EGR 5110: Homework #4

Due on April 20, 2024 at 11:59pm

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Background

A long rectangular fin is attached to a heat source. The fin is much longer (into the page) than its other dimensions, so heat flow is approximately two-dimensional. Its left side is subjected to a constant base temperature of 100 °C and the other three sides experience convection. The fin's initial temperature is 40 °C and the free stream air temperature is 25 °C.

Below is a cross sectional view of the fin:

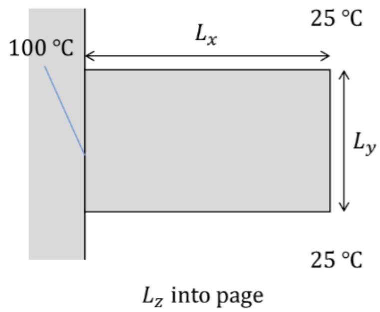


Figure 1: Long Rectangular Fin Attached to Heat Source

The time-dependent temperature distribution is governed by the 2D heat diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{1}$$

where T is temperature and α is the thermal diffusivity coefficient.

Goal: Solve Equation (1) from an initial time t_0 to a final time t_f for the temperature distribution across the 2D rectangular fin in Figure 1 (as a function of time) using a finite-difference method.

The following figure shows the coordinate system and general discretization of the fin:

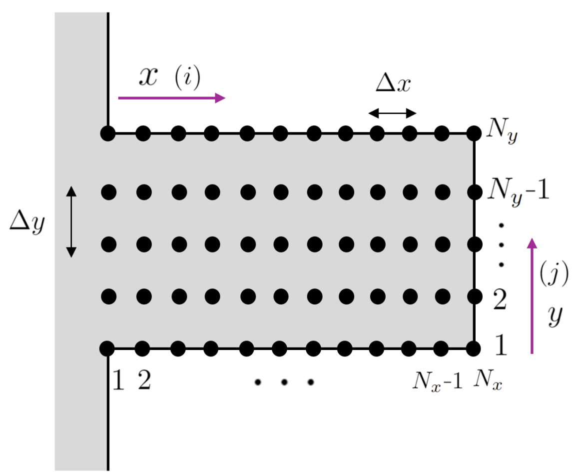


Figure 2: Coordinate system and discretization of a 2D thin rectangular fin

In this setup, the origin is fixed to the bottom-left corner of the fin, Δx & Δy represents the node spacings, N_x & N_y represents the final nodes in the x & y direction, i & j are the node indices for the x & y direction, respectively.

Deriving Node Equations

In the class notes, we derived the following node equations:

Interior Nodes:

$$T_{i,j}^{k+1} = \lambda\left(T_{i-1}^{k} + T_{i,j-1}^{k} + T_{i+1,j}^{k} + T_{i,j+1}^{k}\right) + (1-4\lambda)T_{i,j}^{k} \tag{2}$$

Left Boundary:

$$T_{1,j}^{k+1} = T_{i,j}^{k} = T_b \tag{3}$$

Right Boundary (excluding corner nodes):

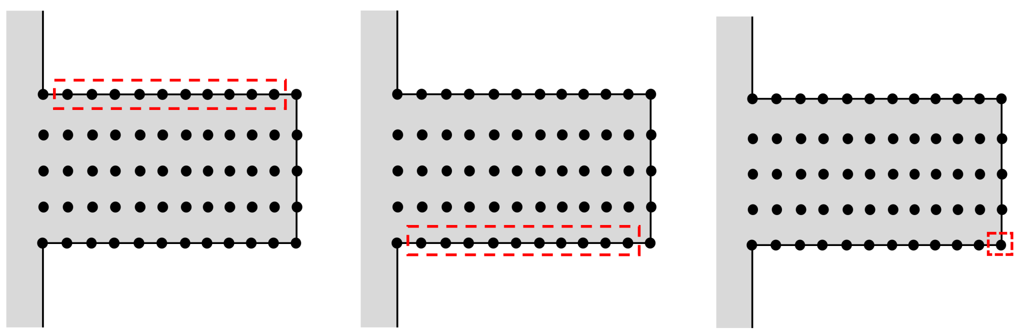
$$T_{N_x,j}^{k+1} = \lambda\left(2T_{N_x-1,j}^{k} + T_{N_x,j+1}^{k} + T_{N_x,j-1}^{k} + 2BT_\infty\right) + (1-4\lambda-2B\lambda)T_{N_x,j}^{k} \tag{4}$$

