

# **EGR 5110: Homework #4**

Due on April 20, 2024 at 11:59pm

*Professor Nissenson*

**Francisco Sanudo**

## Background

A long rectangular fin is attached to a heat source. The fin is much longer (into the page) than its other dimensions, so heat flow is approximately two-dimensional. Its left side is subjected to a constant base temperature of 100 °C and the other three sides experience convection. The fin's initial temperature is 40 °C and the free stream air temperature is 25 °C.

Below is a cross sectional view of the fin:

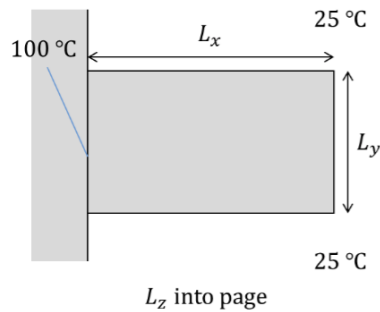


Figure 1: Long Rectangular Fin Attached to Heat Source

The time-dependent temperature distribution is governed by the 2D heat diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (1)$$

where  $T$  is temperature and  $\alpha$  is the thermal diffusivity coefficient.

**Goal:** Solve Equation (1) from an initial time  $t_0$  to a final time  $t_f$  for the temperature distribution across the 2D rectangular fin in Figure 1 (as a function of time) using a finite-difference method.

## Deriving Node Equations

In the class notes, we derived the following node equations:

Interior Nodes:

$$T_{i,j}^{k+1} = \lambda (T_{i-1,j}^k + T_{i,j-1}^k + T_{i+1,j}^k + T_{i,j+1}^k) + (1 - 4\lambda)T_{i,j}^k \quad (2)$$

Left Boundary:

$$T_{1,j}^{k+1} = T_{i,j}^k = T_b \quad (3)$$

Right Boundary (excluding corner nodes):

$$T_{N_x,j}^{k+1} = \lambda (2T_{N_x-1,j}^k + T_{N_x,j+1}^k + T_{N_x,j-1}^k + 2BT_\infty) + (1 - 4\lambda - 2B\lambda)T_{N_x,j}^k \quad (4)$$

Top-right corner node:

$$T_{N_x,N_y}^{k+1} = \lambda (2T_{N_x-1,N_y}^k + T_{N_x,N_y-1}^k + 2BT_\infty) + (1 - 4\lambda - 4B\lambda)T_{N_x,N_y}^k \quad (5)$$

where  $B = \frac{h\Delta x}{k}$ ,  $\lambda = \frac{\alpha\Delta t}{(\Delta x)^2}$ ,  $\alpha = \frac{k}{\rho c_p}$ .

We must derive the remaining node equations for the top boundary, lower boundary, and the bottom-right corner.

The energy balance at all boundary nodes is captured by:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{generated} = \dot{E}_{stored}$$

Since the rate of energy flowing out of the control volume is zero ( $\dot{E}_{out} = 0$ ) and there is no energy generation within the control volume ( $\dot{E}_{generated} = 0$ ), then the equation above simplifies to:

$$\dot{E}_{in} = \dot{E}_{stored}$$

This implies that:

$$\sum \dot{Q}_{cond} + \sum \dot{Q}_{conv} = m c_p \frac{\partial T}{\partial t} \quad (6)$$

This equation represents the balance of heat energy at the node, accounting for both conductive and convective heat transfer rates and the rate of change of stored thermal energy within the fin material. For our numerical simulations, this equation can be discretized further to solve for the temperature distribution over time within the rectangular fin using finite-difference approximations.

## Top Boundary Nodes

Consider the nodes located along the top boundary of the rectangular fin ( $i, N_y$ ), excluding the corner nodes. These nodes are subject to conduction and convection with the free stream air temperature  $T_\infty$ .

