0.0 TOPIC

$$x + y \quad \text{some example equation related to the topic} \tag{1}$$

Mynd? Always try to have a visual representation of the equation

Topic: describes. minute details about the equation

Theorem: *Definition of "Topic"* theorem + tab

Uses: What is it be used for in linear algebra

Example: "We can use **this** to calculate **that**", followed by an example

Mynd? Always try to have a visual along with the example

Exercise: Exercise for the reader

Revision

0.1 Linear equations

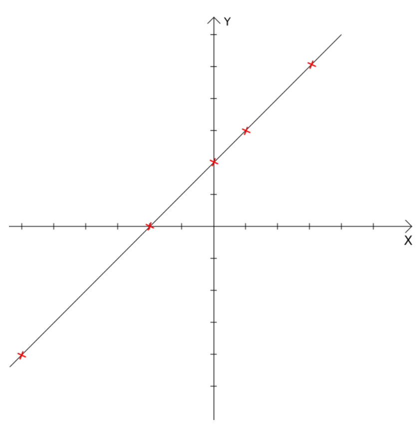
$$x + 2 = y \tag{2}$$

This is an example of a **linear equation**, one that has two variables, x and y, and it describes how the value of one of the variables depends on the value of the other variable.
What makes it **linear** is that every variable is only raised to the first power, so *this*

$$x^2 + 2 = y$$

is **not** a linear equation.

Lets plot (draw) equation (2), for a few diferent values of x, say, -6, -2, 1 and 5.



As you can see, the graph of this equation is a **straight line**, which is true for all linear equations

0.1 Linear equation with multiple variables

$$x + 9y + z = 3 \tag{3}$$

Lets now extend our definition of a linear equation to include more variables. (2) had only x and y, but (3) has 3 variables, x, y and z.
The 9 in front of the y? That is called a *coefficient*, and the 3 on the right-hand-side is called a *constant*

Because we will (eventually) run out of letters in the alphabet, we write our equations like this:

$$c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = b$$

Here the c's are the *coefficients*, the x's are the *variables* and the b is (spoiler) the *constant* .
It may look complex the first time, but you'll get used to reading equations like this.

Lets look at an example: Jimmy goes to the store to buy cokes, snickers and apples. Jimmy knows that a can of coke is $2.2\$$, snickers is $1.6\$$ and an apple is $3.0\$$
If c_1 is price of coke, c_2 is price of snickers and c_3 is price of apples, our equation would look like this

$$2.2x_1 + 1.6x_2 + 3.0x_3 = b$$

Jimmy needs a couple of cokes (x_1) and four apples (x_3). Jimmy has $20\$$.
How many snickers bars can he buy with the leftover money?

$$2.2 \cdot 2 + 1.6x_2 + 3.0 \cdot 4 = 20$$

Do the math and help Jimmy get his snickers by solving for x_2.

0.1 System of equations

$$2x_1 + x_2 = 17$$
$$x_1 + 3x_2 = 26$$

The equations above are an example of a **system of equations**.

We want to solve these systems by finding the correct values for the *variables*, in this case, x_1 and x_2, so that both equations work out.

We could start by *guessing* that $x_1 = 3$ and $x_2 = 5$, which would give us

$$2 \cdot 3 + 2 = 8 \neq 17$$
$$3 + 3 \cdot 5 = 18 \neq 26$$

This is far from correct, we need the **the substitution method**.

The substitution method consists of **two** steps, that you use over and over again until the system has been solved. These steps are:

1. Isolating a variable
2. Substitution
3. Simplification

Lets take another look at our system

$$2x_1 + x_2 = 17 \tag{4}$$
$$x_1 + 3x_2 = 26 \tag{5}$$

and solve it using the substitution method.