Top-right corner node:

$$T_{N_x,N_y}^{k+1} = \lambda\left(2T_{N_x-1,N_y}^{k} + T_{N_x,N_y-1}^{k} + 2BT_\infty\right) + (1-4\lambda-4B\lambda)T_{N_x,N_y}^{k} \tag{5}$$

where $B = \dfrac{h\Delta x}{k}$, $\lambda = \dfrac{\alpha \Delta t}{(\Delta x)^2}$, $\alpha = \dfrac{k}{\rho c_p}$.

We must derive the remaining node equations for the top boundary, lower boundary, and the bottom-right corner:



(a) Top Nodes (b) Bottom Nodes (c) Corner Node

Figure 3: Visualization of node configurations and boundary conditions

The energy balance at all boundary nodes is captured by:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{generated} = \dot{E}_{stored}$$

Since the rate of energy flowing out of the control volume is zero ($\dot{E}_{out}$) and there is no energy generation within the control volume ($\dot{E}_{generated} = 0$), then the equation above simplifies to:

$$\dot{E}_{in} = \dot{E}_{stored}$$

This implies that:

$$\sum \dot{Q}_{cond} + \sum \dot{Q}_{conv} = mc_p \frac{\partial T}{\partial t} \tag{6}$$

This equation represents the balance of heat energy at the node, accounting for both conductive and convective heat transfer rates and the rate of change of stored thermal energy within the fin material. For our numerical simulations, this equation can be discretized further to solve for the temperature distribution over time within the rectangular fin using finite-difference approximations.

Top Boundary Nodes

Consider the nodes located along the top boundary of the rectangular fin (i, N_y), excluding the corner nodes. These nodes are subject to conduction and convection with the free stream air temperature T_∞.

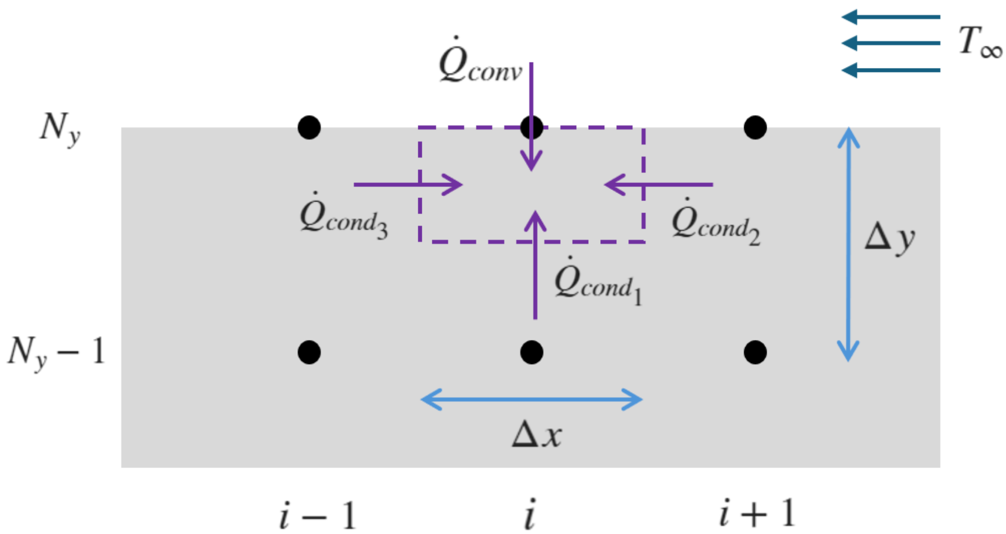


Figure 4: Energy Balance on Top Nodes with Convection Boundary Condition

Using Fourier's Law of Conduction ($\dot{Q}_{cond} = kA\dfrac{\partial T}{\partial x}$) and Newton's Law of Cooling ($\dot{Q}_{conv} = hA\Delta T$), let's look at the flow rates coming into the control volume and their discretization:

- $\dot{Q}_{cond_1} = kA\dfrac{\Delta T}{\Delta y} = k(\Delta x \Delta z)\dfrac{T^k_{i,1-1} - T^k_{i,1}}{\Delta y}$
- $\dot{Q}_{cond_2} = kA\dfrac{\Delta T}{\Delta x} = k(\dfrac{\Delta y}{2}\Delta z)\dfrac{T^k_{i+1,N_y} - T^k_{i,1}}{\Delta x}$
- $\dot{Q}_{cond_3} = kA\dfrac{\Delta T}{\Delta x} = k(\dfrac{\Delta y}{2}\Delta z)\dfrac{T^k_{i-1,N_y} - T^k_{i,1}}{\Delta x}$
- $\dot{Q}_{conv} = hA\Delta T = h(\Delta x \Delta z)\left(T_\infty - T^k_{i,1}\right)$

Subsituting these expressions in Equation (6) leads to:

$$\rho\left(\Delta x\frac{\Delta y}{2}\Delta z\right)c_p\frac{T^{k+1}_{i,1} - T^k_{i,1}}{\Delta t} = k(\Delta x \Delta z)\frac{T^k_{i,1-1} - T^k_{i,1}}{\Delta y} + k(\frac{\Delta y}{2}\Delta z)\frac{T^k_{i+1,N_y} - T^k_{i,1}}{\Delta x} + k(\frac{\Delta y}{2}\Delta z)\frac{T^k_{i-1,N_y} - T^k_{i,1}}{\Delta x} + h(\Delta x \Delta z)\left(T_\infty - T^k_{i,1}\right)$$

Assuming $\Delta x = \Delta y$, then this simplifies to

$$\boxed{T^{k+1}_{i,1} = \lambda\left(2T^k_{i,1-1} + T^k_{i+1,N_y} + T^k_{i-1,N_y} + 2BT_\infty\right) + (1 - 4\lambda - 2B\lambda)T^k_{i,1}}$$

Lower Boundary Nodes

Consider the nodes located along the top boundary of the rectangular fin $(i, 1)$, excluding the corner nodes. These nodes are subject to conduction and convection with the free stream air temperature T_∞.

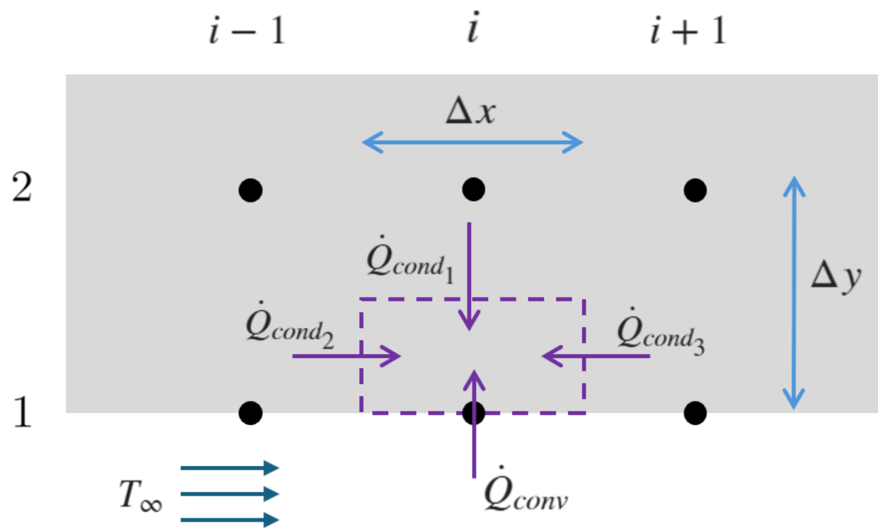


Figure 5: Energy Balance on Bottom Nodes with Convection Boundary Condition

Using Fourier's Law of Conduction and Newton's Law of Cooling, the flow rates coming into the control volume are:

- $\dot{Q}_{cond_1} = kA\dfrac{\Delta T}{\Delta y} = k(\Delta x \Delta z)\dfrac{T_{i,2}^k - T_{i,1}^k}{\Delta y}$
- $\dot{Q}_{cond_2} = kA\dfrac{\Delta T}{\Delta x} = k(\dfrac{\Delta y}{2}\Delta z)\dfrac{T_{i-1,1}^k - T_{i,1}^k}{\Delta x}$
- $\dot{Q}_{cond_3} = kA\dfrac{\Delta T}{\Delta x} = k(\dfrac{\Delta y}{2}\Delta z)\dfrac{T_{i+1,1}^k - T_{i,1}^k}{\Delta x}$
- $\dot{Q}_{conv} = hA\Delta T = h(\Delta x \Delta z)\left(T_\infty - T_{i,1}^k\right)$

Subsituting these expressions in Equation (6) leads to:

$$\begin{aligned}\rho\left(\Delta x \frac{\Delta y}{2} \Delta z\right) c_p \frac{T_{i,N_y}^{k+1} - T_{i,N_y}^k}{\Delta t} = k(\Delta x \Delta z)\frac{T_{i,2}^k - T_{i,1}^k}{\Delta y} \\ + k(\frac{\Delta y}{2}\Delta z)\frac{T_{i-1,1}^k - T_{i,1}^k}{\Delta x} \\ + k(\frac{\Delta y}{2}\Delta z)\frac{T_{i+1,1}^k - T_{i,1}^k}{\Delta x} + h(\Delta x \Delta z)\left(T_\infty - T_{i,1}^k\right)\end{aligned}$$

Assuming $\Delta x = \Delta y$, then this simplifies to

$$\boxed{T_{i,1}^{k+1} = \lambda\left(2T_{i,2}^k + T_{i+1,1}^k + T_{i-1,1}^k + 2BT_\infty\right) + (1 - 4\lambda - 2B\lambda)T_{i,1}^k}$$

Bottom-right Corner Node

Now, consider the node located on the bottom-right corner of the rectangular fin $(N_x, 1)$. This is also subject subject to conduction and convection with the free stream air temperature T_∞.

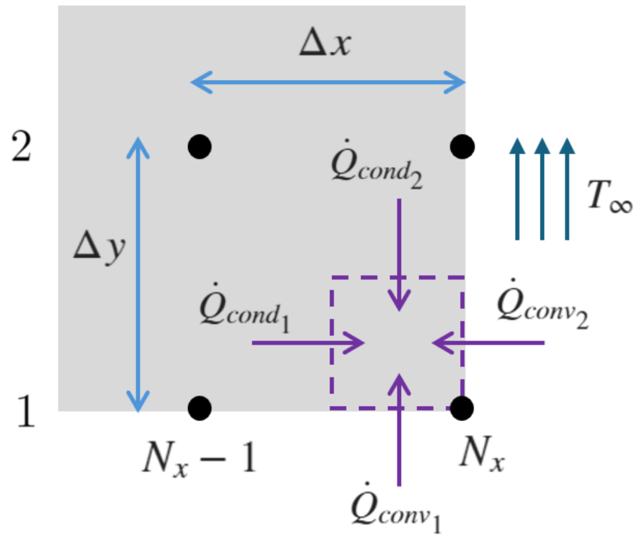


Figure 6: Energy Balance on Bottom-right Corner Node with Convection Boundary Condition

Using Fourier's Law of Conduction and Newton's Law of Cooling, the flow rates coming into the control volume are:

- $\dot{Q}_{cond_1} = kA\dfrac{\Delta T}{\Delta x} = k(\dfrac{\Delta y}{2}\Delta z)\dfrac{T^k_{N_x-1,1} - T^k_{N_x,1}}{\Delta x}$
- $\dot{Q}_{cond_2} = kA\dfrac{\Delta T}{\Delta y} = k(\dfrac{\Delta x}{2}\Delta z)\dfrac{T^k_{N_x,2} - T^k_{N_x,1}}{\Delta y}$
- $\dot{Q}_{conv_1} = hA\Delta T = h(\dfrac{\Delta x}{2}\Delta z)\left(T_\infty - T^k_{N_x,1}\right)$
- $\dot{Q}_{conv_2} = hA\Delta T = h(\dfrac{\Delta y}{2}\Delta z)\left(T_\infty - T^k_{N_x,1}\right)$

Subsituting these expressions in Equation (6) leads to:

$$\rho\left(\frac{\Delta x}{2}\frac{\Delta y}{2}\Delta z\right)c_p\frac{T^{k+1}_{N_x,1} - T^k_{N_x,1}}{\Delta t} = k(\frac{\Delta y}{2}\Delta z)\frac{T^k_{N_x-1,1} - T^k_{N_x,1}}{\Delta x} + k(\frac{\Delta x}{2}\Delta z)\frac{T^k_{N_x,2} - T^k_{N_x,1}}{\Delta y} + h(\frac{\Delta x}{2}\Delta z)\left(T_\infty - T^k_{N_x,1}\right) + h(\frac{\Delta y}{2}\Delta z)\left(T_\infty - T^k_{N_x,1}\right)$$

Assuming $\Delta x = \Delta y$, then this simplifies to

$$\boxed{T^{k+1}_{N_x,1} = 2\lambda\left(T^k_{N_x-1,1} + T^k_{N_x,2} + 2BT_\infty\right) + (1 - 4\lambda - 4B\lambda)T^k_{N_x,1}}$$

Scenarios

Understanding heat transfer and thermal properties is essential in engineering applications. Thermal conductivity (k_{cond}), thermal diffusivity (α), and convective heat transfer coefficient (h) are key parameters that influence heat flow within materials.

Thermal conductivity (k_{cond}) denotes a material's ability to conduct heat, indicating how efficiently heat moves through the material. It depends on the material type and structure. Thermal diffusivity (α) represents how quickly a material responds to temperature changes and is defined as the ratio of thermal conductivity (k_{cond}) to the product of density (ρ) and specific heat capacity (c_p). Convective heat transfer coefficient (h) quantifies heat transfer effectiveness between a solid surface and a fluid (like air or water). This coefficient varies based on fluid properties, flow conditions, and surface characteristics. In the study of temperature distribution in a thin rectangular fin over time, variations in k_{cond}, α, and h play crucial roles in determining transient and steady-state heat transfer behavior.

Let's analyze five scenarios based on the values of thermal conductivity (k_{cond}), thermal diffusivity (α), and convection coefficient (h) listed in the table below:

Table 1: Five Scenarios Using an Explicit Finite-Difference Method

Scenario	k_{cond} $\left(\frac{\text{W}}{\text{m}\,^\circ\text{C}}\right)$	α $\left(\frac{\text{m}^2}{\text{s}}\right)$	h $\left(\frac{\text{W}}{\text{m}^2\,^\circ\text{C}}\right)$	t_{ss} (min)	T_{avg} tip 1D eqn* (°C)	T_{avg} tip sim* (°C)	$\dot{Q}$ 1D eqn* (W)	$\dot{Q}$ sim* (W)
Pure Al, fan high	240	97×10^{-6}	100	0.93	93.94	94.16	133.31	126.32
Pure Al, fan low	240	97×10^{-6}	10	0.99	99.35	99.37	14.02	13.27
AISI 302	15	4×10^{-6}	100	11.23	52.54	53.57	78.49	74.32
Low k, high α	3	100×10^{-6}	100	0.055	28.77	29.40	37.61	34.43
High k, low α	100	3×10^{-6}	100	27.51	86.72	87.18	124.08	117.68

* The average tip temperature and heat rate are the values at the end of the simulation, which are well past the time when the contour lines stop moving.

