CS 135 Spring 2022: Mid-Term Practice Problems.

Solution Sketches

Problem 1. Either prove that the following argument is valid or give a counterexample:

If Bob carries an umbrella, then either it is raining, or it is cloudy

<u>If it is cloudy then Bob does not carry an umbrella</u>

Therefore, if it is raining Bob carries an umbrella

B: Bob carries an umbrella

R: It is raining

C: It is cloudy

$$\begin{array}{l} B \to R \lor C \\ \underline{C \to \neg B \quad} \\ R \to B \end{array}$$

Counterexamples, $R = T, B = F, C = T$ and $R = T, B = F, C = T$

Problem 2. Either prove that the following argument is valid or give a counterexample:

$$\begin{array}{c} A \Rightarrow B \lor \neg D \\ \neg C \Rightarrow \neg B \land D \\ \underline{\quad D \Rightarrow A \lor C \quad} \\ \neg A \Rightarrow (D \Rightarrow B) \end{array}$$

Counterexample: A,B: F and C,D: T.

Problem 3. Prove (using logic or set identities) that for all sets A, B, C:

$$(B - A) \cup (C - A) \subseteq (B \cup C) - A$$

See solutions to PS 6.

Problem 4. Determine all amounts of postage that can be formed using only 4-cent and 11-cent stamps. Prove your answer using strong induction. Be sure to show the base case, the inductive hypothesis, and the inductive step.

First, note that we can make exact postage for:

4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 32, 33 … then by adding 4c we can all greater amounts.

Claim: $\forall n \geq 30$, we can make exact amounts.

Basis: $30 = 11 + 2 \cdot 4$, $31 = 11 + 5 \cdot 4$, $32 = 8 \cdot 4$, $33 = 3 \cdot 11$

I.H.: for arbitrary $k \geq 33$: all amounts 30, 31, … , k can be made exactly.

I.S.: Since $k - 3 \geq 30$, we know $k - 3$ can be made exactly. Adding a 4c stamp gives $k + 1$.

Problem 5. Consider the following Scheme function:

```
(define (mystery L)
  (if (null? L) 0
    (if (null? (cdr L)) 1
          (+ 1 (mystery (cddr L))))))
```

(a) What is (mystery (list 1 2 3 4))? What is (mystery (list 1 2 3 4 5))?
2, 3

(b) What is the function computed by mystery? Justify your answer with a proof.
$\left\lceil \frac{length(L)}{2} \right\rceil$, the ceiling function which is the smallest integer no smaller than $length(L)/2$.

(c) Now change the last line of mystery so the new definition becomes:

```
(define (mystery L)
  (if (null? L) 0
    (if (null? (cdr L)) 1
          (+ (mystery (cdr L)) (mystery (cddr L))))))
```

What function does mystery now compute? Justify your answer with a proof.
Mystery computes the fibonacci numbers. If the length of L is n, (mystery L) is the nth fibonacci number. The fibonacci numbers are defined as:

$$f(0) = 0, f(1) = 1, \forall n > 1: f(n) = f(n-1) + f(n-2).$$

Problem 6. Prove by induction the statement $\forall n \geq 0$: $1 + 2 + 2^2 + \cdots 2^n = 2^{n+1} - 1$

a. State and establish the base case. $n = 0. 1 = 2^{0+1} - 1$, which is true.
b. State the inductive hypothesis. For arbitrary $k \geq 0, 1 + 2 + 2^2 + \cdots 2^k = 2^{k+1} - 1$
c. Establish the inductive step.

$$\begin{aligned} 1 + 2 + 2^2 + \cdots 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Problem 7. Prove the following equality, for all $n \geq 1$:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

BASIS: $1^3 = 1^2$ is true.

I.H.: for arbitrary $k \geq 1, 1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^2$

I.S.: $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + \cdots + k)^2 + (k+1)^3$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= (k+1)^2 \left(\frac{k^2}{4} + k + 1\right)$$

$$= \frac{(k+1)^2}{4}(k^2 + 4k + 4)$$
$$= \frac{(k+1)^2}{4}(k+2)^2$$
$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$
$$= \big(1 + 2 + \cdots (k+1)\big)^2$$

Problem 8. Consider the following Scheme functions:

```
(define (func A)  (myfun A '() ))

(define (myfun A B)
        (if (null? A) B
                  (myfun (cdr A) (cons (car A ) B))))
```

a. What is the value returned by (func '(1 2 3))? (3 2 1)
b. What is the value returned by func on a list of length N? The reverse of the input list.
c. Prove, by induction on the length of list A, the correctness of your assertion in part b.
   CLAIM: The value returned by (myfun A B) is the reverse of A appended to B.
   BASIS: If the input list A is of length 0, then (myfun '() B) = B so the claim is true in this case.
   I.H.: If list A is of length k then (myfun A B) returns the reverse of A appended to B.
   I.S.: If list A is of length k+1, then A is not null and therefore (myfun A B)
   evaluates to (myfun (cdr A) (cons (car A) B).
   By the I.H. since the length of (cdr A) is k+1, the value returned is the reverse of (cdr A) appended to the list into which (car A) has been inserted at the head. But this is the same as appending the reverse of A to list B.

   Finally, (func A) = (myfun A '()) appends the reverse of A to the empty list, which is the reverse of A.