EGR 5110: Homework #3

Due on April 1, 2024 at 11:59pm

Professor Nissenson

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CR3BP Background

The three-body problem is a dynamics problem in which a relatively small object is influenced by two much more massive bodies, and the two massive bodies have circular orbits about their barycenter (combined center of mass). The following nonlinear second-order differential equations describe the motion of a spacecraft in orbit about the Earth and Moon:

$$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} + x - \frac{\tilde{\mu}\,(x+\mu)}{r_1^3} - \frac{\mu\,(x-\tilde{\mu})}{r_2^3} - f_d\frac{dx}{dt} \tag{1}$$

$$\frac{d^2y}{dt^2} = -2\frac{dx}{dt} + y - \frac{\tilde{\mu}y}{r_1^3} - \frac{\mu y}{r_2^3} - f_d\frac{dy}{dt} \tag{2}$$

where

$$\mu = \frac{1}{82.45} \qquad \tilde{\mu} = 1-\mu \qquad r_1^2 = (x+\mu)^2 + y^2 \qquad r_2^2 = (x-\tilde{\mu})^2 + y^2$$

μ is the ratio of the Moon's mass to Earth's mass, r_1 is the distance from the Earth's center to the spacecraft, r_2 is the distance from the Moon's center to the spacecraft, and f_d is a deceleration/acceleration coefficient.

The equations are derived from Newton's law of motion and the inverse square law of gravitation. We are taking the perspective of someone in a reference frame that is rotating at the same angular speed as the Earth and Moon about their barycenter, so the Earth and Moon appear fixed in this reference frame (only the spaceship will appear to move). It is assumed that the three bodies move in a 2D plane and the x-axis forms a line through the Earth and Moon. The origin is located at the barycenter, the Moon's center of mass is located at point (1-μ,0), and the Earth's center of mass is located at point (-μ,0).

The equation uses normalized units for space and time. A normalized spatial unit of 1 in this coordinate system is equal to the characteristic length of the system, which is the distance between the earth and moon (d = 384,400 km). One normalized time unit is equal to the characteristic time:

$$t = \sqrt{\frac{d^3}{G\,(m_{earth} + m_{moon})}}$$

where G = gravitational constant = $6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, $m_{earth} = 5.972 \times 10^{24}$ kg, and $m_{moon} = 7.348 \times 10^{22}$ kg. This means one normalized time unit is 3.75×10^5 s, or 4.34 days.

System of Coupled First-Order ODEs

The goal of this assignment is to build an RK4 method ODE solver that solves a set of coupled first-order ODEs from an initial time t_0 to a final time t_f.

Recall the first order ODE

$$\frac{dy}{dx} = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

Any higher order ODE, that can be solved for the highest derivative term, can be written as a system of first-order ODEs (One n^{th} order ODE $\rightarrow$ n 1st order ODEs). Applying this to Equation (1) and (2):

Let

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= \dot{x} \\ x_4 &= \dot{y} \end{aligned}$$

Thus,

$$\begin{aligned} \dot{x}_1 &= \dot{x} = x_3 \\ \dot{x}_3 &= \ddot{x} = 2\dot{y} + x - \frac{\tilde{\mu}\,(x+\mu)}{r_1^3} - \frac{\mu\,(x-\tilde{\mu})}{r_2^3} - f_d\dot{x} \\ &= 2x_4 + x_1 - \frac{\tilde{\mu}\,(x_1+\mu)}{r_1^3} - \frac{\mu\,(x_1-\tilde{\mu})}{r_2^3} - f_d x_3 \end{aligned}$$

$$\begin{aligned} \dot{x}_2 &= \dot{y} = x_4 \\ \dot{x}_4 &= \ddot{y} = -2\dot{x} + y - \frac{\tilde{\mu}y}{r_1^3} - \frac{\mu y}{r_2^3} - f_d\dot{y} \\ &= -2x_3 + x_2 - \frac{\tilde{\mu}x_2}{r_1^3} - \frac{\mu x_2}{r_2^3} - f_d x_4 \end{aligned}$$

Let $\mathbf{X}$ denote the column vector containing the states x_1, x_2, x_3, x_4. This means that

$$\boxed{\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \dot{\mathbf{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 + x_1 - \dfrac{\tilde{\mu}\,(x_1+\mu)}{r_1^3} - \dfrac{\mu\,(x_1-\tilde{\mu})}{r_2^3} - f_d x_3 \\ -2x_3 + x_2 - \dfrac{\tilde{\mu}\,x_2}{r_1^3} - \dfrac{\mu\,x_2}{r_2^3} - f_d x_4 \end{bmatrix}} \tag{3}$$

This is a system of coupled first-order ordinary differential equations that require a total of 4 initial condtions to solve. The benefit of writing the equations in this form is that two 2nd-order equations are reduced to four 1st-order equations that can be simultaneously solved for the spacecraft's states (position and velocity), and hence provide us with the time histories of the spacecraft's trajectory relative to the Earth and Moon in the rotating coordinate system.

The system $\dot{\mathbf{X}} = f(t, \mathbf{X})$ can now be solved using the RK4 algorithm for the position and velocity of the spacecraft at N discrete times from t_0 to t_f, when provided the vector comprising the components of the states at time t_0:

$$\mathbf{X}(0) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \Big|_{t=t_0} = \begin{bmatrix} x_0 & y_0 & \dot{x}_0 & \dot{y}_0 \end{bmatrix}^T$$

Simulations

It is possible that the number of steps input by the user (N) is too small to get accurate results in the RK4 alogrithm. So how do we determine the number of steps required to obtain an accurate solution?

