# Analytic Solutions of the Jaynes-Cummings Model

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## 1 Moving Into the Rotating Frame

We start with the Jaynes-Cummings Hamiltonian

$$H_{JC} = \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + j(a^\dagger \sigma_- + \sigma_+ a) \tag{1}$$

Where we have set $\hbar = 1$ for simplicity. We consider changing our frame of reference i.e. moving into the rotating frame via the unitary

$$R = \exp\left(i\omega t\left(a^\dagger a + \frac{1}{2}\sigma_z\right)\right) \tag{2}$$

Then, our new Hamiltonian should be of the form

$$H_{RJC} = RH_{JC}R^\dagger + i\dot{R}R^\dagger \tag{3}$$

We start by calculating $i\dot{R}R^\dagger$, which gives

$$i \cdot i\omega\left(a^\dagger a + \frac{1}{2}\sigma_z\right)RR^\dagger \tag{4}$$

$$= -\omega a^\dagger a + -\frac{\omega}{2}\sigma_z \tag{5}$$

### 1.1 Commutatation Relations within $H_{RJC}$

To handle the $RH_{JC}R^\dagger$ term of $H_{RJC}$, we separate $H_{JC}$ into its free and interaction components

$$H_{JC} = H_0 + H_I \tag{6}$$

And we address these components separately.

#### 1.1.1 Commutativity with $H_0$

Recall that some operator $A$ always computes with a function of that same operator. This is easily seen by expanding whatever function $f(A)$ as a Taylor series. Additionally, recall that operators acting on different Hilbert spaces must commute since

$$\begin{aligned}
&[X \otimes I_Y, I_X \otimes Y] \\
&= (X \otimes I_Y)(I_X \otimes Y) - (I_X \otimes Y)(X \otimes I_Y) \\
&= (X \otimes Y) - (X \otimes Y) \\
&= 0
\end{aligned}$$

With these facts in mind, we view $R$ as a function of $a^\dagger a$ and $\sigma_z$, also noting that these two parameters belong to different Hilbert spaces. Thus, we can separate $R$ into a more general, simplified product

$$R = F(a^\dagger a)G(\sigma_z) \tag{7}$$

Where we require both $F$ and $G$ to be unitary, which is clearly attainable by the definition of $R$. But then

$$\begin{aligned}
&R(\omega a^\dagger a)R^\dagger \\
&= F(a^\dagger a)G(\sigma_z) \cdot \omega a^\dagger a \cdot G^\dagger(\sigma_z)F^\dagger(a^\dagger a) && (8)\\
&= G(\sigma_z)F(a^\dagger a) \cdot \omega a^\dagger a \cdot F^\dagger(a^\dagger a)G^\dagger(\sigma_z) && (9)\\
&= G(\sigma_z) \cdot \omega a^\dagger a \cdot F(a^\dagger a)F^\dagger(a^\dagger a)G^\dagger(\sigma_z) && (10)\\
&= G(\sigma_z) \cdot \omega a^\dagger a \cdot G^\dagger(\sigma_z) && (11)\\
&= \omega a^\dagger a \cdot G(\sigma_z)G^\dagger(\sigma_z) && (12)\\
&= \omega a^\dagger a && (13)
\end{aligned}$$

And the same is easily shown for the $\frac{\omega}{2}\sigma_z$ term. This means that

$$RH_0R^\dagger = H_0 \tag{14}$$

### 1.1.2 The Interaction Term

Using the commutativity observation pertaining to operators acting on different Hilbert spaces in the previous section, we can evaluate $RH_IR^\dagger$ as follows, starting with just the first term without the coupling constant

$$\begin{aligned}
&= F(a^\dagger a)G(\sigma_z) \cdot (a^\dagger \sigma_-) \cdot G^\dagger(\sigma_z)F^\dagger(a^\dagger a) && (15)\\
&= \big(F(a^\dagger a) \cdot a^\dagger \cdot F^\dagger(a^\dagger a)\big) \otimes \big(G(\sigma_z) \cdot \sigma_- \cdot G^\dagger(\sigma_z)\big) && (16)
\end{aligned}$$

Let's focus on the first bit. We can replace $F$ with its actual value

$$F(a^\dagger a) = \exp\left(i\omega t\hat{n}\right)$$

Where $\hat{n} = a^\dagger a$. Then we have

$$\exp(i\omega t\hat{n})a^\dagger \exp(-i\omega t\hat{n}) \tag{17}$$

By Hadamard's Lemma, this is equivalent to

$$a^\dagger + [i\omega t\hat{n}, a^\dagger] + \frac{1}{2!}[i\omega t\hat{n}, [i\omega t\hat{n}, a^\dagger]] + \dots \tag{18}$$

Recalling the commutation relation $[\hat{n}, a^\dagger] = a^\dagger$, this simplifies to

$$= \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} a^\dagger \tag{19}$$

$$= a^\dagger e^{i\omega t} \tag{20}$$

We apply the same concept to the $\sigma_z$ portion of this term

$$\exp\left(i\frac{1}{2}\omega t\sigma_z\right) \sigma_- \exp\left(-i\frac{1}{2}\omega t\sigma_z\right) \tag{21}$$

