Infinite Groups

Ishaan Patkar

Presentations of Groups

Groups often have complex definitions that may be difficult to work with, mathematically. Consider, for instance, the dihedral group D_n, defined as the group of all symmetries of a regular n-gon under a group operation of composition. Even simple operations on D_n can be difficult to compute because of the geometric definition of the group. For instance, given an n-gon with a vertex on the positive x-axis, what is the transformation given by a rotation counterclockwise by $\frac{10\pi}{n}$ radians, followed by a reflection across $y = \tan(\frac{2\pi}{n})$, a rotation of $\frac{6\pi}{n}$ radians, and then a reflection across $y = 0$?

In attempting to calculate the result of such a transformation using geometric means, one runs into difficulties, especially when an n-gon cannot be accurately drawn. Even if an accurate answer or picture can be formed, a geometric sketch is not a substitute for a proof, and ultimately we must resort to coordinate geometry in order to answer this question geometrically.

Perhaps a better question, however, is what are the rules that allow us to combine these various operations? Here we see two different types of operations being applied: rotation and reflection.

Let us begin by considering rotations in the dihedral group. How do rotations relate? A rotation by α radians followed by a rotation by β radians will ultimately result in a rotation by $\alpha + \beta$ radians; in other words, rotations combine additively. However, the n-gon cannot be rotated at any arbitrary angle for the dihedral group. We must rotate the n-gon onto itself, meaning that only angles of multiple of $\frac{2\pi}{n}$ are allowed. In this sense, we can represent any rotation as a composition of several of these elementary rotations. Thus, let $r \in D_n$ be the element corresponding to a rotation of $\frac{2\pi}{n}$ radians. We can represent the first step above as r^5 and the second step as r^3. Note also that composing r onto itself n times gives us a rotation by 2π radians, the identity transformation, so $r^n = e$.

Now let us consider reflections. How do reflections relate to each other? The reflections don't combine nicely, so perhaps we should instead think about connecting different types of reflections much in the same way that we did the rotations. Notice that a reflection over $y = \tan(\frac{2\pi}{n})$ is like rotating the n-gon clockwise $\frac{2\pi}{n}$ radians, then reflecting over $y = 0$, and then rotation back counterclockwise by $\frac{2\pi}{n}$ radians. This way, we can rephrase any kind of reflection as a composition of rotations and a reflection over $y = 0$. Like r, we can let this element of D_n be s.

So, the transformation above reduces to: $(r^5)(r^{-1}sr)(r^3)(s)$. Since these are elements of D_n, they are associative, so we can simplify this to r^4sr^4s. Now, however, we need to know how the elementary rotation and reflection relate to one another. So consider rs. This is equivalent to a rotation counterclockwise by $\frac{2\pi}{n}$ radians followed by a reflection across the $y = 0$. This reminds us of the way that we reflect over the line $y = \tan(\frac{2\pi}{n})$, which is by $r^{-1}sr$. Notice that this reflection is the same as reflecting over $y = 0$ followed by a rotation by $\frac{4\pi}{n}$ counterclockwise. So, we have $r^{-1}sr = sr^2$ or $r^{-1}s = sr$. We can also rearrange this to show $rsr = s$ or $rs = sr^{-1}$. We can use this to simplify the above expression to get the identity transformation.

This method of expressing the dihedral group is called a *presentation* of a group.

Definition 1. A *presentation* of a group G is a set S that generates G along with a set of equations, called *relations*, that the elements of S satisfy. The elements of S are sometimes also called generators.

For instance, we can present the dihedral group using r and s. Notice that every isometry of a regular n-gon onto itself can either be a rotation or a reflection, as no translation is possible, and these are the only types of isometries. In addition, r and s generates every possible rotation and reflection, so r and s generate D_n. We might write this as:

$$D_n = \langle r, s \rangle.$$

However, notice that the important identities that we use to simplify expressions in D_n are the relations. For this reason, and others following, we write a presentation as follows:

$$D_n = \langle r, s \,|\, r^n = e, s^2 = e, rs = sr^{-1} \rangle.$$

Note, though, that r and s are not unique in satisfying these relations. Any rotation in D_n, when composed n times onto itself, gives the identity, as does any reflection on itself twice. We can represent any rotation as r^i and any reflection as $r^{-j}sr^j$ so $r^i r^{-j} sr^j = r^{-j} r^i sr^j = r^{-j} sr^{-i} r^j = r^{-j} sr^j r^{-i}$ by the third relation above. Thus, any reflection and rotation will satisfy these relations. But is that enough to present D_n? We shall soon see that it is.

