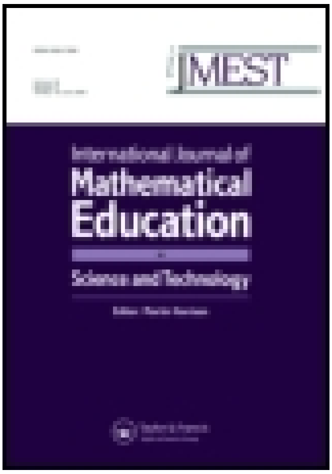

International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tmes20

An algebraic approach to finding the Fermat-Torricelli point

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Published online: 29 Apr 2015.

International Journal of Mathematical Education in Science and Technology, 2015
http://dx.doi.org/10.1080/0020739X.2015.1036947

CLASSROOM NOTE

An algebraic approach to finding the Fermat–Torricelli point

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(*Received 28 January 2015*)

Using a calculus and an algebraic approach, the Cartesian coordinates of the Fermat–Torricelli point are deduced for triangles with no internal angle greater than 120°. Although in theory, the deduction of these coordinates could be made 'by hand', in practice it is very laborious to obtain them without the aid of mathematical computer software, but with human guidance, since there are mathematical artifices not yet incorporated into the software. It is also shown that these coordinates can be conveniently expressed in terms of the side lengths and the area of the triangle. These coordinates are contrasted with the coordinates of a similar point: one whose sum of the squares of the distances to the vertices of an arbitrary triangle is a minimum.

Keywords: fermat point; minimal sum of distances; algebraic solution

Subject classification codes: 14Q99

1. Introduction

The problem we are going to address is to find the point whose distances to the vertices of an arbitrary triangle have a minimum sum, using only a calculus and an algebraic approach. Durocher and Kirkpatrick [1] point out that this problem has been repeatedly rediscovered under a variety of names. We will refer to it as the 'Fermat–Torricelli point'. Cieslik [2] mentions that 'The problem seems disarmingly simple, but is so rich in possibilities and traps that it has generated an enormous literature dating back to the seventeenth century, and continues to do so'. For the interested reader, both Boltyanski et al. [3] and very recently Bruno et al. [4] present an extensive and enriching historic review of this problem. Several geometric solutions are known and an excellent review is presented in the Wikipedia's Fermat point article.[5] Additionally Park and Flores [6] present five approaches to study the Fermat point using amongst others a mechanical device and even soap films.

However, as remarked by Mowaffaw Hajja,[7] 'one wonders why this beautiful extremum problem is not usually presented to students of advanced calculus. Possibly the negative speculation of D.C. Kay in his well-known College Geometry [3, p. 271] has tended to direct instructors away from the problem: "Any attempt to solve this by means of calculus would most probably end in considerable frustration"'. These remarks by Mowaffaw Hajja, although expressed in 1994, seem to be completely valid in present day and were the main reason that moved the authors to explore the algebraic approach. As the authors of this work were able to verify, Kay's assertion was not without base. At least

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that was the situation in 1969, the year when College Geometry was written. At that time, computer software to perform mathematic symbolic computations was incipient.

The aim of this work is to obtain the Cartesian coordinates of the Fermat–Torricelli point for an arbitrary triangle, with no internal angle greater than 120°, using only calculus and algebraic concepts, which in a way complements the work of Park and Flores.[6]

The authors believe that their contribution will be to show that solving the system of two non-linear simultaneous equations, although in theory could be made ‘by hand’, in practice it is very laborious to carry out without the aid of mathematical computer software. Furthermore, it will be also shown that the software alone cannot solve the problem. Mathematical artifices and interactive human guidance is still indispensable, something that without any doubt will attract the interest of students and teachers of mathematics and computer science. Finally, the difficult and laborious operations that are carried out in the pure algebraic approach will allow the readers to better appreciate the enormous simplicity and elegance of the geometric solutions found several centuries ago by Torricelli and Simpson.

