# *Exercise session 15: Extremal graph theory and Szemerédi's regularity lemma · 1MA020*

*Vilhelm Agdur*[1]

[1] vilhelm.agdur@math.uu.se

*29 November 2023*

We introduce the concept of extremal graph theory, starting with Turan's theorem. Then we introduce Szemerédi's regularity lemma as a tool for extremal graph theory.

## *Extremal graphs*

We start with the central definition of extremal graph theory, and then we explain what it actually means through some exercises.

**Definition 1.** Given any graph $H$, we say that a graph $G$ is *$H$-free* if it has no subgraph isomorphic to $H$. We say that it is *maximal $H$-free* if adding any edge to it would create a subgraph isomorphic to $H$, and we say that is is *maximum $H$-free* (or *extremal* among $H$-free graphs) if additionally no other $H$-free graph has more edges than $G$.

For each integer $n$, we define the *extremal function for $H$*, denoted $\mathrm{ex}(n; H)$, to be the number of edges of a maximum $H$-free graph on $n$ vertices.

**Exercise 1.** As a warm-up exercise, if $H$ is the path on three vertices, what is $\mathrm{ex}(n; H)$? What are the extremal graphs for this problem?

Letting $H_k$ be a star graph with $k$ leaves,[2] what is $\mathrm{ex}(n; H_k)$? Which are the extremal graphs here?

[2] That is, a tree with one root with $k$ children, and no other vertices or edges.

Having done this warmup, we can move on to the original question that motivated the start of extremal graph theory: How many edges can a graph have if it does not contain any triangles? This requirement clearly imposes *some* bound on the number of edges – a complete graph certainly contains a triangle – but what is the bound?

**Exercise 2.** Letting $H = K_3$, the triangle graph, what is $\mathrm{ex}(n; K_3)$? What do the extremal graphs look like?[3]

[3] Side exercise: Can you find a graph that is maximal triangle-free but not extremal?

**Exercise 3.** Can you generalize what you just did to finding $\mathrm{ex}(n; K_k)$ for $k > 3$?

In the lecture, we will see several very elegant proofs of Turán's theorem, which is the theorem that gives the answer to this question. Let us close out with an exercise that is definitely rather tough, but has a very elegant solution[4] which points towards more advanced problems.

[4] Which uses a probabilistic method style of argument.

**Exercise 4.** Let us generalize the definition of $\mathrm{ex}(n; H)$ to saying that for every graph $G$,

$$\mathrm{ex}(G; H) = \max\left\{|E(F)| \mid H \not\subseteq F \subseteq G\right\},$$

that is, $\mathrm{ex}(G; H)$ is the largest number of edges of an $H$-free subgraph of $G$. So $\mathrm{ex}(n; H) = \mathrm{ex}(K_n; H)$.

Prove that for all $H$ and all $n$-vertex graphs $G$,

$$\mathrm{ex}(G; H) \geq \mathrm{ex}(n; H)\frac{|E(G)|}{\binom{n}{2}}.$$