1. isolate the x_1 from (4), $x_1 = 26 - 3x_2$
2. substitute x_1 into (3), $2(26 - 3x_2) + x_2 = 17$
3. now the system looks like

$$2(26 - 3x_2) + x_2 = 17 \tag{6}$$

$$x_1 + 3x_2 = 26 \tag{7}$$

4. which simplifies to

$$x_2 = 7 \tag{8}$$

$$x_1 + 3x_2 = 26 \tag{9}$$

5. now we just insert x_2 into (8) to get

$$x_2 = 7 \tag{10}$$

$$x_1 + 21 = 26 \tag{11}$$

6. so (8) simplifies to $\mathbf{x_1 = 5}$

Now its your turn to solve the following

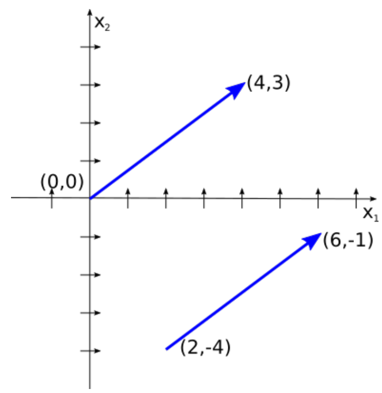
$$6x_1 + 2x_2 = 70$$

$$3x_1 + 3x_2 = 45$$

Vectors

0.2 Vector properties

$$\vec{v} = [4 \;\; 3] = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Vectors: Its easy to think of vectors as a **length** and a **direction**, usually denoted $v = [x_1 \;\; x_2 \;\; x_3 \;\; \ldots \;\; x_n]$, where the **x's** are usually called elements.

Above we have an example of a (**2-dimensional**) vector, written both as a row vector and a column vector. The vector represents a "travel" by 4 steps along the x_1 axis and 3 steps along the x_2 axis. So if you find yourself positioned at the point $(2, -4)$ and someone "applies" this vector to you, you'll be moved to $(2+4, -4+3) = (6, -1)$.

Properties: The **length** (also known as *norm* or *size*) of a vector is written as $|\vec{v}|$ and found with the Pythagorean-theorem:

$$|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

For higher dimensional vectors, the calculations look similar

$$|u| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

A two-dimensional vector also has a **direction** written as $\theta_{\vec{v}}$ and calculated using absolute classic geometry

$$\theta_{\vec{v}} = \tan^{-1}\frac{3}{4}$$

Exercise: Find the length of the 5-dimensional vector $\vec{p} = [6 \;\; 2 \;\; 3 \;\; 9 \;\; 1]$

0.2 Elementary vector operations

$$3 \cdot [1 \;\; 2] \qquad \text{scalar multiplication}$$

$$[1 \;\; 2]^{\mathsf{T}} \qquad \text{transpose}$$

$$[2 \;\; 1] + [1 \;\; 2] \qquad \text{addition}$$

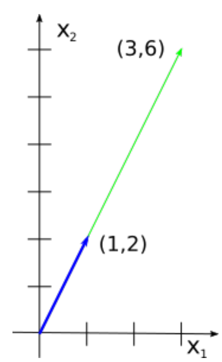
$$[2 \;\; 1] - [1 \;\; 2] \qquad \text{subtraction}$$

Operations: These 4 operations are the most elementary and common operations you will be using in linear algebra. They are

- Scalar multiplication simply multiply every element of the vector with the scalar.

$$3 \cdot [1 \;\; 2] = [3 \;\; 6]$$

All this operation does is lengthen the vector.



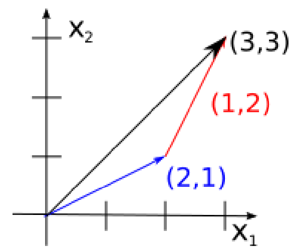
- Transposing a vector is simply converting it from a row-vector to a colum-vector, or a column-vector back to a row-vector.

$$[1 \;\; 2]^{\mathsf{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Adding two vectors is straighforward

$$[2 \;\; 1] + [1 \;\; 2] = [3 \;\; 3]$$

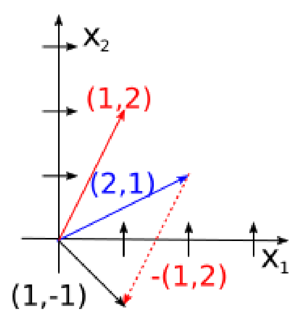
Visually, when adding two vectors you simply add one to the end of the other. Note that it does not matter in which order you add them.



