

## Appendix A. Ray tracing standard formula

### Appendix A.1. Transfer Equations

To begin, light rays originate from the object space point  $P_0$ , collide with each surface in order from left to right and refract at the point of collision to proceed to the next surface. Eventually, the rays reach the image space point  $P_n$ . In the case of the incident light on the  $(i-1)$ th surface, the position  $P_{i-1}(x_{i-1}, y_{i-1}, z_{i-1})$  and the unit direction vector  $\mathbf{d}_{i-1} = L_{i-1}\mathbf{i} + M_{i-1}\mathbf{j} + N_{i-1}\mathbf{k}$  of the incident light are available. We can calculate the subsequent refracted ray projection point  $P_i(x_i, y_i, z_i)$ , and the unit direction vector  $\mathbf{d}_i = L_i\mathbf{i} + M_i\mathbf{j} + N_i\mathbf{k}$  on the  $i$ th surface, by applying transfer Eqs. (A.1)–(A.8) and refraction Eqs. (A.9)–(A.15). Here,  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors respectively along the X, Y and Z axes. In Eqs. (A.1)–(A.15), the variable  $Q_z$  denotes the z-coordinate of vector  $\mathbf{Q}_i$ , while the coefficient  $p_i$  represents the quadratic surface coefficient of the  $i$ th surface,  $c_i = 1/r_i$ .

$$f_{i-1} = D_{i-1}N_{i-1} - (x_{i-1}L_{i-1} + y_{i-1}M_{i-1} + z_{i-1}N_{i-1}) \quad (\text{A.1})$$

$$Q_z = z_{i-1} + f_{i-1}N_{i-1} - D_{i-1} \quad (\text{A.2})$$

$$Q_i^2 = (x_{i-1} + f_{i-1}L_{i-1})^2 + (y_{i-1} + f_{i-1}M_{i-1})^2 + (z_{i-1} + f_{i-1}N_{i-1} - D_{i-1})^2 \quad (\text{A.3})$$

$$f_i = \frac{\Delta_1}{\Delta_4} - \frac{\sqrt{\Delta_2 - \Delta_3}}{\Delta_4} \quad (\text{A.4})$$

$$\text{where } \Delta_1 = [N_{i-1}r_i + (1 - p_i)Q_zN_{i-1}]$$

$$\Delta_2 = [N_{i-1}r_i + (1 - p_i)Q_zN_{i-1}]^2$$

$$\Delta_3 = [1 - (1 - p_i)N_{i-1}^2][Q_i^2 - 2r_iQ_z - (1 - p_i)Q_z^2]$$

$$\Delta_4 = 1 - (1 - p_i)N_{i-1}^2$$

$$t_{i-1} = f_{i-1} + f_i \quad (\text{A.5})$$

$$x_i = x_{i-1} + t_{i-1}L_{i-1} \quad (\text{A.6})$$

$$y_i = y_{i-1} + t_{i-1}M_{i-1} \quad (\text{A.7})$$

$$z_i = z_{i-1} + t_{i-1}N_{i-1} - D_{i-1} \quad (\text{A.8})$$

### Appendix A.2. Refraction Equations

$$A = \sqrt{1 + c_i^2(1 - p_i)(x_i^2 + y_i^2)} \quad (\text{A.9})$$

$$\cos \theta = \left| \frac{(1 - z_i p_i c_i)N_{i-1} - x_i c_i L_{i-1} - y_i c_i M_{i-1}}{A} \right| \quad (\text{A.10})$$

$$\cos \theta' = \sqrt{1 - \frac{n_{i-1}^2}{n_i^2}(1 - \cos^2 \theta)} \quad (\text{A.11})$$

$$g_i = n_i \cos \theta' - n_{i-1} \cos \theta \quad (\text{A.12})$$

$$L_i = \frac{n_{i-1}}{n_i}L_{i-1} - \frac{g_i c_i x_i}{n_i A} \quad (\text{A.13})$$

$$M_i = \frac{n_{i-1}}{n_i}M_{i-1} - \frac{g_i c_i y_i}{n_i A} \quad (\text{A.14})$$

$$N_i = \frac{n_{i-1}}{n_i}N_{i-1} - \frac{g_i(p_i c_i z_i - 1)}{n_i A} \quad (\text{A.15})$$

Eqs. (A.1)–(A.15) are collectively referred to as the standard formula for ray tracing. The correctness of the ray-tracing results is verified using the following validation Eqs. (A.16)–(A.17).

$$x_i^2 + y_i^2 + p_i z_i^2 - 2r_i z_i = 0 \quad (\text{A.16})$$

$$L_i^2 + M_i^2 + N_i^2 = 1 \quad (\text{A.17})$$

### Appendix A.3. Derivation of ray tracing standard formula

The simple derivation of Eqs. (A.1)-(A.15) is as follows. A more comprehensive illustration detailed in references [1]-[2], although their primary focus is on the formula for spherical tracing.

1. To compute  $\mathbf{P}_i$  from  $\mathbf{P}_{i-1}$  and  $\mathbf{d}_{i-1}$ , begin by calculating  $\mathbf{Q}_i$  from  $\mathbf{P}_{i-1}$  and  $\mathbf{d}_{i-1}$ , then utilize  $\mathbf{Q}_i$  and  $\mathbf{d}_{i-1}$  to determine  $\mathbf{P}_i$ .

(1) Derive the vector formula  $\mathbf{P}_{i-1} + f_{i-1}\mathbf{d}_{i-1} = D_{i-1}\mathbf{k} + \mathbf{Q}_i$  from the quadrilateral  $O_{i-1}P_{i-1}Q_iO_i$ . Taking the dot product of both sides of the equation with  $\mathbf{d}_{i-1}$ , we obtain  $\mathbf{P}_{i-1} \cdot \mathbf{d}_{i-1} + f_{i-1} = D_{i-1}\mathbf{k} \cdot \mathbf{d}_{i-1}$ , considering that  $\mathbf{d}_{i-1} \cdot \mathbf{d}_{i-1} = 1$  and  $\mathbf{Q}_i \cdot \mathbf{d}_{i-1} = 0$  (due to the perpendicularity of  $\mathbf{d}_{i-1}$  and  $\mathbf{Q}_i$ ). By substituting the components of  $\mathbf{P}_{i-1}$  and  $\mathbf{d}_{i-1}$  into the equation, we arrive at Eq. (A.1):  $f_{i-1} = D_{i-1}N_{i-1} - (x_{i-1}L_{i-1} + y_{i-1}M_{i-1} + z_{i-1}N_{i-1})$ . Substituting  $f_{i-1}$  into  $\mathbf{P}_{i-1} + f_{i-1}\mathbf{d}_{i-1} = D_{i-1}\mathbf{k} + \mathbf{Q}_i$  leads to the derivation of Eqs. (A.2)-(A.3).

(2) Calculate  $\mathbf{P}_i$  from  $\mathbf{Q}_i$  and  $\mathbf{d}_{i-1}$ . From the triangle  $\triangle O_iQ_iP_i$ , we have the following vector equation:  $\mathbf{P}_i = \mathbf{Q}_i + f_i\mathbf{d}_{i-1}$ . The point  $P_i$  lies on the quadratic surface given by Eq. (2):  $x_i^2 + y_i^2 + p_i z_i^2 - 2r_i z_i = 0$ . Expressing it in vector form, we obtain  $\mathbf{P}_i \cdot \mathbf{P}_i - 2r_i\mathbf{k} \cdot \mathbf{P}_i - (1 - p_i)(\mathbf{k} \cdot \mathbf{P}_i)^2 = 0$ . By using  $\mathbf{P}_i = \mathbf{Q}_i + f_i\mathbf{d}_{i-1}$  to substitute  $\mathbf{P}_i$  in the previous equation with  $\mathbf{Q}_i$ ,  $\mathbf{d}_{i-1}$  and  $f_i$ , we can solve for  $f_i$ , yielding Eq. (A.4). From Fig. (1), we have  $t_{i-1} = f_{i-1} + f_i$ , expressed in Eq. (A.5). Substituting  $f_i$  into  $\mathbf{P}_i = \mathbf{Q}_i + f_i\mathbf{d}_{i-1}$ , we can solve for  $\mathbf{P}_i$ , resulting in scalar Eqs. (A.6)-(A.8).

