Linear Algebra Notes

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Ch 1

1.2 Intro stuff

Thm: Any system of linear eqs has either 0, exactly 1, or ∞ many solutions

How to: check how many solutions there are in a given system of eq:
If there last line is $0 = c$, then there is $no\ solution$
If not, are there free variables? If so, there are $inf\ many\ solutions$, otherwise $1\ solution$.

Thm: Homogenous systems ALWAYS are consistent, aka they have either 1 or inf many solutions.
This is b/c the trivial solution will always exist when $A\mathbf{x} = 0$

Ch 2

2.1 Linear Combinations

How to: find all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t. $c_1 u_1 + c_2 u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$?
> Augment the vector with $\begin{pmatrix} a \\ b \end{pmatrix}$ and row reduce, the remaining things in the augment become the scalars c_1, c_2

How to: show that a certain vector $\mathbf{b}$ cannot be obtained as a linear combination of some other vectors?
> Augment the matrix with b then row reduce, you'll find an inconsistent set with $0 = c$

2.2 Spans

Definition of $span\{\mathbf{u_1 \ldots u_m}\}$ is the set of ALL linear combinations

How to: See if some vector $\mathbf{v}$ is an element of $span\{u_1, u_2, \ldots, u_m\}$?
> iff the linear system w/ $\mathbf{v}$ as the augment has a solution

Thm: IF $\mathbf{u, v}$ are in $span\{\mathbf{u_1 \ldots u_m}\}$, THEN $\mathbf{u}+\mathbf{v}$ & $a\mathbf{u}$ are in that span
(linear combinations of the vectors are in that span)

How to: Find a vector $\mathbf{b}$ not in a given span of vectors?
> Set matrix equal (by augment) to some (a, b, c), and track what happens to it until row reduced.
> If the row is [0 0 0 | f(c)], then any vector that has f(c) $\neq$ 0 is not in the span

Thm: ($span\{u_1 \ldots u_m\} = \mathbb{R}^n$) $\iff$ (*B has a pivot in every row*)

How to: check if a set of vectors span $\mathbb{R}^n$?
> Row reduce the matrix augmented with $[a\ b\ c]$
> Check if there is a row of 0s, which would imply that there could be a vector not in that span
> If there is a pivot in every row, there are either 1 or inf many ways to get to any point in $\mathbb{R}^n$ (depending on if there are free variables in any of the columns)

How to: Find what values of h in a n x m matrix allow it to span $\mathbb{R}^n$?
a. Row reduce the matrix, moving the h down as needed to the last row.
b. The vectors in A span $\mathbb{R}^n$ iff there is a trivial solution, which only occurs when $f(h) \neq 0$
c. The final answer should be all vectors with $\{h | h \neq 27\}$ or something

How to: find $span(\{\mathbf{a_1 \ldots a_m}\})$ (aka the columns of T)?
a.

Thm: For a given set of *m* vectors in $\mathbf{R}^n$:
a. IF $m < n$, THEN the set does not span $\mathbf{R}^n$ (b/c theres no way to have a pivot in every row)
b. IF $m \geq n$, THEN the set could span $\mathbf{R}^n$ (depends on whether or not they are linearly independent)

Thm: $\mathbf{b} \ \epsilon \ span\{\mathbf{a_1 \ldots a_m}\} \ in \ \mathbb{R}^n \iff A\mathbf{x} = \mathbf{b}$ has at least 1 solution

2.3 Linear Independence

Definition: IF the only way to express $\mathbf{0}$ as a linear combination of $\mathbf{A}$ is the trivial solution $\mathbf{0}$, THEN the system is *linearly independent*. Nontrivial solutions imply *linear dependence*.
Thm: ($\mathbf{Ax} = \mathbf{0}$ has only the trivial solution) $\iff$ ($\{\mathbf{a_1 \ldots a_m}\}$ is linearly independent)

How to: Check if a sustem is linearly independent?
> Set the sytem equal to $\mathbf{0}$, and row reduce. The only solution should be $\mathbf{x} = \mathbf{0}$ (which will happen when there is a pivot in every column)

Thm: (*m* vectors in $\mathbf{R}^n$ are linearly independent) $\implies$ ($m \leq n$)

(b/c you can't have pivots in every column if there are too many columns)
(in other words, there are more variables than equations in the system)

