Linear Algebra Notes

Ishaan Dey | Spring 2020

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Ch 1

1.2 Intro stuff

Thm: Any system of linear eqs has either 0, exactly 1, or ∞ many solutions

How to: check how many solutions there are in a given system of eq:
If there last line is $0 = c$, then there is $no\ solution$
If not, are there free variables? If so, there are $inf\ many\ solutions$, otherwise $1\ solution$.

Thm: Homogenous systems ALWAYS are consistent, aka they have either 1 or inf many solutions.
This is b/c the trivial solution will always exist when $A\mathbf{x} = 0$

Ch 2

2.1 Linear Combinations

How to: find all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t. $c_1u_1 + c_2u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$?
> Augment the vector with $\begin{pmatrix} a \\ b \end{pmatrix}$ and row reduce, the remaining things in the augment become the scalars c_1, c_2

How to: show that a certain vector $\mathbf{b}$ cannot be obtained as a linear combination of some other vectors?
> Augment the matrix with b then row reduce, you'll find an inconsistent set with $0 = c$

2.2 Span

Definition of $span\{\mathbf{u_1 \dots u_m}\}$ is the set of ALL linear combinations

How to: See if some vector $\mathbf{v}$ is an element of $span\{\mathbf{u_1 \dots u_m}\}$?
> iff the linear system w/ $\mathbf{v}$ as the augment has a solution

Thm: IF $\mathbf{u}, \mathbf{v}$ are in $span\{\mathbf{u_1 ... u_m}\}$, THEN $\mathbf{u}+\mathbf{v}$ & $a\mathbf{u}$ are in that span
(linear combinations of the vectors are in that span)

How to: Find a vector $\mathbf{b}$ not in a given span of vectors?
> Set matrix equal (by augment) to some (a, b, c), and track what happens to it until matrix is row reduced.
> If the row is $[0\ 0\ 0|f(c)]$, then any vector that has $f(c) \neq 0$ is not in the span

Thm: ($span\{u_1 ... u_m\} = \mathbb{R}^n$) $\iff$ (B has a pivot in every row)

How to: check if a set of vectors span $\mathbb{R}^n$?
> Row reduce the matrix augmented with $[a\ b\ c]$
> Check if there is a row of 0s, which would imply that there could be a vector not in that span
> If there is a pivot in every row, there are either 1 or inf many ways to get to any point in $\mathbb{R}^n$ (depending on if there are free variables in any of the columns)

How to: Find what values of h in a n x m matrix allow it to span $\mathbb{R}^n$?
a. Row reduce the matrix, moving the h down as needed to the last row.
b. The vectors in A span $\mathbb{R}^n$ $\iff$ there is a trivial solution, which only occurs when $f(h) \neq 0$
c. The final answer should be all vectors with $\{h|h \neq 27\}$ or something

How to: find $span(\{\mathbf{a_1 ... a_m}\})$ (aka the columns of T)?

Thm: For a given set of *m* vectors in $\mathbf{R}^n$:
a. IF $m < n$, THEN the set does not span $\mathbf{R}^n$ (b/c theres no way to have a pivot in every row)
b. IF $m \geq n$, THEN the set could span $\mathbf{R}^n$ (depends on whether or not they are linearly independent)

Thm: $\mathbf{b}\ \epsilon\ span\{\mathbf{a_1 ... a_m}\}$ in $\mathbb{R}^n$ $\iff A\mathbf{x} = \mathbf{b}$ has at least 1 solution

2.3 Linear Independence

Definition: IF the only way to express $\mathbf{0}$ as a linear combination of $\mathbf{A}$ is the trivial solution $\mathbf{0}$, THEN the system is *linearly independent*. Nontrivial solutions imply *linear dependence*.
Thm: ($\mathbf{Ax} = \mathbf{0}$ has only the trivial solution) $\iff$ ($\{\mathbf{a_1 ... a_m}\}$ is linearly independent)

How to: Check if a system is linearly independent?
a. Set the sytem equal to $\mathbf{0}$, and row reduce. The only solution should be $\mathbf{x} = \mathbf{0}$ (which will happen when there is a pivot in every column)

Thm: (m vectors in $\mathbb{R}^n$ are linearly independent) $\implies$ ($m \leq n$)
(b/c you can't have pivots in every column if there are too many columns)
(in other words, there are more variables than equations in the system)

Thm: For a given set of m vectors in $\mathbb{R}^n$:

a. ($span\{u_1 \ldots u_m\} = \mathbb{R}^n$) $\iff$ (B has a pivot in every row)

b. ($\{u_1 \ldots u_m\}$ are linearly independent) $\iff$ (B has a pivot in every column)

How to: Check if one vector lies in the span of others in a set?

a. Row reduce the matrix augmented with $\mathbf{0}$

b. If B does not have a pivot in every column $\implies$ the system is linearly dependent $\implies$ one of the vectors lies in the span of the others

General vs. Particular Solutions to Ax=0

Thm: $\mathbf{x_g} = \mathbf{x_p} + \mathbf{x_h}$, where $\mathbf{x_g}$ is the solution to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x_h}$ is the solution to the associated homogenous system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{x_p}$ is a particular solution to $\mathbf{x_g}$

Thm: For a given set of vectors $\{\mathbf{a_1 \ldots a_m}\}$ and $\mathbf{b}$ in $\mathbb{R}^n$:

a. ($\{\mathbf{a_1 \ldots a_m}\}$ are linearly independent) $\iff$ ($\mathbf{Ax} = \mathbf{b}$)

Ch 3

3.1 Linear Transformations

Definition: A transformation $T(\mathbf{x}) = \mathbf{Ax}$ is linear if both:

a. $T(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) \ \ = \ \ \mathbf{A}(\mathbf{u}) + \mathbf{A}(\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v})$

b. $T(r\mathbf{u}) = \mathbf{A}(r\mathbf{u}) \ \ = \ r\mathbf{A}(\mathbf{u}) = rT(\mathbf{u})$