2. The unit normal vector  $\mathbf{N}$  at point  $P_i$  on the  $i$ th surface is determined. The expression for  $\mathbf{N}$  is given as  $\{\frac{-x_i c_i}{A}, \frac{-y_i c_i}{A}, \frac{1 - z_i p_i c_i}{A}\}$ , by employing calculus principles, where  $A = \sqrt{1 + c_i^2(1 - p_i)(x_i^2 + y_i^2)}$ . As a result, Eq. (A.9) is obtained.

3. Calculate  $\mathbf{d}_i$  using the law of refraction. The vector form of the law of refraction indicates that  $(n_i\mathbf{d}_i - n_{i-1}\mathbf{d}_{i-1}) \times \mathbf{N} = 0$ , which implies that  $(n_i\mathbf{d}_i - n_{i-1}\mathbf{d}_{i-1})$  is parallel to  $\mathbf{N}$ , thus  $(n_i\mathbf{d}_i - n_{i-1}\mathbf{d}_{i-1}) = g_i\mathbf{N}$ , where  $g_i$  is a coefficient. By taking the dot product of the above equation with  $\mathbf{N}$ , we can obtain  $g_i = (n_i\mathbf{d}_i \cdot \mathbf{N} - n_{i-1}\mathbf{d}_{i-1} \cdot \mathbf{N})$ . According to the property that the dot product of two unit vectors is equal to the cosine of the angle between the vectors, we have the following formulas:  $g_i = n_i \cos \theta' - n_{i-1} \cos \theta$ ,  $\cos \theta = \left| \frac{(1 - z_i p_i c_i)N_{i-1} - x_i c_i L_{i-1} - y_i c_i M_{i-1}}{A} \right|$ . Applying the law of

refraction yields  $\cos \theta' = \sqrt{1 - \frac{n_{i-1}^2}{n_i^2}(1 - \cos^2 \theta)}$ . Hence, Eqs. (A.10)-(A.12) are derived as mentioned above. From equation  $(n_i\mathbf{d}_i - n_{i-1}\mathbf{d}_{i-1}) = g_i\mathbf{N}$ , we can derive  $\mathbf{d}_i = \frac{n_{i-1}}{n_i}\mathbf{d}_{i-1} + \frac{g_i}{n_i}\mathbf{N}$ , resulting in Eqs. (A.13)-(A.15) in scalar form.

### Appendix B. Ray tracing error model

During the ray tracing process, given the starting point  $P_{i-1} = x_{i-1}\mathbf{i} + y_{i-1}\mathbf{j} + z_{i-1}\mathbf{k}$  and directional vector  $\mathbf{d}_{i-1} = L_{i-1}\mathbf{i} + M_{i-1}\mathbf{j} + N_{i-1}\mathbf{k}$  of a ray on the  $(i-1)$ th surface, the spacing  $D_{i-1}$  between the  $(i-1)$ th and  $i$ th surfaces, the curvature radius  $r_i$  and quadratic surface coefficient  $p_i$  of the  $i$ th surface, the intersection point  $P_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$  of the  $i$ th surface with the ray is calculated using the transfer Eqs. (A.1)-(A.8), and the directional vector  $\mathbf{d}_i = L_i\mathbf{i} + M_i\mathbf{j} + N_i\mathbf{k}$  is calculated using the refraction Eqs. (A.9)-(A.15). According to Theorem (1), the input physical quantities have the following representation errors.

$$fl(V_I) = V_I(1 \pm \delta), \quad 0 \leq \delta \leq u, \quad (\text{B.1})$$

where  $V_I$  is one of  $x_{i-1}, y_{i-1}, z_{i-1}, L_{i-1}, M_{i-1}, N_{i-1}, D_{i-1}, r_i, p_i$ . Meanwhile, when calculating the ray tracing projection point  $P_i$  and direction  $\mathbf{d}_i$  in the order of Eqs. (A.1)-(A.15), it will introduce cumulative errors, such as the calculation of  $Q_z$  in the transfer Eq. (A.2) requires the result of the calculation of  $f_{i-1}$  in the transfer Eq. (A.1), and the calculation of  $f_{i-1}$  will introduce cumulative errors  $(1 \pm \delta_{f_{i-1}})$ , and there exists certain positive integer  $n_{f_{i-1}}$  such that  $f_{i-1}(1 \pm \delta_{f_{i-1}}) \leq f_{i-1}(1 \pm u)^{n_{f_{i-1}}}$ . In general, the following formula holds.

$$V_{cum}(1 \pm \delta_{V_{cum}}) \leq V_{cum}(1 \pm u)^{n_{V_{cum}}} \quad (\text{B.2})$$

where  $V_{cum}$  denotes any of  $f_{i-1}, Q_z, Q_i^2, f_i, t_{i-1}, x_i, y_i, z_i, A, \cos \theta, \cos \theta', g_i$ .

$$\begin{aligned}
fl(Q_i^2) &= (x_{i-1} + f_{i-1}L_{i-1})^2 + (y_{i-1} + f_{i-1}M_{i-1})^2 + (z_{i-1} + f_{i-1}N_{i-1} - D_{i-1})^2 \\
&\subset [x_{i-1} + f_{i-1}(1 \pm \delta_{f_{i-1}})L_{i-1}]^2 + [y_{i-1} + f_{i-1}(1 \pm \delta_{f_{i-1}})M_{i-1}]^2 + [z_{i-1} + f_{i-1}(1 \pm \delta_{f_{i-1}})N_{i-1} - D_{i-1}]^2 \\
&\subset [x_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)L_{i-1}(1 \pm u)]^2 + [y_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)M_{i-1}(1 \pm u)]^2 \\
&\quad + [z_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)N_{i-1}(1 \pm u) - D_{i-1}(1 \pm u)]^2
\end{aligned}$$

