

Linear Algebra Notes

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Ch 1

1.2 Intro stuff

Thm: Any system of linear eqs has either 0, exactly 1, or ∞ many solutions

How to: check how many solutions there are in a given system of eq:

If there last line is $0 = c$, then there is *no solution*

If not, are there free variables? If so, there are *inf many solutions*, otherwise *1 solution*.

Thm: Homogenous systems ALWAYS are consistent, aka they have either 1 or inf many solutions.

This is b/c the trivial solution will always exist when $A\mathbf{x} = 0$

Ch 2

2.1 Linear Combinations

How to: find all vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t. $c_1 u_1 + c_2 u_2 = \begin{pmatrix} a \\ b \end{pmatrix}$?

> Augment the vector with $\begin{pmatrix} a \\ b \end{pmatrix}$ and row reduce, the remaining things in the augment become the scalars c_1, c_2

How to: show that a certain vector \mathbf{b} cannot be obtained as a linear combination of some other vectors?

> Augment the matrix with \mathbf{b} then row reduce, you'll find an inconsistent set with $0 = c$

2.2 Span

Definition of $\text{span}\{\mathbf{u}_1 \dots \mathbf{u}_m\}$ is the set of ALL linear combinations

How to: See if some vector \mathbf{v} is an element of $\text{span}\{u_1, u_2, \dots, u_m\}$?

> iff the linear system w/ \mathbf{v} as the augment has a solution

Thm: IF \mathbf{u}, \mathbf{v} are in $\text{span}\{\mathbf{u}_1 \dots \mathbf{u}_m\}$, THEN $\mathbf{u} + \mathbf{v}$ & $a\mathbf{u}$ are in that span
(linear combinations of the vectors are in that span)

How to: Find a vector \mathbf{b} not in a given span of vectors?

> Set matrix equal (by augment) to some (a, b, c) , and track what happens to it until matrix is row reduced.

> If the row is $[0 \ 0 \ 0 \mid f(c)]$, then any vector that has $f(c) \neq 0$ is not in the span

Thm: $(\text{span}\{u_1 \dots u_m\} = \mathbb{R}^n) \iff (B \text{ has a pivot in every row})$

How to: check if a set of vectors span \mathbb{R}^n ?

> Row reduce the matrix augmented with $[a \ b \ c]$

> Check if there is a row of 0s, which would imply that there could be a vector not in that span

> If there is a pivot in every row, there are either 1 or inf many ways to get to any point in \mathbb{R}^n
(depending on if there are free variables in any of the columns)

How to: Find what values of h in a $n \times m$ matrix allow it to span \mathbb{R}^n ?

a. Row reduce the matrix, moving the h down as needed to the last row.

b. The vectors in A span \mathbb{R}^n iff there is a trivial solution, which only occurs when $f(h) \neq 0$

c. The final answer should be all vectors with $\{h \mid h \neq 27\}$ or something

How to: find $\text{span}(\{\mathbf{a}_1 \dots \mathbf{a}_m\})$ (aka the columns of T)?

Thm: For a given set of m vectors in \mathbb{R}^n :

a. IF $m < n$, THEN the set does not span \mathbb{R}^n (b/c theres no way to have a pivot in every row)

b. IF $m \geq n$, THEN the set could span \mathbb{R}^n (depends on whether or not they are linearly independent)

Thm: $\mathbf{b} \in \text{span}\{\mathbf{a}_1 \dots \mathbf{a}_m\} \text{ in } \mathbb{R}^n \iff A\mathbf{x} = \mathbf{b} \text{ has at least 1 solution}$

2.3 Linear Independence

Definition: IF the only way to express $\mathbf{0}$ as a linear combination of \mathbf{A} is the trivial solution $\mathbf{0}$, THEN the system is *linearly independent*. Nontrivial solutions imply *linear dependence*.

Thm: $(A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution}) \iff (\{\mathbf{a}_1 \dots \mathbf{a}_m\} \text{ is linearly independent})$

How to: Check if a system is linearly independent?

a. Set the sytem equal to $\mathbf{0}$, and row reduce. The only solution should be $\mathbf{x} = \mathbf{0}$ (which will happen when there is a pivot in every column)

Thm: $(m \text{ vectors in } \mathbb{R}^n \text{ are linearly independent}) \implies (m \leq n)$

(b/c you can't have pivots in every column if there are too many columns)
(in other words, there are more variables than equations in the system)

Thm: For a given set of m vectors in \mathbb{R}^n :

- $(\text{span}\{u_1 \dots u_m\} = \mathbb{R}^n) \iff (B \text{ has a pivot in every row})$
- $(\{u_1 \dots u_m\} \text{ are linearly independent}) \iff (B \text{ has a pivot in every column})$

How to: Check if one vector lies in the span of others in a set?