Thm: $T(\mathbf{x}) = \mathbf{Ax} \implies T$ is a linear transformation, where A is a nxm matrix, and T goes from $\mathbf{R^m}$ to $\mathbf{R^n}$

How to: Check if a given transformation is linear:

a. Convert the system into a matrix A

b. Plug in the vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for $\mathbf{u}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ for $\mathbf{v}$ to prove the general case *true*

c. Try the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ independently and check if the output fails to prove *false*

How to: Find $range(T)$, where $\mathbf{T}(\mathbf{x}) = \mathbf{Ax}$:

a. $range(T) = span(\{\mathbf{a_1 \ldots a_m}\})$

b. range is the set of linear combinations of the columns of A

How to: Check if a given vector $\mathbf{w}$ is in $range(\mathbf{T})$:

a. Make matrix of $[\mathbf{A} \ |\mathbf{w}]$ and solve

One-to-One vs Onto

Definition: A transformation is *one-to-one* when there's at most one input that maps to an output
Definition: A transformation is *onto* when no element in the codomain B is left out

Thm: Given $T : \mathbb{R}^m \to \mathbb{R}^n$ and $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where B is A in row-echelon form:
1a. (T is one-to-one) $\iff$ (columns of A are linearly *in*dependent) $\iff$ (B has a pivot in every column)
1b. $n < m \implies$ T is not one-to-one (aka if output space is smaller than input space)

2a. (T is onto) $\iff$ (columns of A span the codomain $\mathbb{R}^n$ aka $range(T) = \mathbb{R}^n$) $\iff$ (B has a pivot in every row)
2b. $n > m \implies$ T is not onto (aka if output space is bigger than input space)

> "No matrix that goes from bigger space to smaller space can be one-to-one"
> "No matrix that goes from small space to bigger space can be onto"

Geometry of transformations

How to: Rotate a vector CCW by θ :

a. $T_r(\mathbf{x}) = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \mathbf{x}$

How to: Shear to the right:

a. $T_r(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

How to: Find the transformation for a parallelogram from a unit square:

a. $\mathbf{a_1}$, or the first column of A defines where the first basis vector, $\mathbf{i}$, lands; same for $\mathbf{a_2}$

3.2 Matrix Algebra

Properties of Elementary Matrices

a. $A(BC) = (AB)C$
b. $A(B + C) = AB + AC$
c. $(A + B)C = AC + BC$
d. $s(AB) = (sA)B = A(sB)$
e. $AI = IA = A$

Non-Properties of Nonzero Matrices

a. It is possible that $AB \neq BA$ (Unless both are square)
b. $AB = 0$ does not imply that $A = 0$ or $B = 0$
c. $AC = BC$ does not imply that $A = B$ or $C = 0$ (unless A is invertible)

Transpose of a Matrix

a. $(A+B)^T = A^T + B^T$
b. $(sA)^T = sA^T$
c. $(AC)^T = C^T A^T$

3.3 Inverses

<u>Definition</u>: If T is a linear transformation, Then

a. T has an inverse $\implies$ $m = n$
b. If T is invertible, then T^{-1} is also a linear transformation

(T is invertible) $\iff$ (T is one-to-one AND onto)

<u>How to:</u> find an invertible matrix A^{-1}?

a. Augment matrix A with I_n , then row reduce until you get I_n augmented with A^{-1}
(aka $[A|I_n] \rightarrow [I_n|A^{-1}]$)

<u>Thm:</u> Elementary matrices are invertible

Properties of Inverses

a. $(A^{-1})^{-1} = A$
b. $(AB)^{-1} = B^{-1}A^{-1}$
c. $AC = AD \implies C = D$
d. $AC = 0_{nm} \implies C = 0_{nm}$

Ch 4

4.1 Subspaces

<u>Definition</u>: A subset of S is a subspace if all three conditions are true:

a. S contains $\mathbf{0}$ (S contains the origin)
b. If $\mathbf{u}$ and $\mathbf{v}$ are both in S, then $(\mathbf{u}+\mathbf{v})$ is in S (S is closed under addition)
c. If $r \in \mathbb{R}$, then $r\mathbf{u}$ is also in S (S is closed under scalar multiplication)

<u>Thm</u>: If S = $span\{\mathbf{u_1} \ldots \mathbf{u_m}\}$ in $\mathbb{R}^n$, then S is a subspace of $\mathbb{R}^n$

<u>How to:</u> Check if S is a subspace?

a. Check if $\mathbf{0}$ is in S, which it must be to be a subspace
b. Try to show S is generated by a set of vectors (See if it can be composed as a matrix of coefficients)

<u>Definition</u>: If $\mathbf{A}$ is a n x m matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ is called $null(\mathbf{A})$

(aka the null space is all linear combinations where $\mathbf{Ax} = \mathbf{0}$)
(aka the null space is the solution to the homogenous system)

Thm: If $\mathbf{A}$ is a n x m matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ forms a subspace of $\mathbb{R}^m$
(aka null space is a subspace)

Thm: Given $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a *linear* transformation:
a. $ker(T)$ is a subspace of the domain $\mathbb{R}^m$
b. $range(T)$ is a subspace of the *co*domain $\mathbb{R}^n$

> "The kernel is the set of vectors that are sent to $\{\mathbf{0}\}$ after applying T"
> "The range of T is the $span$ after applying T"

How to: Find $ker(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?
a. $(T(\mathbf{x}) = \mathbf{Ax}) \implies$ ($ker(T) = null(A)$), so solve for $\mathbf{Ax} = \mathbf{0}$, and $ker(T)$ is the span of that answer
How to: Find $range(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?
a. $range(T) = span(\mathbf{a_1} \ldots \mathbf{a_m})$, so just delete any linearly *dependent* columns of $\mathbf{A}$ and that's your answer

