The capacity of photographic film. Here is a problem with a nice answer that takes a little time. We're interested in the capacity of photographic film. The film consists of silver iodide crystals, Poisson distributed, with a density of λ particles per square inch. The film is illuminated without knowledge of the position of the silver iodide particles. It is then developed and the receiver sees only the silver iodide particles that have been illuminated. It is assumed that light incident on a cell exposes the grain if it is there and otherwise results in a blank response. Silver iodide particles that are not illuminated and vacant portions of the film remain blank. The question is, "What is the capacity of this film?"

We make the following assumptions. We grid the film very finely into cells of area dA. It is assumed that there is at most one silver iodide particle per cell and that no silver iodide particle is intersected by the cell boundaries. Thus, the film can be considered to be a large number of parallel binary asymmetric channels with crossover probability $1 - \lambda dA$.

By calculating the capacity of this binary asymmetric channel to first order in dA (making the necessary approximations) one can calculate the capacity of the film in bits per square inch. It is, of course, proportional to λ. The question is what is the multiplicative constant?

The answer would be λ bits per unit area if both illuminator and receiver knew the positions of the crystals.

Solution: *Capacity of photographic film*

As argued in the problem, each small cell can be modelled as a binary asymmetric Z-channel with probability transition matrix

$$p(y|x) = \begin{bmatrix} 1 & 0 \\ 1 - \lambda\, dA & \lambda\, dA \end{bmatrix} \qquad x, y \in \{0, 1\} \tag{9.55}$$

where $x = 1$ corresponds to shining light on the cell. Let $\beta = \lambda\, dA$.

First we express $I(X;Y)$, the mutual information between the input and output of the Z-channel, as a function of $\alpha = \Pr(X = 1)$:

$$H(Y|X) \;=\; \Pr(X = 0) \cdot 0 + \Pr(X = 1) \cdot H(\beta) = \alpha H(\beta)$$

$$\begin{aligned} H(Y) &= \mathbf{H}(\Pr(Y=1)) = H(\alpha\beta) \\ I(X;Y) &= H(Y) - H(Y|X) = H(\alpha\beta) - \alpha H(\beta) \end{aligned}$$

Since $I(X;Y) = 0$ when $\alpha = 0$ and $\alpha = 1$, the maximum mutual information is obtained for some value of α such that $0 < \alpha < 1$.

Using elementary calculus, we determine that (converting the equation to nats rather than bits),

$$\frac{d}{d\alpha} I(X;Y) = \beta \ln \frac{1-\alpha\beta}{\alpha\beta} - H_e(\beta)$$

To find the optimal value of α, we set this equal to 0, and solve for α as

$$\alpha = \frac{1}{\beta} \frac{1}{1 + e^{\frac{H_e(\beta)}{\beta}}} \tag{9.56}$$

If we let

$$\gamma = \frac{1}{1 + e^{\frac{H_e(\beta)}{\beta}}} \tag{9.57}$$

then $\alpha\beta = \gamma$, and

$$\overline{\gamma} = 1 - \gamma = \frac{e^{\frac{H_e(\beta)}{\beta}}}{1 + e^{\frac{H_e(\beta)}{\beta}}} = \gamma e^{\frac{H_e(\beta)}{\beta}} \tag{9.58}$$

or

$$\ln \overline{\gamma} - \ln \gamma = \frac{H_e(\beta)}{\beta} \tag{9.59}$$

so that

$$\begin{aligned} I(X;Y) &= H_e(\alpha\beta) - \alpha H_e(\beta) && (9.60) \\ &= H_e(\gamma) - \frac{1}{1 + e^{\frac{H_e(\beta)}{\beta}}} \frac{H_e(\beta)}{\beta} && (9.61) \\ &= -\gamma \ln \gamma - \overline{\gamma} \ln \overline{\gamma} - \gamma(\ln \overline{\gamma} - \ln \gamma) && (9.62) \\ &= -\ln \overline{\gamma} && (9.63) \\ &= \ln(1 + e^{-\frac{H_e(\beta)}{\beta}}) && (9.64) \\ &\approx e^{-\frac{H_e(\beta)}{\beta}} && (9.65) \\ &= e^{-\frac{-\beta \ln \beta - (1-\beta)\ln(1-\beta)}{\beta}} && (9.66) \\ &\approx e^{--\ln \beta} && (9.67) \\ &= \beta && (9.68) \end{aligned}$$

Thus the capacity of this channel is approximately β nats when $\beta \to 0$.