Algorithm Efficiency

CS284

Algorithm Efficiency and Big-O

- Getting a precise measure of the performance of an algorithm is difficult
- Big-O notation expresses the performance of an algorithm as a function of the number of items to be processed
- This permits algorithms to be compared for efficiency
- It does so independently of the underlying compiler
- We're going to provide an informal introduction, more in CS 385 Algorithms

Linear Growth Rate

Processing time increases in proportion to the number of inputs n

```
public static int f(int[] x, int target) {
  for(int i=0; i<x.length; i++) {
    if (x[i]==target)
      return i;
  }
  return -1; // target not found
}
```

Linear Growth Rate

Processing time increases in proportion to the number of inputs n

```
public static int f(int[] x, int target) {
  for(int i=0; i<x.length; i++) {
    if (x[i]==target)
      return i;
  }
  return -1; // target not found
}
```

- Let n be `x.length`
- Target not present $\Rightarrow$ for loop will execute n times
- Target present $\Rightarrow$ for loop will execute (on average) $(n+1)/2$ times
- Therefore, the total execution time is directly proportional to n
- This is described as a growth rate of order n or $\mathcal{O}(n)$

$n * m$ Growth Rate

Processing time can be dependent on two different inputs n and m

```
public static boolean g(int[] x, int[] y) {
  for(int i=0; i<x.length; i++) {
    if (f(y, x[i]) != -1)
      return false;
  }
  return true;
}
```

$n * m$ Growth Rate

Processing time can be dependent on two different inputs n and m

```
public static boolean g(int[] x, int[] y) {
  for(int i=0; i<x.length; i++) {
    if (f(y, x[i]) != -1)
      return false;
  }
  return true;
}
```

- The for loop will execute `x.length` times
- But it will call `search`, which will execute `y.length` times
- The total execution time is proportional to `(x.length * y.length)`
- The growth rate has an order of $n * m$ or $\mathcal{O}(n * m)$

Quadratic Growth Rate

Processing time proportional to square of number of inputs n

```
public static boolean h(int[] x) {
  for(int i=0; i<x.length; i++) {
    for(int j=0; j<x.length; j++) {
      if (i != j && x[i] == x[j])
        return false;
    }
  }
  return true;
}
```

Quadratic Growth Rate

Processing time proportional to square of number of inputs n

```
public static boolean h(int[] x) {
  for(int i=0; i<x.length; i++) {
    for(int j=0; j<x.length; j++) {
      if (i != j && x[i] == x[j])
        return false;
    }
  }
  return true;
}
```

- The for loop with `i` as index will execute `x.length` times
- The for loop with `j` as index will execute `x.length` times
- The total number of times the inner loop will execute is `(x.length)`^{2}
- The growth rate has an order of n^2 or $\mathcal{O}(n^2)$

Logarithmic Growth Rate

You must also examine the number of times a loop is executed

```
for(int i=1; i < x.length; i *= 2) {
   System.out.println(x[i]);
}
```

- The loop body will execute k times, with i having the following values:

$$1, 2, 4, 8, 16, ..., 2^k$$

until 2^k is greater or equal to `x.length`

- Lets deduce the value of k

$$\begin{aligned} & 2^{k-1} < x.length \leq 2^k \\ \Rightarrow \quad & k - 1 < \log_2(x.length) \leq k \quad (\text{since } \log_2 2^k \text{ is } k) \\ \Rightarrow \quad & k = \lceil \log_2(x.length) \rceil \end{aligned}$$

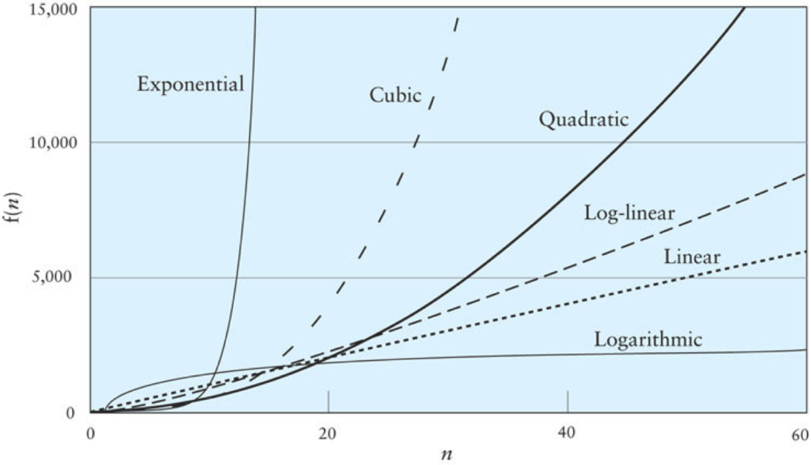
Logarithmic Growth Rate

You must also examine the number of times a loop is executed

```
for(int i=1; i < x.length; i *= 2) {
   System.out.println(x[i]);
}
```