Thm: For a given set of *m* vectors in $\mathbb{R}^n$:

a. ($span\{u_1 ... u_m\} = \mathbb{R}^n$) $\iff$ *(B has a pivot in every row)*

b. ($\{u_1 ... u_m\}$ are linearly independent) $\iff$ *(B has a pivot in every column)*

How to: Check if one vector lies in the span of others in a set?

a. Row reduce the matrix augmented with $\mathbf{0}$

b. If B does not have a pivot in every column $\implies$ the system is linearly dependent $\implies$ one of the vectors lies in the span of the others

General vs. Particular Solutions to Ax=0

Thm: $\mathbf{x_g} = \mathbf{x_p} + \mathbf{x_h}$, where $\mathbf{x_g}$ is the solution to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x_h}$ is the solution to the associated homogenous system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{x_p}$ is a particular solution to $\mathbf{x_g}$

Thm: For a given set of vectors $\{\mathbf{a_1} ... \mathbf{a_m}\}$ and $\mathbf{b}$ in $\mathbb{R}^n$:

a. ($\{\mathbf{a_1} ... \mathbf{a_m}\}$ are linearly independent) $\iff$ ($\mathbf{Ax} = \mathbf{b}$)

Ch 3

3.1 Linear Transformations

Definition: A transformation $T(\mathbf{x}) = \mathbf{Ax}$ is linear if both:

a. $T(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) \quad = \quad \mathbf{A}(\mathbf{u}) + \mathbf{A}(\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v})$

b. $T(r\mathbf{u}) = \mathbf{A}(r\mathbf{u}) \quad = \ r\mathbf{A}(\mathbf{u}) = rT(\mathbf{u})$

Thm: $T(\mathbf{x}) = \mathbf{Ax} \implies T$ is a linear transformation, where A is a nxm matrix, and T goes from $\mathbf{R^m}$ to $\mathbf{R^n}$

How to: Check if a given transformation is linear:

a. Convert the system into a matrix A

b. Plug in the vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for $\mathbf{u}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ for $\mathbf{v}$ to prove the general case *true*

c. Try the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ independently and check if the output fails to prove *false*

How to: Find $range(T)$, where $\mathbf{T}(\mathbf{x}) = \mathbf{Ax}$:

a. $range(T) = span(\{\mathbf{a_1} ... \mathbf{a_m}\})$

b. range is the set of linear combinations of the columns of A

How to: Check if a given vector $\mathbf{w}$ is in $range(\mathbf{T})$:

a. Make matrix of $[\mathbf{A} \mid \mathbf{w}]$ and solve

One-to-One vs Onto

Definition: A transformation is *one-to-one* when there's at most one input that maps to an output
Definition: A transformation is *onto* when no element in the codomain B is left ou\

Thm: Given $T : \mathbb{R}^m \to \mathbb{R}^n$ and $T(\mathbf{x}) = \mathbf{Ax}$, where $\mathbf{B}$ is A in row-echelon form:

1a. (T is one-to-one) $\iff$ (columns of A are linearly independent) $\iff$ (B has a pivot in every column)

1b. $n < m \implies$ T is not one-to-one (aka if output space is smaller than input space)

2a. (T is onto) $\iff$ (columns of A span the codomain $\mathbb{R}^n$ aka $range(T) = \mathbb{R}^n$) $\iff$ (B has a pivot in every row)

2b. $n > m \implies$ T is not onto (aka if output space is bigger than input space)

> "No matrix that goes from bigger space to smaller space can be one-to-one"
> "No matrix that goes from small space to bigger space can be onto"

Geometry of transformations

How to: Rotate a vector CCW by θ :

a. $T_r(\mathbf{x}) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \mathbf{x}$

How to: Shear to the right:

a. $T_r(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

3.2 Matrix Algebra

Properties of Elementary Matrices

a. $A(BC) = (AB)C$
b. $A(B + C) = AB + AC$
c. $(A + B)C = AC + BC$
d. $s(AB) = (sA)B = A(sB)$
e. $AI = IA = A$

Non-Properties of Nonzero Matrices

a. It is possible that $AB \neq BA$
b. $AB = 0$ does not imply that $A = 0$ and $B = 0$
c. $AC = BC$ does not imply that $A = B$ or $C = 0$ (unless A is invertible)

Transpose of a Matrix

a. $(A+B)^T = A^T + B^T$
b. $(sA)^T = sA^T$
c. $(AC)^T = C^T A^T$

3.3 Inverses

<u>Definition</u>: If T is a linear transformation, Then

a. T has an inverse $\implies$ $m = n$
b. If T is invertible, then T^{-1} is also a linear transformation