2. Definition of the point whose sum of the squares of the distances to the vertices of an arbitrary triangle is a minimum

Before studying the coordinates of the Fermat–Torricelli point, we will consider a very closely related but much less complex problem: finding the point Q (x, y), whose sum of square distances to the vertices of an arbitrary triangle is a minimum.

The function to optimize is

$$ssd = (x - \mathrm{xa})^2 + (y - \mathrm{ya})^2 + (x - \mathrm{xb})^2 + (y - \mathrm{yb})^2 + (x - \mathrm{xc})^2 + (y - \mathrm{yc})^2, \quad (1)$$

where xa, ya, xb, yb, xc and yc are the coordinates of the triangle vertices; x and y are the coordinates of the searched point. The coordinates of Q (x, y) are found by equating to zero the partial derivatives of function ssd. Such point coincides with the triangle centroid, i.e. the point where the three medians of the triangle intersect

$$x = \frac{\mathrm{xa} + \mathrm{xb} + \mathrm{xc}}{3} \quad (2)$$

$$y = \frac{\mathrm{ya} + \mathrm{yb} + \mathrm{yc}}{3}. \quad (3)$$

One could expect that finding the coordinates of the Fermat–Torricelli point, where the distances are not squared, would be not much more difficult.

3. Definition of the point whose sum of distances to the vertices of an arbitrary triangle is a minimum

In this case, the function to optimize is

$$sd = \sqrt{(x - \mathrm{xa})^2 + (y - \mathrm{ya})^2} + \sqrt{(x - \mathrm{xb})^2 + (y - \mathrm{yb})^2} + \sqrt{(x - \mathrm{xc})^2 + (y - \mathrm{yc})^2}, \quad (4)$$

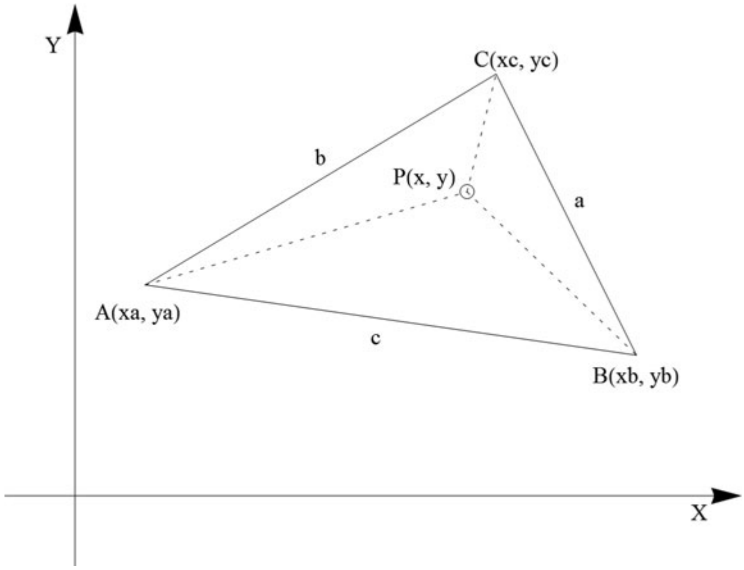

Figure 1. The Fermat point for an arbitrary triangle.

where xa, ya, xb, yb, xc and yc are the coordinates of the triangle vertices, presented in anticlockwise direction and x, y are the coordinates of the searched point (see Figure 1).

In order to solve this problem, once again we have to solve a system of two, in this case, non-linear simultaneous equations, obtained by equating to zero the partial derivatives of the sum of distances.