Thm: (T is one-to-one) $\iff$ ($ker(T) = \{\mathbf{0}\}$)

4.2 Basis vectors

Definition: Set $\mathcal{B} = \{\mathbf{u_1} \ldots \mathbf{u_m}\}$ is a *basis* of subspace S iff:
a. $\mathcal{B}$ spans S
b. $\mathcal{B}$ is linearly independent
"To get to any point in S, you can take a linear combination of the basis vectors to get there"

How to: find a basis for S = $span\{\mathbf{u_1} \ldots \mathbf{u_m}\}$?
Method 1 (Thm 4.10):
a. Create a matrix $\begin{pmatrix} \mathbf{u_1} \\ \ldots \\ \mathbf{u_m} \end{pmatrix}$
b. Row reduce to B
c. The nonzero *rows* of B give a basis of S
Method 2 (Thm 4.11):
a. Create a matrix out of $\{\mathbf{u_1} \ldots \mathbf{u_m}\}$
b. Row reduce to B. The pivot columns of B are linearly independent
(the other cols are dependent on the pivot columns)
c. The *columns of* A corresponding to the *pivot columns of* B form a basis of S.

Dimension

Thm: If S is a subspace of $\mathbb{R}^n$, then every basis of S has the same number of vectors

Definition: If S is a subspace of $\mathbb{R}^n$, then the *dimension* of S is the number of vectors in any basis of S

Thm: $\mathcal{U}$ = $\{\mathbf{u_1} \dots \mathbf{u_m}\}$ is a set of m vectors in subspace S of dimension m.
IF $\mathcal{U}$ is *either* linearly independent or spans S, THEN $\mathcal{U}$ is a basis for S.

How to: expand a set of vectors to become a basis of $\mathbb{R}^n$:

a. Append on all the unit vectors of $\mathbb{R}$, $\mathbf{e_i}$, that you have and then row reduce down to B.
b. The original columns of A that correspond w/ the pivots of B become the basis vectors for $\mathbb{R}^n$

Unifying Theorem: Given S = $\{a_1 \dots a_m\}, \{a_1 \dots a_m\} \epsilon \mathbb{R}^n, A = [\mathbf{a_1} \dots \mathbf{a_m}]$, and $T : \mathbb{R}^m \to \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

a. S spans $\mathbb{R}^n$
b. S is linearly independent
c. $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall\ b\ \epsilon\ \mathbb{R}^n$
d. T is onto
e. T is one-to-one
f. A is invertible
g. $ker\{T\} = \{\mathbf{0}\} \iff null(A) = \{\mathbf{0}\}$
h. S is a basis of $\mathbb{R}^n$

4.3 Row and Column Spaces

Definition: Row vectors of A come from viewing A as a set of rows; Column vectors of A come from viewing A as a set of columns.

Definition: Given A is a n x m matrix:

a. $row(A)$ or row space is the subspace of $\mathbb{R}^{\mathrm{m}}$ spanned by *row vectors* of A
b. $col(A)$ or column space is the subspace of $\mathbb{R}^{\mathrm{n}}$ spanned by *column* vectors of A

Thm: Given matrix A and B in echelon form:

a. Nonzero rows of B form a basis for $row(A)$
(The redundant rows get killed off by row reduction)
b. The cols of A corresponding to pivot columns of B form a basis for $col(A)$
(The pivot columns are linearly independent $\implies$ they form the column space of A)

Thm: For any matrix A, the dimension of $row(A)$ equals the dimension of $col(A)$
(aka the number of basis vectors needed to define $row(A)$ the number needed to define $col(A)$)

Definition: $rank(A)$ is the dimension of $row(A)$ or $col(A)$

Rank-Nullity Thm: IF A is a n x m matrix, THEN $rank(A) + nullity(A) = m$ (# of cols)
($nullity(A)$ is the number of free variables in the system $A\mathbf{x} = 0$)
*Note: $ker(T)$ is $null(A)$, and $range(T)$ is $col(A)$

Thm: IF A is a n x m matrix, and $\mathbf{b}$ is a vector in $\mathbb{R}^n$:

a. The system $A\mathbf{x} = \mathbf{b}$ is consistent (one or many solutions) $\iff$ $\mathbf{b}$ is in $col(A)$

b. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution (exactly 1 solution)

$\iff$ $\mathbf{b}$ is in $col(A)$ AND columns of A are linearly independent

Unifying Theorem: Given S = $\{a_1 ... a_m\}, \{a_1 ... a_m\} \epsilon \mathbb{R}^n, A = [\mathbf{a_1 ... a_m}]$, and $T: \mathbb{R}^m \to \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

a. S spans $\mathbb{R}^n$

b. S is linearly independent

c. $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall\ b\ \epsilon\ \mathbb{R}^n$

d. T is onto

e. T is one-to-one

f. A is invertible

g. $ker\{T\} = \{\mathbf{0}\} \iff null(A) = \{\mathbf{0}\}$

h. S is a basis of $\mathbb{R}^n$

i. $col(A)$ = $\mathbb{R}^n$

j. $col(A)$ = $\mathbb{R}^n$

k. $rank(A) = n$

4.4 Change of Basis

Definition: Suppose that B = $\{\mathbf{u_1 ... u_n}\}$ forms a basis of $\mathbb{R}^n$, and if $\mathbf{y} = y_1\mathbf{u_1} + ... + y_n\mathbf{u_n}$:

Then the *coordinate vector* of y w.r.t. B is $[\mathbf{y}]_B = \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$

"the coordinate vector contains the coeffs required to express y as a linear combination of vectors in basis B"

$$U \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} = y_1\mathbf{u_1} + ... + y_n\mathbf{u_n}$$

$\mathbf{y} = U[\mathbf{y}]_B$; where U is the *change of basis matrix* that transforms the coordinate vector wrt B back to the standard basis

(U is just the n x n matrix containing the basis vectors of set B: $[\mathbf{u_1 ... u_n}]$)

Thm: Let $\mathbf{x}$ be expressed wrt standard basis, and B = $\{\mathbf{u_1 ... u_n}\}$ be any basis for $\mathbb{R}^n$:

If U = $[\mathbf{u_1 ... u_n}]$, then: $\mathbf{x} = U[\mathbf{x}]_B$ and $[\mathbf{x}]_B = U^{-1}\mathbf{x}$

"U takes us from a vector described with a weird basis into our standard definition"

"U^{-1} tells us how to define a vector w/ standard definition into back into weird basis land"

How to: Move from one nonstandard basis to another?

