PROBLEM SET 4 – PHYS 0500

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Solution 6-2.
We define the length of a curve as

$$\int_{(x_1,y_1)}^{(x_2,y_2)} \sqrt{dx^2 + dy^2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

which is the quantity we want to extremize. Since the integrand is strictly real and positive, minimizing the integrand is equivalent to minimizing its integral. So, we let $f = \sqrt{\dot{x}^2 + \dot{y}^2}$. Then, we use Euler's equation with respect to $\dot{x}$ and a second time with respect to $\dot{y}$, which gives

$$\frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0, \quad \frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0$$
$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1, \quad \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_2$$

We can divide these equations by each other to get

$$\frac{\dot{y}}{\dot{x}} = \frac{C_1}{C_2}$$
$$C_2\dot{y} = C_1\dot{x}$$

And now we consider two integrations of this equation–one from t_1 to t, and one from t_1 to t_2 to obtain the values of C_1 and C_2

$$C_2(y - y_1) = C_1(x - x_1), \quad C_2(y_2 - y_1) = C_1(x_2 - x_1)$$
$$\frac{C_1}{C_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

And now plugging this ratio into our general equation gives

$$(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Which is precisely the straight line that passes through (x_1, y_1) and (x_2, y_2).

Solution 6-3.
We set up in a manner similar to problem 6-2. However, we will use a parameterized representation this time, considering the curve $\langle x(t), y(t), z(t)\rangle$. We

write

$$S = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \sqrt{dx^2 + dy^2 + dz^2} = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

And again, we minimize the integrand, letting $f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. Using Euler's equation three times, we have

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_1$$
$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_2$$
$$\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = C_3$$

We can take ratios between the equations, which gives

$$\frac{\dot{x}}{\dot{y}} = \frac{C_1}{C_2}$$
$$\frac{\dot{y}}{\dot{z}} = \frac{C_2}{C_3}$$
$$C_2\dot{x} = C_1\dot{y}$$
$$C_3\dot{y} = C_2\dot{z}$$

Integrating both sides from t_1 to t gives

$$C_2(x - x_1) = C_1(y - y_1)$$
$$C_3(y - y_1) = C_2(z - z_1)$$
$$\frac{y - y_1}{C_2} = \frac{x - x_1}{C_1} = \frac{z - z_1}{C_3}$$

Which is precisely the equation of a line in three dimensions. To solve for the constants C_1, C_2, and C_3, we can integrate instead from t_1 to t_2, which gives

$$C_2(x_2 - x_1) = C_1(y_2 - y_1)$$
$$C_3(y_2 - y_1) = C_2(z_2 - z_1)$$

Then, we can rearrange as follows

$$\frac{y - y_1}{C_2(y_2 - y_1)} = \frac{x - x_1}{C_1(y_2 - y_1)} = \frac{z - z_1}{C_3(y_2 - y_1)}$$
$$\frac{y - y_1}{C_2(y_2 - y_1)} = \frac{x - x_1}{C_2(x_2 - x_1)} = \frac{z - z_1}{C_2(z_2 - z_1)}$$
$$\frac{y - y_1}{(y_2 - y_1)} = \frac{x - x_1}{(x_2 - x_1)} = \frac{z - z_1}{(z_2 - z_1)}$$

Which is exactly the straight line that passes through (x_1, y_1, z_1) and (x_2, y_2, z_2).

Solution 6-4.
Once again, we consider the line element

$$dS = \sqrt{dx^2 + dy^2 + dz^2}$$

As we are considering a right circular cylinder, we switch to cylindrical coordinates, noting the following relations

$$\begin{gathered}
x = r\cos\theta \to dx = -r\sin\theta\, d\theta \\
y = r\sin\theta \to dy = r\cos\theta d\theta \\
z = z \to dz = dz
\end{gathered}$$

Then

$$dS = \sqrt{r^2\sin^2\theta d\theta^2 + r^2\cos^2\theta d\theta^2 + dz^2} = \sqrt{r^2\, d\theta^2 + dz^2} = \sqrt{r^2 + \dot{z}^2}\, d\theta$$

We put this into Euler's equation

$$\begin{gathered}
\frac{d}{d\theta}\frac{\dot{z}}{\sqrt{r^2+\dot{z}^2}} = 0 \\
\frac{\dot{z}}{\sqrt{r^2+\dot{z}^2}} = C \\
\dot{z} = C\sqrt{r^2+\dot{z}^2} \\
\dot{z}^2 = C^2(r^2+\dot{z}^2) \\
\dot{z} = \sqrt{\frac{C^2r^2}{1-C^2}}
\end{gathered}$$

Noting that the RHS is a constant, this implies $\frac{dz}{d\theta}$ is constant. But given that r is constant, the resulting curve must then be a helix, and we are done.

Solution 6-7.
Let the speed of a light ray through a given medium be v. Then, we can define the time it takes to travel from one point to another as

$$T = \int_{x_1}^{x_2} \frac{1}{v}\cdot\sqrt{1+(y')^2}\, dx$$

Note that v is a function of y, as at some y value, it must traverse into a new medium, at which point its value changes. We designate the switch to happen at $y = 0$. We minimize the integrand by inputting it into Euler's equation, yielding

$$\begin{gathered}
\frac{d}{dx}\frac{y'}{v\sqrt{1+(y')^2}} = 0 \\
\frac{y'}{v\sqrt{1+(y')^2}} = C
\end{gathered}$$

At this point we can approximate path of the light ray in the process of switching mediums with a straight line (local linearity) leading us to the substitution $y' = -\tan\theta$, where θ is the angle the ray makes with the horizontal (assuming the ray is coming from above). We also make the refraction index substitution $v = \frac{c}{n}$ to get

$$\begin{aligned} \frac{n}{c} \cdot \frac{-\tan\theta}{\sec\theta} &= C \\ n\sin\theta &= -Cc \end{aligned}$$

And since the RHS would be the same constant for any corresponding n and θ, we must have that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

And we are done.

Solution 6-15.

a. We have $f = (y')^2 - y^2$. Plugging this into Euler's equation gives

$$\begin{aligned} -2y - \frac{d}{dx} \cdot 2y' &= 0 \\ y + y'' &= 0 \\ y'' &= -y \\ y &= A\cos(x) + B\sin(x) \end{aligned}$$

We plug in initial conditions

$$\begin{aligned} 0 &= A \\ 1 &= B\sin(1) \\ y &= \frac{1}{\sin(1)}\sin(x) \end{aligned}$$

b. The minimum value of the integral is given by

$$\frac{1}{\sin^2(1)} \int_0^1 \cos^2 x - \sin^2 x \, dx = \frac{1}{\sin^2(1)} \int_0^1 \cos(2x)\, dx = \frac{\sin(2)}{2\sin^2 1} \equiv 0.6421$$

c. We write

$$I[y] = \int_0^1 1 - x^2 \, dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

And we are done.