- Row reduce the matrix augmented with $\mathbf{0}$
- If B does not have a pivot in every column \implies the system is linearly dependent \implies one of the vectors lies in the span of the others

General vs. Particular Solutions to $A\mathbf{x}=\mathbf{0}$

Thm: $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$, where \mathbf{x}_g is the solution to $A\mathbf{x} = \mathbf{b}$, \mathbf{x}_h is the solution to the associated homogenous system $A\mathbf{x} = \mathbf{0}$, and \mathbf{x}_p is a particular solution to \mathbf{x}_g

Thm: For a given set of vectors $\{\mathbf{a}_1 \dots \mathbf{a}_m\}$ and \mathbf{b} in \mathbb{R}^n :

- $(\{\mathbf{a}_1 \dots \mathbf{a}_m\} \text{ are linearly independent}) \iff (A\mathbf{x} = \mathbf{b})$

Ch 3

3.1 Linear Transformations

Definition: A transformation $T(\mathbf{x}) = A\mathbf{x}$ is linear if both:

- $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$
- $T(r\mathbf{u}) = A(r\mathbf{u}) = rA\mathbf{u} = rT(\mathbf{u})$

Thm: $T(\mathbf{x}) = A\mathbf{x} \implies T$ is a linear transformation, where A is a $n \times m$ matrix, and T goes from \mathbb{R}^m to \mathbb{R}^n

How to: Check if a given transformation is linear:

- Convert the system into a matrix A
- Plug in the vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for \mathbf{u} and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ for \mathbf{v} to prove the general case *true*
- Try the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ independently and check if the output fails to prove *false*

How to: Find $\text{range}(T)$, where $T(\mathbf{x}) = A\mathbf{x}$:

- $\text{range}(T) = \text{span}(\{\mathbf{a}_1 \dots \mathbf{a}_m\})$
- range is the set of linear combinations of the columns of A

How to: Check if a given vector \mathbf{w} is in $\text{range}(T)$:

- Make matrix of $[A \mid \mathbf{w}]$ and solve

One-to-One vs Onto

Definition: A transformation is *one-to-one* when there's at most one input that maps to an output

Definition: A transformation is *onto* when no element in the codomain B is left out

Thm: Given $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where \mathbf{B} is \mathbf{A} in row-echelon form:

1a. (T is one-to-one) \iff (columns of \mathbf{A} are linearly *independent*) \iff (\mathbf{B} has a pivot in every column)

1b. $n < m \implies T$ is not one-to-one (aka if output space is smaller than input space)

2a. (T is onto) \iff (columns of \mathbf{A} span the codomain \mathbb{R}^n aka $\text{range}(T) = \mathbb{R}^n$) \iff (\mathbf{B} has a pivot in every row)

2b. $n > m \implies T$ is not onto (aka if output space is bigger than input space)

> "No matrix that goes from bigger space to smaller space can be one-to-one"

> "No matrix that goes from small space to bigger space can be onto"

Geometry of transformations

How to: Rotate a vector CCW by θ :

a. $T_r(\mathbf{x}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mathbf{x}$

How to: Shear to the right:

a. $T_r(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

3.2 Matrix Algebra

Properties of Elementary Matrices

a. $A(BC) = (AB)C$

b. $A(B + C) = AB + AC$

c. $(A + B)C = AC + BC$

d. $s(AB) = (sA)B = A(sB)$

e. $AI = IA = A$

Non-Properties of Nonzero Matrices

a. It is possible that $AB \neq BA$

b. $AB = 0$ does not imply that $A = 0$ and $B = 0$

c. $AC = BC$ does not imply that $A = B$ or $C = 0$ (unless A is invertible)

Transpose of a Matrix

- a. $(A + B)^T = A^T + B^T$
- b. $(sA)^T = sA^T$
- c. $(AC)^T = C^T A^T$

3.3 Inverses

Definition: If T is a linear transformation, Then

- a. T has an inverse $\implies m = n$
- b. If T is invertible, then T^{-1} is also a linear transformation

$(T \text{ is invertible}) \iff (T \text{ is one-to-one AND onto})$

How to: find an invertible matrix A^{-1} ?

- a. Augment matrix A with I_n , then row reduce until you get I_n augmented with A^{-1}
(aka $[A|I_n] \rightarrow [I_n|A^{-1}]$)

Thm: Elementary matrices are invertible

Properties of Inverses

- a. $(A^{-1})^{-1} = A$
- b. $(AB)^{-1} = B^{-1}A^{-1}$
- c. $AC = AD \implies C = D$
- d. $AC = 0_{nm} \implies C = 0_{nm}$

Ch 4

4.1 Subspaces

Definition: A subset of S is a subspace if all three conditions are true:

- a. S contains $\mathbf{0}$ (S contains the origin)
- b. If \mathbf{u} and \mathbf{v} are both in S , then $(\mathbf{u} + \mathbf{v})$ is in S (S is closed under addition)
- c. If $r \in \mathbb{R}$, then $r\mathbf{u}$ is also in S (S is closed under scalar multiplication)

Thm: If $S = \text{span}\{\mathbf{u}_1 \dots \mathbf{u}_m\}$ in \mathbb{R}^n , then S is a subspace of \mathbb{R}^n

How to: Check if S is a subspace?