If B_1 = $\{\mathbf{u_1} \ldots \mathbf{u_n}\}$ corresponds to U and B_2 = $\{\mathbf{v_1} \ldots \mathbf{v_n}\}$ corresponds to V, then:

$[\mathbf{x}]_{B_2} = V^{-1}U[\mathbf{x}]_{B_1}$

$[\mathbf{x}]_{B_1} = U^{-1}V[\mathbf{x}]_{B_2}$

"To go from basis 1$\rightarrow$ basis 2, apply U to go from basis 1 land into standard basis, then apply V^{-1} to go to basis 2 land"

Ch 5

5.1 Determinant

Definition: Given A is a nxn matrix, each position is defined as a_{ij}:

a. cofactor $C_{ij} = (-1)^{i+j} * det(M_{ij})$, where M_{ij} is the rest of the matrix that doesn't include the ith row & jth column (M_{ij} is called the Minor of i, j)

b. $det(A) = a_{11}C_{ij} + \ldots + a_{1n}C_{1n}$: for each element in the *first* column, multiply each times its cofactor.

c. In general, you can expand down any row or column and apply the same formula (Thm 5.8)

Matrix of signs of cofactors: $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$; where $C_{ij} = (-1)^{i+j}$

Thm 5.5: $det(I_n) = 1$

Thm 5.6: A is invertible $\iff$ $det(A) \neq 0$

> This would invoke unifying theorem, so you could say things like: "cols of A form a basis of $\mathbb{R}^n$ " etc.

Thm 5.9: If A is triangular matrix, then $det(A)$ is product of terms along the diagonal

Thm 5.11: If A is a square matrix, then:

a. If A has a row or column of zeros, then $det(A) = 0$

b. If A has two identical rows or columns, then $det(A) = 0$

5.2 Properties of Determiniants

Thm 5.13:

a. Swap 2 rows of A $\implies$ $-det(A)$

b. Multiply row of A by c $\implies$ $c * det(A)$

c. Add multiple of one row of A to another $\implies$ $det(A)$

Thm 5.10: If A is a square matrix, then $det(A^T) = det(A)$

If A is invertible, $det(A^{-1}) = \frac{1}{det(A)}$

Thm 5.12: $\det(AB) = det(A)det(B)$ if A and B are both n x n matrices

Implication: $det(PDP^{-1}) = det(D) \implies$ product of eigenvalues

Cramer's Rule (Thm 5.17): Let A be invertible; to find unique solution, $\mathbf{x}$, to $A\mathbf{x} = \mathbf{b}$:

$\mathbf{x}_i = \frac{det(A_i)}{det(A)}$, where A_i is the matrix A but with the ith column replaced by $\mathbf{b}$

How To: Find the unique solution to $A\mathbf{x} = \mathbf{b}$:

> Apply Cramer's Rule for each column in A

Thm 5.18: If A is invertible, then:

$A^{-1} = \frac{adj(A)}{det(A)}$, where $\mathrm{adj}(A) = C^T$, the transpose of the cofactor matrix of A

Thm: $A(\mathrm{adj}(A)) = \det(A) * I_n$

Thm 5.20: Let D be a region w finite area in $\mathbf{R}^2$, $T(\mathbf{x}) = A\mathbf{x}$, and $T(D)$ is the image of D under T, then:

$\mathrm{area}(T(D)) = |\det(A)| * \mathrm{area}(D)$

Thm 5.21: Same thing as Thm 5.20 but w/ volume in $\mathbf{R}^3$

Ch 6

6.1 Eigenvalues and Eigenvectors

Definition: Let A be a n x n matrix; $\mathbf{u}$ is an eigen*vector* of A if there exists a scalar λ s.t.

$A\mathbf{u} = \lambda\mathbf{u}$; where λ is an eigen*value* of A

Intuition: If A is a transofrmation that changes the basis vectors of our subspace, then we expect most vectors to also be transformed in some manner (sheared, etc). Eigen*vectors* are the specific vectors that remain parallel after the transformation, and the degree to which its scaled is called the eigen*value*

Thm 6.2: If $\mathbf{u}$ is an eigenvector of A associated with λ, then $c\mathbf{u}$ is also associated w/ λ

How to: Check if a given vector $\mathbf{u}$ is a eigenvector of A?

a. Multiply $A * \mathbf{u}$, and the result should be some multiple $c\mathbf{u}$

How to: Find an eigenvector if you know the eigenvalues for a given nxn matrix A:

For a specific eigenvector, say $\lambda = 6$: $(A\mathbf{u} - 6I_n)\mathbf{u} = \mathbf{0}$

a. Subtract off 6 from each value along the *diagonal* of A

b. Then set equal to $\mathbf{0}$ and solve out

c. You'll get something like $s_1 \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, which means any vector $\mathbf{u}$ that is a linear combination of the

those two is an eigenvector (this is called the eigenspace btw)

<u>Thm 6.3:</u> If A is a nxn matrix w/ eigenvalue λ, and S is the set of all eigenvectors associated w/ λ, including $\mathbf{0}$:

