

Correcting the Coverage Bias of Quantile Regression

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Abstract

We develop a collection of methods for adjusting the predictions of quantile regression to ensure coverage. Our methods are model agnostic and can be used to correct for high-dimensional overfitting bias with only minimal assumptions. Theoretical results show that the estimates we develop are consistent and facilitate accurate calibration in the proportional asymptotic regime where the ratio of the dimension of the data and the sample size converges to a constant. This is further confirmed by experiments on both simulated and real data. One of the key components of our work is a new connection between the leave-one-out coverage and the fitted values of variables appearing in a dual formulation of the quantile regression. This facilitates the use of cross-validation in a variety of settings at significantly reduced computational costs.

Keywords: prediction set, cross-validation, high-dimensional statistics.

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1 Introduction

Quantile regression is a popular tool for bounding the tail of a target outcome. This method has a long history dating back to the foundational work of [Koenker & Bassett \(1978\)](#) and has found widespread applications across a variety of areas ([Koenker & Hallock 2001](#), [Koenker 2017](#)). Classical results demonstrate that as the sample size increases quantile regression estimates are consistent, normally distributed around their population analogs ([Koenker & Bassett 1978](#), [Angrist et al. 2006](#)), and, perhaps most critically, achieve their target coverage level ([Jung et al. 2023](#), [Duchi 2025](#)).

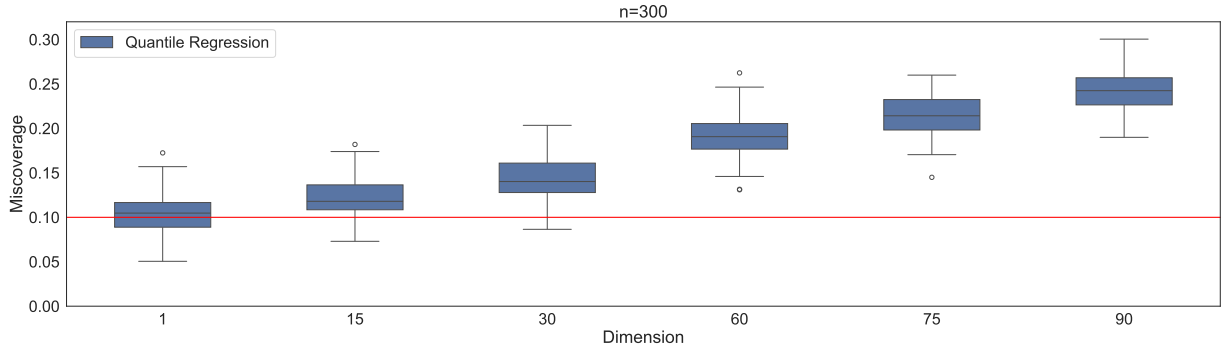


Figure 1: Miscoverage of (unregularized) quantile regression with model $Y_i \sim \beta_0 + X_i^\top \beta$ on i.i.d. data $\{(X_i, Y_i)\}_{i=1}^n$ sampled from the Gaussian linear model $Y_i = X_i^\top \tilde{\beta} + \epsilon_i$ for $X_i \sim \mathcal{N}(0, I_d)$, $\epsilon \sim \mathcal{N}(0, 1)$, and $\epsilon_i \perp\!\!\!\perp X_i$. Boxplots in the figure show the empirical distribution of the training-conditional coverage, $\mathbb{P}(Y_{n+1} \leq \hat{\beta}_0 + X_{n+1}^\top \hat{\beta} \mid \{(X_i, Y_i)\}_{i=1}^n)$ where $(\hat{\beta}_0, \hat{\beta})$ denote the estimated coefficients at quantile level $\tau = 0.9$ and (X_{n+1}, Y_{n+1}) is an independent sample from the same model. The results come from 100 trials where in each trial the coverage is evaluated over a test set of size 2000 and the population coefficients are sampled as $\tilde{\beta} \sim \mathcal{N}(0, I_d/d)$. The red line shows the target miscoverage level of $1 - \tau = 0.1$.

Although the classical theory can be accurate for large datasets, it is often insufficient to fully capture the realities of finite samples. Figure 1 shows the realized miscoverage of quantile estimates fit at target level $\tau = 0.9$ in a well specified linear model $Y_i = X_i^\top \tilde{\beta} + \epsilon_i$ with $\epsilon_i \perp\!\!\!\perp X_i$ and $X_i \in \mathbb{R}^d$. In agreement with the classical theory, we see that when X_i is very low-dimensional (e.g., $d = 1$) quantile regression reliably obtains the target miscoverage rate of $1 - \tau = 0.1$. However, the scope of this theory is limited and the coverage shows visible bias in what might be typically considered to be small or moderate dimensions (e.g., $d \in \{15, 30\}$ compared to a sample size of $n = 300$). Perhaps unsurprisingly, this issue only worsens as the dimension increases and quantile regression exhibits over 2 times the target error rate at dimension $d = 90$.

Formal characterization of the coverage bias of quantile regression was first given in [Bai et al. \(2021\)](#). They eschew classical theory and instead work under a proportional asymptotic framework in which the ratio of the dimension of the data and the sample size converges to a constant. Under a stylized linear model, they show that in this regime the coverage of quantile regression converges to value different from the target level and provide an exact

formula for quantifying this bias. Interestingly, while both under- and overcoverage are possible, they demonstrate that in most settings quantile regression will tend to undercover.* This is consistent with the results in Figure 1 as well as additional empirical results that we will present in Section 6.

Two proposals have been made in the literature for correcting quantile regression’s bias. Under the same linear model assumptions, Bai et al. (2021) derive a simple method for adjusting the nominal level to account for overfitting. While quite effective, this procedure is limited in scope to small aspect ratios and a restrictive model for the data. A more generic procedure that does not require any such modeling assumptions was given in Gibbs et al. (2025). They employ a technique known as full conformal inference, which augments the regression fit with a guess of the unseen test point. This mimics the effect of overfitting the training data on the test point, thereby eliminating the resulting bias. In general, this approach has two main drawbacks. First, it requires randomization in order to obtain the desired coverage level. As we will show shortly in Section 2, this randomization can be significant and may cause the quantile estimate to vary substantially. Second, additional computation is required for every test point in order to accurately incorporate it into the fit. This contrasts sharply with standard quantile regression, which once fitted can issue new predictions at the cost of computing just a single inner product. Depending on the application, significant additional test-time computational complexity of this form may not be permissible.

In this article, we develop three alternative procedures for adjusting the quantile regression fit. All of these methods are deterministic, and two of them require per test point computation that is identical to standard quantile regression. Briefly, our methods can be summarized as 1) a deterministic analog of the procedure proposed in Gibbs et al. (2025), 2) a level-adjustment method that tunes the nominal level of the quantile regression loss, and 3) an additive adjustment that adds a constant bias to the quantile estimates. To tune the parameters of these latter two methods, we will utilize leave-one-out cross-validation. A central contribution of our work is a new connection between the leave-one-out coverage indicators and a set of dual variables to the quantile regression. This will enable us to compute the entire set of leave-one-out coverage values in time identical to that of running a single regression fit and facilitate hyperparameter tuning at significantly reduced computational costs.

The remainder of this article is structured as follows. After giving a brief overview of the work of Gibbs et al. (2025) in Section 2, we formally introduce our main methods in Section 3. Section 4 then gives a formal connection between the quantile dual and leave-one-out coverage. Theoretical results showing the consistency of our proposals in the proportional asymptotic regime are presented in Section 5, while Sections 3.4 and 6 give empirical results demonstrating the accuracy of our estimates in finite samples. Overall, our results show

*As a matter of terminology, if \hat{q}_τ is an estimate of the $\tau \in [1/2, 1]$ quantile of Y we say that \hat{q}_τ undercovers if $\mathbb{P}(Y \leq \hat{q}_\tau) < \tau$ and overcovers if $\mathbb{P}(Y \leq \hat{q}_\tau) > \tau$. For $\tau < 1/2$ this terminology is reversed and we say that \hat{q}_τ undercovers if $\mathbb{P}(Y \leq \hat{q}_\tau) > \tau$ and overcovers otherwise. This is motivated by the fact that for $\tau > 1/2$ (resp. $\tau < 1/2$) the τ -quantile is designed to be a high probability upper (resp. lower) bound on Y . We use the terms undercoverage and overcoverage to reflect these goals.

that all of our proposed methods are robust and provide reliable coverage irrespective of the dimension of the data.

The theoretical results in this paper contribute to a growing literature on characterizing and correcting for overfitting bias in high dimensions (e.g., [Karoui et al. \(2013\)](#), [Donoho & Montanari \(2013\)](#), [Zhang & Zhang \(2013\)](#), [Javanmard & Montanari \(2014\)](#), [van de Geer et al. \(2014\)](#), [Thrampoulidis et al. \(2018\)](#), [Hastie et al. \(2022\)](#)). Of particular relevance to our work are the Gaussian comparison inequalities of [Gordon \(1985, 1988\)](#) and their development for high dimensional M-estimation problems in [Thrampoulidis et al. \(2018\)](#). These tools will allow us to characterize the asymptotic behaviour of the quantile regression dual variables and, through their connection to leave-one-out coverage, to prove the consistency of our cross-validation estimates. There is a large body of literature investigating the performance of cross-validation in high-dimensional parameter tuning (e.g., [Steinberger & Leeb \(2016\)](#), [Rad et al. \(2020\)](#), [Bayle et al. \(2020\)](#), [Austern & Zhou \(2020\)](#), [Xu et al. \(2021\)](#), [Patil et al. \(2021, 2022\)](#), [Steinberger & Leeb \(2023\)](#), [Zou et al. \(2025\)](#)). On a technical level, these articles often require smoothness and/or strong convexity assumptions on the loss in order to derive exact formulas for the leave-one-out coefficients. In contrast, we will be interested in the behaviour of the leave-one-out coverage of quantile regression, which is a discontinuous objective taken over parameter estimates coming from a non-differentiable loss. Here, our connection to the dual program will be critical in allowing us to avoid technical problems present in prior work and facilitate the application of tools which are typically unavailable in studies of cross-validation.

Notation: In the remainder of this article we let $\{(X_i, Y_i)\}_{i=1}^{n+1} \in \mathbb{R}^d \times \mathbb{R}$ denote a set of covariate-response pairs, where the first n points denote the training set and the last entry is the test point for which Y_{n+1} is unobserved. Given a target level $\tau \in (0, 1)$, we will be interested in quantile regression estimates of the form

$$(\hat{\beta}_0, \hat{\beta}) = \underset{(\beta_0, \beta) \in \mathbb{R}^{d+1}}{\operatorname{argmin}} \sum_{i=1}^n \ell_\tau(Y_i - \beta_0 - X_i^\top \beta) + \mathcal{R}(\beta),$$

where $\ell_\tau(r) = \tau r - \min\{r, 0\}$ is the usual pinball loss and $\mathcal{R} : \mathbb{R}^d \rightarrow \mathbb{R}$ is an optional regularization function. For d fixed and n tending to infinity, the quantile regression estimates satisfy the target coverage guarantee $\mathbb{P}(Y_{n+1} \leq \hat{\beta}_0 + X_{n+1}^\top \hat{\beta}) \rightarrow \tau$. Our goal in this article is to adjust the regression procedure to recover this guarantee even in cases where $d/n \rightarrow \gamma \in (0, \infty)$ converges to a constant.

2 Overview of the methods of [Gibbs et al. \(2025\)](#)

As discussed above, the task of removing the coverage bias of quantile regression has been previously considered by [Gibbs et al. \(2025\)](#). They propose to adjust the regression by adding an imputed guess for the test point into the fit. Concretely, they consider unpenalized

regressions of the form

$$(\hat{\beta}_0^{\text{adj.},y}, \hat{\beta}^{\text{adj.},y}) = \underset{(\beta_0, \beta) \in \mathbb{R}^{d+1}}{\operatorname{argmin}} \sum_{i=1}^n \ell_\tau(Y_i - \beta_0 - X_i^\top \beta) + \ell_\tau(y - \beta_0 - X_{n+1}^\top \beta), \quad (2.1)$$

and define the adjusted quantile estimate

$$\hat{q}_{\text{GCC}}(X_{n+1}) = \sup\{y : y \leq \hat{\beta}_0^{\text{adj.},y} + X_{n+1}^\top \hat{\beta}^{\text{adj.},y}\},$$

as the maximum value of y that is covered by the regression fit with y in place of Y_{n+1} . Under no assumptions on the data beyond that they are i.i.d., this adjustment has the conservative coverage guarantee $\mathbb{P}(Y_{n+1} \leq \hat{q}_{\text{GCC}}(X_{n+1})) \geq \tau$.

Unfortunately, the coverage of this method is not typically tight and the authors find that $\hat{q}_{\text{GCC}}(X_{n+1})$ can exhibit significant overcoverage bias in high dimensions. To further correct this estimate, they additionally introduce a smaller, randomized threshold that is constructed using the quantile regression dual. More formally, let $r_{n+1} = y - \beta_0 - X_{n+1}^\top \beta$ and $r_i = Y_i - \beta_0 - X_i^\top \beta$ for $i \in \{1, \dots, n\}$ denote a set of primal variables that are constrained to be equal to the residuals. Let $\eta \in \mathbb{R}^{n+1}$ denote the corresponding dual variables for these constraints. Then, the adjusted quantile regression (2.1) can be equivalently written in its primal form as

$$\begin{aligned} (\hat{\beta}_0^{\text{adj.},y}, \hat{\beta}^{\text{adj.},y}, \hat{r}^{\text{adj.},y}) = & \underset{(\beta_0, \beta) \in \mathbb{R}^{d+1}, r \in \mathbb{R}^{n+1}}{\operatorname{argmin}} \sum_{i=1}^{n+1} \ell_\tau(r_i) \\ & \text{subject to} \quad r_{n+1} = y - \beta_0 - X_{n+1}^\top \beta, \\ & \quad r_i = Y_i - \beta_0 - X_i^\top \beta, \quad \forall i \in \{1, \dots, n\}, \end{aligned}$$

with associated Lagrangian,

$$L(\beta_0, \beta, r, \eta) = \sum_{i=1}^{n+1} \ell_\tau(r_i) + \sum_{i=1}^n \eta_i (Y_i - \beta_0 - X_i^\top \beta - r_i) + \eta_{n+1} (y - \beta_0 - X_{n+1}^\top \beta - r_{n+1}),$$

and dual program,

$$\begin{aligned} \hat{\eta}^{\text{adj.},y} = & \underset{\eta \in \mathbb{R}^{n+1}}{\operatorname{argmax}} \sum_{i=1}^n \eta_i Y_i + \eta_{n+1} y \\ & \text{subject to} \quad \sum_{i=1}^{n+1} \eta_i = 0, \quad \sum_{i=1}^{n+1} \eta_i X_i = 0, \quad -(1 - \tau) \preceq \eta \preceq \tau. \end{aligned}$$

To connect the dual variables to coverage, note that differentiating the Lagrangian with respect to r_{n+1} gives the first-order condition

$$\hat{\eta}_{n+1}^{\text{adj.},y} \in \begin{cases} \{\tau\}, & y > \hat{\beta}_0^{\text{adj.},y} + X_{n+1}^\top \hat{\beta}^{\text{adj.},y}, \\ \{-(1 - \tau)\}, & y < \hat{\beta}_0^{\text{adj.},y} + X_{n+1}^\top \hat{\beta}^{\text{adj.},y}, \\ [-(1 - \tau), \tau], & y = \hat{\beta}_0^{\text{adj.},y} + X_{n+1}^\top \hat{\beta}^{\text{adj.},y}. \end{cases}$$

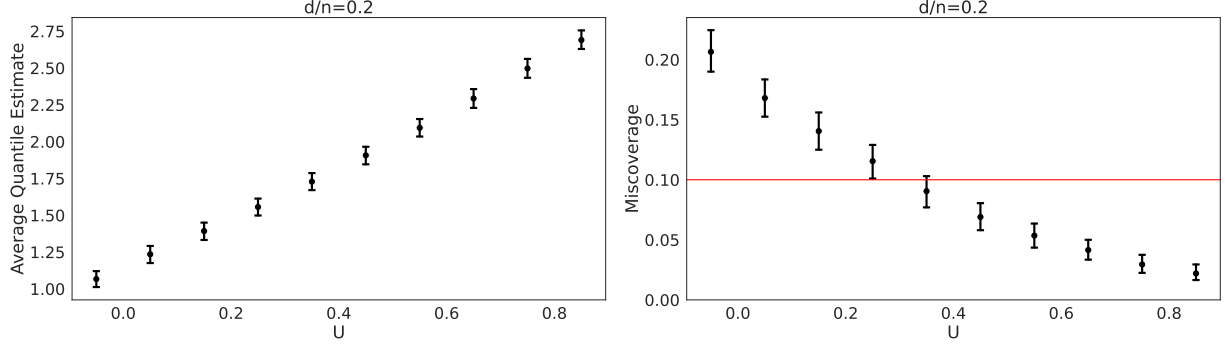


Figure 2: Empirical estimates of the average adjusted quantile (left panel) and miscoverage (right panel) of the randomized method of Gibbs et al. (2025) conditional on the cutoff, U . Data for this experiment are sampled from the Gaussian linear model $Y_i = X_i^\top \beta + \epsilon_i$ where $X_i \sim \mathcal{N}(0, I_d)$ and $\epsilon_i \sim \mathcal{N}(0, 1)$ with $X_i \perp \epsilon_i$. Dots and error bars show means and 95% confidence intervals obtained over 2000 samples of the combined training and test dataset $\{(X_i, Y_i)\}_{i=1}^{n+1}$ where in each sample the population coefficients are generated as $\tilde{\beta} \sim \mathcal{N}(0, I_d/d)$. Throughout, we set $d = 40$ and $n = 200$. The red line in the right panel indicates the target miscoverage level of $1 - \tau = 0.1$.

This connection, along with some additional calculations, motivates the randomized quantile adjustment $\hat{q}_{\text{GCC, rand.}}(X_{n+1}) = \sup\{y : \hat{\eta}_{n+1}^y \leq U\}$, where $U \sim \text{Unif}(-(1 - \tau), \tau)$. Crucially, this method has the desired exact coverage guarantee, $\mathbb{P}(Y_{n+1} \leq \hat{q}_{\text{GCC, rand.}}(X_{n+1})) = \tau$.

As discussed above, this method has two shortcomings. The first is that to compute the cutoff we need to evaluate the solution path of $\hat{\eta}_{n+1}^y$ as y varies. Although Gibbs et al. (2025) give some strategies for accomplishing this in an efficient manner, their methods still typically require additional computational time of at least $\Omega(d^3)^\dagger$ per test point. Adapting their methods to penalized regressions is more challenging and requires even higher computational complexity. This contrasts sharply with standard quantile regression which can issue predictions quickly at the low cost of computing the inner product $X_{n+1}^\top \hat{\beta}$. The second major shortcoming of $\hat{q}_{\text{GCC, rand.}}(X_{n+1})$ is that its value depends heavily on the randomized choice of U . Figure 2 displays estimates of the average conditional cutoff, $\mathbb{E}[\hat{q}_{\text{GCC, rand.}}(X_{n+1}) | U]$ and miscoverage, $\mathbb{P}(Y_{n+1} > \hat{q}_{\text{GCC, rand.}}(X_{n+1}) | U)$ as U varies on data sampled from a Gaussian linear model with $d/n = 0.2$. We see that the average cutoff can change by a factor of almost 2.5 and the miscoverage can vary by over 0.5 – 2 times the target level depending on the sampled value of U . As an aside, we note that the exact magnitude of these values depends directly on the aspect ratio. In the classical case where $d/n \rightarrow 0$ the randomization disappears and the method (asymptotically) produces a fixed cutoff, while larger aspect ratios produce greater variability.

[†]This comes from the cost of inverting a $d \times d$ matrix, which we shorthand as requiring $\Omega(d^3)$ time, although some algorithms with faster scaling are known.

3 Methods

We now introduce three alternative methods for debiasing quantile regression. As shown theoretically in Section 5 and empirically in Sections 3.4 and 6, all of these methods provide (asymptotically) exact coverage. Notably, this does not mean that their performance is identical. In Section 6 we compare the three approaches across a number of additional metrics (e.g. prediction set length, conditional coverage properties) and observe considerable variability. After reading the introduction to each method below, readers who are primarily interested in practical recommendations may choose to skip ahead to these results.

3.1 Fixed dual thresholding

Our first procedure makes a simple adjustment to $\hat{q}_{\text{GCC, rand.}}(X_{n+1})$ by replacing the randomized cutoff with a fixed threshold. This gives us the adjusted quantile estimate

$$\hat{q}_{\text{dual thresh.}}(X_{n+1}; t) = \sup \left\{ y : \hat{\eta}_{n+1}^{\text{adj.}, y} \leq t \right\}.$$

At threshold t , the coverage of this method is given by

$$\mathbb{P}(Y_{n+1} \leq \hat{q}_{\text{dual thresh.}}(X_{n+1}; t)) = \mathbb{P}(\hat{\eta}^{\text{adj.}, Y_{n+1}} \leq t).$$

So, to obtain the target coverage level of τ we see that we should set t as the τ quantile of $\hat{\eta}^{\text{adj.}, Y_{n+1}}$. Since this quantity is unknown, we replace it with the empirical estimate

$$\hat{t} = \text{Quantile} \left(\tau, \frac{1}{n} \sum_{i=1}^n \delta_{\hat{\eta}_i} \right),$$

where $\hat{\eta}$ denotes the dual variables fit using just the training data $\{(X_i, Y_i)\}_{i=1}^n$ and $\text{Quantile}(\tau, P)$ denotes the τ quantile of the distribution P . Corollary 5.1 below verifies that this estimate is consistent in high dimensions and thus that this method provides the desired asymptotic coverage.

While this approach is derandomized, it still retains the same test-time computational complexity as the method of Gibbs et al. (2025). Our next two proposals will address this shortcoming.

3.2 Level adjustment

The second method we will consider is to modify the nominal level used in the quantile regression loss. In particular, let

$$(\hat{\beta}_0(\tau^{\text{adj.}}), \hat{\beta}(\tau^{\text{adj.}})) = \underset{(\beta_0, \beta) \in \mathbb{R}^{d+1}}{\text{argmin}} \sum_{i=1}^n \ell_{\tau^{\text{adj.}}} (Y_i - \beta_0 - X_i^\top \beta),$$

denote the quantile estimates fit at adjusted level $\tau^{\text{adj.}}$. Let $(\hat{\beta}_0^{-i}(\tau^{\text{adj.}}), \hat{\beta}^{-i}(\tau^{\text{adj.}}))$ denote the corresponding leave-one-out coefficients obtained when the i_{th} sample is excluded from the fit. Then, we define

$$\hat{\tau}^{\text{adj.}} = \underset{\tau^{\text{adj.}} \in [0,1]}{\operatorname{argmin}} \left| \frac{1}{n} \sum_{i=1}^n \mathbb{1} \{Y_i \leq \hat{\beta}_0^{-i}(\tau^{\text{adj.}}) + X_i^\top \hat{\beta}^{-i}(\tau^{\text{adj.}})\} - \tau \right|, \quad (3.1)$$

as the level that obtains the smallest leave-one-out coverage gap. This gives us the adjusted quantile estimate,

$$\hat{q}_{\text{level-adj.}}(X_{n+1}) = \hat{\beta}_0(\hat{\tau}^{\text{adj.}}) + X_{n+1}^\top \hat{\beta}(\hat{\tau}^{\text{adj.}}).$$

As an aside, we remark that in practice the leave-one-out coverage is typically a non-decreasing function of $\tau^{\text{adj.}}$. Using this observation, in the experiments that follow we will compute (3.1) using binary search.

A method for adjusting the quantile regression level has also been previously proposed by [Bai et al. \(2021\)](#). They showed that when the aspect ratio is small and the data come from a stylized linear model the value $\hat{\tau}^{\text{adj.}} = (\tau - \frac{1}{2}\frac{d}{n}) / (1 - \frac{1}{2}\frac{d}{n})$ asymptotically provides the desired coverage. The method above can be seen as a generalization of this procedure that replaces their modeling assumptions with a generic leave-one-out cross-validation based approach.