- a. Check if $\mathbf{0}$ is in S , which it must be to be a subspace
- b. Try to show S is generated by a set of vectors (See if it can be composed as a matrix of coefficients)

Definition: If \mathbf{A} is a $n \times m$ matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ is called $\text{null}(\mathbf{A})$

(aka the null space is all linear combinations where $\mathbf{Ax} = \mathbf{0}$)

(aka the null space is the solution to the homogenous system)

Thm: If \mathbf{A} is a $n \times m$ matrix, then the set of solutions to $\mathbf{Ax} = \mathbf{0}$ forms a subspace of \mathbb{R}^m
(aka null space is a subspace)

Thm: Given $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a *linear* transformation:

- $\ker(T)$ is a subspace of the domain \mathbb{R}^m
- $\text{range}(T)$ is a subspace of the codomain \mathbb{R}^n

> "The kernel is the set of vectors that are sent to $\{\mathbf{0}\}$ after applying T "

> "The range of T is the span after applying T "

How to: Find $\ker(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?

a. $(T(\mathbf{x}) = \mathbf{Ax}) \implies (\ker(T) = \text{null}(A))$, so solve for $\mathbf{Ax} = \mathbf{0}$, and $\ker(T)$ is the span of that answer

How to: Find $\text{range}(T)$ of $T(\mathbf{x}) = A(\mathbf{x})$?

a. $\text{range}(T) = \text{span}(\mathbf{a}_1 \dots \mathbf{a}_m)$, so just delete any linearly *dependent* columns of \mathbf{A} and that's your answer

Thm: (T is one-to-one) $\iff (\ker(T) = \{\mathbf{0}\})$

4.2 Basis vectors

Definition: Set $B = \{\mathbf{u}_1 \dots \mathbf{u}_m\}$ is a *basis* of subspace S iff:

- B spans S
- B is linearly independent

"To get to any point in S , you can take a linear combination of the basis vectors to get there"

How to: find a basis for $S = \text{span}\{\mathbf{u}_1 \dots \mathbf{u}_m\}$?

Method 1 (Thm 4.10):

- Create a matrix $\begin{pmatrix} \mathbf{u}_1 \\ \dots \\ \mathbf{u}_m \end{pmatrix}$
- Row reduce to B
- The nonzero *rows* of B give a basis of S

Method 2 (Thm 4.11):

- Create a matrix out of $\{\mathbf{u}_1 \dots \mathbf{u}_m\}$
- Row reduce to B . The pivot columns of B are linearly independent
(the other cols are dependent on the pivot columns)
- The *columns* of A corresponding to the *pivot columns* of B form a basis of S .

Dimension

Thm: If S is a subspace of \mathbb{R}^n , then every basis of S has the same number of vectors

Definition: If S is a subspace of \mathbb{R}^n , then the *dimension* of S is the number of vectors in any basis of S

Thm: Let $U = \{\mathbf{u}_1 \dots \mathbf{u}_m\}$ is a set of m vectors in subspace S of dimension m .

IF U is *either* linearly independent or spans S , THEN U is a basis for S .

How to: expand a set of vectors to become a basis of \mathbb{R}^n :

- Append on all the unit vectors of \mathbb{R}^n , \mathbf{e}_i , that you have and then row reduce down to B .
- The original columns of A that correspond w/ the pivots of B become the basis vectors for \mathbb{R}^n

Unifying Theorem: Given $S = \{\mathbf{a}_1 \dots \mathbf{a}_m\}$, $\{\mathbf{a}_1 \dots \mathbf{a}_m\} \in \mathbb{R}^n$, $A = [\mathbf{a}_1 \dots \mathbf{a}_m]$, and

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

- S spans \mathbb{R}^n
- S is linearly independent
- $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall \mathbf{b} \in \mathbb{R}^n$
- T is onto
- T is one-to-one
- A is invertible
- $\ker\{T\} = \{\mathbf{0}\} \iff \text{null}(A) = \{\mathbf{0}\}$
- S is a basis of \mathbb{R}^n

4.3 Row and Column Spaces

Definition: Row vectors of A come from viewing A as a set of rows; Column vectors of A come from viewing A as a set of columns.

Definition: Given A is a $n \times m$ matrix:

- $\text{row}(A)$ or row space is the subspace of \mathbb{R}^m spanned by *row vectors* of A
- $\text{col}(A)$ or column space is the subspace of \mathbb{R}^n spanned by *column vectors* of A

Thm: Given matrix A and B in echelon form:

- Nonzero rows of B form a basis for $\text{row}(A)$
(The redundant rows get killed off by row reduction)
- The cols of A corresponding to pivot columns of B form a basis for $\text{col}(A)$
(The pivot columns are linearly independent \implies they form the column space of A)

Thm: For any matrix A , the dimension of $\text{row}(A)$ equals the dimension of $\text{col}(A)$
(aka the number of basis vectors needed to define $\text{row}(A)$ the number needed to define $\text{col}(A)$)