Then S is a subspace of $\mathbf{R}^n$

<u>Definition:</u> Eigenspace of A is the subspace of all eigenvectors associated w/ λ together w/ $\mathbf{0}$

<u>Thm 6.5:</u> λ is an eigenvalue of $A \iff det(A - \lambda I_n) = 0$

<u>How to:</u> find eigenvalues of A:

a. Subtract off λ from each diagonal, and take the determinant of that matrix

The result is called the characteristic polynomial btw

b. Set that equal to 0, and solve for lambda

<u>Definition:</u> Multiplicity is the number of times a root of the characteristic equation is repeated, aka multiplicity is equal to its factor's exponent

<u>Thm 6.6:</u> Let A be a square matrix w/ eigenvalue λ: Then $\dim$ of associated eigenspace is $\leq$ multiplicity of λ

<u>Thm 6.7</u>: Unifying thm: $\lambda = 0$ is not an eigenvalue of A

In other words, if $\lambda = 0$ *is* an eigenvalue of A, then one of the columns is linearly dependent or something

6.2 Diagonalization

<u>Definition</u>: A is diagonalizable if there exists nxn matrices D and P, s.t. D is diagonal and P is invertible:

=> $A = PDP^{-1}$

<u>How to:</u> Find P and D to diagonalize matrix A:

a. Find each eigenvalue using $det(A - \lambda I_n) = 0$

b. Find the associateed eigenvector by solving the homogenous system $(A - I_n)\mathbf{u} = \mathbf{0}$

c. $P = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$; $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

<u>Thm</u>:

$det(A) = det(PDP^{-1}) = det(D) =$ product of λ_is

If A is triangular, eigenvalues lie along the diagonal

Thm 6.9: A is diagonalizable $\iff$ A has eigenvectors that form a basis for $\mathbf{R}^n$

$\iff$ A has n linearly *independent* eigenvectors

Thm 6.10: If $\{\lambda_1, ..., \lambda_k\}$ are distinct eigenvalues of A,

then any set of associated eigenvectors $\{\mathbf{u_1}, ..., \mathbf{u}_k\}$ are *linearly independent*

> Means that vectors from *distinct* eigenspaces are linearly independent

> This also means that A can be not invertible and yet diagonalizable

Thm 6.11: If A has only real eigenvalues, then:

A is diagonalizable $\iff$ $\dim$ of each eigenspace *equals* the multiplicity of corresponding eigenvalue

> Means that you'll know that a matrix isn't diagonalizable (for example) when you see that there's one basis vector of an eigenspace when the root had a power of 2

Thm 6.12: If A is a n x n matrix with n *distinct* real eigenvalues, then A is diagonalizable

How to: Get kth power of matrix: $A^k = PD^kP^{-1}$

Ch 7

7.1 Vector Spaces + Subspaces

Definition: Vector space is a set V of vectors, together w/ operations of addition and scalar multiplication:

a. V is closed under addition: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V, then so is $\mathbf{v}_1 + \mathbf{v}_2$

b. V is closed under *scalar* multiplication: If c is a real scalar and $\mathbf{v}$ is in V, then so is $c\mathbf{v}$

c. There exists a zero vector $\mathbf{0}$ in V s.t. $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$ in V

d. There exists an additive inverse $-\mathbf{v}$ in V s.t. $-\mathbf{v} + \mathbf{v} = \mathbf{0}$ for all $\mathbf{v}$ in V

e. $\forall\ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$:

a. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$

b. $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$

c. $c_1(\mathbf{v}_1 + \mathbf{v}_2) = c_1\mathbf{v}_1 + c_1\mathbf{v}_2$

d. $(c_1 + c_2)\mathbf{v}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_1$

e. $(c_1c_2)\mathbf{v}_1 = c_1(c_2\mathbf{v}_1)$

f. $1 * \mathbf{v}_1 = \mathbf{v}_1$

Thm 7.2: Let $\mathbf{v}$ be in vector space V:

a. $\mathbf{0}$ is the zero vector in V $\implies$ $\mathbf{v} + \mathbf{0} = \mathbf{v}$

b. $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$ $\implies$ $-\mathbf{v} + \mathbf{v} = \mathbf{0}$

c. $\mathbf{v}$ has a unique additive inverse $-\mathbf{v}$

d. Zero vector $\mathbf{0}$ is unique

e. $0 * \mathbf{v} = \mathbf{0}$

f. $(-1) * \mathbf{v} = -\mathbf{v}$

Definition: A subset S of a vector space V is a subspace if S:

a. S contains the zero vector, $\mathbf{0}$

b. If $\mathbf{u}, \mathbf{v}$ is in S, then $\mathbf{u} + \mathbf{v}$ is also in S

c. If c is a scalar and $\mathbf{v}$ is in S, then $c\mathbf{v}$ is also in S

7.2 Span and Linear Independence

Definition: span of set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is set of all linear combinations of the form: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$

Thm 7.6: Suppose that $\mathcal{V}$ is a subset of vector space V, and let $S = \mathrm{span}(\mathcal{V})$ then S is a subspace of $\mathcal{V}$

Definition: Set $\mathcal{V}$ = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly *in*dependent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$ has only the trivial solution

Thm 7.8 Set $\mathcal{V}$ = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly *de*pendent $\iff$ at least one vector is in the span of the others

Thm 7.9 If set $\mathcal{V}$ = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a subset of V, then:

a. $\mathcal{V}$ is *linearly independent* $\iff$ $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \mathbf{v}$ has *at most* one solution for each $\mathbf{v}$ in V

b. $\mathcal{V}$ *spans* V $\iff$ $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} = \mathbf{v}$ has *at least* one solution for each $\mathbf{v}$ in V

How to: Check if a system is linearly independent, i.e. given $a(1) + b(sin(x)) + c(cos(x)) = 0$?

a. To generate a system of equations, take the first and second derivatives wrt. x to generate 3 equations

b. Check if the determinant $\neq 0$, $\implies$ the matrix is invertible, $\implies$ the system $A\mathbf{x} = \mathbf{0}$ has a unique solution which is the trivial solution.