The partial derivatives of sd with respect to x and y are

$$\frac{\partial sd}{\partial x} = \frac{x-\mathrm{xa}}{\sqrt{(x-\mathrm{xa})^2+(y-\mathrm{ya})^2}} + \frac{x-\mathrm{xb}}{\sqrt{(x-\mathrm{xb})^2+(y-\mathrm{yb})^2}} + \frac{x-\mathrm{xc}}{\sqrt{(x-\mathrm{xc})^2+(y-\mathrm{yc})^2}}$$

$$\frac{\partial sd}{\partial y} = \frac{y-\mathrm{ya}}{\sqrt{(x-\mathrm{xa})^2+(y-\mathrm{ya})^2}} + \frac{y-\mathrm{yb}}{\sqrt{(x-\mathrm{xb})^2+(y-\mathrm{yb})^2}} + \frac{y-\mathrm{yc}}{\sqrt{(x-\mathrm{xc})^2+(y-\mathrm{yc})^2}}.$$

The next step, after equating to zero these partial derivatives, is to get rid of the radicals. Since there are three different radicals, after eliminating each one of them we obtain the three following equations:

$$\frac{((x-\mathrm{xb})(y-\mathrm{ya})-(x-\mathrm{xa})(y-\mathrm{yb}))^2}{((x-\mathrm{xc})(y-\mathrm{ya})-(x-\mathrm{xa})(y-\mathrm{yc}))^2} = \frac{(x-\mathrm{xb})^2+(y-\mathrm{yb})^2}{(x-\mathrm{xc})^2+(y-\mathrm{yc})^2}$$

$$\frac{((x-\mathrm{xa})(y-\mathrm{yb})-(x-\mathrm{xb})(y-\mathrm{ya}))^2}{((x-\mathrm{xc})(y-\mathrm{yb})-(x-\mathrm{xb})(y-\mathrm{yc}))^2} = \frac{(x-\mathrm{xa})^2+(y-\mathrm{ya})^2}{(x-\mathrm{xc})^2+(y-\mathrm{yc})^2}$$

$$\frac{((x-\mathrm{xa})(y-\mathrm{yc})-(x-\mathrm{xc})(y-\mathrm{ya}))^2}{((x-\mathrm{xb})(y-\mathrm{yc})-(x-\mathrm{xc})(y-\mathrm{yb}))^2} = \frac{(x-\mathrm{xa})^2+(y-\mathrm{ya})^2}{(x-\mathrm{xb})^2+(y-\mathrm{yb})^2}. \tag{5}$$

There are only two unknown coordinates, so one could think that only two equations are needed to solve the problem, which is true for numeric solutions. However, for a pure algebraic solution the third equation becomes indispensable, as will be shown in the next development.

One way to simplify Equations (5), without loss of generality, is to put the origin of the coordinate system onto one of the triangle vertices, say onto vertex A(xa, ya) and to rotate the triangle in such a way that the second vertex 'B' will lie on the abscise axis. In this way, the number of parameters is halved, since it is equivalent to assume that xa = 0, ya = 0 and yb = 0. After this substitution, the equations are factorized retaining only the main factors. In this way Equations (5) become

$$-x^2\mathrm{xb}\,y - x^2\mathrm{xc}\,y + 2x\,\mathrm{xb}\,\mathrm{xc}\,y - \mathrm{xb}\,y^3 - \mathrm{xc}\,y^3 + x^3\mathrm{yc} - x^2\mathrm{xb}\,\mathrm{yc} + x\,y^2\mathrm{yc} + \mathrm{xb}\,y^2\,\mathrm{yc} = 0 \tag{6a}$$

$$2x^2\mathrm{xb}\,y - 2x\,\mathrm{xb}^2y - x^2\mathrm{xc}\,y + \mathrm{xb}^2\mathrm{xc}\,y + 2\mathrm{xb}\,y^3 - \mathrm{xc}\,y^3 + x^3\mathrm{yc} - 2x^2\mathrm{xb}\,\mathrm{yc} + x\,\mathrm{xb}^2\mathrm{yc} + x\,y^2\mathrm{yc} - 2\mathrm{xb}\,y^2\mathrm{yc} = 0 \tag{6b}$$