Free Groups and Presentations

In order to formalize the notion of generators and relations, we must have some way to deal not just with the elements of a group, but with the products of generators that equal elements of the group. This is the idea behind the free group — to represent products of the elements of a set under any arbitrary group operation. If the generator set is S, then the free group on S is denoted by $F(S)$. An element of a free group is called a *word*. Some example elements of the free group $F(\{a, b\})$ are a^2b, aba^{-1} or $ababa$.

To formalize this notion of a free group, we need to do three things: first, construct the group itself, second, ensure the group is well-defined, and third, define a group operation. The purpose of a free group is to contain the possible products of a set of generators in a group, so we know we require a set of inverses. Let this be S^{-1} and let S and S^{-1} have a bijection between them of inverse. That is, every $s \in S$ has an inverse, denoted by $s^{-1} \in S^{-1}$ and conversely, $(s^{-1})^{-1} = s$. For simplicity, we will omit an identity element. Now, from these elements, we can construct words:

Definition 2. A *word* in a free group $F(S)$ of a set S is a finite sequence of elements of either S or S^{-1}. In other words:

$$w = (s_1, s_2, \ldots, s_n)$$

where $s_i \in S \cup S^{-1}$ for every integer $0 < i \leq n$. As a shorthand, we may say that $w = s_1 s_2 \cdots s_n$.

Now, to ensure that the group is well-defined, first we must determine what equality is in the free group. Elements of the free group are similar to products in a group, so equality, like in a group, would mean that the result of the products should be equal. However, we cannot evaluate a word as we can in a group. But we can do something similar: we can simplify.

We will consider a word $w = (s_1, s_2, \ldots, s_n)$ *reduced* if, for every integer $0 < i < n$, $s_i \neq s_{i+1}^{-1}$. If S is a subset of a group, then a word under the group operation with $s_i = s_{i+1}^{-1}$ would simplify as $s_i s_{i+1}^{-1} = e$. Thus, we can consider all reduced words to be distinct, and the free group the set of all reduced words.

There is one final formalization of the free group, and that is the group operation. The group operation will clearly be concatenation of words, but we must prove that the concatenation of two elements of the free group will be another element of the free group. That is, the product of two reduced words will be a reduced word.

Consider words $w = s_1 s_2 \cdots s_n$ and $v = r_1 r_2 \cdots r_m$. Let i be the largest positive integer such that, for every integer $0 \leq j < i$, $s_{n-j} = r_{j+1}^{-1}$ (if $s_n \neq r_1^{-1}$ then let $i = 0$). Then define the product

$$wv = s_1 s_2 \cdots s_{n-i} r_{i+1} r_{i+2} \cdots r_m.$$

Notice that $s_{n-i} \neq r_i^{-1}$ because of the maximality of i, and since w and v are reduced, at all other points, consecutive terms are not inverses. So wv as defined above is a reduced word.

Now all we require is to determine that the free group is indeed a group.

Proposition 1. *If S is a set, then the free group on S, $F(S)$ is a group.*

Proof. The identity of $F(S)$ is called the *empty word*, which is the empty list () and denoted e. Every $w = s_1 s_2 \cdots s_n$ has an inverse $s_n^{-1} s_{n-1}^{-1} \cdots s_1^{-1}$ (notice that this is the same inverse as a product under a group).