Definition: $\text{rank}(A)$ is the dimension of $\text{row}(A)$ or $\text{col}(A)$

Rank-Nullity Thm: IF A is a $n \times m$ matrix, THEN $\text{rank}(A) + \text{nullity}(A) = m$ (# of cols)
($\text{nullity}(A)$ is the number of free variables in the system $A\mathbf{x} = \mathbf{0}$)

*Note: $\ker(T)$ is $\text{null}(A)$, and $\text{range}(T)$ is $\text{col}(A)$

Thm: IF A is a $n \times m$ matrix, and \mathbf{b} is a vector in \mathbb{R}^n :

- a. The system $A\mathbf{x} = \mathbf{b}$ is consistent $\iff \mathbf{b}$ is in $\text{col}(A)$
- b. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\iff \mathbf{b}$ is in $\text{col}(A)$ and columns of A are linearly independent

Unifying Theorem: Given $S = \{\mathbf{a}_1 \dots \mathbf{a}_m\}$, $\{\mathbf{a}_1 \dots \mathbf{a}_m\} \in \mathbb{R}^n$, $A = [\mathbf{a}_1 \dots \mathbf{a}_m]$, and

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$:

- a. S spans \mathbb{R}^n
- b. S is linearly independent
- c. $A\mathbf{x} = \mathbf{b}$ has precisely 1 unique solution $\forall \mathbf{b} \in \mathbb{R}^n$
- d. T is onto
- e. T is one-to-one
- f. A is invertible
- g. $\ker\{T\} = \{\mathbf{0}\} \iff \text{null}(A) = \{\mathbf{0}\}$
- h. S is a basis of \mathbb{R}^n
- i. $\text{col}(A) = \mathbb{R}^n$
- j. $\text{col}(A) = \mathbb{R}^n$
- k. $\text{rank}(A) = n$

4.4 Change of Basis

Definition: Suppose that $B = \{\mathbf{u}_1 \dots \mathbf{u}_n\}$ forms a basis of \mathbb{R}^n , and if $\mathbf{y} = y_1 \mathbf{u}_1 + \dots + y_n \mathbf{u}_n$:

THEN: The *coordinate vector* of y w.r.t. B is $[\mathbf{y}]_B = \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix}$

"the coordinate vector contains the coeffs required to express y as a linear combination of vectors in basis B "

$$U \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} = y_1 \mathbf{u}_1 + \dots + y_n \mathbf{u}_n$$

$\mathbf{y} = U[\mathbf{y}]_B$; where U is the *change of basis matrix* that transforms the coordinate vector wrt B back to the standard basis

(U is just the $n \times n$ matrix containing the basis vectors of set B : $[\mathbf{u}_1 \dots \mathbf{u}_n]$)

Thm: Let \mathbf{x} be expressed wrt standard basis, and $B = \{\mathbf{u}_1 \dots \mathbf{u}_n\}$ be any basis for \mathbb{R}^n :

If $U = [\mathbf{u}_1 \dots \mathbf{u}_n]$, then: $\mathbf{x} = U[\mathbf{x}]_B$ and $[\mathbf{x}]_B = U^{-1}\mathbf{x}$

" U takes us from a vector described in weird basis land into our standard definition"

" U^{-1} tells us how to define a vector w/ standard definition into weird basis land"

How to: Move from one nonstandard basis to another?

If $B_1 = \{\mathbf{u}_1 \dots \mathbf{u}_n\}$ corresponds to U and $B_2 = \{\mathbf{v}_1 \dots \mathbf{v}_n\}$ corresponds to V , then:

$$[\mathbf{x}]_{B_2} = V^{-1}U[\mathbf{x}]_{B_1}$$

$$[\mathbf{x}]_{B_1} = U^{-1}V[\mathbf{x}]_{B_2}$$

> "To go from basis 1 to basis 2, apply U to go into standard basis, then apply V^{-1} to go to basis 2 and"

INSERT PIC

Ch 5

5.1 Determinant

Definition: Given A is a $n \times n$ matrix, each position is defined as a_{ij} :

a. cofactor $C_{ij} = (-1)^{i+j} * \det(M_{ij})$, where M_{ij} is the rest of the matrix that doesn't include the i th row & j th column

b. $\det(A) = a_{11}C_{11} + \dots + a_{1n}C_{1n}$: for each element in the *first* column, multiply each times its cofactor.

c. In general, you can expand down any row or column and apply the same formula (Thm 5.8)

Matrix of signs:
$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Thm 5.5: $\det(I_n) = 1$

Thm 5.6: A is invertible $\iff \det(A) \neq 0$

> This would invoke unifying theorem, so you could say things like: "cols of A form a basis of \mathbb{R}^n " etc.

