**Fisher information and relative entropy.** Show for a parametric family $\{p_\theta(x)\}$ that

$$\lim_{\theta' \to \theta} \frac{1}{(\theta - \theta')^2} D(p_\theta || p_{\theta'}) = \frac{1}{\ln 4} J(\theta). \tag{11.86}$$

**Solution:** *Fisher information and relative entropy.* Let $t = \theta' - \theta$. Then

$$\frac{1}{(\theta - \theta')^2} D(p_\theta || p_{\theta'}) = \frac{1}{t^2} D(p_\theta || p_{\theta+t}) = \frac{1}{t^2 \ln 2} \sum_x p_\theta(x) \ln \frac{p_\theta(x)}{p_{\theta+t}(x)}. \tag{11.87}$$

Let

$$f(t) = p_\theta(x) \ln \frac{p_\theta(x)}{p_{\theta+t}(x)}. \tag{11.88}$$

We will suppress the dependence on $x$ and expand $f(t)$ in a Taylor series in $t$. Thus

$$f'(t) = -\frac{p_\theta}{p_{\theta+t}} \frac{dp_{\theta+t}}{dt}, \tag{11.89}$$

and

$$f''(t) = \frac{p_\theta}{p_{\theta+t}^2}\left(\frac{dp_{\theta+t}}{dt}\right)^2 + \frac{p_\theta}{p_{\theta+t}}\frac{d^2 p_{\theta+t}}{dt^2}. \tag{11.90}$$

Thus expanding in the Taylor series around $t = 0$, we obtain

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2} + O(t^3), \tag{11.91}$$

where $f(0) = 0$,

$$f'(0) = -\frac{p_\theta}{p_\theta}\left.\frac{dp_{\theta+t}}{dt}\right|_{t=0} = \frac{dp_\theta}{d\theta} \tag{11.92}$$

and

$$f''(0) = \frac{1}{p_\theta}\left(\frac{dp_\theta}{d\theta}\right)^2 + \frac{d^2 p_\theta}{d\theta^2} \tag{11.93}$$

Now $\sum_x p_\theta(x) = 1$, and therefore

$$\sum_x \frac{dp_\theta(x)}{d\theta} = \frac{d}{dt}1 = 0, \tag{11.94}$$

and

$$\sum_x \frac{d^2 p_\theta(x)}{d\theta^2} = \frac{d}{d\theta}0 = 0. \tag{11.95}$$

Therefore the sum of the terms of (11.92) sum to 0 and the sum of the second terms in (11.93) is 0.

Thus substituting the Taylor expansions in the sum, we obtain

$$\begin{aligned}
\frac{1}{(\theta-\theta')^2}D(p_\theta||p_{\theta'}) &= \frac{1}{t^2\ln 2}\sum_x p_\theta(x)\ln\frac{p_\theta(x)}{p_{\theta+t}(x)} && (11.96)\\
&= \frac{1}{t^2\ln 2}\left(0 + \sum_x \frac{dp_\theta(x)}{d\theta}t + \sum_x\left(\frac{1}{p_\theta}\left(\frac{dp_\theta}{d\theta}\right)^2 + \frac{d^2 p_\theta}{d\theta^2}\right)\frac{t^2}{2} + O(t^3)\right) && (11.97)\\
&= \frac{1}{2\ln 2}\sum_x \frac{1}{p_\theta(x)}\left(\frac{dp_{\theta(x)}}{d\theta}\right)^2 + O(t) && (11.98)\\
&= \frac{1}{\ln 4}J(\theta) + O(t) && (11.99)
\end{aligned}$$

and therefore

$$\lim_{\theta'\to\theta}\frac{1}{(\theta-\theta')^2}D(p_\theta||p_{\theta'}) = \frac{1}{\ln 4}J(\theta). \tag{11.100}$$