We'll first look at case 1 (Pure Al, fan high), and then show how adjusting these parameters affects the temperature distribution, time to reach steady-state, and heat rate into the fin for scenarios 2-5.

Scenario 1: Pure Aluminum, Fan High

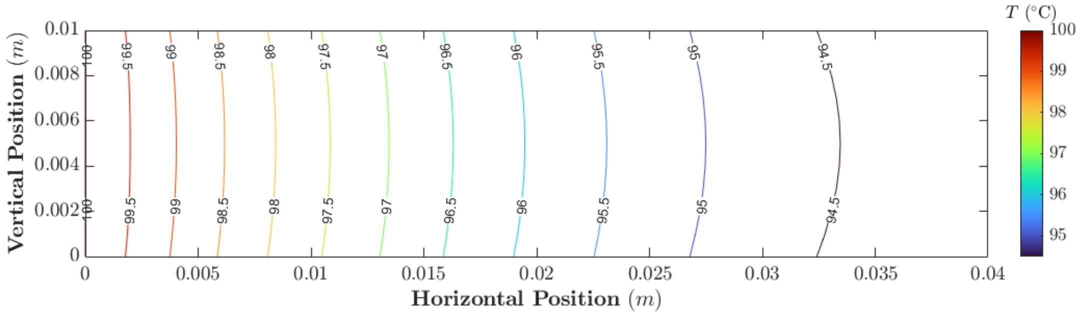


Figure 7: Steady-State Temperature Distrubution for Scenario 1

Simulation Parameters:

- dt = 0.0012
- N_t = 500,000
- $B = 4.167 \times 10^{-4}$
- $\lambda = 0.1164$

Scenario 2: Pure Aluminum, Fan Low

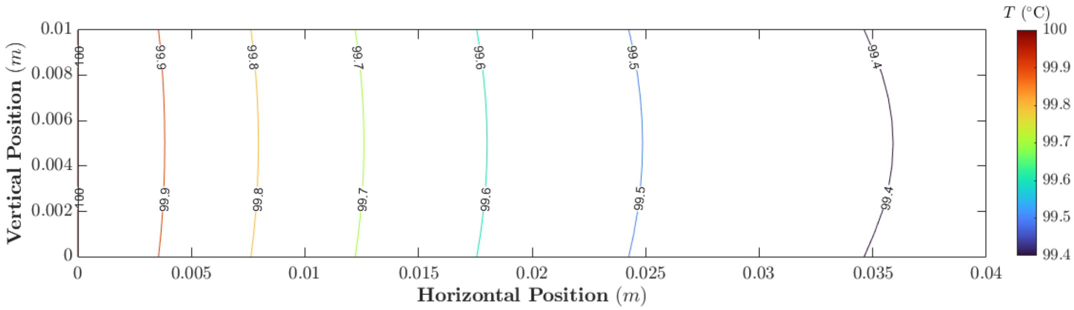


Figure 8: Steady-State Temperature Distrubution for Scenario 2

Simulation Parameters:

- dt = 0.00015
- N_t = 200,000
- $B = 4.167 \times 10^{-4}$
- $\lambda = 0.1455$

Compared to scenario 1, the convective heat transfer coefficient (h) was reduced while holding thermal diffusivity (α) and thermal conducitivity (k_{cond}) constant. A lower h value means slower heat transfer and potentially higher temperatures near the surface due to reduced cooling efficiency.

Effect of Adjustments:

1. **Temperature Distrubution**:
   - Surface temperatures are higher, with a slightly steeper temperature gradient across the fin.
   - Distribution became more non-uniform, with higher temperatures near the base of the fin.
   - Heat dissipation became less efficient
2. **Time to Steady State (t_{ss})**:
   - Slightly longer time to reach steady-state temperature due to slower heat dissipation.
3. **Heat Rate ($\dot{Q}$)**:
   - Reducing h decreased the convective heat transfer rate, $\dot{Q}_{conv}$, which means that less heat was transferred from the fin's surface to the surrounding air per unit time.
   - This resulted in slower cooling of the fin's surface, and thus a smaller $\dot{Q}$ for dissipating heat.