Thm 5.9: If A is triangular matrix, then $\det(A)$ is product of terms along the diagonal

Thm 5.11: If A is a square matrix, then:

a. If A has a row or column of zeros, then $\det(A) = 0$

b. If A has two identical rows or columns, then $\det(A) = 0$

5.2 Properties of Determinants

Thm 5.13:

a. Swap 2 rows of $A \implies -\det(A)$

b. Multiply row of A by $c \implies c * \det(A)$

c. Add multiple of one row of A to another $\implies \det(A)$

Thm 5.10: If A is a square matrix, then $\det(A^T) = \det(A)$

Thm 5.12: $\det(AB) = \det(A)\det(B)$ if A and B are both $n \times n$ matrices

5.3 Applications of Determinants

Cramer's Rule (Thm 5.17): Let A be invertible; to find unique solution, \mathbf{x} , to $A\mathbf{x} = \mathbf{b}$:

$$\mathbf{x}_i = \frac{\det(A_i)}{\det(A)}, \text{ where } A_i \text{ is the matrix } A \text{ but with the } i\text{th column replaced by } \mathbf{b}$$

How To: Find the unique solution to $A\mathbf{x} = \mathbf{b}$:

> Apply Cramer's Rule for each column in A

Thm 5.18: If A is invertible, then:

$$A^{-1} = \text{adj}(A) / \det(A), \text{ where } \text{adj}(A) = C^T, \text{ the transpose of the cofactor matrix of } A$$

Thm: $A(\text{adj}(A)) = \det(A) * I_n$

Thm 5.20: Let D be a region w finite area in \mathbf{R}^2 , $T(\mathbf{x}) = A\mathbf{x}$, and $T(D)$ is the image of D under T , then:

$$\text{area}(T(D)) = |\det(A)| * \text{area}(D)$$

Thm 5.21: Same thing as Thm 5.20 but w/ volume in \mathbf{R}^3

Ch 6

6.1 Eigenvalues and Eigenvectors

Definition: Let A be a $n \times n$ matrix; \mathbf{u} is an eigenvector of A if there exists a scalar λ s.t.

$$A\mathbf{u} = \lambda\mathbf{u}; \text{ where } \lambda \text{ is an eigenvalue of } A$$

> Intuition: If A is a transformation that changes the basis vectors of our subspace, then we expect most vectors to also be transformed.

Eigenvectors are the specific vectors that remain parallel after the transformation, and the degree to which its scaled is called the eigenvalue

Thm 6.2: If \mathbf{u} is an eigenvector of A associated with λ , then $c\mathbf{u}$ is also associated w/ λ

How to: Check if a given vector \mathbf{u} is a eigenvector of A ?

a. Multiply $A\mathbf{u}$, and the result should be some multiple $c\mathbf{u}$

How to: Find an eigenvector if you know the eigenvalues for a given $n \times n$ matrix A :

$$\text{For a specific eigenvalue, say } \lambda = 6: (A\mathbf{u} - 6I_n)\mathbf{u} = \mathbf{0}$$

- Subtract off 6 from each value along the *diagonal* of A
- Then set equal to 0 and solve out

c. You'll get something like $s_1 \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, which means any vector \mathbf{u} that is a linear

combination of the

those two is an eigenvector (this is called the eigenspace btw)

Thm 6.3: If A is a $n \times n$ matrix w/ eigenvalue λ , and S is the set of all eigenvectors associated w/ λ , including $\mathbf{0}$:

Then S is a subspace of \mathbf{R}^n

Definition: Eigenspace of A is the subspace of all eigenvectors associated w/ λ together w/ $\mathbf{0}$

Thm 6.5: λ is an eigenvalue of $A \iff \det(A - \lambda I_n) = 0$

How to: find eigenvalues of A :

- Subtract off λ from each diagonal, and take the determinant of that matrix
> The result is called the characteristic polynomial btw
- Set that equal to 0, and solve for lambda

Definition: Multiplicity is the number of times a root of the characteristic equation is repeated, aka equal to its factor's exponent

Thm 6.6: Let A be a square matrix w/ eigenvalue λ : Then \dim of associated eigenspace is \leq multiplicity of λ

Thm 6.7: Unifying thm: $\lambda = 0$ is not an eigenvalue of A

In other words, if $\lambda = 0$ is an eigenvalue of A , then one of the columns is linearly dependent or something

6.2 Diagonalization

Definition: A is diagonalizable if there exists $n \times n$ matrices D and P , s.t. D is diagonal and P is invertible:

$$\Rightarrow A = PDP^{-1}$$

How to: Find P and D to diagonalize matrix A :

- Find each eigenvalue using $\det(A - \lambda I_n) = 0$
- Find the associated eigenvector by solving the homogenous system $(A - \lambda I_n)\mathbf{u} = \mathbf{0}$
- $P = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]$; $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Thm 6.9: A is diagonalizable $\iff A$ has eigenvectors that form a basis for \mathbf{R}^n

$\iff A$ has n linearly independent eigenvectors

Thm 6.10: If $\{\lambda_1, \dots, \lambda_k\}$ are distinct eigenvalues of A ,

then any set of associated eigenvectors $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ are *linearly independent*

> Means that vectors from *distinct* eigenspaces are linearly independent

> This also means that A can be not invertible and yet diagonalizable

Thm 6.11: If A has only real eigenvalues, then:

A is diagonalizable \iff dim of each eigenspace *equals* the multiplicity of corresponding eigenvalue

> Means that you'll know that a matrix isn't diagonalizable (for example) when you see that there's one basis vector of an eigenspace when the root had a higher power

Thm 6.12: If A is a $n \times n$ matrix with n *distinct* real eigenvalues, then A is diagonalizable

How to: Get k th power of matrix: $A^k = PD^kP^{-1}$

Ch 7

7.1 Vector Spaces + Subspaces

Definition: Vector space is a set V of vectors, together w/ operations of addition and scalar multiplication:

- V is closed under addition: If \mathbf{v}_1 and \mathbf{v}_2 are in V , then so is $\mathbf{v}_1 + \mathbf{v}_2$
- V is closed under *scalar* multiplication: If c is a real scalar and \mathbf{v} is in V , then so is $c\mathbf{v}$
- There exists a zero vector $\mathbf{0}$ in V s.t. $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all \mathbf{v} in V
- There exists an additive inverse $-\mathbf{v}$ in V s.t. $-\mathbf{v} + \mathbf{v} = \mathbf{0}$ for all \mathbf{v} in V
- For all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in V :
 - $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$
 - $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$
 - $c_1(\mathbf{v}_1 + \mathbf{v}_2) = c_1\mathbf{v}_1 + c_1\mathbf{v}_2$
 - $(c_1 + c_2)\mathbf{v}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_1$
 - $(c_1c_2)\mathbf{v}_1 = c_1(c_2\mathbf{v}_1)$
 - $1 * \mathbf{v}_1 = \mathbf{v}_1$

Thm 7.2: Let \mathbf{v} be in vector space V :

- $\mathbf{0}$ is the zero vector in $V \implies \mathbf{v} + \mathbf{0} = \mathbf{v}$
- $-\mathbf{v}$ is the additive inverse of $\mathbf{v} \implies -\mathbf{v} + \mathbf{v} = \mathbf{0}$
- \mathbf{v} has a unique additive inverse $-\mathbf{v}$
- Zero vector $\mathbf{0}$ is unique
- $0 * \mathbf{v} = \mathbf{0}$
- $(-1) * \mathbf{v} = -\mathbf{v}$

Definition: A subset S of a vector space V is a subspace if S :

- S contains the zero vector, $\mathbf{0}$
- If \mathbf{u}, \mathbf{v} is in S , then $\mathbf{u} + \mathbf{v}$ is also in S
- If c is a scalar and \mathbf{v} is in S , then $c\mathbf{v}$ is also in S

7.2 Span and Linear Independence

Definition: Span of set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is set of all linear combinations of the form:

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_m \mathbf{v}_m$$

Thm 7.6: Suppose that \mathcal{V} is a subset of vector space V , and let $S = \text{span}(\mathcal{V})$ then S is a subspace of V

Definition: Set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is linearly independent if $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_m \mathbf{v}_m = \mathbf{0}$ has only the trivial solution

Thm 7.8 Set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is linearly dependent \iff at least one vector is in the span of the others

Thm 7.9 If set $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is a subset of V , then:

- \mathcal{V} is linearly independent $\iff \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \mathbf{v}$ has at most one solution for each \mathbf{v} in V
- \mathcal{V} spans V $\iff \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \mathbf{v}$ has at least one solution for each \mathbf{v} in V

How to: Check if a system is linearly independent, i.e. given $a(1) + b(\sin(x)) + c(\cos(x)) = 0$?

- To generate a system of equations, take the first and second derivatives wrt. x to generate 3 equations
- Check if the determinant $\neq 0$, \implies the matrix is invertible, \implies the system $A\mathbf{x} = \mathbf{0}$ has a unique solution which is the trivial solution.

Application: Decomposition of signal frequencies:

i.e. Given $1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots, \sin(nx), \cos(nx)$, is this system linearly independent?

Definition: \mathbb{P}^n is the set of all polynomials of degree $\leq n$

- Standard bases: $\{1, x, x^2, \dots, x^n\} \implies \dim(\mathbb{P}^n) = n + 1$
- $\dim \mathbb{P} = \infty$, enough to show that there is no finite basis
- $\mathbb{P} \subset C(\mathbb{R}) \implies \dim C(\mathbb{R}) = \infty$

How to: check if a given set of polynomials, i.e. $\{x + 1, x^2 - 1, x^2 + x + 1\}$, is a basis of \mathbb{P}^2 ?

a. Check linear independence: Assign a scalar c_i to each polynomial; each column represents c_i , and each element in each column corresponds to a different degree of that polynomial, i.e. x^2

$$\begin{aligned} c_1(x + 1) + c_2(x^2 - 1) + c_3(x^2 + x + 1) &= \mathbf{0} \\ \implies (c_1 - c_2 + c_3) * 1 + (c_1 + c_3) * x + (c_2 + c_3) * x^2 &= 0 \end{aligned}$$

$$\implies \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

a1. Check that determinant $\neq 0$ which would imply unique trivial solution, which implies linear independence

b. Could also check that they $\text{span } \mathbb{P}^2$

$$\implies \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

c. Could also check matrix of derivatives: if there exists at least a single value of x that makes $\det(A) \neq 0$, \implies linear independence.