Using Fourier's Law of Conduction ( $\dot{Q}_{cond} = kA \frac{\partial T}{\partial x}$ ) and Newton's Law of Cooling ( $\dot{Q}_{conv} = hA\Delta T$ ), let's look at the flow rates coming into the control volume and their discretization:

- $\dot{Q}_{cond_1} = kA \frac{\Delta T}{\Delta y} = k(\Delta x \Delta z) \frac{T_{i,1-1}^k - T_{i,1}^k}{\Delta y}$
- $\dot{Q}_{cond_2} = kA \frac{\Delta T}{\Delta x} = k(\frac{\Delta y}{2} \Delta z) \frac{T_{i+1,N_y}^k - T_{i,1}^k}{\Delta x}$
- $\dot{Q}_{cond_3} = kA \frac{\Delta T}{\Delta x} = k(\frac{\Delta y}{2} \Delta z) \frac{T_{i-1,N_y}^k - T_{i,1}^k}{\Delta x}$

- $\dot{Q}_{conv} = hA\Delta T = h(\Delta x \Delta z) (T_\infty - T_{i,1}^k)$

Substituting these expressions in Equation (6) leads to:

$$\begin{aligned} \rho \left( \Delta x \frac{\Delta y}{2} \Delta z \right) c_p \frac{T_{i,1}^{k+1} - T_{i,1}^k}{\Delta t} &= k(\Delta x \Delta z) \frac{T_{i,1-1}^k - T_{i,1}^k}{\Delta y} \\ &+ k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i+1,N_y}^k - T_{i,1}^k}{\Delta x} \\ &+ k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i-1,N_y}^k - T_{i,1}^k}{\Delta x} + h(\Delta x \Delta z) (T_\infty - T_{i,1}^k) \end{aligned}$$

Assuming  $\Delta x = \Delta y$ , then this simplifies to

$$T_{i,1}^{k+1} = \lambda \left( 2T_{i,1-1}^k + T_{i+1,N_y}^k + T_{i-1,N_y}^k + 2BT_\infty \right) + (1 - 4\lambda - 2B\lambda)T_{i,1}^k$$

## Lower Boundary Nodes

Consider the nodes located along the top boundary of the rectangular fin  $(i, 1)$ , excluding the corner nodes. These nodes are subject to conduction and convection with the free stream air temperature  $T_\infty$ .

Using Fourier's Law of Conduction and Newton's Law of Cooling, the flow rates coming into the control volume are:

- $\dot{Q}_{cond_1} = kA \frac{\Delta T}{\Delta y} = k(\Delta x \Delta z) \frac{T_{i,2}^k - T_{i,1}^k}{\Delta y}$
- $\dot{Q}_{cond_2} = kA \frac{\Delta T}{\Delta x} = k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i-1,1}^k - T_{i,1}^k}{\Delta x}$
- $\dot{Q}_{cond_3} = kA \frac{\Delta T}{\Delta x} = k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i+1,1}^k - T_{i,1}^k}{\Delta x}$
- $\dot{Q}_{conv} = hA\Delta T = h(\Delta x \Delta z) (T_\infty - T_{i,1}^k)$

Substituting these expressions in Equation (6) leads to:

$$\begin{aligned} \rho \left( \Delta x \frac{\Delta y}{2} \Delta z \right) c_p \frac{T_{i,N_y}^{k+1} - T_{i,N_y}^k}{\Delta t} &= k(\Delta x \Delta z) \frac{T_{i,2}^k - T_{i,1}^k}{\Delta y} \\ &+ k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i-1,1}^k - T_{i,1}^k}{\Delta x} \\ &+ k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{i+1,1}^k - T_{i,1}^k}{\Delta x} + h(\Delta x \Delta z) (T_\infty - T_{i,1}^k) \end{aligned}$$

Assuming  $\Delta x = \Delta y$ , then this simplifies to

$$T_{i,1}^{k+1} = \lambda (2T_{i,2}^k + T_{i+1,1}^k + T_{i-1,1}^k + 2BT_\infty) + (1 - 4\lambda - 2B\lambda)T_{i,1}^k$$

## Bottom-right Corner Node

Now, consider the node located on the bottom-right corner of the rectangular fin  $(N_x, 1)$ . This is also subject to conduction and convection with the free stream air temperature  $T_\infty$ .

