# QHO via Dirac

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## 1 Introduction

We study the QHO via the creation and annihilation operators.

## 2 Arriving at Creation and Annihilation

### 2.1 Minimum Energy

A key motivation behind the formulation of the creation and annihilation operators is factoring the Hamiltonian into some $A^\dagger A$. Doing so allows us to make some powerful statements about a system's minimum energy.

The Hamiltonian of interest is

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2$$

We may manipulate this expression as follows

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

$$= \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) \tag{2}$$

At this point, we use the factoring $a^2 + b^2 = (a - bi)(a + bi)$. However, this assumes commutativity. To deal with this, observe $(a - bi)(a + bi) = a^2 - iba + abi + b^2 = a^2 + b^2 + i[a, b]$. Then, we may write

$$= \frac{1}{2}m\omega^2\left(\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) - i\left[\hat{x}, \frac{\hat{p}}{m\omega}\right]\right)$$

For simplicity, we'll denote our factorization terms as $V^\dagger$ and $V$. If it's not obvious that the factorization is done in such a form, note that $\hat{p}$ is Hermitian. Or, you could consider the fact that the adjoint of the derivative is the negative derivative. This yields

$$\frac{1}{2}m\omega^2\left(V^\dagger V + \frac{\hbar}{m\omega}\right)$$

$$= \frac{1}{2}m\omega^2(V^\dagger V) + \frac{1}{2}\hbar\omega \tag{3}$$

The power of this Hamiltonian representation stems from the following observation

$$\begin{aligned}
&\langle\psi|H|\psi\rangle \\
&= \frac{1}{2}m\omega^2 \langle\psi|V^\dagger V|\psi\rangle + \frac{1}{2}\hbar\omega \\
&= \frac{1}{2}m\omega^2 |V\,|\psi\rangle\,|^2 + \frac{1}{2}\hbar\omega
\end{aligned}$$

Since the first term can have a minimum value of 0, we can conclude that the minimum possible energy of the QHO is given by $\frac{1}{2}\hbar\omega$.

## 2.2 Definitions

While it may seem a bit contrived, we will proceed by 'massaging' Eq. (3) so that we are able to factor the $\hbar\omega$ term out. The reason for this will become clear shortly

$$\begin{aligned}
H &= \frac{1}{2}m\omega^2(V^\dagger V) + \frac{1}{2}\hbar\omega \\
&= \hbar\omega\left(\frac{m\omega}{2\hbar}\cdot V^\dagger V + \frac{1}{2}\right)
\end{aligned}$$

We define

$$a = \sqrt{\frac{m\omega}{2\hbar}}V \tag{4}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}V^\dagger \tag{5}$$

These are the *annihilation* and *creation* operators, respectively. With these definitions, we may write

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

It is also convenient to define $N = a^\dagger a$, noting that $N$ is clearly Hermitian since $N^\dagger = (a^\dagger a)^\dagger = a^\dagger a = N$. This yields

$$H = \hbar\omega\left(N + \frac{1}{2}\right) \tag{6}$$

# 3 Energy Eigenvalues

Since $H$ and $N$ have a linear relationship, we may consider the eigenvalues of $N$ as follows

$$\begin{aligned}
N\,|n\rangle &= n\,|n\rangle \\
H\,|n\rangle &= \hbar\omega\left(N + \frac{1}{2}\right)|n\rangle \\
H\,|n\rangle &= \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle
\end{aligned}$$

Since the RHS of the equation is a constant being multiplied by $|n\rangle$, we can conclude that $|n\rangle$ is an eigenvector of $H$ with eigenvalue $\hbar\omega\left(n+\frac{1}{2}\right)$, giving

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{7}$$

## 3.1 Significance of Creation and Annihilation

To see why these operators are important, we look at

$$Na^{\dagger}|n\rangle$$

We can rewrite this as

$$\begin{aligned}&(a^{\dagger}N - a^{\dagger}N + Na^{\dagger})|n\rangle\\ &= (a^{\dagger}N + [N, a^{\dagger}])|n\rangle\end{aligned}$$

This commutator is given in Appendix I

$$\begin{aligned}&= (a^{\dagger}N + a^{\dagger})|n\rangle\\ &= a^{\dagger}(N+1)|n\rangle\\ &= a^{\dagger}(n+1)|n\rangle\\ &= (n+1)a^{\dagger}|n\rangle\end{aligned} \tag{8}$$

Which tells us that $a^{\dagger}|n\rangle$ is an eigenvector of $N$ with eigenvalue $n+1$. We can do similar calculation for $Na|n\rangle$

$$\begin{aligned}&Na|n\rangle\\ &= (aN - aN + Na)|n\rangle\\ &= (aN + [N, a])|n\rangle\\ &= (aN - a)|n\rangle\\ &= a(N-1)|n\rangle\\ &= a(n-1)|n\rangle\\ &= (n-1)a|n\rangle\end{aligned} \tag{9}$$

So $a|n\rangle$ is an eigenvector of $N$ with eigenvalue $n-1$. Then, $a^{\dagger}$ corresponds to adding one unit of energy (given by $\hbar\omega$), and $a$ corresponds to removing one unit of energy. These properties give the operators their names.

