being congruent to $[OE]$. We say that it is obtained by *duplicating* the segment $[OE]$. The segment $[A(0)A(3)]$ is obtained by *triplicating* the segment $[OE]$, the next segment $[A(0)A(4)]$ is obtained by *quadrupling* etc. Let's visualize these words as follows:

$$[A(0)A(2)] \cong 2 \cdot [OE], \ \dots\ , [A(0)A(n)] \cong n \cdot [OE].$$

AXIOM A18. *For any two segments $[AB]$ and $[OE]$ there is a positive integer $n \in \mathbb{N}$ such that $[AB] < [CD] \cong n \cdot [OE]$, where the segment $[CD]$ is obtained as the sum of n replicas of the segment $[OE]$.*

The axiom A18 is known as the Archimedes axiom. It is a geometric version of the axiom R16 considered in § 3.

Let's consider the segment $[OE]$ on the line a again. According to the theorem 7.2 from Chapter III, in the interior of this segment one can find a point E_1 being its center. Then we have $[OE] \cong 2 \cdot [OE_1]$. Let's write this fact as follows:

$$[OE_1] \cong \frac{1}{2} \cdot [OE].$$

Applying the operation of bisection once more, we find the center of the segment $[OE_1]$. We denote it E_2. Repeating this procedure many times, we get a series of segments each of which is twice as small as the previous one:

$$[OE_1] \cong \frac{1}{2} \cdot [OE], \ \dots\ , [OE_m] \cong \frac{1}{2^m} \cdot [OE].$$

The following theorem is a geometric version of the theorem 3.2.

THEOREM 5.1. *For any two segments $[AB]$ and $[OE]$ there is a positive integer m such that*

$$[AB] > [CD] \cong \frac{1}{2^m} \cdot [OE],$$

where the segment $[CD]$ is obtained as a result of m-tuple bisection of the segment $[OE]$.

PROOF. If we assume that the proposition of the theorem is not valid, then for any positive integer m and for $n = 2^m$ we have

$$[AB] < [CD] \cong \frac{1}{2^m} \cdot [OE] \quad \text{or} \quad [OE] > 2^m \cdot [AB],$$

which contradicts the Archimedes axiom A18. This contradiction proves the theorem. □

AXIOM A19. *Let $\{[A_n B_n]\}_{n\in\mathbb{N}}$ be a sequence of segments on some straight line such that*

$$[A_{n+1}B_{n+1}] \subset [A_n B_n] \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then the intersection of these segments is not empty and there is a point X belonging to all of them.

The axiom A19 is called Cantor's axiom. It is a geometric version of Cantor's axiom R17 from § 3.

§ 6. The real axis.

Let's consider again the line a and the segment $[OE]$ on it. In § 5 we have constructed the sequence of points $\{A(n)\}_{n\in\mathbb{Z}}$ on this line enumerated by integer numbers. The sequence $\{A(n)\}_{n\in\mathbb{Z}}$ is equidistant and monotonic, it is such that the inequality $m < n$ implies $A(m) \prec A(n)$. It defines a *coordinate network* (or a *gauge*) on the line a. We bisect each of the segments connecting the neighboring points $A(n)$ and $A(n+1)$, then we enumerate the centers of these segments by means of half-integers. As a result we get the doubly dense gauge of points:

$$\ldots \prec A(-1) \prec A(-1/2) \prec A(0) \prec A(1/2) \prec A(1) \prec \ldots .$$

The procedure of doubling the density of points on the line a can be performed repeatedly again and again. On the second step we use the numbers of the forms $k/4$, then of the form $k/8$, $k/16$ etc. Repeating this procedure infinitely many times, we get the set of points enumerated with binary-rational numbers. Let's denote this set through $\mathcal{A}$, while the set of binary-rational numbers through $\mathbb{D}$. Then, putting into correspondence the point $A(r)$ to a number $r \in \mathbb{D}$, we get a mapping

$$A\colon \mathbb{D} \to \mathcal{A}. \tag{6.1}$$

The mapping (6.1) is bijective. For this mapping

$$p < q \ \text{ implies } \ A(p) \prec A(q). \tag{6.2}$$

The property (6.2) follows from the way in which the mapping A is constructed. It is easy understand it if we write the numbers p and q in the form brought to a common denominator:

$$p = \frac{k_1}{2^m}, \qquad\qquad q = \frac{k_2}{2^m}.$$

Theorem 6.1. *The mapping* (6.1) *can be extended up to a bijective mapping $A\colon \mathbb{R} \to a$ from the set of real numbers $\mathbb{R}$ to the line $a = OE$ preserving the property* (6.2).

Proof. First of all we prove the existence of a mapping that extends (6.1). We do it by constructing this mapping directly. Let $\xi \in \mathbb{R}$ be some real number. According to the results of § 5 for each real number ξ there are two sequences of its binary-rational approximations a_m and b_m, which are fixed by this number uniquely. Applying the mapping (6.1) to the numbers a_m and b_m, we get two sequences of points $A_m = A(a_m)$ and $B_m = A(b_m)$. Due to the lemma 4.1 and due to the property (6.2) of the mapping A for these sequences we have

$$[A_{m+1}B_{m+1}] \subset [A_m B_m].$$

Hence we can apply Cantor's axiom A19. Due to this axiom there is a point X belonging to all segments $[A_m B_m]$.

The point X belonging to all segments $[A_m B_m]$ simultaneously is unique. Indeed, from $b_m - a_m = 2^{-m}$ and by construction of the points A_m and B_m we have

$$[A_m B_m] \cong \frac{1}{2^m} \cdot [OE].$$

The existence of a second point $\tilde{X}$ on the line a belonging to all segments $[A_m B_m]$ would mean

$$[X\tilde{X}] < [A_m B_m] \cong \frac{1}{2^m} \cdot [OE],$$

which contradicts the theorem 5.1. The existence and uniqueness of the point X determined by a real number ξ through its binary-rational approximations defines the required mapping $A\colon \mathbb{R} \to a$ if we set $A(\xi) = X$. If the number ξ binary-rational, then the sequence a_m stabilizes: $a_m = \xi$ for $m > m_0$. Therefore, $A_m = X$ for $m > m_0$, and we get that the restriction of $A\colon \mathbb{R} \to a$ to the set of binary-rational numbers coincides with (6.1).

Let ξ and η be two real numbers and let $\xi < \eta$. Let's denote through a_m, b_m, $\tilde{a}_m$, and $\tilde{b}_m$ their binary-rational approximations. From the theorem 4.2 we derive the existence of a positive integer number m such that

$$a_m < b_m < \tilde{a}_m < \tilde{b}_m.$$

Let $A_m = A(a_m)$, $B_m = A(b_m)$, $\tilde{A}_m = A(\tilde{a}_m)$, and $\tilde{B}_m = A(\tilde{b}_m)$. Let's use the property (6.2) of the mapping (6.1) and get

$$A_m \prec B_m \prec \tilde{A}_m \prec \tilde{B}_m. \tag{6.3}$$

But the point $X = A(\xi)$ by construction belongs to the segment $[A_m B_m]$, while the point $Y = A(\eta)$ belongs to $[\tilde{A}_m \tilde{B}_m]$. Therefore, we can sharpen the relationships (6.3):

$$A_m \preccurlyeq A(\xi) \preccurlyeq B_m \prec \tilde{A}_m \preccurlyeq A(\eta) \preccurlyeq \tilde{B}_m.$$

From these relationships we already can derive the required formula $A(\xi) \prec A(\eta)$. Thus, for the mapping $A : \mathbb{R} \to a$ the following condition is fulfilled:

$$\xi < \eta \text{ implies } A(\xi) \prec A(\eta). \tag{6.4}$$

As an immediate consequence of the condition (6.4) we get that the mapping $A : \mathbb{R} \to a$ is injective. In order to prove its bijectivity now it is sufficient to prove its surjectivity. We formulate and prove this fact as a separate theorem. □

THEOREM 6.2. *For any point X on the line $a = OE$ there is a real number ξ such that $X = A(\xi)$.*

PROOF. Let's consider the intervals $I(m,k)$ with the use of which in § 4 we have defined binary-rational approximations of real numbers. Their boundaries are defined by the numbers

$$a_{mk} = \frac{k}{2^m}, \qquad\qquad b_{mk} = \frac{k+1}{2^m}.$$

Let's consider the analogous semi-open intervals on the line a:

$$\Theta(m,k) = [A(a_{mk})A(b_{mk})). \tag{6.5}$$

For each fixed m the intervals with different numbers $k_1 \neq k_2$ do not intersect, while their union covers the whole line a. This fact follows from the Archimedes axiom A18 (compare it with the theorem 4.1 for real numbers). Hence, for each m there exists exactly one interval $\Theta(m,k)$ that contains the point X. Let's denote by k_m the number of this interval:

$$a_m = \frac{k_m}{2^m}, \qquad\qquad b_m = \frac{k_m+1}{2^m}.$$

The intervals (6.5) are so that each $\Theta(m,k)$ is the union of two intervals of the next order:

$$\Theta(m,k) = \Theta(m+1,2k) \cup \Theta(m+1,2k+1).$$

Hence, $k_{m+1} = 2k_m$ or $k_{m+1} = 2k_m + 1$. For the numbers a_m and b_m this fact means $a_m \leqslant a_{m+1} < b_{m+1} \leqslant b_m$. Therefore, Cantor's axiom R17 is applicable to the sequences a_m and b_m. It yields the existence of a real number ξ satisfying the inequalities $a_m \leqslant \xi \leqslant b_m$. Such a number ξ is unique since the existence of another number $\tilde{\xi}$ satisfying the inequalities $a_m \leqslant \tilde{\xi} \leqslant b_m$ would lead to the relationship that contradicts the theorem 3.2:

$$|\xi - \tilde{\xi}| \leqslant b_m - a_m = \frac{1}{2^m}.$$

The existence and uniqueness of a number ξ given by a point X on the line $a = OE$ means that we have constructed a mapping from the line a to the set of real numbers:

$$\xi \colon a \to \mathbb{R}. \tag{6.6}$$

Let's prove that $A(\xi) = X$. For this purpose we sharpen the inequalities $a_m \leqslant \xi \leqslant b_m$. The coincidence $\xi = b_m$ in these inequalities is excluded since due to $\xi \leqslant b_{m+1} \leqslant b_m$ it would mean that the sequence b_m stabilizes, i. e. $b_m = b = \xi$ for $m > m_0$. For the point $X \in a$ this stabilization would yield

$$X \in [A(b - 2^{-m})A(b)) \ \text{ for all } \ m > m_0, \tag{6.7}$$

which is impossible since the intersection of all semi-open intervals $[A(b - 2^{-m})A(b))$ in (6.7) is empty. Thus, for the number ξ and the binary-rational numbers a_m and b_m the inequalities $a_m \leqslant \xi < b_m$ are fulfilled. Hence, $\xi \in I(m, k_m)$ and the numbers a_m and b_m coincides exactly with the binary-rational approximations of the number ξ, which are used in constructing the points $A(\xi)$. Now from $X \in [A(a_m)A(b_m)]$ for all $m \in \mathbb{N}$ we derive the required relationship $X = A(\xi)$. $\square$

EXERCISE 6.1. *Show that the intersection of all semi-open intervals* $[A(b - 2^{-m})A(b))$ *from* (6.7) *is an empty set. For this purpose consider the segments* $[A(b - 2^{-m})A(b)]$ *and, relying upon*

Cantor's axiom, prove that the intersection of these segments consists of the single point $A(b)$.

In proving the theorem 6.2, we not only have finished the proof of bijectivity of the mapping $A\colon \mathbb{R} \to a$ in the theorem 6.1, but have constructed its inverse mapping (6.6). In constructing both mappings A and ξ we used substantially the point O and the vector $\overrightarrow{OE}$ on the line a.

DEFINITION 6.1. If a point O and some vector $\mathbf{e} = \overrightarrow{OE}$ on a straight line a are given, the define a *Cartesian coordinate system* on this line. The point O is called the *origin*, while the vector $\mathbf{e}$ is called the *basis vector*.

DEFINITION 6.2. Let $(O, \mathbf{e})$ be a Cartesian coordinate system on some line a. Then the number $\xi = \xi(X)$ is called the *coordinate* of a point $X \in a$, while the vector $\overrightarrow{OX}$ connecting the origin with the point X is called the *radius-vector* of this point.

If a point $X \in a$ is given, its coordinate is fixed uniquely and, conversely, if the coordinate of a point X is given, the point $X \in a$ is fixed uniquely. This fact follows from the bijectivity of the mappings A and $\xi = A^{-1}$.

Using the mappings A and ξ, we can identify real numbers with the points of some straight line. This is a visual image for understanding the set $\mathbb{R}$. For this reason the set of real numbers $\mathbb{R}$ itself is sometimes called the *real axis*.

§ 7. Measuring straight line segments.

Assume that a straight line segment $[OE]$ is given. Then we introduce the Cartesian coordinate system with the origin O and with the basis vector $\mathbf{e} = \overrightarrow{OE}$ on the line $a = OE$. We use the line as a ruler and the segment $[OE]$ as a gauge unit on this ruler. Let $[PQ]$ is some arbitrary segment. On the ray $[OE\rangle$ we choose a point M such that $[OM] \cong [PQ]$. Then the point is M associated with the number $\xi(M)$ being its coordinate. From

$O \prec M$ due to the relationship (6.4) we find that $\xi = \xi(M)$ is a positive number. The positive number ξ is called the *length* of the segment $[PQ]$ measured relative to the gauge unit $[OE]$ on the line a. This fact is written as follows:

$$[PQ] \cong \xi \cdot [OE] \quad \text{or} \quad |PQ| = \xi \cdot |OE|.$$

If the reference segment $[OE]$ is fixed, the number ξ can be taken for the length of the segment $[PQ]$. In this case we write

$$|PQ| = \xi.$$

If the segments $[AB]$ and $[CD]$ are congruent, then the points M and $\tilde{M}$ on the ray $[OE\rangle$ of the reference line a determined by the segments $[AB]$ and $[CD]$ do coincide. For their lengths this yields $|AB| = |CD|$. Conversely, the equality of lengths $|AB| = |CD|$ imply $M = \tilde{M}$, $[OM] \cong [AB]$, and $[OM] \cong [CD]$, which yields $[AB] \cong [CD]$. Let's formulate this result as a theorem.

THEOREM 7.1. *Two segments $[AB]$ and $[CD]$ are congruent if and only if their lengths measured with respect to the same reference segment $[OE]$ are equal.*

THEOREM 7.2. *The relationship $[AB] < [CD]$ for two segments $[AB]$ and $[CD]$ is equivalent to the inequality $|AB| < |CD|$ for their lengths measured with respect to some fixed reference segment $[OE]$.*

PROOF. The procedure of measuring associates the segments $[AB]$ and $[CD]$ with two points M and $\tilde{M}$ on the ray $[OB\rangle$ such that $[OM] \cong [AB]$ and $[O\tilde{M}] \cong [CD]$. The relationship $[AB] < [CD]$ is equivalent to $[OM] < [O\tilde{M}]$ and to

$$O \prec M \prec \tilde{M}. \tag{7.1}$$

Hence, due to the property (6.4) of the mapping A we get $\xi(M) < \xi(\tilde{M})$ which means $|AB| < |CD|$.