$$\begin{aligned}
& + [z_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)N_{i-1}(1 \pm u) - D_{i-1}(1 \pm u)]^2 \\
& \subset [[x_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})L_{i-1}(1 \pm u)^3](1 \pm u)]^2(1 \pm u) \\
& + [[y_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})M_{i-1}(1 \pm u)^3](1 \pm u)]^2(1 \pm u) \\
& + [ [z_{i-1}(1 \pm u) + f_{i-1}(1 \pm \delta_{f_{i-1}})N_{i-1}(1 \pm u)^3](1 \pm u) - D_{i-1}(1 \pm u) ]^2(1 \pm u) \\
& \subset [x_{i-1} + f_{i-1}L_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^2]^2(1 \pm u)^5 + [y_{i-1} + f_{i-1}M_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^2]^2(1 \pm u)^5 \\
& + [(z_{i-1} + f_{i-1}N_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^2)(1 \pm u) - D_{i-1}]^2(1 \pm u)^5 \\
& \subset x_{i-1}^2(1 \pm u)^5 + 2x_{i-1}f_{i-1}L_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^7 + f_{i-1}^2L_{i-1}^2(1 \pm \delta_{f_{i-1}})^2(1 \pm u)^9 \\
& + y_{i-1}^2(1 \pm u)^5 + 2y_{i-1}f_{i-1}M_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^7 + f_{i-1}^2M_{i-1}^2(1 \pm \delta_{f_{i-1}})^2(1 \pm u)^9 \\
& + z_{i-1}^2(1 \pm u)^7 + 2z_{i-1}f_{i-1}N_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^9 + f_{i-1}^2N_{i-1}^2(1 \pm \delta_{f_{i-1}})^2(1 \pm u)^{11} \\
& - 2D_{i-1}z_{i-1}(1 \pm u)^6 - 2D_{i-1}f_{i-1}N_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u)^9 + D_{i-1}^2(1 \pm u)^5 \\
& \subset x_{i-1}^2(1 \pm u)^5 + 2x_{i-1}f_{i-1}L_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u)^7 + f_{i-1}^2L_{i-1}^2(1 \pm u)^{2n_{f_{i-1}}}(1 \pm u)^9 \\
& + y_{i-1}^2(1 \pm u)^5 + 2y_{i-1}f_{i-1}M_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u)^7 + f_{i-1}^2M_{i-1}^2(1 \pm u)^{2n_{f_{i-1}}}(1 \pm u)^9 \\
& + z_{i-1}^2(1 \pm u)^7 + 2z_{i-1}f_{i-1}N_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u)^9 + f_{i-1}^2N_{i-1}^2(1 \pm u)^{2n_{f_{i-1}}}(1 \pm u)^{11} \\
& - 2D_{i-1}z_{i-1}(1 \pm u)^6 - 2D_{i-1}f_{i-1}N_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u)^9 + D_{i-1}^2(1 \pm u)^5 \\
& \subset x_{i-1}^2(1 \pm u)^5 + 2x_{i-1}f_{i-1}L_{i-1}(1 \pm u)^{7+n_{f_{i-1}}} + f_{i-1}^2L_{i-1}^2(1 \pm u)^{9+2n_{f_{i-1}}} \\
& + y_{i-1}^2(1 \pm u)^5 + 2y_{i-1}f_{i-1}M_{i-1}(1 \pm u)^{7+n_{f_{i-1}}} + f_{i-1}^2M_{i-1}^2(1 \pm u)^{9+2n_{f_{i-1}}} \\
& + z_{i-1}^2(1 \pm u)^7 + 2z_{i-1}f_{i-1}N_{i-1}(1 \pm u)^{9+n_{f_{i-1}}} + f_{i-1}^2N_{i-1}^2(1 \pm u)^{11+2n_{f_{i-1}}} \\
& - 2D_{i-1}z_{i-1}(1 \pm u)^6 - 2D_{i-1}f_{i-1}N_{i-1}(1 \pm u)^{9+n_{f_{i-1}}} + D_{i-1}^2(1 \pm u)^5 \\
& \subset x_{i-1}^2(1 \pm \gamma_5) + 2x_{i-1}f_{i-1}L_{i-1}(1 \pm \gamma_{7+n_{f_{i-1}}}) + f_{i-1}^2L_{i-1}^2(1 \pm \gamma_{9+2n_{f_{i-1}}}) \\
& + y_{i-1}^2(1 \pm \gamma_5) + 2y_{i-1}f_{i-1}M_{i-1}(1 \pm \gamma_{7+n_{f_{i-1}}}) + f_{i-1}^2M_{i-1}^2(1 \pm \gamma_{9+2n_{f_{i-1}}}) \\
& + z_{i-1}^2(1 \pm \gamma_7) + 2z_{i-1}f_{i-1}N_{i-1}(1 \pm \gamma_{9+n_{f_{i-1}}}) + f_{i-1}^2N_{i-1}^2(1 \pm \gamma_{11+2n_{f_{i-1}}}) \\
& - 2D_{i-1}z_{i-1}(1 \pm \gamma_6) - 2D_{i-1}f_{i-1}N_{i-1}(1 \pm \gamma_{9+n_{f_{i-1}}}) + D_{i-1}^2(1 \pm \gamma_5) \\
& = (x_{i-1} + f_{i-1}L_{i-1})^2 + (y_{i-1} + f_{i-1}M_{i-1})^2 + (z_{i-1} + f_{i-1}N_{i-1} - D_{i-1})^2 \\
& [\pm(x_{i-1}^2 + y_{i-1}^2 + D_{i-1}^2)\gamma_5 \pm 2x_{i-1}f_{i-1}L_{i-1}\gamma_{7+n_{f_{i-1}}} \pm 2y_{i-1}f_{i-1}M_{i-1}\gamma_{7+n_{f_{i-1}}} \pm 2z_{i-1}f_{i-1}N_{i-1}\gamma_{9+n_{f_{i-1}}} \\
& \pm 2D_{i-1}f_{i-1}N_{i-1}\gamma_{9+n_{f_{i-1}}} \pm 2D_{i-1}z_{i-1}\gamma_6 \pm f_{i-1}^2L_{i-1}^2\gamma_{9+2n_{f_{i-1}}} \pm f_{i-1}^2M_{i-1}^2\gamma_{9+2n_{f_{i-1}}} \pm z_{i-1}^2\gamma_7 \pm f_{i-1}^2N_{i-1}^2\gamma_{11+2n_{f_{i-1}}}] \quad (B.6)
\end{aligned}$$

Consequently, the error term (in square brackets) of  $Q_i^2$  is bounded by

$$\begin{aligned}
err_{Q_i^2} &= (x_{i-1}^2 + y_{i-1}^2 + D_{i-1}^2)\gamma_5 + 2|x_{i-1}f_{i-1}L_{i-1}|\gamma_{7+n_{f_{i-1}}} + 2|y_{i-1}f_{i-1}M_{i-1}|\gamma_{7+n_{f_{i-1}}} + 2|z_{i-1}f_{i-1}N_{i-1}|\gamma_{9+n_{f_{i-1}}} \\
&+ 2|D_{i-1}f_{i-1}N_{i-1}|\gamma_{9+n_{f_{i-1}}} + 2|D_{i-1}z_{i-1}|\gamma_6 + f_{i-1}^2L_{i-1}^2\gamma_{9+2n_{f_{i-1}}} + f_{i-1}^2M_{i-1}^2\gamma_{9+2n_{f_{i-1}}} + z_{i-1}^2\gamma_7 + f_{i-1}^2N_{i-1}^2\gamma_{11+2n_{f_{i-1}}} \quad (B.7)
\end{aligned}$$

#### Appendix B.1.4. Floating-Point Error for Equation A.4

To facilitate the derivation of the error model, the following intermediate variables are introduced.

$$\Delta_1 = [N_{i-1}r_i + (1 - p_i)Q_zN_{i-1}] \quad (B.8)$$

$$\Delta_2 = [N_{i-1}r_i + (1 - p_i)Q_zN_{i-1}]^2 \quad (B.9)$$

$$\Delta_3 = [1 - (1 - p_i)N_{i-1}^2][Q_i^2 - 2r_iQ_z - (1 - p_i)Q_z^2] \quad (B.10)$$

$$\Delta_4 = 1 - (1 - p_i)N_{i-1}^2 \quad (B.11)$$

Thus the transfer Eq. (B.12) can be expressed as:

$$f_i = \frac{\Delta_1}{\Delta_4} - \frac{\sqrt{\Delta_2 - \Delta_3}}{\Delta_4} \quad (B.12)$$

First compute the  $\Delta_1$  error expression.

$$\begin{aligned}
 fl(\Delta_1) &= [N_{i-1}r_i + (1 - p_i)Q_z N_{i-1}] \\
 &\subset [N_{i-1}r_i(1 \pm u)^3 + [1 - p_i(1 \pm u)](1 \pm u)Q_z(1 \pm \delta_{Q_z})N_{i-1}(1 \pm u)^4] \\
 &\subset \{[N_{i-1}r_i(1 \pm u)^3 + [1 - p_i(1 \pm u)](1 \pm u)Q_z(1 \pm u)^{n_{Q_z}}N_{i-1}(1 \pm u)^4]\}(1 \pm u) \\
 &\subset [N_{i-1}r_i(1 \pm u)^4 + [1 - p_i(1 \pm u)]Q_z N_{i-1}(1 \pm u)^{6+n_{Q_z}}] \\
 &\subset [N_{i-1}r_i(1 \pm u)^4 + Q_z N_{i-1}(1 \pm u)^{6+n_{Q_z}} - p_i Q_z N_{i-1}(1 \pm u)^{7+n_{Q_z}}] \\
 &\subset [N_{i-1}r_i(1 \pm \gamma_4) + Q_z N_{i-1}(1 \pm \gamma_{6+n_{Q_z}}) - p_i Q_z N_{i-1}(1 \pm \gamma_{7+n_{Q_z}})] \\
 &= [N_{i-1}r_i + (1 - p_i)Q_z N_{i-1}] + [\pm N_{i-1}r_i \gamma_4 \pm Q_z N_{i-1} \gamma_{6+n_{Q_z}} \pm p_i Q_z N_{i-1} \gamma_{7+n_{Q_z}}]
 \end{aligned} \tag{B.13}$$