$$\begin{vmatrix} x+1 & x^2-1 & x^2+x+1 \\ 1 & 2x & 2x+1 \\ 0 & 2 & 2 \end{vmatrix}$$

7.3 Basis and Dimension

Definition: \mathcal{V} is a basis of V if \mathcal{V} is linearly independent *and* spans V

Thm 7.11 Set $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis of vector space V iff:

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = \mathbf{v}$ has a *unique* solution for every \mathbf{v} in V

Thm 7.12 If \mathcal{V}_1 and \mathcal{V}_2 are both bases of vector space V , then \mathcal{V}_1 and \mathcal{V}_2 have the same number of elements

Definition Dimension of V is equal to the number of vectors in any basis of V . *It's possible for dimension to be infinite if the basis of V is infinitely long*

Thm 7.14 Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a subset of *nontrivial, finite, dimensional* vector space V :

- If \mathcal{V} spans V , then either \mathcal{V} is a basis of V or vectors can be removed from \mathcal{V} to form a basis of V
- If \mathcal{V} is linearly independent, then either \mathcal{V} is a basis of V or vectors can be added to \mathcal{V} to form a basis of V

Thm 7.15: If V_1 is a subspace of V_2 , then $\dim(V_1) \leq \dim(V_2)$

Thm 7.16: Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a subset of V with $\dim(V) = n$.

- If $m < n$, then \mathcal{V} does not span V
- If $m > n$, then \mathcal{V} is not linearly independent

Thm 7.17: Let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a subset of V with $\dim(V) = m$.

- If \mathcal{V} is linearly independent *or* spans V , then \mathcal{V} is a basis for V

How to: find $\dim \text{span}\{T_1, T_2, T_3\}$, given something like $T_1 \mathbf{x} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 \end{pmatrix}$, etc.?

a. Check linear independence of T_1, T_2, T_3 (Must hold $\forall x$)

a1. Assign each T_i a scalar c_i which become its own columns;

each element in that matrix has its own row on that particular column

b. Row reduce: if you get a free variable in a certain column, say s_1 , then that column is a linear combination of the others

b. Check how many T 's are needed to maintain linear independence, that is $\dim \text{span}$

Ch 8

8.1 Dot Products and Orthogonal Sets

Definition If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in \mathbf{R}^n : $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = \mathbf{u}^T \mathbf{v}$

Thm 8.2: Properties of dot products:

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$

e. $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$

Definition: Norm or length of $\mathbf{x} = \|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$$\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$$

Definition: Unit vector in direction of \mathbf{x} :

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| = 1$$

Definition: The distance between \mathbf{u} and \mathbf{v} in \mathbf{R}^n is $\|\mathbf{u} - \mathbf{v}\|$

a. Subtract v_i from u_i , etc., then take the magnitude of that resulting vector

b. This is really the distance between the endpoints of the rays

Definition: Vectors \mathbf{u} and \mathbf{v} in \mathbf{R}^n are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$

Pythagorean Thm 8.6: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \mathbf{u} \cdot \mathbf{v} = 0$

a. Think middle school geometry: pythagorean thm applies iff right triangle

Cauchy-Schwarz Inequality Thm 8.7: $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$

a. Define $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

Triangle Inequality Thm 8.8 $|\mathbf{u} + \mathbf{v}| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

a. This says that the line segment between two vectors is the shortest way between those two points

Orthogonal Subspaces

Definition: Vector \mathbf{u} is orthogonal to subspace S if $\mathbf{u} \cdot \mathbf{v} = 0 \forall \mathbf{v} \in S$. The set of all such vectors \mathbf{u} is called the orthogonal complement of S and is denoted by S^\perp

$$S^\perp = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{s} = 0 \forall \mathbf{s} \in S\}$$

Thm 8.10: $S \sim \text{subspace of } \mathbb{R}^n \implies S^\perp \sim \text{subspace of } \mathbb{R}^n$

Thm 8.11: Let $\mathcal{V} = \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ be a basis for subspace S and \mathbf{u} be a vector.

$$\text{Then } \mathbf{u} \cdot \mathbf{s}_1 = 0, \mathbf{u} \cdot \mathbf{s}_2 = 0, \dots, \mathbf{u} \cdot \mathbf{s}_k = 0 \iff \mathbf{u} \in S^\perp$$

How to: find a basis for S^\perp , given $S = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right\}$: (think of this as a 2D plane in \mathbb{R}^3)

a. Make the basis vectors for S into rows of a matrix

b. Because $A\mathbf{u} = \begin{bmatrix} \mathbf{s}_1 \cdot \mathbf{u} \\ \mathbf{s}_2 \cdot \mathbf{u} \end{bmatrix}$, \mathbf{u} is in S^\perp only when $A\mathbf{u} = \mathbf{0}$

c. Solve out $A\mathbf{u} = \mathbf{0}$, and find something like $\mathbf{u} = t_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

d. $S^\perp = \text{span}\left\{\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\right\}$

