Bison-Bear-Wolves Predator-Predator-Prey Model

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Abstract

This paper presents a predator-prey differential model based on the Lotka-Volterra equations that examine the interaction between two predators and a single prey. The model considers population size and interaction between the species, showing that both the predator and prey are affected. The model is a useful tool for understanding the dynamics of predator-prey populations and can be used to inform conservation and management efforts.

1 Introduction

The Lotka-Volterra model of predator-prey interactions is a classic example of a mathematical model used to describe the interactions of species in a given environment. In the case of Yellowstone National Park, this model is particularly relevant as it describes the interactions of bisons, grey wolves, and grizzly bears. Here, the bison are considered the prey, while the grey wolves and grizzly bears are the predators. This model provides insight into the populations of each species, and how they interact with each other, in order to better understand the dynamics of the Yellowstone ecosystem.

We used current or estimated data from Yellowstone National Park to see if our altered model could accurately reflect the year-to-year population changes between these three animals. We had to account for the population growth of each species and their natural death rates which can drastically vary depending on their environment. This approach can be useful to ensure no single predator is over hunting a prey that could affect another predators population by restricting their food supply.

2 Predator-Predator-Prey Model

2.1 Model Proof

Word Formulation:

rate of change of prey = rate of prey birth - rate of prey death - rate of prey killed
rate of change of predator = rate of predator birth - rate of predator natural death
rate of change of predator = rate of predator birth - rate of predator natural death

Using Lotka-Volterra:

$$\begin{cases} \dot{x} = \alpha x - c_1 xy - c_1 xz \\ \dot{y} = -\beta y + c_2 xy \\ \dot{z} = -\gamma z + c_3 xz \end{cases}$$

Using the values:

$$\begin{cases} \dot{x} = 0.4x - 0.01xy - 0.01xz \\ \dot{y} = -0.47y + 0.005xy \\ \dot{z} = -0.12z + 0.001xz \end{cases}$$

Solving for equilibrium points we get three points: (0,0,0), (0.12, 0.4, 0), (0.47, 0, 0.4)

Differential equations in a matrix: $\begin{bmatrix} 0.4x - 0.01xy - 0.01xz \\ -0.47y + 0.005xy \\ -0.12z + 0.001xz \end{bmatrix}$

Jacobian Matrix: $\begin{bmatrix} -0.01y - 0.01z + 0.4 & -0.01x & -0.01x \\ 0.005y & 0.005x + 0.47 & 0 \\ 0.001z & 0 & 0.001x - 0.12 \end{bmatrix}$

Derivatives at equilibrium points:

$$Df((0,0,0) = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & -0.47 & 0 \\ 0 & 0 & -0.12 \end{bmatrix}, Df(0.12, 0.4, 0) = \begin{bmatrix} 0 & -0.47 & -0.47 \\ 0.4 & 0 & 0 \\ 0 & 0 & -0.12 + 0.47 \end{bmatrix},$$
$$Df(0.47, 0, 0.4) = \begin{bmatrix} 0 & -0.12 & -0.12 \\ 0 & -0.47 + 0.12 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}$$