- $k = \lceil \log_2(x.length) \rceil$
- Thus we say the loop is $\mathcal{O}(\log_2 n)$
- Logarithmic functions grow slowly as the number of data items n increases

Different Growth Rates



Growth Rate

Defining Big-O

Big-O Notation

- The $\mathcal{O}()$ in the previous examples can be thought of as an abbreviation of "order of magnitude"
  - $\mathcal{O}(f(n))$ is the set of functions that grow no faster than $f(n)$
- We can thus say that $f(n)$ is an upper bound on the growth rate
- We are next going to define $\mathcal{O}()$ more precisely

Formal Definition of Big-O

- Consider the two snippets of code below
- In order to compare their growth rates, why not just count the number of time units for each?

```
for (int i = 0; i < n; i++){
  for (int j = 0; j < 7; j++){
   System.out.println("Hello");
 }
}
for (int j = 0; j < 50; j++){
 System.out.println("Hello");
}
```

$\mathcal{T}_1(n) = 7n + 50$

```
for (int i = 0; i < n; i++){
 for (int j = 0; j < 100; j++){
   System.out.println("Hello");
 }
}
```

$\mathcal{T}_2(n) = 100n$

- For large values of n independent terms (such as 50) and constant coefficients (such as 7 and 100) are negligible
- Both are considered to have linear growth

Formal Definition of Big-O

$$\mathcal{O}(f(n)) = \{g(n) \mid \text{ there exist two positive constants, } n_0 \text{ and } c \text{ such that, } 0 \leq g(n) \leq c * f(n) \text{ for all } n > n_0\}$$

- $\mathcal{O}(f(n))$ is a set of functions
- It is the set of functions $g(n)$ s.t., as n gets sufficiently large (larger than n_0), there is some constant c for which the processing time will always be less than or equal to $c * f(n)$

Big-O Example 1

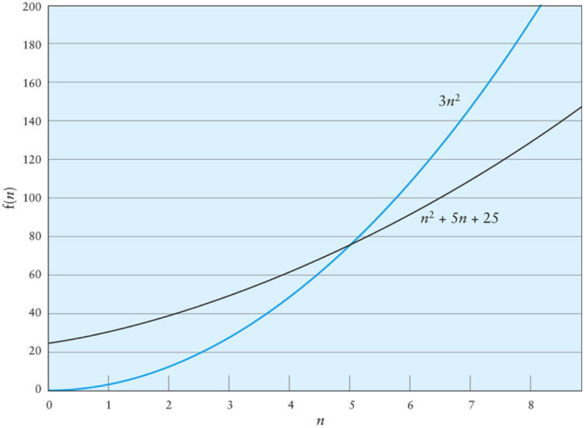
$$n^2 + 5n + 25 \in \mathcal{O}(n^2)$$

- Find constants n_0 and c so that, for all $n > n_0$, $cn^2 > n^2 + 5n + 25$

$$\begin{array}{rcl} cn^2 & > & n^2 + 5n + 25 \\ c & > & \frac{n^2}{n^2} + \frac{5n}{n^2} + \frac{25}{n^2} \\ c & > & 1 + \frac{5}{n} + \frac{25}{n^2} \end{array}$$

- When $n = n_0 = 5$, the RHS is $(1 + \frac{5}{5} + \frac{25}{25})$, c is 3
- Moreover, $\lim_{n \to \infty} 1 + \frac{5}{n} + \frac{25}{n^2} = 1$
- So, $4n^2 > n^2 + 5n + 25$, for all $n > 5$
- Other values of n_0 and c also work

