THE AUTOMORPHISM GROUP OF GRAPH COMPLEMENTS

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Abstract. Automorphism groups of a graph $\mathrm{Aut}(G)$ provide key insight into the structure of a graph G. In this paper, we introduce an equation involving $\mathrm{Aut}(G)$ and the automorphism group of its complement $\mathrm{Aut}(\overline{G})$, which illustrates various properties of G and $\overline{G}$. The main result of this paper is that the group $\mathrm{Aut}(G)$ is *equal* to the group $\mathrm{Aut}(\overline{G})$. We apply this to find automorphism properties of graphs that have better studied or more understandable complements. These properties include group order, automorphism classification, and the relationship between G and $\overline{G}$.

1. Introduction

Graphs are a fundamental structure, with a myriad of applications in math and computer science. Let G be a graph. Let its automorphism group be $\mathrm{Aut}(G)$ and its complement be $\overline{G}$. This paper explores what an automorphism group $\mathrm{Aut}(G)$ can tell us about a graph G and its complement $\overline{G}$. $\mathrm{Aut}(G)$ is the set of all homomorphisms that a graph can have onto itself. $\overline{G}$ is the complement of a G, which takes the graph's vertices and connects them with all edges not in G. Graph homomorphisms, complements, and properties are more properly defined in the Background section with examples. We study the implications of how $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ relate, as given by the Main Theorem.

Main Theorem (As stated in [3])**.** *The automorphism group of* G *is equal to the automorphism group of* $\overline{G}$*.*

$$\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$$

The history of the Main Theorem is unclear. The founding of group theory is often credited to Lagrange, Galois, and Cayley, with the fundamentals of group theory tangibly forming around 1846 [1]. However, the applications of group theory on graphs and algorithmic problems has been sourced to around 1979 [2].

However, the origins of the Main Theorem are undocumented. This is likely because the theorem can be quite intuitive to any who understands graphs and their automorphisms well. The basic idea is that automorphisms preserve adjacency between vertices — thus they should also preserve non-adjacency between vertices. Then, any automorphisms in the graph should be an automorphism for the graph's complement, and vice versa. These definitions and intuition is further developed in the Introduction section.

The first documented reference to the main theorem we can find is a paper from 2000, which simply said "clearly [G] and its complement have the same automorphism group" [4]. Further, the papers that contain the main theorem never cite any source for the theorem, but usually just note the theorem as an accepted truth, or a simple "fact" [4] [3]. Mathematicians that are familiar with graph automorphism theory view the Main Theorem as a fundamental fact that is nicely derived from the definitions of automorphisms and graph components.

The Main Theorem is useful for understanding certain graph and graph complement constructions. In this paper, we show some interesting examples on graph automorphisms, combinatorics problems, and more. The Main Theorem's applications are very relevant; we use the Main Theorem to prove other graph theorems and we show its application to well-known graphs in our examples. The main theme across these examples is that we know more about $\overline{G}$ than G, thus we can use the main theorem to tell us about $\mathrm{Aut}(G)$ and G.

Here is a brief outline of the paper. In Section 2, we give a proper definition of graph automorphisms and complements, with simple graphs as examples. In Section 3 we prove the Main Theorem. Finally, in Section 4, we give various interesting applications for the main theorem; the main purpose of these examples is to expand understanding about graphs and extend statements about graphs to their complements.

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2. Background

We assume the reader is familiar only with graph components, i.e. what vertices and edges represent. Throughout, let G be a graph, let $\mathrm{Aut}(G)$ be its automorphism group, and let $\overline{G}$ be its complement.

Definition 2.1. For a graph G, its complement $\overline{G}$ is a graph with all of the same vertices, with edges being any edge that is not in G.

Example 2.2. Take a graph G, as seen in 1. The complement graph $\overline{G}$ is show in 2.

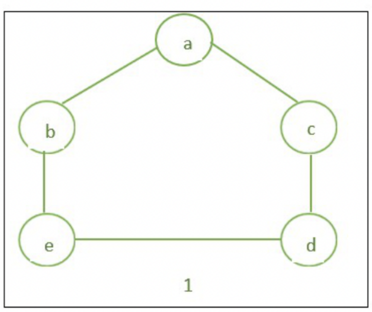

Figure 1. G.

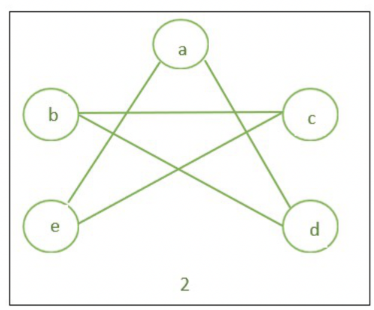

Figure 2. $\overline{G}$.

