CS 135 Spring 2022: Problem Set 5

Problem 1. (10 points) Let R be a reflexive relation over a set A. Show that if the proposition $\forall x, y, z \in A\colon \big((x,y) \in R \wedge (x,z) \in R\big) \rightarrow (y,z) \in R$ is true, then R is an equivalence relation (Note that we are given that $\forall x \in A\colon (x,x) \in R.$)

(a) First, show that R is symmetric. Pick an ordered pair $(a,b) \in R$. In the quantified proposition, substitute $x = a, y = b, z = a$. What can you conclude?

$If\ \forall x, y\ (x \neq y) \rightarrow (x,y) \notin R$, then $R\ is$ symmetric (vacuously true).
$Else, let\ (a,b) \in R \wedge a \neq b.$
$Since\ (a,b) \in R\ \ and\ (a,a) \in$
$R, if\ we\ \ substitute\ b\ for\ y\ and\ a\ for\ x\ and\ a\ for\ z, we\ get:$
$(b,a) \in R$. Since a, b are arbitrary elements of R, it follows from the rule of universal generalization that R is symmetric.

(b) Next, show that R must also be transitive. (Hint: if $(a,b) \in R\ and\ (b,c) \in R$ use the fact that R is symmetric and use the quantified proposition that is given.)

$(a,b) \in R \rightarrow (b,a) \in R \quad (R\ is\ symmetric)$
$(b,a) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \ \ (substitute\ x = b, y = a, z = c).$
$Therefore, R\ is\ transitive\ (universal\ generalization).$

Problem 2. (10 points) If $f\colon S \rightarrow \mathbb{N}$ is injective and S is an infinite set, then describe how to construct a bijective function $g\colon \mathbb{N} \rightarrow S$. (Hint: Start with the range of the function f and use the well-ordering principle.)

Let $X = range(f)$
Define $g(0) = s_0 \in S : f(s_0) = \ \min X,$
Define the rest of $g\ over\ \mathbb{N} - \{0\}$ recursively, so that

$$g(1) = s_1 \in S\colon f(s_1) = \min(X - \{f(s_0)\}), \ldots$$

In general, $g(i) = s_i \in S\colon f(s_i) = \min(X - \{f(s_0), f(s_1), \ldots, f(s_{i-1})\})$

First, note that $g(\cdot)$ is a well-defined function, i.e., $g(x)$ is a unique element of S. Formally, $g(x)$ is defined $\forall x \in \mathbb{N}$ and if $(g(x) = a \wedge g(x) = b)$ then $a = b$. This follows directly by construction and from the fact that $f(\cdot)$ is injective.

Next, $g(\cdot)$ is injective, i.e., $x \neq y \rightarrow g(x) \neq g(y)$. This is true because if $x < y$ then $g(y)$ is chosen from a subset of S that excludes $g(x)$.

Finally, $g(\cdot)$ is surjective because every element in S will be chosen as the range of a number – after all the elements of S with smaller f –values have been assigned as the range of smaller numbers.

Problem 3. (10 points) A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is a monotonically non-decreasing function if $\forall x \forall y \;\; x \leq y \;\Rightarrow f(x) \leq f(y)$.

a) Prove that if f, g are both monotonically non-decreasing functions over $\mathbb{N}$ then $g \circ f$ is also monotonically non-decreasing.
   Let $x \leq y, so\ that\ f(x) \leq f(y).\ \ Since\ g\ is\ monotonically\ non-decreasing, it\ follows\ that\ g\big(f(x)\big) \leq g\big(f(y)\big)$.
b) Give a counterexample to show that the converse is not always true. That is, construct f, g over $\mathbb{N}$ such that $g \circ f$ is monotonically non-decreasing but either f or g is not.
   Define $f(0) = 1, f(x) = 0, \forall x > 0.$ $g(x) = 1 \; \forall x \geq 0.$
   Then $f$is not monotonically non-decreasing, but $g\big(f(x)\big) = 1 \; \forall x$ is.

Problem 4. (20 points) Let A, B, and C be sets, $f: A \rightarrow B\ and\ g: B \rightarrow C$, and $h: A \rightarrow C$ be the composition, $g\ o\ f$, that is, $h(x) = g(f(x))$ for $x \in A$. Prove, or give a counterexample, for each of the following claims:

a) If h is surjective, then g must be surjective.

True. Consider the contrapositive. If g is not surjective, then some element of C does not get mapped onto under$g$; therefore, it cannot get mapped onto under $g\ o\ f$.

b) If h is surjective, then f must be surjective.

False. Let $A = \{0\}, B = \{a, b\}, C = \{100\}, f(0) = a, g(a) = g(b) = 100$. In this case $g\ o\ f$ is surjective, but f is not.

c) If h is injective, then g must be injective.

False. $A = \{0,1\}, B = \{a, b, c\}, C = \{100,200\}, f(0) = a, f(1) = c, g(a) = g(b) = 100, g(c) = 200$. Then $g\ o\ f$ is injective, but g is not.

d) If h is injective, then f must be injective.

True. If f is not injective then h is not injective; the two elements of A that are mapped to the same element in B under f will be mapped to the same element of C under h.