Conversely, from $|AB| < |CD|$ we derive $\xi(M) < \xi(\tilde{M})$, which leads to the relationship (7.1). From (7.1) and from $[OM] \cong [AB]$ with $[O\tilde{M}] \cong [CD]$ we get $[AB] < [CD]$. $\square$

THEOREM 7.3. *Assume that a point B lies in the interior of a segment $[AC]$. Then for the lengths of the segments $[AB]$, $[BC]$, and $[AC]$ measured with respect to some fixed reference segment $[OE]$ we have the equality $|AC| = |AB| + |BC|$.*

PROOF. Let's consider the procedure of measuring the lengths of the segments $[AB]$ and $[AC]$ with respect to some reference segment $[OE]$. On the ray $[OE\rangle$ we mark the points M and $\tilde{M}$ such that $[OM] \cong [AB]$ and $[O\tilde{M}] \cong [AC]$. Then the point M lies in the interior of the segment $[O\tilde{M}]$, which follows from $[AB] < [AC]$. Moreover, $[M\tilde{M}] \cong [BC]$, which is due to the axiom A15.

Let $\xi = \xi(M)$ and $\tilde{\xi} = \xi(\tilde{M})$. Then $\xi < \tilde{\xi}$ and for the lengths of the segments $[AB]$ and $[AC]$ we have

$$|AB| = \xi, \qquad\qquad |AC| = \tilde{\xi}. \tag{7.2}$$

The numbers ξ and $\tilde{\xi}$ are associated with the sequences a_m, b_m, $\tilde{a}_m$,and $\tilde{b}_m$ of their binary-rational approximations. Applying the theorem 4.2, we get $b_m < \tilde{a}_m$ for $m > m_0$. Hence, we write

$$a_m \leqslant \xi < b_m < a_m \leqslant \tilde{\xi} < b_m.$$

Now let's take into account that $A(\xi) = M$ and $A(\tilde{\xi}) = \tilde{M}$, then let's apply the property (6.4) of the mapping A. This yields

$$A(a_m) \preccurlyeq M \prec A(b_m) \prec A(\tilde{a}_m) \preccurlyeq \tilde{M} \prec A(\tilde{b}_m).$$

In other words, the segment $[M\tilde{M}]$ comprises the segment $[A(b_m)A(\tilde{a}_m)]$ and is enclosed into the segment $[A(a_m)A(\tilde{b}_m)]$. Let's apply the theorem 7.2 in this situation. It yields

$$|A(b_m)A(\tilde{a}_m)| < |M\tilde{M}| < |A(a_m)A(\tilde{b}_m)|. \tag{7.3}$$

Binary-rational numbers a_m, b_m, $\tilde{a}_m$, and $\tilde{b}_m$ are determined by two integer numbers k_m and $\tilde{k}_m$ (see formulas (4.1)). From

$$b_m = \frac{k_m + 1}{2^m}, \qquad\qquad \tilde{a}_m = \frac{\tilde{k}_m}{2^m}$$

we conclude that the segment $[A(b_m)A(\tilde{a}_m)]$ is composed of congruent segments each of which can be obtained by mens of the m-fold bisection of the segment $[OE]$. The number of such segments is equal to $\tilde{k}_m - k_m - 1$. Therefore, we have

$$|A(b_m)A(\tilde{a}_m)| = \frac{\tilde{k}_m - k_m - 1}{2^m} = \tilde{a}_m - b_m.$$

Similarly for the segment $[A(a_m)A(\tilde{b}_m)]$ we get

$$|A(a_m)A(\tilde{b}_m)| = \frac{\tilde{k}_m - k_m + 1}{2^m} = \tilde{b}_m - a_m.$$

Now from (7.3) we obtain the following estimates for the length of the segment $[M\tilde{M}]$:

$$\tilde{a}_m - b_m < |\tilde{M}M| < \tilde{b}_m - a_m.$$

Let's remember that $[M\tilde{M}] \cong [BC]$ and use the inequalities $a_m \leqslant \xi < b_m$ and $\tilde{a}_m \leqslant \tilde{\xi} < \tilde{b}_m$. This yields

$$\begin{aligned} &\tilde{\xi} - \xi - |BC| < \tilde{b}_m - a_m - (\tilde{a}_m - b_m) = 2^{m+1}, \\ &\tilde{\xi} - \xi - |BC| > \tilde{a}_m - b_m - (\tilde{b}_m - a_m) = -2^{-m+1}. \end{aligned} \tag{7.4}$$

If we take into account (7.2), the inequalities (7.4) can be written as the following estimates:

$$-2^{-m+1} < |AC| - |AB| - |BC| < 2^{-m+1}.$$

Since $m > m_0$ is an arbitrary integer greater than m_0, these estimates yield $|AC| = |AB| + |BC|$. The theorem is proved. $\square$

Definition 7.1. The function associating each segment $[PQ]$ with some positive number $\xi([PQ])$ is called a *length function*, if the following conditions are fulfilled:

(1) $\xi([OE]) = 1$ for some reference segment $[OE]$;
(2) $[AB] \cong [CD]$ implies $\xi([AB]) = \xi([CD])$;
(3) if a point B lies in the interior of the segment $[AC]$, then $\xi([AC]) = \xi([AB]) + \xi([BC])$.

Theorem 7.4. *A length function satisfying the above conditions* (1)–(3) *in the definition* 7.1 *does exist. It is unique if some reference segment* $[OE]$ *is fixed.*

Proof. Indeed, one of the length functions, satisfying the conditions (1)–(3) from the definition 7.1 was constructed above. Let's denote it ξ. Let η be some other such function satisfying the same conditions (1)-(3). Let's show that these functions do coincide: $\xi([PQ]) = \eta([PQ])$. The property (2) means that it is sufficient to consider the segments of the form $[OM]$, where M is some point on the ray $[OE\rangle$ comprising the segment $[OE]$.

If $M = E$, then the equality $\xi([OE]) = \eta([OE]) = 1$ follows from the condition (1). Assume that the segment $[AB]$ is obtained by m-fold bisection of the segment $[OE]$ and by subsequent composing k replicas of the resulting segment. Then

$$[AB] \cong \frac{k}{2^m} \cdot [OE].$$

Using the condition (3), now one can derive the relationships

$$\xi([AB]) = \frac{k}{2^m}, \qquad\qquad \eta([AB]) = \frac{k}{2^m}.$$

Thus, we have proved $\xi([AB]) = \eta([AB])$ for a segment $[AB]$ being binary-rational multiple of the reference segment $[OE]$.

The condition (3) in the definition 7.1 has another important consequence. It leads to $\eta([AB]) < \eta([AC])$ for any two segments

$[AB]$ and $[AC]$ such that $[AB] < [AC]$. This property of the function η is derived from the formula

$$\eta([AC]) = \eta([AB]) + \eta([BC])$$

since $\eta([BC])$ in this formula is positive. Let's use this property in order to prove $\xi([OM]) = \eta([OM])$ for a point M on the ray $[OE\rangle$. According to the theorem 6.2, we have $M = A(\alpha)$, where α is some positive real number. Let's denote by a_m and b_m the binary-rational approximations of this number. Then $a_m \leqslant \alpha < b_m$ and we have

$$A(a_m) \preccurlyeq M \prec A(b_m).$$

From the inclusions $[OA(a_m)] \subset [OM] \subset [OA(b_m)]$ we derive

$$\eta([OA(a_m)]) \leqslant \eta([OM]) \leqslant \eta([OA(b_m)]).$$

But $\eta([OA(a_m)]) = a_m$ and $\eta([OA(b_m)]) = b_m$ since we have already proved that η and ξ do coincide for segments being binary-rational multiples of the segment $[OE]$. Hence, we get

$$\alpha - \frac{1}{2^m} < a_m \leqslant \eta([OM]) \leqslant b_m \leqslant \alpha + \frac{1}{2^m}.$$

These inequalities yield $\eta([OM]) = \alpha = \xi([OM])$ since $m \in \mathbb{N}$ is an arbitrary positive integer. The theorem is proved. $\square$

Theorem 7.5. *If the length of a segment $[AB]$ measured with respect to a reference segment $[OE]$ is equal to ξ and if the length of the the reference segment $[OE]$ measured with respect to another reference segment $[\tilde{O}\tilde{E}]$ is equal to η, then the length of the segment $[AB]$ measured with respect to the second reference segment $[\tilde{O}\tilde{E}]$ is equal to the product $\tilde{\xi} = \xi \cdot \eta$.*

Exercise 7.1. *Derive the theorem 7.5 from the theorem 7.4.*

§ 8. Similarity mappings for straight lines. Multiplication of vectors by a number.

Assume that on a straight line a a Cartesian coordinate system with the origin O and with the basis vector $\mathbf{e} = \overrightarrow{OE}$ is given. Let's consider another straight line b on which another Cartesian coordinate system with the origin Q and with the basis vector $\mathbf{h} = \overrightarrow{QH}$ is given. We define a mapping $f : a \to b$ as follows. For a point $X \in a$ we take its coordinate $\xi = \xi(X)$ and then we associate to X the point $Y \in b$ with exactly the same coordinate ξ . In the other words, f is the composition of two mappings

$$\xi\colon a \to \mathbb{R}, \qquad\qquad A\colon \mathbb{R} \to b.$$

Such a mapping $f = A \circ \xi$ is called a *similarity mapping.* The ratio of the lengths of reference vectors

$$k = |QH|/|OE|$$

is called the *similarity factor* for this mapping. Any similarity mapping $f\colon a \to b$ is bijective. The inverse mapping $f^{-1}\colon b \to a$ is also a similarity mapping, its similarity factor is k^{-1}. The composition of two similarity mappings $f\colon a \to b$ and $g\colon b \to c$ is a similarity mapping $g \circ f\colon a \to c$. Its similarity factor is equal to the product of similarity factors of the mappings f and g.

Lemma 8.1. *Assume that on a line a a Cartesian coordinate system with the origin O and the basis vector $\mathbf{e} = \overrightarrow{OE}$ is given. Then the following propositions are valid:*

(1) *the vector $\overrightarrow{AB}$ is codirected with the vector $\overrightarrow{OE}$ if and only if for the points B and A the difference of their coordinates is positive: $\xi(B) - \xi(A) > 0$;*

(2) *the length of the segment $[AB]$ measured with respect to the reference segment $[OE]$ is given by the formula*

$$|AB| = |\xi(B) - \xi(A)|.$$

Exercise 8.1. *Consider all possible dispositions of the points A and B relative to the point O on the line a and, using the theorem 7.3, prove the lemma 8.1.*

Theorem 8.1. *Assume that a and b are two straight lines with the vectors $\overrightarrow{OE}$ and $\overrightarrow{QH}$ on them. Let's define the positive directions on the lines a and b by means of the vectors $\overrightarrow{OE}$ and $\overrightarrow{QH}$ and then consider the similarity mapping $f\colon a \to b$ defined by them. This mapping*

(1) *preserves the relation of precedence for points, i.e. $X \prec Y$ implies $f(X) \prec f(Y)$;*
(2) *multiplies the lengths of segments by k, where k is the similarity factor: $|f(X)f(Y)| = k \cdot |XY|$.*

Theorem 8.2. *Let $f\colon a \to b$ be a similarity mapping. Then for some arbitrary points A, B, C, and D on the line a and for their images $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, and $D' = f(D)$ on the line b the equality $\overrightarrow{AB} = \overrightarrow{CD}$ implies the equality $\overrightarrow{A'B'} = \overrightarrow{C'D'}$.*

Exercise 8.2. *Using the theorem 7.5 and the lemma 8.1, prove the theorem 8.1 and derive the theorem 8.2 from it.*

A special sort of similarity mappings arise if the lines a and b do coincide: $a = b$. Assume that on a straight line a two vectors $\overrightarrow{OE}$ and $\overrightarrow{OH}$ are given. Then we have two Cartesian coordinate systems with the common origin O on this line. They define a similarity mapping $f\colon a \to a$. Such a mapping is called a *homothety* with the center O. The mapping f takes a point X with the coordinate $\xi(X)$ in the first coordinate system to the point $Y = f(X)$ with the coordinate $\tilde{\xi}(Y) = \xi(X)$ in the second coordinate system. Using the theorem 8.1, it is easy to calculate the coordinate of the point Y in the first coordinate system:

$$\xi(Y) = \begin{cases} (|OH|/|OE|) \cdot \xi(X) & \text{for } \overrightarrow{OH} \upuparrows \overrightarrow{OE}, \\ -(|OH|/|OE|) \cdot \xi(X) & \text{for } \overrightarrow{OH} \uparrow\downarrow \overrightarrow{OE}. \end{cases}$$

For the homothety mapping f we introduce the numeric parameter, which is called the *homothety factor*:

$$k = \begin{cases} \;\;|OH|/|OE| & \text{if} \quad \overrightarrow{OH} \upuparrows \overrightarrow{OE}, \\ -|OH|/|OE| & \text{if} \quad \overrightarrow{OH} \uparrow\downarrow \overrightarrow{OE}. \end{cases}$$

Then the homothety mapping $f\colon a \to a$ can be defined using only one Cartesian coordinate system: a point X with the coordinate ξ is taken to the point $Y = f(X)$ with the coordinate $k \cdot \xi$.

DEFINITION 8.1. Assume that $\overrightarrow{AB}$ is a vector on some straight line a. A vector $\overrightarrow{CD}$ with the length $|CD| = |k| \cdot |AB|$ on the line a is called the *product of the vector $\overrightarrow{AB}$ by a number* $k \neq 0$ if it is codirected to $\overrightarrow{AB}$ for $k > 0$ and is oppositely directed to $\overrightarrow{AB}$ for $k < 0$. We express this fact by writing $\overrightarrow{CD} = k \cdot \overrightarrow{AB}$.

THEOREM 8.3. *A point Y on a straight line a is the image of a point X on the same line under the homothety mapping with the center at the point O and with the homothety factor k if and only if its radius-vector $\overrightarrow{OY}$ is obtained from the radius vector $\overrightarrow{OX}$ through multiplying it by the number k.*

EXERCISE 8.3. *Prove the theorem 8.3 by choosing some Cartesian coordinate system in the line a.*

The definition 8.1 does not fix a definite position of the vector $\overrightarrow{CD} = k \cdot \overrightarrow{AB}$ on a line, the vector $\overrightarrow{CD}$ is determined up to the replacement of it by any other vector equal to it in the sense of the definition 4.1 from Chapter III. If $\overrightarrow{GF} = \overrightarrow{AB}$, then $k \cdot \overrightarrow{GF} = k \cdot \overrightarrow{AB}$. This means that in the definition 8.1 the product $\overrightarrow{CD} = k \cdot \overrightarrow{AB}$ is defined only as a slipping vector $\mathbf{c} = \overrightarrow{CD}$ obtained through multiplying another slipping vector $\mathbf{b} = \overrightarrow{AB}$ by the number k. If $k = 0$ or if $\mathbf{b} = \mathbf{0}$, the product $k \cdot \mathbf{b}$ is assumed to be equal to zero vector:

$$0 \cdot \mathbf{b} = \mathbf{0} \quad \text{for any vector} \quad \mathbf{b},$$

$$k \cdot \mathbf{0} = \mathbf{0} \quad \text{for any number} \quad k \in \mathbb{R}.$$

THEOREM 8.4. *The operations of addition and multiplication by a number for slipping vectors possess the following properties:*

(1) *commutativity of addition:* $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$*;*
(2) *associativity of addition:* $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$*;*
(3) *there is a zero vector* $\mathbf{0}$ *such that* $\mathbf{a} + \mathbf{0} = \mathbf{a}$ *for an arbitrary vector* $\mathbf{a}$*;*
(4) *for any vector* $\mathbf{a}$ *there is an opposite vector* $\mathbf{a}'$ *such that* $\mathbf{a} + \mathbf{a}' = \mathbf{0}$*;*
(5) *distributivity of multiplication by a number with respect to addition of vectors:* $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$*;*
(6) *distributivity of multiplication by a number with respect to addition of numbers:* $(k + q) \cdot \mathbf{a} = k \cdot \mathbf{a} + q \cdot \mathbf{a}$*;*
(7) *associativity of multiplication:* $(k \cdot q) \cdot \mathbf{a} = k \cdot (q \cdot \mathbf{a})$*;*
(8) *the property of the numeric unity:* $1 \cdot \mathbf{a} = \mathbf{a}$*.*

EXERCISE 8.4. *Prove those propositions of the theorem 8.4 which are not yet proved.*

§ 9. Measuring angles.