Proving associativity is more difficult. We will rely on the fact that function composition is associative, meaning that the symmetric group on the free group is in fact a group. Let $\sigma_w : F(S) \to F(S)$ be a permutation of $F(S)$ such that $\sigma_w(v) = wv$ for all $v \in F(S)$. This is a permutation since there exists an inverse $\sigma_{w^{-1}}$ where $(\sigma_w \circ \sigma_{w^{-1}})(v) = ww^{-1}v = v$ and $(\sigma_{w^{-1}} \circ \sigma_w)(v) = w^{-1}wv = v$ meaning that σ_w is a bijection. Now, let $A = \{\sigma_w \mid w \in F(S)\}$. We can construct a bijection $\phi : F(S) \to A$ such that $\phi(w) = \sigma_w$. Note that if $w \neq v$ then $\phi(w) \neq \phi(v)$ as $\phi(w)(e) = w$ and $\phi(v)(e) = v$. Surjectivity follows from the definition of A.

In addition to being a bijection, f has properties similar to a homomorphism. Notice that $\phi(wv) = \sigma_{wv}$ and, $\sigma_{wv}(u) = wvu = (\sigma_w \circ \sigma_v)(u)$ for all $u \in F(S)$. So, $\sigma_{wv} = \sigma_w \circ \sigma_v = \phi(w)\phi(v)$ meaning $\phi(wv) = \phi(w)\phi(v)$. We can use this, along with the inverse as a bijection, to prove associativity of the group operation in the free group.

$$\begin{aligned}\phi((uv)w) &= \phi(uv)\phi(w)\\ &= (\phi(u)\phi(v))\phi(w)\\ &= \phi(u)(\phi(v)\phi(w))\\ &= \phi(u)\phi(vw)\\ &= \phi(u(vw)).\end{aligned}$$

So, taking ϕ^{-1} on both sides, $(uv)w = u(vw)$. □

There is one more result that allows us to use function notation on a set to denote the free group:

Proposition 2. *There is exactly one free group on a set S.*

Proof. Suppose that $F(S)$ and $F'(S)$ are free groups on S. We will show that $F(S) = F'(S)$. Every element of $F(S)$ can be written in the form $s_1 s_2 \cdots s_n$, where n is a positive integer and $s_i \in S \cup S^{-1}$ for all integers $0 < i \leq n$. But as $F'(S)$ is also a free group of S, $s_1 s_2 \cdots s_n \in F'(S)$. Thus, $F(S) \subseteq F'(S)$, and we can similarly show that $F'(S) \subseteq F(S)$. Thus, $F(S) = F'(S)$. □

Thus, we have formally established the free group. How, now, do we apply the free group to expressing the products of elements of any group? Let $\phi : F(S) \to G$ be a function where G is a group and $S \subseteq G$. Then every word in $F(S)$ will be a product of elements of S or S^{-1}, under which the group operation is defined. So, we will let ϕ map elements of $F(S)$ to their product under the group operation. Notice that for any $s \in S$, $\phi(s) = s$ as $s \in G$, and likewise for every $s^{-1} \in S^{-1}$ $\phi(s^{-1}) = s^{-1}$. Therefore, for any word $w = s_1^{\epsilon_1} s_2^{\epsilon_1} \cdots s_n^{\epsilon_1}$ with $s_i \in S$ and $\epsilon_i = \pm 1$ for all integers $0 < i \leq n$, $\phi(w) = s_1 s_2 \cdots s_n = \phi(s_1)\phi(s_2)\cdots\phi(s_n)$. This gives the idea that perhaps homomorphisms are how we apply the free group to groups. In fact, free groups have an interesting and unique property:

Proposition 3 (The Universal Property)**.** *If S is a set and $F(S)$ the free group on that set, then if there exists a mapping $\phi : S \to G$ where G is a group, then there is a unique homomorphic extension of ϕ, $\Phi : F(S) \to G$.*

Proof. Notice that for every $s \in S$, (s) is a word in $F(S)$, so $S \subseteq F(S)$.

$\implies$ First we show the existence of such a Φ. Let $\Phi(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}) = \phi(s_1)^{\epsilon_1}\phi(s_2)^{\epsilon_2}\cdots\phi(s_n)^{\epsilon_n}$, where $s_1, s_2, \ldots, s_n \in S$ and $\epsilon_i = \pm 1$ for every integer $0 < i \leq n$.

