As an example of a stochastic process, let us consider a random walk on a connected graph (Figure 4.2). Consider a graph with $m$ nodes labeled $\{1, 2, \ldots, m\}$, with weight $W_{ij} \geq 0$ on the edge joining node $i$ to node $j$. (The graph is assumed to be undirected, so that $W_{ij} = W_{ji}$. We set $W_{ij} = 0$ if there is no edge joining nodes $i$ and $j$.)

A particle walks randomly from node to node in this graph. The random walk $\{X_n\}$, $X_n \in \{1, 2, \ldots, m\}$, is a sequence of vertices of the graph. Given $X_n = i$, the next vertex $j$ is chosen from among the nodes connected to node $i$ with a probability proportional to the weight of the edge connecting $i$ to $j$. Thus, $P_{ij} = W_{ij} / \sum_k W_{ik}$.

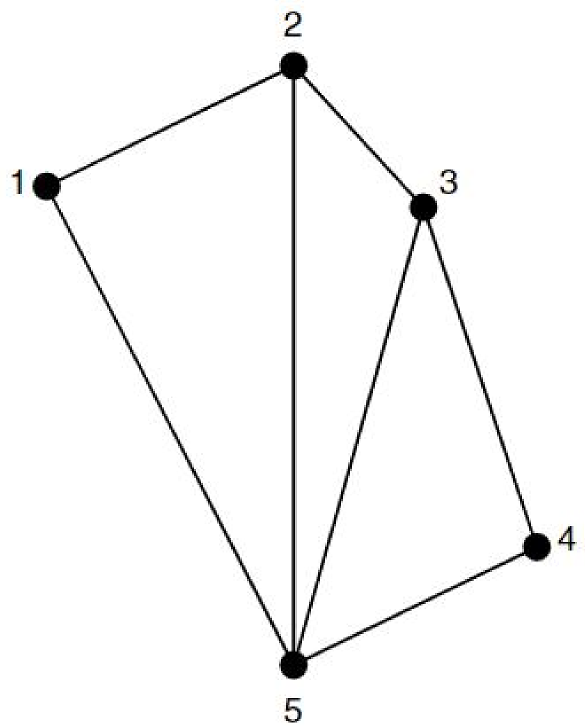


**FIGURE 4.2.** Random walk on a graph.

In this case, the stationary distribution has a surprisingly simple form, which we will guess and verify. The stationary distribution for this Markov chain assigns probability to node $i$ proportional to the total weight of the edges emanating from node $i$. Let

$$W_i = \sum_j W_{ij} \tag{4.29}$$

be the total weight of edges emanating from node $i$, and let

$$W = \sum_{i,j:j>i} W_{ij} \tag{4.30}$$

be the sum of the weights of all the edges. Then $\sum_i W_i = 2W$.

We now guess that the stationary distribution is

$$\mu_i = \frac{W_i}{2W}. \tag{4.31}$$

We verify that this is the stationary distribution by checking that $\mu P = \mu$. Here

$$\sum_i \mu_i P_{ij} = \sum_i \frac{W_i}{2W} \frac{W_{ij}}{W_i} \tag{4.32}$$

$$= \sum_i \frac{1}{2W} W_{ij} \tag{4.33}$$

$$= \frac{W_j}{2W} \tag{4.34}$$

$$= \mu_j. \tag{4.35}$$

Thus, the stationary probability of state $i$ is proportional to the weight of edges emanating from node $i$. This stationary distribution has an interesting property of locality: It depends only on the total weight and the weight of edges connected to the node and hence does not change if the weights in some other part of the graph are changed while keeping the total weight constant. We can now calculate the entropy rate as

$$H(\mathcal{X}) = H(X_2|X_1) \tag{4.36}$$

$$= -\sum_i \mu_i \sum_j P_{ij} \log P_{ij} \tag{4.37}$$

$$= -\sum_i \frac{W_i}{2W} \sum_j \frac{W_{ij}}{W_i} \log \frac{W_{ij}}{W_i} \tag{4.38}$$

$$= -\sum_i \sum_j \frac{W_{ij}}{2W} \log \frac{W_{ij}}{W_i} \tag{4.39}$$

$$= -\sum_i \sum_j \frac{W_{ij}}{2W} \log \frac{W_{ij}}{2W} + \sum_i \sum_j \frac{W_{ij}}{2W} \log \frac{W_i}{2W} \tag{4.40}$$

$$= H\left(\ldots, \frac{W_{ij}}{2W}, \ldots\right) - H\left(\ldots, \frac{W_i}{2W}, \ldots\right). \tag{4.41}$$

If all the edges have equal weight, the stationary distribution puts weight $E_i/2E$ on node $i$, where $E_i$ is the number of edges emanating from node $i$ and $E$ is the total number of edges in the graph. In this case, the entropy rate of the random walk is

$$H(\mathcal{X}) = \log(2E) - H\left(\frac{E_1}{2E}, \frac{E_2}{2E}, \ldots, \frac{E_m}{2E}\right). \tag{4.42}$$

This answer for the entropy rate is so simple that it is almost misleading. Apparently, the entropy rate, which is the average transition entropy, depends only on the entropy of the stationary distribution and the total number of edges.

The entropy rate of a random walk on a graph with equal weights is given by equation 4.41 in the text:

$$H(\mathcal{X}) = \log(2E) - H\left(\frac{E_1}{2E}, \ldots, \frac{E_m}{2E}\right)$$

There are 8 corners, 12 edges, 6 faces and 1 center. Corners have 3 edges, edges have 4 edges, faces have 5 edges and centers have 6 edges. Therefore, the total number of edges $E = 54$. So,

$$\begin{aligned} H(\mathcal{X}) &= \log(108) + 8\left(\frac{3}{108}\log\frac{3}{108}\right) + 12\left(\frac{4}{108}\log\frac{4}{108}\right) + 6\left(\frac{5}{108}\log\frac{5}{108}\right) + 1\left(\frac{6}{108}\log\frac{6}{108}\right) \\ &= 2.03 \text{ bits} \end{aligned}$$