2.2 Variables/Factors

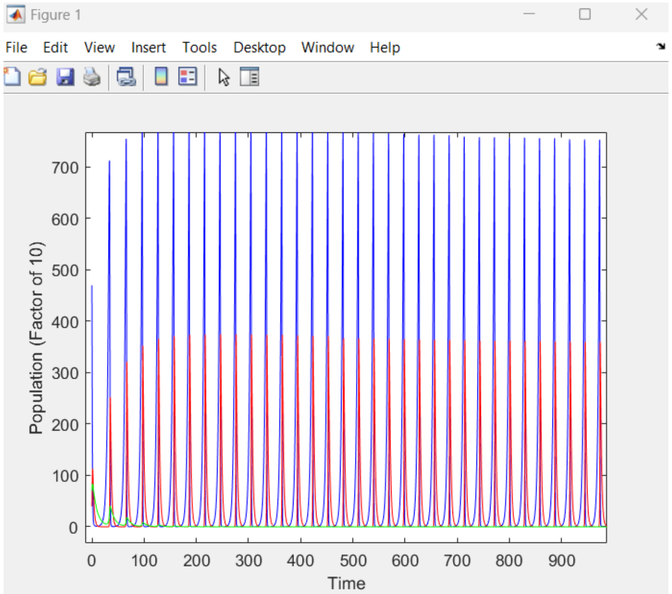
$\dot{x}$ is the change in the population of bison while $\dot{y}$ denotes the change in the population of grey wolves predator and $\dot{z}$ is the change in the grizzly bear population. x is the current estimate of the bison population. At Yellowstone national park there are currently around 4700 bison [2]. y is the current estimate of the grey wolf population which is currently around 400 [2]. z is the current estimate of the grizzly bear population which is around 700 in Yellowstone [7]. α is the rate at which the prey population increases given no predators are present. Given an estimate of bison birth rates at Yellowstone national park, we set α equal to 40% [1]. While β is the natural death rate of grey wolves. Wolves have drastic differences in morality rate depending on if they are monitored or not but for this example, we'll presume they are unmonitored and have a death rate of 47%.[5] γ is the natural death rate of grizzly bears which we found to be around 12% [8]. c_1, c_2, and c_3 are all conversion rates determining the chance the predator kills its prey. This value changes per predator. These variables are small numbers because of how rarely grey wolves and grizzly bear hunt bison.

2.3 Matlab

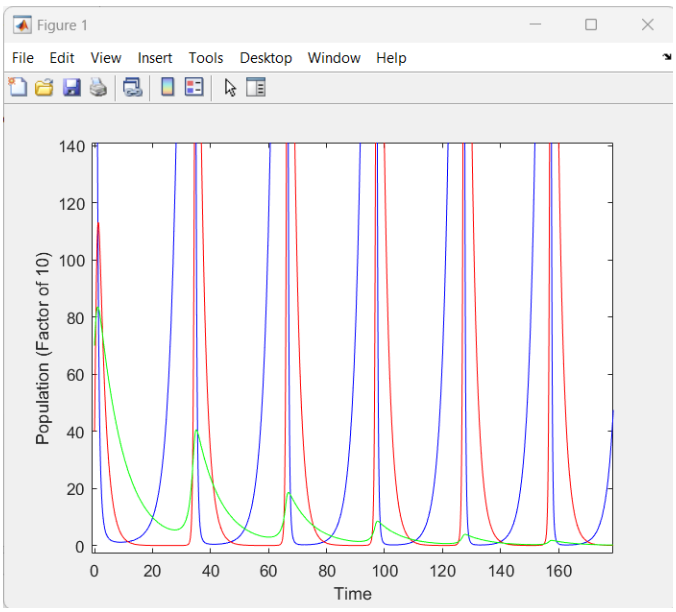
```
function c_cp_predprey
global beta1 alpha2 gamma3 c1 c2 c3;

beta1 = 0.4; alpha2 = 0.47; gamma3 = 0.12; c1 = 0.01; c2 = 0.005; c3= 0.001;
tend = 1000; %set the end time to run the simulation
u0 = [470; 40; 70]; %set initual conditions as column vector
[tsol, usol] = ode45 (@rhs, [0, tend], u0);
Xsol = usol (:, 1); Ysol = usol (:, 2); Zsol = usol (:, 3);
plot (tsol, Xsol, 'b'); hold on; plot (tsol,Ysol, 'r'); plot (tsol, Zsol, 'g');
ylabel('Population (Factor of 10)')
xlabel('Time')

function udot = rhs(t,u)
global beta1 alpha2 gamma3 c1 c2 c3;
X = u(1); Y = u(2); Z = u(3);
Xdot = beta1*X - c1*X*Y - c1*X*Z;
Ydot = -alpha2*Y + c2*X*Y;
Zdot = -gamma3*Z + c3*X*Z;
udot = [Xdot; Ydot; Zdot];
```

This code is for the two predators and one prey model.

2.4 Figures



This model represents the two predator and prey populations.