Definition: A set of vectors \mathcal{V} in \mathbb{R}^n form an orthogonal set if $\mathbf{v}_i \cdot \mathbf{v}_j = 0, \forall \mathbf{v}_i \& \mathbf{v}_j$ in \mathcal{V} , with $i \neq j$

a. All of the vectors in the set must be orthogonal to each other

How to: check if a set of vectors is orthogonal?

a. Take the dot product of each possible pair, and if *all* are 0, then set is orthogonal

Thm 8.13: An orthogonal set of nonzero vectors is linearly *independent*

Definition: Orthogonal basis of a subspace is a *basis* which is also an *orthogonal set*

Thm 8.14: If S is a subspace w/ orthogonal basis $= \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$,

Then $\forall \mathbf{s} \in S: \mathbf{s} = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k$, where $c_i = \frac{\mathbf{s} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}$

How to: express \mathbf{s} as a linear combination of a given orthogonal vector basis:

a. Use Thm 8.14 to get a c_i for each vector in the basis, and the result will be something like

$$\mathbf{s} = -2\mathbf{v}_1 + \mathbf{v}_2$$

8.2 Projection

Definition: Projection of \mathbf{u} onto \mathbf{v} : $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|^2} \mathbf{v}$

Thm 8.16: For a projection when \mathbf{v} is nonzero:

- $\text{proj}_{\mathbf{v}} \mathbf{u}$ lies on $\text{span}\{\mathbf{v}\}$
- $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$ is orthogonal to \mathbf{u}
- if \mathbf{u} is in $\text{span}\{\mathbf{v}\}$, then $\text{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{u}$
- $\text{proj}_{\mathbf{v}} \mathbf{u} = \text{proj}_{c\mathbf{v}} \mathbf{u}$

Definition: Let S be a nonzero subspace w/ orthogonal basis $= \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$:

$$\text{proj}_S \mathbf{u} = \text{proj}_{\mathbf{v}_1} \mathbf{u} + \dots + \text{proj}_{\mathbf{v}_k} \mathbf{u}$$

Thm 8.18: For a projection on subspace when S is nonzero:

- $\text{proj}_S \mathbf{u}$ lies in S
- $\mathbf{u} - \text{proj}_S \mathbf{u}$ is orthogonal to S
- If \mathbf{u} is in S , then $\text{proj}_S \mathbf{u} = \mathbf{u}$
- $\text{proj}_S \mathbf{u}$ is independent of the choice of the orthogonal basis of S

Gram-Schmidt Processes

Thm 8.19: Let S be a subspace with basis $\{\mathbf{s}_1, \dots, \mathbf{s}_k\}$. $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is defined as:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{s}_1 \\ \mathbf{v}_2 &= \mathbf{s}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{s}_2 \\ \mathbf{v}_3 &= \mathbf{s}_3 - \text{proj}_{\mathbf{v}_1} \mathbf{s}_3 - \text{proj}_{\mathbf{v}_2} \mathbf{s}_3 \\ \mathbf{v}_k &= \mathbf{s}_k - \text{proj}_{\mathbf{v}_1} \mathbf{s}_k - \text{proj}_{\mathbf{v}_2} \mathbf{s}_k - \dots - \text{proj}_{\mathbf{v}_{k-1}} \mathbf{s}_k \end{aligned}$$

(The pattern here is that the row defining the i th vector uses the original basis vector \mathbf{s}_i on that row only.

The vertical columns are unique to each orthogonal vector, that is to say, column 1 is always nothing, column 2 is \mathbf{v}_1 , and column k is \mathbf{v}_{k-1} ← you're recursively defining the next vector)

Intuition: The basis vectors of S are already linearly independent, so you know that none of them are parallel to the others. We start off with the first basis vector of S becoming the first orthogonal basis vector \mathbf{v}_1 . We can find a basis vector orthogonal to that by taking the second basis vector, and subtracting off its projection onto the previous vector to define the new orthogonal basis vector \mathbf{v}_2 (because we know that difference will be orthogonal). We can keep going for all the basis vectors, such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$, then not only are the \mathbf{v} vectors defining the same subspace, but they also are orthogonal to each other.

Definition: A set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ is orthonormal IF the set is orthogonal AND

$$\begin{aligned} \|\mathbf{w}_j\| &= 1 \quad \forall j = 1, 2, \dots, k \\ \mathbf{w}_j &= \frac{1}{\|\mathbf{v}_j\|} \mathbf{v}_j \text{ for } j = 1, 2, \dots, k \end{aligned}$$

How to: Find a set of orthonormal basis vectors from a given set of basis vectors i.e. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

- a. Orthogonalize $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ using GS process to o
- b. Normalize to obtain an orthonormal basis of $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$