PROBLEM SET 2 – APMA 0360

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Solution 1.

a. We have

$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{ikx} g(x)\, dx$$
$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x-a)\, dx$$

Letting $u = x - a$, which implies $du = dx$, we have

$$\hat{g}(k) = \int_{-\infty}^{\infty} e^{ik(u+a)} f(u)\, du$$
$$\hat{g}(k) = e^{ika} \int_{-\infty}^{\infty} e^{iku} f(u)\, du$$
$$\hat{g}(k) = e^{ika} \hat{f}(k)$$

And we are done.

b. We start by taking the Fourier transform of both sides, using integration by parts to deal with the transform of u_x.

$$u_t + cu_x = 0$$
$$\int_{-\infty}^{\infty} e^{ikx} u_t\, dx + c\int_{-\infty}^{\infty} e^{ikx} u_x\, dx = 0$$
$$\frac{d}{dt}\hat{u}(k,t) - ikc\int_{-\infty}^{\infty} e^{ikx} u\, dx = 0$$
$$\frac{d}{dt}\hat{u}(k,t) - ikc\hat{u}(k,t) = 0$$

For the sake of simplification, define $p = \hat{u}(k,t)$. Then

$$\dot{p} - ikcp = 0$$
$$\frac{dp}{dt} = ikcp$$
$$\int p^{-1}\,dp = \int ikc\,dt$$
$$\ln(p) = ikct + C(x)$$
$$p = e^{ikct+C(x)}$$
$$\hat{u}(k,t) = e^{ikct}B(x)$$

Note that the Fourier transform of $u(x,0) = f(x)$ is simply $\hat{u}(k,0) = \hat{f}(k)$. So, we have

$$\hat{u}(k,0) = e^{ikc\cdot 0}B(x)$$
$$\hat{u}(k,0) = B(x) = \hat{f}(k)$$
$$\hat{u}(k,t) = e^{ikct}\hat{f}(k)$$

Finally, by part (a) with $a = ct$, we get the desired result

$$u(x,t) = f(x-ct)$$

Solution 2.

a. Given a solution of the form $u(x,t) = e^{\lambda t}e^{ikx}$, we start by plugging it into the heat equation

$$u_t = Du_{xx}$$
$$\lambda e^{\lambda t}e^{ikx} = -Dk^2 e^{\lambda t}e^{ikx}$$

Since e^{ikx} and $e^{\lambda t}$ are necessarily non-zero, we can divide out by them, giving

$$\lambda = -Dk^2$$

And we are done.

b. Using our answer from (a), we write

$$u(x,t) = \int_{-\infty}^{\infty} a(k)e^{\lambda t}e^{ikx}\,dk$$
$$u(x,t) = \int_{-\infty}^{\infty} a(k)e^{-Dk^2 t}e^{ikx}\,dk$$

Now, we use the heat equation to find u_t and u_{xx}

$$u_t = \int_{-\infty}^{\infty} -Dk^2 a(k)e^{-Dk^2 t}e^{ikx}\,dk$$
$$u_{xx} = \int_{-\infty}^{\infty} -k^2 a(k)e^{-Dk^2 t}e^{ikx}\,dk$$

Plugging this into the heat equation gives

$$\begin{aligned}
u_t &= Du_{xx} \\
\int_{-\infty}^{\infty} -Dk^2 a(k) e^{-Dk^2 t} e^{ikx}\, dk &= D \int_{-\infty}^{\infty} -k^2 a(k) e^{-Dk^2 t} e^{ikx}\, dk \\
\int_{-\infty}^{\infty} -Dk^2 a(k) e^{-Dk^2 t} e^{ikx}\, dk &= \int_{-\infty}^{\infty} -Dk^2 a(k) e^{-Dk^2 t} e^{ikx}\, dk
\end{aligned}$$

And since the LHS and the RHS are the exact same expression, $u(x,t)$ is a solution. Note that $a(k)$ being a part of the Schwartz class is important to ensure that the integral is finite.

Solution 3.
We have

$$\begin{aligned}
u_t &= u_{xxxx} \\
\int_{-\infty}^{\infty} e^{ikx} u_t\, dx &= \int_{-\infty}^{\infty} e^{ikx} u_{xxxx}\, dx
\end{aligned}$$

We use integration by parts four times on the RHS to simplify the integral as follows

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{ikx} u_t\, dx &= \int_{-\infty}^{\infty} (ik)^4 e^{ikx} u\, dx \\
\frac{\partial}{\partial t} \hat{u}(k,t) &= k^4 \hat{u}(k,t)
\end{aligned}$$

This is an ODE of the form $y' = ky$, which we can solve easily as follows

$$\hat{u}(k,t) = C(k) \cdot e^{k^4 t}$$

Note that

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} e^{ikx} u(x,0)\, dx = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx = \hat{f}(k)$$

So

$$\hat{u}(k,0) = C(k) \cdot e^{k^4 \cdot 0} = C(k) = \hat{f}(k)$$

Giving us a final solution of

$$\hat{u}(k,t) = \hat{f}(k) \cdot e^{k^4 t}$$

Solution 4.

a. We start by finding v_t and v_{yy}. We have

$$\begin{aligned}
v_t &= \frac{\partial}{\partial t} u(y-ct,t) e^{at} \\
&= a \cdot u(y-ct,t) e^{at} + (-c \cdot u_x(y-ct,t) + u_t(y-ct,t)) e^{at} \\
v_{yy} &= u_{xx}(y-ct,t) e^{at}
\end{aligned}$$

We can plug this into the equation $v_t = Dv_{yy}$ to get

$$e^{at}(au(y-ct,t) - cu_x(y-ct,t) + u_t(y-ct,t)) = De^{at}(u_{xx}(y-ct,t))$$

Since e^{at} is necessarily non-zero, we can divide out by it, giving

$$\begin{aligned} au(y-ct,t) - cu_x(y-ct,t) + u_t(y-ct,t) &= Du_{xx}(y-ct,t) \\ u_t(y-ct,t) &= Du_{xx}(y-ct,t) + cu_x(y-ct,t) - au(y-ct,t) \\ u_t &= Du_{xx} + cu_x - au \end{aligned}$$

Which is given as true, and we are done.

b. Based on the equation $u_t = u_{xx} + 2u_x - 0.5u$, we know $D = 1$, $c = 2$, and $a = 0.5$. Then, we use the fact that $x = y - ct \to y = x + ct$ and subtitute to get

$$\begin{aligned} v(y,t) &= u(y-ct,t)e^{at} \\ &= u(y-2t,t)e^{0.5t} \\ \frac{1}{\sqrt{4\pi t}}e^{\frac{-y^2}{4t}} &= u(y-2t,t)e^{0.5t} \\ \frac{1}{\sqrt{4\pi t}}e^{\frac{-y^2}{4t}}e^{-0.5t} &= u(y-2t,t) \\ x = y - 2t &\to y = x + 2t \\ u(x,t) &= \exp\left(-0.5t - \frac{(x+2t)^2}{4t}\right) \cdot \frac{1}{\sqrt{4\pi t}} \end{aligned}$$

Which is our $u(x,t)$.