Scenario 3: Stainless Steel, AISI 302

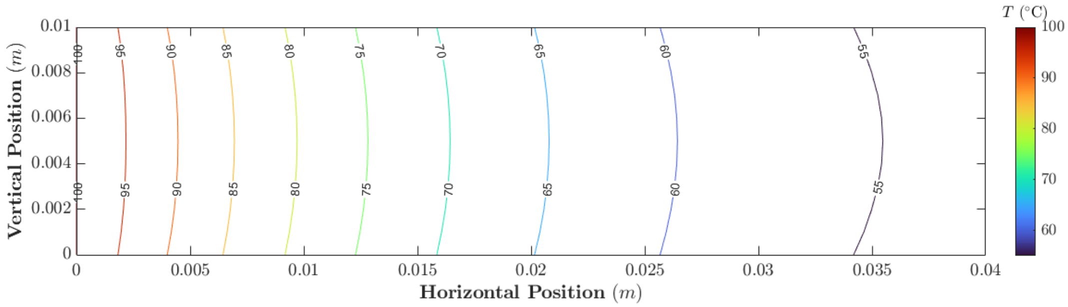


Figure 9: Steady-State Temperature Distrubution for Scenario 3

Simulation Parameters:

- dt = 0.0015
- N_t = 100,000
- $B = 6.667 \times 10^{-3}$
- $\lambda = 0.06$

Compared to scenario 1, the convective heat transfer coefficient (h) was held constant whiile thermal diffusivity (α) and thermal conducitivity (k_{cond}) constant. Decreasing α means the material has a lower ability to respond to changes in temperature, implying slower heat propagation. Meanwhile, a decrease in k means the material is less efficient at conducting heat.

Effect of Adjustments:

1. **Temperature Distrubution**:
   - Decreasing α and k led to a more non-uniform temperature distribution along the fin.
   - Steeper temperature gradients are observed, especially near the heat source and fin base.
2. **Time to Steady State (t_{ss})**:
   - Both decreasing α and k increased the time required for the fin to reach steady state.
   - Slower heat propagation and reduced heat conduction prolonged the transient period, as seen in the animation produced by the solver.
3. **Heat Rate ($\dot{Q}$)**:
   - Decreasing α and k reduced the efficiency of heat transfer into the fin.
   - As a result, the heat rate into the fin decreased because of slower heat propagation and reduced heat conduction through the material.

Scenario 4: Low k, high α

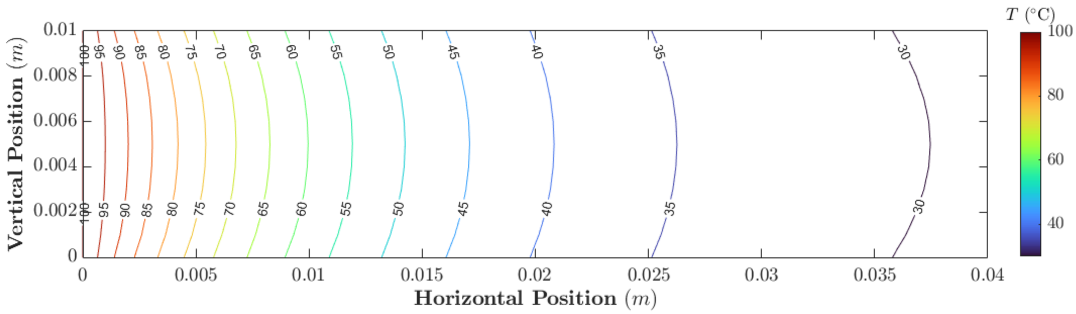


Figure 10: Steady-State Temperature Distrubution for Scenario 4

Simulation Parameters:

- dt = 0.0015
- N_t = 200,000
- B = 0.0333
- λ = 0.15

Compared to scenario 1, the convective heat transfer coefficient (h) and thermal diffusivity (α) were essentially held constant while thermal conductivity (k_{cond}) was reduced. Because k_{cond} represents the material's ability to conduct heat, decreasing this value implies reduced heat conduction capability through the material.

