0.0 Topic

$$x + y \quad \text{some example equation related to the topic} \tag{1}$$

Mynd? Always try to have a visual representation of the equation

Topic: describes. minute details about the equation

Theorem: *Definition of "Topic"* theorem + tab

Uses: What is it be used for in linear algebra

Example: "We can use **this** to calculate **that**", followed by an example

Mynd? Always try to have a visual along with the example

Exercise: Exercise for the reader

Revision

0.1 Linear equations

$$x + 2 = y \tag{2}$$

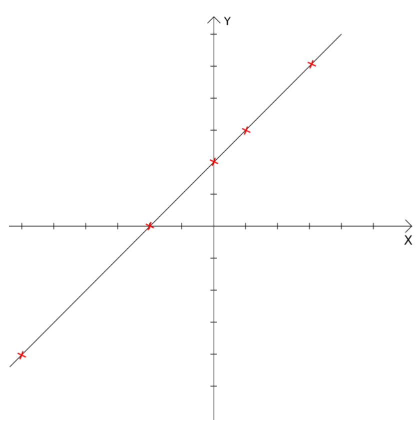
This is an example of a **linear equation**, one that has two variables, x and y, and it describes how the value of one of the variables depends on the value of the other variable.

What makes it **linear** is that every variable is only raised to the first power, so *this*

$$x^2 + 2 = y$$

is **not** a linear equation.

Lets plot (draw) equation (2), for a few diferent values of x, say, -6, -2, 1 and 5.



As you can see, the graph of this equation is a **straight line**, which is true for all linear equations

0.1 Linear equation with multiple variables

$$x + 9y + z = 3 \tag{3}$$

Lets now extend our definition of a linear equation to include more variables. (2) had only x and y, but (3) has 3 variables, x, y and z.
The 9 in front of the y? That is called a *coefficient*, and the 3 on the right-hand-side is called a *constant*

Because we will (eventually) run out of letters in the alphabet, we write our equations like this:

$$c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = b$$

Here the c's are the *coefficients*, the x's are the *variables* and the b is (spoiler) the *constant* .
It may look complex the first time, but you'll get used to reading equations like this.

Lets look at an example: Jimmy goes to the store to buy cokes, snickers and apples. Jimmy knows that a can of coke is 2.2$, snickers is 1.6$ and an apple is 3.0$
If c_1 is price of coke, c_2 is price of snickers and c_3 is price of apples, our equation would look like this

$$2.2x_1 + 1.6x_2 + 3.0x_3 = b$$

Jimmy needs a couple of cokes (x_1) and four apples (x_3). Jimmy has 20\$.

How many snickers bars can he buy with the leftover money?

$$2.2 \cdot 2 + 1.6x_2 + 3.0 \cdot 4 = 20$$

Do the math and help Jimmy get his snickers by solving for x_2.

0.1 System of equations

$$2x_1 + x_2 = 17$$
$$x_1 + 3x_2 = 26$$

The equations above are an example of a **system of equations**.

We want to solve these systems by finding the correct values for the *variables*, in this case, x_1 and x_2, so that both equations work out.

We could start by *guessing* that $x_1 = 3$ and $x_2 = 5$, which would give us

$$2 \cdot 3 + 2 = 8 \neq 17$$
$$3 + 3 \cdot 5 = 18 \neq 26$$

This is far from correct, we need the **the substitution method**.

The substitution method consists of **two** steps, that you use over and over again until the system has been solved. These steps are:

1. Isolating a variable
2. Substitution
3. Simplification

Lets take another look at our system

$$2x_1 + x_2 = 17 \tag{4}$$
$$x_1 + 3x_2 = 26 \tag{5}$$

and solve it using the substitution method.

1. isolate the x_1 from (4), $x_1 = 26 - 3x_2$
2. substitute x_1 into (3), $2(26 - 3x_2) + x_2 = 17$
3. now the system looks like

$$2(26 - 3x_2) + x_2 = 17 \tag{6}$$
$$x_1 + 3x_2 = 26 \tag{7}$$

4. which simplifies to

$$x_2 = 7 \tag{8}$$
$$x_1 + 3x_2 = 26 \tag{9}$$

5. now we just insert x_2 into (8) to get

$$x_2 = 7 \tag{10}$$
$$x_1 + 21 = 26 \tag{11}$$

6. so (8) simplifies to $\mathbf{x_1 = 5}$

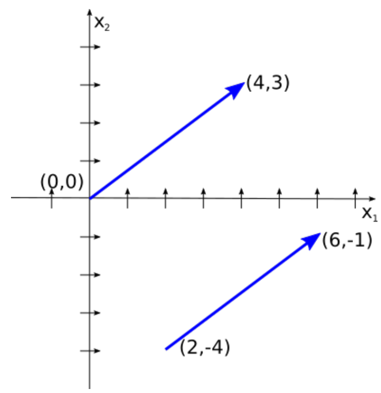
Now its your turn to solve the following

$$6x_1 + 2x_2 = 70$$
$$3x_1 + 3x_2 = 45$$

Vectors

0.2 Vector properties

$$\vec{v} = [4 \;\; 3] = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Vectors: Its easy to think of vectors as a **length** and a **direction**, usually denoted $v = [x_1 \;\; x_2 \;\; x_3 \;\; \ldots \;\; x_n]$, where the **x's** are usually called elements.