The numeric measure of angles is constructed approximately in the same way as the length of segments. The only difference is that here the propositions like the Archimedes axiom A18 and Cantor's axiom A19 should be proved.

THEOREM 9.1. *Let* $\{\angle h_n k_n\}_{n \in \mathbb{N}}$ *be s sequence of angles lying on one plane and having a common vertex* O*. Assume that the following relationships are fulfilled:*

$$\angle h_{n+1} k_{n+1} \subset \angle h_n k_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Then there is a ray l *coming out from the point* O *and belonging to the intersection of all these angles.*

PROOF. Let's mark a point A_1 on the ray h_1 and a point B_1 on the ray k_1. Then we connect them with the segment $[A_1 B_1]$.

According to the lemma 6.2 from Chapter II, the rays h_n and k_n intersect the segment $[A_1 B_1]$. We denote the intersection points through A_n and B_n respectively. Such points form a sequence of segments such that

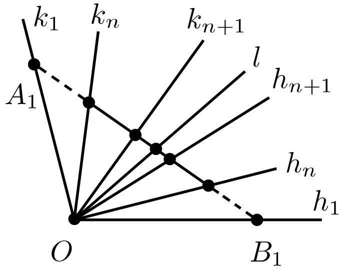


Fig. 9.1

$$[A_{n+1} B_{n+1}] \subset [A_n B_n].$$

In this situation Cantor's axiom A19 is applicable. It yields the existence of a point X belonging to all segments $[A_n B_n]$. We draw the ray $[OX\rangle$ trough this point X and denote it l. According to the lemma 6.2 from Chapter II, this ray lies in the intersection of all angles $\angle h_n k_n$. The theorem is proved. $\square$

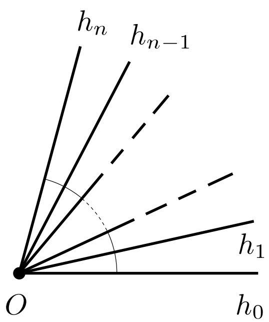


Fig. 9.2

Let's consider an angle $\angle h_0 h_n$ with the vertex at some point O. Assume that inside this angle $\angle h_0 h_n$ the rays $h_1, \ldots, h_{n-1}$ are drawn so that they form the angles

$$\angle h_0 h_1, \ldots, \angle h_{n-1} h_n$$

congruent to each other. In this case we say that the angle $\angle h_0 h_n$ is n times as bigger than the angle $\angle h_0 h_1$:

$$\angle h_0 h_n \cong n \cdot \angle h_0 h_1.$$

Conversely, for the angle $\angle h_0 h_1$ we say that it is obtained from $\angle h_0 h_n$ by dividing into n congruent parts. We write this fact as

$$\angle h_0 h_1 \cong \frac{1}{n} \cdot \angle h_0 h_n.$$

LEMMA 9.1. *Assume that in a triangle ABC the bisector AD is drawn. Then $\angle ABC > \angle ACB$ implies $[CD] > [BD]$.*

PROOF. From the relationship $\angle ABC > \angle ACB$, applying the theorem 2.4, we get $[AC] > [AB]$. On the ray $[AC\rangle$ we mark a point E so that $[AE] \cong [AB]$. Due to $[AC] > [AB]$ the point E is an interior point of the segment $[AC]$. From the congruence $\angle EAD \cong \angle BAD$ and from $[AE] \cong [AB]$ we derive that the triangles EAD and BAD are congruent. Hence, $[DE] \cong [DB]$. The angle $\angle CED$ is adjacent to the angle $\angle AED$ which is congruent to $\angle ABC$. Therefore the angle $\angle CED$ is congruent to the external angle of the triangle ABC at the vertex B. According to the theorem 2.3 the internal angle of this triangle at the vertex C is smaller than its external angle at the vertex B. Hence, we get $\angle CED > \angle ECD$.

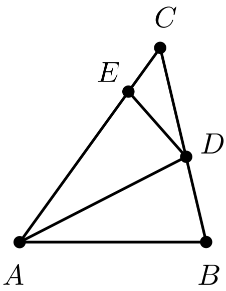


Fig. 9.3

Now, applying the theorem 2.4 to the triangle CED, we get $[CD] > [ED]$ which is equivalent to the relationship $[CD] > [BD]$ due to $[ED] \cong [BD]$. The lemma is proved. $\square$

THEOREM 9.2. *For any two acute angles $\angle hk$ and $\angle lq$ there is a positive integer n such that*

$$\angle hk < \angle rp \cong n \cdot \angle lq,$$

where the angle $\angle rp$ is n times as bigger than the angle $\angle lq$.

PROOF. If the angle $\angle lq$ is bigger th$\angle hk$, then by choosing $n = 1$ we provide the required relationship $\angle hk < n \cdot \angle lq$. If $\angle lq \cong \angle hk$, it is

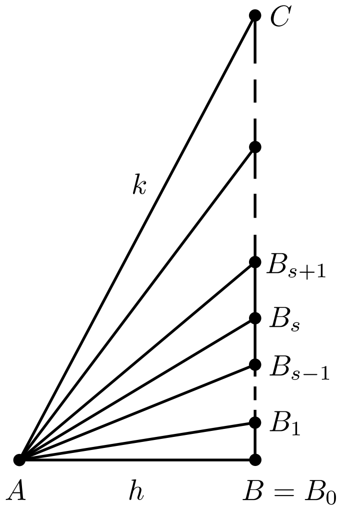


Fig. 9.4

sufficient take $n = 2$. Therefore, we consider the case, where the angle $\angle lq$ is smaller than the angle $\angle hk$. In this case we denote by A the vertex of the angle $\angle hk$, mark some point C on the ray k and drop the perpendicular from the point C onto the line containing the ray h. The foot of this perpendicular lies on the ray h since the angle $\angle hk$ is acute. We denote it through B (see Fig. 9.4 above).

The further proof of the theorem is by contradiction. We denote $h = h_0$ and in the half-plane containing the ray k we draw the series of rays $h_1, h_2, \ldots, h_n$ so that all angles of the form $\angle h_s h_{s+1}$ are congruent to the angle $\angle lq$. If we assume that the relationship $\angle hk < n \cdot \angle lq$ is not fulfilled for all positive integers $n \in \mathbb{N}$, we would be able to draw infinitely many rays $h_1, h_2, \ldots, h_n, \ldots$, all of them lying inside the angle $\angle hk$. Let's denote by $B_1, B_2, \ldots, B_n, \ldots$ the intersection point of these rays and the segment $[BC]$ (such points do exist due to the lemma 6.2 from Chapter II). Let's consider the triangle $AB_{s-1}B_{s+1}$. For $s > 1$ the angle $\angle AB_{s-1}B_{s+1}$ in this triangle is adjacent to the acute angle $\angle AB_{s-1}B$ in the rectangular triangle ABB_{s-1}. Therefore, it is an obtuse angle. For $s = 1$ the angle $\angle AB_{s-1}B_{s+1}$ is a right angle. In each of these two cases we have

$$\angle AB_{s-1}B_{s+1} > \angle AB_{s+1}B_{s-1}.$$

The ray $[AB_s\rangle$ is the bisector of the angle $\angle B_{s-1}AB_{s+1}$. Therefore we can apply the lemma 9.1, which yields

$$[B_{s-1}B_s] < [B_sB_{s+1}].$$

This means that the lengths of the segments $[B_sB_{s+1}]$ form a monotonic increasing sequence of numbers. For the segment $[BB_n]$ this yields the relationship $[BB_n] > n \cdot [BB_1]$. Now the assumption that the relationship $\angle hk < n \cdot \angle lq$ is not fulfilled for all positive integers n leads to $n \cdot [BB_1] < [BC]$ for all $n \in \mathbb{N}$. But this contradicts the Archimedes axiom A18. The contradiction obtained completes the proof of the theorem. $\square$

The construction of a gauge for measuring angles does not differ from that for segments on a straight line. The natural restriction that all angles are enclosed into a straight angle, defines the natural choice of the reference angle. In order to have an acute reference angle we take some straight angle, bisect it twice, and then assign the value of $\pi/4$ radians to the resulting angle. Here $\pi = 3.14\ldots$ is the well-known irrational number arising as the area of a unit circle.

THEOREM 9.3. *Each angle $\angle hk$ is associated with some real number $\xi(\angle hk) = \widehat{hk}$ from the interval $0 < \xi \leqslant \pi$ so that the following conditions are fulfilled:*

(1) *a straight angle is associated with the number π;*
(2) *if $\angle hk \cong \angle lq$, then $\xi(\angle hk) = \xi(\angle lq)$;*
(3) *if a ray l lies inside an angle $\angle hk$ and divides it into two angles, then $\xi(\angle hk) = \xi(\angle hl) + \xi(\angle lk)$.*

EXERCISE 9.1. *Using the analogy to segments, completes the details required for proving the theorem 9.3 and prove it.*

CHAPTER VI

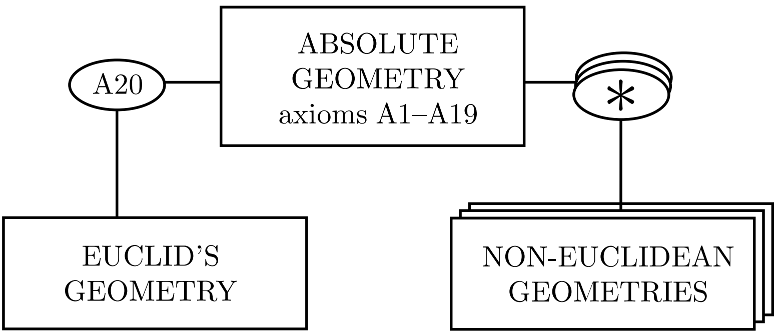
AXIOM OF PARALLELS.

§ 1. The axiom of parallels and the classical Euclidean geometry.

The fifth group of Euclid's axioms consists of one axiom A20. It is formulated as follows.

Axiom A20. *For any point O not lying on a straight line a there is exactly one straight line passing through the point O and being parallel to the line a.*

The axiom A20 played an important role in the history of science. Multiple attempts to prove it by deriving from the other axioms lasted more than 2000 years. However, they did not succeed. To the contrary, giving up the idea to prove it, people had discovered new non-Euclidean geometries. For the firs time this was done by Lobachevsky, Bolyai, and Gauss.



The axioms A1–A19 and their consequences considered in Chapters I–V constitute the so called «absolute geometry». They are valid in classical Euclidean geometry and they remain valid in its non-Euclidean variations, where the axiom A20 is replaced by some propositions not equivalent to it. To the contrary, in Chapter VI we consider those results which are specific to Euclidean geometry only. Non-Euclidean geometries are beyond the scope of this book.

THEOREM 1.1. *Assume that a and b are two straight lines lying on one plane and intersecting with a third straight line c at the points A and B. The lines a and b are parallel if and only if the inner crosswise lying angles at the points A and B are congruent.*

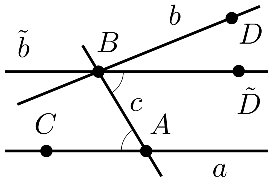


Fig. 1.1

PROOF. The direct proposition of the theorem is already proved (see theorem 8.1 in Chapter III). Let's prove the converse proposition. Assume that $a \parallel b$. We mark some point $D \neq B$ on the line b. The line c divides the plane of the lines a and b into two half-planes. On that half-plane which contains the point D we draw the ray $[B\tilde{D}\rangle$ so that $\angle AB\tilde{D} \cong \angle BAC$. Applying the theorem 8.1 from Chapter III to the line $\tilde{b} = B\tilde{D}$, we get $\tilde{b} \parallel a$. If the line $\tilde{b}$ would be different from b, we would have two lines passing through the point B and being parallel to the line a, which contradicts the axiom A20. Hence, $\tilde{b} = b$ and $\angle ABD \cong \angle BAC$. The theorem 1.1 is proved. $\square$

THEOREM 1.2. *Let $a \neq b$ be two parallel lines lying on a plane α. If a line $c \neq b$ lying on the plane α intersect the line b at some point B, then it intersects the line a at some other point A.*

PROOF. If we assume that c does not intersect the line a, then, according to the definition 8.1 from Chapter III, these lines are parallel: $c \parallel a$. As a result we get two lines b and c, passing through the point B and parallel to the line a, which contradicts the axiom A20. Hence, the line c intersects the line a at some point $A \neq B$. $\square$

THEOREM 1.3. *Assume that a and b are two parallel lines lying on a p[lane α. Then a perpendicular to the line a drawn on the plane α is a perpendicular to the line b either.*

The theorem 1.3 is a simple consequence of the theorems 1.1 and 1.2. It does not require a separate proof.

THEOREM 1.4. *In Euclidean geometry the relation of parallelism of straight lines is reflexive, symmetric, and transitive, because of which it is an equivalence relation.*

PROOF. Reflexivity and symmetry of parallelism of straight lines follows immediately from its definition (see definition 8.1 in Chapter III). Let's prove its transitivity. Assume that $a \parallel b$ and $b \parallel c$. If $a = b$, $b = c$, or $c = a$, the relationship $a \parallel c$ is obviously fulfilled. Therefore, we consider the case, where the lines a, b, and c are mutually distinct. From $a \parallel b$ we conclude that there is a plane containing the lines a and b. We denote this plane by α. Similarly, we denote by β the plane containing the lines b and c.

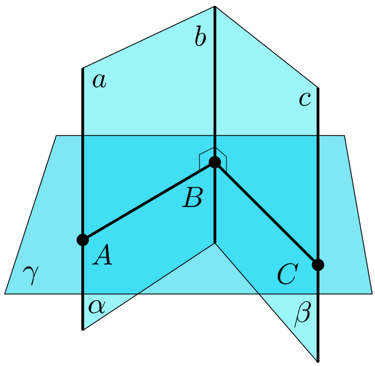


Fig. 1.2

Let's choose a point B on the line b and draw the plane γ perpendicular to the line b through this point. At the intersection of the planes α and γ we get the perpendicular to the line b. According to the theorem 1.2, this perpendicular crosses the line a at some point A. According to the theorem 1.3 the line AB is perpendicular to the line a. By construction the line b is a perpendicular to the plane γ, while the plane α contains this line b. Hence, $\alpha \perp \gamma$ (see definition 3.1 in Chapter IV). Due to the theorem 3.1 from Chapter IV the plane contains the perpendicular to the plane γ passing through the point A. This perpendicular coincides with the line a due to the theorem 6.3 from Chapter III since $a \perp AB$.