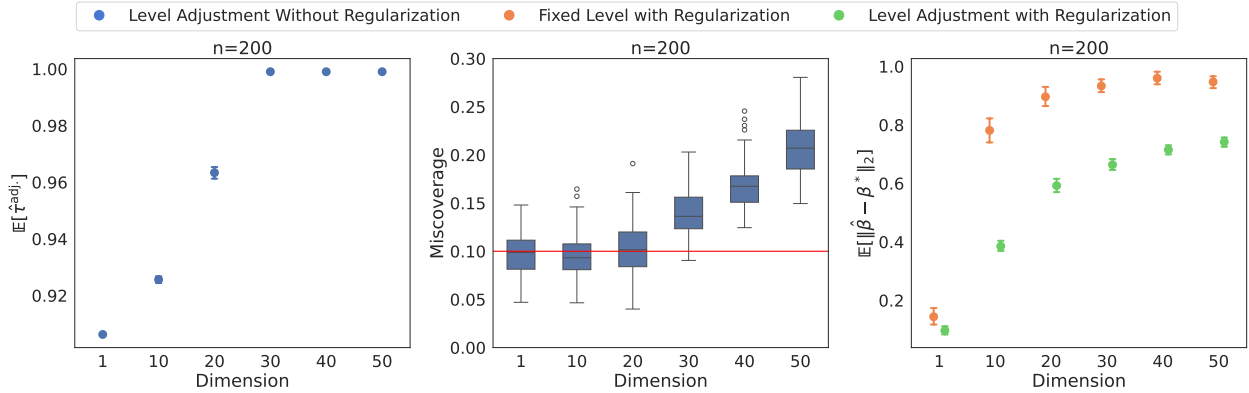


Figure 3: Average value of $\hat{\tau}^{\text{adj.}}$ (left panel), empirical miscoverage (center panel), and mean coefficient estimation error (right panel) of quantile regression fit with an adjusted level (blue), adjusted regularization (orange) and a joint level and regularization adjustment (green) as the dimension of the data varies. Data for these experiments are sampled from the Gaussian linear model $Y_i = X_i^\top \tilde{\beta} + \epsilon$ with $X_i \sim \mathcal{N}(0, I_d)$, $\epsilon_i \sim \mathcal{N}(0, 1)$, and $\epsilon_i \perp X_i$. Dots and error bars in the left and right panel show estimated means and 95% confidence intervals from 100 trials where in each trial the population coefficients are sampled as $\tilde{\beta} \sim \mathcal{N}(0, I_d/d)$. Boxplots in the center panel show the empirical distribution of the training-conditional miscoverage evaluated over the same 100 trials where in each trial the miscoverage is estimated on a test set of size 2000. The red line shows the target miscoverage of $1 - \tau = 0.1$. All regularization levels are chosen from the grid $\Lambda = \{0, 0.005, 0.01, \dots, 0.1\}$

Unfortunately, tuning the level alone is not sufficient to regain coverage at higher aspect ratios. The center panel of Figure 3 shows the realized miscoverage of $\hat{q}_{\text{level-adj.}}(X_{n+1})$ for increasing values of d/n on data generated from the Gaussian linear model. We see that

for $d/n \leq 0.1$ leave-one-out cross-validation successfully finds an adjusted level that restores coverage. On the other hand, for larger aspect ratios all values of $\hat{\tau}^{\text{adj.}}$ undercover. As a result, despite selecting the largest possible adjustment of $\hat{\tau}^{\text{adj.}} \approx 1$,[‡] this method still realizes a significant bias.

To obtain uniform coverage across higher aspect ratios, we will add regularization to the regression. For simplicity, we focus our experiments on ridge regularization, though we anticipate that other choices would also be effective. Proceeding as above, let

$$(\hat{\beta}_0(\lambda, \tau^{\text{adj.}}), \hat{\beta}(\lambda, \tau^{\text{adj.}})) = \underset{(\beta_0, \beta) \in \mathbb{R}^{d+1}}{\operatorname{argmin}} \sum_{i=1}^n \ell_{\tau^{\text{adj.}}}(Y_i - \beta_0 - X_i^\top \beta) + \lambda \|\beta\|_2^2.$$

denote the coefficients fit with regularization λ and adjusted level $\tau^{\text{adj.}}$ and $(\hat{\beta}_0^{-i}(\lambda, \tau^{\text{adj.}}), \hat{\beta}^{-i}(\lambda, \tau^{\text{adj.}}))$ denote the corresponding coefficients obtained when the i_{th} sample is omitted from the fit. Let

$$\text{LOOCov}(\lambda, \tau^{\text{adj.}}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1} \left\{ Y_i \leq \hat{\beta}_0^{-i}(\lambda, \tau^{\text{adj.}}) + X_i^\top \hat{\beta}^{-i}(\lambda, \tau^{\text{adj.}}) \right\},$$

denote the leave-one-out coverage at parameters $(\lambda, \tau^{\text{adj.}})$. Then, our goal is to find a specific choice $(\hat{\lambda}, \hat{\tau}^{\text{adj.}})$ such that $\text{LOOCov}(\hat{\lambda}, \hat{\tau}^{\text{adj.}})$ is close to τ .

In general, there will be more than one setting of $(\lambda, \tau^{\text{adj.}})$ that provides valid coverage. To choose amongst these values, we will use an auxiliary multiaccuracy target. Briefly, we aim to ensure that the miscoverage of quantile regression is uncorrelated with the covariates. We defer a detailed discussion of the motivation behind this metric to Section 6.1 where we discuss other goals for quantile regression beyond marginal coverage. Now, given a discrete grid of candidate values Λ for λ , we select the parameters using the following two-step procedure:

1. For $\lambda \in \Lambda$ define

$$\hat{\tau}^{\text{adj.}}(\lambda) = \underset{\tau^{\text{adj.}} \in [0,1]}{\operatorname{argmin}} \left| \text{LOOCov}(\lambda, \tau^{\text{adj.}}) - \tau \right|,$$

as the adjusted level that gives the smallest leave-one-out coverage error,

2. Let $\Lambda_\tau = \{\lambda \in \Lambda : |\text{LOOCov}(\lambda, \hat{\tau}^{\text{adj.}}(\lambda)) - \tau| \leq 2/n\}$ denote the set of regularization levels that provide approximate leave-one-out coverage and

$$\hat{\lambda} = \underset{\lambda \in \Lambda_\tau}{\operatorname{argmin}} \max_{j \in \{1, \dots, d\}} \frac{\left| \frac{1}{n} \sum_{i=1}^n X_{i,j} (\mathbb{1} \{ Y_i \leq \hat{\beta}_0^{-i}(\lambda, \hat{\tau}^{\text{adj.}}(\lambda)) + X_i^\top \hat{\beta}^{-i}(\lambda, \hat{\tau}^{\text{adj.}}(\lambda)) \}) - \tau \right|}{\frac{1}{n} \sum_{i=1}^n |X_{i,j}|}, \quad (3.2)$$

as the regularization level that minimizes the leave-one-out multiaccuracy error (see Section 6.1 for a detailed explanation of this error metric).

As above, in our experiments we implement the first step of this procedure using binary search. This gives us the final adjusted quantile estimate,

$$\hat{q}_{\text{level-reg.}}(X_{n+1}) = \hat{\beta}_0(\hat{\lambda}, \hat{\tau}^{\text{adj.}}(\hat{\lambda})) + X_{n+1}^\top \hat{\beta}(\hat{\lambda}, \hat{\tau}^{\text{adj.}}(\hat{\lambda})).$$

[‡]In this case, we set $\hat{\tau}^{\text{adj.}}$ to be slightly less than 1 to have a well-defined quantile regression fit.

Before moving on, it is worthwhile to ask if level-tuning is necessary or if coverage could be more easily obtained by simply holding $\tau^{\text{adj.}} = \tau$ fixed and adjusting the regularization alone. Empirically, we find that while such a strategy is feasible, it typically leads to over regularization. To illustrate this, the right panel of Figure 3 compares the estimation error of $\hat{\beta}(\hat{\lambda}, \hat{\tau}^{\text{adj.}}(\hat{\lambda}))$ against that of $\hat{\beta}(\hat{\lambda}_\tau, \tau)$ where

$$\hat{\lambda}_\tau = \min \{ \lambda \in [0, \infty) : \text{LOOCov}(\lambda, \tau) \geq \tau - 2/n \}$$

denotes the smallest regularization level that obtains a leave-one-out coverage of at least $\tau - 2/n$. Data for this experiment are sampled from a well-specified Gaussian linear model and in both cases, we target a coverage level of $\tau = 0.9$. We find that joint regularization and level tuning gives a smaller estimation error uniformly across all aspect ratios. As a result, we will prefer this method in the sections that follow and omit further investigation of sole regularization tuning.

3.3 Additive adjustment

The final method we will consider is applying an additive adjustment to the quantile estimate. One way to implement such an adjustment would be to fit the parameters $(\hat{\beta}_0, \hat{\beta})$ using a standard quantile regression and then, at prediction time, output the corrected estimate $c + \hat{\beta}_0 + X_{n+1}^\top \hat{\beta}$ for some constant $c \in \mathbb{R}$. This approach has been previously considered by Romano et al. (2019) under the name conformalized quantile regression. They propose to fit the parameter c using a held out subset of the training data that is not used in the quantile regression. In high-dimensional problems where data is scarce, withholding data from the initial regression may lead to a considerable drop in efficiency. In the following section, we will develop a computationally efficient leave-one-out cross-validation procedure that facilitates accurate parameter tuning without data splitting. To leverage that theory here, we now introduce an alternative method for computing an additive adjustment.

For any $c \in \mathbb{R}$, let $\hat{\beta}^c$ denote the coefficients fit in the intercept-less quantile regression,

$$\hat{\beta}^c = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^n \ell_\tau(Y_i - c - X_i^\top \beta). \quad (3.3)$$

Let $\hat{\beta}^{c,-i}$ denote the corresponding coefficients obtained when the i_{th} sample is excluded from the fit. Similar to the previous section, one reasonable proposal is to select the adjustment

$$\hat{c} = \underset{c \in C}{\operatorname{argmin}} \left| \frac{1}{n} \sum_{i=1}^n \mathbb{1} \{ Y_i \leq c + X_i^\top \hat{\beta}^c \} - \tau \right|, \quad (3.4)$$

that provides the smallest leave-one-out coverage gap over some appropriate set of candidate values C . This would then give us the adjusted quantile estimate $\hat{q}_{\text{add.-adj.}}(X_{n+1}) = c + X_{n+1}^\top \hat{\beta}^{\hat{c}}$. Unfortunately, as with the level adjustment procedure, we find that at larger aspect ratios this is insufficient to ensure coverage. Figure 4 demonstrates this on simulated data from a

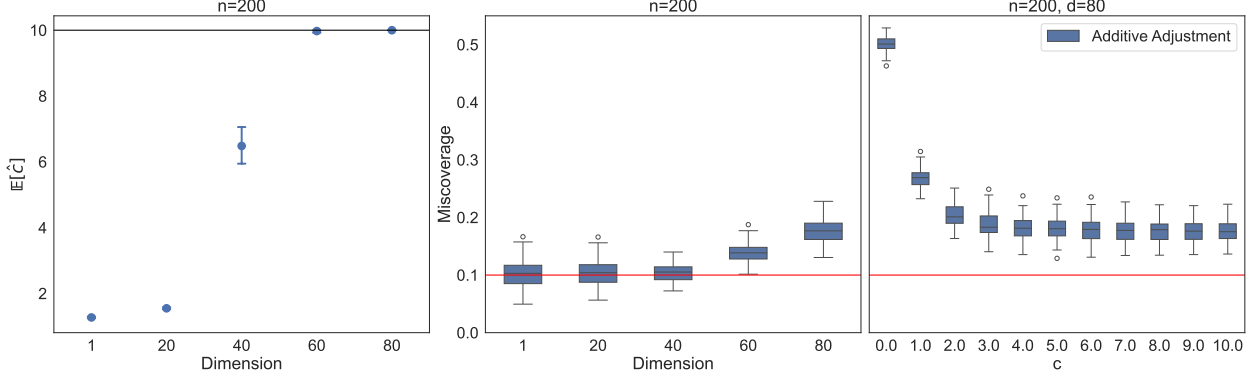


Figure 4: Empirical estimate of the mean selected value of \hat{c} (left panel), realized miscoverage for varying dimension (center panel), and realized miscoverage as c varies (right panel) of the unregularized additive adjustment. Data for this experiment are sampled from the Gaussian linear model $Y_i = X_i^\top \tilde{\beta} + \epsilon$ with $X_i \sim \mathcal{N}(0, I_d)$, $\epsilon_i \sim \mathcal{N}(0, 1)$, and $\epsilon_i \perp X_i$. Dots and error bars in the left panel show estimated means and 95% confidence intervals taken over 100 trials. Boxplots in the center and right panel show the empirical distribution of the training-conditional miscoverage evaluated over the same 100 trials where in each trial the miscoverage is estimated on a test set of size 2000 and the population coefficients are sampled as $\tilde{\beta} \sim \mathcal{N}(0, I_d/d)$. The black line in the left panel shows the maximum allowable value for \hat{c} , while red lines in the center and right panel show the target miscoverage of $1 - \tau = 0.1$.

Gaussian linear model. For simplicity, in this experiment we restrict the set of candidate values for c to $C = [-10, 10]$. Similar to the previous section, we find empirically that the leave-one-out coverage is non-decreasing in c and thus we solve (3.4) using binary search. We find that for $d/n \geq 0.3$ this method almost always selects the maximum value of $\hat{c} = 10$ (left panel) and, despite selecting such a large value, still undercovers (center panel). This issue cannot be alleviated by increasing the cap on \hat{c} as larger values do not change the coverage (right panel).

To overcome this shortcoming, we will once again add regularization to the regression. Let

$$\hat{\beta}^{\lambda, c} = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^n \ell_\tau(Y_i - c - X_i^\top \beta) + \lambda \|\beta\|_2^2,$$

denote the coefficients fit with regularization level λ and additive adjustment c , and $\hat{\beta}^{\lambda, c, -i}$ denote the corresponding coefficients when the i_{th} sample is excluded from the fit. Let $\text{LOOCov}^{\text{add}}(\lambda, c) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{Y_i \leq c + X_i^\top \hat{\beta}^{\lambda, c, -i}\}$ denote the corresponding leave-one-out coverage estimate. As above, we search for a pair (λ, c) that obtains the desired leave-one-out coverage while minimizing multiaccuracy error. Namely, we fix a grid Λ of possible values for λ and consider the two-step procedure:

1. For $\lambda \in \Lambda$ define

$$\hat{c}(\lambda) = \underset{c \in C}{\operatorname{argmin}} \left| \text{LOOCov}^{\text{add}}(\lambda, c) - \tau \right|,$$

as the adjusted level that gives the smallest leave-one-out coverage error,

2. Let $\Lambda_\tau = \{\lambda \in \Lambda : |\text{LOOCov}^{\text{add}}(\lambda, \hat{c}(\lambda)) - \tau| \leq 2/n\}$ denote the set of regularization levels that provide approximate leave-one-out coverage and

$$\hat{\lambda} = \underset{\lambda \in \Lambda_\tau}{\operatorname{argmin}} \max_{j \in \{1, \dots, d\}} \frac{\left| \frac{1}{n} \sum_{i=1}^n X_{i,j} (\mathbb{1}\{Y_i \leq \hat{c}(\lambda) + X_i^\top \hat{\beta}^{\lambda, \hat{c}(\lambda), -i}\} - \tau) \right|}{\frac{1}{n} \sum_{i=1}^n |X_{i,j}|}, \quad (3.5)$$

as the regularization level that minimizes the leave-one-out multiaccuracy error.

As above, in our experiments step one of this procedure is computed using binary search. This gives us the final quantile adjustment,

$$\hat{q}_{\text{add.-reg.}}(X_{n+1}) = \hat{c}(\hat{\lambda}) + X_{n+1}^\top \hat{\beta}^{\hat{c}(\hat{\lambda}), \hat{\lambda}}.$$

3.4 Simulated example

We conclude this section with a brief simulated example demonstrating that all of the methods proposed above give accurate coverage in high dimensions. More extensive comparisons that evaluate these procedures across a number of additional metrics are given in Section 6. Similar to Figure 1 from the introduction, we generate data from the Gaussian linear model $Y_i = X_i^\top \tilde{\beta} + \epsilon$ with $X_i \sim \mathcal{N}(0, I_d)$, $\epsilon_i \sim \mathcal{N}(0, 1)$, and $\epsilon_i \perp X_i$. Figure 5 shows the resulting coverage of both our methods and that of standard quantile regression. We see that all three of our methods offer robust coverage irrespective of dimension (left panel). As anticipated by the theory presented below in Section 5, this coverage becomes more tightly concentrated on the target level as n and d increase (right panel).

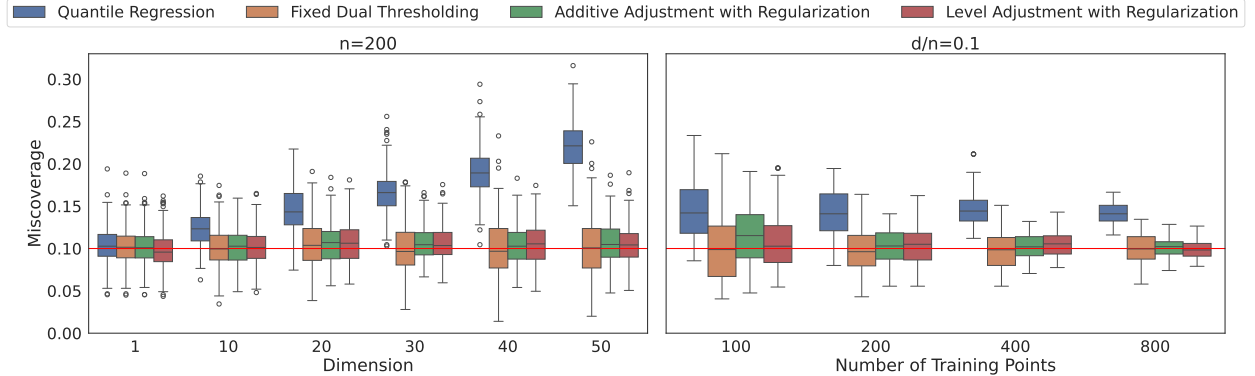


Figure 5: Empirical distribution of the training-conditional miscoverage of quantile regression (blue) and our fixed dual thresholding (orange), additive adjustment (green), and level adjustment (red) methods. The left panel shows results obtained with varying dimension and a fixed sample size of $n = 200$, while the right panel varies n and d together at a fixed aspect ratio of $d/n = 0.1$. Boxplots show results from 200 trials where in each trial the miscoverage is evaluated empirically over a test set of size 2000 and the population coefficients are sampled as $\tilde{\beta} \sim \mathcal{N}(0, I_d/d)$. The additive adjustment procedure is implemented with range $C = [-10, 10]$ for c and all regularization levels are chosen from the grid $\Lambda = \{0, 0.005, 0.01, \dots, 0.1\}$.

4 Efficient leave-one-out cross-validation

Two of the methods developed in the previous section use leave-one-out cross-validation to select their hyperparameters. The typical implementation of these procedures requires fitting n quantile regressions across a range of hyperparameter settings. In this section, we derive a connection between the leave-one-out coverage and the quantile regression dual variables that allows us to obtain all n leave-one-out coverage indicators with just a single fit. Hyperparameter tuning can then be performed at the cost of just a few regression fits across different parameter values.

To introduce this method, we define a few pieces of additional notation. Throughout this section, we consider quantile regressions of the form

$$\hat{w} = \operatorname{argmin}_{w \in \mathbb{R}^p} \sum_{i=1}^n \ell_\tau(\tilde{Y}_i - \tilde{X}_i^\top \hat{w}) + \mathcal{R}(w). \quad (4.1)$$

Note that unlike the previous sections, here we have chosen to omit an explicit intercept parameter. This allows us to unify the notation to encompass both our level-based adjustment, in which $\tilde{Y}_i = Y_i$, $\tilde{X}_i = (1, X_i)$, and $p = d+1$, and our additive adjustment, in which $\tilde{Y}_i = Y_i - c$, $\tilde{X}_i = X_i$, and $p = d$. Following the same steps as in Section 2, a useful dual for this regression can be obtained by defining the additional primal variables $r_i = \tilde{Y}_i - \tilde{X}_i^\top w$ for $i \in \{1, \dots, n\}$ and corresponding dual variables $\eta \in \mathbb{R}^n$ for these constraints. This gives the dual program

$$\begin{aligned} \hat{\eta} = \operatorname{argmax}_{\eta \in \mathbb{R}^n} & \sum_{i=1}^n \eta_i \tilde{Y}_i - \mathcal{R}^* \left(\sum_{i=1}^n \eta_i \tilde{X}_i \right) \\ \text{subject to} & \quad -(1 - \tau) \preceq \eta_i \preceq \tau, \end{aligned} \quad (4.2)$$

where \mathcal{R}^* denotes the convex conjugate of \mathcal{R} . Finally, in what follows we let \hat{w}^{-i} denote the corresponding primal solution when the i _{th} sample is omitted from the fit.

Our first result derives a general connection between the leave-one-out coverage and the sign of the dual variables.

Proposition 4.1. *Assume that \mathcal{R} is convex. Then, all dual solutions $\hat{\eta}$ and leave-one-out primal solutions \hat{w}^{-i} satisfy the conditions*

$$\tilde{Y}_i < \tilde{X}_i^\top \hat{w}^{-i} \implies \hat{\eta}_i \leq 0,$$

and

$$\tilde{Y}_i > \tilde{X}_i^\top \hat{w}^{-i} \implies \hat{\eta}_i \geq 0.$$

Now, recall that our goal is to compute the leave-one-out coverage, $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\tilde{Y}_i \leq \tilde{X}_i^\top \hat{w}^{-i}\}$. The above proposition suggests that this quantity should be comparable to $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\}$. Unfortunately however, deriving an exact equivalence between these two quantities is not possible due to the ambiguity around the edge cases $\tilde{Y}_i = \tilde{X}_i^\top \hat{w}^{-i}$ and $\hat{\eta}_i = 0$. We are not aware of any simple method for resolving these cases in full generality. One of the key

difficulties is that without additional assumptions both the primal and dual solutions are not unique and at these edge cases the coverage can vary depending on which solution we select. The following example illustrates one such instance where this occurs.

Example 4.1. *Consider fitting an intercept only quantile regression with $\tau = 1/2$ to find the median of the three data points (Y_1, Y_2, Y_3) . For simplicity, assume that $Y_1 < Y_2 < Y_3$. The primal solution is $\hat{w} = Y_2$ with corresponding dual variables $\hat{\eta} = (-1/2, 0, 1/2)$. Critically, we have that $\hat{\eta}_2 = 0$. Now, consider the leave-one-out problem when Y_2 is omitted. Then, the median is any point $\hat{w}^{-2} \in [Y_1, Y_3]$ and it is ambiguous whether Y_2 is covered.*

We will now introduce two different techniques for modifying the regression to avoid the above ambiguity. For simplicity, we focus specifically on cases where \mathcal{R} is a quadratic regularizer, although we expect similar results to hold for other choices. Our first method is to perturb the covariates by adding a small amount of independent noise to each of their values. The magnitude of this noise is not critical and can be made arbitrarily small such that it has a vanishing impact on the quantile regression objective. Our insight is that even a small amount of noise is sufficient to push the dual solutions away from zero and, correspondingly, to enforce a unique value for the leave-one-out coverage. To illustrate this, the following demonstrates how added noise removes the ambiguity observed in Example 4.1.

Example 4.2. *Consider adding noise to the intercept parameter in Example 4.1, i.e., consider fitting an intercept-less quantile regression on the three data points $\{(1 + \xi_1, Y_1), (1 + \xi_2, Y_2), (1 + \xi_3, Y_3)\}$ where ξ_1, ξ_2 , and ξ_3 are i.i.d. continuously distributed random variables independent of (Y_1, Y_2, Y_3) . As before, assume for simplicity that $Y_1 < Y_2 < Y_3$. For sufficiently small values of (ξ_1, ξ_2, ξ_3) , the dual solution is uniquely specified as $\hat{\eta} = (-1/2, \frac{\xi_1 - \xi_3}{2(1 + \xi_2)}, 1/2)$ and the leave-one-out primal solution with point $(1 + \xi_2, Y_2)$ omitted is (with probability one) unique and given by*

$$\hat{w}^{-2} = \begin{cases} \frac{Y_1}{1 + \xi_1}, & \text{if } |1 + \xi_1| > |1 + \xi_3|, \\ \frac{Y_3}{1 + \xi_3}, & \text{if } |1 + \xi_1| < |1 + \xi_3|. \end{cases}$$

For (ξ_1, ξ_2, ξ_3) sufficiently small, we see that $Y_2 \neq (1 + \xi_2)\hat{w}^{-2}$ and thus there is no ambiguity in the coverage of the leave-one-out solution.

The second method we will consider is to add non-zero L_2 regularization to all of the primal variables. Similar to the added noise, the magnitude of this regularization can be arbitrary and, in particular, can be taken to be vanishingly small such that it has almost no impact on the regression. The only important consideration is that the regularization makes the fitted leave-one-out solutions unique and thus removes ambiguity in the coverage.

Assumptions 1 and 2 give more formal statements of our two approaches for ensuring leave-one-out uniqueness. We note that both of these assumptions require that the distribution of $\hat{Y}_i \mid \hat{X}_i$ is continuous. This can always be guaranteed by adding a small amount of noise to \hat{Y}_i . The main result of this section is stated in Theorem 4.1 which shows that these assumptions are sufficient to ensure a one-to-one equivalence between the leave-one-out coverage and the signs of the dual variables.