Big-O Example 1



Big-O Example 2

- Consider the following loop

```
for (int i = 0; i < n; i++) {
  for (int j = i + 1; j < n; j++) {
      3 simple statements
  }
}
```

$$\mathcal{T}(n) = 3(n-1) + 3(n-2) + \ldots + 3$$

- Question:

$$\mathcal{T}(n) \in \mathcal{O}(n^2)?$$

Big-O Example 2

$$\mathcal{T}(n) = 3(n-1) + 3(n-2) + ... + 3$$

- Factoring out the 3,

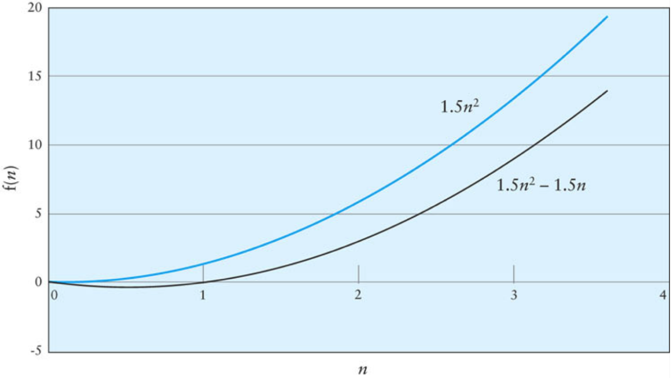
$$3(n - 1 + n - 2 + ... + 1)$$

- $1 + 2 + ... + n - 1 = \frac{(n*(n-1))}{2}$
- Therefore $\mathcal{T}(n) = 1.5n^2 - 1.5n$

$$\begin{aligned} cn^2 &> 1.5n^2 - 1.5n \\ c &> 1.5 - \frac{1.5}{n} \\ c &> 1.5 - \frac{1.5}{n_0}, n_0 > 1 \end{aligned}$$

- Therefore $\mathcal{T}(n) \in \mathcal{O}(n^2)$ when n_0 is 2 and c is 1.5

Big-O Example 2



Exercises

- Show that $\mathcal{T}(n) = n^3 - 5n^2 + 20n - 20 \in \mathcal{O}(n^3)$.
- Show that $\mathcal{T}(n) = 7n^4 + 5n^2 - 50n \in \mathcal{O}(n^4)$.

Symbols Used in Quantifying Performance

Symbol	Meaning
$T(n)$	The time that a method or program takes as a function of the number of inputs, n. We may not be able to measure or determine this exactly.
$f(n)$	Any function of n. Generally, $f(n)$ will represent a simpler function than $T(n)$, for example, n^2 rather than $1.5n^2 - 1.5n$.
$\mathbf{O}(f(n))$	Order of magnitude. $\mathbf{O}(f(n))$ is the set of functions that grow no faster than $f(n)$. We say that $T(n) = \mathbf{O}(f(n))$ to indicate that the growth of $T(n)$ is bounded by the growth of $f(n)$.

Common Growth Rates

Big-O	Name
$\mathcal{O}(1)$	Constant
$\mathcal{O}(\log n)$	Logarithmic
$\mathcal{O}(n)$	Linear
$\mathcal{O}(n \log n)$	Log-linear
$\mathcal{O}(n^2)$	Quadratic
$\mathcal{O}(n^3)$	Cubic
$\mathcal{O}(2^n)$	Exponential
$\mathcal{O}(n!)$	Factortial

Effects of Different Growth Rates

$O(f(n))$	$f(50)$	$f(100)$	$f(100)/f(50)$
$O(1)$	1	1	1
$O(\log n)$	5.64	6.64	1.18
$O(n)$	50	100	2
$O(n \log n)$	282	664	2.35
$O(n^2)$	2500	10,000	4
$O(n^3)$	12,500	100,000	8
$O(2^n)$	1.126×10^{15}	1.27×10^{30}	1.126×10^{15}
$O(n!)$	3.0×10^{64}	9.3×10^{157}	3.1×10^{93}

Algorithms with Exponential and Factorial Growth Rates

- Algorithms with exponential and factorial growth rates have an effective practical limit on the size of the problem they can be used to solve
- With an $\mathcal{O}(2^n)$ algorithm, if 100 inputs takes an hour then,
  - 101 inputs will take 2 hours
  - 105 inputs will take 32 hours
  - 114 inputs will take 16,384 hours (almost 2 years!)