One crude approach is to look at the global error. This means doubling the number of steps (cut the step size in half) until the relative difference in the final position falls below a tolerance tol. The termination criterion is:

$$\left| \frac{y_2(x_f) - y_1(x_f)}{y_2(x_f)} \right| < \text{tol}$$

where $y_1(x_f)$ is the solution at the final step for a certain Δx, and $y_2(x_f)$ is the solution at the final step for $\Delta x/2$.

I will explore two scenarios. For both scenarios, I will create two plots (described below) and include a table of N, dt, r_1, and the test criterion below:

$$\left| \frac{r_1(\text{end})_{2N} - r_1(\text{end})_N}{r_1(\text{end})_{2N}} \right|$$

where $r_1(\text{end})_N$ is the radial distance from the earth's center to the spaceship at the end of the simulation using a certain step size, and $r_1(\text{end})_{2N}$ is the radial distance from the earth's center to the spaceship at the end of the simulation using the smaller step size (one-half the step size used to calculate $r_1(\text{end})_N$).

We shall start with $N = 1000$ and use a tolerance of tol $= 0.01$.

Scenario 1

Initial Conditions:

- $x(0) = 1.2$
- $v_x(0) = 0$
- $v_y(0) = $ -1.0493571
- $y(0) = 0$
- $f_d = 0$

Note: All quantities are in normalized units.

These initial conditions correspond to a spacecraft initially on the far side of the moon. The trajectory will be calculated from $t = 0$ to 15 normalized time units.

The results are shown below:

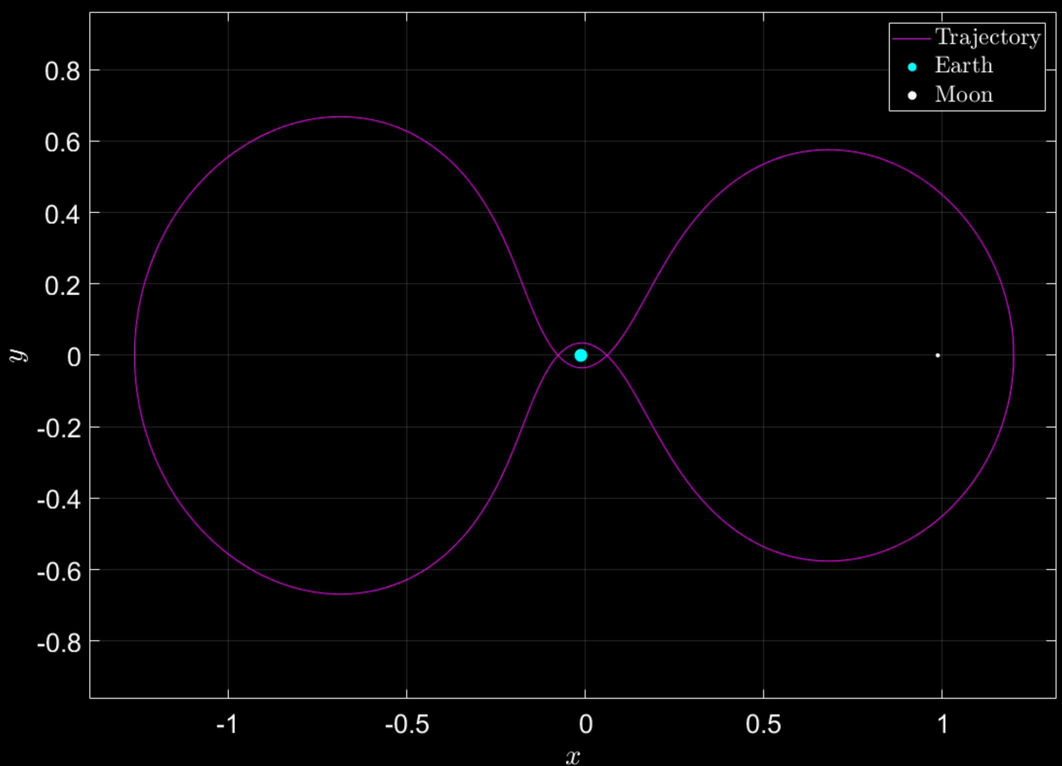


Figure 1: Spacecraft Trajectory about the Earth & Moon in the Synodic Frame

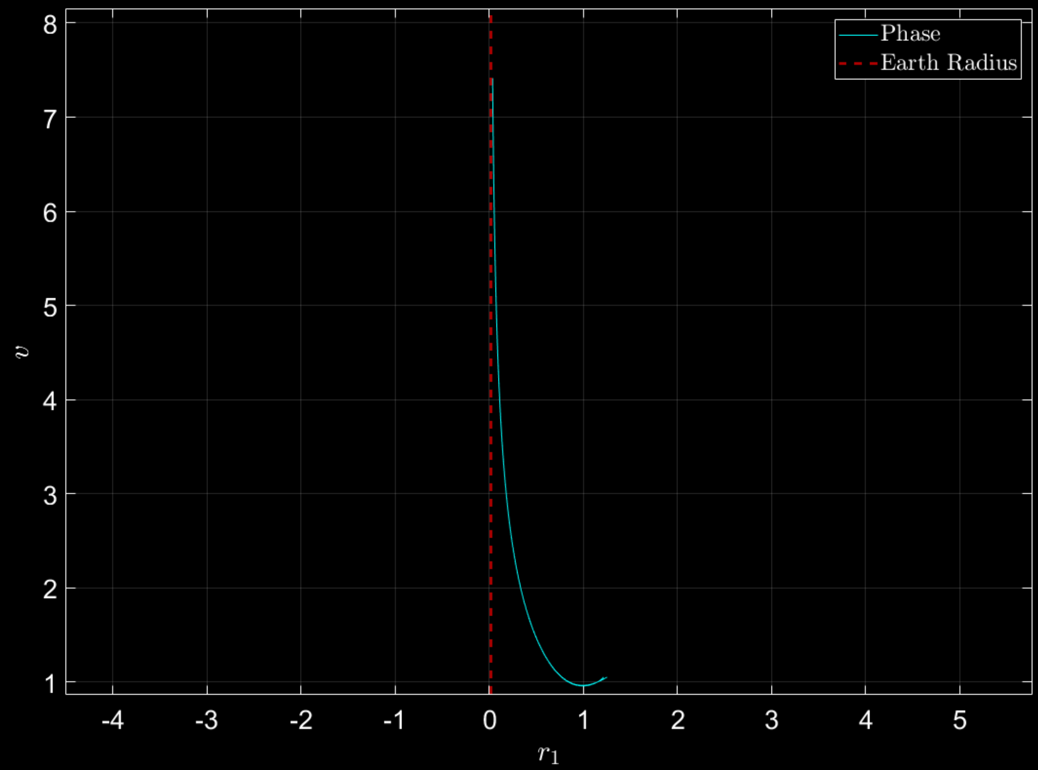


Figure 2: Phase Space, *Speed vs. Radial Distance from Earth*

Table 1: Effects of Increasing the Number of Steps for Scenario 1

N	dt	r_1	Test Criterion
1000	0.0150	14.7302	N/A
2000	0.0075	212.1128	0.9306
4000	0.0038	7.0135	29.2437
8000	0.0019	1.2056	4.8173
16000	9.3750×10^{-4}	1.1816	0.0203
32000	4.6875×10^{-4}	1.1808	$6.3125 \times 10^{-4} <$ tol

Upon completion of the simulation, the closest proximity to the Earth's surface was determined to be $r_1 =$ 6947.40 km.

Scenario 2

Initial Conditions:

- $x(0) = 1.2$
- $v_x(0) = 0$
- $v_y(0) =$ -1.0493571
- $y(0) = 0$
- $f_d = 1$

Note: All quantities are in normalized units.

These initial conditions are identical to the ones from the Scenario 1, but we set $f_d = 1$, which corresponds to rockets firing in the retrograde direction with constant thrust through the entire flight. The trajectory will be calculated from $t = 0$ to 4 normalized time units.

The results are shown below:

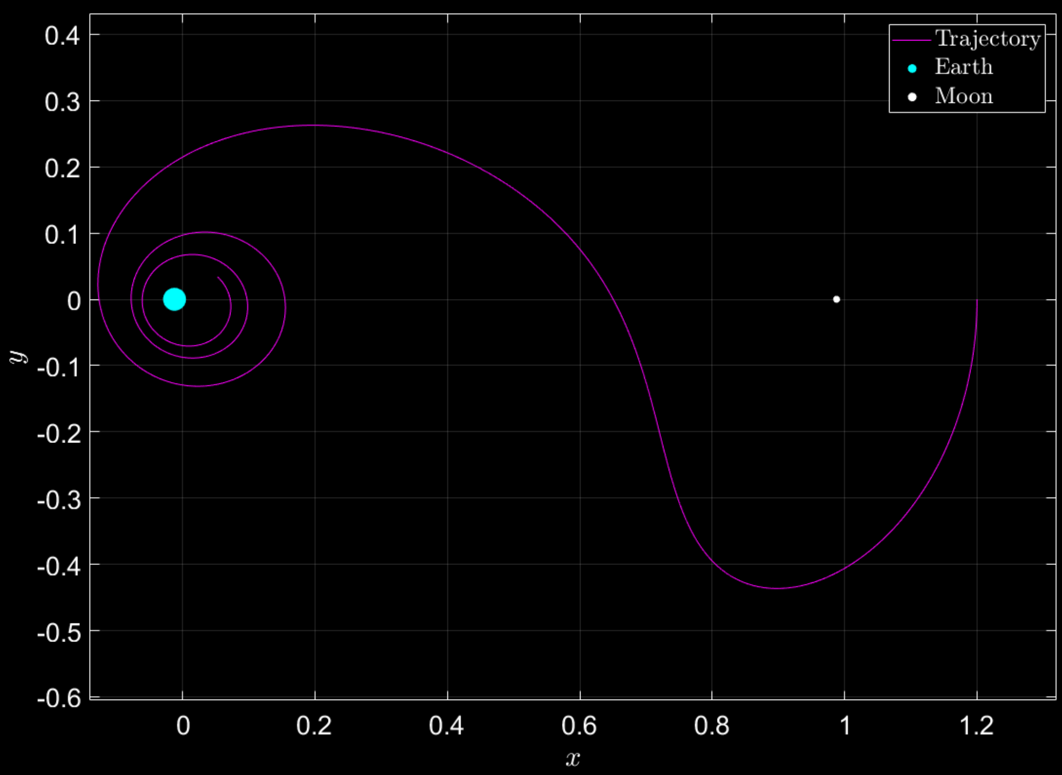


Figure 3: Spacecraft Trajectory about the Earth & Moon in the Synodic Frame

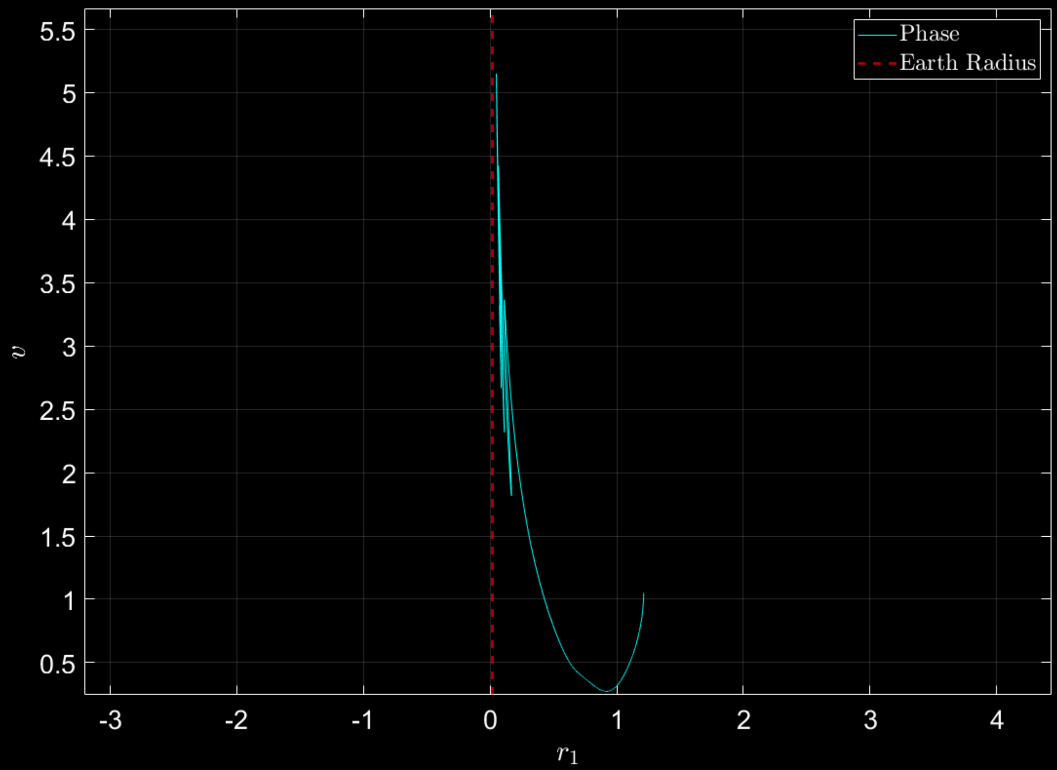


Figure 4: Phase Space, *Speed vs. Radial Distance from Earth*

Table 2: Effects of Increasing the Number of Steps for Scenario 2

N	dt	r_1	Test Criterion
1000	0.0040	0.0767	N/A
2000	0.0020	0.0748	0.0249
4000	0.0010	0.0738	0.0144
8000	0.0005	0.0732	0.0075 $<$ tol

Extending the simulation duration would lead to the solution diverging towards infinity. This outcome is anticipated, as the spacecraft gradually spirals in towards Earth under a sustained deceleration, causing the radial distance r_1 to approach zero. A close examination of the equations of motion, as shown in (3), reveals that as r_1 diminishes towards zero, both $\dot{v}_x$ and $\dot{v}_y$ tend towards negative infinity. To keep things in check, it's a good idea to add a quick check in the numerical code, making sure r_1 doesn't dip below Earth's radius. This would stop the simulation from spiraling out of control (my code doesn't do this, yet).

Lagrange Points

Lagrange points are special locations in the CR3BP rotating reference frame where if one were to place an object with negligible mass at a Lagrange point, with zero velocity, it will remain stationary with respect to the rotating frame.

In vector equation terms, we can define Lagrange points as locations where:

$$\vec{r}(t) = \vec{r}(0) \rightarrow \frac{d\vec{r}}{dt} = \vec{0} \rightarrow \frac{d^2\vec{r}}{dt^2} = \vec{0}$$

i.e., the position vector is constant for all time and equal to it's initial position. Therefore, the first and second deivatives of the position vector must be zero at all times.

Recall our state vector and the derivative of our states:

$$\mathbf{X} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}, \qquad \dot{\mathbf{X}} = \begin{bmatrix} v_x \\ v_y \\ 2v_y + x - \frac{\tilde{\mu}\,(x+\mu)}{r_1^3} - \frac{\mu\,(x-\tilde{\mu})}{r_2^3} - f_d v_x \\ -2v_x + y - \frac{\tilde{\mu} y}{r_1^3} - \frac{\mu y}{r_2^3} - f_d v_y \end{bmatrix}$$

If $\frac{d\vec{r}}{dt} = \vec{v} = \vec{0}$ and $\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{0}$, then the derivative of our state vector $\dot{\mathbf{X}}$ must be zero:

$$\dot{\mathbf{X}} = \begin{bmatrix} v_x \\ v_y \\ 2v_y + x - \frac{\tilde{\mu}\,(x+\mu)}{r_1^3} - \frac{\mu\,(x-\tilde{\mu})}{r_2^3} - f_d v_x \\ -2v_x + y - \frac{\tilde{\mu} y}{r_1^3} - \frac{\mu y}{r_2^3} - f_d v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since we asserted that $\frac{d\vec{v}}{dt} = [v_x \ v_y]^T = \vec{0}$, then we must find the values of x and y such that:

$$\begin{bmatrix} x - \frac{\tilde{\mu}\,(x+\mu)}{r_1^3} - \frac{\mu\,(x-\tilde{\mu})}{r_2^3} \\ y - \frac{\tilde{\mu} y}{r_1^3} - \frac{\mu y}{r_2^3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

From Equation (4), we have

$$y\left(1 - \frac{(1-\mu)}{{r_1}^3} - \frac{\mu}{{r_2}^3}\right) = 0 \tag{5}$$

while the individual terms μ, $1-\mu$, r_1^3, and r_2^3 cannot be zero based on the given constraints, the value of the entire expression within the parentheses could potentially be zero if the terms subtract to zero. However, let's look at the case where $y = 0$ – corresponding to solutions that are collinear with the x-axis – and later revisit the case where $y \neq 0$.

