Limit Rules in $\mathbb{R}^n$

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Limits in $\mathbb{R}$ interact nicely with the operations of addition and multiplication: in particular, limits of sums and products are sums and products of limits respectively. This begs the question: what other operations have similar limit rules? In this paper, we attempt to answer that question and present a technique allowing us to systematically prove these limit properties. Do note while we illustrate this technique primarily with the spaces $\mathbb{R}$ and $\mathbb{R}^n$, these ideas can be generalized to other spaces as well.

1 Fundamental Properties of Limits in $\mathbb{R}^n$

We begin by discussing the basic limit properties of $\mathbb{R}^n$. First, the formal definition of a limit is:

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. For any $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, we say that $\lim_{x\to a} f(x) = b$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $x \in \mathbb{R}^n$, $|f(x) - b| < \epsilon$ whenever $0 < |x - a| < \delta$.

Here, we use the notation $|y|$, where $y \in \mathbb{R}^n$, to denote the Euclidean length of the vector y. Notice that this definition of limits is similar to the definition in $\mathbb{R}$, with the exception that we use $|y|$ to mean absolute value in $\mathbb{R}$. Both definitions of limits are also consistent with the intuitive idea behind limits: they both encode the notion of what it means for a function to approach a point.

Now, it turns out that limits in $\mathbb{R}^n$ behave similarly to limits in $\mathbb{R}$. There are two fundamental properties which allow us to talk about limits in $\mathbb{R}^n$ as limits in $\mathbb{R}$. Both are discussed at length in [SCW20], and thus we will omit their proofs. The first property tells us that limits in $\mathbb{R}^n$ are nothing more than limits in $\mathbb{R}$:

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be functions such that for all $x \in \mathbb{R}^n$, $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$. Then, for any $a \in \mathbb{R}^n$ and $b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$, $\lim_{x\to a} f(x) = b$ if and only if $\lim_{x\to a} f_i(x) = b_i$ for all $i = 1, 2, \ldots, m$.*

That is, a function f approaches a point b if and only if each of its components f_i approaches the respective component b_i. This is a particularly important theorem as it lets us reduce limits in $\mathbb{R}^n$ to limits in $\mathbb{R}$, which we know how to deal with.

The second limit property is a "limit chain rule" of sorts:

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$. If $\lim_{x\to a} f(x) = b$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, and g is continuous at b, then $\lim_{x\to a} g(f(x)) = g(b)$.*

In particular, this means the composition of continuous functions is continuous.

This is a property that often gets glossed over while learning how to compute limits, although it is a fundamental and useful property. Do note that this theorem can be generalized to all spaces on which we can define limits in a sensible manner. In this way, the "chain rule" is a central property of limits.

2 Generalizing the Basic Limit Properties

With these two theorems and a definition, we are ready to present the main idea of this article (based off a trick in Chapter 2 of [Spi65]). Suppose we have an operation that takes in several variables, and we want to prove the limit rule for this operation. The idea is to first show the operation, when interpreted as a suitable multivariable function, is continuous. Then, we can apply Theorem 1 and Theorem 2 to show the limit property by substituting functions in for the variables.

This is all very abstract, so we demonstrate this technique by proving the tried and tested additive property of limits:

Corollary 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be functions such that for some $a \in \mathbb{R}^n$, and $b, c \in \mathbb{R}$, $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. Then $\lim_{x\to a}(f(x) + g(x)) = b + c$.*

This can be proven in various ways. Our idea is that this result can be viewed as a consequence of this more general theorem:

Theorem 3. *Addition is continuous.*

Proof. In order to show the continuity of addition, we first interpret it as a multivariable function. Let $s : \mathbb{R}^2 \to \mathbb{R}$ be the addition function; so, $s(x, y) = x + y$. We will prove s is continuous using the formal definition of limits.

