PROBLEM SET 1 - APMA 0360

Ishaan Ganti
Brown University
Applied PDEs

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Solution 1.

a. We calculate the discriminant

$$u_{xx} - 3u_{xy} + 2u_{yy} + u_y + 5u = 0, \quad D = b^2 - 4ac$$
$$D = ((-3)^2 - 4 \cdot 1 \cdot 2) = 1$$

so the PDE is hyperbolic.

b. As in part (a), we have

$$9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$$
$$D = 6^2 - 4 \cdot 9 \cdot 1 = 0$$

so the PDE is parabolic.

Solution 2.
Recall that the solution to the transport equation of the form

$$u_t + cu_x = 0$$

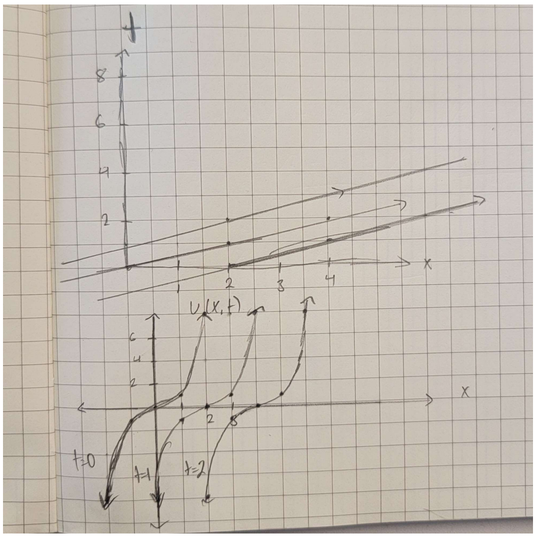
Is of the form

$$u(x,t) = u_0(x - ct)$$

We are given $u_0 = x^3$ and $c = 2$, so we get the solution

$$u(x,t) = (x - 2t)^3$$

The characteristic lines and the graphs of $u(x,0), u(x,1), u(x,2)$ are as follows



Solution 3.
We plug the solution into the differential equation. We have

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}, \quad u_t = Du_{xx}$$

$$u_t = -\frac{1}{2\sqrt{4\pi Dt^3}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{4Dt^2} e^{-\frac{x^2}{4Dt}} \cdot \frac{1}{\sqrt{4\pi Dt}}$$

$$u_{xx} = -\frac{1}{2Dt\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{4D^2t^2} \cdot \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Now, we can plug these partial derivatives into the equation and simplify

$$-\frac{1}{2\sqrt{4\pi Dt^3}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{4Dt^2} e^{-\frac{x^2}{4Dt}} \cdot \frac{1}{\sqrt{4\pi Dt}} = D \cdot \left(-\frac{1}{2Dt\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} + \frac{x^2}{4D^2t^2} \cdot \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \right)$$

$$-\frac{1}{2\sqrt{4\pi Dt^3}} + \frac{x^2}{4Dt^2} \cdot \frac{1}{\sqrt{4\pi Dt}} = -\frac{1}{2t\sqrt{4\pi Dt}} + \frac{x^2}{4Dt^2} \cdot \frac{1}{\sqrt{4\pi Dt}}$$

$$-\frac{1}{2\sqrt{4\pi Dt^3}} + \frac{x^2}{4Dt^2} \cdot \frac{1}{\sqrt{4\pi Dt}} = -\frac{1}{2\sqrt{4\pi Dt^3}} + \frac{x^2}{4Dt^2} \cdot \frac{1}{\sqrt{4\pi Dt}}$$

$$0 = 0$$

Thus, $u(x,t)$ is a solution on the specified intervals.
Solution 4.

$$P = \int_{-\infty}^{\infty} e^{-Qx^2}\, dx$$

$$P^2 = \int_{-\infty}^{\infty} e^{-Qx^2}\, dx \cdot \int_{-\infty}^{\infty} e^{-Qy^2}\, dy$$

$$P^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Qx^2} e^{-Qy^2}\, dx\, dy$$

$$P^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Q(x^2+y^2)}\, dx\, dy$$

$$P^2 = \int_0^{2\pi} \int_0^{\infty} e^{-Qr^2} r \, dr \, d\theta$$
$$P^2 = \int_0^{2\pi} \left[\frac{e^{-Qr^2}}{-2Q} \right]_0^{\infty} d\theta$$
$$P^2 = \int_0^{2\pi} \frac{1}{2Q} \, d\theta$$
$$P^2 = \frac{2\pi}{2Q} = \frac{\pi}{Q}$$
$$P = \sqrt{\frac{\pi}{Q}}$$

Solution 5.

a. We have

$$u(x, t + \tau)h = u(x - h, t)h$$

where h is a small distance interval and τ is a small time interval. Essentially, the equation states that the amount of particles at position x at some time $t + \tau$ is equal to the amount of particles at position $x + h$ at earlier time t.

b. Using Taylor series expansions, we have

$$u(x, t + \tau) = u(x, t) + u_t(x, t)\tau + O(\tau^2)$$
$$u(x - h, t) = u(x, t) - u_x(x, t)h + O(h^2)$$

Substituting, we get

$$(u(x, t) + u_t(x, t)\tau + O(\tau^2))h = (u(x, t) - u_x(x, t)h + O(h^2))h$$
$$(u_t(x, t)\tau + O(\tau^2))h = (-u_x(x, t)h + O(h^2))h$$
$$u_t(x, t)\tau h + O(\tau^2)h = -u_x(x, t)h^2 + O(h^2)h$$

c. Now, we let the higher order terms go to 0 and divide both sides by $h\tau$

$$\frac{u_t(x, t)\tau h}{h\tau} = \frac{-u_x(x, t)h^2}{h\tau}$$
$$u_t(x, t) = -u_x(x, t)\frac{h}{\tau}$$

And we let $\frac{-h}{\tau} = c$, the rate at which particles move to the right, giving us the desired

$$u_t(x, t) = cu_x(x, t)$$