Exercise session 13: The probabilistic method, edge-colourings, and Ramsey theory · 1MA020

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We consider the notion of an edge-colouring, and some results of Ramsey theory. We see how the probabilistic method can be used to prove various results in graph theory.

Edge-colourings

In our last lecture, we looked at the notion of a vertex colouring of a graph, and derived a few results about it. Of course, vertices are not the only thing we could be colouring – we could also look at colouring the *edges* of a graph.

Definition 1. Let $G = (V, E)$ be a graph. A *proper*[2] *k-edge-colouring* is a function $c : E \to [k]$ such that no two edges which are incident to each other (i.e. share an endpoint) are assigned the same colour. If we do not have this restriction on incident edges, we call it just an (improper) edge colouring.

The *edge-chromatic number* of G, denoted $\chi_1(G)$,[3] is the smallest integer k such that G has a proper k-edge-colouring.

[2] Unlike for vertex colourings, we will actually be interested in improper edge colourings more often than proper ones, so we choose the opposite convention of including the word proper and omitting the word improper for them.

[3] This is sometimes also called the *chromatic index* of G. The 1 in the notation indicates that edges are one-dimensional – if we ever need to refer to both the chromatic number and the edge-chromatic number at the same time, we may thus denote the chromatic number by $\chi_0(G)$, since vertices are zero-dimensional. In some texts the edge-chromatic number is denoted by $\chi'(G)$, but χ and χ' look way too similar in LaTeX and on a blackboard, so let us avoid that notation.

Exercise 2. We observed for vertex colourings that there are trivial bounds for it in terms of the clique number and the independence number, and that each colour class is an independent set.

Now, observe that a proper edge-colouring of G is just a vertex colouring on the line graph of G. Use this observation to derive bounds on $\chi_1(G)$. What are the colour classes of a proper edge-colouring, using a term we've already defined?

Exercise 1. Given this little discussion, can you offer a definition of what $\chi_2(G)$ might refer to for a plane graph?

In our study of vertex colourings, we proved that all planar graphs can be coloured with five colours, using a trick known as Kempe changes.[4] You can do something similar for proper edge-colourings, considering edge-induced components of edges with two colours and swapping those.

[4] We defined exactly what these are in the previous exercise session.

Exercise 3. Use a trick like this to prove the following theorem by König:[5]

[5] This exercise is probably a little bit tough, but should be doable if you give it some time. It definitely isn't as quick as the other ones could be, though.

Theorem 2 (König, 1916)**.** *For every bipartite graph G with maximal degree Δ, we have $\chi_1(G) = \Delta$.*

Ramsey theory

Ramsey theory, named after remarkable British mathematician Frank P. Ramsey,[6] studies the question of how large a graph has to be in order to always contain a given structure. The simplest case is that of the eponymous *Ramsey number*. Let us give two definitions of it:

[6] Seriously, read his Wikipedia page if you get bored, he was almost a modern day Newton in terms of being British and inventing tons of stuff across fields.

Definition 3. For any integer k, the k-th *Ramsey number* $R(k)$ is the smallest integer n such that every edge-colouring of K_n contains a monochromatic K_k.[7]

[7] Clearly, we mean improper edge colourings here, since there is no proper 2-edge-colouring of any K_n other than K_2. By "containing a monochromatic K_k" we mean that for some colour i, the edge-induced subgraph $K_n\langle c^{-1}(i)\rangle$ contains a k-clique.

Definition 4. For any integer k, the k-th *Ramsey number* $R(k)$ is the smallest integer n such that every graph on n vertices contains either a k-clique or an independent set of size k.

Exercise 4. Prove that the above two definitions are equivalent.

Of course, it is a non-trivial fact that these Ramsey numbers in fact even exist – apriori it could be the case that for some k, there exist arbitrarily big graphs that contain neither a k-clique nor an independent set of size k. We will prove that this is not the case later, using the probabilistic method. For now, let us just show that one Ramsey number is finite:

Exercise 5. Prove that $R(3) = 6$.

The Rado graph

In almost all of this course, we only care about finite graphs. On the few occasions we have mentioned infinite graphs, it has been to demonstrate that things can turn weird if we allow infinite graphs, because infinity is strange.

In this section, we will see perhaps the strangest example of all, the *Rado graph*, also known as THE *random graph*, with emphasis on THE.

Definition 5. The *Rado graph* is the unique[8] countably infinite homogeneous graph G such that, for any finite graph H, H is isomorphic to an induced subgraph of G.[9]

[8] Up to isomorphism.

[9] In fact, the Rado graph contains every *countably infinite* graph as an induced subgraph as well, but let's skip that in the definition.

This definition should leave you with two large questions hovering in your mind, and one smaller one:

1. Can such a thing even exist?!

2. Even if such a thing exists, how can it be unique?

3. Wait, what does it mean for a graph to be "homogeneous"?

We will explore some of the strangeness of this graph, but the definition we gave of it is a bit hard to work with. So let us offer two definitions – first, what did we mean by homogeneous?

Definition 6. Let $G = (V, E)$ be a finite or infinite graph. We say that G is *homogeneous* if, for any two subsets $A, B \subseteq V$ such that the induced subgraphs $G[A]$ and $G[B]$ are isomorphic, the isomorphism between them can be extended to an automorphism of the entire graph.

Concretely, this means that if $f : A \to B$ is the isomorphism between $G[A]$ and $G[B]$, there is an isomorphism $g : V \to V$ between G and itself, such that $f(a) = g(a)$ whenever $a \in A$.

Exercise 6. Find at least two different examples of families of homogeneous graphs.[10]

[10] There are two somewhat "trivial" examples that come to mind immediately for me, among the graph families we have already defined in the course. Plus a funny construction of an infinite graph with this property.

The version of the definition of the Rado graph that we can actually work with is in terms of *saturation*.

Definition 7. A graph $G = (V, E)$, finite or infinite, is *k-saturated* if, for any two subsets $U, W \subseteq V$, each of size at most k, there exists a vertex $v \in V$ such that $v \sim u \in E$ for every $u \in U$, and $v \sim w \notin E$ for every $w \in W$.

Exercise 7. Consider the graph G whose vertices are the two-element subsets of the set $\{1, 2, \ldots, 6\}$, and where there is an edge between two such subsets if their intersection is non-empty. Prove that this graph is 2-saturated.

Exercise 8. Prove that if G is k-saturated, then for any graph H on at most k vertices, G contains H as an induced subgraph.[11]

[11] **Hint:** Use induction in the size of H.

Having done all this, let us give the definition of the Rado graph that we will actually be using:

Definition 8. The *Rado graph* is the unique countably infinite graph which is k-saturated for every $k \in \mathbb{N}$.

Notice how the previous exercise we did proves that the Rado graph by this definition does contain every finite graph as an induced subgraph. We leave the proof of the existence of the Rado graph, and of the equivalence of the two definitions, to the actual lecture.[12]

[12] Though proving homogeneity of the Rado graph according to the second definition is actually not too hard, so if you like you can try to find a proof of this yourself.