Chapter 1 Reading Notes

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1 Introduction

We start by discussing an 'incompleteness' of the rationals that motivates the study of the reals. Put bluntly, there exists no such $p \in \mathbb{Q}$ that satisfies $p^2 = 2$. The proof of this is pretty straightforward and is widely found, hence omitted. Instead, I will prove a related result.

Theorem 1

Let A be the set of all $a \in Q$ such that $a^2 < 2$. Similarly, let B be the set of all $b \in Q$ such that $b^2 > 2$. Then A has no largest element and B has no smallest element.

Proof

For any $p \in Q$, consider the value given by

$$\gamma = p - \frac{p^2 - 2}{p + 2}$$

If $p \in A$, then either $p^2 - 2 < 0$ and $p + 2 > 0$ or $p^2 - 2 > 0$ and $p + 2 < 0$, so $\gamma > p$. Then, we calculate $\gamma^2 - 2$

$$\gamma = \frac{p^2 + 2p - p^2 + 2}{p + 2} = \frac{2p + 2}{p + 2}$$
$$\gamma^2 - 2 = \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p + 2)^2} = \frac{2p^2 - 4}{(p + 2)^2} = \frac{2(p^2 - 2)}{(p + 2)^2}$$

But then $p^2 - 2$ in the numerator is negative, so $\gamma^2 - 2 < 0$ and $\gamma \in A$. Practically the same logic applies for showing that B has no smallest element.

The bit of this proof that was tricky for me was defining the expression that explicitly shows that A has no largest element and B has no smallest element. The proof statement itself, however, felt pretty intuitive.

Theorem 2: Least upper bound property implies greatest lower bound

Suppose S is an ordered set with the least upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S and $\alpha = \inf B$.

Proof

First, we show that L is bounded above. Since for all $b \in B$, $l \in L$, we know $l \leq b$, all b serve as upper bounds for L. Then, since S possesses the least upper-bound property, there exists some $\alpha = \sup L$ in S. This proves the first part of the theorem.

Now, consider some other γ in S. Assume $\gamma > \alpha$. Then, since α is the supremum of L, $\gamma \notin L$. By the definition of L, γ is not a lower bound of B. However, if $\gamma < \alpha$, then it cannot be the infimum of B as α bounds B from below as $\alpha \in L$ and $\alpha > \gamma$. Thus, α must be the infimum of B.

Note: paying attention to definitions is important. B being bounded below as a subset of S implies that all values that bound B below *are within* S by the definition of bounded below. So, we must have $L \subset S$.

2 Fields

The text goes on to discuss fields in some length. I will skip over the introductory propositions as I have experience in algebra, but I will prove one set of propositions just as a sort of checkpoint.

Theorem 1

The following statements are true in every ordered field.

1. If $x > 0$ then $-x < 0$ and vice versa.
2. If $x > 0$ and $y < z$ then $xy < xz$.
3. If $x < 0$ and $y < z$ then $xy > xz$.
4. If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
5. If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof

1. Since $x > 0$, $-x + x > -x$, but this is just $0 > -x$ or $-x < 0$.
2. $xz = x(z - y) + xy > 0 + xy = xy$.
3. $xy = x(y - z) + xz > 0 + xz > xz$.

I got too lazy to do the rest; they follow from the definitions and some of the earlier propositions.

3 The Real Field