Next we show that Φ is a homomorphism. Let $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}$ and $r_1^{\delta_1} r_2^{\delta_2} \cdots r_m^{\delta_m}$ be elements of $F(S)$, where $s_i, r_j \in S$ and $\epsilon_i, \delta_j = \pm 1$ for every integer $0 < i \leq n$ and $0 < j \leq m$. Furthermore, let i be the greatest nonnegative integer such that for every integer $0 \leq j < i$, $s_{n-j}^{\epsilon_{n-j}} = r_{i+1}^{-\delta_{i+1}}$ (and if $s_n \neq r_1^{-1}$ then let $i = 0$). Therefore:

$$(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n})(r_1^{\delta_1} r_2^{\delta_2} \cdots r_m^{\delta_m}) = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{n-i}^{\epsilon_{n-i}} r_{i+1}^{\delta_{i+1}} r_{i+2}^{\delta_{i+2}} \cdots r_m^{\delta_m}.$$

Notice that

$$\Phi(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n})\Phi(r_1^{\delta_1} r_2^{\delta_2} \cdots r_m^{\delta_m}) = \phi(s_1)^{\epsilon_1}\phi(s_2)^{\epsilon_2}\cdots\phi(s_n)^{\epsilon_n}\phi(r_1)^{\delta_1}\phi(r_2)^{\delta_2}\cdots\phi(r_m)^{\delta_m}$$

by definition. For every integer $0 \leq j < i$, $s_{n-j}^{\epsilon_{n-j}} = r_{i+1}^{\delta_{i+1}}$ so $\Phi(s_{n-j}^{\epsilon_{n-j}}) = \Phi(r_{i+1}^{-\delta_{i+1}})$ meaning $\phi(s_{n-j})^{\epsilon_{n-j}} = \phi(r_{i+1})^{-\delta_{i+1}}$. Therefore, $\phi(s_{n-j})^{\epsilon_{n-j}}\phi(r_{i+1})^{\delta_{i+1}} = e$ meaning that we can simplify the above expression to simply:

$$\phi(s_1)^{\epsilon_1}\phi(s_2)^{\epsilon_2}\cdots\phi(s_{n-i})^{\epsilon_{n-i}}\phi(r_{i+1})^{\delta_{i+1}}\phi(r_{i+2})^{\delta_{i+2}}\cdots\phi(r_m)^{\delta_m}$$

But this is equal to $\Phi(s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_{n-i}^{\epsilon_{n-i}}r_{i+1}^{\delta_{i+1}}r_{i+2}^{\delta_{i+2}}\cdots r_m^{\delta_m})$ meaning that:

$$\Phi(s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_n^{\epsilon_n})\Phi(r_1^{\delta_1}r_2^{\delta_2}\cdots r_m^{\delta_m}) = \Phi(s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_{n-i}^{\epsilon_{n-i}}r_{i+1}^{\delta_{i+1}}r_{i+2}^{\delta_{i+2}}\cdots r_m^{\delta_m}).$$

Thus Φ is a homomorphism.

$\Longleftarrow$ Suppose there exists two homomorphisms Φ and Φ' that extend ϕ. Then $\Phi(s) = \phi(s) = \Phi'(s)$ for every $s \in S$ and $\Phi(s^{-1}) = \phi(s)^{-1} = \Phi'(s^{-1})$ by the inverse property of homomorphisms. Thus, take any word $s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_n^{\epsilon_n}$ where $s_1, s_2, \ldots, s_n \in S$ and $\epsilon_i = \pm 1$ for every integer $0 < i \leq n$. We have $\Phi(s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_n^{\epsilon_n}) = \phi(s_1)^{\epsilon_1}\phi(s_2)^{\epsilon_2}\cdots\phi(s_n)^{\epsilon_n}$ by the homomorphism property. But we also have $\Phi'(s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_n^{\epsilon_n}) = \phi(s_1)^{\epsilon_1}\phi(s_2)^{\epsilon_2}\cdots\phi(s_n)^{\epsilon_n}$ meaning that $\Phi = \Phi'$. □

The universal property of free groups allows us to characterize the free groups.

