

## 0.0 TOPIC

$$x + y \quad \text{some example equation related to the topic} \quad (1)$$

**Mynd?** Always try to have a visual representation of the equation

**Topic:** describes. minute details about the equation

**Theorem:** *Definition of "Topic"* theorem + tab

**Uses:** What is it be used for in linear algebra

**Example:** "We can use **this** to calculate **that**", followed by an example

**Mynd?** Always try to have a visual along with the example

**Exercise:** Exercise for the reader

## Revision

### 0.1 LINEAR EQUATIONS

$$x + 2 = y \quad (2)$$

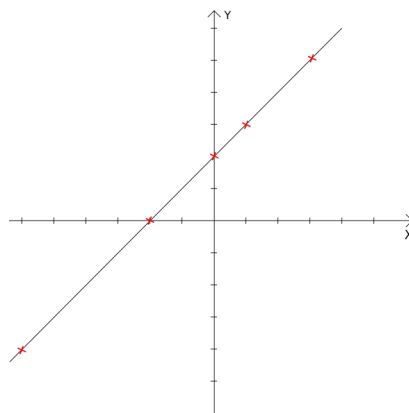
This is an example of a **linear equation**, one that has two variables,  $x$  and  $y$ , and it describes how the value of one of the variables depends on the value of the other variable.

What makes it **linear** is that every variable is only raised to the first power, so *this*

$$x^2 + 2 = y$$

is **not** a linear equation.

Lets plot (draw) equation (2), for a few diferent values of  $x$ , say,  $-6$ ,  $-2$ ,  $1$  and  $5$ .



As you can see, the graph of this equation is a **straight line**, which is true for all linear equations

## 0.1 LINEAR EQUATION WITH MULTIPLE VARIABLES

$$x + 9y + z = 3 \quad (3)$$

Lets now extend our definition of a linear equation to include more variables. (2) had only  $x$  and  $y$ , but (3) has 3 variables,  $x$ ,  $y$  and  $z$ .

The 9 in front of the  $y$ ? That is called a *coefficient*, and the 3 on the right-hand-side is called a *constant*

Because we will (eventually) run out of letters in the alphabet, we write our equations like this:

$$c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = b$$

Here the  $c$ 's are the *coefficients*, the  $x$ 's are the *variables* and the  $b$  is (spoiler) the *constant*.

It may look complex the first time, but you'll get used to reading equations like this.

**Lets look at an example:** Jimmy goes to the store to buy cokes, snickers and apples. Jimmy knows that a can of coke is 2.2\$, snickers is 1.6\$ and an apple is 3.0\$

If  $c_1$  is price of coke,  $c_2$  is price of snickers and  $c_3$  is price of apples, our equation would look like this

$$2.2x_1 + 1.6x_2 + 3.0x_3 = b$$

Jimmy needs a couple of cokes ( $x_1$ ) and four apples ( $x_3$ ). Jimmy has 20\$.

*How many snickers bars can he buy with the leftover money?*

$$2.2 \cdot 2 + 1.6x_2 + 3.0 \cdot 4 = 20$$

Do the math and help Jimmy get his snickers by solving for  $x_2$ .

## 0.1 SYSTEM OF EQUATIONS

$$2x_1 + x_2 = 17$$

$$x_1 + 3x_2 = 26$$

The equations above are an example of a **system of equations**.

We want to solve these systems by finding the correct values for the *variables*, in this case,  $x_1$  and  $x_2$ , so that both equations work out.

We could start by *guessing* that  $x_1 = 3$  and  $x_2 = 5$ , which would give us

$$2 \cdot 3 + 2 = 8 \neq 17$$

$$3 + 3 \cdot 5 = 18 \neq 26$$

This is far from correct, we need the **the substitution method**.

The substitution method consists of **two** steps, that you use over and over again until the system has been solved. These steps are:

1. Isolating a variable

2. Substitution

3. Simplification

Lets take another look at our system

$$2x_1 + x_2 = 17 \tag{4}$$

$$x_1 + 3x_2 = 26 \tag{5}$$

and solve it using the substitution method.