Thus, $a \perp \gamma$. A similar situation arises on the plane β. The line BC obtained as the intersection of the planes γ and β appears to be perpendicular to the lines b and c. Then from $b \perp \gamma$ we derive $\beta \perp \gamma$ and $c \perp \gamma$. But any two perpendiculars to the same plane are parallel (see theorem 3.3 in Chapter IV). Therefore from $a \perp \gamma$ and $c \perp \gamma$ we derive the required relationship $a \parallel c$.

Note that there is a special case, where the planes α and β do coincide. In this case the line AB coincides with the line BC. However, the above considerations remain valid for this case too. The theorem is proved. $\square$

§ 2. Parallelism of a straight line and a plane.

Definition 2.1. A straight line a is said to be *parallel* to a plane α if it lies on the plane α or if it does not intersect this plane. The parallelism of a and α is written as $a \parallel \alpha$.

Theorem 2.1. *A straight line a is parallel to a plane α if and only if it is parallel to some straight line b lying on the plane α.*

Proof. The case where the line a lies on the plane α is trivial. In tis case we can choose $b = a$, upon which both propositions of the theorem (direct and inverse ones) appear to be obviously valid.

Let's consider the case where the line a does not lies on the plane α. Assume that $a \parallel \alpha$. Let's choose some arbitrary point A on the plane α. Then let's draw a plane β through the line a and the point A. At the intersection of the planes β and α we get a straight line b. The line b does not intersect the line a since it lies on the plane α that has no common points with the line a.

On the other hand, the line b lies on the same plane β as the line a. Hence, we have $b \parallel a$.

Conversely, assume that the line a is parallel to some line b lying on the plane α. Let's draw a plane β through these two parallel lines. Then $\alpha \cap \beta = b$. If the line a would intersect the plane α, the intersection point would lie on the line b. But the lines a and b have no common points since they are parallel and do not coincide. Hence, $a \parallel \alpha$. $\square$

Note that the definition 2.1 and the theorem 2.1 can be formulated in absolute geometry either. The proof of the theorem 2.1 does not use the axiom A20. However, all other theorems below in this section are valid only in Euclidean geometry.

THEOREM 2.2. *For two straight lines a and b and for a plane α the conditions $a \parallel b$ and $b \parallel \alpha$ imply $a \parallel \alpha$.*

EXERCISE 2.1. *Derive the theorem 2.2 as a consequence of the theorems 1.4 and 2.1.*

THEOREM 2.3. *Let a be some line parallel to a plane α and let O be some point on this plane. If a line b passes through the point O and if $b \parallel a$, then the line b lies on the plane α.*

PROOF. Let's begin with the case where the line a lies on the plane α. If $O \in a$, then the lines a and b have the common point O. In this case $b \parallel a$ implies $b = a$ (see definition 8.1 in Chapter III). Hence, $b \subset \alpha$.

If $a \subset \alpha$, but $O \notin a$, the lines a and b do not coincide. Two parallel lines a and b, according to the definition 8.1 in Chapter III, lie on one plane. Let's denote this plane by β. The planes α and β both contain the line a and the point O not lying on this line. Therefore they coincide: $\beta = \alpha$. Hence, $b \subset \alpha$.

And finally, let's consider the case where the line a does not lie on the plane α. Since $a \parallel \alpha$, it does not intersect the plane α. The line b has the common point O with the plane α. Hence, $a \neq b$. Since $b \parallel a$, there is a plane β passing through the lines a and b (see definition 8.1 in Chapter III). In this case the plane

β is different from the plane α. These two planes α and β have the common point O. Therefore, they intersect along some line $\tilde{b}$ that contains the point O and has no common points with the line a. Hence, $\tilde{b} \parallel a$ since non-intersecting straight lines a and $\tilde{b}$ lie on one plane β. For the last step in our proof we apply the axiom A20 which says that there is a unique line b passing through the point O and being parallel to a. Hence, $b = \tilde{b}$ and $b \subset \alpha$ since $\tilde{b} \subset \alpha$ by construction. $\square$

Theorem 2.4. *Assume that a and b are arbitrary two non-parallel straight lines. Then there is a unique plane β passing through the line b and being parallel to the line a.*

Proof. For the beginning we prove the existence of a required plane. From $a \nparallel b$ we derive $a \neq b$. On the line b we choose a point B not lying on the line a. Then we draw the line $\tilde{a}$ parallel to a through this point B. The lines $\tilde{a}$ and b have one common point B, but they do not coincide (since $\tilde{a} = b$ would mean $a \parallel b$). There is a plane containing both of two such lines. Let's denote this plane by β. Since $\tilde{a} \parallel a$ and $\tilde{a} \subset \beta$, applying the theorem 2.1, we derive that $a \parallel \beta$.

Now let's prove the uniqueness of the plane β constructed just above. Assume that $\tilde{\beta}$ is some other plane containing the line b and being parallel to a. Then $B \in \tilde{\beta}$. The line $\tilde{a}$ by construction is parallel to the line a and it passes through the point B. Due to the theorem 2.3 we have $\tilde{a} \subset \tilde{\beta}$. Hence, the plane $\tilde{\beta}$ contains two intersecting at the point B but not coinciding straight lines $\tilde{a}$ and b. So does the plane β. Hence, we have $\tilde{\beta} = \beta$. The theorem is proved. $\square$

Theorem 2.5. *Let a be some straight line. If two distinct planes α and β are parallel to the line a and if they intersect along a line b, then $b \parallel a$.*

Proof. If b is the line at the intersection of the planes α and β. As stated in the theorem, there are two distinct planes α and β passing through the line b and being parallel to the line a. If the line would be not parallel to a, then, according to the theorem 2.4, there would be only one such plane. These considerations show that $b \parallel a$. $\square$

§ 3. Parallelism of two planes.

DEFINITION 3.1. Two planes α and β are called *parallel*, if they coincide $\alpha = \beta$ or if they have no common points.

For denoting the binary relation of parallelism of planes we use the same sign as for the parallelism of straight lines. We write $\alpha \parallel \beta$.

THEOREM 3.1. *Assume that a plane α is parallel to a plane β. If a plane γ intersects both planes α and β along two lines a and b respectively, then $a \parallel b$.*

PROOF. If $\alpha = \beta$, the parallelism of the lines $a \parallel b$ follows from their coincidence $a = b$.

Now assume that $\alpha \neq \beta$. Then from $\alpha \parallel \beta$ we get that the planes α and β do not intersect. Hence, the lines $a = \alpha \cap \gamma$ and $b = \beta \cap \gamma$ also do not intersect. They lie on one plane γ, therefore they are parallel. The theorem is proved. $\square$

THEOREM 3.2. *Let $\alpha \neq \beta$ be two planes intersecting a line c at the points A and B. Then $c \perp \alpha$ and $c \perp \beta$ imply $\alpha \parallel \beta$.*

PROOF. If $A = B$ we would have two planes $\alpha \neq \beta$ passing through the point $A \in c$ and being perpendicular to the line c. However, this is impossible due to the theorem 1.2 from Chapter IV. Hence, $A \neq B$.

If we assume that the planes α and β intersect and if we denote by C some point from their intersection $\alpha \cap \beta$, then would have two perpendiculars $[CB]$ and $[CA]$ dropped from the point $C \notin c$ onto the line c. However, this contradicts the theorem 6.5 from Chapter III. Hence, the planes α and β are parallel. The theorem is proved. $\square$

The definition of parallelism for two planes can be formulated in absolute geometry either. The theorems 3.1 and 3.2 do not use the axiom A20, they are valid in absolute geometry. But the other theorems below in this section can be proved only in Euclidean geometry.

THEOREM 3.3. *If two intersecting straight lines a and b on a plane α are parallel to intersecting lines $\tilde{a}$ and $\tilde{b}$ on another plane $\tilde{\alpha}$, then the planes α and $\tilde{\alpha}$ are parallel.*

PROOF. If $\alpha = \tilde{\alpha}$, the parallelism of α and $\tilde{\alpha}$ follows from their coincidence (see definition 3.1). Therefore, it is sufficient to consider the case where the planes α and $\tilde{\alpha}$ do not coincide.

Since the line a lies on the plane α, we have $a \parallel \alpha$ (see definition 2.1). The parallelism $a \parallel \tilde{\alpha}$ follows from the parallelism of the lines a and $\tilde{a}$ and from the fact that the line $\tilde{a}$ lies on the plane $\tilde{\alpha}$ (see theorem 2.1). Hence, α and $\tilde{\alpha}$ are two non-coinciding planes parallel to the line a. If we assume that they intersect along some line c, then from the theorem 2.5 we derive $a \parallel c$. In a similar way we derive $b \parallel c$, hence, applying the theorem 1.4, we get $a \parallel b$. However, two intersecting, but not coinciding lines cannot be parallel (see definition 8.1 in Chapter III). The contradiction obtained shows that the planes α and $\tilde{\alpha}$ do not coincide. Hence, they are parallel. The theorem is proved. □

THEOREM 3.4. *For any point O not lying on a plane α there is exactly one plane passing through this point and being parallel to the plane α.*

PROOF. Let's choose some point B on the plane α and draw two different lines a and$b$ through this point on the plane α. Then through the point O we draw two lines $\tilde{a}$ and $\tilde{b}$ parallel to a and b respectively. The existence and uniqueness of such lines $\tilde{a}$ and $\tilde{b}$ follow from the axiom A20. We know, that there is a plane β passing through the pair of non-coinciding lines $\tilde{a}$ and $\tilde{b}$ crossing at the point O. This plane β is parallel to the plane α due to the theorem 3.3.

Let's prove the uniqueness of the plane β constructed just above. Let's consider some plane $\tilde{\beta}$ passing through the point O and being parallel to the plane α. Such a plane has no common points with α. Therefore, for two lines a and b lying on the plane α we have $a \parallel \tilde{\beta}$ and $b \parallel \tilde{\beta}$. Therefore, we can apply the theorem 2.3 to the plane $\tilde{\beta}$, to the point O and to the lines a and $\tilde{a}$. According to this theorem, $\tilde{a} \parallel a$ implies $\tilde{a} \subset \tilde{\beta}$. Similarly, $\tilde{b} \parallel b$ implies $\tilde{b} \subset \tilde{\beta}$. Hence, the plane $\tilde{\beta}$ passes through the lines $\tilde{a}$ and $\tilde{b}$ which define the plane β. This yields $\tilde{\beta} = \beta$. Thus, the theorem is proved. $\square$

THEOREM 3.5. *Let $\alpha \neq \beta$ be two parallel planes. If a plane $\gamma \neq \beta$ intersects the plane β along a line b, then it it intersects the plane α along some line a.*

PROOF. If we assume that γ does not intersect α, then γ is parallel to α according to the definition 3.1. Assume that B is some point on the line b produced as the intersection of the planes β and γ. Then we have two planes β and γ passing through the point B and being parallel to the plane α, which contradicts the theorem 3.4. The contradiction obtained shows that the plane γ intersects the plane α along some line a. $\square$

THEOREM 3.6. *Let $\alpha \neq \beta$ be two parallel planes. If the line c intersects the plane β, but does not lies on it, then the line c intersects the plane α too.*

PROOF. Let's denote by B the intersection point for the plane β and the line c. Then we choose some point $C \neq B$ on the plane β and draw a plane γ through the line c and the point C. At the intersection of the planes β and γ we get the line $b = BC$. According to the theorem 3.5, the plane γ intersects the plane α along some line a. From $\alpha \parallel \beta$ by applying the theorem 3.1 we get $a \parallel b$. The lines a, b, and c lie on one plane γ, therefore we can apply the theorem 1.2 to them. According to this theorem, the line c intersecting the line b at the point B intersects the line a at some point A. But $a \subset \alpha$, hence, the point A is the intersection point for the line c and the plane α. $\square$

THEOREM 3.7. *Assume that $a \neq b$ are two parallel straight lines. If a plane γ intersects the line b, but does not contain this line, then it intersects the line a too.*

EXERCISE 3.1. *Derive the theorem 3.7 as a direct consequence of the theorem 2.2.*

THEOREM 3.8. *Let $\alpha \neq \beta$ be two parallel planes. Then any perpendicular to the plane α is a perpendicular to the plane β.*

PROOF. Assume that the line c is a perpendicular to the plane α at a point $A \in \alpha$. According to the theorem 3.7, the line c crosses the plane β at some point B.

Let's prove that $c \perp \beta$. For this purpose we consider some line b lying on the plane β and passing through the point B. There is a plane γ passing through the lines b and c. It intersects the plane β along the line b. Let's denote by a the line produced at the intersection of the planes γ and α. The line a passes through the point A. From the theorem 3.1 we derive $a \parallel b$. The lines a, b, and c lie on one plane γ, therefore, we can apply the theorem 1.3 to them. Due to this theorem $a \perp c$ implies $b \perp c$. Thus, the line c appears to be perpendicular to an arbitrary line b lying on the plane β and passing through the point B. This fact yields the required result $c \perp \beta$. $\square$

THEOREM 3.9. *Let $\alpha \neq \beta$ be two parallel planes. If a plane γ is perpendicular to the plane α, then it is perpendicular to the plane β either.*

EXERCISE 3.2. *Derive the theorem 3.9 as a consequence of the theorems 3.5 and 3.8.*

§ 4. The sum of angles of a triangle.

THEOREM 4.1. *The sum of angles in an arbitrary triangle is equal to a straight angle.*

Proof. Let's choose some arbitrary triangle ABC. Let's draw a line parallel to the side AC through the vertex B in this triangle. On this line we mark two point D and E on different sides of the point B. The points A and C and the triangle ABC in whole lie on one side with respect to the lune DE since the segment $[AC]$ lies on the line parallel to the line DE and, therefore, cannot intersect this line. Let's apply the theorem 1.1 to the line AB intersecting two parallel lines AC and DE. It yields the congruence of the angles $\angle DBA \cong \angle CAB$. Similarly, applying the theorem 1.1 to the line CB, we get $\angle EBC \cong \angle BCA$. The angles $\angle DBA$, $\angle ABC$, and $\angle EBC$ have the common vertex B and compose the straight angle $\angle DBE$. Now, taking into account the above congruence relationships for angles, we can write

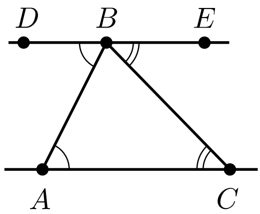


Fig. 4.1

$$\angle CAB + \angle ABC + \angle BCA \cong \angle DBE.$$

This is the very relationship which means that the sum of internal angles of the triangle ABC coincide with a straight angle. $\square$

§ 5. Midsegment of a triangle.

Let ABC be a triangle. Let's mark the centers of the sides $[AB]$ and $[BC]$ in this triangle. Let's denote them by M and N respectively. The segment $[MN]$ is called the *midsegment* of the triangle ABC.