Collinear Solutions ($y = 0$)

From Equation (4), we have

$$x - \frac{(1-\mu)(x+\mu)}{{r_1}^3} - \frac{\mu(x-1+\mu)}{{r_2}^3} = 0 \tag{6}$$

Recall that

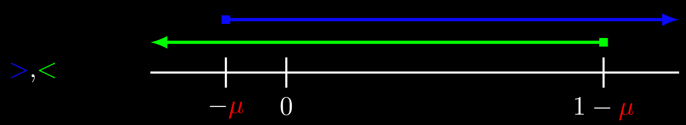
$${r_1}^2 = (x+\mu)^2 + y^2 \qquad {r_2}^2 = (x-1+\mu)^2 + y^2$$

By setting $y = 0$, Equation (6) becomes

$$x - \frac{(1-\mu)(x+\mu)}{\left(\sqrt{(x+\mu)^2}\right)^3} - \frac{\mu(x-1+\mu)}{\left(\sqrt{(x-1+\mu)^2}\right)^3} = 0 \tag{7}$$

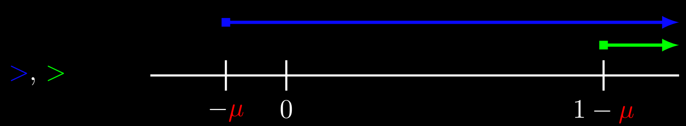
In Equation (7), there may be a temptation to cancel terms prematurely but this could potentially eliminate valid solutions. Let's consider four scenarios that correspond to the second and third terms in (7) being either positive or negative before further simplifying the equation.

<u>Scenario 1</u>: $x+\mu > 0, \quad x-1+\mu < 0 \quad \longrightarrow \quad x > -\mu, \quad x < 1-\mu$



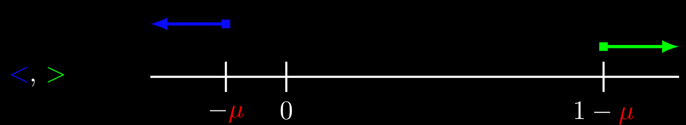
This corresponds to a solution that lies between the two bodies.

<u>Scenario 2</u>: $x+\mu > 0, \quad x-1+\mu > 0 \quad \longrightarrow \quad x > -\mu, \quad x > 1-\mu$



From above, we can see that both inequalities are true when $x > 1-\mu$. We don't know what this value is yet, but we know that it must lie on the right side of the smaller body (Moon).

<u>Scenario 3</u>: $x+\mu < 0, \quad x-1+\mu > 0 \quad \longrightarrow \quad x < -\mu, \quad x > 1-\mu$



This leads to an invalid solution since these conditions do not overlap (there are no values of x that are both less than μ and greater than $1-\mu$).

<u>Scenario 4</u>: $x+\mu < 0, \quad x-1+\mu < 0 \quad \longrightarrow \quad x < -\mu, \quad x < 1-\mu$

This corresponds to a solution that lies to the left of the larger body (Earth).

Now, we can simplify Equation (7) corresponding to each scenario we looked at above:

$$\boxed{f_1(x) = x - \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0}$$

$$\boxed{f_2(x) = x - \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0}$$

$$\boxed{f_3(x) = x + \frac{1-\mu}{(x+\mu)^2} - \frac{\mu}{(x-1+\mu)^2} = 0}$$

$$\boxed{f_4(x) = x + \frac{1-\mu}{(x+\mu)^2} + \frac{\mu}{(x-1+\mu)^2} = 0}$$

Having completed that process, the simplified equations above can be plotted, with careful attention to the positive and negative signs, to confirm the expected solutions.