1. isolate the  $x_1$  from (4),  $x_1 = 26 - 3x_2$
2. substitute  $x_1$  into (3),  $2(26 - 3x_2) + x_2 = 17$
3. now the system looks like

$$2(26 - 3x_2) + x_2 = 17 \quad (6)$$

$$x_1 + 3x_2 = 26 \quad (7)$$

4. which simplifies to

$$x_2 = 7 \quad (8)$$

$$x_1 + 3x_2 = 26 \quad (9)$$

5. now we just insert  $x_2$  into (8) to get

$$x_2 = 7 \quad (10)$$

$$x_1 + 21 = 26 \quad (11)$$

6. so (8) simplifies to  $x_1 = 5$

Now its your turn to solve the following

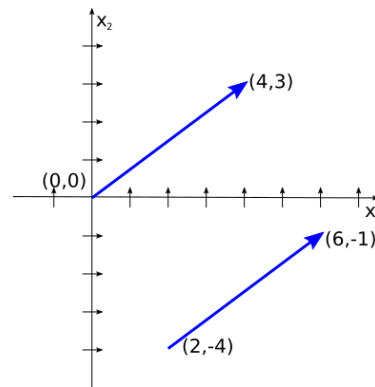
$$6x_1 + 2x_2 = 70$$

$$3x_1 + 3x_2 = 45$$

# Vectors

## 0.2 VECTOR PROPERTIES

$$\vec{v} = [4 \ 3] = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



**Vectors:** Its easy to think of vectors as a **length** and a **direction**, usually denoted  $v = [x_1 \ x_2 \ x_3 \ \dots \ x_n]$ , where the **x's** are usually called **elements**.

Above we have an example of a **(2-dimensional)** vector, written both as a **row vector** and a **column vector**. The vector represents a "travel" by 4 steps along the  $x_1$  axis and 3 steps along the  $x_2$  axis. So if you find yourself positioned at the point  $(2, -4)$  and someone "applies" this vector to you, you'll be moved to  $(2 + 4, -4 + 3) = (6, -1)$ .

**Properties:** The **length** (also known as *norm* or *size*) of a vector is written as  $|\vec{v}|$  and found with the Pythagorean-theorem:



$$|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

For higher dimensional vectors, the calculations look similar

$$|\mathbf{u}| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

A two-dimensional vector also has a **direction** written as  $\theta_{\vec{v}}$  and calculated using absolute classic geometry

$$\theta_{\vec{v}} = \tan^{-1} \frac{3}{4}$$

**Exercise:** Find the length of the 5-dimensional vector  $\vec{p} = [6 \ 2 \ 3 \ 9 \ 1]$

## 0.2 ELEMENTARY VECTOR OPERATIONS

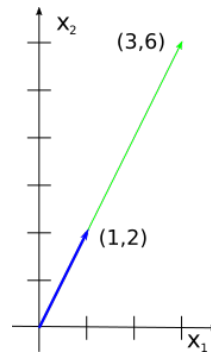
$3 \cdot [1 \ 2]$	scalar multiplication
$[1 \ 2]^T$	transpose
$[2 \ 1] + [1 \ 2]$	addition
$[2 \ 1] - [1 \ 2]$	subtraction

**Operations:** These 4 operations are the most elementary and common operations you will be using in linear algebra. They are

- **Scalar multiplication** simply multiply every element of the vector with the scalar.

$$3 \cdot [1 \ 2] = [3 \ 6]$$

All this operation does is  
lengthen the vector.



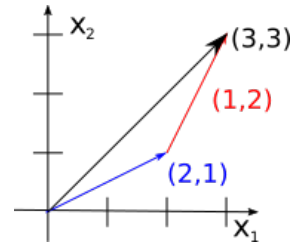
- **Transposing** a vector is simply converting it from a row-vector to a column-vector, or a column-vector back to a row-vector.

$$[1 \ 2]^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Adding two vectors is straightforward

$$\begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \end{bmatrix}$$

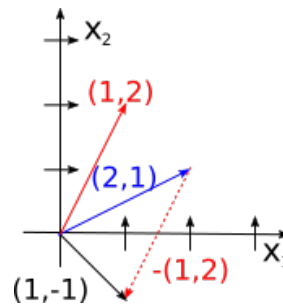
Visually, when adding two vectors you simply add one to the end of the other. Note that it does not matter in which order you add them.