Definition 2.3. For a graph G, an automorphism on G is an isomorphism from G to itself that preserves adjacency. Two vertices v_1 and v_2 are adjacent if there is an edge between them. An edge between the two vertices denoted v_1v_2. Preserving adjacency in an automorphism means that if two vertices v_1, v_2 that are connected by an edge in G, then $\phi(v_1)$ and $\phi(v_2)$ is connected by an edge in $\phi(G)$. In mathematical terms, an automorphism is a bijection $\phi(G) : G \to G$ such that $v_1v_2 \in E(G)$ if and only if $\phi(v_1)\phi(v_2) \in E(G)$.

Example 2.4. Below is a graph G and a possible automorphism for G 3 [3].

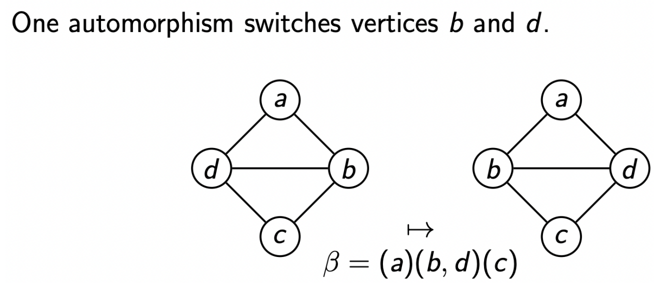

FIGURE 3. One possible automorphism for a graph G.

Example 2.5. Here is another example to develop the intuition of automorphisms and why they are important. Consider the graph in 4 [5]. The vertices 5,6,7,8 play the same role in the graph. Similarly, 0 and 2 are structurally the same vertices. Automorphisms can be thought of ways to permute vertices that share a "role" or "structure" in the graph. Then, any neighbors of those vertices must remain neighbors in the transformed graph, so those are also permuted with each other. For example, if you want to construct an automorphism that swaps vertices 0 and 2 in figure 4, it must also swap vertices 4 and 7 to preserve adjacency.

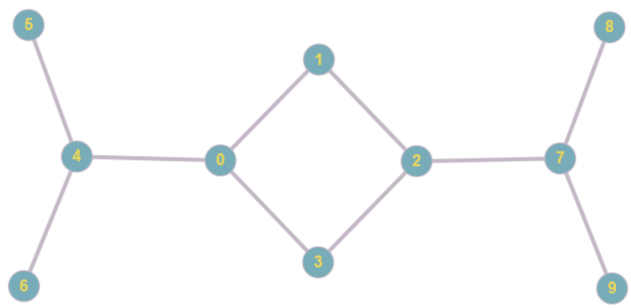

FIGURE 4. One possible automorphism for a graph G.

Definition 2.6. For a graph G, the automorphism group denoted $\mathrm{Aut}(G)$ is the set of all possible automorphisms.

Proposition 2.7. $\mathrm{Aut}(G)$ *is a group under composition*

Example 2.8. Below is an example of a graph G that has a corresponding automorphism group equal to the alternating group A4 5 [3]. A4 is generated by $(1,2,3)$ and $(1,2)(3,4)$; multiplication by (1,2,3) is represented by a blue edge and multiplication by (1,2)(3,4) is represented by a red edge [3]. The graph G has *every combination* of these operations for its automorphisms. Note that the red edges are bidirectional since (1,2)(3,4) is its own inverse. In fact, though we do not follow this path in this paper, Frucht proved the theorem below.

Theorem 2.9 ([6])**.** *Every group is the automorphism group of a graph. If the group is finite, the graph can be taken to be finite.*

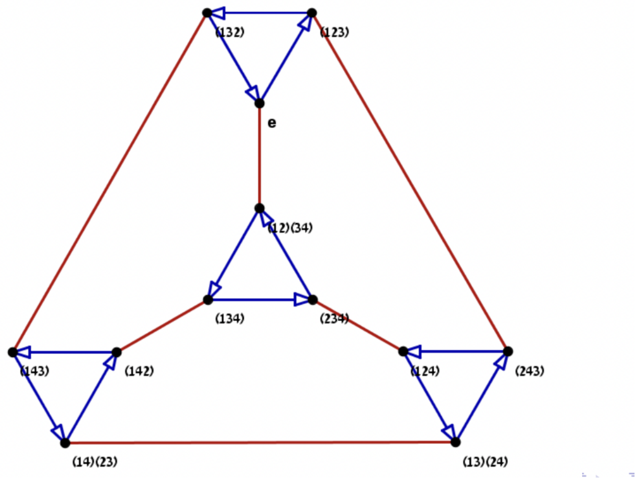

FIGURE 5. A graph, G, that has A4 as its automorphism group

3. MAIN THEOREM

Now that we've introduced the flavor of graph automorphisms and complements we are ready to move to the Main Theorem. The Main Theorem combines the idea of graph complements and graph automorphism groups.