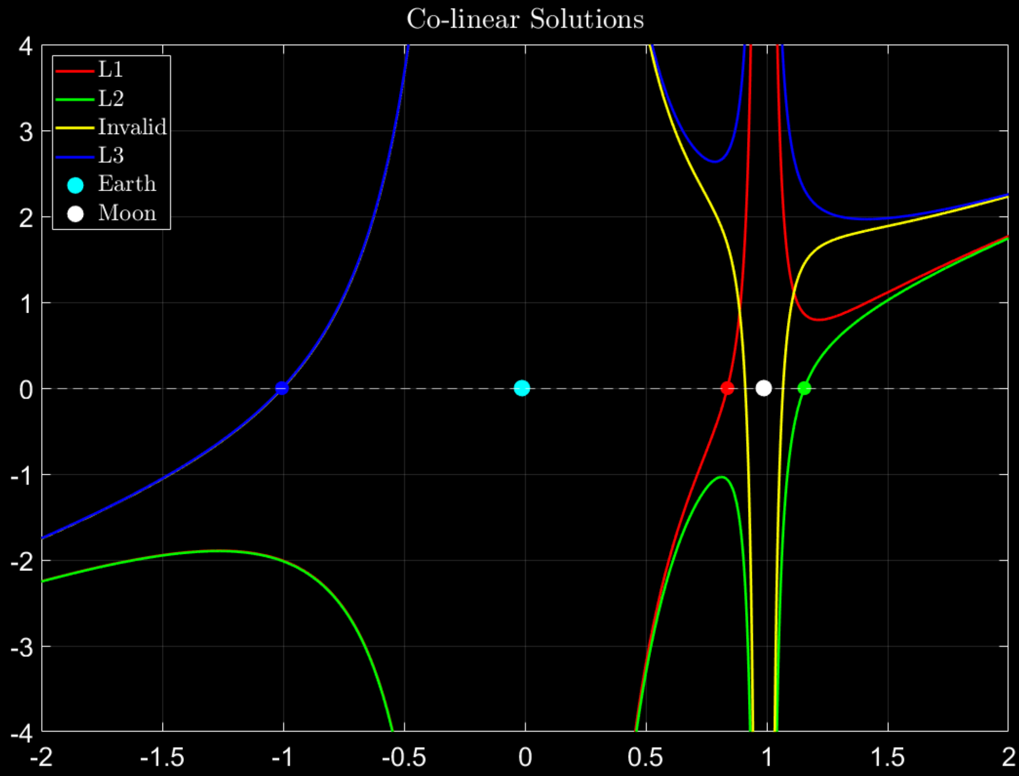


Figure 5: Graphical Representation of the Collinear Solutions for the Lagrange points

It is noted that these collinear solutions are determined using a numerical root-solving method (secant method) that utilizes finite differences for the estimation of derivatives and iteratively converges on the solutions. These solutions correspond to the x-coordinates of the Lagrange points (L_1–L_3), and the Earth and Moon are marked to indicate their relative positions in this configuration.

The coordinates of L_1, L_2, and L_3 are summarized below:

Table 3: Collinear Lagrange Points for Earth-Moon System

Lagrange Point	x	y
L_1	0.837023544523	0
L_2	1.155597402589	0
L_3	-1.005053470159	0

Non-Collinear Solutions ($\mathbf{y \neq 0}$)

Let's bring back Equations (5) & (6):

$$x - \frac{(1-\mu)(x+\mu)}{{r_1}^3} - \frac{\mu(x-1+\mu)}{{r_2}^3} = 0$$
$$y\left(1 - \frac{(1-\mu)}{{r_1}^3} - \frac{\mu}{{r_2}^3}\right) = 0$$

Recalling the process for the collinear solutions, we looked at the trivial case where the variable y is zero. The next step is to consider what occurs when y is not assumed to be zero, which leads to setting the expression inside the parentheses of Equation (5) to zero:

$$1 - \frac{(1-\mu)}{{r_1}^3} - \frac{\mu}{{r_2}^3} = 0 \tag{8}$$

However, this introduces a multivariable problem because the norms of the position vectors depend on both x and y. Consequently, there are now two unknowns, x and y, and two equations. To find a solution, the system of equations can be transformed into a linear system by implementing variable substitutions. This is done by defining new variables, α and β, to represent the reciprocal of the cube of the norm of the position vectors:

$$\alpha = \frac{1}{{r_1}^3}, \quad \beta = \frac{1}{{r_2}^3} \tag{9}$$

Subsituting these new variables into our system of non-linear equations:

$$x - (1-\mu)(x+\mu)\alpha - \mu(x-1+\mu)\beta = 0 \tag{10}$$
$$1 - (1-\mu)\alpha - \mu\beta = 0 \tag{11}$$

To solve the given system of equations for α and β, we can use the method of substitution or elimination Given Equation (11) is simpler, we can start by solving it for one of the variables, say α, and then substitute the value of α into the Equation (10) to solve for β.

Let's rearrange Equation (11) for α:

$$1 - (1-\mu)\alpha - \mu\beta = 0$$
$$\Rightarrow \alpha = \frac{1-\mu\beta}{1-\mu} \tag{12}$$

Now, we substitute this expression for α into Equation (10) and solve for β.

$$\begin{aligned}
x - (1-\mu)(x+\mu)\left(\frac{1-\mu\beta}{1-\mu}\right) - \mu\,(x-1+\mu)\,\beta &= 0 \\
x - (x+\mu)(1-\mu\beta) - \mu\,(x-1+\mu)\,\beta &= 0 \\
x - (x - \mu\beta x + \mu - \mu^2\beta) - \mu(x-1\mu)\beta &= 0 \\
\mu\beta x - \mu + \mu^2\beta - \mu(x-1\mu)\beta &= 0 \\
\mu\beta\left[x+\mu-(x-1+\mu)\right] - \mu &= 0 \\
\mu\beta - \mu &= 0 \\
\implies \beta &= 1
\end{aligned}$$

With $\beta = 1$, we can substitute back into the expression for α in Equation (12) to find:

$$\begin{aligned}
\alpha &= \frac{1-\mu}{1-\mu} \\
\implies \alpha &= 1
\end{aligned}$$

With $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}1 \\ 1\end{bmatrix}$, this implies that:

$$\alpha = \frac{1}{{r_1}^3} = \beta = \frac{1}{{r_2}^3} = 1$$

Which means $\boxed{r_1 = r_2 = 1}$. In our non-dimensionalized system, this result indicates that r_1 & r_2 are equal to the characteristic length of the system, which is one spatial unit and also the distance between the two main bodies. This also means that the remaining Lagrange points are equidistant from each of the two bodies and form equilateral triangles.