This shows that one of the predator populations quickly approaches 0.

2.5 Analysis

While it is easy to say that this model may not be the most accurate, there are some glimmers of hope. Our model shows that one predator eventually dies out, which does make some sense. By assuming an equal population for both predators as well as a common food source, there must be some facts that illustrate who kills more bison, and in turn, has their population grow. By including an estimated kill conversion rate, we aimed to account for that possible discrepancy, but the problem lies with the fact that there are still many other variables that we do not take into account. For example, bison do not actually have an unlimited food supply as well as various migration patterns. This would cause problems for our predators, but most likely they do have other sources of food. If we do go with our model, however, we can convert this problem into a single prey-predator model between wolves and bison and focus on their ever-changing population.

3 Predator-Prey Model

3.1 Variables/Factors

$$\dot{x} = \alpha x - c_1 xy$$

$$\dot{y} = -\beta y + c_2 xy$$

Using the same variables as before we get:

$$\dot{x} = 0.4x - 0.01xy$$

$$\dot{y} = -0.47y + 0.005xy$$

3.2 Matlab

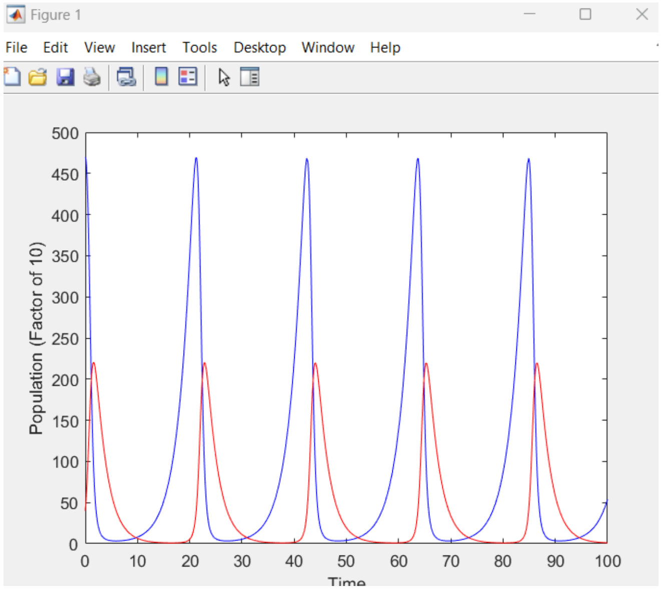
```
function PPModel
global beta1 alpha2 c1 c2;

beta1 = 0.4; alpha2 = 0.47;  c1 = 0.01; c2 = 0.005;
tend = 100; %set the end time to run the simulation
u0 = [470; 40]; %set initual conditions as column vector
[tsol, usol] = ode45 (@rhs, [0, tend], u0);
Xsol = usol (:, 1); Ysol = usol (:, 2);
plot (tsol, Xsol, 'b'); hold on; plot (tsol,Ysol, 'r');
ylabel('Population (Factor of 10)')
xlabel('Time')

function udot = rhs(t,u)
global beta1 alpha2 c1 c2;
X = u(1); Y = u(2);
Xdot = beta1*X - c1*X*Y
Ydot = -alpha2*Y + c2*X*Y;
udot = [Xdot; Ydot];
```

This code is the model after one of the predators becomes extinct.

