

EGR 5110: Homework #2

Due on March 10, 2024 at 11:59pm

Professor Nissenson

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Derivation of $y'(fr)$

The Newton-Raphson method, a numerical root-solving method, requires determining an expression for f' and calculating it at each iteration. The Colebrook Equation is a non-linear algebraic equation that can be solved using the Newton-Raphson method to determine the friction factor for steady, incompressible, isothermal pipe flow with friction. It is defined below:

$$\frac{1}{\sqrt{fr}} = -2 \log_{10} \left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\text{Re}\sqrt{fr}} \right)$$

Moving all terms to the LHS:

$$\frac{1}{\sqrt{fr}} + 2 \log_{10} \left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\text{Re}\sqrt{fr}} \right) = 0$$

Thus,

$$y(fr) = 0$$

Now we just have to find the value of fr that make the value of the function y equal to zero.

The algorithm employed by Newton-Raphson method is a result of a Taylor-series approximation of the first order about an initial root estimate x_0 , and is summarized below:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

A consequence of this method is that the derivative of $f(x)$ is required at each step. Therefore, to solve the Colebrook Equation for the friction factor using the Newton-Raphson method, we must know $y'(fr)$:

$$\begin{aligned} y'(fr) &= \frac{d}{d(fr)} \left[\frac{1}{\sqrt{fr}} + \frac{2}{\ln(10)} \cdot \ln \left(\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right) \right] && \text{where differentiation is linear} \\ &= \frac{d}{d(fr)} \left[\frac{1}{\sqrt{fr}} \right] + \frac{2}{\ln(10)} \cdot \frac{d}{d(fr)} \left[\ln \left(\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right) \right] && \text{apply power rule \& chain rule} \\ &= \left(-\frac{1}{2} \right) fr^{(-1/2-1)} + \frac{2}{\ln(10)} \cdot \frac{\frac{d}{d(fr)} \left[\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right]}{\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}}} && \text{where differentiation is linear} \\ &= \frac{2 \left(\frac{251}{100\text{Re}} \cdot \frac{d}{d(fr)} \left[\frac{1}{\sqrt{fr}} \right] + \frac{d}{d(fr)} \left[\frac{10\epsilon}{37D} \right] \right)}{\ln(10) \left(\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right)} - \frac{1}{2fr^{3/2}} && \text{apply power rule} \\ &= \frac{2 \left(251 \left(-\frac{1}{2} \right) fr^{-1/2-1} \cdot \frac{1}{100\text{Re}} + 0 \right)}{\ln(10) \left(\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right)} - \frac{1}{2fr^{3/2}} && \text{simplify} \end{aligned}$$

Solution

$$y'(fr) = -\frac{251}{100 \ln(10) \text{Re} \cdot \left(\frac{10\epsilon}{37D} + \frac{251}{100\text{Re}\sqrt{fr}} \right) fr^{3/2}} - \frac{1}{2fr^{3/2}}$$

Pipe Network Problem

Initial guess

For the given pipe network below:

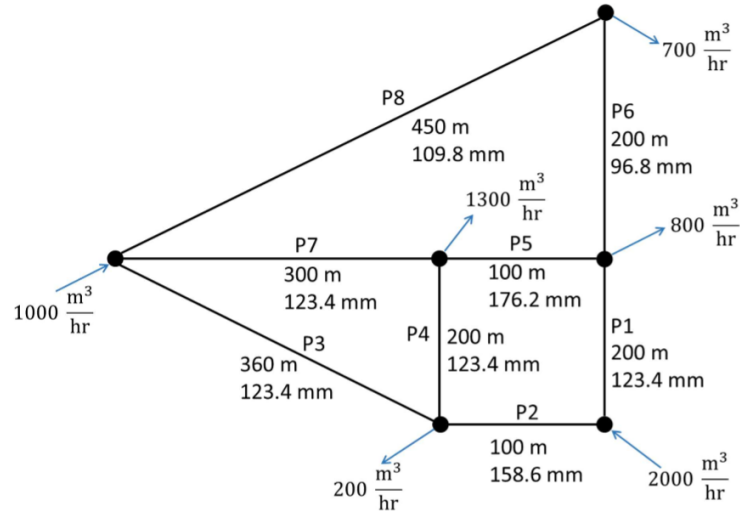


Figure 1: Pipe network with three loops

By applying conservation of mass through each node, we arrive at the initial guess for the direction and magnitude of the flowrates Q_j through each pipe j :

