EGR 5110: Homework #3

Due on April 1, 2024 at 11:59pm

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Derivation of $y'(fr)$

The Newton-Raphson method, a numerical root-solving method, requires determining an expresssion for f' and calculating it at each iteration. The Colebrook Equation is a non-linear algebraic equation that can be solved using the Newton-Raphson method to determine the friction factor for steady, incompressible, isothermal pipe flow with friction. It is defined below:

$$\frac{1}{\sqrt{fr}} = -2\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{fr}}\right)$$

Moving all terms to the LHS:

$$\frac{1}{\sqrt{fr}} + 2\log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{fr}}\right) = 0$$

Thus,

$$y(fr) = 0$$

Now we just have to find the value of fr that make the value of the function y equal to zero.

The algorithm employed by Newton-Raphson method is a result of a Taylor-series approximation of the first order about an initial root estimate x_0, and is summarized below:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

A consequence of this method is that the derivative of $f(x)$ is required at each step. Therefore, to solve the Colebrook Equation for the friction factor using the Newton-Raphson method, we must know $y'(fr)$:

$$\begin{aligned}
y'(fr) &= \frac{d}{d(fr)}\left[\frac{1}{\sqrt{fr}} + \frac{2}{\ln(10)}\cdot\ln\left(\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right)\right] && \text{where differentiation is linear} \\
&= \frac{d}{d(fr)}\left[\frac{1}{\sqrt{fr}}\right] + \frac{2}{\ln(10)}\cdot\frac{d}{d(fr)}\left[\ln\left(\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right)\right] && \text{apply power rule \& chain rule} \\
&= \left(-\frac{1}{2}\right) fr^{(-1/2-1)} + \frac{2}{\ln(10)}\cdot\frac{\frac{d}{d(fr)}\left[\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right]}{\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}} && \text{where differentiation is linear} \\
&= \frac{2\left(\frac{251}{100\mathrm{Re}}\cdot\frac{d}{d(fr)}\left[\frac{1}{\sqrt{fr}}\right] + \frac{d}{d(fr)}\left[\frac{10\epsilon}{37D}\right]\right)}{\ln(10)\left(\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right)} - \frac{1}{2fr^{3/2}} && \text{apply power rule} \\
&= \frac{2\left(251\left(-\frac{1}{2}\right) fr^{-1/2-1}\cdot\frac{1}{100\mathrm{Re}} + 0\right)}{\ln(10)\left(\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right)} - \frac{1}{2fr^{3/2}} && \text{simplify}
\end{aligned}$$

Solution

$$y'(fr) = -\frac{251}{100\ln(10)\mathrm{Re}\cdot\left(\frac{10\epsilon}{37D} + \frac{251}{100\mathrm{Re}\sqrt{fr}}\right)fr^{3/2}} - \frac{1}{2fr^{3/2}}$$

Pipe Network Problem

Initial guess

For the given pipe network below:

pics/network.png

Figure 1: Pipe network with three loops

By applying conservation of mass through each node, we arrive at the initial guess for the direction and magnitude of the flowrates Q_j through each pipe j:

pics/network2.png

Figure 2: Updated pipe network

Loop and Node Equations

Labeling each loop and node:
Using Figure **??** and applying both mass conservation at each node and conservation of energy, we arrive at the 5 node and 3 loop equations, yielding 8 total:

Solution – (**Note**: Flowrates are in m^3/hr)

$$
\begin{aligned}
F_1 &= Q_1 + Q_2 - 2000 \\
F_2 &= Q_3 + Q_7 + Q_8 - 1000 \\
F_3 &= -Q_4 - Q_5 - Q_7 + 1300 \\
F_4 &= -Q_1 + Q_5 + Q_6 + 800 \\
F_5 &= -Q_6 - Q_8 + 700 \\
F_6 &= -fr_1\frac{8L_1}{\pi^2 g D_1^5}Q_1|Q_1| + fr_2\frac{8L_2}{\pi^2 g D_2^5}Q_2|Q_2| + fr_4\frac{8L_4}{\pi^2 g D_4^5}Q_4|Q_4| - fr_5\frac{8L_5}{\pi^2 g D_5^5}Q_5|Q_5| \\
F_7 &= -fr_4\frac{8L_4}{\pi^2 g D_4^5}Q_4|Q_4| - fr_3\frac{8L_3}{\pi^2 g D_3^5}Q_3|Q_3| + fr_7\frac{8L_7}{\pi^2 g D_7^5}Q_7|Q_7| \\
F_8 &= fr_5\frac{8L_5}{\pi^2 g D_5^5}Q_5|Q_5| - fr_6\frac{8L_6}{\pi^2 g D_6^5}Q_6|Q_6| - fr_7\frac{8L_7}{\pi^2 g D_7^5}Q_7|Q_7| + fr_8\frac{8L_8}{\pi^2 g D_8^5}Q_8|Q_8|
\end{aligned}
$$

pics/network3.png

Figure 3: Updated network with loops and nodes labeled

The goal is to search for **Q** – a column vector containing the flowrates in each branch – such that $\mathbf{F}(\mathbf{Q}) = 0$, where **F** is a vector-valued function – containing the set of non-linear algebraic equations – evaluated at **Q**. My MATLAB code `RootFinder_P2.m` uses this set of non-linear equations, and the other parameters given in Problem 2, to solve for the flowrates using the modified-secant method.