Assumption 1. *The distribution of $\tilde{Y}_i \mid \tilde{X}_i$ is continuous. Moreover, the regularization can be written as $\mathcal{R}(w) = \sum_{j=1}^p \lambda_j w_j^2$ for some non-negative constants $\lambda_1, \dots, \lambda_p \geq 0$ and the covariates can be written as $\tilde{X}_i = Z_i + \xi_i$ where $\xi_i \perp (Z_i, Y_i)$ has independent, continuously distributed entries. Finally, we have that $p < n$.*

Assumption 2. *The distribution of $\tilde{Y}_i \mid \tilde{X}_i$ is continuous. Moreover, the regularization can be written as $\mathcal{R}(w) = \sum_{j=1}^p \lambda_j w_j^2$ for some positive constants $\lambda_1, \dots, \lambda_p > 0$.*

Theorem 4.1. *Assume that $\{(\tilde{X}_i, \tilde{Y}_i)\}_{i=1}^n$ are i.i.d. and that the conditions of either Assumption 1 or 2 are satisfied. Then, with probability one we have that for all $i \in \{1, \dots, n\}$ either all dual solutions satisfy $\hat{\eta}_i < 0$ or all dual solutions satisfy $\hat{\eta}_i > 0$. Similarly, with probability one either all leave-one-out primal solutions satisfy $\tilde{Y}_i < \tilde{X}_i^\top \hat{w}^{-i}$ or all leave-one-out primal solutions satisfy $\tilde{Y}_i > \tilde{X}_i^\top \hat{w}^{-i}$. Finally, letting $\hat{\eta}$ and $\{\hat{w}^{-i}\}_{i=1}^n$ denote any such solutions we have that*

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\} \stackrel{a.s.}{=} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\tilde{Y}_i \leq \tilde{X}_i^\top \hat{w}^{-i}\}.$$

In general, on real data we find that the conditions outlined in Assumptions 1 and 2 tend to be redundant. In our experiments in Sections 6 and 3.4 we ignore these assumptions and use the dual variables to estimate the leave-one-out coverage and perform hyperparameter selection across a variety of different datasets and regularization settings that do not satisfy these conditions. In all cases, we find that the dual estimate is accurate and facilitates the selection of hyperparameter values that yield reliable coverage. As a result, outside of rare edge cases we find that $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\}$ can typically be used to estimate the leave-one-out coverage without the need to modify the data or estimation procedure.

5 High-dimensional consistency

We now develop our main theoretical results establishing the high-dimensional consistency of the estimates proposed in the previous sections. Throughout, we will work in a stylized linear model with Gaussian covariates that is commonly used in work in this area (e.g., Bayati & Montanari (2012), Donoho & Montanari (2013), Thrampoulidis et al. (2015, 2018), Dicker (2016), Sur & Candès (2019)). While we will not pursue this in detail, universality results derived for similar problems suggest that one should expect our results to also hold under more relaxed assumptions (e.g. X_i having i.i.d. entries) (Han & Shen 2023). This is validated by the empirical results presented in the following section which demonstrate the robustness of our methods on real datasets.

Assumption 3. *The data $\{(X_i, Y_i)\}_{i=1}^n$ are i.i.d. and distributed as $Y_i = X_i^\top \tilde{\beta} + \epsilon_i$ with $X_i \sim \mathcal{N}(0, I_d)$, $\epsilon_i \sim P_\epsilon$, and $\epsilon_i \perp X_i$. Moreover, the error distribution P_ϵ is continuous, mean zero, and has at least two bounded moments. Additionally, the density of P_ϵ is bounded, continuous, and positive on \mathbb{R} . Finally, the population coefficients are themselves random and sampled as $(\sqrt{d}\tilde{\beta}_j)_{j=1}^d \stackrel{i.i.d.}{\sim} P_\beta$ independent of $\{(X_i, \epsilon_i)\}_{i=1}^n$.*

We will focus on quantile regressions of the form

$$\min_{(\beta_0, \beta) \in \mathbb{R}^{d+1}} \sum_{i=1}^n \ell_\tau(Y_i - \beta_0 - X_i^\top \beta) + \mathcal{R}_d(\beta). \quad (5.1)$$

We make two remarks about this set-up. First, for simplicity, we have chosen to focus on regressions containing an intercept. To obtain results for our additive adjustment method we will also need to consider cases where β_0 is replaced by a fixed constant. This extension is stated at the end of this section in Theorem 5.2. Second, here we have allowed the regularization function \mathcal{R}_d to depend explicitly on the dimension. This is done to account for the fact that the regularization level may need to be rescaled as n and d increase. Our formal assumptions on the regularizer are stated in Assumption 4 in the appendix. At a high-level, we require that \mathcal{R}_d is convex and that the data have enough bounded moments to ensure that various functions of \mathcal{R}_d satisfy the law of large numbers. As an example, Lemma B.1 verifies that our assumptions are met if P_β has four bounded moments and $\mathcal{R}_d(\beta) = \sqrt{d}\lambda\|\beta\|_1$ or $\mathcal{R}_d(\beta) = d\lambda\|\beta\|_2^2$ is L_1 or L_2 regularization.

We now state the main result of this section, which establishes that the coordinate-wise empirical distribution of the dual variables converges to an asymptotic limit. Although we only state this result for aspect ratios $d/n \rightarrow \gamma \in (0, 2/\pi)$, we expect similar conclusions to hold for $\gamma \geq 2/\pi$ under appropriate assumptions on the regularization.

Theorem 5.1. *Fix any $\tau \in (0, 1)$ and suppose that the data and regularizer satisfy the conditions of Assumptions 3 and 4. Suppose that $d, n \rightarrow \infty$ with $d/n \rightarrow \gamma \in (0, 2/\pi)$. Then, there exists a limiting distribution P_η such that for any bounded, Lipschitz function ψ and any $\delta > 0$,*

$$\mathbb{P} \left(\text{For all dual solution } \hat{\eta} \text{ to (5.1), } \left| \frac{1}{n} \sum_{i=1}^n \psi(\hat{\eta}_i) - \mathbb{E}_{Z \sim P_\eta} [\psi(Z)] \right| \leq \delta \right) \rightarrow 1.$$

Moreover, the distribution P_η is supported on $[-(1-\tau), \tau]$ with discrete masses at $-(1-\tau)$ and τ and a continuous distribution on $(-(1-\tau), \tau)$.

As an aside, we note that an explicit description of the asymptotic distribution P_η can be found in equation (B.7) in the appendix. Briefly, P_η is a truncated Gaussian distribution with mean and variance parameters dictated by the solutions to the asymptotic optimization program specified in (B.2).

Theorem 5.1 has two critical corollaries for our debiasing methods. The first shows that the quantile estimate used by our fixed dual thresholding method is consistent.

Corollary 5.1. *Consider unregularized quantile regression with $\mathcal{R}_d(\beta) = 0$. Under the assumptions of Theorem 5.1, it holds that for any $\delta > 0$,*

$$\mathbb{P} \left(\text{For all dual solution } \hat{\eta} \text{ to (5.1), } \left| \text{Quantile} \left(\tau, \frac{1}{n} \sum_{i=1}^n \delta_{\hat{\eta}_i} \right) - \text{Quantile}(\tau, P_\eta) \right| \leq \delta \right) \rightarrow 1.$$

Our second corollary establishes the consistency of our leave-one-out coverage estimates. As in Section 4 above, we focus on the case of L_2 regularization though we expect similar results to hold for other choices.

Corollary 5.2. *Let (X_{n+1}, Y_{n+1}) denote an independent sample from the same distribution as $\{(X_i, Y_i)\}_{i=1}^n$. Suppose that $\mathcal{R}_d(\beta) = \sum_{i=1}^d \lambda_i^d \beta_i^2$ for some $\lambda_1^d, \dots, \lambda_d^d \geq 0$. Let $(\hat{\beta}_0, \hat{\beta})$ denote any primal solution to (5.1) chosen independently of (X_{n+1}, Y_{n+1}) . Then, under the assumptions of Theorem 5.1, it holds that for any $\delta > 0$,*

$$\mathbb{P} \left(\text{For all dual solutions } \hat{\eta} \text{ to (5.1), } \left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\} - \mathbb{P}(Y_{n+1} \leq \hat{\beta}_0 + X_{n+1}^\top \hat{\beta}) \right| \leq \delta \right) \rightarrow 1.$$

Proofs of Theorem 5.1 as well as Corollaries 5.1 and 5.2 are given in the appendix. Our arguments build heavily on Gordon’s comparison inequalities (Gordon 1985, 1988) and their application to high-dimensional regression developed in Thrampoulidis et al. (2018). At a high level, these results allow us to derive a correspondence between the dual quantile regression program and a simplified auxiliary optimization problem that replaces the covariate matrix with vector-valued random variables. The main technical difficulty is then to characterize the solutions of this auxiliary program. One key difference between our result and that of the original work of Thrampoulidis et al. (2018) is that we consider the behaviour of the solutions under arbitrary bounded, Lipschitz functions. We are not the first to derive an extension of this type. However, to the best of our knowledge previous extensions typically rely on strong convexity of the auxiliary optimization (see e.g., Abbasi et al. (2016), Miolane & Montanari (2021), Celentano et al. (2023)). Here, we derive a similar result under weaker conditions.

It is worthwhile to contrast Theorem 4.1 with the results of Bai et al. (2021). In that paper, the authors derive a number of asymptotic consistency results for the primal quantile regression estimates $(\hat{\beta}_0, \hat{\beta})$. Here, we provide a set of complementary asymptotics for the dual. In addition, we also treat a more general setting that removes the restrictions to small aspect ratios and unregularized regressions present in their work. While not our main focus, a corollary of our analysis is that the intercept, $\hat{\beta}_0$ and estimation error, $\|\hat{\beta} - \tilde{\beta}\|_2$ both converge to constants under the assumptions of Theorem 5.1. This is formally stated in Theorem B.1 in the appendix, which directly generalizes Theorem C.1 of Bai et al. (2021).

Finally, as a last remark, we note that all of the conclusions stated above also hold for the intercept-less regressions used in the additive adjustment procedure. The proof of this result is nearly identical to that of Theorem 5.1 and thus is omitted.

Theorem 5.2. *Under identical assumptions, the conclusions of Theorem 5.1 and Corollaries 5.1 and 5.2 remain true when the intercept β_0 is replaced by a fixed, real-valued constant.*

6 Real data experiments

6.1 Methods and metrics

We now undertake a series of empirical comparisons of our proposed methods. As baselines, we also evaluate the performance of standard quantile regression, the randomized method of [Gibbs et al. \(2025\)](#), and the conformalized quantile regression (CQR) method of [Romano et al. \(2019\)](#). In all experiments we implement conformalized quantile regression so that 75% of the data is used to fit the quantile regression and 25% is used to calibrate its coverage.

To evaluate these methods, we compare the quality of prediction sets constructed using their estimated quantiles. More precisely, for a given miscoverage level $\alpha \in (0, 1/2)$ (taken to be 0.1 in our experiments) we compute the (adjusted) estimates $\hat{q}^{\alpha/2}(X_{n+1})$ and $\hat{q}^{1-\alpha/2}(X_{n+1})$ of the $\alpha/2$ and $1 - \alpha/2$ quantiles using each of the methods. We then evaluate the resulting prediction interval $[\hat{q}^{\alpha/2}(X_{n+1}), \hat{q}^{1-\alpha/2}(X_{n+1})]$ in terms of three criteria: 1) marginal coverage, $\mathbb{P}(\hat{q}^{\alpha/2}(X_{n+1}) \leq Y_{n+1} \leq \hat{q}^{1-\alpha/2}(X_{n+1}))$, 2) interval length, $\max\{\hat{q}^{1-\alpha/2}(X_{n+1}) - \hat{q}^{\alpha/2}(X_{n+1}), 0\}$, and 3) maximum multiaccuracy error.

Multiaccuracy as introduced in [Hébert-Johnson et al. \(2018\)](#) and [Kim et al. \(2019\)](#) is a general criteria for measuring the bias of a predictor over reweightings of the covariate space. In our context, we will consider linear reweightings and thus our goal will be to obtain quantile estimates whose miscoverage events are uncorrelated with the features. This is motivated by results from the classical regime in which $d \log(n)/n \rightarrow 0$. There, [Duchi \(2025\)](#) showed that (under appropriate tail bounds on the data) the canonical quantile regression estimates $(\hat{q}_{\text{QR}}^{\alpha/2}, \hat{q}_{\text{QR}}^{1-\alpha/2})$ satisfy the multiaccuracy condition,

$$\sup_{\|v\|_2 \leq 1} \mathbb{E} \left[X_{n+1}^\top v (\mathbb{1}\{Y_{n+1} \in [\hat{q}_{\text{QR}}^{\alpha/2}(X_{n+1}), \hat{q}_{\text{QR}}^{1-\alpha/2}(X_{n+1})]\} - (1 - \alpha)) \mid \{(X_i, Y_i)\}_{i=1}^n \right] \xrightarrow{\mathbb{P}} 0. \quad (6.1)$$

As a concrete example to motivate the utility of this condition, consider fitting quantile regression with a feature $X_{i,j} = \mathbb{1}\{X_i \in G\}$ that indicates whether sample i falls into group G . Then, applying (6.1) with $v = e_i$ gives the conditional coverage statement,

$$\mathbb{P}(Y_{n+1} \in [\hat{q}_{\text{QR}}^{\alpha/2}(X_{n+1}), \hat{q}_{\text{QR}}^{1-\alpha/2}(X_{n+1})] \mid X_{n+1} \in G, \{(X_i, Y_i)\}_{i=1}^n) \xrightarrow{\mathbb{P}} 1 - \alpha.$$

More generally, by designing the features appropriately multiaccuracy conditions of this form can be used to ensure that the prediction set provides accurate performance across sensitive attributes of the population.

Motivated by this, [Gibbs et al. \(2025\)](#) extend (6.1) to the high-dimensional setting and show that their randomized adjustment satisfies

$$\mathbb{E} \left[X_{n+1}^\top v (\mathbb{1}\{Y_{n+1} \in [\hat{q}_{\text{GCC, rand.}}^{\alpha/2}(X_{n+1}), \hat{q}_{\text{GCC, rand.}}^{1-\alpha/2}(X_{n+1})]\} - (1 - \alpha)) \right] = 0, \quad \forall v \in \mathbb{R}^d.$$

Notably, this statement is not directly comparable to (6.1) since here the expectation is taken marginally over the random draw of the training set. In general, one cannot expect to

obtain training-conditional convergence uniformly over v in high dimensions. Nevertheless, as we will see shortly, empirically $\hat{q}_{\text{GCC, rand.}}(X_{n+1})$ can provide approximate validity when v is restricted to a smaller set (e.g. to the coordinate axes).

The methods developed in the previous section are not designed to explicitly guarantee multiaccuracy. Regardless, since they are built on top of quantile regression one may hope that they still approximately satisfy these conditions. To evaluate this, we will examine the coordinatewise multiaccuracy error of each method defined as

$$\max_{j \in \{1, \dots, d\}} \frac{\mathbb{E}[X_{n+1,j}(\mathbb{1}\{Y_{n+1} \in [\hat{q}^{\alpha/2}(X_{n+1}), \hat{q}^{1-\alpha/2}(X_{n+1})]\} - (1 - \alpha)) \mid \{(X_i, Y_i)\}_{i=1}^n]}{\mathbb{E}[|X_{n+1,j}|]}. \quad (6.2)$$

In order to improve the performance on this metric, we recall that in Section 3 we defined the parameters for our regularized level and additive adjustment procedures to minimize a leave-one-out estimate of (6.2) (cf. equations (3.2) and (3.5)).

6.2 Results

We compare the methods on three datasets in which the goals are to predict the per capita violent crime rate of various communities (Redmond & Baveja 2002), the critical temperature of superconductors (Hamidieh 2018), and the number of times a news article was shared online (Fernandes et al. 2015). All three datasets are publicly available from the University of California, Irvine machine learning repository (Dua & Graff 2017). After filtering out features with missing values and removing (approximately) linearly dependent columns, the datasets have 99, 81, and 55 covariates, respectively. We normalize both the features and the target to have mean zero and variance one and then compare the methods discussed above in terms of their miscoverage, median length, and multiaccuracy error.

Figure 6 shows the outcome of this experiment. Results in the figure summarize 20 trials where in each trial the data are randomly split into a training set of size 400 and a test set of size 2000[§] and a random subset of the features are selected for use. As shown in the top row, all of our methods provide the desired marginal coverage. The randomized method of Gibbs et al. (2025) also provides the desired coverage (green), while conformalized quantile regression (orange) has a very slight bias due to the fact that after data splitting only a small number (≤ 100) of points are available to estimate its bias correction.

Among the methods with the desired coverage, our level adjustment procedure (purple) yields the smallest intervals (center row). The additive adjustment procedure (brown) produces almost identical interval lengths on the communities and crime (left) and superconductivity (center) datasets and slightly larger intervals on the news (right) dataset. The largest intervals are output by the randomized method of Gibbs et al. (2025), which obtains a median interval length of up to two times that of the level adjustment procedure in higher dimensions (green).

[§]The communities and crimes dataset only has 1994 samples, so we utilize a smaller test set of size 1594 for its experiments.

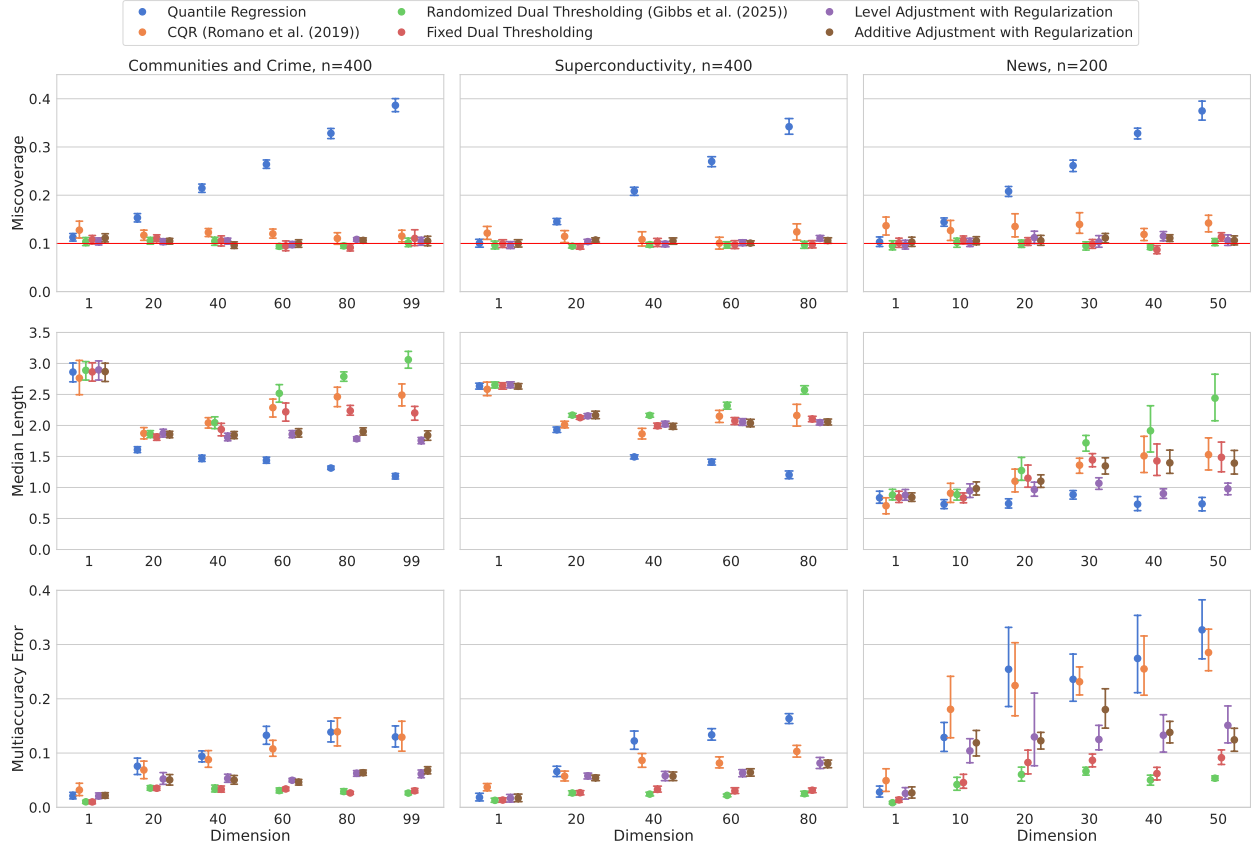


Figure 6: Empirical miscoverage (top row), median length (center row), and multiaccuracy error (bottom row) of quantile regression (blue), the baseline methods of Romano et al. (2019) (orange) and Gibbs et al. (2025) (green) and our fixed dual thresholding (red), level adjustment (purple), and additive adjustment (brown) methods on the communities and crime (left panels), superconductivity (center panels), and news (right panels) datasets. Dots and error bars show means and 95% confidence intervals obtained over 20 trials. The red lines in the top panels show the target level of $\alpha = 0.1$. In all experiments, the additive adjustment procedure is implemented with range $C = [-10, 10]$ for c and regularization levels are chosen from the grid $\Lambda = \{0, 0.005, 0.01, \dots, 0.2\}$.

In terms of multiaccuracy, the lowest error is obtained by the dual thresholding methods (bottom row). Interestingly, while randomization is necessary to obtain a theoretical multiaccuracy bias of zero, we find that the fixed thresholding method (red) offers nearly identical performance in practice. On the other hand, our level and additive adjustment procedures realize a higher multiaccuracy error (purple and brown). This is to be expected since by adding regularization to these methods we have introduced bias. To see this, note that letting $(\hat{\beta}_0(\lambda), \hat{\beta}(\lambda))$ denote the fitted coefficients at quantile level τ with L_2 regularization λ and $\tilde{\beta}$ denote the population quantile regression coefficients, we have that in the classical regime where $d \log(n)/n \rightarrow 0$,

$$\mathbb{E} \left[X_{n+1}^\top v(\mathbb{1}\{Y_{n+1} \leq \hat{\beta}_0(\lambda) + X_{n+1}^\top \hat{\beta}(\lambda)\} - \tau) \mid \{(X_i, Y_i)\}_{i=1}^n \right] \xrightarrow{\mathbb{P}} -2\lambda v^\top \tilde{\beta}.$$

This follows directly from the first-order conditions of quantile regression and the arguments

of [Duchi \(2025\)](#). Notably, while non-negligible, we find that this error is small relative to the effects of overfitting bias and our level and additive adjustment procedures still produce much lower multiaccuracy error than the baseline approaches of quantile regression and conformalized quantile regression.

7 Conclusions

In this paper we developed three methods for correcting the coverage bias of quantile regression. Theoretical and empirical results show that all of these procedures provide robust coverage irrespective of the dimension of the data. In terms of prediction interval length and multiaccuracy error, none of these three methods dominate. Across our empirical results we find that the fixed dual thresholding method offers the lowest multiaccuracy error. However, this comes at the cost of wider prediction intervals and greater test-time computational complexity as compared to the level and additive adjustment procedures.

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A Proofs for Section 4

We will now give formal proofs for the results stated in Section 4. Throughout, we use the same notation as was defined in the main text. Namely, we let $\{(\tilde{X}_i, \tilde{Y}_i)\}_{i=1}^n \subseteq \mathbb{R}^p \times \mathbb{R}$ denote the training data and we consider quantile regressions of the form

$$\min_{w \in \mathbb{R}^p} \sum_{i=1}^n \ell_\tau(\tilde{Y}_i - \tilde{X}_i^\top w) + \mathcal{R}(w).$$

We use \hat{w} and $\hat{\eta}$ to denote primal and dual solutions to this regression, and \hat{w}^{-i} and $\hat{\eta}^{-i}$ to denote corresponding leave-one-out primal and dual solutions when the i_{th} sample is omitted from the fit. To begin, we prove a useful technical lemma that relates a dual value of zero to interpolation of the leave-one-out prediction.

Lemma A.1. *Fix any $i \in \{1, \dots, n\}$ and suppose there exists a leave-one-out primal solution with $\tilde{Y}_i = \tilde{X}_i^\top \hat{w}^{-i}$. Then, there exists a dual solution to the full program with $\hat{\eta}_i = 0$. Symmetrically, if there exists a dual solution to the full program with $\hat{\eta}_i = 0$, then there exists a leave-one-out primal solution with $\tilde{Y}_i = \tilde{X}_i^\top \hat{w}^{-i}$.*

Proof. For notational simplicity, we will focus on the case $i = n$. Suppose there exists a leave-one-out primal solution with $\tilde{Y}_n = \tilde{X}_n^\top \hat{w}^{-n}$. For $i \in \{1, \dots, n-1\}$, let $\hat{r}_i^{-n} = \tilde{Y}_i - \tilde{X}_i^\top \hat{w}^{-n}$ denote the additional primal variables. Let $\hat{\eta}^{-n} \in \mathbb{R}^{n-1}$ denote a corresponding dual solution to the leave-one-out program. The Lagrangian for the leave-one-out program is

$$L_{-n}(w^{-n}, r^{-n}, \eta^{-n}) = \sum_{j=1}^{n-1} \ell_\tau(r_j^{-n}) + \sum_{j=1}^{n-1} \eta_j^{-n} (\tilde{Y}_j - \tilde{X}_j^\top w^{-n} - r_j^{-n}) + \mathcal{R}(w^{-n}),$$

and the Lagrangian for the full program is

$$L(w, r, \eta) = \sum_{j=1}^n \ell_\tau(r_j) + \sum_{j=1}^n \eta_j (\tilde{Y}_j - \tilde{X}_j^\top w - r_j) + \mathcal{R}(w). \quad (\text{A.1})$$

By assumption, $(\hat{w}^{-n}, \hat{r}^{-n}, \hat{\eta}^{-n})$ is a saddle point of L_{-n} . Using this fact, and taking first-order derivatives, it is straightforward to verify that $(\hat{w}^{-n}, (\hat{r}^{-n}, 0), (\hat{\eta}^{-n}, 0))$ is a saddle point of L . Thus, $\hat{\eta} = (\hat{\eta}^{-n}, 0)$ is a solution to the full dual program, as desired.