Application: Decomposition of signal frequencies:

I.e. Given $1, sin(x), cos(x), sin(2x), cos(2x), \ldots, sin(nx), cos(nx)$, is this system linearly independent?

Definition: $\mathbb{P}^n$ is the set of all polynomials of degree $\leq n$

a. Standard bases: $\{1, x, x^2, \ldots, x^n\} \implies dim(\mathbb{P}^n) = n + 1$

b. $\dim \mathbb{P} = \infty$, enough to show that there is no finite basis

b1. $\mathbb{P} \subseteq C(\mathbb{R}) \implies \dim C(\mathbb{R}) = \infty$

How to: check if a given set of polynomials, i.e. $\{x+1, x^2-1, x^2+x+1\}$, is a basis of $\mathbb{P}^2$?

a. Check linear independence: Assign a scalar c_i to each polynomial; each column represents c_i, and each element in each *column* corresponds to a different degree of that polynomial, i.e. x^2

$$c_1(x+1) + c_2(x^2-1) + c_3(x^2+x+1) = \mathbf{0}$$
$$\implies (c_1 - c_2 + c_3) * 1 + (c_1 + c_3) * x + (c_2 + c_3) * x^2 = 0$$
$$\implies \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

a1. Check that determinant $\neq 0$ which would imply unique trivial solution, which implies linear independence

b. Could also check that they $\text{span } \mathbb{P}^2$

$$\implies \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

c. Could also check matrix of derivatives: if there exists at least a single value of x that makes $det(A) \neq 0, \implies$ linear independence.

$$\begin{vmatrix} x+1 & x^2-1 & x^2+x+1 \\ 1 & 2x & 2x+1 \\ 0 & 2 & 2 \end{vmatrix}$$

7.3 Basis and Dimension

Definition: $\mathcal{V}$ is a basis of V if $\mathcal{V}$ is linearly independent *and* spans V

Thm 7.11 Set $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis of vector space V iff:

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \mathbf{v}$ has a *unique* solution for every $\mathbf{v}$ in V

Thm 7.12 If $\mathcal{V}_1$ and $\mathcal{V}_2$ are both bases of vector space V, then $\mathcal{V}_1$ and $\mathcal{V}_2$ have the same number of elements

Definition Dimension of V is equal to the number of vectors in any basis of V. *It's possible for dimension to be infinte if the basis of V is infinitely long*

Thm 7.14 Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a subset of *nontrivial, finite, dimensional* vector space V:

a. If $\mathcal{V}$ spans V, then either $\mathcal{V}$ is a basis of V or vectors can be removed from $\mathcal{V}$ to form a basis of V

b. If $\mathcal{V}$ is linearly independent, then either $\mathcal{V}$ is a basis of V or vectors can be added to $\mathcal{V}$ to form a basis of V

Thm 7.15: If V_1 is a subspace of V_2, then $\dim(V_1) \leq \dim(V_2)$

Thm 7.16: Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a subset of V with $\dim(V) = n$.

a. If $m < n$, then $\mathcal{V}$ does not span V

b. If $m > n$, then $\mathcal{V}$ is not linearly independent

Thm 7.17: Let $\mathcal{V}$ = $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a subset of V with $\dim(V) = m$.

a. If $\mathcal{V}$ is linearly independent *or* spans V, then $\mathcal{V}$ is a basis for V

How to: find dim $\text{span}\{T_1, T_2, T_3\}$, given something like $T_1\mathbf{x} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 \end{pmatrix}$, etc.?

a. Check linear independence of T_1, T_2, T_3 (Must hold $\forall$ x)

a1. Assign each T_i a scalar c_i, each of which becomes its own column; each element in that matrix from T_i has its own row on column i

a2. Row reduce: if you get a free variable in a certain column, say s_1, then the corresponding T is a linear comination of the others

b. Check how many Ts are needed to maintain linear independence, that is dim span

Ch 8

8.1 Dot Products and Orthogonal Sets

Definition If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbf{R}^n$: $\quad \mathbf{u} \cdot \mathbf{v} = u_1v_1 + \cdots + u_nv_n = \mathbf{u}^T\mathbf{v}$

Thm 8.2: Properties of dot products:

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq= 0$

e. $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$

Definition: Norm or length of $\mathbf{x} = ||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$\implies ||c\mathbf{x}|| = |c|\ ||\mathbf{x}||$

Definition: Unit vector in direction of $\mathbf{x}$:

$||\frac{\mathbf{x}}{||\mathbf{x}||}|| = 1$

Definition: The distance between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ is $||\mathbf{u} - \mathbf{v}||$

a. Subtract $v_!$ from u_1, etc., then take the magnitude of that resulting vector

b. This is really the distance between the endpoints of the rays

Definition: Vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$

Pythagorean Thm 8.6: $\quad ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \iff \mathbf{u} \cdot \mathbf{v} = 0$

a. Think middle school geometry: pythagorean thm applies iff right triangle

Cauchy-Schwarz Inequality Thm 8.7: $|\mathbf{u}\cdot\mathbf{v}| \le ||\mathbf{u}||\ ||\mathbf{v}||$

a. Define $cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$

b. $\implies -||\mathbf{u}||\ ||\mathbf{v}|| \le \mathbf{u}\cdot\mathbf{v} \le ||\mathbf{u}||\ ||\mathbf{v}||$

Triangle Inequality Thm 8.8 $||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$

a. This says that the line segment between two vectors is the shortest way between those two points

Orthogonal Subspaces

Definition: Vector $\mathbf{u}$ is orthogonal to subspace S if $\mathbf{u}\cdot\mathbf{v} = 0\ \forall\ \mathbf{s} \in S$. The set of all such vectors $\mathbf{u}$ is called the orthogonal complement of S and is denoted by $S^{\perp}$

$$S^{\perp} = \{\mathbf{u}\ \in\ \mathbb{R}^n \mid \mathbf{u}\cdot\mathbf{s} = 0\ \forall\ \mathbf{s} \in S\}$$

Thm 8.10: S ~ subspace of $\mathbb{R}^n \implies S^{\perp}$ ~ subspace of $\mathbb{R}^n$

Thm 8.11: Let $\mathcal{V} = \{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$ be a basis for subspace S and $\mathbf{u}$ be a vector.