Given this constraint, let's determine what the values of x & y are. Recall that:

$${r_1}^2 = (x+\mu)^2 + y^2 \qquad {r_2}^2 = (x-1+\mu)^2 + y^2$$

Since the norms are equal to one spatial unit, then we have:

$$(x+\mu)^2 + y^2 = 1 \tag{13}$$
$$(x-1+\mu)^2 + y^2 = 1 \tag{14}$$

This is a system of two equations with two unknowns. Setting the LHS of Equations (13) and (14) equal, the y^2 terms cancel and we have:

$$(x+\mu)^2 = (x-1+\mu)^2$$

Expanding both sides:

$$x^2 + 2\mu x + \mu^2 = x^2 - 2x + 2\mu x - 2\mu + \mu^2 + 1$$

Cancelling terms on both sides and solving for x:

$$\begin{aligned}
0 &= -2x - 2\mu + 1 \\
\Rightarrow x &= 1/2 - \mu
\end{aligned}$$

Subsituting this result for x into Equation (13) leads us to:

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$
$$y^2 = \frac{3}{4}$$
$$\Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

Thus, the x and y coordinates of the remaining Lagrange points are:

$$\boxed{x = \frac{1}{2} - \mu, \quad y = \pm\frac{\sqrt{3}}{2}}$$

We have now solved for all 5 lagrange points, summarized below in Table 4:

Table 4: Location of the Lagrange Points for Earth-Moon System

Lagrange Point	x	y	Units
L_1	0.837023544523946	0	Nondimensional
	321751.850515005	0	Kilometers
L_2	1.155597402589	0	Nondimensional
	444211.641555549	0	Kilometers
L_3	-1.005053470159	0	Nondimensional
	-386342.553929263	0	Kilometer
L_4	0.487871437234688	0.866025403784439	Nondimensional
	187537.780473014	332900.165214738	Kilometers
L_5	0.487871437234688	-0.866025403784439	Nondimensional
	187537.780473014	-332900.165214738	Kilometers

Stability of Lagrange Points

To assess the stability of a spacecraft's trajectory near the Lagrange points, we need to consider the equations of motion linearized near these points of equilibrium in the CR3BP.

For a nonlinear time-invariant system $\dot{x} = f(x)$ where $x, f \in \mathbb{R}^n$, and $\bar{x}$ represents an equilibrium point, i.e. $f(\bar{x}) = 0$, we investigate the localized behavior by considering small deviations δx from the equilibrium. Consider a point x near $\bar{x}$, i.e. $x = \bar{x} + \delta x$. Expanding the system about $\bar{x}$ through a Taylor series and disregarding higher-order terms yields:

$$\delta\dot{x} = J(\bar{x})\,\delta x \tag{15}$$

where $J(\bar{x}) = \dfrac{\mathrm{D}f}{\mathrm{D}x}(\bar{x})$ is the Jacobian matrix of f evaluated at the equilibrium $\bar{x}$. This differential equation represents a linear approximation governing the evolution of deviations δx in a small neighborhood.

Let L denote any of the Lagrange points L_1 to L_5. Let's find the linearized CR3BP equations around the Lagrange point L with the generalized coordinates $\bar{x} = (x_L, y_L, \dot{x}_L, \dot{y}_L) = (x_L, y_L, 0, 0)$:

Linearization of Equations of Motion

Recall the set of ODEs:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 2v_y + x - \dfrac{(1-\mu)\,(x+\mu)}{r_1^3} - \dfrac{\mu\,(x-1+\mu)}{r_2^3} - f_d v_x \\ -2v_x + y - \dfrac{1-\mu y}{r_1^3} - \dfrac{\mu y}{r_2^3} - f_d v_y \end{pmatrix} = f(x, y, v_x, v_y)$$

The associated Jacobian for these dynamics is a 4 × 4 matrix:

$$J(\mathbf{X}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial v_x} & \dfrac{\partial f_1}{\partial v_y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial v_x} & \dfrac{\partial f_2}{\partial v_y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial v_x} & \dfrac{\partial f_3}{\partial v_y} \\ \dfrac{\partial f_4}{\partial x} & \dfrac{\partial f_4}{\partial y} & \dfrac{\partial f_4}{\partial v_x} & \dfrac{\partial f_4}{\partial v_y} \end{pmatrix}$$

Under the assumption $f_d = 0$, this simplifies to:

$$J(\mathbf{X}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & 0 & 2 \\ \dfrac{\partial f_4}{\partial x} & \dfrac{\partial f_4}{\partial y} & -2 & 0 \end{pmatrix} \tag{16}$$

where

$$\frac{\partial f_3}{\partial x} = 1 - \frac{(1-\mu)}{{r_1}^3} - \frac{\mu}{{r_2}^3} + \frac{3\,(1-\mu)\,(x+\mu)^2}{{r_1}^5} + \frac{3\,\mu\,(x-1+\mu)^2}{{r_2}^5}$$

$$\frac{\partial f_3}{\partial y} = \frac{\partial f_4}{\partial x} = \frac{3\,(1-\mu)\,(\mu+x)\,y}{r_1^5} + \frac{3\,\mu\,(x-1+\mu)\,y}{r_2^5}$$

$$\frac{\partial f_4}{\partial y} = 1 - \frac{1-\mu}{{r_1}^3} - \frac{\mu}{{r_2}^3} + \frac{3\,(1-\mu)\,y^2}{{r_1}^5} + \frac{3\,\mu\,y^2}{{r_2}^5}$$

