CS 135 Spring 2022: Problem Set 6

Problem 1. (10 points) The factorial function $fac(n)$ over the natural numbers is defined as:

$$fac(0) = 1, \qquad \forall n > 0: fac(n) = n \times fac(n-1)$$

$fac(n)$ is also denoted by $n!$.

Prove by induction that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ for all positive integers n.

BASIS: $n = 1$. $1 \cdot 1! = 1 = (1+1)! - 1$.

I.H.: for arbitrary $k > 0$, $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$

I.S.: $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$ (Using the I.H.)

$= (k+1)!\,(1 + k + 1) - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$

Problem 2. (10 points) Prove by induction that

$$\forall n \geq 1: (A_1 - B) \cup (A_2 - B) \cup \ldots \cup (A_n - B) = (A_1 \cup A_2 \cup \ldots \cup A_n) - B$$

BASIS: $n = 1$, $(A_1 - B) = (A_1 - B)$, which is true.

$n = 2, (A_1 - B) \cup (A_2 - B) = (A_1 \cap \bar{B}) \cup (A_2 \cap \bar{B}) = (A_1 \cup A_2) - B$

I.H.: for arbitrary $k \geq 1, (A_1 - B) \cup (A_2 - B) \cup \ldots \cup (A_k - B) = (A_1 \cup A_2 \cup \ldots \cup A_k) - B$

I.S.: $(A_1 - B) \cup (A_2 - B) \cup \ldots \cup (A_{k+1} - B)$

$= \big((A_1 \cup A_2 \cup \ldots \cup A_k) - B\big) \cup (A_{k+1} - B)$ Using the I.H.

$= (A_1 \cup A_2 \cup \ldots \cup A_{k+1}) - B$ Using the Base Case $n = 2$.

Problem 3. (15 points) On his quiz Ben Bitdiddle must prove the claim that $\forall n \geq 0\, P(n)$, where $P(x)$ is a predicate over the natural numbers. Ben can establish that $P(0)$ is true, but he is unable to establish the inductive step $\forall k \geq 0\, (P(k) \Rightarrow P(k+1))$.

After some thinking, Ben is finally able to prove that $\forall k \geq 0\, (P(k) \Rightarrow P(k+2))$

a. Has Ben succeeded in proving the claim? Explain why or why not.

   No, Ben has established $P(n)$ is true for all even numbers only.

b. If not, and Ben remains unable to establish that $P(k) \Rightarrow P(k+1)$, what would you suggest Ben try to establish to complete his inductive proof?

   Ben should establish $P(1)$ is true as a second base case.

Problem 4. (15 points) Prove by induction the statement $\forall n \geq 5: 2^n > n^2$

BASIS: $n = 5$. $2^5 = 32 > 5^2 = 25$.

I.H.: for arbitrary $k \geq 5, 2^k > k^2$.

I.S.: First, note that $k(k-2) > 1$ when $k \geq 3$. From this it follows that $k^2 > 2k + 1, k \geq 3$.

Therefore, $2^{k+1} = 2^k + 2^k > k^2 + k^2$ (Using the I.H.)

$> k^2 + 2k + 1$ (from the observation above)

$= (k+1)^2$