Main Theorem. *For any graph G, the automorphism group of* G *is equal to the automorphism group of* $\overline{G}$.

$$\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$$

We follow the proof given in [7].

Proof. We prove by set inclusion in both directions.

Suppose that for an arbitrary graph G, we have an automorphism $\phi \in \mathrm{Aut}(G)$ and an edge $e \notin E(G)$. By the definition of graph complements, $e \in E(\overline{G})$. Recall that adjacency between two vertices means that there is an edge between them. Since the edge e is not in G, and an automorphism must preserve adjacency (and non-adjacency), we know $\phi(e) \notin G$. Thus, we also know $\phi(e) \in E(\overline{G})$. This is enough to show that ϕ is also an automorphism of $\overline{G}$ because any arbitrary edge $e \in E(\overline{G})$ also has $\phi(e) \in E(\overline{G})$ — thus all adjacency is preserved.

By definition, the complement of $\overline{G}$ is isomorphic to G. Thus, by using the proof above, we can substitute $\overline{G}$ for G, and susbstitute $\overline{G}$'s complement (which is just G) for $\overline{G}$ to prove the other direction. So, any automorphism $\phi \in \mathrm{Aut}(\overline{G})$ also satisfies $\phi \in \mathrm{Aut}(G)$.

By mutual containment, $\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$ □

Let's return to an example from Section 2 to demonstrate the Main Theorem.

Example 3.1. Consider example 2.2. One sample automorphism ϕ of the graph G in 1 is the following:

$$a \mapsto b, b \mapsto e, e \mapsto d, d \mapsto c, c \mapsto e.$$

We will formalize this kind of automorphism in Section 4; for illustrative purposes it is clear that this permutation will preserve adjacencies.

If you look at $\overline{G}$ in 2, you can see the same automorphism ϕ preserves adjacency. If this is not clear at first, consider ϕ as a rotation of the vertices counterclockwise. It becomes clear that any adjacency in $\overline{G}$ is preserved in $\phi(\overline{G})$.

4. Applications + Example Problems

The Main Theorem is often very informative when more is known about the complement of a graph than the graph itself. We study the applications of the Main Theorem by looking at a series of examples.

Example 4.1. The first example we examine is the complete graph.

Definition 4.2. A complete graph is a graph in which each vertex is connected to all other vertices by an edge.

See below (6) for an example of a complete graph with 5 vertices. Each vertex is connected to all other vertexes. [3].

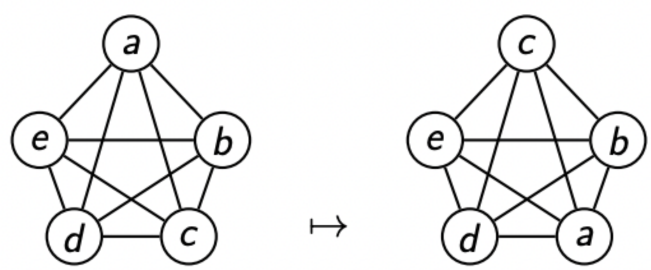

Figure 6. A complete graph with 5 vertices, and one possible automorphism.

Call the graph shown above G. How many permutations of vertices are there that preserve adjacency? How is $\mathrm{Aut}(G)$ characterized?

One way to answer these questions is to apply the Main Theorem.

Proposition 4.3. *The complement of a complete graph is an edgeless graph.*

This should be clear: the complete graph contains all possible edges, so its complement cannot contain any edges. In this case, $\overline{G}$ is a graph consisting of just 5 vertices.

Now we can answer the questions. Clearly, since there are no adjacencies to preserve, the number of permutations becomes a trivial counting problem! For a graph of 5 vertices, the number permutations is just 5!. Similarly, these automorphisms are clearly isomorphic to S_5.

Theorem 4.4. *If G is a complete graph,* $\mathrm{Aut}(G) \cong S_n$ [3].

Indeed, this is a known theorem. Perhaps the fastest and most intuitive proof is using the Main Theorem, like how we do above!

Example 4.5. Call the graph shown above G 7.

G is characterized each vertex being connected to every other except its two neighbors. Say we are curious about the same questions. How many ways can we permute vertices while preserving adjacency? What elements comprise the set $\mathrm{Aut}(G)$?