Using Fourier's Law of Conduction and Newton's Law of Cooling, the flow rates coming into the control volume are:

- $\dot{Q}_{cond_1} = kA \frac{\Delta T}{\Delta x} = k(\frac{\Delta y}{2} \Delta z) \frac{T_{N_x-1,1}^k - T_{N_x,1}^k}{\Delta x}$
- $\dot{Q}_{cond_2} = kA \frac{\Delta T}{\Delta y} = k(\frac{\Delta x}{2} \Delta z) \frac{T_{N_x,2}^k - T_{N_x,1}^k}{\Delta y}$
- $\dot{Q}_{conv_1} = hA \Delta T = h(\frac{\Delta x}{2} \Delta z) (T_\infty - T_{N_x,1}^k)$
- $\dot{Q}_{conv_2} = hA \Delta T = h(\frac{\Delta y}{2} \Delta z) (T_\infty - T_{N_x,1}^k)$

Substituting these expressions in Equation (6) leads to:

$$\begin{aligned} \rho \left( \frac{\Delta x}{2} \frac{\Delta y}{2} \Delta z \right) c_p \frac{T_{N_x,1}^{k+1} - T_{N_x,1}^k}{\Delta t} &= k \left( \frac{\Delta y}{2} \Delta z \right) \frac{T_{N_x-1,1}^k - T_{N_x,1}^k}{\Delta x} \\ &\quad + k \left( \frac{\Delta x}{2} \Delta z \right) \frac{T_{N_x,2}^k - T_{N_x,1}^k}{\Delta y} \\ &\quad + h \left( \frac{\Delta x}{2} \Delta z \right) (T_\infty - T_{N_x,1}^k) + h \left( \frac{\Delta y}{2} \Delta z \right) (T_\infty - T_{N_x,1}^k) \end{aligned}$$

Assuming  $\Delta x = \Delta y$ , then this simplifies to

$$T_{N_x,1}^{k+1} = 2\lambda (T_{N_x-1,1}^k + T_{N_x,2}^k + 2BT_\infty) + (1 - 4\lambda - 4B\lambda)T_{N_x,1}^k$$

## Scenarios

Let's analyze each scenario based on the values of thermal conductivity ( $k_{\text{cond}}$ ), thermal diffusivity ( $\alpha$ ), and convection coefficient ( $h$ ) listed in the table below:

Table 1: Five Scenarios Using an Explicit Finite-Difference Method

Scenario	$k_{\text{cond}}$ ( $\frac{\text{W}}{\text{m}^\circ\text{C}}$ )	$\alpha$ ( $\frac{\text{m}^2}{\text{s}}$ )	$h$ ( $\frac{\text{W}}{\text{m}^2^\circ\text{C}}$ )	$t_{ss}$ (min)	$T_{\text{avg tip}}$ 1D eqn* ( $^\circ\text{C}$ )	$T_{\text{avg tip}}$ sim* ( $^\circ\text{C}$ )	$\dot{Q}$ 1D eqn* (W)	$\dot{Q}$ sim* (W)
Pure Al, fan high	240	$97 \times 10^{-6}$	100	0.93	93.94	94.16	133.31	126.32
Pure Al, fan low	240	$97 \times 10^{-6}$	10	0.99	99.35	99.37	14.02	13.27
AISI 302	15	$4 \times 10^{-6}$	100	11.23	52.54	53.57	78.49	74.32
Low $k$ , high $\alpha$	3	$100 \times 10^{-6}$	100	0.055	28.77	29.40	37.61	34.43
High $k$ , low $\alpha$	100	$3 \times 10^{-6}$	100	27.51	86.72	87.18	124.08	117.68

\* The average tip temperature and heat rate are the values at the end of the simulation, which are well past the time when the contour lines stop moving.

We'll look at how adjusting these parameters affects the temperature distribution, time to reach steady-state, and heat rate into the fin.