Thus, the  $\Delta_1$  error expression is:

$$err_{\Delta_1} = |N_{i-1}r_i|\gamma_4 + |Q_z N_{i-1}|\gamma_{6+n_{Q_z}} + |p_i Q_z N_{i-1}|\gamma_{7+n_{Q_z}} \tag{B.14}$$

The  $\Delta_2$  error expression is derived as follows:

$$\begin{aligned}
 fl(\Delta_2) &= [N_{i-1}r_i + (1 - p_i)Q_z N_{i-1}]^2 \\
 &= N_{i-1}^2 r_i^2 + 2N_{i-1}r_i Q_z N_{i-1} - 2p_i N_{i-1}r_i Q_z N_{i-1} + Q_z^2 N_{i-1}^2 - 2p_i Q_z^2 N_{i-1}^2 + p_i^2 Q_z^2 N_{i-1}^2 \\
 &\subset N_{i-1}^2 r_i^2 (1 \pm u)^{4+3} + 2N_{i-1}r_i Q_z N_{i-1} (1 \pm u)^{4+4} (1 \pm \delta_{Q_z}) \\
 &\quad - 2p_i N_{i-1}r_i Q_z N_{i-1} (1 \pm u)^{5+5} (1 \pm \delta_{Q_z}) + Q_z^2 N_{i-1}^2 (1 \pm u)^{4+3} (1 \pm \delta_{Q_z})^2 \\
 &\quad - 2p_i Q_z^2 N_{i-1}^2 (1 \pm u)^{5+5} (1 \pm \delta_{Q_z})^2 + p_i^2 Q_z^2 N_{i-1}^2 (1 \pm u)^{6+5} (1 \pm \delta_{Q_z})^2 \\
 &\subset N_{i-1}^2 r_i^2 (1 \pm u)^{7+5} + 2N_{i-1}r_i Q_z N_{i-1} (1 \pm u)^{8+5+n_{Q_z}} - 2p_i N_{i-1}r_i Q_z N_{i-1} (1 \pm u)^{10+4+n_{Q_z}} + Q_z^2 N_{i-1}^2 (1 \pm u)^{7+3+2n_{Q_z}} \\
 &\quad - 2p_i Q_z^2 N_{i-1}^2 (1 \pm u)^{10+2+2n_{Q_z}} + p_i^2 Q_z^2 N_{i-1}^2 (1 \pm u)^{11+1+2n_{Q_z}} \\
 &\subset N_{i-1}^2 r_i^2 (1 \pm \gamma_{12}) + 2N_{i-1}r_i Q_z N_{i-1} (1 \pm \gamma_{13+n_{Q_z}}) - 2p_i N_{i-1}r_i Q_z N_{i-1} (1 \pm \gamma_{14+n_{Q_z}}) + Q_z^2 N_{i-1}^2 (1 \pm \gamma_{10+2n_{Q_z}}) \\
 &\quad - 2p_i Q_z^2 N_{i-1}^2 (1 \pm \gamma_{12+2n_{Q_z}}) + p_i^2 Q_z^2 N_{i-1}^2 (1 \pm \gamma_{12+2n_{Q_z}}) \\
 &= [N_{i-1}r_i + (1 - p_i)Q_z N_{i-1}]^2 + [\pm N_{i-1}^2 r_i^2 \gamma_{12} \pm 2N_{i-1}r_i Q_z N_{i-1} \gamma_{13+n_{Q_z}} \pm 2p_i N_{i-1}r_i Q_z N_{i-1} \gamma_{14+n_{Q_z}} \\
 &\quad \pm Q_z^2 N_{i-1}^2 \gamma_{10+2n_{Q_z}} \pm 2p_i Q_z^2 N_{i-1}^2 \gamma_{12+2n_{Q_z}} \pm p_i^2 Q_z^2 N_{i-1}^2 \gamma_{12+2n_{Q_z}}]
 \end{aligned} \tag{B.15}$$

Therefore, the  $\Delta_2$  error expression is:

$$\begin{aligned}
 err_{\Delta_2} &= N_{i-1}^2 r_i^2 \gamma_{12} + 2|N_{i-1}r_i Q_z N_{i-1}|\gamma_{13+n_{Q_z}} + 2|p_i N_{i-1}r_i Q_z N_{i-1}|\gamma_{14+n_{Q_z}} \\
 &\quad + Q_z^2 N_{i-1}^2 \gamma_{10+2n_{Q_z}} + 2|p_i Q_z^2 N_{i-1}^2 \gamma_{12+2n_{Q_z}} + p_i^2 Q_z^2 N_{i-1}^2 \gamma_{12+2n_{Q_z}}
 \end{aligned} \tag{B.16}$$

The  $\Delta_3$  error expression is:

$$\begin{aligned}
 fl(\Delta_3) &= [1 - (1 - p_i)N_{i-1}^2][Q_i^2 - 2r_i Q_z - (1 - p_i)Q_z^2] \\
 &\subset Q_i^2(1 \pm u)^3(1 \pm \delta_{Q_i})^2 - 2r_i Q_z(1 \pm u)^4(1 \pm \delta_{Q_z}) - Q_z(1 \pm u)(1 \pm \delta_{Q_z}) \\
 &\quad + p_i Q_z^2(1 \pm u)^5(1 \pm \delta_{Q_z})^2 - N_{i-1}^2 Q_i^2(1 \pm u)^{4+3}(1 \pm \delta_{Q_i})^2 \\
 &\quad + 2r_i N_{i-1}^2 Q_z(1 \pm u)^{4+4}(1 \pm \delta_{Q_z}) + N_{i-1}^2 Q_z^2(1 \pm u)^{4+3}(1 \pm \delta_{Q_z})^2 \\
 &\quad - p_i N_{i-1}^2 Q_z^2(1 \pm u)^{5+4}(1 \pm \delta_{Q_z})^2 + p_i N_{i-1}^2 Q_i^2(1 \pm u)^{5+4}(1 \pm \delta_{Q_i})^2 \\
 &\quad - 2p_i r_i N_{i-1}^2 Q_z(1 \pm u)^{5+5}(1 \pm \delta_{Q_z}) - p_i N_{i-1}^2 Q_z(1 \pm u)^{5+4}(1 \pm \delta_{Q_z})^2 + p_i^2 N_{i-1}^2 Q_z(1 \pm u)^{6+5}(1 \pm \delta_{Q_z})^2 \\
 &\subset Q_i^2(1 \pm u)^{3+2n_{Q_i}} - 2r_i Q_z(1 \pm u)^{4+n_{Q_z}} - Q_z(1 \pm u)^{1+n_{Q_z}} + p_i Q_z^2(1 \pm u)^{5+2n_{Q_i}} - N_{i-1}^2 Q_i^2(1 \pm u)^{7+2n_{Q_i}} \\
 &\quad + 2r_i N_{i-1}^2 Q_z(1 \pm u)^{8+n_{Q_z}} + N_{i-1}^2 Q_z^2(1 \pm u)^{7+2n_{Q_i}} - p_i N_{i-1}^2 Q_z^2(1 \pm u)^{9+2n_{Q_z}} + p_i N_{i-1}^2 Q_i^2(1 \pm u)^{9+2n_{Q_i}} \\
 &\quad - 2p_i r_i N_{i-1}^2 Q_z(1 \pm u)^{10+n_{Q_z}} - p_i N_{i-1}^2 Q_z(1 \pm u)^{9+2n_{Q_z}} + p_i^2 N_{i-1}^2 Q_z(1 \pm u)^{11+2n_{Q_z}}
 \end{aligned}$$