Continuity, which we want to show, means that for any $(a, b) \in \mathbb{R}^2$, $\lim_{(x,y)\to(a,b)} s(x, y) = s(a, b)$. To do this, take any $\epsilon > 0$. We want to find a $\delta > 0$ such that whenever $(x, y) \in \mathbb{R}^2$ such that $0 < |(x, y)-(a, b)| < \delta$ then $|s(x, y) - s(a, b)| < \epsilon$. Notice that if $0 < |(x, y) - (a, b)| < \delta$, then

$$|x - a|, |y - b| \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Hence, if $\delta = \frac{\epsilon}{2}$, then $|x - a| + |y - b| < 2\delta = \epsilon$. Now, by the Triangle inequality, $|x - a + y - b| \leq |x - a| + |y - b| < \epsilon$, so, rearranging, $|(x + y) - (a + b)| < \epsilon$. Thus, $|s(x, y) - (a + b)| < \epsilon$. So, for any $\epsilon > 0$, if $0 < |(x, y) - (a, b)| < \delta = \frac{\epsilon}{2}$, then $|s(x, y) - s(a, b)| < \epsilon$. This means $\lim_{(x,y)\to(a,b)} s(x, y) = s(a, b)$ and so s is continuous. □

Remark 1. The Triangle Inequality used above is a particularly important inequality in analysis. It says that for any real numbers x and y, $|x + y| \leq |x| + |y|$. We can see this by noticing that $z \leq |z|$ for all $z \in \mathbb{R}$, meaning that $xy \leq |xy| = |x||y|$. So, $x^2 + 2xy + y^2 \leq x^2 + 2|x||y| + y^2$ and so $(x + y)^2 \leq (|x| + |y|)^2$. Taking the square root gives the Triangle Inequality $|x + y| \leq |x| + |y|$, as desired.

Now with this fact, we can begin assembling a proof of Corollary 1. The idea is to use Theorem 1 to construct a function that outputs 2-dimensional vectors, and compose this function with addition using the continuity of addition and Theorem 2.

Proof of Corollary 1. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions defined as in the statement of the corollary; so $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. Now, by Theorem 1, $\lim_{x\to a}(f(x), g(x)) = (b, c)$. Since the addition function s is continuous, by Theorem 2,

$$\lim_{x\to a} s(f(x), g(x)) = s(b, c).$$

From this, it follows that $\lim_{x\to a}(f(x) + g(x)) = b + c$. □

While an interesting proof of the addition rule, this result is by no means novel. A reader familiar with other proofs of Corollary 1 might notice that the proof of Theorem 3 is very similar to other proofs of this theorem. These similarities indicate that every proof implicitly shows continuity of addition. Explicitly proving that addition is continuous, as we have done, makes our results more general: if we define limits for functions $f : X \to \mathbb{R}^n$ in a sensible way, where X is a different type of space, then we will be able to generalize Theorems 1 and 2, which allows us to generalize Corollary 1.

More concretely, this technique gives us easier ways of proving limit rules for more complicated operations. For a final example, we will prove the limit law for the dot product. To do so, we first need to establish continuity for the other fundamental binary operation on $\mathbb{R}$:

Theorem 4. *Multiplication is continuous.*

The proof of this theorem is considerably more technical than that of Theorem 3. As it is not crucial to the problem at hand, nor particularly enlightening, we will forego the proof and instead point to [Spi08] or [SCW20], where the multiplication rule is proven; the proof of Theorem 4 is quite similar.