For the reverse direction, suppose there exists a solution to the full dual program with $\hat{\eta}_n = 0$. Let $(\hat{w}, \hat{r}, \hat{\eta})$ denote the corresponding saddle point of L . By differentiating L with respect to r_n , we see that we must have $\hat{r}_n = 0$. Moreover, differentiating L with respect to $\hat{\eta}_n$, we then also find that $\tilde{Y}_n - \tilde{X}_n^\top \hat{w} = \hat{r}_n = 0$. So, using the notation $v_{1:(n-1)}$ to denote the first $n-1$ entries of a vector $v \in \mathbb{R}^n$ and taking first-order derivatives of L_{-n} , it is straightforward to verify that $(\hat{w}, \hat{r}_{1:(n-1)}, \hat{\eta}_{1:(n-1)})$ is a saddle point of L_{-n} . Since $\tilde{Y}_n = \tilde{X}_n^\top \hat{w}$, this proves the desired result. \square

To prove Proposition 4.1, we will need one additional technical result demonstrating that the i_{th} coordinate of the dual solution, $\hat{\eta}_i$, behaves monotonically in \tilde{Y}_i . This result was originally

derived in [Gibbs et al. \(2025\)](#) where it was used to obtain efficient algorithms for computing $\hat{q}_{\text{GCC, rand.}}(\cdot)$. To state the result formally, let

$$\hat{\eta}^{\tilde{Y}_i \rightarrow y} = \operatorname{argmax}_{\eta \in [-(1-\tau), \tau]^n} \sum_{j \neq i} \eta_j \tilde{Y}_j + \eta_i y - \mathcal{R}^* \left(\sum_{j=1}^n \eta_j \tilde{X}_j \right), \quad (\text{A.2})$$

denote the dual solution obtained when \tilde{Y}_i is replaced with $y \in \mathbb{R}$. We have the following lemma.

Lemma A.2. *[Theorem 4 of [Gibbs et al. \(2025\)](#)] Fix any $i \in \{1, \dots, n\}$ and let $\{\hat{\eta}^{\tilde{Y}_i \rightarrow y}\}_{y \in \mathbb{R}}$ denote any collection of solutions to (A.2). Then, $y \mapsto \hat{\eta}^{\tilde{Y}_i \rightarrow y}$ is non-decreasing.*

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Fix any $i \in \{1, \dots, n\}$. Suppose there exists a leave-one-out primal solution with $\tilde{Y}_i < \tilde{X}_i^\top \hat{w}^{-i}$. By Lemma A.1, when $y = \tilde{X}_i^\top \hat{w}^{-i}$ there exists a solution to (A.2) with $\hat{\eta}_i^{\tilde{Y}_i \rightarrow \tilde{X}_i^\top \hat{w}^{-i}} = 0$. By the monotonicity of the dual solutions (Lemma A.2), this immediately implies that any dual solution to the full program must satisfy $\hat{\eta}_i \leq \hat{\eta}_i^{\tilde{Y}_i \rightarrow \tilde{X}_i^\top \hat{w}^{-i}} = 0$, as desired.

The case where $\tilde{Y}_i > \tilde{X}_i^\top \hat{w}^{-i}$ follows by an identical argument. \square

We conclude this section with a proof of Theorem 4.1. To aid in this proof, we introduce a number of additional pieces of notation. We let $\tilde{X} \in \mathbb{R}^{n \times p}$ denote the matrix with rows $\tilde{X}_1, \dots, \tilde{X}_n$ and $\tilde{Y} \in \mathbb{R}^n$ denote the vector with entries $\tilde{Y}_1, \dots, \tilde{Y}_n$. For any vector $v \in \mathbb{R}^k$ and set $I \subseteq \{1, \dots, k\}$ we let $(v_I) = (v_i)_{i \in I}$ denote the subvector consisting of the entries in I . Similarly, for any $I \subseteq \{1, \dots, n\}$ and $J \subseteq \{1, \dots, p\}$ we let $\tilde{X}_{I,J} = (\tilde{X}_{i,j})_{i \in I, j \in J}$ denote the submatrix consisting of the rows in I and columns in J . Finally, for any $k \in \mathbb{N}$ we let $[k]$ denote the set $\{1, \dots, k\}$.

We begin by presenting a preliminary lemma which controls the rank of the submatrix of \tilde{X} corresponding to the interpolated points of the quantile regression.

Lemma A.3. *Assume that $\{(\tilde{X}_i, \tilde{Y}_i)\}_{i=1}^n$ are i.i.d. and that the distribution of $\tilde{Y}_i \mid \tilde{X}_i$ is continuous. Then, with probability one all primal solutions \hat{w} satisfy,*

$$\operatorname{rank} \left(\tilde{X}_{\{i: \tilde{Y}_i = \tilde{X}_i^\top \hat{w}\}, [p]} \right) = |\{i : \tilde{Y}_i = \tilde{X}_i^\top \hat{w}\}|.$$

Proof. Fix any primal solution \hat{w} . Let $I_=(\hat{w}) = \{i \in \{1, \dots, n\} : \tilde{Y}_i = \tilde{X}_i^\top \hat{w}\}$ denote the set of interpolated points. By definition, we have that

$$\tilde{Y}_{I_=(\hat{w})} = \tilde{X}_{I_=(\hat{w}), [p]} \hat{w}.$$

For the sake of deriving a contradiction, suppose that $\operatorname{rank} \left(\tilde{X}_{\{i: \tilde{Y}_i = \tilde{X}_i^\top \hat{w}\}, [p]} \right) < |I_=(\hat{w})|$. Let $I(\hat{w}) \subset I_=(\hat{w})$ be such that $\tilde{X}_{I(\hat{w}), [p]}$ is of maximal rank. Then, there exists a matrix $A(\tilde{X})$ such that $\tilde{X}_{I_=(\hat{w}) \setminus I(\hat{w}), [p]} = A(\tilde{X}) \tilde{X}_{I(\hat{w}), [p]}$ and, in particular,

$$\tilde{Y}_{I_=(\hat{w}) \setminus I(\hat{w})} = \tilde{X}_{I_=(\hat{w}) \setminus I(\hat{w}), [p]} \hat{w} = A(\tilde{X}) \tilde{X}_{I(\hat{w}), [p]} \hat{w} = A(\tilde{X}) \tilde{Y}_{I(\hat{w})}$$

However, for any fixed sets $I' \subset I'_\pm \subseteq [n]$, the distribution of $\tilde{Y}_{I'_\pm \setminus I'} \mid (\tilde{X}, \tilde{Y}_{I'})$ is continuous. So, taking a union bound over all choices of $I_\pm(\hat{w})$ and $I(\hat{w})$ we find that this occurs with probability zero, as claimed. \square

We now prove Theorem 4.1.

Proof of Theorem 4.1. We split into two cases corresponding to the two sets of assumptions.

Case 1, Assumption 2 holds: In this case the primal program is strongly convex. Thus, for any $i \in \{1, \dots, n\}$, the leave-one-out solutions \hat{w}^{-i} is unique and by the continuity of the distribution of $\tilde{Y}_i \mid \tilde{X}_i$ we must have that $\mathbb{P}(\tilde{Y}_i = \tilde{X}_i^\top \hat{w}^{-i}) = 0$. By Lemma A.1, this implies that $\mathbb{P}(\text{Any dual solution satisfies } \hat{\eta}_i = 0) = 0$. The desired result then follows from the convexity of the space of primal and dual solutions.

Case 2, Assumption 1 holds: This case is considerably more involved. For notational simplicity, let us restrict to the case $i = n$. First, note that by the convexity of the space of primal and dual solutions and the results of Proposition 4.1 and Lemma A.1, it is sufficient to show that $\mathbb{P}(\text{There exists a dual solution with } \hat{\eta}_n = 0) = 0$. Fix a dual solution $\hat{\eta}$ and define the sets

$$I_{-(1-\tau)}(\hat{\eta}) = \{i \in \{1, \dots, n\} : \hat{\eta}_i = -(1-\tau)\}, \quad I_\tau(\hat{\eta}) = \{i \in \{1, \dots, n\} : \hat{\eta}_i = \tau\},$$

and $I_{\text{int.}}(\hat{\eta}) = \{i \in \{1, \dots, n\} : -(1-\tau) < \hat{\eta}_i < \tau\}.$

Let (\hat{w}, \hat{r}) denote any corresponding primal solution. The Lagrangian for this optimization problem is

$$L(w, r, \eta) = \sum_{i=1}^n \ell_\tau(r_i) + \sum_{i=1}^n \eta_i (\tilde{Y}_i - \tilde{X}_i^\top w - r_i) + \sum_{j=1}^p \lambda_j w_j^2.$$

Let $J_+ = \{j \in \{1, \dots, d\} : \lambda_j > 0\}$ denote the set of coordinates which receive positive regularization. Let $\Lambda_{J_+} = \text{diag}((\lambda_j)_{j \in J_+})$ be the diagonal matrix with diagonal entries $(\lambda_j)_{j \in J_+}$. Differentiating L with respect to w gives us that

$$\tilde{X}^\top \hat{\eta} = (2\lambda_j \hat{w}_j)_{j=1}^p \iff \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = -\tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \text{ and } \hat{w}_{J_+} = \frac{1}{2} \Lambda_{J_+}^{-1} \tilde{X}_{[n], J_+}^\top \hat{\eta}.$$

Moreover, considering the first-order conditions of L in η and r gives us that

$$\begin{aligned} \tilde{Y}_{I_{\text{int.}}(\hat{\eta})} &= \tilde{X}_{I_{\text{int.}}(\hat{\eta}), [p]} \hat{w} = \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} \hat{w}_{J_+^c} + \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{[n], J_+}^\top \hat{\eta} \\ &= \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} \hat{w}_{J_+^c} + \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} + \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}. \end{aligned}$$

Combining these equations, we arrive at the system of equations

$$\begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top & \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top & \mathbf{0}_{|J_+^c|, |J_+^c|} \end{bmatrix} \begin{bmatrix} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} \\ \hat{w}_{J_+^c} \end{bmatrix} = \begin{bmatrix} \tilde{Y}_{I_{\text{int.}}(\hat{\eta})} - \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \\ -\tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \end{bmatrix}.$$

We claim that the matrix appearing on the left hand side above is invertible. To see this, suppose that $(v_1, v_2) \in \mathbb{R}^{|I_{\text{int.}}(\hat{\eta})| \times |J_+^c|}$ is in the kernel of this matrix. In particular, suppose that

$$\frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top v_1 + \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} v_2 = 0 \quad \text{and} \quad \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top v_1 = 0. \quad (\text{A.3})$$

Taking the inner product of the first equation with v_1 we find that

$$\begin{aligned} 0 &= v_1^\top \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top v_1 + v_1^\top \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} v_2 = v_1^\top \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top v_1 \\ &\implies \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top v_1 = 0. \end{aligned}$$

Combining this fact with the second equation in (A.3) gives that $v_1^\top \tilde{X}_{I_{\text{int.}}(\hat{\eta}), [p]} = 0$ and so applying the result of Lemma A.3 we find that $v_1 = 0$. Returning (A.3), this implies that $\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} v_2 = 0$. Lemma A.4 shows that $\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}$ is of rank $|J_+^c|$. Thus, we must have that $v_2 = 0$. This proves the desired claim.

Applying this claim, we find that

$$\begin{aligned} &\begin{bmatrix} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} \\ \hat{w}_{J_+^c} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top & \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c} \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top & \mathbf{0}_{|J_+^c|, |J_+^c|} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{Y}_{I_{\text{int.}}(\hat{\eta})} - \frac{1}{2} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^c \\ -\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^c \end{bmatrix} \end{aligned}$$

Now, let us consider the behavior of the random variable appearing on the last line above when $I_{\text{int.}}(\hat{\eta})$ is a fixed set and $\hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^c$ is a fixed vector, i.e., fix any set $I_{\text{int.}}[n]$ and vector $\eta_{I_{\text{int.}}}^c \in \{-(1-\tau), \tau\}^{|I_{\text{int.}}^c|}$ with the property that the matrix inverse appearing above exists and consider the behaviour of the random variable

$$\begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{int.}}, J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}, J_+}^\top & \tilde{X}_{I_{\text{int.}}, J_+^c} \\ \tilde{X}_{I_{\text{int.}}, J_+}^\top & \mathbf{0}_{|J_+^c|, |J_+^c|} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{Y}_{I_{\text{int.}}} - \frac{1}{2} \tilde{X}_{I_{\text{int.}}, J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}, J_+}^\top \hat{\eta}_{I_{\text{int.}}}^c \\ -\tilde{X}_{I_{\text{int.}}, J_+^c}^\top \hat{\eta}_{I_{\text{int.}}}^c \end{bmatrix}. \quad (\text{A.4})$$

Now, by our assumptions the distribution of (\tilde{X}, \tilde{Y}) , we have that conditional on the values of $(\tilde{X}_{I_{\text{int.}}, J_+}$ and $\tilde{X}_{I_{\text{int.}}, J_+^c}^\top$, the vector appearing in (A.4) is continuously distributed. So, condition on the event that the matrix appearing in this expression is invertible, the entire random variable appearing in (A.4) has a continuous distribution. In particular, we find that with probability one this random variables has all non-zero entries. Taking a union bound over the values of $I_{\text{int.}}$ and $\hat{\eta}_{I_{\text{int.}}}^c$ gives the desired result. \square

We close this section with a proof a lemma regarding the rank of the submatrix of interpolated points that was helpful in the proof in Theorem 4.1.

Lemma A.4. *Suppose that Assumption 1 holds. For any dual solution $\hat{\eta}$, let $I_{\text{int.}}(\hat{\eta}) = \{i : -(1 - \tau) < (\hat{\eta})_i < \tau\}$ denote the set of coordinates in which $\hat{\eta}$ lies in the interior of its range. Let $J_+^c = \{j : \lambda_j = 0\}$ denote the set of unregularized coordinates of the primal variables. Then,*

$$\mathbb{P}(\text{For all dual solutions, } \text{rank}(\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}) = J_+^c) = 1.$$

Proof. By our assumptions on the distribution of \tilde{X} we have that with probability one $\text{rank}(X_{A,B}) = \min\{|A|, |B|\}$ for all $A \subseteq [n]$ and $B \subseteq [d]$. Thus, it is sufficient to show that with probability one all dual solutions satisfy $|J_+^c| \leq |I_{\text{int.}}(\hat{\eta})|$.

Fix any dual solution $\hat{\eta}$ and corresponding primal solution (\hat{w}, \hat{r}) . Recall that $(\hat{w}, \hat{r}, \hat{\eta})$ is a saddle point of the Lagrangian

$$L(w, r, \eta) = \sum_{i=1}^n \ell_\tau(r_i) + \sum_{i=1}^n \eta_i(\tilde{Y}_i - \tilde{X}_i^\top w - r_i) + \sum_{j=1}^p \lambda_j w_j^2.$$

Differentiating L with respect to r gives us that $\hat{\eta} \in [-(1 - \tau), \tau]^n$. Moreover, differentiating L with respect to w gives us that

$$\tilde{X}_{[n], J_+^c}^\top \hat{\eta} = 0 \iff \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}.$$

For the sake of deriving a contradiction, suppose that $|I_{\text{int.}}(\hat{\eta})| < |J_+^c|$. Let $J_{\text{sub.}}(\hat{\eta}) \subseteq J_+^c$ be a subset of size $|J_{\text{sub.}}(\hat{\eta})| = |I_{\text{int.}}(\hat{\eta})|$. Rearranging the above, we have that

$$\begin{aligned} & \left(\hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = (\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub.}}(\hat{\eta})}^\top)^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_{\text{sub.}}(\hat{\eta})}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}, \right. \\ & \quad \text{and } \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub.}}(\hat{\eta})^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_{\text{sub.}}(\hat{\eta})^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \Big) \\ & \implies \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub.}}(\hat{\eta})^c}^\top (\tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub.}}(\hat{\eta})}^\top)^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_{\text{sub.}}(\hat{\eta})}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} = \tilde{X}_{I_{\text{int.}}(\hat{\eta})^c, J_{\text{sub.}}(\hat{\eta})^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \quad (\text{A.5}) \end{aligned}$$

Now for any fixed sets $I_{\text{int.}} \subseteq [n]$ and $J_{\text{sub.}} \subseteq [d]$ with $|J_{\text{sub.}}| = |I_{\text{int.}}|$ and any fixed vector $\hat{\eta}_{I_{\text{int.}}}^c \in \{-(1 - \tau), \tau\}^n$, the random variable $\tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top \hat{\eta}_{I_{\text{int.}}}^c$ has a continuous distribution conditional on $\tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top (\tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top)^{-1} \tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top \hat{\eta}_{I_{\text{int.}}}^c$ and so, in particular,

$$\mathbb{P} \left(\tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top (\tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top)^{-1} \tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top \hat{\eta}_{I_{\text{int.}}}^c = \tilde{X}_{I_{\text{int.}}, J_{\text{sub.}}}^\top \hat{\eta}_{I_{\text{int.}}}^c \right) = 0.$$

Taking a union bound over all possible choices of the sets $I_{\text{int.}}$ and $J_{\text{sub.}}$ and vector $\hat{\eta}_{I_{\text{int.}}}^c$, we find that with probability one no dual solution can satisfy (A.5). This proves the desired result. \square

B Proofs for Section 5

The bulk of this section is devoted to a proof of Theorem 5.1. Proofs of Corollaries 5.1 and 5.2 are then given at the end of the section. In what follows, we use $X \in \mathbb{R}^{n \times p}$ to denote

the matrix with rows X_1, \dots, X_n and $Y \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}^n$ to denote the vectors with entries (Y_1, \dots, Y_n) and $(\epsilon_1, \dots, \epsilon_n)$, respectively. Additionally, for any convex function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $x \in \mathbb{R}$, and $\rho > 0$ we recall the definition of the Moreau envelope,

$$e_f(x; \rho) = \min_{v \in \mathbb{R}} \frac{1}{2\rho} \|x - v\|^2 + f(v).$$

For ease of notation, we will also define the Moreau envelope at $\rho = 0$ using the continuous extension $e_f(x; 0) = f(x)$ (cf. Theorem 1.25 of [Rockafellar & Wets \(1997\)](#)). Finally, for $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we recall the definition of the convex conjugate,

$$f^*(x) = -\inf_{v \in \mathbb{R}} f(v) - v^\top x.$$

We have the following assumptions on the regularizer and population coefficients.

Assumption 4. *The distribution of population coefficients, P_β has two bounded moments. Moreover, the regularization function and P_β are such that:*

1. \mathcal{R}_d is convex. Moreover, for all $\beta \in \mathbb{R}^d$, $\mathcal{R}(\beta) \geq 0$ and $\mathcal{R}(0) = 0$.
2. For any $C > 0$, the subderivatives of \mathcal{R}_d are bounded as

$$\sup_{d \in \mathbb{N}} \sup_{\|\beta\|_2 \leq C} \frac{1}{d} \|\partial \mathcal{R}_d(\beta)\|_2 < \infty.$$

3. There exists a convex function $\nu : \mathbb{R} \rightarrow \mathbb{R}$ with the property that for any $c \in \mathbb{R}$, $\rho \geq 0$, and $h \sim N(0, I_d)$,

$$\frac{1}{d} e_{\tilde{\mathcal{R}}_d}(ch_i + \sqrt{n}\tilde{\beta}_i; \rho) \xrightarrow{\mathbb{P}} \mathbb{E}[e_\nu(ch_1 + \gamma\sqrt{d}\tilde{\beta}_1; \rho)] < \infty,$$

as $d/n \rightarrow \gamma \in (0, \infty)$.

4. The function $(c, \rho) \mapsto \mathbb{E}[e_\nu(ch_1 + \gamma\sqrt{d}\tilde{\beta}_1; \rho)]$ is jointly continuous on $\mathbb{R} \times \mathbb{R}_{\geq 0}$.
5. For any $\rho > 0$, $\partial_x e_{\nu^*}(\cdot; \rho)$ and $\partial_x^2 e_{\nu^*}(\cdot; \rho)$ exist almost everywhere and satisfy the equations

$$\frac{d}{dc} \mathbb{E}[e_{\nu^*}(ch_1 + \rho\gamma\sqrt{d}\tilde{\beta}_1; \rho)] = \mathbb{E}[h_1 \partial_x e_{\nu^*}(ch_1 + \rho\gamma\sqrt{d}\tilde{\beta}_1; \rho)] = c \mathbb{E}[\partial_x^2 e_{\nu^*}(ch_1 + \rho\gamma\sqrt{d}\tilde{\beta}_1; \rho)].$$

6. For any compact set $A \subseteq \mathbb{R}_{>0}$ and constant $c > 0$,

$$\inf_{c \in A, 0 < \rho \leq C} \mathbb{E}[\partial_x^2 e_{\nu^*}(ch_1 + \rho\gamma\sqrt{d}\tilde{\beta}_1; \rho)] > 0.$$

All of the assumptions above are fairly generic and will hold for most common separable regularizers. In particular, the first two conditions are mild convexity and boundedness

conditions. The third part of Assumption 4 will follow for separable regularizers by the law of large numbers and the fourth part will follow by the dominated convergence theorem. The fifth part of the assumption will follow by using the dominated convergence theorem to swap the derivative and expectation and using Stein's lemma to compare the expectations involving the first and second derivative. Since the Moreau envelope is a convex function, the sixth condition of Assumption 4 is fairly weak and will hold as long as $\partial_x^2 e_{\nu^*}(\cdot; \rho)$ is sufficiently bounded away from zero. As an example, the following lemma verifies that all these conditions are satisfied by L_1 and L_2 regularization.

Lemma B.1. *Assume P_β has four bounded moments. Then, for any $\lambda \geq 0$ the conditions of Assumption 4 are met for $\mathcal{R}_d(\beta) = \sqrt{d}\lambda\|\beta\|_1$ and $\mathcal{R}_d(\beta) = d\lambda\|\beta\|_2^2$.*

Proof. For $\mathcal{R}_d(\beta) = d\lambda\|\beta\|_2^2$ define $\nu(b) = \lambda b^2$. Then, by a direct calculation we have that

$$e_\nu(x; \rho) = \frac{\lambda x^2}{1 + 2\lambda\rho}, \quad \nu^*(b) = \frac{b^2}{4\lambda}, \quad e_{\nu^*}(x; \rho) = \frac{x^2}{4\lambda + 2\rho}.$$

Parts one, two, and three of Assumption 4 are immediate. Part four follows by Chernoff's bound and part five follows by the dominated convergence theorem and Stein's lemma (Lemma 1 of Stein (1981)). Part six is also immediate since $\partial_x^2 e_{\nu^*}(x; \rho) = (2\lambda + \rho)^{-1} > 0$.

On the other hand, suppose $\mathcal{R}_d(\beta) = \sqrt{d}\lambda\|\beta\|_1$. Define $\nu(b) = \lambda|b|$. Then,

$$e_\nu(x; \rho) = \begin{cases} \lambda x - \frac{\lambda^2 \rho}{2}, & x > \lambda\rho, \\ \frac{x^2}{2\rho}, & |x| \leq \lambda\rho, \\ -x\lambda - \frac{\lambda^2 \rho}{2}, & x < -\lambda\rho, \end{cases} \quad \nu^*(b) = \begin{cases} 0, & |x| \leq \lambda, \\ \infty, & |x| > \lambda, \end{cases} \quad e_{\nu^*}(x; \rho) = \begin{cases} 0, & |x| \leq \lambda, \\ \frac{(|x| - \lambda)^2}{2\rho}, & |x| > \lambda. \end{cases}.$$

Moreover, one can verify that $e_{\nu^*}(x; \rho)$ is twice piecewise continuously differentiable with

$$\partial_x e_{\nu^*}(x; \rho) = \begin{cases} 0, & |x| \leq \lambda, \\ \frac{x - \text{sgn}(x)\lambda}{\rho}, & |x| > \lambda, \end{cases} \quad \partial_x^2 e_{\nu^*}(x; \rho) = \begin{cases} 0, & |x| < \lambda, \\ \frac{1}{\rho}, & |x| > \lambda. \end{cases}$$

The desired results once again follow by the dominated convergence theorem and Stein's lemma. □

For notational convenience, let $\tilde{\mathcal{R}}_d = n^{-1/2}\mathcal{R}_d$ denote a rescaling of the regularizer. Our main point of study is the joint min-max formulation of the quantile regression,

$$\max_{\eta} \min_{\beta_0, \beta, r} \frac{1}{n} \sum_{i=1}^n \ell(r_i) + \frac{1}{n} \eta^\top (Y - \beta_0 \mathbf{1}_n - X\beta) + \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d(\beta).$$

Letting $u = \beta - \tilde{\beta}$ and re-writing $\tilde{\mathcal{R}}_d(\beta)$ in terms of the convex conjugate, this can be equivalently formulated as

$$\max_{\eta, s} \min_{\beta_0, u, r} \frac{1}{n} \sum_{i=1}^n \ell(r_i) + \frac{1}{n} \eta^\top (\epsilon - \beta_0 \mathbf{1}_n + Xu - r) + \frac{1}{\sqrt{n}} s^\top (\tilde{\beta} + u) - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s). \quad (\text{B.1})$$

To prove Theorem 5.1 we will need to study the solutions of this optimization program. We proceed in four main steps. First, in Section B.1 we give a number of preliminary simplifications of the optimization program. Section B.2 then begins our main study of (B.1). We show that the solutions to this problem are characterized by an auxiliary optimization program in which the matrix X is replaced by vector-valued Gaussian random variables. Moreover, we additionally demonstrate that the solutions to this auxiliary problem are themselves characterized by the deterministic asymptotic program

$$\min_{(|\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u, 0 \leq \rho_1 \leq C_1)} \max_{(0 < \rho_2 < C_2, c_\eta \leq M_\eta \leq C_\eta)} \left(\mathbb{E} \left[e_{\ell_\tau} \left(M_u g + \epsilon - \beta_0; \frac{\rho_1}{M_\eta} \right) \right] - \frac{M_\eta^2 M_u \gamma}{2\rho_2} \right. \\ \left. + \frac{M_\eta \rho_1}{2} - \frac{M_u \rho_2}{2} - \mathbb{E} \left[e_\nu \left(\frac{M_u M_\eta}{\rho_2} h_1 + \gamma \sqrt{d} \tilde{\beta}_1; \frac{M_2}{\rho_2} \right) \right] \right), \quad (\text{B.2})$$

where $(\beta_0, M_u, \rho_1, \rho_2, M_\eta)$ are the optimization variables, $(C_{\beta_0}, C_u, C_1, C_2, c_\eta, C_\eta)$ are constants that we will define shortly, and $g_1, h_1 \sim \mathcal{N}(0, 1)$ independent of $\tilde{\beta}_1$ and ϵ_1 . The solutions to this asymptotic program are characterized in Section B.3. Section B.4 then gives a proof of Theorem 5.1 and Section B.5 gives proofs of Corollaries 5.1 and 5.2.