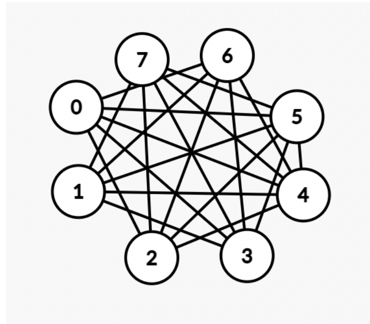

FIGURE 7. A graph where each vertex is connected to all vertices except its immediate neighbors

Let's apply the Main Theorem. In this case, $\overline{G}$ is characterized as a graph with 8 vertices, each of which are only connected to its 2 immediate neighbors. This gives us a well-studied graph known as the cycle graph, denoted C_8.

Proposition 4.6. *The automorphism group of a cycle graph,* C_n *is characterized by two operations: a flip and a rotation. Thus* $\mathrm{Aut}(C_n) \cong D_n$.

So, let's look at $\overline{G}$ and its possible automorphism operations as shown in 8 and 9. Since Aut(G) = $\mathrm{Aut}(\overline{G})$, then $\mathrm{Aut}(G) \cong D_8$ by the proposition. G has the same automorphisms as C_8: rotations and a flips. Further, now we know that |Aut(G)| = $|D_n|$ 2*n = 16. Using the Main Theorem, we were able to smoothly derive properties of G from its better-studied complement.

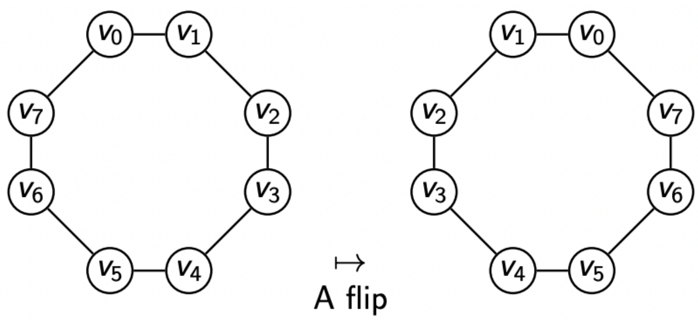

FIGURE 8. One possible flip for the graph C_8

Original Example. Consider a graph that is nearly complete, and call it G. G and its complement $\overline{G}$ is shown in **??**. The difference between G and a complete graph is that vertex 10 and 11 are not connected by an edge.

How many ways can we permute vertices in G while preserving adjacency? What automorphisms comprise the set $\mathrm{Aut}(G)$? By applying the main theorem, $\mathrm{Aut}(G) = \mathrm{Aut}(\overline{G})$, so we now consider $\overline{G}$. Permuting vertices while preserving adjacency again becomes a simple counting problem! We can permute any of the vertices 0-9 freely, which gives 10! permutations. For each of these permutations, we can either swap 10 and 11 or let them remain as they are. Thus, the total number of permutations is 10!*2. Further, Aut(G) $\cong S_10 \times S_2$, because we freely permute 10 vertices, then we independently freely permute 2 vertices.

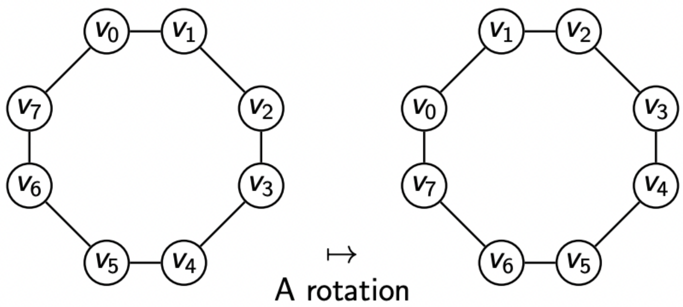

FIGURE 9. One possible rotation for the graph C_8

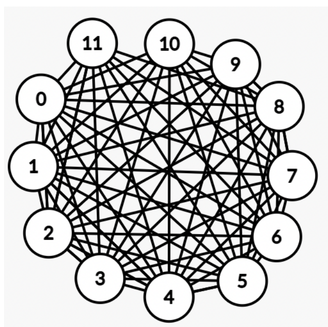

FIGURE 10. G, a complete graph with one edge missing.

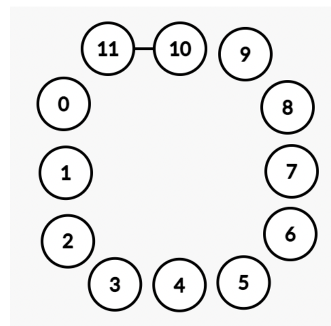

FIGURE 11. $\overline{G}$.

Theorem 4.7. [7] *The automorphism group of a complete graph of n vertices with any edge removed is isomorphic to* $S_n - 2 \times S_2$.

Indeed, this example is generalizable to all complete graphs with one edge removed. The Main Theorem serves as another way prove this theorem, and illustrates a picture of why this theorem works — revealing a visual and intuitive insight into the graph's properties.

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