(T is invertible) $\iff$ (T is one-to-one AND onto)

<u>How to:</u> find an invertible matrix A^{-1}?

a. Augment matrix A with I_n , then row reduce until you get I_n augmented with A^{-1}
(aka $[A|I_n] \to [I_n|A^{-1}]$)

<u>Thm:</u> Elementary matrices are invertible

Properties of Inverses

a. $(A^{-1})^{-1} = A$
b. $(AB)^{-1} = B^{-1}A^{-1}$
c. $AC = AD \implies C = D$
d. $AC = 0_{nm} \implies C = 0_{nm}$

Ch 4

4.1 Subspaces

<u>Definition</u>: A subset of S is a subspace if all three conditions are true:

a. S contains $\mathbf{0}$ (S contains the origin)
b. If $\mathbf{u}$ and $\mathbf{v}$ are both in S, then $(\mathbf{u} + \mathbf{v})$ is in S (S is closed under addition)
c. If $r \in \mathbb{R}$, then $r\mathbf{u}$ is also in S (S is closed under scalar multiplication)

<u>Thm</u>: If S = $span\{\mathbf{u_1 \ldots u_m}\}$ in $\mathbb{R}^n$, then S is a subspace of $\mathbb{R}^n$

<u>How to:</u> Check if S is a subspace?

a. Check if $\mathbf{0}$ is in S, which it must be to be a subspace
b. Try to show S is generated by a set of vectors (See if it can be composed as a matrix of coefficients)

<u>Definition</u>: If $\mathbf{A}$ is a n x m matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ is called $null(\mathbf{A})$
(aka the null space is all linear combinations where $\mathbf{Ax} = \mathbf{0}$)
(aka the null space is the solution to the homogenous system)

Thm: If $\mathbf{A}$ is a n x m matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ forms a subspace of $\mathbb{R}^m$

(aka null space is a subspace)

Thm: Given $T: \mathbb{R}^m \to \mathbb{R}^n$ is a *linear* transformation:

a. $ker(T)$ is a subspace of the domain $\mathbb{R}^m$

b. $range(T)$ is a subspace of the codomain $\mathbb{R}^n$

> "The kernel is the set of vectors that are sent to $\{\mathbf{0}\}$ after applying T"

> "The range of T is the span after applying T"

How to: Find $ker(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?

a. $(T(\mathbf{x}) = \mathbf{Ax}) \implies (ker(T) = null(A))$, so solve for $\mathbf{Ax} = \mathbf{0}$, and $ker(T)$ is the span of that answer

How to: Find $null(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?

a. $range(T) = span(\mathbf{a_1 \dots a_m})$, so just delete any linearly dependent columns of $\mathbf{A}$ and that's your answer

Thm: (T is one-to-one) $\iff (ker(T) = \{\mathbf{0}\})$

4.2 Basis vectors

Definition: Set *B* $= \{\mathbf{u_1 \dots u_m}\}$ is a *basis* of subspace S iff:

a. *B* spans S

b. *B* is linearly independent

"To get to any point in S, you can take a linear combination of the basis vectors to get there"

How to: find a basis for S = span$\{\mathbf{u_1 \dots u_m}\}$?

Method 1 (Thm 4.10):

a. Create a matrix $\begin{pmatrix} \mathbf{u_1} \\ \dots \\ \mathbf{u_m} \end{pmatrix}$

b. Row reduce to B

c. The nonzero *rows* of B give a basis of S

Method 2 (Thm 4.11):

a. Create a matrix out of $\{\mathbf{u_1 \dots u_m}\}$

b. Row reduce to B. The pivot columns of B are linearly independent

(the other cols will depend on the pivot columns)

c. The *columns of A* corresponding to the *pivot columns of B* form a basis of S.

Dimension

Thm: If *S* is a subspace of $\mathbb{R}^n$, then every basis of *S* has the same number of vectors

Definition: If *S* is a subspace of $\mathbb{R}^n$, then the dimension of *S* is the number of vectors in any basis of *S*

Thm: If $U = \{\mathbf{u_1 ... u_m}\}$ is a set of m vectors in subspace S of dimension m, if U is *either* linearly independent or spans S, THEN U is a basis for S.