Proposition 4. *If $|X| = |Y|$ then $F(X) \cong F(Y)$.*

Proof. Let ϕ be a bijection $X \to Y$. Then ϕ is an injection $X \to F(Y)$, so let Φ be the extension $F(X) \to F(Y)$ of ϕ. Now, let Φ' be the extension $F(Y) \to F(X)$ of ϕ^{-1}. For every word $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$, where $x_i \in X$ and $\epsilon_i = \pm 1$ for every integer $0 < i \leq n$, $\Phi'(\Phi(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n})) = \Phi'(\phi(x_1)^{\epsilon_1}\phi(x_2)^{\epsilon_2}\cdots\phi(x_n)^{\epsilon_n}) = \phi^{-1}(\phi(x_1))^{\epsilon_1}\phi^{-1}(\phi(x_2))^{\epsilon_2}\cdots\phi^{-1}(\phi(x_n))^{\epsilon_n} = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$. Likewise, $\Phi(\Phi'(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n})) = x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ meaning that Φ and Φ' are inverses and so bijections. Thus, $F(X) \cong F(Y)$. □

Now, the converse of this theorem is true given finite sets, as follows:

Proposition 5. *If $F(X) \cong F(Y)$ and X and Y are finite sets, then $|X| = |Y|$.*

Proof. Consider the set of all homomorphisms from $F(X)$ to $\mathbb{Z}/2\mathbb{Z}$. By the universal property, each homomorphism corresponds to a unique map $X \to \mathbb{Z}/2\mathbb{Z}$, and there are exactly $2^{|X|}$ such maps. Likewise, there are $2^{|Y|}$ such maps from $F(Y)$ to $\mathbb{Z}/2\mathbb{Z}$. But since $F(X) \cong F(Y)$, every map $F(X) \to \mathbb{Z}/2\mathbb{Z}$ is a map $F(Y) \to \mathbb{Z}/2\mathbb{Z}$ and vice versa, so the number of homomorphisms from $F(X)$ to $\mathbb{Z}/2\mathbb{Z}$ is equal to the number of homomorphisms from $F(Y)$ to $\mathbb{Z}/2\mathbb{Z}$. Thus, $2^{|X|} = 2^{|Y|}$, and since X and Y have finite cardinalities, $|X| = |Y|$. □

This same technique works for infinite sets, but we must assume the General Continuum Hypothesis to show that the cardinalities of the two sets are equal.

The importance of the size of the set which the free group is defined on gives us the following definition:

Definition 3. The *rank* of a free group $F(S)$ is $|S|$.

Proposition 5 tells us that the definition of the rank of a free group is well-defined, and Proposition 4 tells us that all free groups of the same rank are congruent. This fact will be important when we use free groups to find homomorphisms of groups.

Now, the importance of the free group is that is the group of all possible products of a set of elements. We are able to simplify these products to a certain extent by making sure there are no consecutive inverses, but we cannot fully simplify every element of the free group, unlike a regular group.

Take the dihedral group as an instance of this phenomenon. The dihedral group D_n is generated by the elementary rotation r and the elementary reflection s. If we consider the free group $F(\{r, s\})$, we have the set of all products of r and s. But this group includes products like r^{2n} and rs^4 which can clearly be simplified, based on the properties of r and s. These "properties", which we will give a formal definition of, are called relations. For the dihedral group, one set of relations on the generators r and s is: $r^n = e$, $s^2 = e$, and $rs = sr^{-1}$. These properties are what we need to take products of r and s — words of $F(\{r, s\})$ — and simplify them to get the elements of the dihedral group.

So the ideas of generators, relations, and presentations are inherently linked to the free group. This "link" is simply a homomorphism.

Definition 4. A *presentation* of a group G is a pair of sets (S, R) such that $G = \langle S \rangle$, $R \subseteq F(S)$ and there exists a homomorphism $\pi : F(S) \to G$ such that $\ker \phi$ is equal to the normal closure of R in $F(S)$. (The normal closure of a subset of a group is equal to the smallest normal subgroup of the group containing the subset.)

Elements of S are known as *generators* of G, and elements of R are known as *relations*. We say G is *finitely generated* if S is finite, and that G is *finitely presented* if S and R are both finite.