Theorem 5.1. *The midsegment $[MN]$ connecting the centers of the sides $[AB]$ and $[BC]$ in a triangle ABC is parallel to the side $[AC]$ of this triangle and $[AC] \cong 2 \cdot [MN]$.*

Proof. Let M be the center of the side $[AB]$ in a triangle ABC. We draw the line parallel to the side $[AC]$ through this point M. Applying Pasch's axiom A12, it is easy to show that such line crosses the side $[BC]$ at some interior point N. Then we draw the line parallel to the side $[BC]$ through the point M. At the intersection of this line with the side $[AC]$ we get a point K lying in the interior of the segment $[AC]$. Let's connect the points N and K with the segment $[NK]$. On the lines MN and MK we mark two points D and E for the sake of convenience.

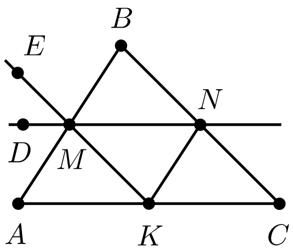


Fig. 5.1

The angles $\angle MBN$ and $\angle BME$ are inner crosswise lying angles at the intersections of the line AB with two parallel lines BC and MK. The angles $\angle BME$ and $\angle AMK$ are vertical angles. Hence, applying the theorem 1.1, we get $\angle MBN \cong \angle AMK$. Now let's consider the angles $\angle MAK$ and $\angle DMA$. They are inner crosswise lying angles at the intersections of the line AB with two parallel lines AC and MN. If we take into account that $\angle DMA$ and $\angle BMN$ are vertical angles, we get $\angle MAK \cong \angle BMN$. From the following three relationships

$$[AM] \cong [MB], \quad \angle AMK \cong \angle MBN, \quad \angle MAK \cong \angle BMN$$

we derive the congruence of the triangles AMK and MBN (see theorem 5.2 in Chapter III). Due to this congruence we get

$$[AK] \cong [MN], \qquad [MK] \cong [BN]. \tag{5.1}$$

Let's complement the congruences (5.1) with one more relationship $\angle NMK \cong \angle MNB$. This relationship is derived if we consider the inner crosswise lying angles at the intersections of the line MN with two parallel lines BC and MK. Now, applying the theorem 5.1 from Chapter III, we find that the triangles MBN and NKM are congruent.

At the intersections of the line NK with two parallel lines BC and MK we get the inner crosswise lying angles $\angle MKN$ and

$\angle KNC$. Similarly, at the intersections of the line NK wih two parallel lines AC and MN we get the inner crosswise lying angles $\angle CKN$ and $\angle KNM$. Now, from the relationships

$$[NK] \cong [KN], \quad \angle MKN \cong \angle KNC, \quad \angle CKN \cong \angle KNM$$

we derive the congruence of the triangles NKM and KNC. Thus, we see that the four triangles AMK, MBN, NKM, and KNC on Fig. 5.1 are congruent to each other. Hence, we have

$$[BN] \cong [NC], \qquad [AK] \cong [KC] \cong [MN].$$

The first of these relationships means that the segment $[MN]$ lying on the line parallel to the line AC is the midsegment of the initial triangle ABC. The second relationship is equivalent to $[AC] \cong 2 \cdot [MN]$. The theorem is proved. $\square$

In a triangle ABC there are three midsegments $[MN]$, $[NK]$, and $[KM]$ parallel to the sides $[AC]$, $[BA]$, and $[CB]$ of this triangle respectively. They divide the triangle ABC into four triangles BMN, NKC, MAK, and KNM, whose sides are twice as smaller than the corresponding sides of the triangle ABC. The angles of these triangles are congruent to the corresponding angles of the triangle ABC.

§ 6. Midsegment of a trapezium.

Assume that on two parallel lines $a \neq b$ two segments $[AB]$ and $[CD]$ are marked. Let's connect the points A, B, C, and D with four segments $[DA]$, $[DB]$, $[CA]$, and $[CB]$. The points A and B lies on one side of the line CD since the segment does not intersect the line CD (see § 5 in Chapter II). The rays $[DA\rangle$ and $[DB\rangle$ cannot coincide since in this case the segment $[AB]$ would lie on the line DA intersecting the line CD. Hence, we conclude: one of the two angles $\angle CDA$ and $\angle CDB$ lies inside the other. For the sake of certainty assume that $\angle CDB < \angle CDA$ as it is shown on Fig. 6.1. In this case we can apply the lemma 6.2 from Chapter II. From this lemma we derive that the ray $[DB\rangle$ intersects the segment $[AC]$ at some its interior point O.

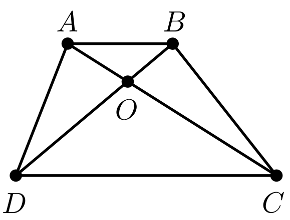


Fig. 6.1

The point O lies on the ray $[DB\rangle$, therefore, the following two mutual dispositions of the points D, B, and O are possible:

$$(D \blacktriangleright O \blacktriangleleft B),$$
$$(D \blacktriangleright B \blacktriangleleft O). \tag{6.1}$$

The coincidence $O = B$ is impossible since in this case the segment $[AB]$ would lie on the line AC, which is not parallel to the line CD. If we assume that thew point B lies between the points D and O, then the line AB intersects the side $[DO]$ in the triangle DOC, but it does not intersect the side $[OC]$. Applying Pasch's axiom A12 in this situation, we find that the line AB intersects the cide $[CD]$ in the triangle DOC, which contradicts the parallelism of the lines AB and CD. This contradiction means that only the first disposition of points D, B, and O in (6.1) can be actually implemented, i. e. O is an interior point for both segments $[AC]$ and $[BD]$.

Now it is easy to show that the segments $[DA]$ and $[CB]$ do not intersect. For this purpose it is sufficient to consider the triangle DAO and the line CB. The line CB does not intersect the sides $[DO]$ and $[OA]$ in the triangle DAO. Hence, according to Pasch's axiom A12 it cannot intersect the third side $[DA]$ of this triangle. These considerations prove the following theorem.

Theorem 6.1. *For any two segments $[AB]$ and $[CD]$ lying on two parallel lines $a \neq b$ exactly one of two segments $[DA]$ or $[DB]$ intersects exactly one of two segments $[CA]$ or $[CB]$ at some interior point O for both intersecting segments.*

According to the theorem 6.1, the segments $[AB]$ and $[CD]$, lying on two parallel lines can be complemented up to a closed polygonal line with no self-intersections. For the situation shown on Fig. 6.2 this line is $ABCD$. It bounds a part of the plane consisting of two triangles ABC and ADC, which intersect along their common side $[AC]$. In general case any closed polygonal line without self-intersections bounds some set of points on a plane which can be represented as the union of finite number of triangles none two of which have common interior points. This fact is known as Jordan's theorem. Its proof can be found in the book [7].

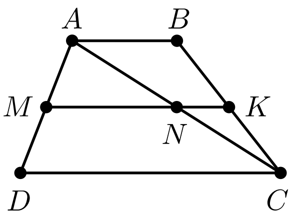


Fig. 6.2

A set of points on a plane bounded by a closed polygonal line is called a *polygon*. The segments of such a line are called *sides* of a polygon. By the number of sides polygons are divided into triangles, quadrangles, pentagons, hexagons etc.

DEFINITION 6.1. A quadrangle two sides of which lie on two parallel lines is called a *trapezium*. Parallel sides of a trapezium are called *bases*, other two sides are called *lateral sides*.

DEFINITION 6.2. The segment connecting centers of lateral sides of a trapezium is called the *midsegment* of this trapezium.

Let $ABCD$ be a trapezium. The segments $[AC]$ and $[BD]$ are called *diagonals* of this trapezium. According to the theorem 6.1 they intersect at some O lying in the interior of each of these two segments (see Fig. 6.1).

THEOREM 6.2. *The midsegment $[MK]$ of a trapezium $ABCD$ is parallel to its bases $[AB]$ and $[CD]$ and such that the relationship $2 \cdot [MK] \cong [AB] + [CD]$ is fulfilled.*

PROOF. Let's consider the pair of triangles ABC and CDA, which compose the trapezium $ABCD$ (see Fig. 6.2). Let's draw their midsegments $[NK]$ and $[NM]$. Then we apply the theorem 5.1 to them. This yields the relationships

$$NK \parallel AB, \qquad\qquad NM \parallel CD.$$

Now from $AB \parallel CD$ we derive that both lines NM and NK passing through the point N are parallel to the line CD. Due to the axiom A20 these two lines should coincide. This means that the points M, N, and K lie on one straight line MK parallel to the bases of our trapezium.

Note that the segment $[MK]$ is composed of two segments $[MN]$ and $[NK]$. For these segments from the theorem 5.1 we derive $2 \cdot [MN] \cong [CD]$ and $2 \cdot [NK] \cong [AB]$. This yields the required relationship $2 \cdot [MK] \cong [AB] + [CD]$ for the segment $[MK]$. The theorem is proved. □

§ 7. Parallelogram.

Definition 7.1. A trapezium lateral sides of which are parallel is called a *parallelogram*.

Theorem 7.1. *A trapezium is a parallelogram if and only if its bases are congruent.*

Proof. Let's consider a trapezium $ABCD$. Its diagonals $[AC]$ and $[BD]$ intersect at some point O lying in the interior of both of these two segments (see Fig. 6.1). From the parallelism of bases $AB \parallel CD$ in our trapezium we derive the congruence of the following inner crosswise lying angles:

$$\angle BAC \cong \angle ACD, \qquad\qquad \angle ABD \cong \angle BDC.$$

If we complement these relationships with the congruence of the bases $[AB] \cong [CD]$, then due to the theorem 5.2 from Chapter III we find that the triangles AOB and COD are congruent. Hence, we have the following relationships:

$$[AO] \cong [CO], \qquad\qquad [BO] \cong [DO]. \tag{7.1}$$

The angles $\angle AOD$ and $\angle BOC$ are congruent since they are vertical angles. Applying the theorem 5.1 from Chapter III and taking into account the relationships (7.1), we find that the triangles AOD and COB are congruent. Hence, for the inner crosswise lying angles $\angle ADB$ and $\angle DBC$ we get $\angle ADB \cong \angle DBC$, which yields $AD \parallel BC$. Hence, $ABCD$ is a parallelogram.

Conversely, assume that the quadrangle $ABCD$ is a parallelogram. Then from $AB \parallel CD$ and $AD \parallel BC$ we derive

$$\angle BAC \cong \angle ACD, \qquad\qquad \angle BCA \cong \angle CAD.$$

We complement these relationships with the trivial relationship $[AC] \cong [CA]$ and we apply the theorem 5.2 from Chapter III. As a result we derive the congruence of the triangles ABC and CDA. Hence, we have $[AB] \cong [CD]$ and $[AD] \cong [BC]$. The theorem is proved. $\square$

In proving the theorem 7.1 we have proved the following two important additional facts:

(1) opposite sides in any parallelogram are congruent, i. e. if $ABCD$ is a parallelogram, then $[AB] \cong [CD]$ and $[AD] \cong [BC]$;
(2) diagonals of any parallelogram intersect each other and the intersection point divides them into halves, i. e. if $ABCD$ is a parallelogram and if $[AC] \cap [BD] = O$, then $[AO] \cong [OC]$ and $[BO] \cong [OD]$.

Theorem 7.2. *A quadrangle $ABCD$ is a parallelogram if and only if the opposite sides of this quadrangle are congruent, i. e. if $[AB] \cong [CD]$ and if $[AD] \cong [BC]$.*

Theorem 7.3. *A quadrangle $ABCD$ is a parallelogram if and only if its diagonals intersect each other at some interior point O that divides them into halves.*

Exercise 7.1. *Considering various dispositions of the points A and C relative to the line BD, prove the theorems 7.2 and 7.3.*

§ 8. Codirected and equal vectors in the space.

The concept of codirectedness was introduced above in Chapter II. However, it was applicable only to vectors lying on one straight line. The concept of equality was also applicable only to vectors lying on one straight line (see definition 4.2 in Chapter II and definition 4.1 in Chapter III). Here we extend these concepts for the case of vectors not lying on one line.

Definition 8.1. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ not lying on one straight line are called *codirected* if

(1) they lie on parallel straight lines;
(2) the segment $[BD]$ connecting their ending points does not intersect the segment $[AC]$ connecting their initial points.

Definition 8.2. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called *equal* if they are codirected and if the segment $[AB]$ is congruent to the segment $[CD]$.

The definitions 8.1 and 8.2 can be formulated in absolute geometry either. However, only in Euclidean geometry the axiom A20 providing the transitivity of the parallelism relation (see theorem 1.4) makes these definitions reasonable.

It is easy to see that the relation of codirectedness of vectors introduced by the definition 8.1 and by the definition 4.2 in Chapter II is reflexive and symmetric. In order to prove its transitivity one should study several special cases.

Lemma 8.1. *Let $A_0 \prec A_1 \prec A_2 \prec A_3$ be a monotonic sequence of points on a straight line a. If a vector $\overrightarrow{MN}$ lying on another line $b \neq a$ is codirected to the vector $\overrightarrow{A_1A_3}$, then it is codirected to each of the vectors $\overrightarrow{A_0A_3}$ and $\overrightarrow{A_2A_3}$.*

Proof. The codirectedness of the vectors $\overrightarrow{MN}$ and $\overrightarrow{A_1A_3}$ means that the quadrangle MNA_3A_1 is a trapezium. Lateral sides of a trapezium does not intersect each other, therefore, the line NA_3 has no common points with the side $[MA_1]$ in the triangle MA_1A_2. This line has no common points with the side $[A_1A_2]$ in the triangle MA_1A_2 either. Applying Pasch's axiom A12, we conclude that the line NA_3 cannot intersect the side $[MA_2]$ in this triangle. In other words, the segments $[MA_2]$ and $[NA_3]$ do not intersect each other. Hence, the vectors $\overrightarrow{MN}$ and $\overrightarrow{A_2A_3}$ are codirected. The relationship $\overrightarrow{MN} \upuparrows \overrightarrow{A_0A_3}$ is proved in a similar way by considering the triangle MA_0A_1. The lemma is proved. $\square$

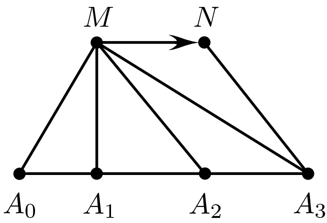


Fig. 8.1

LEMMA 8.2. *Let $A_1 \prec A_2 \prec A_3 \prec A_4$ be a monotonic sequence of points on a line a. If a vector $\overrightarrow{MN}$ lying on another line $b \neq a$ is codirected to the vector $\overrightarrow{A_1A_3}$, then it is codirected to each of the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_4}$.*

EXERCISE 8.1. *Prove the lemma 8.2 using considerations similar to those used in proving the lemma 8.1.*

LEMMA 8.3. *Assume that a vector $\overrightarrow{MN}$ lies on a line b, while the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lie on another line $a \neq b$. Then the relationships $\overrightarrow{MN} \upuparrows \overrightarrow{AB}$ and $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ imply $\overrightarrow{MN} \upuparrows \overrightarrow{CD}$.*