Above we have an example of a (**2-dimensional**) vector, written both as a row vector and a column vector. The vector represents a "travel" by 4 steps along the x_1 axis and 3 steps along the x_2 axis. So if you find yourself positioned at the point $(2, -4)$ and someone "applies" this vector to you, you'll be moved to $(2 + 4, -4 + 3) = (6, -1)$.

Properties: The **length** (also known as *norm* or *size*) of a vector is written as $|\vec{v}|$ and found with the Pythagorean-theorem:

$$|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

For higher dimensional vectors, the calculations look similar

$$|u| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

A two-dimensional vector also has a **direction** written as $\theta_{\vec{v}}$ and calculated using absolute classic geometry

$$\theta_{\vec{v}} = \tan^{-1}\frac{3}{4}$$

Exercise: Find the length of the 5-dimensional vector $\vec{p} = [6\ \ 2\ \ 3\ \ 9\ \ 1]$

0.2 Elementary vector operations

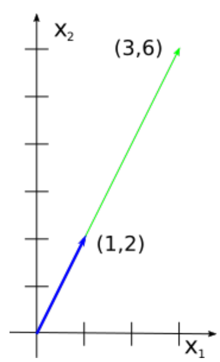
$3 \cdot [1\ \ 2]$	scalar multiplication
$[1\ \ 2]^{\mathsf{T}}$	transpose
$[2\ \ 1] + [1\ \ 2]$	addition
$[2\ \ 1] - [1\ \ 2]$	subtraction

Operations: These 4 operations are the most elementary and common operations you will be using in linear algebra. They are

- Scalar multiplication simply multiply every element of the vector with the scalar.

  $$3 \cdot [1\ \ 2] = [3\ \ 6]$$

  All this operation does is lengthen the vector.

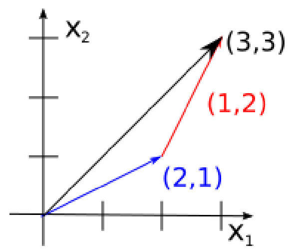
- Transposing a vector is simply converting it from a row-vector to a colum-vector, or a column-vector back to a row-vector.

  $$[1\ \ 2]^{\mathsf{T}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Adding two vectors is straighforward

$$[2\ \ 1] + [1\ \ 2] = [3\ \ 3]$$

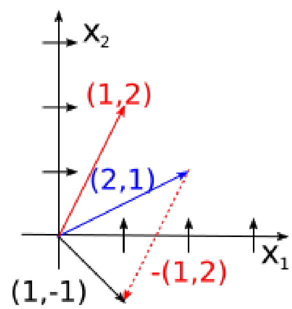
Visually, when adding two vectors you simply add one to the end of the other. Note that it does not matter in which order you add them.



- Subtraction, like addition, is intuitive

$$[2\ \ 1] - [1\ \ 2] = [1\ \ -1]$$

Here we negate the vector we are subtracting (the red one in this case) and just like addition we add it to the end of the other.



Exercice: Here are 3 vectors, $u = [3\ \ 5], v = [8\ \ 10], p = [10\ \ 1\ \ 1]$, can you add them together?

0.2 Vector dot product

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1 \cdot 4 + 3 \cdot 1 = 7$$

Dot product: Another useful **vector operation** is the **dot product**. Take the product of corresponding elements, and then add together the result.

Definition: *Two vectors* $u = [u_1 \ u_2 \ \ldots \ u_n]$ *and* $v = [v_1 \ v_2 \ \ldots \ v_n]$*, of the same dimension (**n**) have the dot product*

$$\vec{u} \bullet \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots u_n \cdot v_n$$

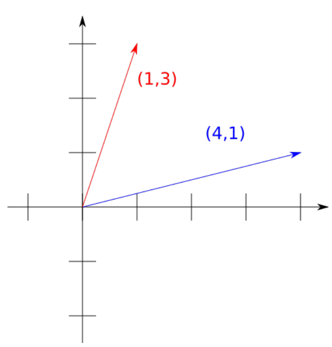
or

$$\vec{u} \bullet \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos \theta_{uv}$$

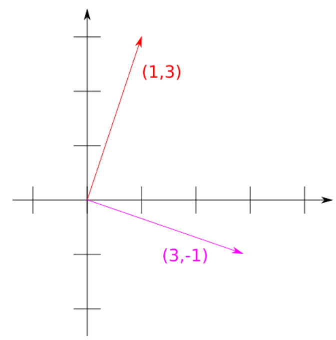
where θ_{uv} *is the angle between* $\vec{u}$ *and* $\vec{v}$.