- Subtraction, like addition, is intuitive

$$\begin{bmatrix} 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Here we negate the vector we are subtracting (the red one in this case) and just like addition we add it to the end of the other.



**Exercise:** Here are 3 vectors,  $u = \begin{bmatrix} 3 & 5 \end{bmatrix}$ ,  $v = \begin{bmatrix} 8 & 10 \end{bmatrix}$ ,  $p = \begin{bmatrix} 10 & 1 & 1 \end{bmatrix}$ , can you add them together?

## 0.2 VECTOR DOT PRODUCT

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1 \cdot 4 + 3 \cdot 1 = 7$$

**Dot product:** Another useful **vector operation** is the **dot product**. Take the product of corresponding elements, and then add together the result.

**Definition:** Two vectors  $\mathbf{u} = [u_1 \ u_2 \ \dots \ u_n]$  and  $\mathbf{v} = [v_1 \ v_2 \ \dots \ v_n]$ , of the same dimension ( $n$ ) have the dot product

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots u_n \cdot v_n$$

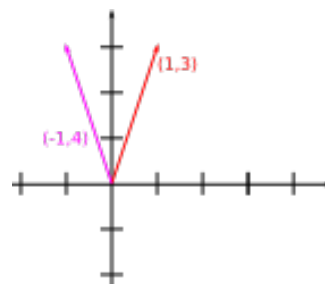
or

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = |\vec{\mathbf{u}}| \cdot |\vec{\mathbf{v}}| \cos \theta_{uv}$$

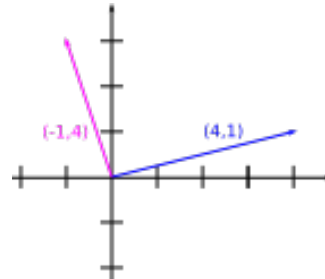
where  $\theta_{uv}$  is the angle between  $\vec{\mathbf{u}}$  and  $\vec{\mathbf{v}}$ .

**Uses:** The dot product tells us a lot about the direction of the vectors relative to each other:

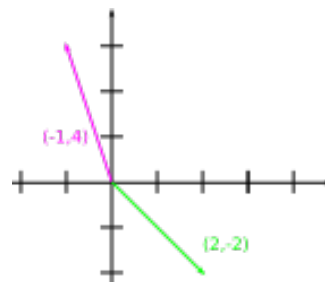
*Positive dot product:* The angle between the vectors is less than  $90^\circ$ .



*Dot product is 0:* The vectors are perpendicular to each other.



*Negative dot product:* The angle between the vectors is more than  $90^\circ$ .



The dot product can even help us find the exact angle between the vectors using the equation in the definition above. If we isolate  $\cos \theta_{uv}$  in the equation we get

$$\cos \theta_{uv} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

**Examples:** Let  $\vec{u} = [-4 \ 2 \ 8]$  and  $\vec{v} = [-3 \ 5 \ 1]$ .

1. Calculate the dot product  $\vec{u} \bullet \vec{v}$ .

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-4 \cdot -3) + (2 \cdot 5) + (8 \cdot 1) \\ &= 12 + 10 + 8 \\ &= 30\end{aligned}$$

2. *What does the dot product tell us about the angle between the vectors?*

The dot product is positive so the angle between the vectors is less than  $90^\circ$

3. *Calculate the exact angle between these two vectors.*

To do this we need the dot product of the vectors, which is 30, as well as the product of their lengths. Let's find that:

$$|\vec{u}| = \sqrt{(-4)^2 + 2^2 + 8^2} = \sqrt{84}$$

$$|\vec{v}| = \sqrt{(-3)^2 + 5^2 + 1^2} = \sqrt{35}$$

$$\text{so } |\vec{u}| \cdot |\vec{v}| = \sqrt{84} \cdot \sqrt{35}.$$

Plugging these values into our equation gives us

$$\cos \theta_{uv} = \frac{30}{\sqrt{84} \cdot \sqrt{35}}$$

Now we can find  $\theta_{uv}$ :

$$\theta_{uv} = \arccos \left( \frac{30}{\sqrt{84} \cdot \sqrt{35}} \right) \approx 56.4^\circ$$

**Exercise:** Do the same for  $\vec{u} = [5 \ 3 \ -2]$  and  $\vec{v} = [2 \ -2 \ -1]$ .