An Introduction to the Metric Topology of $\mathbb{R}^n$

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Topology is the study of the shape of certain sets. In this multivariable calculus course, we have encountered a number of theorems that utilize the topology of the real numbers, $\mathbb{R}$. Here are a few such theorems:

- (The Extreme Value Theorem.) Let $f : D \to \mathbb{R}$ be a continuous function, where $D \subseteq \mathbb{R}^2$ is closed and bounded. Then there exists a global maximum and a global minimum of f on the domain D.

- Let $F : D \to \mathbb{R}^2$ be a path-independent vector field, where $D \subseteq \mathbb{R}^2$ is open and connected. Then F is conservative.

- Let $F : D \to \mathbb{R}^2$ be a vector field with $F = P\mathbf{i} + Q\mathbf{j}$, where $D \subseteq \mathbb{R}^2$ is open and simply-connected. If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, then F is conservative.

Each of these theorems requires the domain D to satisfy certain requirements: open, closed, bounded, connected, or simply-connected. All of these properties are topological in nature, in that they all have something to do with the shape of the set D. It is important to note these conditions are crucial to why the theorems work, which highlights the importance of topology when solving analytic problems.

In this paper, we will discuss the basic topological properties of $\mathbb{R}^n$. These properties are closely related to the notion of distance and length in $\mathbb{R}^n$. We will develop the standard metric or distance function of $\mathbb{R}^n$ use it to discuss the concepts of *open* and *closed* sets:

Definition. Let $E \subseteq \mathbb{R}^n$. We say that E is open if E does not contain any points on its boundary. We say that E is closed if E contains all points on its boundary.

Open and closed sets are a fundamental notion of topology, and more complex topological phenomena such as connectedness, simply-connectedness, and compactness are based on the notion of open and closed sets. We hope this paper will not only serve as an introduction to point-set topology, but also as a guide to how one uses sets and numbers to analyze shapes.

Finally, we note this discussion will take place in the space $\mathbb{R}^n$ of n-dimensional real vectors. For any $x \in \mathbb{R}^n$, we will use the notation x_k to refer to the kth component of x, so $x = (x_1, x_2, \ldots, x_n)$. We use this more general space because the theory in $\mathbb{R}^n$ is not much more complicated than in $\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$. However, should the reader be uncomfortable dealing with $\mathbb{R}^n$, we invite them to replace all instances of $\mathbb{R}^n$ with a more familiar space, such as $\mathbb{R}^2$ or $\mathbb{R}^3$.

Much of the material for this paper was gathered from [Spi65], [Rud76], and [Tao15]. The interested reader is encouraged to consult those books for more information.

1 The Metric of $\mathbb{R}^n$

Shapes are closely related to the notions of length and distance. One way in which we can develop topology, a notion of shape, is to use what we know about length and distance.

Recall for a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the *length* of x is:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1}$$

Observe also that $|x| = \sqrt{x \cdot x}$ (where $x \cdot x$ refers to the dot product of x with itself). In the case of $\mathbb{R} = \mathbb{R}^1$, for any $x \in \mathbb{R}$ this length becomes $\sqrt{x^2} = |x|$ which is just the absolute value.

Additionally, this notion of length naturally gives us a notion of distance. Given two points $x, y \in \mathbb{R}^n$, the *distance* from x to y is the length of $y - x$, which is the vector from x to y. That is:

Definition. Let $x, y \in \mathbb{R}^n$ be points. The *distance* from x to y is $|y - x|$. We define a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d(x, y) = |y - x|$ is the distance from x to y. We call this the *distance function* or *metric* of $\mathbb{R}^n$.

The distance function allows us to describe certain regions very easily:

Example. Consider the sphere centered at $x = (0, 1, 2)$ with radius 3 in $\mathbb{R}^3$. All $y \in \mathbb{R}^3$ that are contained in this sphere satisfy the following equation:

$$(y_1 - 0)^2 + (y_2 - 1)^2 + (y_3 - 2)^2 \leq 3^2.$$

We observe that the left-hand side is simply $d(x, y)^2$, or the square of the distance from x to y. Thus, the distance function allows us to write the equation for a sphere as the set

$$\{y \in \mathbb{R}^3 \mid d(x, y) \leq 3\}$$

The distance function is thus closely to spheres in $\mathbb{R}^3$. More generally in $\mathbb{R}^n$, the distance function gives us the notion of *balls*. We will see more of this in the next section.

2 Interior, Exterior, and Boundary

We now return to the notions of open and closed sets. Recall that our criteria for open and closed sets used the notion of a boundary of a set. This is the question we begin with. Given a set $E \subseteq \mathbb{R}^n$, what is its boundary?

To simplify this problem, observe that the notion of boundary is closely related to the concepts of interior and exterior. Intuitively, we might understand the boundary of a set E to be the all the points not in the interior of E nor the exterior of E. This reduces our problem to determining what interiors and exteriors are.

So what does it mean for a point x to be on the exterior of E? Well, if a point x is in the exterior of E, then it must be in the interior of $\mathbb{R}^n \setminus E$, the region outside E (that is, the set of all points not inside E). So, we need only determine when a point is in the interior of a set E, and exterior and boundary points should follow from that.

Here is one idea. Given a set E, it would be clear that a point x is in the interior of E if we can squeeze a small region U around x that fits fully in E. That is, if there exists a $U \subseteq E$ such that $x \in U$.