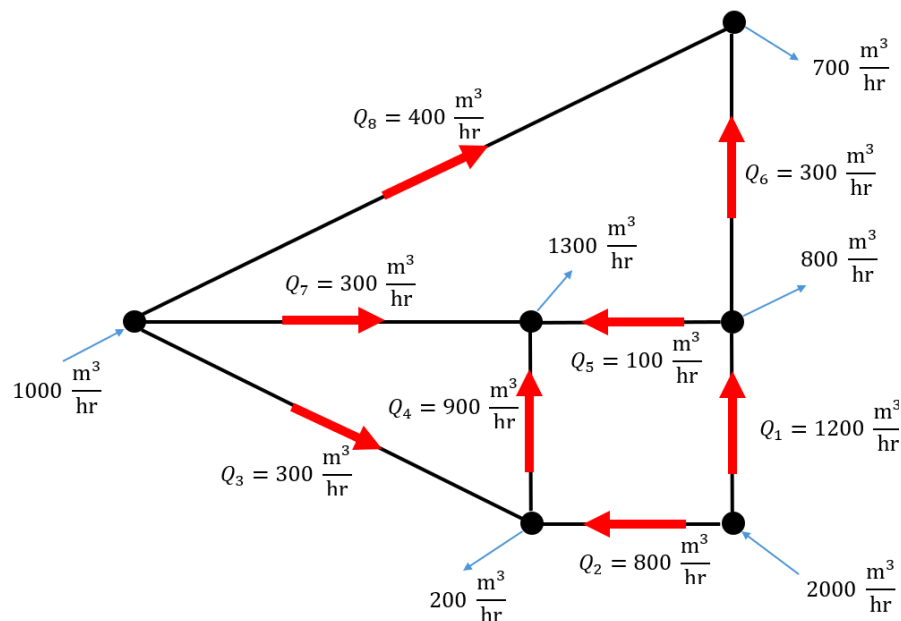


Figure 2: Updated pipe network

Loop and Node Equations

Labeling each loop and node:

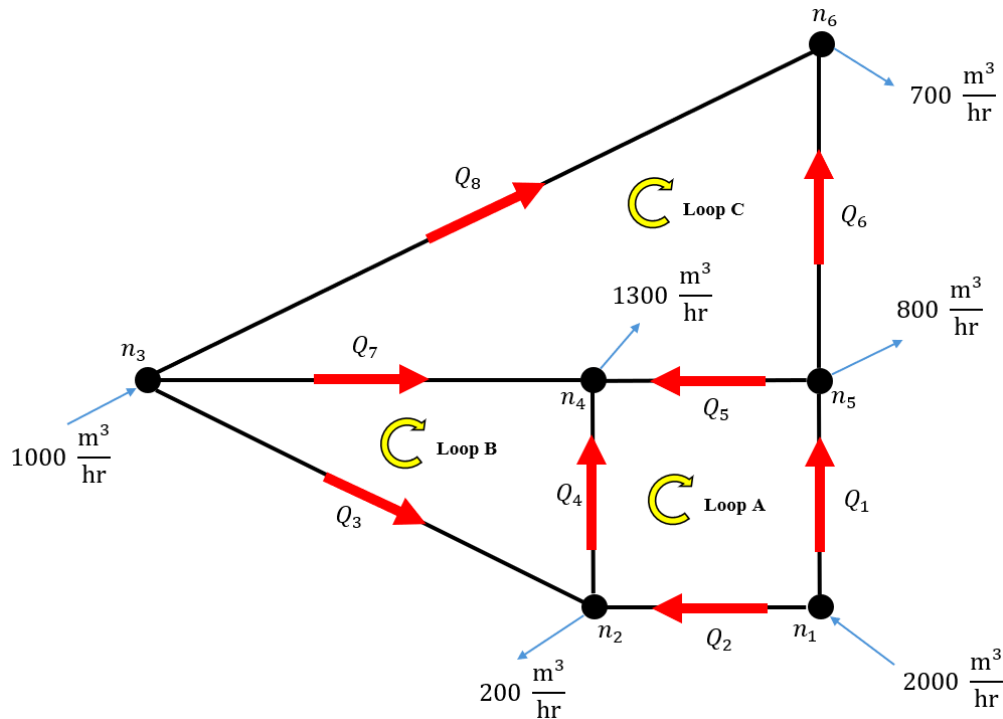


Figure 3: Updated network with loops and nodes labeled

Using Figure 3 and applying both mass conservation at each node and conservation of energy, we arrive at the 5 node and 3 loop equations, yielding 8 total:

Solution – (Note: Flowrates are in m^3/hr)

$$F_1 = Q_1 + Q_2 - 2000$$

$$F_2 = Q_3 + Q_7 + Q_8 - 1000$$

$$F_3 = -Q_4 - Q_5 - Q_7 + 1300$$

$$F_4 = -Q_1 + Q_5 + Q_6 + 800$$

$$F_5 = -Q_6 - Q_8 + 700$$

$$F_6 = -fr_1 \frac{8L_1}{\pi^2 g D_1^5} Q_1 |Q_1| + fr_2 \frac{8L_2}{\pi^2 g D_2^5} Q_2 |Q_2| + fr_4 \frac{8L_4}{\pi^2 g D_4^5} Q_4 |Q_4| - fr_5 \frac{8L_5}{\pi^2 g D_5^5} Q_5 |Q_5|$$

$$F_7 = -fr_4 \frac{8L_4}{\pi^2 g D_4^5} Q_4 |Q_4| - fr_3 \frac{8L_3}{\pi^2 g D_3^5} Q_3 |Q_3| + fr_7 \frac{8L_7}{\pi^2 g D_7^5} Q_7 |Q_7|$$

$$F_8 = fr_5 \frac{8L_5}{\pi^2 g D_5^5} Q_5 |Q_5| - fr_6 \frac{8L_6}{\pi^2 g D_6^5} Q_6 |Q_6| - fr_7 \frac{8L_7}{\pi^2 g D_7^5} Q_7 |Q_7| + fr_8 \frac{8L_8}{\pi^2 g D_8^5} Q_8 |Q_8|$$

The goal is to search for \mathbf{Q} – a column vector containing the flowrates in each branch – such that $\mathbf{F}(\mathbf{Q}) = 0$, where \mathbf{F} is a vector-valued function – containing the set of non-linear algebraic equations – evaluated at \mathbf{Q} . My MATLAB code `RootFinder_P2.m` uses this set of non-linear equations, and the other parameters given in Problem 2, to solve for the flowrates using the modified-secant method.