## Scenario 1: Pure Aluminum, Fan High

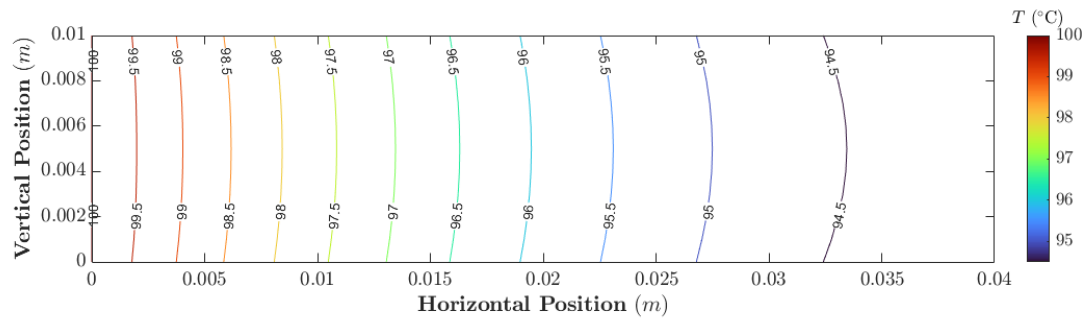


Figure 2: Steady-State Temperature Distribution for Scenario 1

Simulation Parameters:

- $dt = 0.0012$
- $N_t = 500,000$
- $B = 4.167 \times 10^{-4}$
- $\lambda = 0.1164$

Effect of Adjustments:

- **Temperature Distribution:**
- **Time to Steady State ( $t_{ss}$ ):**
- **Heat Rate ( $\dot{Q}$ ):**

## Scenario 2: Pure Aluminum, Fan Low

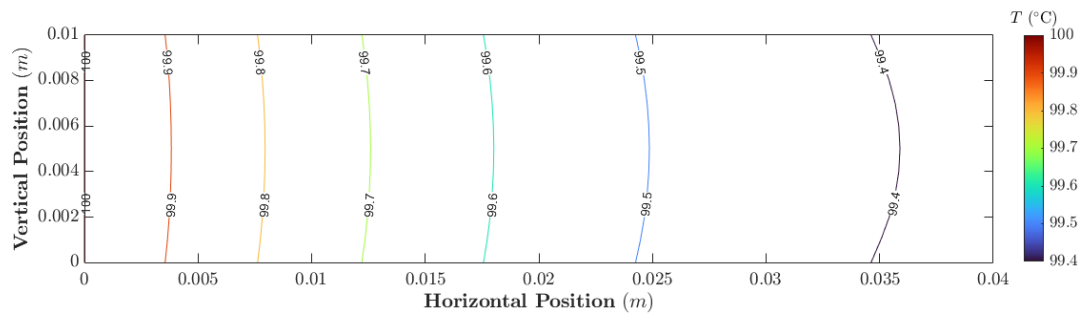


Figure 3: Steady-State Temperature Distribution for Scenario 2

Simulation Parameters:

- $dt = 0.00015$
- $N_t = 200,000$
- $B = 4.167 \times 10^{-4}$
- $\lambda = 0.1455$

Effect of Adjustments:

- **Temperature Distribution:**
- **Time to Steady State ( $t_{ss}$ ):**
- **Heat Rate ( $\dot{Q}$ ):**



### Scenario 3: Stainless Steel, AISI 302

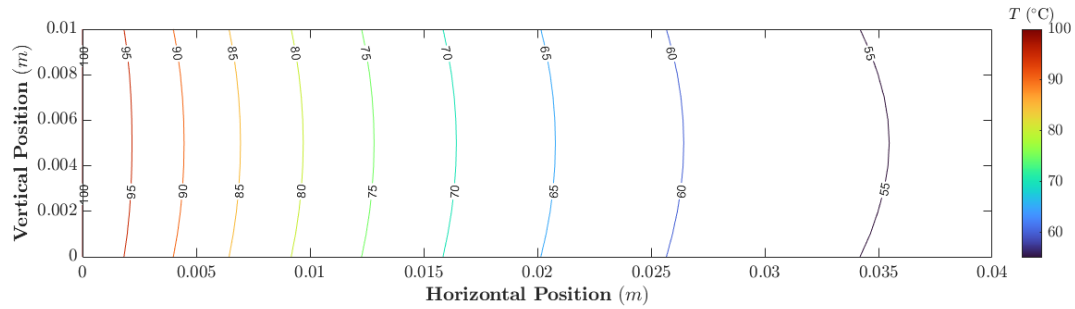


Figure 4: Steady-State Temperature Distribution for Scenario 3

Simulation Parameters:

- $dt = 0.0015$
- $N_t = 100,000$
- $B = 6.667 \times 10^{-3}$
- $\lambda = 0.06$

Effect of Adjustments:

- **Temperature Distribution:**
- **Time to Steady State ( $t_{ss}$ ):**
- **Heat Rate ( $\dot{Q}$ ):**

### Scenario 4: Low $k$ , high $\alpha$

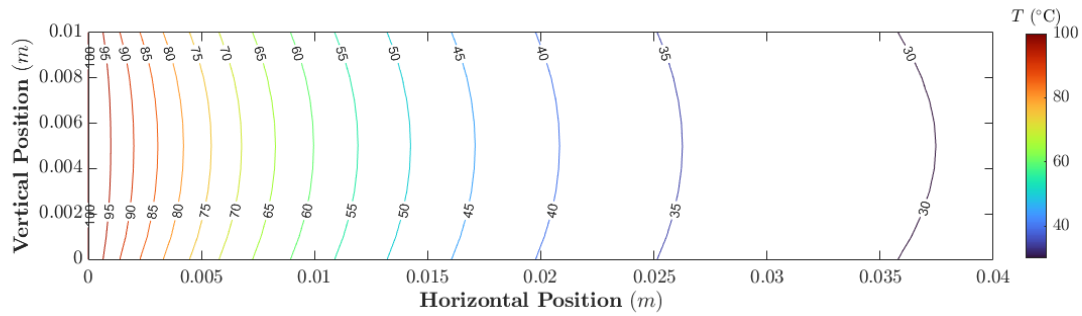


Figure 5: Steady-State Temperature Distribution for Scenario 4

Simulation Parameters:

- $dt = 0.0015$
- $N_t = 200,000$
- $B = 0.0333$
- $\lambda = 0.15$

Effect of Adjustments:

- **Temperature Distribution:**
- **Time to Steady State ( $t_{ss}$ ):**
- **Heat Rate ( $\dot{Q}$ ):**

### Scenario 5: High $k$ , low $\alpha$

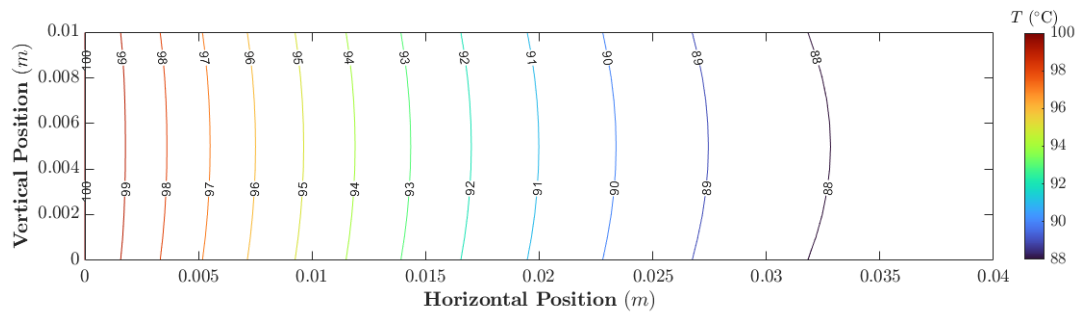


Figure 6: Steady-State Temperature Distribution for Scenario 5

s

Simulation Parameters:

- $dt = 0.0120$
- $N_t = 200,000$
- $B = 0.036$
- $\lambda = 0.01$

Effect of Adjustments:

- **Temperature Distribution:**
- **Time to Steady State ( $t_{ss}$ ):**
- **Heat Rate ( $\dot{Q}$ ):**