PROBLEM SET 2 – PHYS 0500

Ishaan Ganti
Brown University
Advanced Classical Mechanics

21 februari 2024

Solution 3-6.
Let the coordinate of m_1 be x_1, and let the coordinate of m_2 be x_2. We also let the equilibrium length of the spring be l. Then, we have the equations of motion

$$m_1\ddot{x_1} = -k(x_2 - x_1 - l)$$
$$m_2\ddot{x_2} = k(x_2 - x_1 - l)$$

With the only difference between the equations being the sign difference due to Newton's third law. Notice that the $-l$ term is essentially nothing more than a coordinate shift, so for the sake of simplifying calculations, we can leave it out, since only relative position is relevant. Then, we have

$$m_1 a_1 = -k(x_2 - x_1)$$
$$m_2 a_2 = k(x_2 - x_1)$$

Let $x_2 - x_1 = x$, then

$$m_1 a_1 = -kx$$
$$m_2 a_2 = kx$$
$$a = a_2 - a_1 = \frac{kx}{m_1} + \frac{kx}{m_2}$$
$$\frac{a}{x} = \omega^2 = \frac{k}{m_2} + \frac{k}{m_1}$$
$$\omega^2 = k\frac{m_1 + m_2}{m_1 m_2}$$
$$\omega = \sqrt{k\frac{m_1 + m_2}{m_1 m_2}}$$

Putting in the numbers, we have

$$\omega = \sqrt{0.5 \cdot \frac{0.1 + 0.2}{0.1 \cdot 0.2}} = 2.7386 \text{ Hz}$$

And we are done.

Solution 3-13.
We have, via Newton's second law,

$$
\begin{aligned}
ma &= -kx - bv \\
ma + bv + kx &= 0 \\
\ddot{x} + 2\beta\dot{x} + \omega^2 x &= 0
\end{aligned}
$$

Now, if we assume $x = y(t)\exp(-\beta t)$, we first calculate x' and x''

$$
\begin{aligned}
x' &= y'e^{-\beta t} + -\beta y e^{-\beta t} \\
x'' &= y''e^{-\beta t} - \beta y' e^{-\beta t} - \beta y' e^{-\beta t} + \beta^2 y e^{-\beta t}
\end{aligned}
$$

We plug this into our Newton's law equation to get

$$
\begin{aligned}
y''e^{-\beta t} - \beta y'e^{-\beta t} - \beta y'e^{-\beta t} + \beta^2 ye^{-\beta t} + 2\beta y'e^{-\beta t} + -2\beta^2 ye^{-\beta t} + \omega^2 ye^{-\beta t} &= 0 \\
y'' - \beta y' - \beta y' + \beta^2 y + 2\beta y' + -2\beta^2 y + \omega^2 y &= 0 \\
y'' + \beta^2 y - 2\beta^2 y + \omega^2 y &= 0 \\
y''(\omega^2 - \beta^2)y &= 0 \\
y'' &= 0 \\
y &= A + Bt
\end{aligned}
$$

Which is precisely the form of $y(t)$ in equation 3.43.

Solution 3-22.

a. We proceed by using the known form of the overdamping equation and its first derivative. We start with

$$
\begin{aligned}
x(t) &= e^{-\beta t}\left(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right) \\
x(0) &= A_1 + A_2 = x_0
\end{aligned}
$$

Which is our first equation. We then differentiate the very first equation and proceed similarly

$$
\begin{aligned}
x'(t) = (\omega_2 - \beta)A_1 e^{\omega_2 t - \beta t} + (-\omega_2 - \beta)A_2 e^{-\omega_2 t - \beta t} = v(t) \\
v(0) = (\omega_2 - \beta)A_1 - (\omega_2 + \beta)A_2 = v_0
\end{aligned}
$$

We make the appropriate substitutions given in the problem and then solve the system normally

$$
\begin{aligned}
v_0 &= -\beta_1 A_1 - \beta_2 A_2 \\
A_1 &= x_0 - A_2 \\
v_0 &= -\beta_1(x_0 - A_2) - \beta_2 A_2 \\
\beta_2 A_2 - \beta_1 A_2 &= -v_0 - \beta_1 x_0 \\
A_2 &= -\frac{v_0 + \beta_1 x_0}{\beta_2 - \beta_1} \\
A_1 &= x_0 + \frac{v_0 + \beta_1 x_0}{\beta_2 - \beta_1} = \frac{v_0 + x_0\beta_2}{\beta_2 - \beta_1}
\end{aligned}
$$

And we are done.

b. If $A_1 = 0$, we have

$$\begin{aligned} x(t) &= e^{-\beta t} \cdot A_2 e^{-\omega_2 t} \\ x'(t) &= -(\beta + \omega_2) A_2 e^{-\beta t} e^{-\omega_2 t} \\ x'(t) &= -\beta_2 x(t) \\ \dot{x} &= -\beta_2 x \end{aligned}$$