$$\begin{aligned}
& \subset Q_i^2(1 \pm \gamma_{3+2n_{Q_i}}) - 2r_i Q_z(1 \pm \gamma_{4+n_{Q_z}}) - Q_z(1 \pm \gamma_{1+n_{Q_z}}) + p_i Q_z^2(1 \pm \gamma_{5+2n_{Q_i}}) \\
& - N_{i-1}^2 Q_i^2(1 \pm \gamma_{7+2n_{Q_i}}) + 2r_i N_{i-1}^2 Q_z(1 \pm \gamma_{8+n_{Q_z}}) + N_{i-1}^2 Q_z^2(1 \pm \gamma_{7+2n_{Q_i}}) - p_i N_{i-1}^2 Q_z^2(1 \pm \gamma_{9+2n_{Q_z}}) \\
& + p_i N_{i-1}^2 Q_i^2(1 \pm \gamma_{9+2n_{Q_i}}) - 2p_i r_i N_{i-1}^2 Q_z(1 \pm \gamma_{10+n_{Q_z}}) - p_i N_{i-1}^2 Q_z(1 \pm \gamma_{9+2n_{Q_z}}) + p_i^2 N_{i-1}^2 Q_z(1 \pm \gamma_{11+2n_{Q_z}}) \\
& = [1 - (1 - p_i)N_{i-1}^2][Q_i^2 - 2r_i Q_z - (1 - p_i)Q_z^2] + [\pm Q_i^2 \gamma_{3+2n_{Q_i}} \pm 2r_i Q_z \gamma_{4+n_{Q_z}} \pm Q_z \gamma_{1+n_{Q_z}} \pm p_i Q_z^2 \gamma_{5+2n_{Q_i}} \\
& \pm N_{i-1}^2 Q_i^2 \gamma_{7+2n_{Q_i}} \pm 2r_i N_{i-1}^2 Q_z \gamma_{8+n_{Q_z}} \pm N_{i-1}^2 Q_z^2 \gamma_{7+2n_{Q_i}} \pm p_i N_{i-1}^2 Q_z^2 \gamma_{9+2n_{Q_z}} \pm p_i N_{i-1}^2 Q_i^2 \gamma_{9+2n_{Q_i}} \pm 2p_i r_i N_{i-1}^2 Q_z \gamma_{10+n_{Q_z}} \\
& \pm p_i N_{i-1}^2 Q_z \gamma_{9+2n_{Q_z}} \pm p_i^2 N_{i-1}^2 Q_z \gamma_{11+2n_{Q_z}}] \quad (B.17)
\end{aligned}$$

Therefore, the  $\Delta_3$  error expression is:

$$\begin{aligned}
err_{\Delta_3} &= Q_i^2 \gamma_{3+2n_{Q_i}} + 2|r_i Q_z| \gamma_{4+n_{Q_z}} + |Q_z| \gamma_{1+n_{Q_z}} + |p_i| Q_z^2 \gamma_{5+2n_{Q_i}} + N_{i-1}^2 Q_i^2 \gamma_{7+2n_{Q_i}} + 2|r_i N_{i-1}^2 Q_z| \gamma_{8+n_{Q_z}} + N_{i-1}^2 Q_z^2 \gamma_{7+2n_{Q_i}} \\
&+ |p_i| N_{i-1}^2 Q_z^2 \gamma_{9+2n_{Q_z}} + |p_i| N_{i-1}^2 Q_i^2 \gamma_{9+2n_{Q_i}} + 2|p_i r_i N_{i-1}^2 Q_z| \gamma_{10+n_{Q_z}} + |p_i N_{i-1}^2 Q_z| \gamma_{9+2n_{Q_z}} + p_i^2 N_{i-1}^2 |Q_z| \gamma_{11+2n_{Q_z}} \quad (B.18)
\end{aligned}$$

The  $\Delta_4$  error expression is:

$$\begin{aligned}
fl(\Delta_4) &= 1 - (1 - p_i)N_{i-1}^2 \\
&\subset 1 - N_{i-1}^2(1 \pm u)^{2+1} - p_i N_{i-1}^2(1 \pm u)^{3+2} \\
&\subset (1 \pm u)^2 - N_{i-1}^2(1 \pm u)^5 - p_i N_{i-1}^2(1 \pm u)^6 \\
&\subset 1 \pm \gamma_2 - N_{i-1}^2(1 \pm \gamma_5) - p_i N_{i-1}^2(1 \pm \gamma_6) \\
&= 1 - (1 - p_i)N_{i-1}^2 + [\pm \gamma_2 \pm N_{i-1}^2 \gamma_5 \pm p_i N_{i-1}^2 \gamma_6] \quad (B.19)
\end{aligned}$$

Similarly, the  $\Delta_4$  error threshold is:

$$err_{\Delta_4} = \gamma_2 + N_{i-1}^2 \gamma_5 + |p_i| N_{i-1}^2 \gamma_6 \quad (B.20)$$

#### Appendix B.1.5. Floating-Point Error for Equation A.5

$f_{i-1}$  denotes the distance between the intersection points of rays between two adjacent surfaces, which is derived as follows:

$$\begin{aligned}
fl(t_{i-1}) &= f_{i-1} + f_i \\
&\subset f_{i-1}(1 \pm \delta_{f_{i-1}})(1 \pm u) + f_i(1 \pm \delta_{f_i})(1 \pm u) \\
&\subset f_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u) + f_i(1 \pm u)^{n_{f_i}}(1 \pm u) \\
&\subset [f_{i-1}(1 \pm u)^{n_{f_{i-1}}}(1 \pm u) + f_i(1 \pm u)^{n_{f_i}}(1 \pm u)](1 \pm u) \\
&\subset f_{i-1}(1 \pm u)^{2+n_{f_{i-1}}} + f_i(1 \pm u)^{2+n_{f_i}} \\
&\subset f_{i-1}(1 \pm \gamma_{2+n_{f_{i-1}}}) + f_i(1 \pm \gamma_{2+n_{f_i}}) \\
&\subset f_{i-1} + f_i + [\pm f_{i-1} \gamma_{2+n_{f_{i-1}}} \pm f_i \gamma_{2+n_{f_i}}] \quad (B.21)
\end{aligned}$$

Therefore, the upper and lower bounds of the error are given by:

$$err_{t_{i-1}} = |f_{i-1}| \gamma_{2+n_{f_{i-1}}} + |f_i| \gamma_{2+n_{f_i}} \quad (B.22)$$

#### Appendix B.1.6. Floating-Point Error for Equation A.6-A.8

$$\begin{aligned}
fl(x_i) &= x_{i-1} + t_{i-1} L_{i-1} \\
&\subset x_{i-1}(1 \pm u) + t_{i-1}(1 \pm \delta_{t_{i-1}})(1 \pm u) L_{i-1}(1 \pm u) \\
&\subset [x_{i-1}(1 \pm u) + t_{i-1}(1 \pm u)^{n_{t_{i-1}}}(1 \pm u) L_{i-1}(1 \pm u)(1 \pm u)](1 \pm u) \\
&\subset x_{i-1}(1 \pm u)^2 + t_{i-1} L_{i-1}(1 \pm u)^{4+n_{t_{i-1}}}
\end{aligned}$$

6

$$\begin{aligned}
&\subset x_{i-1}(1 \pm \gamma_2) + t_{i-1}L_{i-1}(1 \pm \gamma_{4+n_{i-1}}) \\
&= x_{i-1} + t_{i-1}L_{i-1} + [\pm x_{i-1}\gamma_2 \pm t_{i-1}L_{i-1}\gamma_{4+n_{i-1}}]
\end{aligned} \tag{B.23}$$