Then $\mathbf{u}\cdot\mathbf{s}_1 = 0, \mathbf{u}\cdot\mathbf{s}_2 = 0, \ldots, \mathbf{u}\cdot\mathbf{s}_k = 0 \iff \mathbf{u} \in S^{\perp}$

How to: find a basis for $S^{\perp}$, given $S = span\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}\}$: (think of this as a 2D plane in $\mathbb{R}^3$)

a. Make the basis vectors for S into rows of a matrix

b. Because $A\mathbf{u} = \begin{bmatrix}\mathbf{s}_1\cdot\mathbf{u}\\ \mathbf{s}_2\cdot\mathbf{u}\end{bmatrix}$, $\mathbf{u}$ is in $S^{\perp}$ only when $A\mathbf{u} = \mathbf{0}$

c. Solve out $A\mathbf{u} = \mathbf{0}$, and find something like $\mathbf{u} = t_1\begin{bmatrix}2\\-3\\1\end{bmatrix}$

d. $S^{\perp} = \text{span}\{\begin{bmatrix}2\\-3\\1\end{bmatrix}\}$

Definition: A set of vectors $\mathcal{V}$ in $\mathbb{R}^n$ form an orthogonal set if $\mathbf{v}_i\cdot\mathbf{v}_j = 0,\ \forall\ \mathbf{v}_i$ & $\mathbf{v}_j$ in $\mathcal{V}$, with $i \ne j$

a. All of the vectors in the set must be orthogonal to each other

How to: check if a set of vectors is orthogonal?

a. Take the dot product of each possible pair, and if *all* are 0, then set is orthogonal

Thm 8.13: An orthogonal set of nonzero vectors is linearly *in*dependent

Definition: Orthogonal basis of a subspace is a *basis* which is also an *orthogonal* set

Thm 8:14: If S is a subspace w/ orthogonal basis $= \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$,

Then $\forall\ \mathbf{s} \in S$: $\mathbf{s} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$, where $c_i = \frac{\mathbf{s}\cdot\mathbf{v}_i}{||\mathbf{v}_i||^2}$

How to: express $\mathbf{s}$ as a linear combination of a given orthogonal vector basis:

a. Use Thm 8.14 to get a c_i for each vector in the basis, and the result will be something like $\mathbf{s} = -2\mathbf{v}_1 + \mathbf{v}_2$

8.2 Projection

Definition: Projection of $\mathbf{u}$ onto $\mathbf{v}$: $\mathrm{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$

Thm 8.16: For a projection when $\mathbf{v}$ is nonzero:

a. $\mathrm{proj}_v \mathbf{u}$ lies on $span\{\mathbf{v}\}$

b. $\mathbf{u} - \mathrm{proj}_v \mathbf{u}$ is orthogonal to $\mathbf{u}$

c. if $\mathbf{u}$ is in $\mathrm{span}\{\mathbf{v}\}$, then $\mathrm{proj}_v \mathbf{u} = \mathbf{u}$

d. $\mathrm{proj}_v \mathbf{u}$ = $\mathrm{proj}_{cv} \mathbf{u}$

Definition: Let S be a nonzero subspace w/ orthogonal basis $= \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. For some vector $\mathbf{u}$:

$\mathrm{proj}_s \mathbf{u} = \mathrm{proj}_{\mathbf{v}_1} \mathbf{u} \ + \ \ldots \ + \ \mathrm{proj}_{\mathbf{v}_k} \mathbf{u}$

$\implies$ if basis is not yet orthogonal, you MUST first orthogonalize it

Thm 8.18: For a projection on subspace when S is nonzero:

a. $\mathrm{proj}_s \mathbf{u}$ lies in S

b. $\mathbf{u} - \mathrm{proj}_s \mathbf{u}$ is orthogonal to S

c. If $\mathbf{u}$ is in S, then $\mathrm{proj}_S \mathbf{u} = \mathbf{u}$

d. $\mathrm{proj}_s \mathbf{u}$ is independent of the choice of the orthogonal basis of S

Gram-Schmidt Processes

Thm 8.19: Let S be a subspace with basis $\{\mathbf{s}_1, \ldots, \mathbf{s}_k\}$. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is defined as:

$\mathbf{v}_1 = \mathbf{s}_1$

$\mathbf{v}_2 = \mathbf{s}_2 - \mathrm{proj}_{\mathbf{v}_1} \mathbf{s}_2$

$\mathbf{v}_3 = \mathbf{s}_3 - \mathrm{proj}_{\mathbf{v}_1} \mathbf{s}_3 - \mathrm{proj}_{\mathbf{v}_2} \mathbf{s}_3$

$\mathbf{v}_k = \mathbf{s}_k - \mathrm{proj}_{\mathbf{v}_1} \mathbf{s}_k - \mathrm{proj}_{\mathbf{v}_2} \mathbf{s}_k - \ldots - \mathrm{proj}_{\mathbf{v}_{k-1}} \mathbf{s}_k$

(The pattern here is that the row defining the ith vector uses the original basis vector $\mathbf{s}_i$ on that row only.