This definition of a presentation relies on a homomorphism ϕ to conpute the product of reduced words. However, we can simplify these products use several identities, which are the relations. Specifically, computing the product of each relation gives the identity of G, which is why $\ker \phi$ is generated by R (noting that we must have $\ker \phi \lhd F(S)$).

This definition of a presentation is thus consistent with our prior notion of a presentation, but, for an example, consider again the dihedral group D_n. Let $S = \{r, s\}$ and R be the set $\{r^n, s^2, rsrs\}$, which is our set of chosen relations. Furthermore, let N be the normal closure of R in $F(S)$. We will then show that $D_n \cong F(S)/N$, with which we can use the canonical projection $F(S) \to F(S)/N$ to find a homomorphism $F(S) \to D_n$ with kernal N. We claim that $F(S)/N = \{N, rN, r^2N, \ldots, r^{n-1}N, sN, srN, sr^2N, \ldots, sr^{n-1}N\}$.

For any word $w \in F(S)$, consider wN. Notice that if $uv \in wN$ that, as N is normal, $v^{-1}Nv = N$ meaning that $uNv = uvN = wN$, meaning that substituting an element of N into an element of wN produces an element of wN. For any rs in w, we can substitute in $sr^{-1}sr^{-1}$ after rs to give us sr^{-1}. This is similar to commutativity, meaning that there exists an element in wN in the form $s^x r^y$. Using the fact that $r^n, s^2 \in N$ we can substitute in r^n, s^2, and any inverses of either to ensure that $0 \leq x < 2$ and $0 \leq y < n$. Thus, every element of $F(S)$ is represented in some element of the above set.

Furthermore, every element in this representation of $F(S)/N$ is distinct, since, $s^i r^j s^{-x} r^{-y} \notin N$ for any $0 \leq i, x < 2$ and $0 \leq j, y < n$ such that $i \neq x$ or $j \neq y$.

In general, this is how we prove presentations: considering the normal closure of the set R of relations, N, and proving that $G \cong F(S)/N$. In this same way, we can prove the following theorem.

Theorem 6. *Every finite group G is finitely presented.*

Proof. Let $S = G$ and R be the set of words ghk^{-1}, where $gh = k$ in G. Let N be the normal closure of R. Now consider $F(S)/N$. Let $f : G \to F(S)/N$ be a function such that $f(g) = gN$. Then notice that $f(g)f(h) = (gN)(hN) = ghN$, and if $gh = k$ in G, that $ghk^{-1} \in N$, so $(kh^{-1}g^{-1})gh = k \in ghN$. Thus, $f(g)f(h) = ghN = kN = f(k) = f(gh)$, and so f is a homomorphism.

First, we will show f is surjective. If $w = g_1 g_2 \cdots g_n \in F(S)$, then consider the coset wN. Notice that, if $g_{n-1}g_n = k$ in G, that $g_{n-1}g_n k^{-1} \in N$, so $kg_n^{-1}g_{n-1}^{-1} \in N$. By the substitution property above, $w' = g_1 g_2 \cdots g_{n-2}(kg_n^{-1}g_{n-1}^{-1})g_{n-1}g_n = g_1 g_2 \cdots g_{n-1}k \in wN$. Notice that w' has has length one less than w, and so repeating this procedure multiple times, we can write wN as gN, where $g \in G = S$. Thus, f is surjective.

Next, for injectivity. Suppose that $gN = hN$, where $g, h \in G$. Then $h \in gN$ meaning $h^{-1}g \in N$. If $h^{-1}g = k$ in G, then $h^{-1}gk \in N$ meaning that $(h^{-1}g)^{-1}h^{-1}gk = k \in N$. But this is impossible as every word in N must either be the empty word or a word of at least length 3. So $g = h$, proving injectivity.

Thus, $F(S)/N \cong G$, meaning that there exists a homomorphism $\phi : F(S) \to G$ with $\ker \phi = N$. As R and S are finite, G is finitely presented by (S, R). □

References

[1] Dummit, David S. and Richard M. Foote. *Abstract Algebra*. 3rd ed, John Wiley and Sons Inc, 2004.