PROOF. Let's enumerate the points A, B, C, D on the line a so that they form a monotonic sequence $A_1 \prec A_2 \prec A_3 \prec A_4$. Let $A = A_i$, $B = A_j$, $C = A_p$, $D = A_q$ so that $i < j$. Then $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ implies $p < q$. Let's denote $r = \min\{i, p\}$ and $s = \max\{j, q\}$. Then we have the following implications:

$$\overrightarrow{MN} \upuparrows \overrightarrow{A_iA_j} \quad \Rightarrow \quad \overrightarrow{MN} \upuparrows \overrightarrow{A_rA_j} \quad \Rightarrow \quad \overrightarrow{MN} \upuparrows \overrightarrow{A_rA_s},$$
$$\overrightarrow{MN} \upuparrows \overrightarrow{A_rA_s} \quad \Rightarrow \quad \overrightarrow{MN} \upuparrows \overrightarrow{A_rA_q} \quad \Rightarrow \quad \overrightarrow{MN} \upuparrows \overrightarrow{A_pA_q}.$$

This sequence of implications is obtained as a result of applying the lemmas 8.1 and 8.2 and taking into account the inequalities

$r \leqslant i < j \leqslant s$ and $r \leqslant p < q \leqslant s$. In the end of this sequence of implications we get the required relationship $\overrightarrow{MN} \upuparrows \overrightarrow{CD}$. Thus, the lemma is proved. $\square$

LEMMA 8.4. *Assume that a vector $\overrightarrow{MN}$ lies on a line b, while $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are two vectors lying on another line $a \neq b$. Then $\overrightarrow{AB} \upuparrows \overrightarrow{MN}$ and $\overrightarrow{MN} \upuparrows \overrightarrow{CD}$ imply $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$.*

PROOF. Let's prove the lemma by contradiction. Assume that the conditions $\overrightarrow{AB} \upuparrows \overrightarrow{MN}$ and $\overrightarrow{MN} \upuparrows \overrightarrow{CD}$ are fulfilled, while the condition $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ is not fulfilled. Then the vector $\overrightarrow{CD}$ is codirected to the vector $\overrightarrow{BA}$, which is an opposite vector for the vector $\overrightarrow{AB}$. From $\overrightarrow{MN} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{CD} \upuparrows \overrightarrow{BA}$, applying the lemma 8.3, we get $\overrightarrow{MN} \upuparrows \overrightarrow{BA}$. But two relationships $\overrightarrow{AB} \upuparrows \overrightarrow{MN}$ and $\overrightarrow{MN} \upuparrows \overrightarrow{BA}$ cannot be fulfilled simultaneously because of the theorem 6.1. The contradiction obtained proves the lemma. $\square$

THEOREM 8.1. *The relation of codirectedness of vectors in the space is transitive, i. e. the relationships $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{CD} \upuparrows \overrightarrow{EF}$ imply the relationship $\overrightarrow{AB} \upuparrows \overrightarrow{EF}$.*

PROOF. The case where all of the vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$ lie on one straight line was considered in § 2 of Chapter II. The case where some two of these three vectors lie on one straight line is described by the lemmas 8.3 and 8.4. The rest is the case of general position where these vectors lie on three distinct straight lines a, b, and c. In this case the relationships $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{CD} \upuparrows \overrightarrow{EF}$ imply the parallelism of the corresponding lines $a \parallel b$ and $b \parallel c$. Hence, due to the theorem 1.4 we have $a \parallel c$.

Let's consider the vector $\overrightarrow{AB}$ lying on the line a. Let's draw the plane α perpendicular to the line a through the point A. According to the theorem 3.7, this plane intersects the lines b and c. We denote the intersection points through C' and E'. respectively. In a similar way, we draw the plane $\beta \perp a$ through the point B. At the intersections of this plane β with the lines b and c we find the point D' and F' respectively. Due to the theorem 3.2 the planes α and β are parallel. Therefore, the segment $[AC']$ does not intersect the segment $[BD']$, the segment $[C'E']$ does not intersect the segment $[D'F']$, and the segment $[AE']$ does not intersect the segment $[BF']$. As a result we get the following three relationships:

$$\overrightarrow{AB} \upuparrows \overrightarrow{C'D'}, \qquad \overrightarrow{C'D'} \upuparrows \overrightarrow{E'F'}, \qquad \overrightarrow{AB} \upuparrows \overrightarrow{E'F'}. \tag{8.1}$$

Let's combine the relationship $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ with the first relationship (8.1) and take into account that the vectors $\overrightarrow{CD}$ and $\overrightarrow{C'D'}$ lie on one straight line b. Applying the lemma 8.4, we get $\overrightarrow{C'D'} \upuparrows \overrightarrow{CD}$. Let's combine this relationship with the second relationship in (8.1) and apply the lemma 8.3. As a result we get $\overrightarrow{CD} \upuparrows \overrightarrow{E'F'}$. Being combined with $\overrightarrow{CD} \upuparrows \overrightarrow{EF}$, upon applying the lemma 8.4, this relationship yields $\overrightarrow{EF} \upuparrows \overrightarrow{E'F'}$. The last step is to combine $\overrightarrow{EF} \upuparrows \overrightarrow{E'F'}$ with the third relationship (8.1) and apply the lemma 8.3. As a result we get the required relationship $\overrightarrow{AB} \upuparrows \overrightarrow{EF}$. The theorem is proved. $\square$

Thus, we have proved that the relation of codirectedness for vectors in the space is reflexive, symmetric, and transitive, i. e. it is an equivalence relation. The relation of congruence for segments possesses the same properties. Now from the definition 8.2 we derive the following theorem.

THEOREM 8.2. *In Euclidean geometry the relation of equality for vectors in the space is reflexive, symmetric, and transitive, because of which it is an equivalence relation.*

The classes of mutually equal vectors in Euclidean geometry are called *free vectors*. Geometric vectors composing a class are called *geometric realizations* of a free vector.

THEOREM 8.3. *For any free vector* $\mathbf{a}$ *and for any point* A *there is a geometric realization* $\overrightarrow{AB}$ *of* $\mathbf{a}$ *with the initial point* A.

The theorem 8.3 approved the above terminology. A free vector is called «free» since it can be realized at any point of the space without any limitations.

Exercise 8.2. *Let* $\overrightarrow{CD}$ *be some geometric realization of a free vector* $\mathbf{a}$. *Using this vector* $\overrightarrow{CD}$, *prove the theorem 8.3 through constructing a required geometric realization* $\overrightarrow{AB}$ *of the vector* a *with the initial point* A

Definition 8.3. Free vectors $\mathbf{a}$ and $\mathbf{b}$ are called *collinear* and are written $\mathbf{a} \parallel \mathbf{b}$ if some of their geometric realizations $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lie on parallel straight lines. If $\overrightarrow{AB} \upuparrows \overrightarrow{AB}$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *codirected*. In the case where the vectors $\mathbf{a}$ and $\mathbf{a}$ are collinear, but not codirected, they are said to be *oppositely directed*.

Exercise 8.3. *Prove the correctness of the definition 8.3 by showing that the properties of collinearity and codirectedness of free vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *do not depend on any particular choice of their geometric realizations* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$.

Theorem 8.4. *For any four points* A, B, C, *and* D *the equality* $\overrightarrow{AB} = \overrightarrow{CD}$ *implies* $\overrightarrow{AC} = \overrightarrow{BD}$ *and, conversely, the equality* $\overrightarrow{AC} = \overrightarrow{BD}$ *implies* $\overrightarrow{AB} = \overrightarrow{CD}$.

Proof. In the case where the points A, B, C, and D lie on one straight line the theorem 8.4 reduces to the theorem 4.1 from Chapter III. Therefore, we consider the case where the points A, B, C, and D do not lie on one straight line. The condition $\overrightarrow{AB} = \overrightarrow{CD}$ yields the relationships

$$AB \parallel CD, \qquad\qquad [AB] \cong [CD]$$

and the condition $[AC] \cap [BD] = \varnothing$. In this case we can apply the theorem 7.1 which means that the quadrangle $ACDB$ is a parallelogram. Hence, with the use of the theorem 7.2 we get $\overrightarrow{AC} = \overrightarrow{BD}$. Conversely, the relationship $\overrightarrow{AB} = \overrightarrow{CD}$ is derived from the relationship $\overrightarrow{AC} = \overrightarrow{BD}$ in a quite similar way. $\square$

§ 9. Vectors and parallel translations.

Translations by some vectors along straight lines were defined in § 14 of Chapter IV, In Euclidean geometry due to the axiom A20 one can specify the properties of these mappings making their description substantially more detailed.

THEOREM 9.1. *If* $\mathbf{c}$ *is a slipping vector on a straight line* a *and if* $\overrightarrow{AB}$ *is some its geometric realization lying on the line* a*, then the relationship* $p_{a\mathbf{c}}(C) = D$ *is fulfilled if and only if the vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *are equal in the sense of the definition 8.2.*

PROOF. If the point C lies on the line a, then the point D also lies on the line a. In this case the theorem 9.1 is reduced to the theorem 4.2 from Chapter III.

Let's consider the case where the point C does not lie on the line a. The points A, B, C, and D lie on one plane (see theorem 14.1 in Chapter IV). Let's denote this plane by α. Taking into account that $B = p_{a\mathbf{c}}(A)$, we mark one more point $F = p_{a\mathbf{c}}(B)$ on the line a. Then we draw the lines AC and BD on the plane α and mark a point E on the line BD as shown on Fog. 9.1. Since $p_{a\mathbf{c}}$ is a mapping of congruent translation, now from the relationships

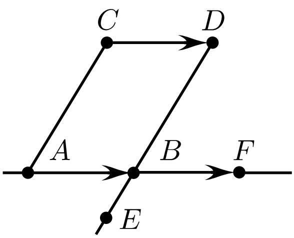


Fig. 9.1

$$\begin{aligned} &p_{a\mathbf{c}}(A) = B,\\ &p_{a\mathbf{c}}(C) = D,\\ &p_{a\mathbf{c}}(B) = F \end{aligned} \tag{9.1}$$

we derive the congruence of angles $\angle CAB \cong \angle DBF$ and the congruence of segments $[AC] \cong [BD]$. Then, using the congruence of the vertical angles $\angle DBF$ and $\angle ABE$, we get $\angle CAB \cong \angle ABE$. But $\angle CAB$ and $\angle DBF$ are the inner crosswise lying angles arising at the intersections of the lines AC and BD with the line a.

Hence, from $\angle CAB \cong \angle ABE$ we derive the parallelism of lines $AC \parallel BD$. By construction of the mapping $p_{a\mathbf{c}}$ the points C and D lie on one side of the line a (see § 14 in Chapter IV). Therefore the segments $[AB]$ and $[CD]$ do not intersect and the quadrangle $ACDB$ is the trapezium with the bases $[AC]$ and $[BD]$. Moreover, let's take into account the relationship $[AC] \cong [BD]$ derived from (9.1) and apply the theorem 7.1. According to this theorem the quadrangle $ACDB$ is a parallelogram. Hence, we immediately get $\overrightarrow{AB} = \overrightarrow{CD}$.

Now, conversely, assume that $\overrightarrow{AB} = \overrightarrow{CD}$. Then the quadrangle $ACDB$ is a parallelogram, because of which $[AC] \cong [BD]$ and $\angle CAB \cong \angle DBF$. Remember that the mapping $p_{a\mathbf{c}}$ is constructed as an extension of the mapping $p_{\mathbf{c}}\colon a \to a$ from the line a to the plane α, then from this plane to the whole space (see § 14 in Chapter IV). It maps the half-planes a_+ to a_+ and a_- to a_-. Then from the relationships

$$
\begin{aligned}
&[AC] \cong [BD], && \angle CAB \cong \angle DBF,\\
&p_{a\mathbf{c}}(A) = B, && p_{a\mathbf{c}}(B) = F
\end{aligned}
$$

and from the fact that the point C and D lies on one side with respect to the line a, we get $p_{a\mathbf{c}}(C) = D$. Both propositions of the theorem are proved. □

Theorem 9.2. *Let $\overrightarrow{AB}$ be a geometric realization of a slipping vector $\mathbf{c}$ on a line a and let $\overrightarrow{CD}$ be a geometric realization of another slipping vector $\mathbf{d}$ on a line b. Then the relationship $p_{a\mathbf{c}} = p_{b\mathbf{d}}$ is fulfilled if and only if $\overrightarrow{AB} = \overrightarrow{CD}$.*

The theorem 9.2 is easily derived from the theorem 9.1 if we take into account the symmetry and transitivity of the relation of equality of vectors. This theorem shows that in Euclidean geometry the a plays an auxiliary role in constructing the mapping $p_{a\mathbf{c}}$. This line cal be replaced by any other line b parallel to a if we replace the slipping vector $\mathbf{c}$ by the free vector $\mathbf{c}$. Therefore, in Euclidean geometry the mapping $p_{a\mathbf{c}}$ is denoted as $p_{\mathbf{c}}$. Here it is called the *parallel translation* by the vector c.

THEOREM 9.3. *For arbitrary two points A and B in the space there is exactly one parallel translation $p_{\mathbf{c}}$ taking the point A to the point B.*

PROOF. In order to prove the existence of the required parallel translation $p_{\mathbf{c}}$ it is sufficient to take the free vector $\mathbf{c}$ whose geometric realization is $\overrightarrow{AB}$. Now, if we assume that $p_{\mathbf{d}}(A) = B$ for some free vector $\mathbf{d}$ with geometric realization $\overrightarrow{CD}$, then from the theorem 9.1 we derive $\overrightarrow{AB} = \overrightarrow{CD}$. Hence, $\mathbf{c} = \mathbf{d}$, which proves the uniqueness of the required parallel translation $p_{\mathbf{c}}$. $\square$

Due to the theorem 9.3 we can use the notation p_{AB} in order to designate the parallel translation $p_{\mathbf{c}}$ taking the point A to the point B. In this case $\overrightarrow{AB}$ appears to be a geometric realization for the free vector $\mathbf{c}$.

§ 10. The group of parallel translations.

THEOREM 10.1. *The mapping $f\colon \mathbb{E} \to \mathbb{E}$ is a parallel translation if and only if for any two points X and Y the equality $\overrightarrow{Xf(X)} = \overrightarrow{Yf(Y)}$ is valid.*

PROOF. If $f = p_{\mathbf{c}}$ is a parallel translation by a vector $\mathbf{c}$ taking a point A to another point B, then the equality $\overrightarrow{Xf(X)} = \overrightarrow{Yf(Y)}$ is derived from the equalities

$$\overrightarrow{Xf(X)} = \overrightarrow{AB}, \qquad \qquad \overrightarrow{Yf(Y)} = \overrightarrow{AB},$$

which follow from the theorem 9.1.

