PROBLEM SET 8 – APMA 0360

Ishaan Ganti

27 april 2024

Solution 1.

a. We choose $g(t) + xh(t)$. We check

$$v(0,t) = g(t) + 0 \cdot h(t) = g(t) \tag{1}$$
$$v_x(\pi,t) = h(t) \tag{2}$$

b. Since both conditions are derivatives, we can't treat this problem as we did in the previous part. We make use of sines and cosines, noting the boundary condition x-values are 0 and π. We consider $v(x,t) = 2\sin(0.5x)g(t) - 2\cos(0.5x)h(t)$. We check

$$v_x(0,t) = \cos(0.5 \cdot 0)g(t) + \sin(0.5 \cdot 0)h(t) = g(t) \tag{3}$$
$$v_x(\pi,t) = \cos(0.5\pi)g(t) + \sin(0.5\pi)h(t) = h(t) \tag{4}$$

Solution 2.
We use the heat equation with a source formula. We know that our solution will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx) \tag{5}$$

where $A_n = e^{-n^2 t} \int_0^t e^{n^2 s} q_n(s)\, ds + e^{-n^2 t} a_n$. We solve for q_n and a_n first

$$q_n(t) = \frac{2}{\pi} \int_0^{\pi} e^{-t} \sin(3x)\sin(nx)\, dx \tag{6}$$
$$q_n(t) = \frac{2e^{-t}}{\pi} \int_0^{\pi} \sin(3x)\sin(nx)\, dx \tag{7}$$

This integral is 0 for all n except for $n = 3$; then, it simplifies to $\pi/2$. So,

$$q_3(t) = e^{-t} \tag{8}$$

We solve for a_n

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx = -\frac{2}{n\pi} \left[\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi} \cdot (1 - (-1)^n) \tag{9}$$

Now, we evaluate the integral term within A_n, noting that it is only non-zero for $n = 3$

$$\int_0^t e^{9s} e^{-s} \, ds = \frac{1}{8} \left[e^{8s} \right]_0^t = \frac{1}{8}(e^{8t} - 1) \tag{10}$$

Putting everything together yields

$$u(x,t) = e^{-9t} \cdot \frac{1}{8}(e^{8t} - 1) \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \cdot \frac{2}{n\pi} \cdot (1 - (-1)^n) \sin(nx) \tag{11}$$

$$u(x,t) = \frac{1}{8}(e^{-t} - e^{-9t}) \sin(3x) + \sum_{n=1}^{\infty} e^{-n^2 t} \cdot \frac{2}{n\pi} \cdot (1 - (-1)^n) \sin(nx) \tag{12}$$

Solution 3.
We define

$$u(x,t) = v(x,t) + \frac{t}{\pi} x \tag{13}$$

And we note that $v(x,t)$ solves

$$v_t - v_{xx} = -\frac{x}{\pi} \tag{14}$$

$$v(0,t) = 0, \quad v(\pi,t) = 0, \quad v(x,0) = 0 \tag{15}$$

So, we use the same procedure as in the previous problem to proceed. We calculate q_n

$$q_n(t) = \frac{2}{\pi} \int_0^{\pi} -\frac{x}{\pi} \sin(nx) \, dx = \frac{2(-1)^n}{n\pi} \tag{16}$$

And $a_n = 0$ since $u(x,0) = 0$. A_n is then given by

$$A_n = e^{-n^2 t} \int_0^t e^{n^2 s} q_n(s) \, ds \tag{17}$$

$$A_n = e^{-n^2 t} \cdot \frac{2(-1)^n}{n\pi} \cdot \left[\frac{1}{n^2} e^{n^2 s} \right]_0^t = \frac{2(-1)^n}{n^3 \pi} \left(1 - e^{-n^2 t} \right) \tag{18}$$

Giving the final solution

$$u(x,t) = \frac{t}{\pi} x + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3 \pi} \left(1 - e^{-n^2 t} \right) \sin(nx) \tag{19}$$

Solution 4.
We will find a particular solution and then solve a IVP with slightly altered boundary conditions to account for the $u(x,0) = 0$ boundary condition. First, we consider a particular solution of the form $f(x) = ax^2 + bx + c$. To satisfy the first two boundary conditions, we need $a = 0.5$, $b = -\frac{\pi}{2}$, and $c = 0$. So, we have

$$u_p(x) = \frac{1}{2} x^2 - \frac{\pi}{2} x \tag{20}$$

Now, we can solve the wave equation with the same boundary conditions *except* we need $u(x,0) = -\left(\frac{1}{2}x^2 - \frac{\pi}{2}x\right)$ to cancel out the particular solution. We solve for the homogeneous solution

$$X = A\cos(kx) + B\sin(kx), \quad T = C\cos(kt) + D\sin(kt) \tag{21}$$

We plug in boundary conditions. For X, this means that $A = 0$ and $k \in \mathbb{N}$. For T, the second boundary condition implies $D = 0$. The first condition implies

$$\sum_{n=1}^{\infty} A_n \sin(nx) = \frac{1}{2}x^2 - \frac{\pi}{2}x \tag{22}$$

We calculate A_n

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx) \cdot \left(\frac{1}{2}x^2 - \frac{\pi}{2}x\right) dx = \frac{2((-1)^n - 1)}{\pi n^3} \tag{23}$$

Which gives the final solution

$$u(x,t) = \frac{1}{2}x^2 - \frac{\pi}{2}x - \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{\pi n^3} \sin(nx)\cos(nt) \tag{24}$$

And we are done.