In particular, in 2-dimensions, we might imagine we can fit a small disk U in E. In 3-dimensions, perhaps a small sphere will do. In n-dimensions, this generalizes to the notion of a *ball*:

Definition (Metric Balls)**.** Let $x \in \mathbb{R}^n$ and $r > 0$ a real number. We define the *open ball centered at x with radius r* to be the following set:

$$B_r(x) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}.$$

A related, though less important notion is the *closed ball centered at x with radius r*, which is the set

$$\{y \in \mathbb{R}^n \mid d(x, y) \leq r\}.$$

Notice that open and closed balls are ways to generalize the notions of a sphere (or disk) centered at a point $x \in \mathbb{R}^3$ with radius r. In particular, in $\mathbb{R}$, the open ball $B_r(x)$ is the interval $(x - r, x + r)$. Also, note that $x \in B_r(x)$ since $r > 0$, and $d(x, x) = 0$.

Using this notion of an open ball, we can now define what it means to be in the interior of a set E.

Definition (Interior)**.** Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say x is an interior point of E if there exists real number $r > 0$ such that $B_r(x) \subseteq E$. We denote the set of interior points of E to be $\operatorname{Int} E$.

Observe that if x is an interior point of E, then as $x \in B_r(x)$ and $B_r(x) \subseteq E$, we have $x \in E$. So, interior points are always in E, which is a nice sanity check on our definition.

From this, we have the following definition of exterior:

Definition (Exterior)**.** Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that x is an exterior point of E if x is an interior point of $\mathbb{R}^n \setminus E$. That is, there exists an $r > 0$ such that $B_r(x) \subseteq \mathbb{R}^n \setminus E$.

Notice that $B_r(x) \subseteq \mathbb{R}^n \setminus E$ if and only if $B_r(x)$ shares no points in common with E. This is another common criteria we can use for exterior. Also, as with interior, if x is an exterior point of E, then $x \in B_r(x) \subseteq \mathbb{R}^n \setminus E$. So, $x \in \mathbb{R}^n \setminus E$, meaning $x \notin E$.

Finally, we have the following definition of boundary

Definition (Boundary)**.** Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that x is an exterior point of E if x is neither an interior point of E nor an exterior point of E. We define the boundary of E to be the set ∂E.

The boundary of E, ∂E, is the same boundary that appears in Green's Theorem and the Divergence Theorem.

Let us now see an example of interior, exterior, and boundary.

Example. Consider the interval $E = (1, 2) \subseteq \mathbb{R}$. Any $x < 1$ is in the exterior of E, since if we let $r = 1 - x > 0$, the open ball $B_r(x) = (2x - 1, 1)$ has no points in common with E. Likewise, any $x > 2$ is also in the exterior of E, since letting $r = x - 2$, the open ball $B_r(x) = (2, 2x - 2)$ has no points in common with E as well.

Now, consider the interior of E. For this, we need only consider points $x \in (1, 2)$, since interior points of E are necessarily in E. We can now let r be the minimum of $x-1$ and $2-x$. In that case, $B_r(x) = (x - r, x + r)$, so $x - r > x - (x - 1) = 1$ and $x + r < x + (2 - x) = 2$, so $B_r(x) \subseteq (1, 2)$. So, every point in E is an interior point of E.

Finally, we are left with the points 1 and 2. We claim that these two points form the boundary of E. These are not interior points, since they do not lie in E. So, they may be exterior points; we claim they are not. Consider 1. Then for any $r > 0$, we can look at $B_r(1) = (1 - r, 1 + r)$. But then $1 + r > 1$, so there exists some $y \in (1, 1 + r)$ which will lie both in E and in $B_r(1)$. So, $B_r(1)$ always has common points with E, which means 1 cannot be in the exterior. Similar reasoning holds for 2. Thus, 1 and 2 are not exterior points, and so are the boundary.

Therefore, $\operatorname{Int} E = (1, 2)$, $\partial E = \{1, 2\}$, and all the points $x < 1$ and $x > 2$ are in the exterior of E.

We now return to the notion of open and closed sets with greater insight, having a notion for boundary:

Definition. Let $E \subseteq \mathbb{R}^n$. We say E is open if ∂E has no points in common with E. We say that E is closed if $\partial E \subseteq E$.

So, in our example above, $(1, 2)$ would be an open set. Do note that there are sets which are neither open nor closed. In particular, some sets can contain some of their boundary points, but not all of them, making them neither open nor closed. For instance, a set such as the interval $(1, 2]$ would be neither closed nor open.

There are also some sets which are both closed and open. The following proposition gives us two important open and closed sets:

Proposition 1. *Both $\emptyset$ and $\mathbb{R}^n$ are open and closed in $\mathbb{R}^n$.*

Proof. To see this is true, we will show that $\emptyset$ and $\mathbb{R}^n$ have no points in their boundary. This will imply that $\emptyset$ and $\mathbb{R}^n$ contain all of their boundary points, and so are closed, and yet do not contain any boundary points, and so are open.

We start with $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$, x is an interior point of $\mathbb{R}^n$ since $B_r(x) \subseteq \mathbb{R}^n$ for any $r > 0$. So, every point in $\mathbb{R}^n$ is an interior point of $\mathbb{R}^n$, and therefore $\mathbb{R}^n$ has no boundary points.

Next, consider $\emptyset$. For every $x \in \mathbb{R}^n$, x is an exterior point of $\emptyset$ as it is an interior point of $\mathbb{R}^n = \mathbb{R}^n \setminus \emptyset$. Since all points in $\mathbb{R}^n$ are exterior points of $\emptyset$, it cannot have any boundary.

Therefore, it follows that $\mathbb{R}^n$ and $\emptyset$ have no boundary, and therefore are both open and closed. $\square$

More importantly, this proposition and the example prior to it should illustrate how one works with open and closed sets as well as interior, boundary, and exterior points. These notions give us powerful tools for understanding and thinking about shape. More importantly, we hope to have communicated the power of of thinking precisely and deeply about simple things.

References

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