Stability of L_4 and L_5

The points L_4 and L_5 are recognized as stable equilibria for specific values of μ. To demonstrate this, let's consider L_4, where $\bar{x} = (x_{L4}, y_{L4}, 0, 0) = (1/2 - \mu, \sqrt{3}/2, 0, 0)$. Subsituting these values into the Jacobian from (16) yields:

$$J(\bar{x}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.75 & 1.2675 & 0 & 2 \\ 1.2675 & 2.25 & -2 & 0 \end{pmatrix}$$

The dynamics in the vicinity of the equilibrium point are governed by J. By conducting an eigen-analysis and examining the characteristic equation $\det(J - \lambda\mathbf{I}) = \mathbf{0}$, where $\mathbf{I}$ denotes the 4×4 identity matrix, we can assess the stability. The characteristic equation is given by:

$$\det(J - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0.75 & 1.2675 & -\lambda & 2 \\ 1.2675 & 2.25 & -2 & -\lambda \end{vmatrix}$$

This reduces to the following polynomial:

$$p(\lambda) = \lambda^4 + \lambda^2 + 0.08087 = 0 \tag{17}$$

The roots to $p(\lambda)$ correspond to the eigenvalues of $J(\bar{x})$:

$$\lambda = \pm 9.546i, \pm 0.2979i$$

Given that the real parts of λ equal zero, implying purely imaginary eigenvalues, it indicates $\boxed{\text{neutral stability}}$ concerning the Lagrange point. A similar result can be derived for L_5. In practice, however, the stability remains undetermined due to the omission of higher-order terms in the Taylor series expansion. Nevertheless, it is observed that for small deviations from these two equilibrium points, bounded oscillatory motion is generally expected within the framework of the Circular Restricted Three-Body Problem (CR3BP) in a rotating reference frame.

The same analytical approach applies to Lagrange points L_1, L_2, and L_3. These points are widely acknowledged as unstable, as the eigenvalues of their linearized dynamics reveal at least one positive real part. Below, a figure illustrates the stability/instability of all five Lagrange points with small perturbations as initial conditions for simulating each trajectory.

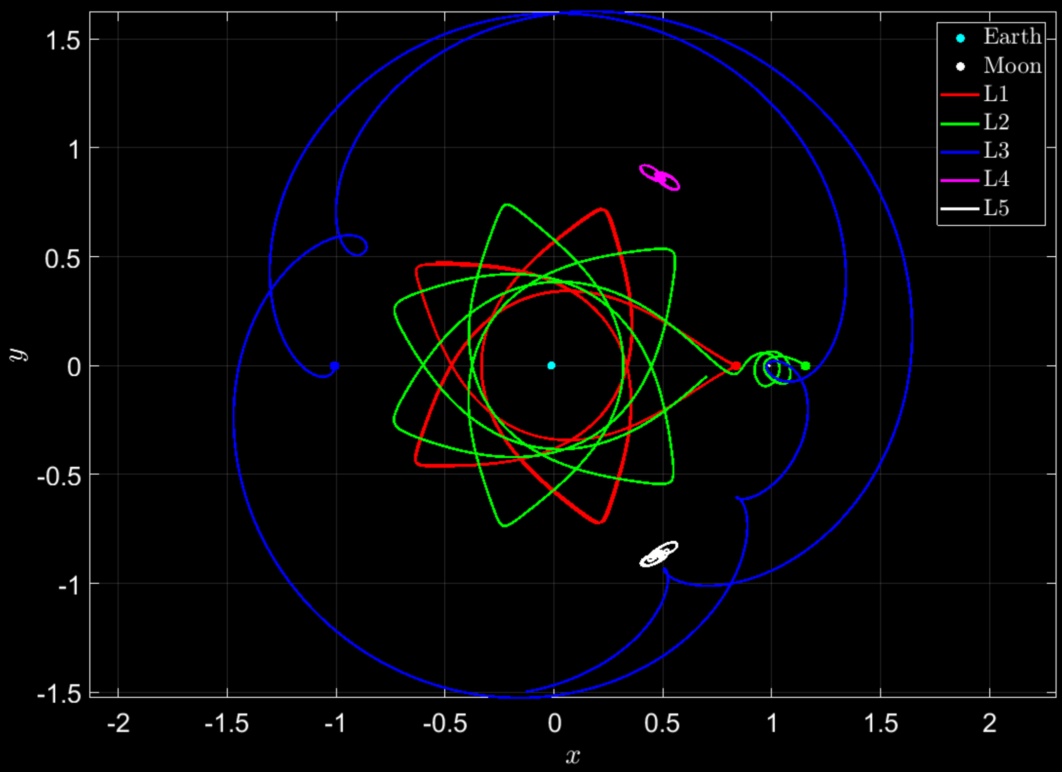


Figure 6: Illustration of Lagrange Point Stability Considering Small Perturbations as Initial Conditions

The figure above reaffirms the instability of L_1 to L_3, while indicating neutral stability for L_4 and L_5, as anticipated.

References

[1] Jacob A. Dahlke. Optimal trajectory generation in a dynamic multi-body environment using a pseudospectral method. Master's thesis, Theses and Dissertations, 2018.

[2] W.S. Koon, M.W. Lo, J.E. Marsden, and S.D. Ross. *Dynamical Systems, the Three-Body Problem and Space Mission Design*. Marsden Books, 2011.