Our overall analysis framework is based on the work of Thrampoulidis et al. (2018). In what follows, we will focus on the aspects of the analysis that are new to our work and omit the proofs of some results that are minor variations of those appearing in Thrampoulidis et al. (2018).

B.1 Preliminaries

We begin our proof of Theorem 5.1 by giving three lemmas which bound the ranges of the optimization variables appearing in (B.1). In what follows, we use the notation $(\hat{\beta}_0, \hat{u}, \hat{r}, \hat{\eta}, \hat{s})$ to denote a generic primal-dual solution to (B.1), where $(\hat{\beta}_0, \hat{u}, \hat{r})$ are the primal solutions and $(\hat{\eta}, \hat{s})$ are the dual solutions. Our first result bounds the range of $\hat{\eta}$.

Lemma B.2. *Under the assumptions of Theorem 5.1, there exist $C_\eta > c_\eta > 0$ such that*

$$\mathbb{P} \left(\text{For all dual solutions to (B.1), } \sqrt{n} c_\eta \leq \|\hat{\eta}\|_2 \leq \sqrt{n} C_\eta \right) \rightarrow 1.$$

Proof. Let $\hat{\eta}$ denote any dual solution. First, note that by considering the first-order conditions in r of (B.1) we have that $\hat{\eta} \in [-(1 - \tau), \tau]^n$ and $Y_i \neq \hat{\beta}_0 + X_i^\top \hat{\beta} \implies \hat{\eta} \in \{-(1 - \tau), \tau\}$. Taking $C_\eta = \max\{(1 - \tau), \tau\}$ gives the upper bound. To get the lower bound, note that by Lemma A.3 any primal solution can interpolate at most $d + 1$ of the data points. Thus, we must have $\|\hat{\eta}\|_2 \geq \sqrt{n - d - 1} \min\{(1 - \tau), \tau\}$ and so setting $c_\eta = (1/2)\sqrt{1 - \gamma} \min\{(1 - \tau), \tau\}$ gives the desired result. □

Our next lemma gives a similar set of bound on \hat{u} and $\hat{\beta}_0$. For ease of notation, we state this result in terms of the original primal variables $\hat{\beta} = \hat{u} + \hat{\beta}_0$.

Lemma B.3. *Suppose the assumptions of Theorem 5.1 hold. Then, there exist constants $C_u, C_{\beta_0} > 0$ such that*

$$\mathbb{P}\left(\text{For all primal solutions to (B.1), } \|\hat{\beta} - \tilde{\beta}\|_2 \leq C_u \text{ and } |\hat{\beta}_0| \leq C_{\beta_0}\right) \rightarrow 1.$$

Proof. Let $(\hat{\beta}_0, \hat{\beta})$ denote any primal solution. By the law of large numbers and the optimality of $(\hat{\beta}_0, \hat{\beta})$, we have that

$$\begin{aligned} \mathbb{E}[\ell_\tau(Y_1)] &\geq \frac{1}{n} \sum_{i=1}^n \ell_\tau(Y_i) - o_{\mathbb{P}}(1) \geq \frac{1}{n} \sum_{i=1}^n \ell_\tau(Y_i - X_i^\top \hat{\beta} - \hat{\beta}_0) + \mathcal{R}_d(\hat{\beta}) - o_{\mathbb{P}}(1) \\ &\geq \min\{1 - \tau, \tau\} \frac{1}{n} \sum_{i=1}^n |X_i^\top (\hat{\beta} - \tilde{\beta}) + \hat{\beta}_0| - \min\{1 - \tau, \tau\} \frac{1}{n} \sum_{i=1}^n |\epsilon_i| - o_{\mathbb{P}}(1) \\ &\geq \min\{1 - \tau, \tau\} \max\{\|\hat{\beta} - \tilde{\beta}\|_2, |\hat{\beta}_0|\} \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \frac{1}{n} \sum_{i=1}^n |X_i^\top u + \beta_0| \\ &\quad - \min\{1 - \tau, \tau\} \mathbb{E}[\epsilon_1] - o_{\mathbb{P}}(1). \end{aligned}$$

Lemma C.1 below shows that

$$\liminf_{n, d \rightarrow \infty} \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \frac{1}{n} \sum_{i=1}^n |X_i^\top u + \beta_0| \xrightarrow{\mathbb{P}} \sqrt{\frac{2}{\pi}} - \sqrt{\gamma}.$$

Applying this to the above, we conclude that

$$\max\{\|\hat{\beta} - \tilde{\beta}\|_2, |\hat{\beta}_0|\} \leq \frac{\mathbb{E}[\ell_\tau(Y_1)] + \min\{1 - \tau, \tau\} \mathbb{E}[\epsilon_1]}{\min\{1 - \tau, \tau\} (\sqrt{2/\pi} - \sqrt{\gamma})} + o_{\mathbb{P}}(1),$$

where it should be understood that the $o_{\mathbb{P}}(1)$ term on the right hand side is uniform over all primal solutions. Taking $C_u = C_{\beta_0} = 2 \frac{\mathbb{E}[\ell_\tau(Y_1)] + \min\{1 - \tau, \tau\} \mathbb{E}[\epsilon_1]}{\min\{1 - \tau, \tau\} (\sqrt{2/\pi} - \sqrt{\gamma})}$ gives the desired result. \square

Our final preliminary lemma bounds the size of solutions for r and s .

Lemma B.4. *Suppose the assumptions of Theorem 5.1 hold. Then, there exist constants $C_r > 0$ and $C_s > 0$ such that*

$$\mathbb{P}\left(\text{For all solutions to (B.1), } \|\hat{s}\|_2 \leq C_s \sqrt{n} \text{ and } \|\hat{r}\|_2 \leq C_r \sqrt{n}\right) \rightarrow 1.$$

Proof. Fix any solution $(\hat{\eta}, \hat{s}, \hat{\beta}_0, \hat{u}, \hat{r})$ to (B.1). By the first-order conditions of (B.1) in η we must have

$$\|\hat{r}\|_2 = \|\epsilon - \hat{\beta}_0 \mathbf{1}_n - X \hat{u}\|_2 \leq \|\epsilon\|_2 + \sqrt{n} |\hat{\beta}_0| + \lambda_{\max}(X) \|\hat{u}\|_2.$$

By standard results (e.g. Theorem 3.1 of Yin et al. (1988)) we have that $\lambda_{\max}(X)/\sqrt{n}$ is converging in probability to a constant. Moreover, by the law of large numbers, $\|\epsilon\|_2/\sqrt{n} \xrightarrow{\mathbb{P}}$

$\sqrt{\mathbb{E}[\epsilon_i^2]}$. Combining these facts with the bounds on $|\hat{\beta}_0|$ and $\|\hat{u}\|_2$ given by Lemma B.3 gives the desired bound on $\|\hat{r}\|_2$.

To bound $\|\hat{s}\|_2$, note that by standard facts regarding the convex conjugate (e.g. Proposition 5.4.3 of Bertsekas (2009)), we have $\hat{s} \in \partial \tilde{\mathcal{R}}_d(\tilde{\beta} + \hat{u})$. Moreover, by Lemma B.3 and the law of large numbers there exists $C > 0$ such that with probability converging to one, $\|\tilde{\beta} + \hat{u}\|_2 \leq C$. So,

$$\frac{1}{\sqrt{n}} \|\hat{s}\|_2 \leq \sup_{\|v\|_2 \leq C} \left\| \frac{1}{\sqrt{n}} \partial \tilde{\mathcal{R}}_d(v) \right\|_2 = \sup_{\|v\|_2 \leq C} \left\| \frac{1}{n} \partial \mathcal{R}_d(v) \right\|_2.$$

This last quantity is bounded by our part 2 of Assumption 4. \square

B.2 Reduction to the auxiliary optimization problem

We will now reduce (B.1) to a simpler asymptotic program that is easier to study. Our main tool will be the Gaussian comparison inequalities of Gordon (1985, 1988) and their application to regression problems developed in Thrampoulidis et al. (2018). In particular, we will apply the following proposition. This result is a minor extension of Theorem 3 of Thrampoulidis et al. (2018) (see also Theorem 3 of Thrampoulidis et al. (2015)) and thus its proof is omitted.

Proposition B.1 (Extension of Theorem 3 of Thrampoulidis et al. (2018)). *Fix any $d, n \in \mathbb{R}$. Let $X \in \mathbb{R}^{n \times d}$ be distributed as $(X_{ij})_{i \in [n], j \in [d]} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ and define $g \sim \mathcal{N}(0, I_n)$ and $h \sim \mathcal{N}(0, I_d)$ to be independent Gaussian vectors. Let $Q(\beta_0, u, r, \eta, s) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ be continuous and jointly convex in (r, β_0, u) and concave in (s, η) . Fix any compact sets $A \subseteq \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n \times \mathbb{R}^d$. Define the values*

$$\begin{aligned} \Phi &= \max_{(\eta, s) \in B} \min_{(\beta_0, u, r) \in A} \eta^\top Xu + Q(\beta_0, u, r, \eta, s), \\ \phi &= \max_{(\eta, s) \in B} \min_{(\beta_0, u, r) \in A} \|u\|_2 \eta^\top g + \|\eta\|_2 u^\top h + Q(\beta_0, u, r, \eta, s). \end{aligned}$$

Then, for all $c \in \mathbb{R}$,

$$\mathbb{P}(\Phi > c) \leq 2\mathbb{P}(\phi \geq c).$$

If in addition A and B are convex, then for all $c \in \mathbb{R}$,

$$\mathbb{P}(\Phi < c) \leq 2\mathbb{P}(\phi \leq c).$$

To apply this result in our context, let

$$\begin{aligned} \Phi(S) &= \max_{(\eta \in S, \|s\|_2 \leq C_s \sqrt{n})} \min_{(|\beta_0| \leq C_{\beta_0}, \|u\|_2 \leq C_u, \|r\|_2 \leq C_r \sqrt{n})} \frac{1}{n} \sum_{i=1}^n \ell(r_i) + \frac{1}{n} \eta^\top (\epsilon - \beta_0 \mathbf{1}_n - Xu - r) \\ &\quad + \frac{1}{\sqrt{n}} s^\top (\tilde{\beta} + u) - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s), \end{aligned}$$

where the constants $C_s, C_{\beta_0}, C_u, C_r$ are defined as in Lemmas B.3 and B.4. Let C_η be defined as in Lemma B.2. We know that asymptotically the solutions of $\Phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\})$ agrees with those of (B.1). Our goal will be to compare the value of $\Phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\})$ to that of $\Phi(S)$ when S is a more restricted set. The key insight of Thrampoulidis et al. (2018) is that for this purpose it is sufficient to study the value of the auxiliary optimization,

$$\begin{aligned} \phi(S) := & \min_{(\|r\|_2 \leq C_r\sqrt{n}, |\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \max_{(\|s\|_2 \leq C_s\sqrt{n}, \eta \in S)} \min_{(u: \|u\|_2 = M_u)} \left(\frac{1}{n} \|u\|_2 \eta^\top g + \frac{1}{n} \|\eta\|_2 u^\top h \right. \\ & \left. + \frac{1}{n} \eta^\top \epsilon - \frac{1}{n} \beta_0 \eta^\top \mathbf{1}_n - \frac{1}{n} \eta^\top r + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top (\tilde{\beta} + u) - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right), \end{aligned}$$

where $h \sim \mathcal{N}(0, I_d)$ and $g \sim \mathcal{N}(0, I_n)$ are Gaussian vectors sampled such that $(g, h, \epsilon, \tilde{\beta})$ is jointly independent. The following proposition formalizes this.

Proposition B.2 (Corollary of Lemma 7 in Thrampoulidis et al. (2018)). *Suppose the assumptions of Theorem 5.1 hold and let C_η be a constant satisfying the conclusion of Lemma B.2. Let S be any set such that*

1. S is compact,
2. There exists $V \in \mathbb{R}$ and $\xi, \delta > 0$ such that $\min\{\mathbb{P}(\phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\}) \geq V + \delta), \mathbb{P}(\phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\}) \leq V - \delta)\} \geq 1 - \xi$. Then,

$$\mathbb{P}(\text{For all dual solutions to (B.2), } \hat{\eta} \notin S) \geq 1 - 4\xi.$$

Proof. This result follows immediately by applying Proposition B.1 and repeating the steps of Lemma 7 in Thrampoulidis et al. (2018). \square

Our goal now is to lower bound $\phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\})$ and upper bound $\phi(S)$ for a more restricted set S . We will focus initially on $\phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\})$. Let c_η and C_η be the constants appearing in Lemma B.2. We have that

$$\begin{aligned} \phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\}) & \geq \phi(\{\eta : c_\eta\sqrt{n} \leq \|\eta\|_2 \leq C_\eta\sqrt{n}\}) \\ & = \min_{(\|r\|_2 \leq C_r\sqrt{n}, |\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \max_{(\|s\|_2 \leq C_s\sqrt{n}, c_\eta\sqrt{n} \leq \|\eta\|_2 \leq C_\eta\sqrt{n})} \left(-M_u \left\| \frac{1}{n} \|\eta\|_2 h + \frac{1}{\sqrt{n}} s \right\|_2 \right. \\ & \quad \left. + \frac{1}{n} M_u \eta^\top g + \frac{1}{n} \eta^\top \epsilon - \frac{1}{n} \beta_0 \eta^\top \mathbf{1}_n - \frac{1}{n} \eta^\top r + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right) \\ & = \min_{(\|r\|_2 \leq C_r\sqrt{n}, |\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \max_{(\|s\|_2 \leq C_s\sqrt{n}, c_\eta \leq M_\eta \leq C_\eta)} \left(-M_u \left\| \frac{1}{\sqrt{n}} M_\eta h + \frac{1}{\sqrt{n}} s \right\|_2 \right. \\ & \quad \left. + M_\eta \left\| \frac{1}{\sqrt{n}} M_u g + \frac{1}{\sqrt{n}} \epsilon - \frac{1}{\sqrt{n}} \beta_0 \mathbf{1}_n - \frac{1}{\sqrt{n}} r \right\|_2 + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right). \end{aligned}$$

Notably, this last optimization problem is convex-concave. Now, note that for any vector x and $C \geq \|x\|_2$, $\|x\|_2 = \min_{0 < \tau \leq C} \frac{\|x\|_2^2}{2\tau} + \frac{\tau}{2}$. Moreover, by the weak law of large numbers, there exist constants $C_1, C_2 > 0$ such that for any with probability converging to one,

$$\begin{aligned} \max_{(\|r\|_2 \leq C_r \sqrt{n}, |\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \left\| \frac{1}{\sqrt{n}} M_u g + \frac{1}{\sqrt{n}} \epsilon - \frac{1}{\sqrt{n}} \beta_0 \mathbf{1}_n - \frac{1}{\sqrt{n}} r \right\|_2 &\leq C_1, \\ \max_{(\|s\|_2 \leq C_s \sqrt{n}, c_\eta \leq M_\eta \leq C_\eta)} \left\| \frac{1}{\sqrt{n}} M_\eta h + \frac{1}{\sqrt{n}} s \right\|_2 &\leq C_2. \end{aligned}$$

So, applying these facts and using Sion's minimax theorem to swap the order of minimization and maximization ([Sion 1958](#)), we have that the above can be rewritten as

$$\begin{aligned} \min_{(|\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \max_{(c_\eta \leq M_\eta \leq C_\eta)} \min_{0 < \rho_1 \leq C_1} \max_{0 < \rho_2 \leq C_2} \min_{\|r\|_2 \leq C_r \sqrt{n}} \max_{\|s\|_2 \leq C_s \sqrt{n}} &\left(\frac{M_\eta}{2n\rho_1} \|M_u g + \epsilon - \beta_0 \mathbf{1}_n - r\|_2^2 \right. \\ &\left. + \frac{M_\eta \rho_1}{2} - \frac{M_u}{2n\rho_2} \|M_\eta h + s\|_2^2 - \frac{M_u \rho_2}{2} + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right). \end{aligned}$$

We will now rewrite the optimizations over r and s in terms of the Moreau envelope.

Lemma B.5. *Fix any constants $C_{\beta_0}, C_u, C_\eta, c_\eta > 0$ with $C_\eta > c_\eta$. Under the assumptions of Theorem 5.1, there exist constants $\tilde{C}_s, \tilde{C}_r > 0$, such that with probability tending to 1, it holds that for any $|\beta_0| \leq C_{\beta_0}$, $0 \leq M_u \leq C_u$, $c_\eta \leq M_\eta \leq C_\eta$, and $\rho_2, \rho_1 > 0$,*

$$\min_{\|r\|_2 \leq \tilde{C}_r \sqrt{n}} \frac{M_\eta}{2n\rho_1} \|M_u g + \epsilon - \beta_0 \mathbf{1}_n - r\|_2^2 + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) = \frac{1}{n} \sum_{i=1}^n e_{\ell_\tau} \left(M_u g_i + \epsilon_i - \beta_0; \frac{\rho_1}{M_\eta} \right),$$

and

$$\begin{aligned} \max_{\|s\|_2 \leq \tilde{C}_s \sqrt{n}} & - \frac{M_u}{2n\rho_2} \|M_\eta h + s\|_2^2 + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \\ &= - \frac{M_\eta^2 M_u}{2n\rho_2} \|h\|_2^2 + \frac{1}{n} e_{\mathcal{R}_d} \left(\frac{M_\eta M_u}{\rho_2} h + \sqrt{n} \tilde{\beta}; \frac{M_u}{\rho_2} \right). \end{aligned}$$

Proof. Recall the definition of the proximal function,

$$\text{prox}_f(x; \tau) = \underset{v}{\operatorname{argmin}} \frac{1}{2\tau} \|x - v\|_2^2 + f(v).$$

By definition of ℓ_τ , we have $\text{prox}_{\ell_\tau}(0; \tau) = 0$ for any $\tau > 0$. Since the proximal function is non-expansive (Proposition 12.28 of [Bauschke & Combettes \(2017\)](#)) it holds that for any $x \in \mathbb{R}$ and $\tau > 0$,

$$\|\text{prox}_{\ell_\tau}(x; \tau)\|_2 = \|\text{prox}_{\ell_\tau}(x; \tau) - \text{prox}_{\ell_\tau}(0; \tau)\|_2 \leq \|x - 0\|_2 = \|x\|_2.$$

So, in particular,

$$\text{prox}(M_u g + \epsilon - \beta_0 \mathbf{1}_n; \rho_1/M_\eta) \leq \|M_u g + \epsilon - \beta_0 \mathbf{1}_n\|_2.$$

This last quantity must be bounded by the law of large numbers and our restrictions on M_u and β_0 . This proves the first part of the lemma. The second part of the lemma follows from equations 86-88 of [Thrampoulidis et al. \(2018\)](#) and an identical argument. \square

Now, without loss of generality we may assume that $C_r \geq \tilde{C}_r$ and $C_s \geq \tilde{C}_s$. So, applying Lemma B.5 and taking a continuous extension at $\rho_1 = 0$, our previous calculations gives us the optimization problem

$$\min_{(|\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u, 0 \leq \rho_1 \leq C_1)} \max_{(c_\eta \leq M_\eta \leq C_\eta, 0 < \rho_2 \leq C_2)} \left(\frac{1}{n} \sum_{i=1}^n e_{\ell_\tau} \left(M_u g_i + \epsilon_i - \beta_0; \frac{\rho_1}{M_\eta} \right) - \frac{M_\eta^2 M_u}{2n\rho_2} \|h\|_2^2 + \frac{1}{n} e_{\mathcal{R}_d} \left(\frac{M_\eta M_u}{\rho_2} h + \sqrt{n} \tilde{\beta}; \frac{M_u}{\rho_2} \right) + \frac{M_\eta \rho_1}{2} - \frac{M_u \rho_2}{2} \right), \quad (\text{B.3})$$

Our final step is to replace all the random quantities above with their asymptotic limits. To do this, we will employ the following lemma which states that pointwise convergence can be converted to convergence of the minimum value of a convex function. This result is a minor variant of Lemma 10 of [Thrampoulidis et al. \(2018\)](#) and we include a partial proof for completeness.

Lemma B.6 (Extension of Lemma 10 of [Thrampoulidis et al. \(2018\)](#)). *Fix $b > a$ and let $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of random convex functions converging pointwise in probability to $f : [a, b] \rightarrow \mathbb{R}$. Then,*

$$\inf_{x \in [a, b]} f_n(x) \xrightarrow{\mathbb{P}} \inf_{x \in [a, b]} f(x).$$

Similarly, if $f_n : (a, b] \rightarrow \mathbb{R}$ is a sequence of random convex functions converging pointwise in probability to function $f : (a, b] \rightarrow \mathbb{R}$, then

$$\inf_{x \in (a, b]} f_n(x) \xrightarrow{\mathbb{P}} \inf_{x \in (a, b]} f(x).$$

Proof. We will prove the first part of the lemma. Proof of the second part is similar and is omitted. For any $x' \in [a, b]$ we have that

$$\limsup_{n \rightarrow \infty} \inf_{x \in [a, b]} f_n(x) \leq \limsup_{n \rightarrow \infty} f_n(x') \stackrel{\mathbb{P}}{=} f(x'),$$

and taking the infimum over x' we conclude that $\limsup_{n \rightarrow \infty} \inf_{x \in [a, b]} f_n(x) \leq \inf_{x \in [a, b]} f(x)$.

It suffices to prove a matching lower bound. If $\inf_{x \in [a, b]} f(x) = -\infty$ there is nothing to show. So, assume that $\inf_{x \in [a, b]} f(x) > -\infty$. By Lemma 7.75 of [Miescke & Liese \(2008\)](#) we have that for any interval $a < x_1 < x_2 < b$,

$$\sup_{x \in [x_1, x_2]} |f_n(x) - f(x)| \xrightarrow{\mathbb{P}} 0,$$

and thus also,

$$\inf_{x \in [x_1, x_2]} f_n(x) \xrightarrow{\mathbb{P}} \inf_{x \in [x_1, x_2]} f(x). \quad (\text{B.4})$$

So, we just need to check what happens on the boundary. We will focus on the lower boundary. First, suppose that $\liminf_{x \rightarrow a} f(x) > \inf_{x \in [a, b]} f(x)$. Let $x^* \in (a, b]$ be such that $f(x^*) < \inf_{x \in [a, b]} f(x) + \frac{\liminf_{x \rightarrow a} f(x) - \inf_{x \in [a, b]} f(x)}{2}$. Fix any $a < x_1 < x_2 < x^*$ with $f(x_1), f(x_2) > f(x^*)$. For any $x \in [a, x_1]$ let $\lambda_x > 0$ be such that $x_2 = \lambda_x x + (1 - \lambda_x)x^*$. Then,

$$f_n(x_2) \leq \lambda_x f_n(x) + (1 - \lambda_x) f_n(x^*) \implies f_n(x) \geq f_n(x^*) + \frac{1}{\lambda_x} (f_n(x_2) - f_n(x^*)).$$

Asymptotically, we have that $\lim_{n \rightarrow \infty} f_n(x_2) - f_n(x^*) \stackrel{\mathbb{P}}{=} f(x_2) - f(x^*) > 0$ and thus $\liminf_{n \rightarrow \infty} \inf_{x \in [a, x_1]} f_n(x) \stackrel{\mathbb{P}}{\geq} f(x^*) \geq \inf_{x \in [a, b]} f(x)$.

On the other hand, suppose $\liminf_{x \rightarrow a} f(x) = \inf_{x \in [a, b]} f(x)$. Fix any $\delta > 0$. We claim that there exists $a < x_1 < x_2 < b$ such that $f(x_1), f(x_2) < \inf_{x \in [a, b]} f(x) + \delta$. To see this, let $x_\delta \in [a, (b+a)/2]$ be such that $f(x_\delta) < \inf_{x \in [a, b]} f(x) + \delta/2$. Fix any $0 < \xi < b - x_\delta$ and for any $x \in [x_\delta, x_\delta + \xi]$ write

$$f(x) \leq \left(1 - \frac{x - x_\delta}{b - x_\delta}\right) f(x_\delta) + \frac{x - x_\delta}{b - x_\delta} f(b) \leq f(x_\delta) + \frac{\xi}{b - x_\delta} f(b).$$

Taking ξ sufficiently small we find that $\sup_{x \in [x_\delta, x_\delta + \xi]} f(x) \leq \inf_{x \in [a, b]} f(x) + \delta$ and so setting $x_1 < x_2$ to be any points in $(x_\delta, x_\delta + \xi)$ gives the desired claim.