Unifying Theorem: Given $S = \{a_1 ... a_m\}, \{a_1 ... a_m\} \epsilon \mathbb{R}^n$, $A = [\mathbf{a_1 ... a_m}]$, and $T : \mathbb{R}^m \to \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

a. S spans $\mathbb{R}^n$
b. S is linearly independent
c. $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall \; b \; \epsilon \; \mathbb{R}^n$
d. T is onto
e. T is one-to-one
f. A is invertible
g. $ker\{T\} = \{\mathbf{0}\} \iff null(A) = \{\mathbf{0}\}$
h. S is a basis of $\mathbb{R}^n$

Ch 4.3 Row and Column Spaces

Definition: Given A is a n x m matrix:

a. $row(A)$ or row space is the subspace of $\mathbb{R}^{\mathrm{m}}$ spanned by *row vectors* of A
b. $col(A)$ or column space is the subspace of $\mathbb{R}^{\mathrm{n}}$ spanned by column vectors of A

How to:

Thm: Given matrix A and B in echelon form:

a. Nonzero rows of B form a basis for $row(A)$
b. The cols of A corresponding to pivot columns of B form a basis for $col(A)$

Thm: For any matrix A, the dimension of $row(A)$ equals the dimension of $col(A)$

Definition: $rank(A)$ is the dimension of $row(A)$ or $col(A)$

Rank-Nullity Thm: IF A is a n x m matrix, THEN $rank(A) + nullity(A) = m$
*Note: $ker(T)$ is $null(A)$, and $range(T)$ is $col(A)$

Thm: IF A is a n x m matrix, and $\mathbf{b}$ is a vector in $\mathbb{R}^{\mathrm{n}}$:

a. The system $A\mathbf{x} = \mathbf{b}$ is consistent $\iff$ $\mathbf{b}$ is in $col(A)$
b. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\iff$ $\mathbf{b}$ is in $col(A)$ *and* columns of A are linearly independent

Unifying Theorem: Given $S = \{a_1 ... a_m\}, \{a_1 ... a_m\} \epsilon \mathbb{R}^n$, $A = [\mathbf{a_1 ... a_m}]$, and $T : \mathbb{R}^m \to \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

a. S spans $\mathbb{R}^n$
b. S is linearly independent

c. $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall\ b\ \epsilon\ \mathbb{R}^n$

d. T is onto

e. T is one-to-one

f. A is invertible

g. $ker\{T\} = \{\mathbf{0}\} \iff null(A) = \{\mathbf{0}\}$

h. S is a basis of $\mathbb{R}^n$

i. $col(A)$ = $\mathbb{R}^{\mathrm{n}}$

j. $col(A)$ = $\mathbb{R}^{\mathrm{n}}$

k. $rank(A) = n$

Ch 4.4 Change of Basis

Definition: Suppose that B = $\{\mathbf{u_1 ... u_n}\}$ forms a basis of $\mathbb{R}^{\mathrm{n}}$, and if $\mathbf{y} = y_1\mathbf{u_1} + ... + y_n\mathbf{u_n}$:

THEN: The *coordinate vector of y w.r.t. *B** is $[\mathbf{y}]_B = \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$

"the coordinate vector contains the coeffs required to express y as a linear combination of vectors in basis B"

$$U \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} = y_1\mathbf{u_1} + ... + y_n\mathbf{u_n}$$

$\mathbf{y} = U[\mathbf{y}]_B$; where U is the *change of basis matrix* that transforms the coordinate vector wrt B back to the standard basis

(U is just the n x n matrix containing the basis vectors of set B: $[\mathbf{u_1 ... u_n}]$)

Thm: Let $\mathbf{x}$ be expressed wrt standard basis, and B = $\{\mathbf{u_1 ... u_n}\}$ be any basis for $\mathbb{R}^{\mathrm{n}}$:

If U = $[\mathbf{u_1 ... u_n}]$, then: $\mathbf{x} = U[\mathbf{x}]_B$ and $[\mathbf{x}]_B = U^{-1}\mathbf{x}$

How to: Move from one nonstandard basis to another?

If B_1 = $\{\mathbf{u_1 ... u_n}\}$ corresponds to U and B_2 = $\{\mathbf{v_1 ... v_n}\}$ corresponds to V, then:

$[\mathbf{x}]_{B_2} = V^{-1}U[\mathbf{x}]_{B_1}$

$[\mathbf{x}]_{B_1} = U^{-1}V[\mathbf{x}]_{B_2}$

> "To go from basis 1 to basis 2, apply U to go into standard basis, then apply V^{-1} to return to basis 2 land"