And otherwise, when $t \to \infty$, we note that the A_2 term goes to 0. So then we have

$$\begin{aligned} x(t) &= e^{-\beta t}(A_1 e^{\omega_2 t}) \\ v(t) &= -(\beta - \omega_2) e^{-\beta t} A_1 e^{\omega_2 t} \\ v(t) &= -\beta_1 x(t) \\ \dot{x} &= -\beta_1 x \end{aligned}$$

Solution 3-42.

a. The problem statement gives us the EOM in the form of a second order differential equation, which we can solve normally. We have

$$m(\ddot{x} + \omega_0^2 x) = F_0 \sin(\omega t)$$

We know that the homogenous solution for $x(t)$ is of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

And we guess our particular solution $x_p(t)$ to be of the form $C\sin(\omega t)$, which we may plug back into our original equation to get

$$\begin{aligned} m(-C\omega^2 \sin(\omega t) + C(\omega_0^2)\sin(\omega t)) &= F_0 \sin(\omega t) \\ m(-C\omega^2 + C(\omega_0^2)) &= F_0 \\ C &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \end{aligned}$$

So far, then, we have

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin(\omega t)$$

We plug in initial conditions to solve for A and B, starting with $x(0) = 0$

$$\begin{aligned} 0 &= A(1) + B(0) + 0 \to A = 0 \\ x(t) &= B\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\sin(\omega t) \\ x'(t) &= B\omega_0 \cos(\omega_0 t) + \frac{F_0 \omega}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \\ 0 &= B\omega_0 + \frac{F_0 \omega}{m(\omega_0^2 - \omega^2)} \\ B &= \frac{F_0 \omega}{m\omega_0(\omega^2 - \omega_0^2)} \end{aligned}$$

Giving us the final EOM

$$x(t) = \frac{F_0\omega}{m\omega_0(\omega^2-\omega_0^2)}\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2-\omega^2)}\sin(\omega t)$$
$$x(t) = \frac{F_0}{m\omega_0(\omega_0^2-\omega^2)}(\omega_0\sin(\omega t) - \omega\sin(\omega_0 t))$$

b. If we let $\omega = \omega_0$, we have the limit

$$\lim_{\omega\to\omega_0}\frac{F_0}{m\omega_0(\omega_0^2-\omega^2)}(\omega_0\sin(\omega t) - \omega\sin(\omega_0 t))$$

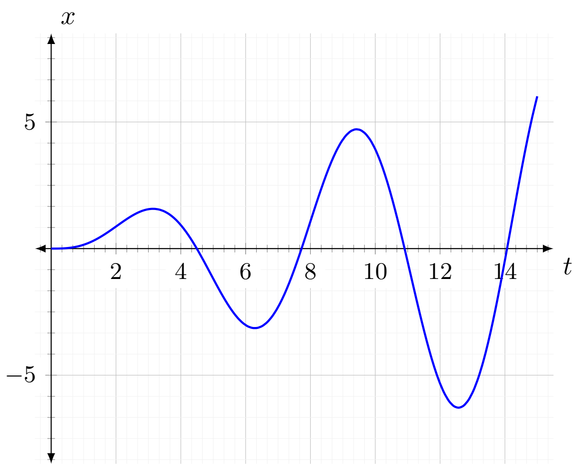
Which is indeterminate, so we use L'Hopital's rule to get

$$\lim_{\omega\to\omega_0}\frac{F_0\omega_0 t\cos(\omega t) - F_0\sin(\omega_0 t)}{-2m\omega_0\omega}$$

Now, we can plug in values directly, giving

$$\frac{F_0}{2m\omega_0^2}(\sin(\omega_0 t) - t\omega_0\cos(\omega_0 t))$$

Which, when sketched out, looks something like this



Solution 3-44.
Since there is actually oscillation, the damping must be underdamping. So, we work with the equation

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

Since ω_1 is the frequency of oscillation, we must have that four cycles corresponds to $\omega_1 t = 8\pi$, so $t = \frac{8\pi}{\omega_1}$, giving a new amplitude of

$$A\exp\left(-\beta\frac{8\pi}{\omega_1}\right)$$

Then, by taking the ratio of the two amplitudes, we have the relationship

$$\exp\left(-\beta\frac{8\pi}{\omega_1}\right) = e^{-1}$$

Expanding β as $\sqrt{\omega_0^2 - \omega_1^2}$ gives

$$\begin{gathered}
\exp\left(\sqrt{\omega_0^2 - \omega_1^2}\frac{8\pi}{\omega_1}\right) = e^{-1} \\
\sqrt{\omega_0^2 - \omega_1^2}\frac{8\pi}{\omega_1} = -1 \\
(\omega_0^2 - \omega_1^2) \cdot \frac{64\pi^2}{\omega_1^2} = 1 \\
\frac{64\omega_0^2\pi^2}{\omega_1^2} - 64\pi^2 = 1 \\
\frac{\omega_0^2}{\omega_1^2} = (1 + 64\pi^2) \cdot \frac{1}{64\pi^2} \\
\frac{\omega_1}{\omega_0} = \frac{8\pi}{\sqrt{1 + 64\pi^2}}
\end{gathered}$$

And we are done.