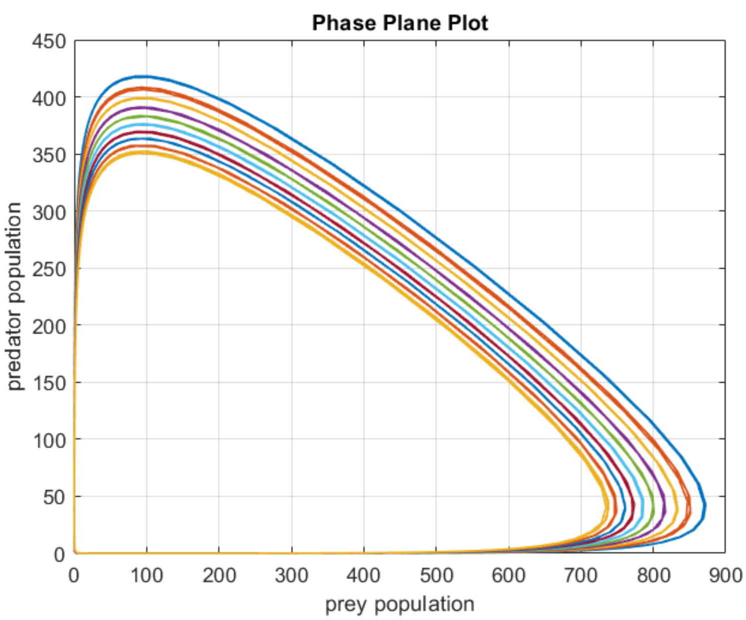
```
% Parameters
beta1 = 0.4;
c1 = 0.01;
c2 = 0.005;
alpha2 = 0.47;


% Lotka-Volterra equations

f = @(t, x) [beta1*x(1) - c1*x(1)*x(2);

            -alpha2*x(2) + c2*x(1)*x(2)];

% Plotting multiple phase portraits

for x0 = 0.9:0.1:1.8

    tspan = [0 100];

    initc = [x0 x0];    % initial condition

    [t, x] = ode45(f, tspan, initc);

    plot(x(:,1), x(:,2), 'linewidth', 1), hold on

end

title ('Phase Plane Plot')
grid on, xlabel('prey population'), ylabel('predator population'), hold off
```

This code creates the phase plane graph for the one predator one prey model after one of the predators becomes extinct.

3.3 Figures



After one of the predators becomes extinct the model reverts to a normal predator-prey model.



This is the phase plane plot of the grey wolves and bison population.

3.4 Analysis

After one of the predators becomes extinct in our two-predator model. The model reverts back to a simpler one predator-prey model. The graph oscillates as the two populations increase and decrease in tandem. While this doesn't entirely represent the true nature of the bison and wolves coexisting in Yellowstone with the few factors we can account for it's possible for them to continue surviving indefinitely. Given the variables we had, we attempted to draw a phase plane plot of the predator and prey but that graph doesn't truly represent what we expect to happen or in actuality what would happen. At no point should either population reach zero while the other grows.

References

[1] Fuller, J., Garrot, R., White, P.J, Aune, K., Roffe, T., Rhyan, J., "Reproduction and Survival of Yellowstone Bison", September 2007, https://www.researchgate.net/publication/227726229_Reproduction_and_Survival_of_Yellowstone_Bison

[2] https://www.yellowstonevacations.com/discover/attractions/wildlife-viewing

[3] Demidov, V., Zemlyanko, I., Stefutin, A., "Mortality of European Bison, American Bison and Complex Hybrids in the Prioksko - Terrasny Reserve Breeding Center", November 2021, https://link.springer.com/chapter/10.1007/978-3-030-91405-9_12

[4] https://billingsgazette.com/news/state-and-regional/wyoming/wolf-packs-attack-the-toughest-prey-in-yellowstone/article_d0deedab-0a8b-5e29-9edb-fb79fefcfe60.html#:~:text=Overall%2C%20bison%20are%20a%20small,were%20adults%20of%20unknown%20sex

[5] Treves, A., Langenberg, J., López-Bao, J., Rabenhorst, M., "Gray wolf mortality patterns in Wisconsin from 1979 to 2012", February 2017, https://academic.oup.com/jmammal/article/98/1/17/2977342

[6] Vito Volterra, Variations and Fluctuations of the Number of Individuals in Animal Species living together, ICES Journal of Marine Science, Volume 3, Issue 1, April 1928, Pages 3–51, https://doi.org/10.1093/icesjms/3.1.3

[7] https://www.nps.gov/yell/learn/nature/grizzlybear.htm

[8] https://www.vitalground.org/life-on-the-brink-how-grizzly-bears-die/#:~:text=Just%2012%20percent%20of%20adult,it's%20less%20common%20for%20cubs.