Now, for any $x \in [a, x_1]$ we have

$$\begin{aligned} f_n(x_1) &= f_n\left(\frac{x_2 - x_1}{x_2 - x}x + \left(1 - \frac{x_2 - x_1}{x_2 - x}\right)x_2\right) \leq \frac{x_2 - x_1}{x_2 - x} f_n(x) + \left(1 - \frac{x_2 - x_1}{x_2 - x}\right) f_n(x_2) \\ \implies f_n(x) &\geq \frac{x_2 - x}{x_2 - x_1} f_n(x_1) - \frac{x_2 - x}{x_2 - x_1} \left(1 - \frac{x_2 - x_1}{x_2 - x}\right) f_n(x_2) \\ &\geq \min\{f_n(x_1), f_n(x_2)\} - \frac{x_2 - x}{x_2 - x_1} \left(1 - \frac{x_2 - x_1}{x_2 - x}\right) (f_n(x_2) - \min\{f_n(x_1), f_n(x_2)\}) \\ &\geq \min\{f_n(x_1), f_n(x_2)\} - \frac{x_1 - a}{x_2 - x_1} (f_n(x_2) - \min\{f_n(x_1), f_n(x_2)\}) \\ &\stackrel{\mathbb{P}}{\geq} \inf_{x' \in [a, b]} f(x') - \frac{x_1 - a}{x_2 - x_1} \delta, \end{aligned}$$

where the probability in the last inequality holds uniformly over x . Thus,

$$\liminf_{n \rightarrow \infty} \inf_{x \in [a, x_1]} f_n(x) \stackrel{\mathbb{P}}{\geq} \inf_{x \in [a, b]} f(x).$$

So, in total, we find that in all cases we may find $x_1 \in (a, b]$ such that

$$\liminf_{n \rightarrow \infty} \inf_{x \in [a, x_1]} f_n(x) \stackrel{\mathbb{P}}{\geq} \inf_{x \in [a, b]} f(x).$$

By a matching argument, we may also find $x'_1 \in [a, b)$ such that

$$\liminf_{n \rightarrow \infty} \inf_{x \in [x'_1, b]} f_n(x) \stackrel{\mathbb{P}}{\geq} \inf_{x \in [a, b]} f(x).$$

Combining these two facts and using (B.4) to get convergence on the interior gives the desired result. \square

Combining all of the previous results we arrive at the following.

Proposition B.3. *Suppose the assumptions of Theorem 5.1 hold. Let $(C_{\beta_0}, C_u, c_\eta, C_\eta, C_1, C_2)$ be constants satisfying the conditions of Lemmas B.2, B.3, B.4, and B.5. Let V denote the value of the asymptotic program defined in (B.2). Then,*

$$\liminf_{n, d \rightarrow \infty} \Phi(\{\eta : \|\eta\|_2 \leq \sqrt{n}C_\eta\}) \stackrel{\mathbb{P}}{\geq} V.$$

Proof. This lemma follows immediately by the law of large numbers, part 3 of Assumption 4, and repeated applications of Lemma B.6 to (B.3). \square

B.3 Analysis of the asymptotic program

In this section we prove a number of useful results regarding the asymptotic auxiliary program defined in (B.3). In what follows we use

$$\begin{aligned} A(\beta_0, M_u, \rho_1, M_\eta, \rho_2) = & \mathbb{E} \left[e_{\ell_\tau} \left(M_u g_1 + \epsilon_1 - \beta_0; \frac{\rho_1}{M_\eta} \right) \right] - \frac{M_\eta^2 M_u \gamma}{2\rho_2} \\ & + \gamma \mathbb{E} \left[e_\nu \left(\frac{M_\eta M_u}{\rho_2} h_1 + \gamma \sqrt{d} \tilde{\beta}_1; \frac{M_u}{\rho_2} \right) \right] + \frac{M_\eta \rho_1}{2} - \frac{M_u \rho_2}{2}, \end{aligned}$$

to denote the objective of this optimization.

Lemma B.7. *Under the assumptions of Theorem 5.1, $A(\beta_0, M_u, \rho_1, M_\eta, \rho_2)$ is jointly continuous, jointly convex in (β_0, M_u, ρ_1) , and jointly concave in (M_η, ρ_2) on the domain $\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0}$.*

Proof. The last four terms of A are jointly continuous by part three of Assumption 4. Moreover, joint continuity of the first term follows immediately by inspecting the form of e_{ℓ_τ} (Lemma C.3) and applying the dominated convergence theorem. The fact that A is convex-concave follows directly from the fact that it is the pointwise limit of a sequence of convex-concave functions. \square

Lemma B.8. *Fix any $C_\eta > c_\eta > 0$ and $C_2 > 0$. Under the conditions of Theorem 5.1, the function*

$$(\beta_0, M_u, \rho_1) \mapsto \max_{c_\eta \leq M_\eta \leq C_\eta, 0 \leq \rho_2 \leq C_2} A(\beta_0, M_u, \rho_1, M_\eta, \rho_2),$$

is jointly strictly convex on $\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0}$. Moreover, for $\rho_1 = 0$ this function is jointly strictly convex in (β_0, M_u) .

Proof. We first consider the case where $\rho_1 > 0$. Fix any $M_\eta \in [c_\eta, C_\eta]$ and pair of distinct points $(\beta_0, M_u, \rho_1), (\beta'_0, M'_u, \rho'_1) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0}$. For $\theta \in [0, 1]$ define the function

$$w(\theta) = \mathbb{E} \left[e_{\ell_\tau} \left(((1-\theta)M_u + \theta M'_u)g_1 + \epsilon_1 - (1-\theta)\beta_0 - \theta\beta'_0; \frac{(1-\theta)\rho_1 + \theta\rho'_1}{M_\eta} \right) \right].$$

For ease of notation, let

$$\begin{aligned} (\xi_1, \xi_2, \xi_3) &= (M'_u - M_u, \beta'_0 - \beta_0, \rho'_1 - \rho_1), \quad \rho_\theta = ((1-\theta)\rho_1 + \theta\rho'_1), \\ \text{and } Z_\theta &= ((1-\theta)M_u + \theta M'_u)g_1 + \epsilon_1 - (1-\theta)\beta_0 - \theta\beta'_0. \end{aligned}$$

By the dominated convergence theorem and a direct calculation using the form of e_{ℓ_τ} (see Lemma C.3), we have

$$\begin{aligned} w'(\theta) &= \mathbb{E} \left[(g_1\xi_1 - \xi_2)\tau \mathbb{1} \left\{ Z_\theta > \tau \frac{\rho_\theta}{M_\eta} \right\} + (g_1\xi_1 - \xi_2) \frac{Z_\theta M_\eta}{\rho_\theta} \mathbb{1} \left\{ -(1-\tau) \frac{\rho_\theta}{M_\eta} \leq Z_\theta \leq \tau \frac{\rho_\theta}{M_\eta} \right\} \right. \\ &\quad \left. - (g_1\xi_1 - \xi_2)(1-\tau) \mathbb{1} \left\{ Z_\theta < -(1-\tau) \frac{\rho_\theta}{M_\eta} \right\} - \frac{\tau^2 \xi_3}{2M_\eta} \mathbb{1} \left\{ Z_\theta > \tau \frac{\rho_\theta}{M_\eta} \right\} \right. \\ &\quad \left. - \frac{Z_\theta^2 M_\eta \xi_3}{2\rho_\theta^2} \mathbb{1} \left\{ -(1-\tau) \frac{\rho_\theta}{M_\eta} \leq Z_\theta \leq \tau \frac{\rho_\theta}{M_\eta} \right\} - \frac{(1-\tau)^2 \xi_3}{2M_\eta} \mathbb{1} \left\{ Z_\theta < -(1-\tau) \frac{\rho_\theta}{M_\eta} \right\} \right], \end{aligned}$$

and

$$\begin{aligned} w''(\theta) &= \mathbb{E} \left[\left((g_1\xi_1 - \xi_2)^2 \frac{M_\eta}{\rho_\theta} - 2(g_1\xi_1 - \xi_2)\xi_3 \frac{Z_\theta M_\eta}{\rho_\theta^2} + \xi_3^2 \frac{Z_\theta^2 M_\eta}{\rho_\theta^3} \right) \mathbb{1} \left\{ -(1-\tau) \frac{\rho_\theta}{M_\eta} \leq Z_\theta \leq \tau \frac{\rho_\theta}{M_\eta} \right\} \right. \\ &\quad \left. + \frac{M_\eta}{\rho_\theta} \mathbb{E} \left[\left(g_1\xi_1 - \xi_2 - \xi_3 \frac{Z_\theta}{\rho_\theta} \right)^2 \mathbb{1} \left\{ -(1-\tau) \frac{\rho_\theta}{M_\eta} \leq Z_\theta \leq \tau \frac{\rho_\theta}{M_\eta} \right\} \right] \right]. \end{aligned}$$

Recall that ϵ_1 has positive support on \mathbb{R} . Thus, Z_θ has positive support on \mathbb{R} and $g_1\xi_1 - \xi_2 - \xi_3 \frac{Z_\theta}{\rho_\theta}$ has positive support on \mathbb{R} if $\xi_3 \neq 0$. Moreover, if $\xi_3 = 0$, then $(\xi_1, \xi_2) \neq (0, 0)$ and we clearly have that $\mathbb{P}(g_1\xi_1 - \xi_2 = 0) = 0$. In either case, we conclude that $w''(\theta) > 0$ and thus that

$$(\beta_0, M_u, \rho_1) \mapsto \mathbb{E} \left[e_{\ell_\tau} \left(M_u g_1 + \epsilon_1 - \beta_0; \frac{\rho_1}{M_\eta} \right) \right],$$

is strictly convex. Since this term does not involve ρ_2 and the remainder of the objective is convex (it is the pointwise limit of a convex function), we conclude that the function

$$(\beta_0, M_u, \rho_1) \mapsto \max_{0 < \rho_2 \leq C_2} A(\beta_0, M_u, \rho_1, M_\eta, \rho_2),$$

is strictly convex. Finally, since A is convex-concave we have that for any $(\beta_0, M_u, \rho_1) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0}$, $M_\eta \mapsto \max_{0 < \rho_2 \leq C_2} A(\beta_0, M_u, \rho_1, M_\eta, \rho_2)$ is concave on $\mathbb{R}_{> 0}$ and thus continuous on $[c_\eta, C_\eta]$. The desired result then follows by Lemma C.2.

Now, consider the case $\rho_1 = 0$. Once again, fix $M_\eta \in [c_\eta, C_\eta]$ and a pair of distinct points $(M_u, \beta_0), (M'_u, \beta'_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. For $\theta \in [0, 1]$ consider the function

$$\tilde{w}(\theta) = \mathbb{E}[\ell_\tau(((1 - \theta)M_u + \theta M'_u)g_1 + \epsilon - (1 - \theta)\beta_0 - \theta\beta'_0)].$$

Let $Z_\theta := ((1 - \theta)M_u + \theta M'_u)g_1 + \epsilon - (1 - \theta)\beta_0 - \theta\beta'_0$. By a direct calculation,

$$\begin{aligned} \tilde{w}'(\theta) &= \mathbb{E}[\tau((M'_u - M_u)g_1 - (\beta'_0 - \beta_0))\mathbb{1}\{Z_\theta > 0\} - (1 - \tau)((M'_u - M_u)g_1 - (\beta'_0 - \beta_0))\mathbb{1}\{Z_\theta \leq 0\}] \\ &= \mathbb{E}[((M_u - M'_u)g_1 - (\beta_0 - \beta'_0))\mathbb{1}\{Z_\theta \leq 0\} + \tau((M'_u - M_u)g_1 - (\beta'_0 - \beta_0))], \end{aligned}$$

and

$$\begin{aligned} \tilde{w}''(\theta) &= \frac{d}{d\theta} \mathbb{E}[\mathbb{E}[(M_u - M'_u)g_1 - (\beta_0 - \beta'_0))\mathbb{1}\{\epsilon \leq (1 - \theta)(\beta_0 - M_u g_1) + \theta(\beta'_0 - M'_u g_1)\} \mid g_1]] \\ &= \mathbb{E}[(M_u - M'_u)g_1 - (\beta_0 - \beta'_0))^2 p_\epsilon((1 - \theta)(\beta_0 - M_u g_1) + \theta(\beta'_0 - M'_u g_1))] \\ &> 0, \end{aligned}$$

where p_ϵ denotes the density of ϵ . Since this last term is positive we find that \tilde{w} is strictly convex. The desired result then follows by arguing as above. \square

Lemma B.9. *Suppose the assumptions of Theorem 5.1 hold. Fix any constants $C_{\beta_0}, C_u, C_1, C_2, C_\eta, c_\eta > 0$ with $C_\eta > c_\eta$ and $c_\eta < (1/2) \min\{\tau^2, (1 - \tau)^2\}$. Then, the asymptotic optimization program (B.2) admits a unique solution for (β_0, M_u, ρ_1) . Moreover, letting $(\beta_0^*, M_u^*, \rho_1^*)$ denote this solution we have that $M_u^* > 0 \implies \rho_1^* > 0$.*

Proof. Since the optimization domain for (β_0, M_u, ρ_1) is compact and A is jointly continuous and convex-concave (Lemma B.7) the optimization program (B.2) must obtain its minimum in (β_0, M_u, ρ_1) (cf. Proposition 1.26 and Theorem 1.9 of Rockafellar & Wets (1997)). The fact that this minimizer is unique then follows directly from Lemma B.8.

Now, let $(\beta_0^*, M_u^*, \rho_1^*)$ denote this unique solution and suppose that $M_u^* > 0$. Recall the identity (Lemma C.5 below),

$$e_f(x; \rho) + e_{f^*}(x/\rho; 1/\rho) = \frac{x^2}{2\rho}.$$

Applying this to our optimization problem, we have that for any $\rho_2 > 0$ and $0 \leq \rho_1 \leq C_1$,

$$A(\beta_0^*, M_u^*, \rho_1, M_\eta, \rho_2) = \mathbb{E}\left[e_{\ell_\tau}\left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta}\right)\right] - \frac{M_\eta^2 M_u^* \gamma}{2\rho_2}$$

$$\begin{aligned}
& -\gamma \mathbb{E} \left[e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \\
& + \gamma \frac{\rho_2}{2M_u^*} \mathbb{E} \left[\left(\frac{M_\eta M_u^*}{\rho_2} h_1 + \sqrt{d} \tilde{\beta}_1 \right)^2 \right] \\
& + \frac{M_\eta \rho_1}{2} - \frac{M_u^* \rho_2}{2} \\
& = \mathbb{E} \left[e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right] \\
& - \gamma \mathbb{E} \left[e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \\
& + \gamma \frac{\rho_2}{2M_u^*} \mathbb{E}[(\sqrt{d} \tilde{\beta}_1)^2] + \frac{M_\eta \rho_1}{2} - \frac{M_u^* \rho_2}{2}.
\end{aligned}$$

So,

$$\begin{aligned}
& \max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} A(\beta_0^*, M_u^*, 0, M_\eta, \rho_2) \geq \max_{(0 < \rho_2 \leq C_2)} A(\beta_0^*, M_u^*, 0, c_\eta, \rho_2) \\
& = \mathbb{E}[\ell_\tau(M_u^* g_1 + \epsilon_1 - \beta_0^*)] - \gamma \mathbb{E} \left[e_{\nu^*} \left(c_\eta h_1 + \gamma \frac{\rho_2}{M_u} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \\
& \quad + \gamma \frac{\rho_2}{2M_u^*} \mathbb{E}[(\sqrt{d} \tilde{\beta}_1)^2] - \frac{M_u^* \rho_2}{2}.
\end{aligned} \tag{B.5}$$

We will now compare this lower bound against an equivalent upper bound when $\rho_1 > 0$ is small and positive. Fix $\rho_1, \rho_2 > 0$. By directly examining the definition of e_{ℓ_τ} , (Lemma C.3) we have the pointwise inequality

$$\begin{aligned}
& e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \\
& \leq \ell_\tau(M_u^* g_1 + \epsilon_1 - \beta_0^*) - \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{2M_\eta} \mathbb{1} \left\{ M_u^* g_1 + \epsilon_1 - \beta_0^* \notin \left[-\frac{\rho_1}{M_\eta} (1-\tau), \frac{\rho_1}{M_\eta} \tau \right] \right\},
\end{aligned}$$

and thus for any $M_\eta \in [c_\eta, C_\eta]$,

$$\begin{aligned}
& \mathbb{E} \left[e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right] \\
& \leq \mathbb{E} \left[\ell_\tau \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right] \\
& \quad - \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{2M_\eta} \left(1 - \mathbb{P} \left(M_u^* g_1 + \epsilon_1 - \beta_0^* \in \left[-\frac{\rho_1}{M_\eta} (1-\tau), \frac{\rho_1}{M_\eta} \tau \right] \right) \right) \\
& \leq \mathbb{E} \left[\ell_\tau \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right]
\end{aligned}$$

$$- \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{2M_\eta} \left(1 - \mathbb{P} \left(M_u^* g_1 + \epsilon_1 - \beta_0^* \in \left[-\frac{\rho_1}{c_\eta} (1-\tau), \frac{\rho_1}{c_\eta} \tau \right] \right) \right).$$

Now, let ρ_1 be sufficiently small such that $\mathbb{P} \left(M_u^* g_1 + \epsilon_1 - \beta_0^* \in \left[-\frac{\rho_1}{c_\eta} (1-\tau), \frac{\rho_1}{c_\eta} \tau \right] \right) \leq 1/2$. Then, the above implies that

$$\mathbb{E} \left[e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right] \leq \mathbb{E} \left[\ell_\tau \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta} \right) \right] - \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{4M_\eta}.$$

On the other hand, by part five of Assumption 4 we also have that

$$\begin{aligned} \frac{d}{dM_\eta} \mathbb{E} \left[e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] &= \mathbb{E} \left[h_1 \partial_x e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \\ &= M_\eta \mathbb{E} \left[\partial_x^2 e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right]. \end{aligned}$$

Let $c = \min_{c_\eta \leq M_\eta \leq C_\eta, 0 < \rho_2 \leq C_2} M_\eta \mathbb{E} \left[\partial_x^2 e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right]$ and note that by part 6 of Assumption 4 we have $c > 0$. Then, the mean value theorem gives

$$\mathbb{E} \left[e_{\nu^*} \left(M_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \geq \mathbb{E} \left[e_{\nu^*} \left(c_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] + c(M_\eta - c_\eta).$$

Putting this all together, we find that for ρ_1 sufficiently close to 0,

$$\begin{aligned} &\max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} A(\beta_0^*, M_u^*, \rho_1, M_\eta, \rho_2) \\ &\leq \max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} \mathbb{E} [\ell_\tau (M_u^* g_1 + \epsilon_1 - \beta_0^*)] - \gamma \mathbb{E} \left[e_{\nu^*} \left(c_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] \\ &\quad + \gamma \frac{\rho_2}{2M_u^*} \mathbb{E}[(\sqrt{d} \tilde{\beta}_1)^2] - \frac{M_u^* \rho_2}{2} + \frac{M_\eta \rho_1}{2} - \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{4M_\eta} - c(M_\eta - c_\eta). \end{aligned}$$

We claim that for ρ_1 sufficiently small the term

$$w(M_\eta, \rho_1) = \frac{M_\eta \rho_1}{2} - \min\{\tau^2, (1-\tau)^2\} \frac{\rho_1}{4M_\eta} - c(M_\eta - c_\eta),$$

is always negative. Indeed, by our choice of c_η we have that $\frac{c_\eta}{2} - \min\{\tau^2, (1-\tau)^2\} \frac{1}{4c_\eta} < 0$. So, we may find $\delta > 0$ such that for all $c_\eta \leq M_\eta \leq c_\eta + \delta$, $\frac{M_\eta}{2} - \min\{\tau^2, (1-\tau)^2\} \frac{1}{4M_\eta} < 0$, and thus also $w(M_\eta, \rho_1) < 0$ for all $\rho_1 > 0$. On the other hand, for $c_\eta + \delta \leq M_\eta \leq C_\eta$ we have

$$w(M_\eta, \rho_1) \leq \rho_1 \left(\frac{C_\eta}{2} - \min\{\tau^2, (1-\tau)^2\} \frac{1}{4C_\eta} \right) - c\delta,$$

which is negative for ρ_1 sufficiently small. This proves the desired claim and thus shows that for ρ_1 sufficiently small

$$\max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} A(\beta_0^*, M_u^*, \rho_1, M_\eta, \rho_2) < \max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} \mathbb{E} [\ell_\tau (M_u^* g_1 + \epsilon_1 - \beta_0^*)]$$

$$- \gamma \mathbb{E} \left[e_{\nu^*} \left(c_\eta h_1 + \gamma \frac{\rho_2}{M_u^*} \sqrt{d} \tilde{\beta}_1; \frac{\rho_2}{M_u^*} \right) \right] + \gamma \frac{\rho_2}{2M_u^*} \mathbb{E}[(\sqrt{d} \tilde{\beta}_1)^2] - \frac{M_u^* \rho_2}{2}.$$

Comparing the above to our bounds in (B.5) for the case $\rho_1 = 0$ we find that

$$\max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} A(\beta_0^*, M_u^*, \rho_1, M_\eta, \rho_2) < \max_{(0 < \rho_2 \leq C_2, c_\eta \leq M_\eta \leq C_\eta)} A(\beta_0^*, M_u^*, 0, M_\eta, \rho_2).$$

Thus, $\rho_1^* \neq 0$, as desired. □

Lemma B.10. *Suppose the assumptions of Theorem 5.1 hold. Fix any constants $C_{\beta_0}, C_u, C_1, C_2, C_\eta, c_\eta > 0$ with $C_\eta > c_\eta$ and $c_\eta < (1/2) \min\{\tau^2, (1 - \tau)^2\}$. Let $(\beta_0^*, M_u^*, \rho_1^*)$ denote the unique solution to the asymptotic program (B.2) defined in Lemma B.9 and suppose that $M_u^* > 0$. Then, the asymptotic program (B.2) obtains a unique solution for M_η .*

Proof. Since A is jointly convex-concave (Lemma B.7) we know that the function

$$M_\eta \mapsto \max_{(0 < \rho_2 \leq C_2)} \min_{(|\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u, 0 \leq \rho_1 \leq C_1)} A(\beta_0, M_u, \rho_1, M_\eta, \rho_2), \quad (\text{B.6})$$

is concave on $\mathbb{R}_{>0}$ and thus continuous on $[c_\eta, C_\eta]$. Thus, this function obtains its maximum.

It remains to show that the maximizer is unique. For ease of notation let $Z = M_u^* g_1 + \epsilon_1 - \beta_0^*$ and define the function

$$w(M_\eta) = \mathbb{E} \left[e_{\ell_\tau} \left(Z; \frac{\rho_1^*}{M_\eta} \right) \right].$$

Recall that by Lemma B.9 we must have that $\rho_1^* > 0$. We claim that w is strongly concave on $[c_\eta, C_\eta]$. To see this, note that by a direct calculation using the form of e_{ℓ_τ} (see Lemma C.3), we have

$$\begin{aligned} w'(M_\eta) = \mathbb{E} & \left[\frac{\tau^2 \rho_1^*}{2M_\eta^2} \mathbb{1} \left\{ Z > \tau \frac{\rho_1^*}{M_\eta} \right\} + \frac{Z^2}{2\rho_1^*} \mathbb{1} \left\{ -(1 - \tau) \frac{\rho_1^*}{M_\eta} \leq Z \leq \tau \frac{\rho_1^*}{M_\eta} \right\} \right. \\ & \left. + \frac{(1 - \tau)^2 \rho_1^*}{2M_\eta^2} \mathbb{1} \left\{ Z < -(1 - \tau) \frac{\rho_1^*}{M_\eta} \right\} \right], \end{aligned}$$

and

$$w''(M_\eta) = \mathbb{E} \left[-\frac{\tau^2 \rho_1^*}{M_\eta^3} \mathbb{1} \left\{ Z > \tau \frac{\rho_1^*}{M_\eta} \right\} - \frac{(1 - \tau)^2 \rho_1^*}{M_\eta^3} \mathbb{1} \left\{ Z < -(1 - \tau) \frac{\rho_1^*}{M_\eta} \right\} \right].$$

So, in particular,

$$\sup_{c_\eta \leq M_\eta \leq C_\eta} w''(M_\eta) \leq \mathbb{E} \left[-\frac{\tau^2 \rho_1^*}{C_\eta^3} \mathbb{1} \left\{ Z > \tau \frac{\rho_1^*}{c_\eta} \right\} - \frac{(1 - \tau)^2 \rho_1^*}{C_\eta^3} \mathbb{1} \left\{ Z < -(1 - \tau) \frac{\rho_1^*}{c_\eta} \right\} \right] < 0,$$

where the get the last inequality we have applied the fact that ϵ_1 has support on all of \mathbb{R} (and thus that Z has support on all of \mathbb{R}).