Effect of Adjustments:

1. **Temperature Distrubution**:
   - Decreasing k_{cond} resulted in a more non-uniform temperature distribution along the fin's length.
   - The fin's base and areas near the heat source experienced higher temperatures due to reduced heat dissipation.
   - Temperature gradients across the fin became steeper, indicating slower heat propagation through the material.
2. **Time to Steady State (t_{ss})**:
   - A lower k_{cond} would typically prolong the time required for the fin to reach steady state.
   - Due to a high α, this facilitated rapid heat propagation.
   - Slower heat conduction due to low k_{cond} combined with a high α and convective effects allowed for thermal equilibrium to be achieved rather quickly.
3. **Heat Rate ($\dot{Q}$)**:
   - Decreasing k_{cond} reduced the efficiency of heat transfer into the fin, since heat transferred more slowly through the material.

Scenario 5: High k, low α

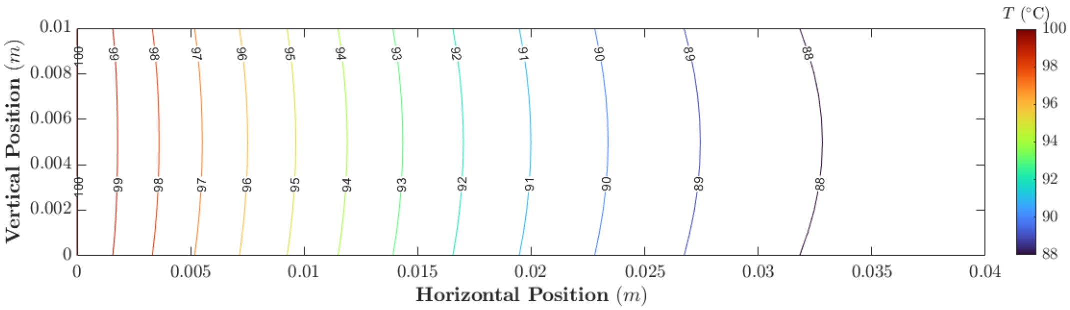


Figure 11: Steady-State Temperature Distrubution for Scenario 5
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Simulation Parameters:

- dt = 0.0120
- N_t = 200,000
- B = 0.036
- λ = 0.01

Instead of comparing to scenario 1, let's compare to scenario 4. The convective heat transfer coefficient (h) was held constant, while the thermal conductivity (k_{cond}) was increased and the thermal diffusivity (α) were reduced.

Effect of Adjustments:

1. **Temperature Distrubution**:
   - Decreasing α and increasing k_{cond} led to a more uniform temperature distribution across the fin.
   - While lower α slows down heat propagation and causes steeper temperature gradients, higher k_{cond} counteracted this by enhancing heat conduction and promoting a more even temperature profile.
2. **Time to Steady State (t_{ss})**:
   - Increasing k_{cond} reduced the time required for the fin to reach steady state by accelerating heat conduction through the material.
   - Despite the slower heat propagation due to lower α, the combined effect of increasing k_{cond} resulted in a shorter transient period before achieving thermal equilibrium, as seen in the animation.
3. **Heat Rate ($\dot{Q}$)**:
   - The heat rate into the fin is influenced by both α and k_{cond}.
   - Decreasing α reduces the efficiency of heat transfer into the fin by slowing down heat propagation.
   - However, increasing k_{cond} enhanced heat conduction and overall heat transfer efficiency, mitigating the impact of lower α on the heat rate. This is seen with the significantly higher heat rate value, compared to those in scenario 4.