Algorithms with Exponential and Factorial Growth Rates (cont.)

- Encryption algorithms take advantage of this characteristic
- Some cryptographic algorithms can be broken in $\mathcal{O}(2^n)$ time, where n is the number of bits in the key
- A key length of 40 is considered breakable by a modern computer, but a key length of 100 bits will take a billion-billion (1018) times longer than a key length of 40

Example: Two Sum

Given a sorted array of integers, return indices of the two numbers such that they add up to a specific target. Assuming there exists only one solution.

```
Given nums = [2, 7, 11, 15], target = 9,

Because nums[0] + nums[1] = 2 + 7 = 9,
return [0, 1].
```

What is your algorithm's time complexity using the big-O

Example: Two Sum

Exhaustive search:

```java
/** two sum that takes quadratic running time
 *
 * @param nums: a sorted increasing array
 * @param target: the target value for two sum
 * @return an int array which contains the indices
 * of the two numbers, e.g., [0, 1]
 */
public int[] twoSum_quadratic(int[] nums, int target) {

    for (int i = 0; i < nums.length; i ++)
        for (int j = i + 1; j < nums.length; j ++) {
            if (nums[i] + nums[j] == target) {
                return new int[]{i, j};
            }
        }
    return new int[]{};
}
```

Example: Two Sum

```
class TwoSumTest {

    @Test
    public void test() {
        TwoSum example = new TwoSum();

        int[] nums = {1, 2, 4, 8, 16, 32};
        int target = 12;

        int[] index = example.twoSum_quadratic(nums, target);

        Assert.assertArrayEquals(index, new int[] {2, 3});
    }

}
```

Improving the Time Complexity

Can we do better than quadratic?

Two Sum: Using Two Pointers

```java
public int[] twoSum_linear(int[] nums, int target) {
    int left = 0;
    int right = nums.length - 1;

    while(left < right) {
        int sum = nums[left] + nums[right];
        if (sum == target) {
            return new int[]{left, right};
        } else if (sum < target) {
            left++;
        } else {
            right--;
        }
    }
   return new int[] {};
}
```

What is the time complexity using two pointers?

Algorithmic Soundness vs. Completeness

- **Soundness**: Any answer returned by the algorithm is a correct answer;
- **Completeness**: If there exists at least one answer, the algorithm will be able to find that answer;

Algorithmic Soundness vs. Completeness

Proof that the two pointer algorithm for two sum is *complete*:

- If there exists two numbers that sums to the target value, the two pointer algorithm will *not* return the empty list;

```
int[] nums = {1, 2, 4, 8, 16, 32};
int target = 12;
```

	1	2	3	4	5
0	3	5	9	17	33
1		6	10	18	34
2			12	20	36
3				24	40
4					48

Algorithmic Soundness vs. Completeness

Observations:

- Moving left → the sum decreases;
- Moving down → the sum increases;
- The exists one unique *shortest path* that first moves left then moves down;

```java
int[] nums = {1, 2, 4, 8, 16, 32};
int target = 12;
```

	1	2	3	4	5
0	3	5	9	17	33
1		6	10	18	34
2			12	20	36
3				24	40
4					48

Algorithmic Soundness vs. Completeness

Observations:

- The exists one unique *shortest path* that first moves left then moves down;
- If we run the two pointers algorithm, it always follow the direction of the *shortest path*, i.e., it first moves left then moves down;

```java
int[] nums = {1, 2, 4, 8, 16, 32};
int target = 12;
```

	1	2	3	4	5
0	3	5	9	17	33
1		6	10	18	34
2			12	20	36
3				24	40
4					48

Algorithmic Soundness vs. Completeness

Observations:

- The exists one unique *shortest path* that first moves left then moves down;
- If we run the two pointers algorithm, it always follow the direction of the *shortest path*, i.e., it first moves left then moves down;
- As a result, the two pointer algorithm can guarantee to find a solution if it exists (i.e., completeness);

Discussions

- What happens if we do not start from $\{0, \text{nums.length - } 1\}$? Is the algorithm still sound? Is it still complete?
- Suppose the starting position for the two pointers can be anywhere, what is the longest path length?
- Suppose the starting position for the two pointers can be anywhere, by knowing the target value, can you shift the starting position to a different location to reduce the path length?