Now, assume by contradiction that there exist distinct maximizers M_η^1 and M_η^2 for (B.6) on the domain $[c_\eta, C_\eta]$. Since this is a convex-concave problem we must that that M_η^1 and M_η^2 are maximizers of the function

$$M_\eta \mapsto \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta, \rho_2),$$

on $[c_\eta, C_\eta]$. Additionally, we must have $\max_{c_\eta \leq M_\eta^1 \leq C_\eta} \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta^k, \rho_2) < \infty$ for $k \in \{1, 2\}$. This follows immediately from the fact that

$$\begin{aligned} \max_{c_\eta \leq M_\eta \leq C_\eta} \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta, \rho_2) &\leq \max_{c_\eta \leq M_\eta \leq C_\eta} A(0, 0, 0, M_\eta, \rho_2) \\ &= \mathbb{E}[\ell_\tau(\epsilon_1)] + \gamma \mathbb{E}[\nu(\gamma \sqrt{d} \tilde{\beta}_1)] < \infty. \end{aligned}$$

Fix $\delta > 0$ small and let $\rho_2^1, \rho_2^2 \in (0, C_2]$ be any two values such that

$$\min_{k \in \{1, 2\}} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta^k, \rho_2^k) \geq \max_{c_\eta \leq M_\eta \leq C_\eta} \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta, \rho_2) - \delta.$$

By the strong concavity of $w(\eta)$ and the joint concavity of the remainder of the terms in A in (M_η, ρ_2) we have

$$\begin{aligned} \max_{c_\eta \leq M_\eta \leq C_\eta} \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta, \rho_2) &\geq A\left(\beta_0^*, M_u^*, \rho_1^*, \frac{1}{2}M_\eta^1 + \frac{1}{2}M_\eta^2, \frac{1}{2}\rho_2^1 + \frac{1}{2}\rho_2^2\right) \\ &\geq \frac{1}{2}A(\beta_0^*, M_u^*, \rho_1^*, M_\eta^1, \rho_2^1) + \frac{1}{2}A(\beta_0^*, M_u^*, \rho_1^*, M_\eta^2, \rho_2^2) + \frac{\inf_{c_\eta \leq M_\eta \leq C_\eta} |w''(M_\eta)|}{8} (M_\eta^1 - M_\eta^2)^2 \\ &= \max_{c_\eta \leq M_\eta \leq C_\eta} \max_{0 < \rho_2 \leq C_2} A(\beta_0^*, M_u^*, \rho_1^*, M_\eta, \rho_2) - \delta + \frac{\inf_{c_\eta \leq M_\eta \leq C_\eta} |w''(M_\eta)|}{8} (M_\eta^1 - M_\eta^2)^2, \end{aligned}$$

as so rearranging,

$$(M_\eta^1 - M_\eta^2)^2 \leq \frac{8}{\inf_{c_\eta \leq M_\eta \leq C_\eta} |w''(M_\eta)|}.$$

Sending $\delta \rightarrow 0$ gives the desired result. \square

Our last result of this section gives a first-order condition for ρ_1^* .

Lemma B.11. *Suppose the conditions of Theorem 5.1 hold. Fix any $C_{\beta_0}, C_u, C_\eta, c_\eta, C_1, C_2 > 0$ with $C_\eta > c_\eta$, $c_\eta < (1/2) \min\{\tau^2, (1 - \tau)^2\}$, and $C_1 > \sqrt{C_u^2 + \mathbb{E}[\epsilon_1^2]} + C_{\beta_0}^2$. Let $(M_u^*, \rho_1^*, \beta_0^*)$ denote the unique optimal solutions to the asymptotic program (B.2) defined in Lemmas B.9 and assume that $M_u^* > 0$. Let M_η^* denote the unique optimal solution in M_η of (B.2) defined in Lemma B.10. Then, ρ_1^* satisfies the first-order condition*

$$\rho_1^* = \sqrt{\mathbb{E} \left[\left(M_u^* g + \epsilon_1 - \beta_0^* - \text{prox}_{\ell_\tau} \left(M_u^* g + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta^*} \right) \right)^2 \right]}.$$

Proof. Under the given assumptions we must have that ρ_1^* minimizes the function

$$w(\rho_1) = \mathbb{E} \left[e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1}{M_\eta^*} \right) \right] + \frac{M_\eta^* \rho_1}{2},$$

on the interval $[0, C_1]$. For ease of notation, let $Z = M_u^* g_1 + \epsilon_1 - \beta_0^*$. By, Lemma 15(iii) of [Thrampoulidis et al. \(2018\)](#) we have that $\frac{d}{d\rho} e_{\ell_\tau}(x; \rho) = -\frac{1}{2\rho^2} (x - \text{prox}_{\ell_\tau}(x; \rho))^2$ (see also the calculations in Lemma C.3 below). Applying this fact alongside the dominated convergence theorem gives

$$w'(\rho_1) = -\frac{M_\eta^*}{2\rho_1^2} \mathbb{E} \left[\left(Z - \text{prox}_{\ell_\tau} \left(Z; \frac{\rho_1}{M_\eta^*} \right) \right)^2 \right] + \frac{M_\eta^*}{2}.$$

So,

$$w'(\rho_1) = 0 \iff \rho_1 = \sqrt{\mathbb{E} \left[\left(Z - \text{prox}_{\ell_\tau} \left(Z; \frac{\rho_1}{M_\eta^*} \right) \right)^2 \right]}.$$

Finally, recall that the function $h(z) = z - \text{prox}_{\ell_\tau}(z; \rho)$ is 1-Lipschitz (cf. Proposition 12.28 of [Bauschke & Combettes \(2017\)](#)) with $h(z) = 0$. Thus, the right-hand-side above is at most

$$\sqrt{\mathbb{E} \left[\left(Z - \text{prox}_{\ell_\tau} \left(Z; \frac{\rho_1}{M_\eta^*} \right) \right)^2 \right]} \leq \sqrt{\mathbb{E} [(Z)^2]} \leq \sqrt{C_u^2 + \mathbb{E}[\epsilon_1^2] + C_{\beta_0}^2}.$$

By assumption this last quantity is strictly below C_1 and thus ρ_1^* must satisfy the given first-order condition. □

B.4 Final steps

In this section we prove Theorem 5.1. We begin by stating a convergence result for the primal variables.

Theorem B.1. *Let $C_u, C_{\beta_0}, C_\eta, c_\eta, C_1, C_2$ be constants satisfying Lemmas B.2, B.3, B.4, and B.5. Let M_u^* and β_0^* denote the unique solutions for M_u and β_0 in the asymptotic program (B.2) defined in Lemma B.9. Then, under the assumptions of Theorem 5.1, it holds that for all $\delta > 0$,*

$$\mathbb{P} \left(\text{For all primal solutions to (B.1), } \|\hat{\beta} - \tilde{\beta}\|_2 - M_u^* < \delta \text{ and } |\hat{\beta}_0 - \beta_0^*| < \delta \right) \rightarrow 1.$$

Proof. The proof of this result follows similar steps to the proof of Theorem 5.1 and, in particular, is very similar to the proof of Proposition B.5 below. Namely, following similar arguments to those presented in Section B.2 for the dual variables, one can show that to prove this result it is sufficient to bound the value of the program

$$\phi^{\text{primal}}(S) := \max_{(\|s\|_2 \leq C_s \sqrt{n}, c_\eta \leq M_\eta) \cap (\|r\|_2 \leq C_r \sqrt{n}, (\beta_0, u) \in S)} \min_{(\eta: \|\eta\|_2 = M_\eta)} \left(\frac{1}{n} \|u\|_2 \eta^\top g + \frac{1}{n} \|\eta\|_2 u^\top h \right)$$

$$\frac{1}{n}\eta^\top \epsilon - \frac{1}{n}\beta_0\eta^\top \mathbf{1}_n - \frac{1}{n}\eta^\top r + \frac{1}{n}\sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}}s^\top(\tilde{\beta} + u) - \frac{1}{\sqrt{n}}\tilde{\mathcal{R}}_d^*(s)\Big),$$

for various choices of S . Arguing as above, the values of this program are completely characterized by values of the asymptotic program (B.2). Convergence of $\|\hat{\beta} - \tilde{\beta}\|_2$ and $\hat{\beta}_0$ then follows from similar arguments to those presented in Proposition B.5 where we show a convergence result for $\|\hat{\eta}\|_2$. Since the details of this proof closely mirror our other arguments, they are omitted. \square

We now turn to the proof of Theorem 5.1. Our first result considers the case $M_u^* = 0$.

Proposition B.4. *Suppose the conditions of Theorem 5.1 hold. Let (M_u^*, β_0^*) be defined as in Theorem B.1 and suppose that $M_u^* = 0$. Let $p := \mathbb{P}(\epsilon_1 - \beta_0^* < 0)$. Then, for all $\xi > 0$, with probability tending to one, all dual solutions $\hat{\eta}$ to B.1 satisfy*

$$\left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i = -(1-\tau)\} - p \right|, \left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i = \tau\} - (1-p) \right|, \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\} < \xi.$$

In particular, the result of Theorem 5.1 goes through for $P_\eta = p\delta_{-(1-\tau)} + (1-p)\delta_\tau$.

Proof. We will focus on the bound on $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\}$. The bounds on the other two terms are similar. By the first-order conditions of the optimization in r , we have that for any joint primal-dual solution $(\hat{\beta}_0, \hat{\beta}, \hat{\eta})$,

$$\hat{\eta}_i \in \begin{cases} \{\tau\}, & Y_i > \hat{\beta}_0 + X_i^\top \hat{\beta}, \\ [(1-\tau), \tau], & Y_i = \hat{\beta}_0 + X_i^\top \hat{\beta}, \\ \{-(1-\tau)\}, & Y_i < \hat{\beta}_0 + X_i^\top \hat{\beta}, \end{cases} = \begin{cases} \{\tau\}, & \epsilon_i > \hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta}), \\ -[\tau, 1-\tau], & \epsilon_i = \hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta}), \\ \{-(1-\tau)\}, & \epsilon_i < \hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta}). \end{cases}$$

Now, by standard results (e.g. Theorem 3.1 of Yin et al. (1988)) we have that $\sigma_{\max}(X)/\sqrt{n}$ is converging in probability to a constant $c > 0$. In particular, this implies that with probability converging to one, $\|X(\hat{\beta} - \tilde{\beta})\|_1 \leq \sqrt{n}\|X(\hat{\beta} - \tilde{\beta})\|_2 \leq n2c\|\hat{\beta} - \tilde{\beta}\|_2$. So, for any $\rho > 0$ we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\} &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\epsilon_i - \hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta})| \leq \rho\} \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\epsilon_i - \hat{\beta}_0| \leq 2\rho\} + \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta})| > \rho\} \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\epsilon_i - \hat{\beta}_0| \leq 2\rho\} + \frac{1}{n\rho} \|\hat{\beta}_0 + X_i^\top (\hat{\beta} - \tilde{\beta})\|_1 \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\epsilon_i - \beta_0^*| \leq 3\rho\} + \mathbb{1}\{|\beta_0^* - \hat{\beta}_0| > \rho\} + \frac{2c}{\rho} \|\hat{\beta} - \tilde{\beta}\|_2. \end{aligned}$$

So,

$$\sup_{\hat{\eta}} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\} \leq \sup_{\hat{\beta}_0, \hat{\beta}} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{|\epsilon_i - \beta_0^*| \leq 3\rho\} + \mathbb{1}\{|\beta_0^* - \hat{\beta}_0| > \rho\}$$

$$+ \frac{2c}{\rho} \|\hat{\beta} - \tilde{\beta}\|_2,$$

where the suprema are over all dual solutions for η and all primal solutions for (β_0, β) , respectively. Applying the law of large numbers and the results of Lemma B.3 we find that

$$\sup_{\hat{\eta}} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\} \leq \mathbb{P}(|\epsilon_1 - \beta_0^*| \leq 3\rho) + o_{\mathbb{P}}(1).$$

Since ϵ_1 has a continuous distribution, the desired result follows by sending $\rho \rightarrow 0$. \square

We now turn to the main proof of Theorem 5.1, which focuses on the more difficult case in which $M_u^* > 0$. To begin, we first show that $\|\hat{\eta}_2\|_2$ converges.

Proposition B.5. *Assume the conditions of Theorem 5.1 hold. Let $C_u, C_{\beta_0}, C_{\eta}, c_{\eta}, C_1, C_2$ be constants satisfying Lemmas B.2, B.3, B.4, and B.5 as well as the conditions of Lemma B.10. Let M_u^* denote the unique solutions for M_u (B.2) defined in Lemma B.9 and assume that $M_u^* > 0$. Let M_{η}^* denote the unique solutions M_{η} in the asymptotic program (B.2) defined in Lemma B.10. Then, for all $\delta > 0$,*

$$\mathbb{P}\left(\text{For all dual solutions of (B.1), } |\|\hat{\eta}\|_2 - \sqrt{n}M_{\eta}^*| < \delta\right) \rightarrow 1.$$

Proof. Let V denote the value of the asymptotic optimization program (B.2). By Propositions B.2 and B.3, it is sufficient to show that there exists $\xi > 0$ such that with probability converging to one,

$$\phi(\{\eta : \sqrt{n}c_{\eta} \leq \|\hat{\eta}\|_2 \leq \sqrt{n}C_{\eta}, |\|\hat{\eta}\|_2 - \sqrt{n}M_{\eta}^*| \geq \delta\}) < V - \xi.$$

For ease of notation, let $S_{M,\delta} = \{M \in [c_{\eta}, C_{\eta}] : M \geq M_{\eta}^* + \delta \text{ or } M \leq M_{\eta}^* - \delta\}$. By a direct calculation following the arguments of Section B.2, we have that

$$\begin{aligned} & \phi(\{\eta : \sqrt{n}c_{\eta} \leq \|\hat{\eta}\|_2 \leq \sqrt{n}C_{\eta}, |\|\hat{\eta}\|_2 - \sqrt{n}M_{\eta}^*| \geq \delta\}) \\ & \xrightarrow{\mathbb{P}} \max_{(M_{\eta} \in S_{M,\delta}, 0 < \rho_2 \leq C_2)} \min_{(|\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u, 0 \leq \rho_1 \leq C_1)} A(\beta_0, M_u, \rho_1, M_{\eta}, \rho_2). \end{aligned}$$

By Lemma B.10, the right hand side is strictly less than V , as desired. \square

We now prove Theorem 5.1.

Proof of Theorem 5.1. Let $C_u, C_{\beta_0}, C_{\eta}, c_{\eta}, C_1, C_2$ be constants satisfying Lemmas B.2, B.3, B.4, B.5 as well as the conditions of Lemmas B.9 and B.11. Let $(M_u^*, \beta_0^*, M_{\eta}^*, \rho_1^*)$ denote the unique solutions to the associated asymptotic program (B.2) defined in Lemmas B.9 and B.10. If $M_u^* = 0$, the result follows from Proposition B.4. So, suppose $M_u^* > 0$. Recall that by Lemma B.9 we must have that $\rho_1^* > 0$ and let P_{η} denote the distribution of

$$\frac{M_{\eta}^*(M_u^*g_1 + \epsilon_1 - \beta_0^* - \text{prox}_{\ell_{\tau}}(M_u^*g_1 + \epsilon_1 - \beta_0^*; \rho_1^*/M_{\eta}^*))}{\rho_1^*}. \quad (\text{B.7})$$

Fix any bounded L -lipschitz function ψ . Let V denote the value of the asymptotic optimization program (B.2) and fix any $\kappa > 0$ small. Let $S_{\kappa,\delta}$ denote the set

$$\left\{ \eta : \sqrt{n} \max\{c_\eta, M_\eta^* - \kappa\} \leq \|\hat{\eta}\|_2 \leq \sqrt{n} \min\{C_\eta, M_\eta^* + \kappa\}, \left| \frac{1}{n} \sum_{i=1}^n \psi(\eta_i) - \mathbb{E}_{Z \sim P_\eta}[\psi(Z)] \right| \geq \delta \right\}.$$

By Propositions B.2, B.3, and B.5 it is sufficient to show that there exists $\xi > 0$ such that with probability converging to one,

$$\phi(S_{\kappa,\delta}) < V - \xi.$$

First, note since ψ is Lipschitz we may assume that κ is sufficiently small so that $S_{\kappa,\delta} \subseteq S_{0,\delta/2}$. So,

$$\begin{aligned} \phi(S_{\kappa,\delta}) &\leq \phi(S_{0,\delta/2}) \\ &= \min_{(\|r\|_2 \leq C_r \sqrt{n}, |\beta_0| \leq C_{\beta_0}, 0 \leq M_u \leq C_u)} \max_{(\|s\|_2 \leq C_s \sqrt{n}, \eta \in S_{0,\delta/2})} \left(\frac{1}{n} M_u \eta^\top g - M_u \left\| \frac{1}{n} \|\eta\|_2 h + \frac{1}{n} s \right\|_2 \right. \\ &\quad \left. + \frac{1}{n} \eta^\top \epsilon - \frac{1}{n} \beta_0 \eta^\top \mathbf{1}_n - \frac{1}{n} \eta^\top r + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right) \\ &\leq \min_{(\|r\|_2 \leq C_r \sqrt{n})} \max_{(\|s\|_2 \leq C_s \sqrt{n}, \eta \in S_{0,\delta/2})} \left(\frac{1}{n} M_u^* \eta^\top g - M_u^* \left\| \frac{1}{n} M_\eta^* h + \frac{1}{n} s \right\|_2 + \frac{1}{n} \eta^\top \epsilon \right. \\ &\quad \left. - \frac{1}{n} \beta_0^* \eta^\top \mathbf{1}_n - \frac{1}{n} \eta^\top r + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \right) \\ &= \min_{(\|r\|_2 \leq C_r \sqrt{n})} \max_{(\eta \in S_{0,\delta/2})} \left(\frac{1}{n} \eta^\top (M_u^* g + \epsilon - \beta_0^* \mathbf{1}_n - r) + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) \right) \\ &\quad + \max_{\|s\|_2 \leq C_s \sqrt{n}} -M_u^* \left\| \frac{1}{n} M_\eta^* h + \frac{1}{n} s \right\|_2 + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s). \end{aligned} \tag{B.8}$$

Arguing as in Section B.2 (and in particular applying Lemmas B.5 and B.6 along with the law of large numbers), the second term converges as

$$\begin{aligned} &\max_{\|s\|_2 \leq C_s \sqrt{n}} -M_u^* \left\| \frac{1}{n} M_\eta^* h + \frac{1}{n} s \right\|_2 + \frac{1}{\sqrt{n}} s^\top \tilde{\beta} - \frac{1}{\sqrt{n}} \tilde{\mathcal{R}}_d^*(s) \\ &\xrightarrow{\mathbb{P}} \max_{0 < \rho_2 \leq C_2} -\frac{(M_\eta^*)^2 M_u^* \gamma}{2} + \gamma \mathbb{E} \left[e_\nu \left(\frac{M_\eta^* M_u^*}{\rho_2} h + \gamma \sqrt{d} \tilde{\beta}; \frac{M_u^*}{\rho_2} \right) \right] - \frac{M_u^* \rho_2}{2}. \end{aligned} \tag{B.9}$$

It remains to consider the first term. Let $r^* \in \mathbb{R}^n$ be the vector given by $r_i^* = \text{prox}_{\ell_\tau}(M_u^* g_i + \epsilon_i - \beta_0^*; \rho_1^*/M_\eta^*)$. By the law of large numbers, we have

$$\frac{1}{n} \sum_{i=1}^n \psi \left(\frac{M_\eta^* (M_u^* g_i + \epsilon_i - \beta_0^* - r_i^*)}{\rho_1^*} \right) \xrightarrow{\mathbb{P}} \mathbb{E}_{Z \sim P_\eta}[Z],$$

and that

$$\frac{\|M_u^* g_i + \epsilon_i - \beta_0^* - r_i^*\|_2}{\sqrt{n}} \xrightarrow{\mathbb{P}} \rho_1^*,$$

where we recall that by Lemma B.11, $\rho_1^* = \sqrt{\mathbb{E}_{Z \sim P_\eta}[Z^2]}$. Since ψ is Lipschitz, this implies that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \min_{\eta \in S_{0, \delta/2}} \frac{1}{\sqrt{n}} \left\| \eta - \frac{M_\eta^* (M_u^* g_i + \epsilon_i - \beta_0^* - r_i^*)}{\rho_1^*} \right\|_2 \\ & \geq \liminf_{n \rightarrow \infty} \min_{\eta \in S_{0, \delta/2}} \frac{1}{L} \left| \frac{1}{n} \sum_{i=1}^n \psi \left(\frac{M_\eta^* (M_u^* g_i + \epsilon_i - \beta_0^* - r_i^*)}{\rho_1^*} \right) - \frac{1}{n} \sum_{i=1}^n \psi(\eta_i) \right| \geq \frac{\delta}{2L}. \end{aligned}$$

For ease of notation, let $Z^* = M_u^* g + \epsilon - \beta_0^* \mathbf{1}_n - r^*$. Applying these calculations, we find that the optimization appearing on line (B.8) can be bounded as,

$$\begin{aligned} & \min_{(\|r\|_2 \leq C_r \sqrt{n})} \max_{(\eta \in S_{0, \delta/2})} \left(\frac{1}{n} \eta^\top (M_u^* g + \epsilon - \beta_0^* \mathbf{1}_n - r) + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i) \right) \\ & \leq \max_{(\eta \in S_{0, \delta/2})} \left(\frac{1}{n} \eta^\top Z^* + \frac{1}{n} \sum_{i=1}^n \ell_\tau(r_i^*) \right) \\ & = \max_{(\eta \in S_{0, \delta/2})} \left(\frac{1}{n} \frac{\eta^\top Z^*}{M_\eta^* \|Z^*\|_2} M_\eta^* \|Z^*\|_2 \right) + \mathbb{E} \left[\ell_\tau \left(\text{prox}_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1^*}{M_\eta^*} \right) \right) \right] + o_{\mathbb{P}}(1) \\ & = \max_{(\eta \in S_{0, \delta/2})} \left(1 - \frac{1}{2} \left\| \frac{1}{\sqrt{n}} \frac{\eta}{M_\eta^*} - \frac{Z^*}{\|Z^*\|_2} \right\|_2^2 \right) \frac{M_\eta^* \|Z^*\|_2}{\sqrt{n}} \\ & \quad + \mathbb{E} \left[\ell_\tau \left(\text{prox}_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1^*}{M_\eta^*} \right) \right) \right] + o_{\mathbb{P}}(1) \\ & \leq \left(1 - \frac{\delta^2}{8L^2(M_\eta^*)^2} \right) M_\eta^* \rho_1^* + \mathbb{E} \left[\ell_\tau \left(\text{prox}_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1^*}{M_\eta^*} \right) \right) \right] + o_{\mathbb{P}}(1) \\ & = \frac{M_\eta^* \rho_1^*}{2} + \mathbb{E} \left[e_{\ell_\tau} \left(M_u^* g_1 + \epsilon_1 - \beta_0^*; \frac{\rho_1^*}{M_\eta^*} \right) \right] - \frac{\delta^2}{8L^2(M_\eta^*)^2} + o_{\mathbb{P}}(1), \end{aligned}$$

where the last line applies the formula for ρ_1^* given in Lemma B.11 alongside the definition of the Moreau envelope. Combining this with (B.9) we conclude that

$$\phi(S_{\kappa, \delta}) \leq V - \frac{\delta^2}{8L^2(M_\eta^*)^2} + o_{\mathbb{P}}(1).$$

As discussed above, this proves the desired result. □

B.5 Corollaries of Theorem 5.1

We now prove Corollaries 5.1 and 5.2.

Proof of 5.1. Let $C_u, C_{\beta_0}, C_\eta, c_\eta, C_1, C_2$ be constants satisfying Lemmas B.2, B.3, B.4, B.5 as well as the conditions of Lemmas B.9 and B.11 and B.11. Let $(M_u^*, \beta_0^*, M_\eta^*, \rho_1^*)$ denote the unique solutions to the associated asymptotic program (B.2) defined in Lemmas B.9 and B.10.

To begin, we will first show that the unregularized quantile regression program must have $M_u^* > 0$. Let $(\hat{\beta}_0, \hat{\beta}, \hat{r}, \hat{\eta})$ denote any primal-dual solutions to the quantile regression (B.1). Recall that the first-order conditions of this optimization in r imply that

$$\hat{\eta}_i \in \begin{cases} \tau, & Y_i > \hat{\beta}_0 + X_i^\top \hat{\beta} \\ [-(1-\tau), \tau], & Y_i = \hat{\beta}_0 + X_i^\top \hat{\beta}, \\ -(1-\tau), & Y_i < \hat{\beta}_0 + X_i^\top \hat{\beta}. \end{cases}$$

By Proposition B.4, if $M_u^* = 0$ we must have that with probability converging to one,

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \in (-(1-\tau), \tau)\} < d.$$

We will show that this is not possible.