From Eq. (B.23), we know:

$$err_{x_i} = |x_{i-1}|\gamma_2 + |t_{i-1}L_{i-1}|\gamma_{4+n_{i-1}} \tag{B.24}$$

Similarly, the cumulative error of the intersection of the ray with the  $(i + 1)$ -th surface at  $y_i$  and  $z_i$  coordinates can be deduced and rewritten as follows:

$$err_{x_i} = |x_{i-1}|\gamma_2 + |t_{i-1}L_{i-1}|\gamma_{4+n_{i-1}} \tag{B.25}$$

$$err_{y_i} = |y_{i-1}|\gamma_2 + |t_{i-1}M_{i-1}|\gamma_{4+n_{i-1}} \tag{B.26}$$

$$err_{z_i} = |z_{i-1}|\gamma_3 + |t_{i-1}N_{i-1}|\gamma_{5+n_{i-1}} + |D_{i-1}|\gamma_2 \tag{B.27}$$

$$\tag{B.28}$$

## Appendix B.2. Refractive equation error model

### Appendix B.2.1. Floating-Point Error for Equation A.9

$$\begin{aligned}
fl(A) &= \sqrt{1 + c_i^2(1 - p_i)(y_i^2 + z_i^2)} \\
&\subset \sqrt{1 + c_i^2(1 \pm u)^3[1 - p_i(1 \pm u)][y_i^2(1 \pm \delta_{y_i})^2(1 \pm u)^3 + z_i^2(1 \pm \delta_{z_i})^2(1 \pm u)^3]} \\
&\subset \sqrt{1 + c_i^2(1 \pm u)^3[1 - p_i(1 \pm u)](1 \pm u)[y_i^2(1 \pm u)^{4+2n_{y_i}} + z_i^2(1 \pm u)^{4+2n_{z_i}}]} \\
&\subset \sqrt{1 + c_i^2[1 - p_i(1 \pm u)][y_i^2(1 \pm u)^{2n_{y_i}} + z_i^2(1 \pm u)^{2n_{z_i}}](1 \pm u)^8(1 \pm u)^2} \\
&\subset \sqrt{\{1 + c_i^2[1 - p_i(1 \pm u)][y_i^2(1 \pm u)^{2n_{y_i}} + z_i^2(1 \pm u)^{2n_{z_i}}](1 \pm u)^{10}\}(1 \pm u)} \\
&\subset \sqrt{(1 \pm u) + c_i^2(1 \pm u)^{11}[y_i^2(1 \pm u)^{2n_{y_i}} + z_i^2(1 \pm u)^{2n_{z_i}}][1 - p_i(1 \pm u)]} \\
&\subset \sqrt{1 + c_i^2(1 - p_i)(y_i^2 + z_i^2) + err_A}, \\
err_A &= [\pm \gamma_1 \pm c_i^2 y_i^2 \gamma_{11+2n_{y_i}} + c_i^2 z_i^2 \gamma_{11+2n_{z_i}} \pm p_i c_i^2 y_i^2 \gamma_{12+2n_{y_i}} \pm p_i c_i^2 z_i^2 \gamma_{12+2n_{z_i}}]
\end{aligned} \tag{B.29}$$

Therefore, the A error threshold can be written as:

$$err_A = \gamma_1 + c_i^2 y_i^2 \gamma_{11+2n_{y_i}} + c_i^2 z_i^2 \gamma_{11+2n_{z_i}} + |p_i| c_i^2 y_i^2 \gamma_{12+2n_{y_i}} + |p_i| c_i^2 z_i^2 \gamma_{12+2n_{z_i}} \tag{B.30}$$

### Appendix B.2.2. Floating-Point Error for Equation A.10

$$\begin{aligned}
fl(\cos \theta) &= \left| \frac{(1 - z_i p_i c_i) N_{i-1} - x_i c_i L_{i-1} - y_i c_i M_{i-1}}{A} \right| \\
&\subset \left| \frac{[1 - z_i(1 \pm \delta_{z_i})(1 \pm u)p_i(1 \pm u)c_i(1 \pm u)]N_{i-1}(1 \pm u)}{A(1 \pm \delta_A)(1 \pm u)} - \frac{x_i(1 \pm \delta_{x_i})(1 \pm u)c_i(1 \pm u)L_{i-1}(1 \pm u)}{A(1 \pm \delta_A)(1 \pm u)} \right. \\
&\quad \left. - \frac{y_i(1 \pm \delta_{y_i})(1 \pm u)c_i(1 \pm u)M_{i-1}(1 \pm u)}{A(1 \pm \delta_A)(1 \pm u)} \right| \\
&\subset \left| \frac{[1 - z_i(1 \pm \delta_{z_i})p_i c_i(1 \pm u)^3]N_{i-1}(1 \pm u)}{A(1 \pm \delta_A)(1 \pm u)} - \frac{x_i(1 \pm \delta_{x_i})c_i L_{i-1}(1 \pm u)^3}{A(1 \pm \delta_A)(1 \pm u)} - \frac{y_i(1 \pm \delta_{y_i})c_i M_{i-1}(1 \pm u)^3}{A(1 \pm \delta_A)(1 \pm u)} \right| \\
&\subset \left| \frac{[1 - z_i(1 \pm \delta_{z_i})p_i c_i(1 \pm u)^3(1 \pm u)^2]N_{i-1}(1 \pm u)(1 \pm u)^2}{A(1 \pm \delta_A)(1 \pm u)} - \frac{x_i(1 \pm \delta_{x_i})c_i L_{i-1}(1 \pm u)^3(1 \pm u)^2}{A(1 \pm \delta_A)(1 \pm u)} \right| (1 \pm u)
\end{aligned}$$

$$\begin{aligned}
& - \frac{y_i(1 \pm \delta_{y_i})c_i M_{i-1}(1 \pm u)^3(1 \pm u)^2}{A(1 \pm \delta_A)(1 \pm u)}(1 \pm u) \Big| \\
& \subset \left| \frac{[1 - z_i(1 \pm u)^{n_{z_i}} p_i c_i(1 \pm u)^3(1 \pm u)^2] N_{i-1}(1 \pm u)^4}{A(1 \pm u)^{n_A}(1 \pm u)} - \frac{x_i(1 \pm u)^{n_{x_i}} c_i L_{i-1}(1 \pm u)^6}{A(1 \pm u)^{n_A}(1 \pm u)} - \frac{y_i(1 \pm u)^{n_{y_i}} c_i M_{i-1}(1 \pm u)^6}{A(1 \pm u)^{n_A}(1 \pm u)} \right| \\
& \subset \left| \left[ \frac{[1 - z_i p_i c_i(1 \pm u)^{5+n_{z_i}}] N_{i-1}(1 \pm u)^4}{A(1 \pm u)^{1+n_A}} - \frac{x_i c_i L_{i-1}(1 \pm u)^{6+n_{x_i}}}{A(1 \pm u)^{1+n_A}} \right] (1 \pm u) - \frac{y_i c_i M_{i-1}(1 \pm u)^{6+n_{y_i}}}{A(1 \pm u)^{1+n_A}} \right| (1 \pm u) \Big| \\
& \subset \left| \frac{[1 - z_i p_i c_i(1 \pm u)^{5+n_{z_i}}] N_{i-1}(1 \pm u)^6}{A(1 \pm u)^{1+n_A}} - \frac{x_i c_i L_{i-1}(1 \pm u)^{8+n_{x_i}}}{A(1 \pm u)^{1+n_A}} - \frac{y_i c_i M_{i-1}(1 \pm u)^{7+n_{y_i}}}{A(1 \pm u)^{1+n_A}} \right| \\
& \subset \left| \frac{N_{i-1}(1 \pm u)^6 - z_i p_i c_i N_{i-1}(1 \pm u)^{11+n_{z_i}}}{A(1 \pm u)^{1+n_A}} - \frac{x_i c_i L_{i-1}(1 \pm u)^{8+n_{x_i}}}{A(1 \pm u)^{1+n_A}} - \frac{y_i c_i M_{i-1}(1 \pm u)^{7+n_{y_i}}}{A(1 \pm u)^{1+n_A}} \right| \\
& \subset \frac{N_{i-1}}{A}(1 \pm \gamma_{7+n_A}) - \frac{z_i p_i c_i N_{i-1}}{A}(1 \pm \gamma_{12+n_A+n_{z_i}}) - \frac{x_i c_i L_{i-1}}{A}(1 \pm \gamma_{9+n_A+n_{x_i}}) - \frac{y_i c_i M_{i-1}}{A}(1 \pm \gamma_{8+n_A+n_{y_i}}) \\
& \subset \left| \frac{(1 - z_i p_i c_i) N_{i-1} - x_i c_i L_{i-1} - y_i c_i M_{i-1}}{A} + \left[ \pm \frac{N_{i-1}}{A} \gamma_{7+n_A} \right. \right. \\
& \quad \left. \left. \pm \frac{z_i p_i c_i N_{i-1}}{A} \gamma_{12+n_A+n_{z_i}} \right] \pm \frac{x_i c_i L_{i-1}}{A} \gamma_{9+n_A+n_{x_i}} \pm \frac{y_i c_i M_{i-1}}{A} \gamma_{8+n_A+n_{y_i}} \right| \Big| \tag{B.31}
\end{aligned}$$