$$(-\mathrm{xb} + 2\mathrm{xc})\,y^3 - 2x^3\mathrm{yc} + y^2\,(2\mathrm{xb}\,\mathrm{yc} - 2\mathrm{xc}\,\mathrm{yc}) + x^2((-\mathrm{xb} + 2\mathrm{xc})\,y + 2\mathrm{xb}\,\mathrm{yc} + 2\mathrm{xc}\,\mathrm{yc}) + x(-2\mathrm{xb}\,\mathrm{xc}\,\mathrm{yc} - 2y^2\mathrm{yc}) + x\,y\left(-2\mathrm{xc}^2 + 2\mathrm{yc}^2\right) + y\left(\mathrm{xb}\,\mathrm{xc}^2 - \mathrm{xb}\,\mathrm{yc}^2\right) = 0. \tag{6c}$$

At first glance, one could think that it is not so difficult to solve simultaneously any two of Equations (6). However, from the several mathematical computer programs available to authors, only Wolfram Research Mathematica® 6 and 7 (but not 8, 9 or 10) were able to find a solution, although such a solution is so enormously long (about 50 notebook pages, even for the case when xa, ya and yb are zero) that the authors did not try to simplify it. Instead they used other artifices to simplify the equations to be solved.

The key to the problem of obtaining a more manageable solution is to lower the degree of the simultaneous equations. Here is where the third of Equations (5) enters. We will eliminate the terms with x^3 from Equations (6a) and (6b) to obtain Equation (7a), and then from Equations (6b) and (6c), to obtain (7b)

$$-3x^2\mathrm{xb}\,y - \mathrm{xb}^2\mathrm{xc}\,y - 3\mathrm{xb}\,y^3 + x^2\mathrm{xb}\,\mathrm{yc} + 3\mathrm{xb}\,y^2\mathrm{yc} + x\left(\left(2\mathrm{xb}^2 + 2\mathrm{xb}\,\mathrm{xc}\right)y - \mathrm{xb}^2\mathrm{yc}\right) = 0 \tag{7a}$$

$$3x^2\mathrm{xb}\,y + 3\mathrm{xb}\,y^3 + y^2\,(-2\mathrm{xb}\,\mathrm{yc} - 2\mathrm{xc}\,\mathrm{yc}) + x^2\,(-2\mathrm{xb}\,\mathrm{yc} + 2\mathrm{xc}\,\mathrm{yc}) + y\left(2\mathrm{xb}^2\mathrm{xc} + \mathrm{xb}\,\mathrm{xc}^2 - \mathrm{xb}\,\mathrm{yc}^2\right) + x(2\mathrm{xb}^2\mathrm{yc} - 2\mathrm{xb}\,\mathrm{xc}\,\mathrm{yc} + y\left(-4\mathrm{xb}^2 - 2\mathrm{xc}^2 + 2\mathrm{yc}^2\right)) = 0. \tag{7b}$$

Now from Equations (7a) and (7b), we eliminate the term with y^3, so we obtain (8), a quadratic in x and y

$$y^2\,(\mathrm{xb}\,\mathrm{yc} - 2\mathrm{xc}\,\mathrm{yc}) + x^2\,(-\mathrm{xb}\,\mathrm{yc} + 2\mathrm{xc}\,\mathrm{yc}) + y\left(\mathrm{xb}^2\mathrm{xc} + \mathrm{xb}\,\mathrm{xc}^2 - \mathrm{xb}\,\mathrm{yc}^2\right) + x\,\left(\mathrm{xb}^2\mathrm{yc} - 2\mathrm{xb}\,\mathrm{xc}\,\mathrm{yc} + y\left(-2\mathrm{xb}^2 + 2\mathrm{xb}\,\mathrm{xc} - 2\mathrm{xc}^2 + 2\mathrm{yc}^2\right)\right) = 0. \tag{8}$$