The vertical columns are unique to each orthogonal vector, that is to say, column 1 is always nothing, column 2 is $\mathbf{v}_1$, and column k is $\mathbf{v}_{k-1}$ $\leftarrow$ you're recursively defining the next vector)

Intuition: The basis vectors of S are already linearly independent, so you know that none of them are parallel to the others. We start off with the first basis vector of S becoming the first orthogonal basis vector $\mathbf{v}_1$. We can find a basis vector orthogonal to that by taking the second basis vector, and subtracting off it's projection onto the previous vector to define the new orthogonal basis vector $\mathbf{v}_2$ (because we know that difference will be orthogonal) . We can keep going for all the basis vectors, such that $span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = span\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, then not only are the $\mathbf{v}$ vectors defining the same subspace, but they also are orthogonal to each other.

Definition: A set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is orthonormal IF the set is orthogonal AND $||\mathbf{w}_j|| = 1 \; \forall \; j = 1, 2, \ldots, k$

$\mathbf{w}_j = \frac{1}{||\mathbf{v}_j||}\mathbf{v}_j$ for $j = 1, 2, \ldots, k$

How to: Find a set of orthonormal basis vectors from a given set of basis vectors i.e. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

a. Orthogonalize $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ using GS process s

b. Normalize to obtain an orthonormal basis of $span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

8.3 Diagonalizing Symmetric Matrices

Thm 8.21 If A ~ symmetric matrix (when $A = A^T$) $\implies$ then the eigenvectors a ssociated w/ distinct eigenvalues are orthogonal

Definition: P ~ square, nxn with orthonormal columns is called an orthogonal matrix

*Cols must be ortho*normal*, not just orthogonal

$\implies P^T P = I_n$

Thm 8.23: If P ~ nxn, orthonormal, $\implies$ then $P^{-1} = P^T$

How to: Check if a given matrix A is orthogonal:

a. Check that dot products of the columns are 0

b. Check that the norm of each column is 1

Definition A ~ *orthogonally* diagonalizable if there exists an *orthogonal* matrix P and a diagonal matrix D s.t. $A = PDP^{-1} = PDP^T$

P represents the *change of basis matrix* from the standard basis into the one defined by the eigenvectors

Spectral Thm 8.26 A ~ orthogonally diagonalizable $\iff$ A ~ symmetric

a. $\implies$ All eigenvalues of a symmetric matrix A are real

b. Each eigenspace of a symmetric matrix A has dimension *equal* to the multiplicity of the associated eigenvalue

How to: Orthogonally diagonalize a matrix A:

a. Get characteristic equation of $det(A - \lambda I_n)$, find eigenvalues

b. Find eigenvectors for each eigenvalue (these are already orthogonal to the other eigenvectors, but may not necessarily be orthogonal within the same eigenspace)

c. Apply Gram-Schmidt process to orthogonalize $\mathbf{u}_1, \mathbf{u}_2$ in a common eigenspace (do this step for every eigenspace)

d. Normalize every orthogonal set of vectors (even the ones from $\dim 1$)

e. Construct D normally, construct P using the corresponding orthonormal eigenvectors, and write inverse of P as P^T. (Make sure to position eigenvectors s.t. P is symmetrical)

Thm 8.27: If A ~ real matrix, $\implies$ then A^TA has nonnegative eigenvalues

Note: this works on 2x3 matrices, for example

A^TA is always symmetric $\implies$ all the stuff from Ch 8 holds, plus the fact that the eigenvalues are nonnegative

Ch 10

10.1 Inner Products

Definition: Inner product on V is a function that takes two *vectors* in V as input and produces a *scalar* as output, and is denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. It must satisfy:

a. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

b. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

c. $\langle c\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{cv} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

d. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} = 0 \rangle$ only when $\mathbf{u} = 0$

Dot product is an inner product on $\mathbb{R}^n$, it could also be "weighted" by some third param t_i for every u_i and v_i in $\mathbf{u} \cdot \mathbf{v}$

Definition: $\mathbf{u} \text{ and } \mathbf{v}$ are orthogonal $\iff$ $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Definition: The norm of $\mathbf{v}$: $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

a. Distance between vectors: $||\mathbf{u} - \mathbf{v}||$

b. Norm of $\mathbf{cv} = |c| \; ||\mathbf{u}||$

Pythagorean Thm 10.4: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $\iff$ $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$

Definition: Projection of $\mathbf{u}$ onto $\mathbf{v}$: $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}\mathbf{v}$

Thm 10.6: For a projection when $\mathbf{v} \text{ and } c$ is nonzero:

a. $\text{proj}_v\mathbf{u}$ lies on $span\{\mathbf{v}\}$

b. $\mathbf{u} - \text{proj}_v\mathbf{u}$ is orthogonal to $\mathbf{v}$

c. if $\mathbf{u}$ is in $\text{span}\{\mathbf{v}\}$, then $\text{proj}_v\mathbf{u} = \mathbf{u}$

d. $\text{proj}_v\mathbf{u}$ = $\text{proj}_{cv}\mathbf{u}$

Cauchy-Schwarz Inequality Thm 10.7: $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \; ||\mathbf{v}||$

Triangle Inequality Thm 10.8 $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$

10.2 Gram-Schmidt Revisted

<u>Definition:</u> Vectors $\{\mathbf{v}_1, \ldots, \mathbf{v_2}\}$ in an inner product space V form an orthogonal set if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$

<u>Definition:</u> If $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v_2}\}$ is an orthogonal set of nonzero vectors in inner product space V, $\implies$ then $\mathcal{V}$ is linearly *in*dependent

<u>How to:</u> find orthonormal basis of a given basis:

a. Do GS process to a new set of orthogonal basis vectors
b. Normalize each one by dividing by the norms