Introduce the notation X_A to denote the submatrix of X consisting of the rows in $A \subseteq \{1, \dots, n\}$ and $X_{A,B}$ to denote the submatrix with rows in $A \subseteq \{1, \dots, n\}$ and columns in $B \subseteq \{1, \dots, d\}$. Let $\hat{\eta}_A$ denote the subvector of $\hat{\eta}$ with entries in A . Let $S_{\text{inter.}} = \{i \in \{1, \dots, n\} : -(1-\tau) < \hat{\eta}_i < \tau\}$ denote the set of entries of $\hat{\eta}$ which lie in the interior. By the first-order conditions of (B.2) in β , we have that

$$\hat{\eta}^\top X = 0 \iff \hat{\eta}_{S_{\text{inter.}}}^\top X_{S_{\text{inter.}}^c} = \hat{\eta}_{S_{\text{inter.}}}^\top X_{S_{\text{inter.}}}. \quad (\text{B.10})$$

On the other hand, for any fixed set $\tilde{S} \subseteq \{1, \dots, n\}$ with $|\tilde{S}| < d-1$ and vector $v \in \{-(1-\tau), \tau\}^{n-|\tilde{S}|}$ we have that with probability one $v^\top X_{\tilde{S}^c}$ is in the kernel of the row space of $X_{\tilde{S}}$. This follows immediately from the fact that for any $u \in \mathbb{R}^{|\tilde{S}|}$,

$$u^\top X_{\tilde{S}} = v^\top X_{\tilde{S}^c} \implies v^\top X_{\tilde{S}^c, \{1, \dots, |\tilde{S}|\}} (X_{\tilde{S}, \{1, \dots, |\tilde{S}|\}}^\top)^{-1} X_{\tilde{S}, \{|\tilde{S}|+1, \dots, d\}} = v^\top X_{\tilde{S}^c, \{|\tilde{S}|+1, \dots, d\}},$$

which occurs with probability zero since $v^\top X_{\tilde{S}^c, \{|\tilde{S}|+1, \dots, d\}}$ is a continuous random vector independent of $v^\top X_{\tilde{S}^c, \{1, \dots, |\tilde{S}|\}} (X_{\tilde{S}, \{1, \dots, |\tilde{S}|\}}^\top)^{-1} X_{\tilde{S}, \{|\tilde{S}|+1, \dots, d\}}$. So, by (B.10) we must have that with probability one, $|S_{\text{inter.}}| > d-1$. As discussed above, this implies that $M_u^* > 0$, as claimed.

With the above claim in hand we are now ready to prove the main result of Corollary (5.1). Fix any $\delta > 0$. Let q^* denote the τ quantile of the asymptotic distribution P_η defined in (B.7). We will show that with probability converging to one the empirical quantile of $\hat{\eta}$ lies below $q^* + 2\delta$. Proof of a matching lower bound is identical. If $q^* = \tau$ then the result is immediate. So, suppose that $q^* < \tau$. Let ψ_δ be the step function

$$\psi_\delta(x) = \begin{cases} 0, & x > q^* + 2\delta, \\ \frac{q^* + 2\delta - x}{\delta}, & q^* + \delta \leq x \leq q^* + 2\delta \\ 1, & x < q^* + \delta. \end{cases}$$

Fix a small value $\xi > 0$ to be specified shortly. By Theorem 5.1, we have that with probability converging to one all dual solutions satisfy

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq q^* + 2\delta\} &\geq \frac{1}{n} \sum_{i=1}^n \psi_\delta(\hat{\eta}_i) \geq \mathbb{E}_{Z \sim P_\eta}[\psi_\delta(Z)] - \xi \\ &\geq \mathbb{P}_{Z \sim P_\eta}(Z \leq q^*) + \mathbb{P}_{Z \sim P_\eta}(q^* < Z \leq q^* + \delta) - \xi. \end{aligned}$$

Since $M_u^* > 0$ we must have that $\rho_1^* > 0$ and thus that P_η has point masses at $-(1 - \tau)$ and τ and a continuous distribution with positive density on $(-(1 - \tau), \tau)$. In particular, by choosing ξ sufficiently small we may guarantee that with probability converging to one, all dual solutions satisfy

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq q^* + 2\delta\} \geq \mathbb{P}_{P_\eta}(Z \leq q^*) + \mathbb{P}_{P_\eta}(q^* < Z \leq q^* + \delta) - \xi \geq \tau,$$

and thus that

$$\text{Quantile}\left(\tau, \frac{1}{n} \sum_{i=1}^n \delta_{\hat{\eta}_i}\right) \leq q^* + 2\delta,$$

as claimed. \square

The proof of Corollary 5.2 will rely on the following lemma which establishes uniqueness of the dual solution.

Lemma B.12. *Suppose the assumptions of Theorem 5.1 hold and that in addition the regularization is of the form $\mathcal{R}_d(w) = \sum_{j=1}^d \lambda_j^d w_j^2$ for some $\lambda_1^d, \dots, \lambda_d^d \geq 0$. Then, with probability one the dual solution $\hat{\eta}$ to (B.1) is unique.*

Proof. Let $Y \in \mathbb{R}^n$ denote the vector with entries Y_1, \dots, Y_n and $X \in \mathbb{R}^{n \times d}$ denote the matrix with rows X_1, \dots, X_n . Let $J_+ = \{j : \lambda_j^d > 0\}$ and $\Lambda_{J_+} = \text{diag}(\lambda_1^d, \dots, \lambda_d^d)$ denote the diagonal matrix with diagonal entries $\lambda_1^d, \dots, \lambda_d^d$. Fix any primal-dual solution pair $(\hat{w}, \hat{\tau}, \hat{\eta})$. Let $I_{\text{int.}}(\hat{\eta}) = \{i : -(1 - \tau) < \hat{\eta}_i < \tau\}$. By repeating the arguments given in the proof of Theorem 4.1 under Assumption 1 one can show that

$$\begin{bmatrix} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} \\ \hat{w}_{J_+^c} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} X_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} X_{I_{\text{int.}}(\hat{\eta}), J_+}^\top & X_{I_{\text{int.}}(\hat{\eta}), J_+^c} \\ X_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top & \mathbf{0}_{|J_+^c|, |J_+^c|} \end{bmatrix}^{-1} \begin{bmatrix} Y_{I_{\text{int.}}(\hat{\eta})} - \frac{1}{2} X_{I_{\text{int.}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} X_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \\ -X_{I_{\text{int.}}(\hat{\eta}), J_+^c}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \end{bmatrix},$$

where the matrix inverse above is guaranteed to exist. Now, following the calculations of (??) in the main text, the dual optimization program can be written as

$$\begin{aligned} &\text{maximize}_{\eta \in [-\tau, 1-\tau]^n} \eta^\top Y - \frac{1}{4} \eta^\top X_{[n], J_+} \Lambda_{J_+}^{-1} X_{[n], J_+}^\top \eta \\ &\text{subject to } \eta^\top X_{[n], J_+^c} = 0. \end{aligned}$$

For any set $I_{\text{int.}} \subseteq [n]$ and vector $\eta_{I_{\text{int.}}} \in \{-(1-\tau), \tau\}^n$ define the matrix

$$A(I_{\text{int.}}) = \begin{bmatrix} \frac{1}{2}X_{I_{\text{int.}}, J_+} \Lambda_{J_+}^{-1} X_{I_{\text{int.}}, J_+}^\top & X_{I_{\text{int.}}, J_+^c} \\ X_{I_{\text{int.}}, J_+}^\top & \mathbf{0}_{|J_+^c|, |J_+^c|} \end{bmatrix}_{|I_{\text{int.}}|, |I_{\text{int.}}| + |J_+^c|}^{-1},$$

and vector $b(I_{\text{int.}}, \eta_{I_{\text{int.}}}) = \frac{1}{2}X_{I_{\text{int.}}, J_+} \Lambda_{J_+}^{-1} X_{I_{\text{int.}}, J_+}^\top \hat{\eta}_{I_{\text{int.}}}$. For any sets $I, I' \subseteq [n]$ let $C(I, I') = X_{I, J_+} \Lambda_{J_+}^{-1} X_{I', J_+}^\top$. Using our expression for $\hat{\eta}_{I_{\text{int.}}(\hat{\eta})}$ we find that the value of the dual optimization program must be equal to

$$\begin{aligned} & \hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^\top Y_{I_{\text{int.}}(\hat{\eta})} + \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}^\top Y_{I_{\text{int.}}(\hat{\eta})^c} - \frac{1}{4} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^\top C(I_{\text{int.}}(\hat{\eta}), I_{\text{int.}}(\hat{\eta})) \hat{\eta}_{I_{\text{int.}}(\hat{\eta})} \\ & - \frac{1}{2} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})}^\top C(I_{\text{int.}}(\hat{\eta}), I_{\text{int.}}(\hat{\eta})^c) \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} - \frac{1}{4} \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}^\top C(I_{\text{int.}}(\hat{\eta})^c, I_{\text{int.}}(\hat{\eta})^c) \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}. \end{aligned}$$

Now, we know that

$$\hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = A(I_{\text{int.}}(\hat{\eta})) \begin{bmatrix} Y_{I_{\text{int.}}(\hat{\eta})} - b(I_{\text{int.}}(\hat{\eta}), \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c}) \\ -X_{I_{\text{int.}}(\hat{\eta})^c, J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \end{bmatrix}.$$

Substituting this into the previous expression we find that the dual program is a quadratic polynomial in Y with coefficients that depend on the covariates X , the set of interior points $I_{\text{int.}}(\hat{\eta})$, and the vector $\hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \in \{-(1-\tau), \tau\}^{|I_{\text{int.}}(\hat{\eta})^c|}$.

Suppose the dual program is non-unique and let $\hat{\eta}(1)$ and $\hat{\eta}(2)$ denote two dual solutions. Since the distribution of $Y \mid X$ is continuous we have that with probability one $\hat{\eta}(1)$ and $\hat{\eta}(2)$ yield the same objective value if and only if they yield same quadratic polynomial in Y . Now, by the continuity of the distribution of X , we have that with probability one the matrices $C(I_{\text{int.}}(\hat{\eta}(1)), I_{\text{int.}}(\hat{\eta}(1)))$ and $C(I_{\text{int.}}(\hat{\eta}(2)), I_{\text{int.}}(\hat{\eta}(2)))$ have all non-zero entries. In particular, this implies that the only way for the two polynomials to have the same second order terms is if

□

Proof of Corollary 5.2. By Lemma B.13 below we have that with probability one all dual solutions satisfy $\{i \in \{1, \dots, n\} : \hat{\eta}_i = 0\} = \emptyset$. By Proposition 4.1 and Lemma A.1 this implies that all dual solutions and all leave-one-out primal solutions satisfy

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{Y_i \leq \hat{\beta}_0^{-i} + X_i^\top \hat{\beta}^{-i}\}.$$

Notably, this means that the values of $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\}$ and $\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{Y_i \leq \hat{\beta}_0^{-i} + X_i^\top \hat{\beta}^{-i}\}$ are unique and, in particular, do not depend on the specific primal and dual solutions. Moreover, by applying Theorem 5.1 and the continuity of the distribution of P_η at zero it is straightforward to show that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{Y_i \leq \hat{\beta}_0^{-i} + X_i^\top \hat{\beta}^{-i}\} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\hat{\eta}_i \leq 0\} \xrightarrow{\mathbb{P}} \mathbb{P}_{Z \sim P_\eta}(Z \leq 0).$$

Since all these random variables above are bounded, we then also have that

$$\mathbb{P}(Y_1 \leq \hat{\beta}_0^{-1} + X_1^\top \hat{\beta}^{-1}) = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{Y_i \leq \hat{\beta}_0^{-1} + X_i^\top \hat{\beta}^{-1}\} \right] \rightarrow \mathbb{P}_{Z \sim P_\eta}(Z \leq 0),$$

or equivalently that $\mathbb{P}(Y_{n+1} \leq \hat{\beta}_0 + X_{n+1}^\top \hat{\beta}) \rightarrow \mathbb{P}_{Z \sim P_\eta}(Z \leq 0)$, as desired. \square

Lemma B.13. *Under the conditions of Corollary (5.2),*

$$\mathbb{P}(\text{For all dual solutions } \hat{\eta} \text{ to (B.2), } \{i \in \{1, \dots, n\} : \hat{\eta}_i = 0\} = \emptyset) = 1.$$

Proof. The proof of this lemma is essentially identical to the proof of Theorem 4.1 under Assumption 1. We will give a short calculation here emphasizing only the aspects of the argument that are new and leaving many details to the proof of Theorem 4.1.

Fix any dual solution $\hat{\eta}$ and primal solution $(\hat{\beta}_0, \hat{\beta})$. For ease of notation let $\tilde{X} = [X \mid \mathbf{1}_n]$ denote the matrix obtained by adding a column of ones to X . For any sets $A \subseteq \{1, \dots, n\}$ and $B \subseteq \{1, \dots, d+1\}$ let $\tilde{X}_{A,B}$ denote the sub-matrix of \tilde{X} given by the rows with indices in A and columns with indices in B . Let $\hat{\eta}_A$ be the subvector of $\hat{\eta}$ given by the entries with indices in A . Let $I_{\text{int.}}(\hat{\eta}) = \{i \in \{1, \dots, n\} : -(1-\tau) < \hat{\eta}_i < \tau\}$ be the set of indices of $\hat{\eta}$ which lie in the interior. Let $\lambda_{d+1} = 0$ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{d+1})$ be the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_{d+1}$. Assume that $I_{\text{int.}}(\hat{\eta}) \neq \emptyset$ (otherwise there is nothing to prove). Let $J_+ = \{j : \lambda_j > 0\}$. Following the calculations of Theorem 4.1, there exist sets $I_{\text{sub}}(\hat{\eta}) \subseteq I_{\text{int.}}(\hat{\eta})$ and $J_{\text{sub}}(\hat{\eta}) \subseteq \{j : \lambda_j = 0\}$ such that

$$\hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = \begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{sub}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub}}(\hat{\eta})}^\top \end{bmatrix}^{-1} \begin{pmatrix} Y_{I_{\text{sub}}(\hat{\eta})} - \frac{1}{2} \tilde{X}_{I_{\text{sub}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_{\text{sub}}(\hat{\eta})}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \end{pmatrix}.$$

Now, if $d+1 \notin J_{\text{sub}}(\hat{\eta})$, then the proof of Theorem 4.1 immediately tell us that w.p.1 $\hat{\eta}_{I_{\text{int.}}(\hat{\eta})}$ has no non-zero entries. So, suppose that $d+1 \in J_{\text{sub}}(\hat{\eta})$. Let $\tilde{J}_{\text{sub}}(\hat{\eta}) = J_{\text{sub}}(\hat{\eta}) \setminus \{d+1\}$ and rewrite the above as

$$\hat{\eta}_{I_{\text{int.}}(\hat{\eta})} = \begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{sub}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), \tilde{J}_{\text{sub}}(\hat{\eta})}^\top \\ \mathbf{1}_{|I_{\text{int.}}(\hat{\eta})|}^\top \end{bmatrix}^{-1} \begin{pmatrix} Y_{I_{\text{sub}}(\hat{\eta})} - \frac{1}{2} \tilde{X}_{I_{\text{sub}}(\hat{\eta}), J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}(\hat{\eta}), J_+}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \\ \tilde{X}_{I_{\text{int.}}(\hat{\eta}), \tilde{J}_{\text{sub}}(\hat{\eta})}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \\ \mathbf{1}_{|I_{\text{int.}}(\hat{\eta})^c|}^\top \hat{\eta}_{I_{\text{int.}}(\hat{\eta})^c} \end{pmatrix}.$$

Note that by construction, we may assume without loss of generality that $|I_{\text{sub}}(\hat{\eta})| + |\tilde{J}_{\text{sub}}(\hat{\eta})| \neq 0$ (otherwise one may simply make different choices for these sets). Now, take any fixed sets $I_{\text{sub}} \subseteq I_{\text{int.}} \subseteq \{1, \dots, n\}$, and $\tilde{J}_{\text{sub}} \subseteq \{1, \dots, d\}$ with $I_{\text{int.}} \neq \emptyset$ and $|I_{\text{sub}}| + |\tilde{J}_{\text{sub}}| \neq 0$. Fix any vector $\eta_{I_{\text{int.}}}^c \in \{-(1-\tau), \tau\}^{|I_{\text{int.}}^c|}$ and consider the behaviour of the random variable

$$\begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{sub}}, J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}, J_+}^\top \\ \tilde{X}_{I_{\text{int.}}, \tilde{J}_{\text{sub}}}^\top \\ \mathbf{1}_{|I_{\text{int.}}|}^\top \end{bmatrix}^{-1} \begin{pmatrix} Y_{I_{\text{sub}}} - \frac{1}{2} \tilde{X}_{I_{\text{sub}}, J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}, J_+}^\top \eta_{I_{\text{int.}}}^c \\ \tilde{X}_{I_{\text{int.}}, \tilde{J}_{\text{sub}}}^\top \eta_{I_{\text{int.}}}^c \\ \mathbf{1}_{|I_{\text{int.}}|}^\top \eta_{I_{\text{int.}}}^c \end{pmatrix}. \quad (\text{B.11})$$

The first $|I_{\text{int.}}| - 1$ entries of the vector above are continuously distributed. Write the matrix inverse above in block form as

$$\begin{bmatrix} \frac{1}{2} \tilde{X}_{I_{\text{sub.}}, J_+} \Lambda_{J_+}^{-1} \tilde{X}_{I_{\text{int.}}, J_+}^\top \\ \tilde{X}_{I_{\text{int.}}, \tilde{J}_{\text{sub.}}}^\top \\ \mathbf{1}_{|I_{\text{int.}}|} \end{bmatrix}^{-1} = \begin{bmatrix} T & S \\ \mathbf{1}_{|I_{\text{int.}}|-1}^\top & 1 \end{bmatrix}^{-1}$$

By the continuity of the distribution of X , the first $|I_{\text{int.}}| - 1$ columns of the matrix inverse above are equal to

$$\begin{bmatrix} (T - S \mathbf{1}_{|I_{\text{int.}}|-1}^\top)^{-1} \\ -\mathbf{1}_{|I_{\text{int.}}|-1}^\top (T - S \mathbf{1}_{|I_{\text{int.}}|-1}^\top)^{-1} \end{bmatrix}.$$

Notably, every row of this matrix must have at least one non-zero entry. Since T and S are independent of the vector appearing in (B.11) we conclude that with probability one none of the entries of (B.11) are zero. Taking a union bound over the choices of $I_{\text{sub.}}$, $I_{\text{int.}}$, $\tilde{J}_{\text{sub.}}$, and $\eta_{I_{\text{int}}}^c$ gives the desired result. □

C Additional technical lemmas

In this section, we state and prove a number of additional results that are useful in the main proofs.

Lemma C.1. *Let $\{X_i\}_{i=1}^n \stackrel{i.i.d.}{\sim} \mathcal{N}(0, I_d)$. Then, as $d/n \rightarrow \gamma \in [0, \infty)$,*

$$\liminf_{d, n \rightarrow \infty} \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \frac{1}{n} \sum_{i=1}^n |X_i^\top u + \beta_0| \stackrel{\mathbb{P}}{\geq} \sqrt{\frac{2}{\pi}} - \sqrt{\gamma}.$$

Proof. Let $X \in \mathbb{R}^{n \times d}$ denote the matrix with rows X_1, \dots, X_n . Write

$$\begin{aligned} & \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \frac{1}{n} \sum_{i=1}^n |X_i^\top u + \beta_0| \\ &= \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \max_{(v \in \{\pm 1\}^n)} \frac{1}{n} v^\top X u + \frac{1}{n} \beta_0 v^\top \mathbf{1}_n. \end{aligned}$$

By the convex Gaussian min-max theorem (Proposition B.2 above), we have that for any $c > 0$,

$$\begin{aligned} & \mathbb{P} \left(\inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \max_{(v \in \{\pm 1\}^n)} \frac{1}{n} v^\top X u + \frac{1}{n} \beta_0 v^\top \mathbf{1}_n \leq c \right) \\ & \leq 2\mathbb{P} \left(\inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \max_{(v \in \{\pm 1\}^n)} \frac{1}{n} \|v\|_2 u^\top h + \frac{1}{n} \|u\|_2 v^\top g + \frac{1}{n} \beta_0 v^\top \mathbf{1}_n \leq c \right), \end{aligned} \tag{C.1}$$

where $h \sim \mathcal{N}(0, I_d)$ and $g \sim \mathcal{N}(0, I_n)$ are independent. Now,

$$\begin{aligned}
& \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \max_{(v \in \{\pm 1\}^n)} \frac{1}{n} \|v\|_2 u^\top h + \frac{1}{n} \|u\|_2 v^\top g + \frac{1}{n} \beta_0 v^\top \mathbf{1}_n \\
&= \inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \frac{1}{\sqrt{n}} u^\top h + \frac{1}{n} \sum_{i=1}^n \|u\|_2 g_i + \beta_0 \\
&= \inf_{(0 \leq c_u \leq 1, \beta_0 \leq 1, \max\{c_u, \beta_0\} = 1)} \frac{1}{n} \sum_{i=1}^n |c_u g_i + \beta_0| - c_u \frac{\|h\|_2}{\sqrt{n}} \\
&\xrightarrow{\mathbb{P}} \inf_{(0 \leq c_u \leq 1, \beta_0 \leq 1, \max\{c_u, \beta_0\} = 1)} \mathbb{E}[|c_u g_1 + \beta_0|] - c_u \sqrt{\gamma},
\end{aligned}$$

where the limit follows standard uniform concentration arguments (e.g. Lemma 7.75 of [Miescke & Liese \(2008\)](#)) applied to the set $\{(c_u, \beta_0) : 0 \leq c_u \leq 1, |\beta_0| \leq 1\}$.

Finally, note that for any $c_u, \beta_0 \mapsto \mathbb{E}[|c_u g_1 + \beta_0|]$ is a convex, even function and thus obtains its minimum at 0. So,

$$\inf_{(c_u=1, |\beta_0| \leq 1)} \mathbb{E}[|c_u g_1 + \beta_0|] - c_u \sqrt{\gamma} = \mathbb{E}[|g_1|] - \sqrt{\gamma} = \sqrt{\frac{2}{\pi}} - \sqrt{\gamma}.$$

On the other hand, by Jensen's inequality,

$$\inf_{(0 \leq c_u \leq 1, |\beta_0| = 1)} \mathbb{E}[|c_u g_1 + \beta_0|] - c_u \sqrt{\gamma} \geq \inf_{(0 \leq c_u \leq 1)} |c_u \mathbb{E}[g_1] + 1| - c_u \sqrt{\gamma} = 1 - \sqrt{\gamma}.$$

Combining the above, we conclude that

$$\inf_{(\|u\|_2 \leq 1, |\beta_0| \leq 1, \max\{\|u\|_2, |\beta_0|\} = 1)} \max_{(v \in \{\pm 1\}^n)} \frac{1}{n} \|v\|_2 u^\top h + \frac{1}{n} \|u\|_2 v^\top g + \frac{1}{n} \beta_0 v^\top \mathbf{1}_n \xrightarrow{\mathbb{P}} \sqrt{\frac{2}{\pi}} - \sqrt{\gamma},$$

and applying (C.1) gives the desired result. \square

Our next lemma gives sufficient conditions under which partial optimization preserves strict convexity.

Lemma C.2. *[Lemma 19 of [Thrampoulidis et al. \(2018\)](#)] Let \mathcal{A} and \mathcal{B} be convex sets and $\Psi : \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$ be strictly convex in its first argument. Assume that $\Psi(a, \cdot)$ obtains its maximum for all $a \in \mathcal{A}$. Then, $a \mapsto \max_{b \in \mathcal{B}} \Psi(a, b)$ is strictly convex.*

Our next result computes the value of the Moreau envelope of the pinball loss.

Lemma C.3. *For any $x \in \mathbb{R}$ and $\rho \geq 0$ the Moreau envelope of the pinball loss is given by*

$$e_{\ell_\tau}(x; \rho) = \begin{cases} \frac{\tau^2 \rho}{2} + \tau(x - \rho\tau), & x - \rho\tau > 0, \\ \frac{x^2}{2\rho}, & x \in [-\rho(1 - \tau), \rho\tau], \\ \frac{(1 - \tau)^2 \rho}{2} - (1 - \tau)(x + \rho(1 - \tau)), & x + \rho(1 - \tau) < 0. \end{cases}$$

Proof. The case $\rho = 0$ is given by Lemma C.6. Now, consider the case $\rho > 0$. We begin by computing the proximal function. Let $f(v) = \frac{1}{2\rho}(v - x)^2 + \ell_\tau(v)$ denote the objective appearing in the definition of the Moreau envelope and the proximal function. We have that

$$\partial f(v) = \begin{cases} \{\frac{v-x}{\rho} + \tau\}, & v > 0, \\ [\frac{v-x}{\rho} - (1 - \tau), \frac{v-x}{\rho} + \tau], & v = 0, \\ \{\frac{v-x}{\rho} - (1 - \tau)\}, & v < 0. \end{cases}$$

Setting this to zero we find that

$$\text{prox}_{\ell_\tau}(x; \rho) \in \begin{cases} x - \rho\tau, & x - \rho\tau > 0, \\ 0, & x \in [-\rho(1 - \tau), \rho\tau], \\ x + \rho(1 - \tau), & x + \rho(1 - \tau) < 0. \end{cases}$$

Plugging this into the definition of the Moreau envelope gives the result. \square

The next three lemmas state a number facts from convex analysis that are useful in the proofs above.

Lemma C.4 (Proposition 13.13 in [Bauschke & Combettes \(2017\)](#)). *Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$. Then f^* is lower semicontinuous.*

Lemma C.5 (Part i of Theorem 14.3 in [Bauschke & Combettes \(2017\)](#)). *Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous convex function. Then, for any $x \in \mathbb{R}$ and $\rho > 0$ we have the identity*

$$e_f(x; \rho) + e_{f^*}(x/\rho; 1/\rho) = \frac{x^2}{2\rho}.$$

Lemma C.6 (Theorem 1.25 in [Rockafellar & Wets \(1997\)](#)). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be convex, lower semicontinuous, and prox-bounded. Then for all $x \in \mathbb{R}$,*

$$\lim_{\rho \downarrow 0} e_f(x; \rho) = f(x).$$