Solving this quadratic for x, we obtain

$$x = -(\mathrm{xb}^2 (2y - \mathrm{yc}) + 2y (\mathrm{xc} - \mathrm{yc}) (\mathrm{xc} + \mathrm{yc}) + 2\mathrm{xb}\,\mathrm{xc} (-y + \mathrm{yc}) \\ + \sqrt{}(4 (\mathrm{xb} - 2\mathrm{xc})\, y\, \mathrm{yc} \left(\mathrm{xb}\,\mathrm{xc} (\mathrm{xb} + \mathrm{xc}) + (\mathrm{xb} - 2\mathrm{xc})\, y\, \mathrm{yc} - \mathrm{xb}\,\mathrm{yc}^2\right) \\ + \left(-2 \left(\mathrm{xb}^2 - \mathrm{xb}\,\mathrm{xc} + \mathrm{xc}^2\right) y + \mathrm{xb} (\mathrm{xb} - 2\mathrm{xc}) \mathrm{yc} + 2y\, \mathrm{yc}^2\right)^2))/ (2 (\mathrm{xb} - 2\mathrm{xc}) \mathrm{yc})\,. \tag{9}$$

This value of x is substituted in one of Equations (7) to obtain an expression just in y, which constitutes the most arduous part of all the process, the reason being the long and complex radical in the numerator of expression (9) that would make practically impossible to handle it without the aid of computer software.

In order to obtain such expression just in y, we have to interactively with the computer software

(1) isolate the radical on one side of the equation,
(2) square both sides of it,
(3) collect the like powers of y,
(4) factorize the polynomial in y, and finally
(5) equate to zero each factor and solve for y.

At the end of this process, expression (10) is obtained for the ordinate

$$y = \mathrm{xb}(3\mathrm{xb}^3\mathrm{yc} - 12\mathrm{xb}^2\mathrm{xc}\,\mathrm{yc} + 12\mathrm{xb}\,\mathrm{xc}^2\mathrm{yc} + \sqrt{3}(3\mathrm{xb}^3\mathrm{xc} - 3\mathrm{xc}^4 - 2\mathrm{xc}^2\mathrm{yc}^2 + \mathrm{yc}^4 \\ - 2\mathrm{xb}^2(3\mathrm{xc}^2 + \mathrm{yc}^2) + 2\mathrm{xb}\,\mathrm{xc}(3\mathrm{xc}^2 + \mathrm{yc}^2)))/(6(\mathrm{xb}^4 - 2\mathrm{xb}^3\mathrm{xc} + \mathrm{xb}^2(3\mathrm{xc}^2 - \mathrm{yc}^2) \\ - 2\mathrm{xb}\,\mathrm{xc}(\mathrm{xc}^2 + \mathrm{yc}^2) + (\mathrm{xc}^2 + \mathrm{yc}^2)^2)). \tag{10}$$

This expression can be simplified to obtain

$$y = \frac{\mathrm{xb}(\sqrt{3}\mathrm{xc} + \mathrm{yc})(\sqrt{3}\,(\mathrm{xb} - \mathrm{xc}) + \mathrm{yc})}{2\sqrt{3}(\mathrm{xb}^2 - \mathrm{xb}\,\mathrm{xc} + \mathrm{xc}^2 + \mathrm{yc}^2 + \sqrt{3}\mathrm{xb}\,\mathrm{yc})}. \tag{11}$$

Substituting the value of y in expression (9), after some simplification, we obtain an expression quite similar for the abscissa

$$x = \frac{\mathrm{xb}(\sqrt{3}\mathrm{xc} + \mathrm{yc})(\mathrm{xb} + \mathrm{xc} + \sqrt{3}\mathrm{yc})}{2\sqrt{3}(\mathrm{xb}^2 - \mathrm{xb}\,\mathrm{xc} + \mathrm{xc}^2 + \mathrm{yc}^2 + \sqrt{3}\mathrm{xb}\,\mathrm{yc})}. \tag{12}$$