Conversely, assume that f is a mapping such that for any two points X and Y the equality $\overrightarrow{Xf(X)} = \overrightarrow{Yf(Y)}$ is valid. Let's fix some point A and denote $B = f(A)$. Let $\mathbf{c} = \overrightarrow{AB}$. Then for any point $X \in \mathbb{E}$ we have the equalities

$$\overrightarrow{Xf(X)} = \overrightarrow{AB}, \qquad \overrightarrow{Xp_{\mathbf{c}}(X)} = \overrightarrow{AB}. \tag{10.1}$$

The first of the equalities (10.1) follows from $\overrightarrow{Xf(X)} = \overrightarrow{Yf(Y)}$ by substituting $Y = A$, the second one is the consequence of the theorem 9.1. From (10.1) we derive $\overrightarrow{Xf(X)} = \overrightarrow{Xp_{\mathbf{c}}(X)}$, which yields $f(X) = p_{\mathbf{c}}(X)$ for all $X \in \mathbb{E}$. Hence, the mappings f and $p_{\mathbf{c}}$ do coincide. The theorem is proved. □

THEOREM 10.2. *The composition of two parallel translations is a parallel translation.*

PROOF. Let $\mathbf{a}$ and $\mathbf{b}$ be two free vectors defining two parallel translations $p_{\mathbf{a}}$ and $p_{\mathbf{b}}$. Let X and X' be two arbitrary points in the space. We denote

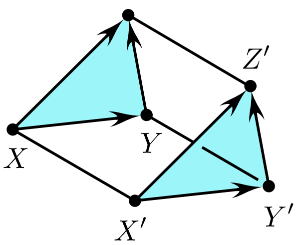


Fig. 10.1

$$Y = p_{\mathbf{a}}(X), \qquad Z = p_{\mathbf{b}}(Y),$$
$$Y' = p_{\mathbf{a}}(X'), \qquad Z' = p_{\mathbf{b}}(Y').$$

If $f = p_{\mathbf{b}} \circ p_{\mathbf{a}}$, then $Z = f(X)$ and $Z' = f(X')$. Let's apply the theorem 10.1 to the mapping $p_{\mathbf{a}}$. This yields the equality $\overrightarrow{XY} = \overrightarrow{X'Y'}$. Similarly, applying the theorem 10.1 to the mapping $p_{\mathbf{b}}$, we get $\overrightarrow{YZ} = \overrightarrow{Y'Z'}$. Now let's use the theorem 8.4. Due to this theorem $\overrightarrow{XY} = \overrightarrow{X'Y'}$ implies $\overrightarrow{XX'} = \overrightarrow{YY'}$ and $\overrightarrow{YZ} = \overrightarrow{Y'Z'}$ implies $\overrightarrow{YY'} = \overrightarrow{ZZ'}$. Using the transitivity of the relation of equality for vectors, we get $\overrightarrow{XX'} = \overrightarrow{ZZ'}$. Applying the theorem 8.4 once more, we derive the relationship $\overrightarrow{XZ} = \overrightarrow{X'Z'}$. Let's write this relationship as

$$\overrightarrow{Xf(X)} = \overrightarrow{X'f(X')}. \tag{10.2}$$

Now, since X and X' are two arbitrary points, applying the theorem!10.1 to the relationship (10.2), we prove that the mapping $f = p_{\mathbf{b}} \circ p_{\mathbf{a}}$ is a parallel translation. $\square$

THEOREM 10.3. *Any two mappings of parallel translation are commutative:* $p_{\mathbf{a}} \circ p_{\mathbf{b}} = p_{\mathbf{b}} \circ p_{\mathbf{a}}$.

PROOF. Let's choose some point A in the space and denote $B = p_{\mathbf{b}}(A)$, $D = p_{\mathbf{a}}(B)$, $C = p_{\mathbf{a}}(A)$. Applying the theorem 10.1 to the mapping $p_{\mathbf{a}}$, we get $\overrightarrow{AC} = \overrightarrow{BD}$. Hence, we can apply the theorem 8.4. From this theorem we derive $\overrightarrow{AB} = \overrightarrow{CD}$. Then $p_{\mathbf{b}}(C) = D$, which follows from the theorem 9.1. As a result for the mappings $p_{\mathbf{a}} \circ p_{\mathbf{b}}$ and $p_{\mathbf{b}} \circ p_{\mathbf{a}}$ we get

$$p_{\mathbf{a}} \circ p_{\mathbf{b}}(A) = D, \qquad\qquad p_{\mathbf{b}} \circ p_{\mathbf{a}}(A) = D. \tag{10.3}$$

According to the theorem 10.3 the composition $p_{\mathbf{a}} \circ p_{\mathbf{b}}$ and the composition $p_{\mathbf{b}} \circ p_{\mathbf{a}}$ both are parallel translations and, as we see in (10.3), they both take the point A to the point D. Due to the theorem 9.3 they should coincide: $p_{\mathbf{a}} \circ p_{\mathbf{b}} = p_{\mathbf{b}} \circ p_{\mathbf{a}}$. $\square$

The theorem 10.2 shows that in Euclidean geometry the set of parallel translations is closed with respect to the composition. It is a group with respect to this operation (see definition 4.2 in Chapter III). The unity of this group is the identical mapping $\mathrm{id} = p_{\mathbf{0}}$ interpreted as the parallel translation by the zero vector. According to the theorem 10.3 the group of parallel translations is an Abelian group.

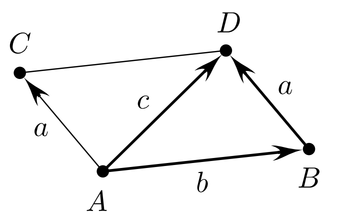


Fig. 10.2

The theorems 9.1, 9.2, and 9.3 establishing one-to-one correspondence between free vectors and parallel translations enable us to define the operation of addition for vectors by means of the formula

$$p_{\mathbf{a}} \circ p_{\mathbf{b}} = p_{\mathbf{a}+\mathbf{b}}.$$

Since $p_{\mathbf{a}}$ commute with $p_{\mathbf{b}}$, the addition of vectors is a commutative operation, i. e. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

Passing from the free vectors $\mathbf{b}$ and $\mathbf{a}$ to their geometric realizations $\overrightarrow{AB}$ and $\overrightarrow{BD}$, we obtain

$$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}, \tag{10.4}$$

where $\overrightarrow{AD}$ is a geometric realization of the vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$. The relationship (10.4) is called the *triangle rule* for adding vectors. The triangle ABC on Fig. 10.2 can be complemented up to a parallelogram, which yields

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}. \tag{10.5}$$

The relationship (10.5) is known as the *parallelogram rule* for adding vectors.

In Chapter IV we have formulated three theorems describing the properties of congruent translations by vectors along straight lines in absolute geometry. These are the theorems 14.3, 14.4, and 14.5. In Euclidean geometry they are formulated as follows.

THEOREM 10.4. *Let $p_{\mathbf{c}}$ and $p_{\mathbf{d}}$ be two parallel translations. If their composition $f = p_{\mathbf{c}} \circ p_{\mathbf{d}}$ has a stable point O, then it is the identical mapping: $f = \mathrm{id}$.*

THEOREM 10.5. *If the composition of three or more parallel translation has a stable point, then it is the identical mapping.*

The rotations mentioned in the theorems 14.3, 14.4, and 14.5 in Euclidean geometry turn to the trivial rotation by the zero angle, which coincide with the identical mapping. The proof of the theorems 10.4 and 10.5 is obvious since the composition of any number of parallel translations is a parallel translation. A parallel translation with a stable point is the translation by the zero vector. It coincides with the identical mapping.

§ 11. Homothety and similarity.

The homothety and similarity mappings on straight lines were introduced in § 8 of Chapter V. The Homothety mapping can be defined in the whole space either. Let's choose some point O, which is called the *center* of homothety, and some real number, which is called the *homothety factor*. The homothety mapping itself $h_{kO}\colon \mathbb{E} \to \mathbb{E}$ is defined as follows:

(1) for the point O we set $h_{kO}(O) = O$;
(2) if $X \neq O$, we draw the line OX, take the vector $\overrightarrow{OX}$ on this line, multiply it by the number k, lay the vector $\overrightarrow{OY} = k \cdot \overrightarrow{OX}$ on the line OX, and then assign $Y = h_{kO}(X)$.

The above manipulations defining a homothety mapping can be performed in Euclidean and in absolute geometries. However, in absolute geometry we could not prove any properties of a homothety mapping that make it worth considering.

Let h_{kO} be the homothety mapping with the center at the point O and with the factor k. For $k = 1$ this mapping coincides with the identical mapping $h_{kO} = \mathrm{id}$, while for $k = -1$ it turns to be the inversion $h_{kO} = i_O$. Moreover, if $k = p \cdot q$, then $h_{kO} = h_{pO} \circ h_{qO}$. In particular, if $k = -q$, then $h_{kO} = h_{qO} \circ i_O$. Therefore, we often can consider homotheties with positive factors $k > 0$ only.

THEOREM 11.1. *Let $f = h_{kO}$ be the homothety with the factor k and with the center at a point O. Then for any two points X and Y the relationship $|f(X)f(Y)| = |k| \cdot |XY|$ is fulfilled and the line connecting the points $f(X)$ and $f(Y)$ is parallel to the line connecting the points X and Y.*

PROOF. In the case where the points X, Y, and O lie on one straight line the proposition of the theorem 11.1 follows from the theorem 8.1 in Chapter V.

Let's consider the case where the points X and Y do not lie on one straight line. Let's draw the lines OX and OY. To each real number k we associate two points $X(k)$ and $Y(k)$. We define these points in the following way:

$$X(k) = h_{kO}(X), \qquad\qquad Y(k) = h_{kO}(Y).$$

Then $X = X(1)$ and $Y = Y(1)$. For the beginning we consider the integer values $k = 1, 2, 4, 8, \ldots$ and the rational values $k = 1/2, 1/4, 1/8, \ldots$ being integer exponentials of the number two. Note that the segment $[X(1/2)Y(1/2)]$ is the midsegment

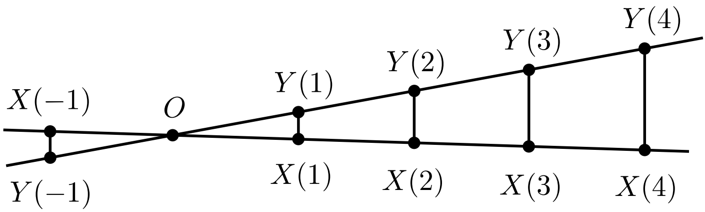


Fig. 11.1

of the triangle XOY, the segment $[XY]$ is the midsegment in the triangle $X(2)OY(2)$, the segment $[X(2)Y(2)]$ is the midsegment in the triangle $X(4)OY(4)$ and so on. Hence, the segments of the form $[X(2^q)Y(2^q)]$ are parallel, their lengths are given by the formula $|X(2^q)Y(2^q)| = 2^q \cdot |XY|$. In the other words, the proposition of the theorem is valid for the homotheties with the factors $k = 2^q$, where $q \in \mathbb{Z}$.

In the second step we prove the proposition of the theorem for all binary-rational values $k > 0$. Each such k is represented as $k = 2^q \cdot n$, where$n$ is some odd positive integer. Let's convert n into the binary format, i. e. into the system with the base 2:

$$n = a_m \cdot 2^m + a_{m-1} \cdot 2^{m-1} + \ldots + a_1 \cdot 2 + a_0.$$

Here $a_0 = 1$, wile the numbers $a_1, \ldots, a_m$ take the values 0 or 1. Let's denote my $\mu(n)$ the number of unities in the binary representation of the number n and perform the induction on this number. If $\mu(n) = 1$, then $n = 1$ since n is odd. In this case $k = 2^q$. For such a case the proposition of the theorem was already proved in the first step.

Assume that the proposition of the theorem is proved for all $k = 2^q \cdot n$, such that $\mu(n) < s$. Let's consider some odd positive integer n such that $\mu(n) = s$. Then we have

$$n = 2^p \cdot \tilde{n} + 1 = \frac{2^{p+1} \cdot \tilde{n} + 2}{2}, \tag{11.1}$$

where $p > 0$, while $\tilde{n}$ is odd and $\mu(\tilde{n}) = s - 1$. Let's denote $k_1 = 2^{q+1} \cdot 1$ and $k_2 = 2^{q+p+1} \cdot \tilde{n}$. For k from (11.1) we derive

$$k = \frac{k_1 + k_2}{2}. \tag{11.2}$$

For the numbers k_1 and k_2 the proposition of the theorem is fulfilled by the inductive hypothesis. Due to (11.2) we conclude that the segment $[X(k)Y(k)]$ is the midsegment of the trapezium $X(k_1)X(k_2)Y(k_2)Y(k_1)$. Hence, the line $X(k)Y(k)$ is parallel to the line XY, while the length of the segment $[X(k)Y(k)]$ is calculated in the following way:

$$\begin{aligned} |X(k)Y(k)| &= \frac{|X(k_1)Y(k_1)| + |X(k_2)Y(k_2)|}{2} = \\ &= \frac{k_1 \cdot |XY| + k_2 \cdot |XY|}{2} = \frac{(k_1 + k_2) \cdot |XY|}{2}. \end{aligned} \tag{11.3}$$

From (11.2) and (11.3) we derive $|X(k)Y(k)| = k \cdot |XY|$. Thus, the inductive step from $\mu(n) < s$ to $\mu(n) = s$ is performed. It means that we have proved the proposition of the theorem for all positive binary-rational values of k.

Now let's consider some positive real value of k. Let a_m and b_m be the binary-rational approximations of the number k:

$$\frac{p_m}{2^m} = a_m \leqslant k < b_m = \frac{p_m + 1}{2^m}. \tag{11.4}$$

Then for $|X(k)Y(k)|$ from the triangle inequality we derive

$$|X(k)Y(k)| < |X(k)X(b_m)| + |X(b_m)Y(b_m)| + |Y(b_m)Y(k)|$$

(see theorem 2.5 in Chapter V). But the lengths of the segments in the right hand side of the above inequality are known:

$$\begin{aligned} &|X(k)X(b_m)| = (b_m - k) \cdot |OX|,\\ &|X(b_m)Y(b_m)| = b_m \cdot |XY|, \\ &|Y(b_m)Y(k)| = (b_m - k) \cdot |OY|. \end{aligned} \tag{11.5}$$

Due to (11.4) we have $b_m - k < 2^{-m}$. Combining this inequality with (11.5), we can transform the above estimate for the length of the segment $[X(k)Y(k)]$ to the following form:

$$|X(k)Y(k)| < k \cdot |XY| + 2^{-m} \cdot (|OX| + |XY| + |YO|).$$

In a similar way, using the triangle inequality, we derive

$$|X(a_m)Y(a_m)| \leqslant |X(a_m)X(k)| + |X(k)Y(k)| + |Y(k)Y(a_m)|,$$

which can be then transformed to

$$|X(k)Y(k)| \geqslant k \cdot |XY| - 2^{-m} \cdot (|OX| + |XY| + |YO|).$$

Now let's take into account that m is an arbitrary positive integer. For $|X(k)Y(k)|$. This yields the required relationship $|X(k)Y(k)| = k \cdot |XY|$.

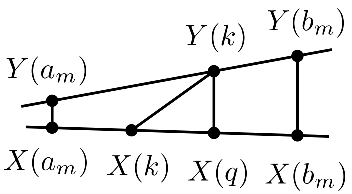


Fig. 11.2

The rest is to prove that the lines $X(k)Y(k)$ and XY are parallel. We do it by contradiction. Assume that the lines $X(k)Y(k)$ and XY are not parallel. Let's draw the line parallel to the line XY through the point $Y(k)$ and denote by $\tilde{X}$ the intersection point of the lines a and OX. The point $\tilde{X}$ is associated with some number q such that $\tilde{X} = X(q)$. From the inequalities (11.4) we find that the points $X(k)$ and $Y(k)$ lie on the lateral sides of the trapezium $X(a_m)X(b_m)Y(b_m)Y(a_m)$. The line a parallel to the bases of this trapezium and crossing the lateral side $[Y(b_m)Y(a_m)]$ should necessarily cross the other lateral side $[X(a_m)X(b_m)]$. This fact follows from the Pasch's axiom A12 applied to the triangles $Y(a_m)X(b_m)Y(b_m)$ and $X(a_m)Y(a_m)X(b_m)$. Hence, the point $X(q)$, like the point $X(k)$, lies within the segment $[X(a_m)X(b_m)]$. This yields the following estimate:

$$|X(k)X(q)| \leqslant |X(a_m)X(b_m)| = 2^{-m} \cdot |OX|.$$

If our assumption is valid, then $X(k) \neq X(q)$ and we come to a contradiction with the theorem 5.1 from Chapter V. This contradiction proves the coincidence $X(q) = X(k)$, which means that the line $X(k)Y(k)$ is parallel to the line XY.