From Eq. (B.31), the  $\cos \theta$  upper and lower bound error threshold expressions are:

$$err_{\cos \theta} = \left| \frac{N_{i-1}}{A} \right| \gamma_{7+n_A} + \left| \frac{z_i p_i c_i N_{i-1}}{A} \right| \gamma_{12+n_A+n_{z_i}} + \left| \frac{x_i c_i L_{i-1}}{A} \right| \gamma_{9+n_A+n_{x_i}} + \left| \frac{y_i c_i M_{i-1}}{A} \right| \gamma_{8+n_A+n_{y_i}} \tag{B.32}$$

### Appendix B.2.3. Floating-Point Error for Equation A.11

$$\begin{aligned}
f_l(\cos \theta') &= \sqrt{1 - \frac{n_{i-1}^2}{n_i^2}(1 - \cos^2 \theta)} \\
&\subset \sqrt{1 - \frac{n_{i-1}^2(1 \pm u)^3}{n_i^2(1 \pm u)^3} [1 - \cos^2 \theta(1 \pm \delta_{\cos \theta})^2(1 \pm u)^3]} \\
&\subset \sqrt{\left[ 1 - \frac{n_{i-1}^2(1 \pm u)^3}{n_i^2(1 \pm u)^3} \right] (1 \pm u) \left[ 1 - \cos^2 \theta(1 \pm \delta_{\cos \theta})^2(1 \pm u)^3 \right] (1 \pm u)^2} (1 \pm u) \\
&\subset (1 \pm u) \sqrt{(1 \pm u) - \frac{n_{i-1}^2(1 \pm u)^7}{n_i^2(1 \pm u)^3} + \frac{n_{i-1}^2 \cos^2 \theta(1 \pm \delta_{\cos \theta})^2(1 \pm u)^{10}}{n_i^2(1 \pm u)^3}} \\
&\subset \sqrt{(1 \pm u)^3 - \frac{n_{i-1}^2(1 \pm u)^9}{n_i^2(1 \pm u)^3} + \frac{n_{i-1}^2 \cos^2 \theta[(1 \pm u)^{n_{\cos \theta}}]^2(1 \pm u)^{12}}{n_i^2(1 \pm u)^3}} \\
&\subset \sqrt{1 \pm \gamma_3 - \frac{n_{i-1}^2}{n_i^2}(1 \pm \gamma_{12}) + \frac{n_{i-1}^2 \cos^2 \theta}{n_i^2}(1 \pm \gamma_{15+2n_{\cos \theta}})} \\
&\subset \sqrt{1 - \frac{n_{i-1}^2}{n_i^2}(1 - \cos^2 \theta) + \left[ \pm \gamma_3 \pm \frac{n_{i-1}^2}{n_i^2} \gamma_{12} \pm \frac{n_{i-1}^2 \cos^2 \theta}{n_i^2} \gamma_{15+2n_{\cos \theta}} \right]} \tag{B.33}
\end{aligned}$$

From the above equation, the threshold value for the upper and lower bounds of  $\cos \theta'$  error can be set as:

$$err_{\cos \theta'} = \gamma_3 + \frac{n_{i-1}^2}{n_i^2} \gamma_{12} + \frac{n_{i-1}^2}{n_i^2} \cos^2 \theta \gamma_{15+2n_{\cos \theta}} \tag{B.34}$$



## Appendix B.2.4. Floating-Point Error for Equation A.12

$$\begin{aligned}
fl(g_i) &= n_i \cos \theta' - n_{i-1} \cos \theta \\
&\subset n_i(1 \pm u) \cos \theta' (1 \pm \delta_{\cos \theta'}) (1 \pm u) - n_{i-1}(1 \pm u) \cos \theta (1 \pm \delta_{\cos \theta}) (1 \pm u) \\
&\subset n_i(1 \pm u) \cos \theta' (1 \pm u)^{n_{\cos \theta'}} (1 \pm u) - n_{i-1}(1 \pm u) \cos \theta (1 \pm u)^{n_{\cos \theta}} (1 \pm u) \\
&\subset [n_i \cos \theta' (1 \pm u)^{n_{\cos \theta'}} (1 \pm u)^3 - n_{i-1} \cos \theta (1 \pm u)^{n_{\cos \theta}} (1 \pm u)^3] (1 \pm u) \\
&\subset n_i \cos \theta' (1 \pm u)^{4+n_{\cos \theta'}} - n_{i-1} \cos \theta (1 \pm u)^{4+n_{\cos \theta}} \\
&\subset n_i \cos \theta' (1 \pm \gamma_{4+n_{\cos \theta'}}) - n_{i-1} \cos \theta (1 \pm \gamma_{4+n_{\cos \theta}}) \\
&\subset n_i \cos \theta' - n_{i-1} \cos \theta + [\pm n_i \cos \theta' \gamma_{4+n_{\cos \theta'}} \pm n_{i-1} \cos \theta \gamma_{4+n_{\cos \theta}}]
\end{aligned} \tag{B.35}$$

Therefore, the upper and lower error thresholds of  $g_i$  can be set in the following form:

$$err_{g_i} = |n_i \cos \theta'| \gamma_{4+n_{\cos \theta'}} + |n_{i-1} \cos \theta| \gamma_{4+n_{\cos \theta}} \tag{B.36}$$