Expressions (11) and (12) allow us to calculate the coordinates of the Fermat–Torricelli point when the origin of the coordinate system is located on vertex ‘A’ (that is xa = ya = 0) of the analysed triangle and the next vertex ‘B’ lies on the abscissa axis (that is yb = 0). Before finding the coordinates of the Fermat point for the general case, it seems natural to express the ordinate (11) as a function of the only vertex ordinate that is different from zero, yc, and the rest of the expression we will write as a function of the triangle invariant parameters a, b and c, the triangle side lengths, and the triangle area (At). For this purpose, we need to consider the following equivalences:

$$\begin{aligned}
\mathrm{xb} &= c \\
\mathrm{yc} &= \frac{2\mathrm{At}}{c} \\
\mathrm{xc} &= \sqrt{b^2 - \mathrm{yc}^2} = \sqrt{b^2 - \frac{4\mathrm{At}^2}{c^2}} = \frac{-a^2 + b^2 + c^2}{2c}.
\end{aligned}$$

Using these equivalences, the denominator of Equations (11) or (12) can be written as

$$2\sqrt{3}\left(\mathrm{xb}^2 - \mathrm{xb}\,\mathrm{xc} + \mathrm{xc}^2 + \mathrm{yc}^2 + \sqrt{3}\mathrm{xb}\,\mathrm{yc}\right) = 12\mathrm{At} + \sqrt{3}\left(a^2 + b^2 + c^2\right).$$

In this way, we obtain

$$y = \frac{(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))(4\mathrm{At} + \sqrt{3}(a^2 - b^2 + c^2))}{8\mathrm{At}(12\mathrm{At} + \sqrt{3}(a^2 + b^2 + c^2))}\,\mathrm{yc}. \tag{13}$$

For the abscissa, we first apply the same equivalences amongst the vertex coordinates and the invariant parameters and obtain

$$x = \frac{(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))(4\sqrt{3}\mathrm{At} + (-a^2 + b^2 + 3c^2))\mathrm{yc}}{(8\mathrm{At}(12\mathrm{At} + \sqrt{3}(a^2 + b^2 + c^2)))}.$$

Secondly, we will express this abscissa as a function of the two abscissa that are different from zero, xb and xc. For this purpose, it is necessary to take into consideration the following identity:

$$\begin{aligned}
&(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))(4\sqrt{3}\mathrm{At} + (-a^2 + b^2 + 3c^2))yc \\
&\quad = (4\mathrm{At} + \sqrt{3}(a^2 + b^2 - c^2))(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))x\mathrm{b} \\
&\qquad + (4\mathrm{At} + \sqrt{3}(a^2 - b^2 + c^2))(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))\mathrm{xc}.
\end{aligned}$$

The abscissa x becomes

$$\begin{aligned}
x = {} & \frac{(4\mathrm{At} + \sqrt{3}(a^2 + b^2 - c^2))(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))}{8\mathrm{At}(12\mathrm{At} + \sqrt{3}(a^2 + b^2 + c^2))}\,\mathrm{xb} \\
& + \frac{(4\mathrm{At} + \sqrt{3}(a^2 - b^2 + c^2))(4\mathrm{At} + \sqrt{3}(-a^2 + b^2 + c^2))}{8\mathrm{At}(12\mathrm{At} + \sqrt{3}(a^2 + b^2 + c^2))}\mathrm{xc}.
\end{aligned} \tag{14}$$

Observe that the coefficient of xc in formula (14) is identical to the coefficient of yc in formula (13).