Thus, we have proved the theorem 11.1 for any real number $k > 0$. It is easy to extend it to the case $k < 0$ since for $k = -q$ the homothety h_{kO} with the negative factor $k < 0$ is the composition of the homothety h_{qO} and the inversion i_O. □

THEOREM 11.2. *Let $f = h_{k_O}$ be the homothety with the factor k and with the center at some point O. In this case*

(1) *if some three points X, Y, and Z lie on one straight line, then their images $f(X)$, $f(Y)$, and $f(Z)$ lie on one straight line and $(X \blacktriangleright Y \blacktriangleleft Z)$ implies $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$;*
(2) *if the points X, Y, and Z do not lie on one straight line then the points $f(X)$, $f(Y)$, and $f(Z)$ also do not lie on one straight line.*

PROOF. Assume that the points X, Y, and Z lie on one straight line so that $(X \blacktriangleright Y \blacktriangleleft Z)$. Then we have the equality $|XY| + |YZ| = |XZ|$, which follows from the theorem 7.3 in Chapter V. Applying the theorem 11.1, we derive the equality $|f(X)f(Y)| + |f(Y)f(Z)| = |f(X)f(Z)|$. If we assume that the points $f(X)$, $f(Y)$, and $f(Z)$ do not lie on one straight line, then we get the triangle inequality

$|f(X)f(Y)| + |f(Y)f(Z)| > |f(X)f(Z)|$, which is not compatible with $|f(X)f(Y)| + |f(Y)f(Z)| = |f(X)f(Z)|$.

Let's prove that the relationship $(X \blacktriangleright Y \blacktriangleleft Z)$ implies the analogous relationship $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$. If it is not so, the following two dispositions of points are possible:

$$(f(Y) \blacktriangleright f(X) \blacktriangleleft f(Z)), \qquad (f(X) \blacktriangleright f(Z) \blacktriangleleft f(Y)).$$

In the first case we have $|f(X)f(Z)| < |f(Y)f(Z)|$, while in the second we have $|f(X)f(Z)| < |f(X)f(Y)|$. It is easy to see that none of these inequalities can be fulfilled simultaneously with the equality $|f(X)f(Y)| + |f(Y)f(Z)| = |f(X)f(Z)|$, which follows from $(X \blacktriangleright Y \blacktriangleleft Z)$. Therefore, we actually have the required disposition of points where $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$.

Now, conversely, assume that the points X, Y, and Z do not lie on one straight line and assume that their images $f(X)$, $f(Y)$, and $f(Z)$ lie on one straight line so that $(f(X) \blacktriangleright f(Y) \blacktriangleleft f(Z))$. Then $|f(X)f(Y)| + |f(Y)f(Z)| = |f(X)f(Z)|$. Using the equalities $|f(X)f(Y)| = |k| \cdot |XY|$, $|f(Y)f(Z)| = |k| \cdot |YZ|$, $|f(X)f(Z)| = |k| \cdot |XZ|$, which follow from the theorem 11.1, for the initial points X, Y, and Z we get $|XY| + |YZ| = |XZ|$. However, the points X, Y, and Z do not lie on one straight line and satisfy the triangle inequality $|XY| + |YZ| > |XZ|$. This inequality is not compatible with $|XY| + |YZ| = |XZ|$. The contradiction obtained proves that the points $f(X)$, $f(Y)$, and $f(Z)$ do not lie on one line either. $\square$

THEOREM 11.3. *Let $f = h_{kO}$ be the homothety with the factor k and with the center at a point O. If three points X, Y, and Z do no lie on one straight line, then $\angle XYZ \cong \angle f(X)f(Y)f(Z)$.*

PROOF. Assume that $k > 0$. Let's draw some arbitrary ray coming out from the center of the homothety h_{kO} and a point A on it so that the relationship $[OA] \cong [XY]$ is fulfilled. Then we draw another ray q coming out from the point O so that $\angle hq \cong \angle XYZ$. Let's mark a point B on the ray k so that

$[OB] \cong [YZ]$. The three points O, A, and B form a triangle AOB congruent to the triangle XYZ (see theorem 5.1 in Chapter III). This fact yields the following relationships:

$$|XY| = |OA|, \qquad |YZ| = |OB|, \qquad |XZ| = |AB|. \tag{11.6}$$

Let's apply the homothety mapping $f = h_{kO}$ to the points O, A, B, X, Y, and Z. The point O is a stable point, i. e. $f(O) = O$, while the points A and B are taken to the points $f(A)$ and $f(B)$ lying on the same rays h and k as the initial points A and B. Hence, we have the following relationship:

$$\angle f(A)Of(B) = \angle hk = \angle AOB \cong \angle XYZ. \tag{11.7}$$

The images of the points X, Y, and Z form the triangle $f(X)f(Y)f(Z)$ whose sides are congruent to the sides of the triangle $f(A)Of(B)$. This fact follows from

$$\begin{aligned} &|f(X)f(Y)| = |f(A)O|,\\ &|f(Y)f(Z)| = |f(B)O|,\\ &|f(X)f(Z)| = |f(A)f(B)|, \end{aligned}$$

which, in turn, are derived by applying the theorem 11.1 to the relationships (11.6). Now, applying the theorem 5.5 from Chapter III, we get the congruence of the triangles $f(X)f(Y)f(Z)$ and $f(A)Of(B)$. Then $\angle f(X)f(Y)f(Z) \cong \angle f(A)Of(B)$. Combining this fact with (11.7), we get the required relationship $\angle f(X)f(Y)f(Z) \cong \angle XYZ$.

The case $k < 0$ is reduced to the case $k > 0$ since for $k = -q$ we have the the relationship $h_{kO} = h_{qO} \circ i_O$, where i_O is the inversion mapping. The inversion i_O is a congruent translation, it maps each angle to a congruent angle. $\square$

The theorems 11.1, 11.2, and 11.3 show that under a homothety straight lines are mapped to straight lines (being parallel to the initial ones), segments are mapped to segments, and rays are mapped to rays. The quantitative measures of angles are preserved. Therefore, the homothety preserves the orthogonality of lines. Applying the theorem 1.5 from Chapter IV, we can prove that that under a homothety a plane is mapped to a plane, a half-plane is mapped to a half-plane, and a half-space — to a half-space. These properties of a homothety are identical to those of a congruent translation.

Note that the mapping $f = h_{kO}$ is bijective, the inverse mapping f^{-1} for it is the homothety h_{qO} with the factor $q = 1/k$.

Definition 11.1. A mapping $f\colon \mathbb{E} \to \mathbb{E}$ is called a *similarity mapping* if it admits an expansion $f = h_{kO} \circ \varphi$, where φ is a mapping of congruent translation, while h_{kO} is a homothety with the coefficient $k \neq 0$. The number $|k|$ is called the *similarity factor* of such a mapping.

Exercise 11.1. *Show that the composition of two similarity mappings is a similarity mapping.*

Exercise 11.2. *Show that the mapping inverse to a similarity mapping is a similarity mapping as well.*

Definition 11.2. Two geometric forms Φ_1 and Φ_2 are called *similar* if there is a similarity mapping $f\colon \mathbb{E} \to \mathbb{E}$ establishing a one-to-one correspondence betwen the points of these two forms.

Theorem 11.4. *If for two triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ the conditions $|AB| : |\tilde{A}\tilde{B}| = |AC| : |\tilde{A}\tilde{C}|$ and $\angle BAC \cong \angle\tilde{B}\tilde{A}\tilde{C}$ are fulfilled, then the triangle ABC is similar to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

Theorem 11.5. *If some two angles of a triangle ABC are congruent to the corresponding angles of another triangle $\tilde{A}\tilde{B}\tilde{C}$, then the triangle ABC is congruent to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

Theorem 11.6. *If for triangles ABC and $\tilde{A}\tilde{B}\tilde{C}$ the conditions $|AB| : |\tilde{A}\tilde{B}| = |AC| : |\tilde{A}\tilde{C}| = |BC| : |\tilde{B}\tilde{C}|$ are fulfilled, then the triangle ABC is similar to the triangle $\tilde{A}\tilde{B}\tilde{C}$.*

Exercise 11.3. *The theorems 11.4, 11.5, and 11.6 are known as the similarity criteria for triangles. Prove these theorems relying upon the theorems 5.1, 5.2, and 5.5 from Chapter III.*

§ 12. Multiplication of vectors by a number.

The multiplication of slipping vectors by a number is given by the definition 8.1 in Chapter V. In Euclidean geometry this operation can be extended to the set of free vectors.

Definition 12.1. Assume that a geometric vector $\overrightarrow{AB}$ is given. The vector $\overrightarrow{CD}$ with the length $|CD| = |k| \cdot |AB|$ is called the *product of the vector $\overrightarrow{AB}$ by the number* $k \neq 0$ and written as $\overrightarrow{CD} = k \cdot \overrightarrow{AB}$ if it is codirected to $\overrightarrow{AB}$ for $k > 0$ and if it is oppositely directed to $\overrightarrow{AB}$ for $k < 0$.

The result of multiplying a vector $\overrightarrow{AB}$ by a number k is not unique since the definition 12.1 does not fix the position of the vector $\overrightarrow{CD}$ in the space. However, there is the following theorem.

Theorem 12.1. *If two vectors $\overrightarrow{CD}$ and $\overrightarrow{C'D'}$ are obtained through multiplying some vector $\overrightarrow{AB}$ by a number k, then they are equal in the sense of the definition 8.2.*

Proof. Indeed, the lengths of the segment $[CD]$ and $[C'D']$ are equal to each other since $|CD| = |k| \cdot |AB|$ and $|C'D'| = |k| \cdot |AB|$. Hence, we have the congruence of segments $[CD] \cong [C'D']$.

For $k > 0$ we have $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{AB} \upuparrows \overrightarrow{C'D'}$. These two relationships yield $\overrightarrow{CD} \upuparrows \overrightarrow{C'D'}$. For $k < 0$ from $\overrightarrow{AB} \updownarrows \overrightarrow{CD}$ and $\overrightarrow{AB} \updownarrows \overrightarrow{C'D'}$ we get the relationships $\overrightarrow{BA} \upuparrows \overrightarrow{CD}$ and $\overrightarrow{BA} \upuparrows \overrightarrow{C'D'}$, which also yield $\overrightarrow{CD} \upuparrows \overrightarrow{C'D'}$. The conditions $\overrightarrow{CD} \upuparrows \overrightarrow{C'D'}$ and $[CD] \cong [C'D']$ are the very conditions that mean $\overrightarrow{CD} = \overrightarrow{C'D'}$ in the sense of the definition 8.2. □

Theorem 12.2. *The equality $\overrightarrow{AB} = \overrightarrow{A'B'}$ implies the equality $\overrightarrow{CD} = \overrightarrow{C'D'}$ for vectors $\overrightarrow{CD}$ and $\overrightarrow{C'D'}$ obtained through multiplying $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ by a number k.*

Proof. The equality $\overrightarrow{AB} = \overrightarrow{A'B'}$ means that the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are codirected, while their lengths are equal:

$$\overrightarrow{AB} \upuparrows \overrightarrow{A'B'}, \qquad\qquad |AB| = |A'B'|. \tag{12.1}$$

From the relationships $\overrightarrow{CD} = k \cdot \overrightarrow{AB}$ and $\overrightarrow{C'D'} = k \cdot \overrightarrow{A'B'}$ for $k > 0$ we derive the relationships

$$\begin{aligned} &\overrightarrow{CD} \upuparrows \overrightarrow{AB}, &\qquad &|CD| = |k| \cdot |AB|,\\ &\overrightarrow{C'D'} \upuparrows \overrightarrow{A'B'}, &\qquad &|C'D'| = |k| \cdot |A'B'| \end{aligned} \tag{12.2}$$

for the vectors $\overrightarrow{CD}$ and $\overrightarrow{C'D'}$. Then from (12.1) and (12.2) we derive that the vectors $\overrightarrow{CD}$ and $\overrightarrow{C'D'}$ are codirected, while their lengths are equal. This fact yields $\overrightarrow{CD} = \overrightarrow{C'D'}$. The case $k < 0$ differs from the case $k > 0$ only in that the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ in (12.2) are replaced by the opposite vectors $\overrightarrow{BA}$ and $\overrightarrow{B'A'}$. The theorem is proved. $\square$

The theorems 12.1 and 12.2 show that the multiplication by a number is a correctly defined unambiguous operation in the set of free vectors. To a vector $\mathbf{b}$ and to a number $k \in \mathbb{R}$ it associates some definite vector $\mathbf{c} = k \cdot \mathbf{b}$. For $k = 0$ of for $\mathbf{b} = \mathbf{0}$ the product $k \cdot \mathbf{b}$ is taken to be equal to the zero vector:

$$\begin{aligned} 0 \cdot \mathbf{b} = \mathbf{0} &\quad \text{for any vector} \quad \mathbf{b},\\ k \cdot \mathbf{0} = \mathbf{0} &\quad \text{for any number} \quad k \in \mathbb{R}. \end{aligned}$$

Theorem 12.3. *The operation of addition and the operation of multiplication by a number in the set of free vectors possesses the following properties:*

(1) *commutativity of addition:* $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$*;*
(2) *associativity of addition:* $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$*;*

(3) *there is a zero vector* $\mathbf{0}$ *such that* $\mathbf{a} + \mathbf{0} = \mathbf{a}$ *for an arbitrary vector* $\mathbf{a}$*;*
(4) *for any vector* $\mathbf{a}$ *there is an opposite vector* $\mathbf{a}'$ *such that* $\mathbf{a} + \mathbf{a}' = \mathbf{0}$*;*
(5) *distributivity of multiplication by a number with respect to the addition of vectors:* $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$*;*
(6) *distributivity of multiplication by a number with respect to the addition of numbers:* $(k + q) \cdot \mathbf{a} = k \cdot \mathbf{a} + q \cdot \mathbf{a}$*;*
(7) *associativity of multiplication:* $(k \cdot q) \cdot \mathbf{a} = k \cdot (q \cdot \mathbf{a})$*;*
(8) *the property of the numeric unity:* $1 \cdot \mathbf{a} = \mathbf{a}$.

Exercise 12.1. *Prove those propositions of the theorem 12.3 which are not already proved.*

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CONTACTS

Address:

Ruslan A. Sharipov,
Math. Department,
Bashkir State University,
32 Frunze street,
Ufa 450074, Russia

Home address:

Ruslan A. Sharipov,
5 Rabochaya street,
Ufa 450003, Russia

Phone:

+7-(347)-273-67-18 (Office)
+7-(917)-476-93-48 (Cell)

E-mails:

r-sharipov@mail.ru
R_Sharipov@ic.bashedu.ru

URL's:

http://www.geocities.com/r-sharipov
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http://sovlit2.narod.ru

APPENDIX

List of publications by the author for the period 1986–2006.

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