- Subtraction, like addition, is intuitive

$$[2 \;\; 1] - [1 \;\; 2] = [1 \;\; -1]$$

Here we negate the vector we are subtracting (the red one in this case) and just like addition we add it to the end of the other.



Exercice: Here are 3 vectors, $u = [3 \;\; 5], v = [8 \;\; 10], p = [10 \;\; 1 \;\; 1]$, can you add them together?

0.2 Vector dot product

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1 \cdot 4 + 3 \cdot 1 = 7$$

Dot product: Another useful **vector operation** is the **dot product**. Take the product of corresponding elements, and then add together the result.

Definition: *Two vectors* $u = [u_1 \ \ u_2 \ \ \dots \ \ u_n]$ *and* $v = [v_1 \ \ v_2 \ \ \dots \ \ v_n]$, *of the same dimension (**n**) have the dot product*

$$\vec{u} \bullet \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots u_n \cdot v_n$$

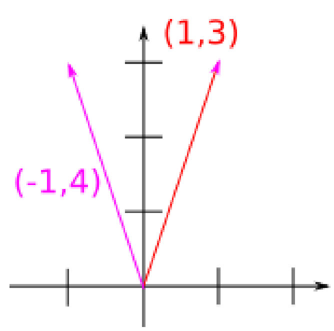
or

$$\vec{u} \bullet \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos \theta_{uv}$$

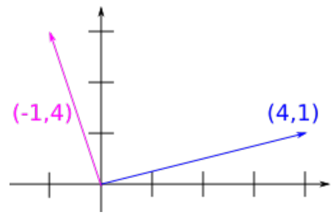
where θ_{uv} *is the angle between* $\vec{u}$ *and* $\vec{v}$.

Uses: The dot product tells us a lot about the direction of the vectors relative to each other:

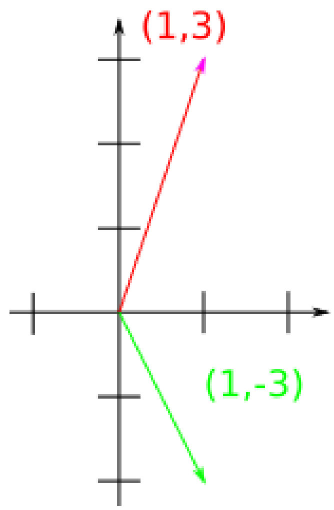
Positive dot product: The angle between the vectors is less than $90°$.



Dot product is 0: The vectors are perpendicular to each other.



Negative dot product: The angle between the vectors is more than $90°$.



The dot product can even help us find the exact angle between the vectors using the equation in the definition above. If we isolate $\cos\theta_{uv}$ in the equation we get

$$\cos\theta_{uv} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

Examples: Let $\vec{u} = [-4 \;\; 2 \;\; 8]$ and $\vec{v} = [-3 \;\; 5 \;\; 1]$.

1. Calculate the dot product $\vec{u} \bullet \vec{v}$.

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-4 \cdot -3) + (2 \cdot 5) + (8 \cdot 1) \\ &= 12 + 10 + 8 \\ &= 30\end{aligned}$$

2. What does the dot product tell us about the angle between the vectors?

   The dot product is positive so the angle between the vectors is less than $90°$

3. Calculate the exact angle between these two vectors.

   To do this we need the dot product of the vectors, which is 30, as well as the product of their lengths. Let's find that:

$$|\vec{u}|$$

Mynd? Always try to have a visual along with the example

Exercise: Exercise for the reader