## 3.2 A Lower Bound

One may consider repeatedly applying the annihilation operator to a given energy eigenstate to keep getting lower and lower energy states. However, Note that

$$\langle n|N|n\rangle = \langle n|a^{\dagger}a|n\rangle \geq 0 \tag{10}$$

This minimum is achieved if $n = 0$, meaning that the minimum energy eigenvalue is then

$$E_0 = \left(0 + \frac{1}{2}\right)\hbar\omega = \frac{1}{2}\hbar\omega \tag{11}$$

And subsequent energy levels are generated via plugging in $n = 1, 2, \ldots$.

# 4 Energy Eigenvectors and Matrix Representations

## 4.1 Eigenvectors

In realizing the significance of the creation and annihilation operators, astute readers may have noticed that equations (8) and (9) imply equivalence between $a\,|n\rangle$ ($a^\dagger\,|n\rangle$) and $|n-1\rangle$ ($|n+1\rangle$) up to some multiplicative constant. We write

$$a\,|n\rangle = c\,|n-1\rangle$$

Since $|n\rangle$ and $|n-1\rangle$ must be normalized, we can write

$$\begin{aligned}
\langle n|a^\dagger a|n\rangle &= \langle n-1||c|^2|n-1\rangle \\
\langle n|N|n\rangle &= |c|^2 \\
n &= |c|^2 \\
c &= \sqrt{n}
\end{aligned}$$

Thus

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{12}$$

We do a similar calculation for the creation operator

$$\begin{aligned}
a^\dagger\,|n\rangle &= c\,|n+1\rangle \\
\langle n|aa^\dagger|n\rangle &= |c|^2 \\
\langle n|a^\dagger a - a^\dagger a + aa^\dagger|n\rangle &= |c|^2 \\
\langle n|a^\dagger a + [a, a^\dagger]|n\rangle &= |c|^2 \\
n+1 &= |c|^2 \\
c &= \sqrt{n+1}
\end{aligned}$$

So

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{13}$$

Using this relation we can write

$$\begin{aligned}
|1\rangle &= a^\dagger\,|0\rangle \\
|2\rangle &= a^\dagger\,|1\rangle \cdot \frac{1}{\sqrt{2}} = (a^\dagger)^2\,|0\rangle \cdot \frac{1}{\sqrt{2}} \\
|3\rangle &= a^\dagger\,|2\rangle \cdot \frac{1}{\sqrt{3}} = (a^\dagger)^3\,|0\rangle \cdot \frac{1}{\sqrt{3\cdot 2}} \\
&\vdots \\
|n\rangle &= (a^\dagger)^n\,|0\rangle \cdot \frac{1}{\sqrt{n!}} \qquad (14)
\end{aligned}$$

### 4.2 Matrix Representations

The matrix representation of our creation and annihilation operators can be clearly seen by starting with equation (12)

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\langle n-1|a|n\rangle = \sqrt{n}$$
$$\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \sqrt{n} & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

And, starting with equation (13) yields

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\langle n+1|a^\dagger|n\rangle = \sqrt{n+1}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \sqrt{1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & \sqrt{n} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

# 5 Appendix I: Fundamental Commutation Relations

We cover the calculation for several key commutation relations used in the Dirac formalism of the QHO.

1. Position and momentum

$$\begin{gathered} [\hat{x}, \hat{p}] \\ = \hat{x}\hat{p} - \hat{p}\hat{x} \\ \to -i\hbar x \frac{\partial}{\partial x}\psi + i\hbar \frac{\partial}{\partial x}(x\psi) \\ = -i\hbar x \frac{\partial}{\partial x}\psi + i\hbar\psi + i\hbar x \frac{\partial}{\partial x}\psi \\ = i\hbar\psi \\ \to i\hbar \end{gathered}$$

2. Annihilation and creation

$$\begin{gathered}
[a, a^\dagger] \\
= \frac{m\omega}{2\hbar} \cdot \frac{1}{m\omega} \left(-i[\hat{x}, \hat{p}] - i[\hat{x}, \hat{p}]\right) \\
= \frac{1}{2\hbar} \left(2\hbar\right) \\
= 1
\end{gathered}$$

3. Number operator and annhilation operator

$$\begin{gathered}
[N, a] \\
= Na - aN \\
= a^\dagger aa - aa^\dagger a \\
= [a^\dagger, a] \cdot a \\
= -a
\end{gathered}$$

4. Number operator and creation operator

$$\begin{gathered}
[N, a^\dagger] \\
= a^\dagger aa^\dagger - a^\dagger a^\dagger a \\
= a^\dagger \cdot [a, a^\dagger] \\
= a^\dagger
\end{gathered}$$