## Appendix B.2.5. Floating-Point Error for Equation A.13-A.15

$$\begin{aligned}
L_i &= \frac{n_{i-1}}{n_i} L_{i-1} - \frac{g_i c_i x_i}{n_i A} \\
&\subset \frac{n_{i-1}(1 \pm u)}{n_i(1 \pm u)} L_{i-1}(1 \pm u) - \frac{g_i(1 \pm \delta_{g_i})(1 \pm u) c_i(1 \pm u) x_i(1 \pm \delta_{x_i})(1 \pm u)}{n_i(1 \pm u) A(1 \pm \delta_A)(1 \pm u)} \\
&\subset \frac{n_{i-1}(1 \pm u)}{n_i(1 \pm u)} L_{i-1}(1 \pm u)^3 - \frac{g_i(1 \pm \delta_{g_i})(1 \pm u) c_i(1 \pm u) x_i(1 \pm \delta_{x_i})(1 \pm u)}{n_i A(1 \pm \delta_A)(1 \pm u)^3} (1 \pm u)^3 \\
&\subset \left[ \frac{n_{i-1}}{n_i(1 \pm u)} L_{i-1}(1 \pm u)^4 - \frac{g_i(1 \pm \delta_{g_i}) c_i x_i(1 \pm \delta_{x_i})}{n_i A(1 \pm \delta_A)(1 \pm u)^3} (1 \pm u)^6 \right] (1 \pm u) \\
&\subset \frac{n_{i-1}}{n_i(1 \pm u)} L_{i-1}(1 \pm u)^5 - \frac{g_i(1 \pm \delta_{g_i}) c_i x_i(1 \pm \delta_{x_i})}{n_i A(1 \pm \delta_A)(1 \pm u)^3} (1 \pm u)^7 \\
&\subset \frac{n_{i-1}}{n_i(1 \pm u)} L_{i-1}(1 \pm u)^5 - \frac{g_i(1 \pm u)^{n_{g_i}} c_i x_i(1 \pm u)^{n_{x_i}}}{n_i A(1 \pm u)^{n_A} (1 \pm u)^3} (1 \pm u)^7 \\
&\subset \frac{n_{i-1}}{n_i} L_{i-1}(1 \pm \gamma_6) - \frac{g_i c_i x_i}{n_i A} (1 \pm \gamma_{10+n_{g_i}+n_A+n_{x_i}}) \\
&= \frac{n_{i-1}}{n_i} L_{i-1} - \frac{g_i c_i x_i}{n_i A} + \left[ \pm \frac{n_{i-1}}{n_i} L_{i-1} \gamma_6 \pm \frac{g_i c_i x_i}{n_i A} \gamma_{10+n_{g_i}+n_A+n_{x_i}} \right]
\end{aligned} \tag{B.37}$$

From the above equation, the upper and lower bounds for the computational error of the direction cosine  $L_i$  are:

$$err_{L_i} = \left| \frac{n_{i-1}}{n_i} L_{i-1} \right| \gamma_6 + \left| \frac{g_i c_i x_i}{n_i A} \right| \gamma_{10+n_{g_i}+n_A+n_{x_i}} \tag{B.38}$$

Therefore the error range for the calculation of the direction cosine  $L_i$  is:

$$L_i \subset \left[ \frac{n_{i-1}}{n_i} L_{i-1} - \frac{g_i c_i x_i}{n_i A} - err_{L_i}, \frac{n_{i-1}}{n_i} L_{i-1} - \frac{g_i c_i x_i}{n_i A} + err_{L_i} \right] \tag{B.39}$$

Referring to  $L_i$ , the  $M_i$  error expression can be written directly as:

$$\begin{aligned}
M_i &= \frac{n_{i-1}}{n_i} M_{i-1} - \frac{g_i c_i y_i}{n_i A} \\
&\subset \frac{n_{i-1}}{n_i} M_{i-1} - \frac{g_i c_i y_i}{n_i A} + \left[ \pm \frac{n_{i-1}}{n_i} M_{i-1} \gamma_6 \pm \frac{g_i c_i y_i}{n_i A} \gamma_{10+n_{g_i}+n_A+n_{y_i}} \right]
\end{aligned} \tag{B.40}$$

The upper and lower bounds for the computational error of the direction cosine  $M_i$  are:

$$err_{M_i} = \left| \frac{n_{i-1}}{n_i} M_{i-1} \right| \gamma_6 + \left| \frac{g_i c_i y_i}{n_i A} \right| \gamma_{10+n_{g_i}+n_A+n_{y_i}} \quad (B.41)$$

The computational error range of the direction cosine  $M_i$  is:

$$M_i \subset \left[ \frac{n_{i-1}}{n_i} M_{i-1} - \frac{g_i c_i y_i}{n_i A} - err_{M_i}, \frac{n_{i-1}}{n_i} M_{i-1} - \frac{g_i c_i y_i}{n_i A} + err_{M_i} \right] \quad (B.42)$$

Derive the error equation for  $N_i$  as follows:

$$\begin{aligned} N_i &= \frac{n_{i-1}}{n_i} N_{i-1} - \frac{g_i(p_i c_i z_i - 1)}{n_i A} \\ &\subset \frac{n_{i-1}(1 \pm u)}{n_i(1 \pm u)} N_{i-1}(1 \pm u) - \frac{g_i(1 \pm \delta_{g_i})(1 \pm u)[p_i(1 \pm u)c_i(1 \pm u)z_i(1 \pm \delta_{z_i})(1 \pm u) - 1]}{n_i(1 \pm u)A(1 \pm \delta_A)(1 \pm u)} \\ &\subset \frac{n_{i-1}(1 \pm u)}{n_i(1 \pm u)} N_{i-1}(1 \pm u) - \frac{g_i(1 \pm u)^{n_{g_i}}(1 \pm u)[p_i(1 \pm u)c_i(1 \pm u)z_i(1 \pm u)^{n_{z_i}}(1 \pm u) - 1]}{n_i(1 \pm u)A(1 \pm u)^{n_A}(1 \pm u)} \\ &\subset \frac{n_{i-1}}{n_i(1 \pm u)} N_{i-1}(1 \pm u)^2 - \frac{g_i(1 \pm u)^{1+n_{g_i}}[p_i c_i z_i(1 \pm u)^{3+n_{z_i}} - 1]}{n_i A(1 \pm u)^{2+n_A}} \\ &\subset \frac{n_{i-1}}{n_i(1 \pm u)} N_{i-1}(1 \pm u)^4 - \left[ \frac{[g_i(1 \pm u)^{1+n_{g_i}}[p_i c_i z_i(1 \pm u)^{5+n_{z_i}} - 1](1 \pm u)](1 \pm u)}{n_i A(1 \pm u)^{3+n_A}} \right] (1 \pm u) \\ &\subset \frac{n_{i-1}}{n_i(1 \pm u)} N_{i-1}(1 \pm u)^4 - \frac{g_i[p_i c_i z_i(1 \pm u)^{5+n_{z_i}} - 1]}{n_i A(1 \pm u)^{3+n_A}} (1 \pm u)^{4+n_{g_i}} \\ &\subset \frac{n_{i-1}}{n_i(1 \pm u)} N_{i-1}(1 \pm u)^5 - \frac{g_i[p_i c_i z_i(1 \pm u)^{5+n_{z_i}} - 1]}{n_i A(1 \pm u)^{3+n_A}} (1 \pm u)^{5+n_{g_i}} \\ &\subset \frac{n_{i-1}}{n_i(1 \pm u)} N_{i-1}(1 \pm u)^5 - \frac{g_i p_i c_i z_i(1 \pm u)^{10+n_{z_i}+n_{g_i}}}{n_i A(1 \pm u)^{3+n_A}} + \frac{g_i(1 \pm u)^{5+n_{g_i}}}{n_i A(1 \pm u)^{3+n_A}} \\ &\subset \frac{n_{i-1}}{n_i} N_{i-1}(1 \pm \gamma_6) - \frac{g_i p_i c_i z_i}{n_i A} (1 \pm \gamma_{13+n_{z_i}+n_{g_i}+n_A}) + \frac{g_i}{n_i A} (1 \pm \gamma_{8+n_{g_i}+n_A}) \\ &= \frac{n_{i-1}}{n_i} N_{i-1} - \frac{g_i(p_i c_i z_i - 1)}{n_i A} + \left[ \pm \frac{n_{i-1}}{n_i} N_{i-1} \gamma_6 \pm \frac{g_i p_i c_i z_i}{n_i A} \gamma_{13+n_{z_i}+n_{g_i}+n_A} \pm \frac{g_i}{n_i A} \gamma_{8+n_{g_i}+n_A} \right] \end{aligned} \quad (B.43)$$

The upper and lower bounds for the computational error of the direction cosine  $N_i$  are:

$$err_{N_i} = \left| \frac{n_{i-1}}{n_i} N_{i-1} \right| \gamma_6 + \left| \frac{g_i p_i c_i z_i}{n_i A} \right| \gamma_{13+n_{z_i}+n_{g_i}+n_A} + \left| \frac{g_i}{n_i A} \right| \gamma_{8+n_{g_i}+n_A} \quad (B.44)$$

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