We could repeat the whole procedure locating the coordinate origin in vertex B (or C) and rotating the triangle in such a way to locate vertex C (or A) on the horizontal axis. In this way, we would have: xb = yb = yc = 0 (or alternatively xc = yc = ya = 0) and would obtain formulae completely similar to (13) and (14), but with another permutation of sides a, b and c. The coordinates of the Fermat point for the general case are

$$x\text{Fermat} = \text{tc}\,[a, b, c]\,\text{xa} + \text{tc}\,[b, c, a]\,\text{xb} + \text{tc}\,[c, a, b]\,\text{xc} \tag{15}$$

$$y\text{Fermat} = \text{tc}\,[a, b, c]\,\text{ya} + \text{tc}\,[b, c, a]\,\text{yb} + \text{tc}\,[c, a, b]\,\text{yc} \tag{16}$$

$$\text{tc}\,[s\_, t\_, u\_] = \frac{(4\text{At} + \sqrt{3}(s^2 + t^2 - u^2))(4\text{At} + \sqrt{3}(s^2 - t^2 + u^2))}{8\text{At}(12\text{At} + \sqrt{3}(s^2 + t^2 + u^2))}. \tag{17}$$

The reader surely has realized that the coefficients tc[a,b,c], tc[b,c,a] and tc[c,a,b] are the normalized barycentric coordinates.[8] The sum of the three coefficients is equal to unity. It is also clear that the coordinates of the Fermat point are triangle interpolated, where the weights are the ratio of the area of the small triangles formed when joining the Fermat point with the vertices, divided by the whole area of the triangle.

4. Final remarks

Formulae (15) and (16) and function (17) are probably the most compact way to express the Cartesian coordinates of the Fermat point. It is interesting to note that the X(13) centre in the Clark Kimberling's Encyclopaedia of Triangle Centres [9], which corresponds to the Fermat point, has a barycentric function similar but less symmetric than function (17) and without the denominator, which means that the coefficients are not normalized. Likewise, in the related Internet site 'Triangle Centers with C.a.R.' (http://www.uff.br/trianglecenters/etcwc.html), one can see the formulae used to calculate the Cartesian coordinates of the Fermat point, similar to ours in the sense that the coordinates of the vertices have coefficients expressed in terms of the sides and area of the triangle. However those formulae are not compacted or structured, resulting in very long sequences (several hundred of characters for the 'x' and for the 'y').

For many readers, as for the authors, it was a little bit unexpected that the coordinates of the Fermat–Torricelli point result so more complex in comparison with the coordinates of the point whose sum of the squared distances to the vertices of an arbitrary triangle is a minimum, although both points tend to coincide as the form of the triangle gets closer to an equilateral triangle. However, the more elongated the triangles, the greater the separation between the points.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Durocher S, Kirkpatrick D. The projection median of a set of points. Comput Geom . 2009;42(5):364–375.

[2] Cieslik D. Shortest connectivity: an introduction with applications in phylogeny. New York, NY: Springer Science + Business Media, Inc.; 2005; p. 268.

[3] Boltyanski V, Martini H, Soltan V. Geometric methods and optimization problems. Dordrecht, The Netherlands: Kluwer Academic Publishers; 1999; p. 444.

[4] Bruno G, Genovese A, Improta G. A historical perspective on location problems. Br Soc Hist Math Bull . 2014;29(2):83–97.

[5] Wikipedia: the free encyclopedia [Internet]. St. Petersburg, FL: Wikimedia Foundation, Inc.; 2001 [cited 2015 February 26]. Available from: http://en.wikipedia.org/wiki/Fermat_point

[6] Park J, Flores A. Fermat's point from five perspectives. Int J Math Educ Sci Technol. 2015;46(3):425–441.
[7] Mowaffaw H. An advanced calculus approach to finding the Fermat point. Math Mag. 1994;67(1):29–34.
[8] MathWorld: a Wolfram web resource [Internet]. Champaign, IL: Wolfram Research, Inc.; 1987 [cited 2015 February 26]. Available from: http://mathworld.wolfram.com/BarycentricCoordinates.html.
[9] ETC: Encyclopedia of Triangle Centers [Internet]. [cited 2015 February 26]. University of Evansville, Evansville, Indiana. Available from: http://faculty.evansville.edu/ck6/encyclopedia/ETC.html