Now, continuity of addition and multiplication allows us to prove the following theorem:

Theorem 5. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be functions such that for $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}^m$, $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. Then $\lim_{x\to a}(f(x) \cdot g(x)) = b \cdot c$. (Here, $\cdot$ refers to the dot product.)*

Proof. We will apply the same trick as before. We first interpret the dot product as a suitable multivariable function, and show continuity. To do this, we consider the dot product on $\mathbb{R}^n$ to be the function $p : \mathbb{R}^{2n} \to \mathbb{R}$ such that:

$$p(x_1, x_2, \dots, x_{2m}) = x_1 x_{m+1} + x_2 x_{m+2} + \cdots + x_m x_{2m}.$$

Notice that this function just takes the dot product of $(x_1, x_2, \dots, x_{2m})$ and $(x_{2m+1}, x_{2m+2}, \dots, x_{2m})$. While this is not exactly the dot product (since the dot product takes in two n-dimensional vectors), this is an equivalent multivariable function which we can work with.

Next, we show continuity of p. But observe that it is just a composition of addition and multiplication, which we know to be continuous. So, by Theorem 2, p is a continuous function. (Alternatively, this is a good place to use the addition and multiplication limit rules.)

Now we want to apply our dot product function p on f and g. So, we take the components: let $f_1, f_2, \dots, f_m, g_1, g_2, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ be functions such that

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \qquad\qquad g(x) = (g_1(x), g_2(x), \dots, g_m(x)).$$

Observe that $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$. Let $b = (b_1, b_2, \dots, b_m)$ and $c = (c_1, c_2, \dots, c_n)$. Then by Theorem 1, $\lim_{x \to a} f_i(x) = b_i$ and $\lim_{x \to a} g_i(x) = c_i$ for all $i = 1, 2, \dots, m$. Hence, by Theorem 1 again,

$$\lim_{x \to a} (f_1(x), f_2(x), \dots, f_m(x), g_1(x), g_2(x), \dots, g_m(x)) = (b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_m).$$

Thus, applying Theorem 2 and the continuity of p gives

$$\begin{aligned}
\lim_{x \to a} p(f_1(x), f_2(x), \dots, f_m(x), g_1(x), g_2(x), \dots, g_m(x)) &= p(b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_m) \\
\lim_{x \to a} (f_1(x)g_1(x) + f_2(x)g_2(x) + \ldots + f_m(x)g_m(x)) &= b_1c_1 + b_2c_2 + \cdots + b_mc_m \\
\lim_{x \to a} (f(x) \cdot g(x)) &= b \cdot c.
\end{aligned}$$

So we are done. $\square$

This final example should illustrate how the interested reader might apply this technique to proving other limit rules in $\mathbb{R}^n$, including the rules for cross products and matrix multiplication (after interpreting the matrix as a suitable vector).

One might notice that in the proof of this theorem, we were forced to convert the dot product function into a multivariable function to make this work. Why could we not just consider the dot product to be a function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$? (Here, $\mathbb{R}^n \times \mathbb{R}^n$ refers to pairs of n-dimensional vectors.)

Well, one immediate reason is that we do not know how to talk about limits or continuity in $\mathbb{R}^n \times \mathbb{R}^n$. However, this is not a very satisfactory answer, since we can consider the elements of $\mathbb{R}^n \times \mathbb{R}^n$ to be $2n$-dimensional vectors and define limits and continuity just as we do in $\mathbb{R}^{2n}$. But we also could define limits in a different way. So what is so special about limits in $\mathbb{R}^{2n}$ that allows this to be possible?

The issue with other types of limits in $\mathbb{R}^n$ is that Theorem 1 may not hold, and Theorem 1 factors extensively into our proofs. If we defined limits in the space $\mathbb{R}^n \times \mathbb{R}^n$ analogously to $\mathbb{R}^{2n}$, then Theorem 1 implies that a function in $\mathbb{R}^n \times \mathbb{R}^n$ converges if and only if the two n-dimensional vector components of the function converge. This may not be true in other spaces and ways of defining limits in those spaces.

In this way, the limit structure on $\mathbb{R}^n$ is an extension of the limit structure on $\mathbb{R}$. It is thus that the continuity of addition, multiplication, and other operations on $\mathbb{R}$ or $\mathbb{R}^n$ reflects a deeper intrinsic compatibility between limits and the nature of the real numbers.

References

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