Uses: The dot product tells us a lot about the direction of the vectors relative to each other:

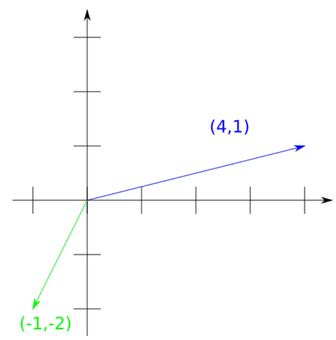
Positive dot product: The angle between the vectors is less than 90°.



Dot product is 0: The vectors are perpendicular to each other, i.e. the angle between them is exactly $90°$.



Negative dot product: The angle between the vectors is more than $90°$.



The dot product can even help us find the exact angle between the vectors using the equation in the definition above. If we isolate $\cos\theta_{uv}$ in the equation we get

$$\cos\theta_{uv} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}| \cdot |\vec{v}|} \tag{12}$$

Example: We're gonna find the angle between $\vec{u} = \begin{bmatrix} -4 \\ 2 \\ 8 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$.

First we find the dot-product between them

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-4 \cdot -3) + (2 \cdot 5) + (8 \cdot 1) \\ &= 12 + 10 + 8 = 30\end{aligned}$$

We also need their length

$$\begin{aligned}|\vec{u}| &= \sqrt{(-4)^2 + 2^2 + 8^2} = \sqrt{84} \\ |\vec{v}| &= \sqrt{(-3)^2 + 5^2 + 1^2} = \sqrt{35}\end{aligned}$$

using (12) we get

$$\cos\theta_{uv} = \frac{30}{\sqrt{84} \cdot \sqrt{35}}$$

so the angle is

$$\theta_{uv} = \cos^{-1}\left(\frac{30}{\sqrt{84} \cdot \sqrt{35}}\right) \approx 56.4^\circ$$

Exercise: Do the same for $\vec{u} = [5 \;\; 3 \;\; -2]$ and $\vec{v} = [2 \;\; -2 \;\; -1]$.

0.2 Cross Product

$$[1 \;\; 3 \;\; 2] \times [4 \;\; 1 \;\; 1]$$

Mynd

Cross Product: When taking the cross product of two vectors, $\vec{u}$ and $\vec{v}$, the result is a **vector** that is perpendicular to both $\vec{u}$ and $\vec{v}$. Also, the length of the cross product vector is equal to the area of the parallelogram bordered by $\vec{u}$ and $\vec{v}$ as shown on the diagram above.

Definition: *The Cross Product of two 3-dimensional vectors,* $\vec{u} = [u_1 \;\; u_2 \;\; u_3]$ *and* $\vec{v} = [v_1 \;\; v_2 \;\; v_3]$, is defined as the vector

$$\vec{u} \times \vec{v} = [u_2 \cdot v_3 - u_3 \cdot v_2 \quad u_3 \cdot v_1 - u_1 \cdot v_3 \quad u_1 \cdot v_2 - u_2 \cdot v_1] \tag{13}$$

Uses: As mentioned above the cross product is a vector perpendicular to both vectors which is useful in many scenarios, for example ....
Area.

Example: Let's calculate the cross product of $\vec{u} = [1 \;\; 7 \;\; 5]$ and $\vec{v} = [2 \;\; 11 \;\; 4]$. We simply plug the correct values into 13

$$\begin{aligned} [1 \;\; 7 \;\; 5] \times [2 \;\; 11 \;\; 4] &= [7 \cdot 4 - 5 \cdot 11 \quad 5 \cdot 2 - 1 \cdot 4 \quad 1 \cdot 11 - 7 \cdot 2] \\ &= [-27 \;\; 6 \;\; -3] \end{aligned}$$

This means that the area of the parallelogram bordered by $\vec{u}$ and $\vec{v}$ is

$$\begin{aligned}
|\vec{u} \times \vec{v}| &= |[-27 \;\; 6 \;\; -3]| \\
&= \sqrt{(-27)^2 + 6^2 + (-3)^2} \\
&= \sqrt{774} \\
&\approx 27.82
\end{aligned}$$

Exercise: Do the same for $\vec{u} = [6 \;\; 3 \;\; 1]$ and $\vec{v} = [0 \;\; 2 \;\; 5]$.

0.2 Linear Combinations of Vectors

$$5\begin{bmatrix}2\\3\end{bmatrix}+3\begin{bmatrix}3\\1\end{bmatrix}+2\begin{bmatrix}4\\2\end{bmatrix} \tag{14}$$

Mynd

Linear Combination: A linear combination of vectors is simply

Definition: *Definition of "Topic"*

Uses: What is it be used for in linear algebra

Example:

Mynd

Exercise: sum of vectors multiplied by constants. sqrt(774)