Recalling the commutation relation $[\sigma_z, \sigma_-] = -2\sigma_-$, we again use Hadamard's Lemma and simplify

$$= \sigma_- + [0.5i\omega t\sigma_z, \sigma_-] + \frac{1}{2!}[0.5i\omega t\sigma_z, [0.5i\omega t\sigma_z, \sigma_-]] + \dots \tag{22}$$

$$= \sum_{k=0}^{\infty} \frac{(-i\omega t)^k}{k!} \sigma_- \tag{23}$$

$$= e^{-i\omega t}\sigma_- \tag{24}$$

Meaning that the first term becomes

$$a^\dagger \sigma_-$$

Exactly what we began with! Doing the same for the second term in $H_I$ yields similar results, giving the transformed interation Hamiltonian to be

$$RH_IR^\dagger = H_I$$

### 1.1.3 Completely Transformed Hamiltonian

Putting everything together, the transformed Hamiltonian is then

$$H_{RJC} = \omega a^\dagger a + \frac{\omega_0}{2}\sigma_z + j(a^\dagger\sigma_- + \sigma_+ a) - \omega a^\dagger a + \frac{\omega}{2}\sigma_z \tag{25}$$

$$= \frac{\omega_0 - \omega}{2}\sigma_z + j(a^\dagger\sigma_- + \sigma_+ a) \tag{26}$$

# 2 Solving the Schrodinger Equation

Now, we must solve the Schrodinger equation with this simplified Hamiltonian. We construct a general ansatz

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} A_n(t) |n\rangle |e\rangle + B_n(t) |n+1\rangle |g\rangle \tag{27}$$

Whose construction is centered around the notion that the loss of one unit of energy (a photon) in the TSS must be reflected as an increase of one unit of energy in the cavity (and vice versa). We plug the $n$th term of the wave function into the Schrodinger equation, yielding

$$i\frac{\partial\left|\psi(t)\right\rangle}{\partial t} = H_I\left|\psi(t)\right\rangle \tag{28}$$

$$= i\dot{A}_n(t)\left|n\right\rangle\left|e\right\rangle + i\dot{B}_n(t)\left|n+1\right\rangle\left|g\right\rangle \tag{29}$$

$$= \left(\frac{\Delta}{2}\sigma_z + j(a^\dagger\sigma_- + \sigma_+ a)\right)\left(A_n(t)\left|n\right\rangle\left|e\right\rangle + B_n(t)\left|n+1\right\rangle\left|g\right\rangle\right) \tag{30}$$

Where Eq. 29 is the LHS of Schrodinger's equation and Eq. 30 is the RHS. We expand Eq. 30, focusing on the $A_n$ portion first.

$$\left(\frac{\Delta}{2}\sigma_z + j(a^\dagger\sigma_- + \sigma_+ a)\right)\left(A_n(t)\left|n\right\rangle\left|e\right\rangle\right) \tag{31}$$

$$= \frac{\Delta}{2}A_n\left|n\right\rangle\left|e\right\rangle + j\sqrt{n+1}A_n\left|n+1\right\rangle\left|g\right\rangle \tag{32}$$

And the $B_n$ portion becomes

$$\left(\frac{\Delta}{2}\sigma_z + j(a^\dagger\sigma_- + \sigma_+ a)\right)\left(B_n(t)\left|n+1\right\rangle\left|g\right\rangle\right) \tag{33}$$

$$= -\frac{\Delta}{2}B_n\left|n+1\right\rangle\left|g\right\rangle + B_n j\sqrt{n+1}\left|n\right\rangle\left|e\right\rangle \tag{34}$$

Equating the coefficients of the corresponding eigenstates then yields the system

$$\begin{cases} i\dot{A}_n = jB_n\sqrt{n+1} + \frac{\Delta}{2}A_n \\ i\dot{B}_n = jA_n\sqrt{n+1} - \frac{\Delta}{2}B_n \end{cases} \tag{35}$$

Which we may write as

$$i\frac{d}{dt}\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{2} & j\sqrt{n+1} \\ j\sqrt{n+1} & -\frac{\Delta}{2} \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{36}$$

To simplify things, we will write this system of coupled differential equations as

$$i\frac{d}{dt}W(t) = M\cdot W(t) \tag{37}$$

We know the solution to this will be of the form

$$W(t) = e^{-iMt}\cdot W(0) \tag{38}$$

We could try to solve the matrix system using eigenvectors and eigenvalues, but notice that

$$M^2 = \begin{pmatrix} \frac{\Delta}{2} & j\sqrt{n+1} \\ j\sqrt{n+1} & -\frac{\Delta}{2} \end{pmatrix}^2 = \begin{pmatrix} \frac{\Delta^2}{4} + j(n+1) & 0 \\ 0 & \frac{\Delta^2}{4} + j(n+1) \end{pmatrix} \tag{39}$$

We can call these diagonal terms $\beta$ for simplicity. In general, then, we note

$$M^{2k} = \begin{pmatrix} \beta^{2k} & 0 \\ 0 & \beta^{2k} \end{pmatrix} = \beta^{2k}I \tag{40}$$

$$M^{1+2k} = M\cdot M^{2k} = M\begin{pmatrix} \beta^{2k} & 0 \\ 0 & \beta^{2k} \end{pmatrix} = \beta^{2k}M \tag{41}$$

Taylor expanding $e^{-iMt}$ gives

$$
\begin{aligned}
e^{-iMt} &= \sum_{k=0}^{\infty} \frac{(-iMt)^k}{k!} && (42)\\
&= \sum_{k \text{ even}} \frac{(-iMt)^k}{k!} + \sum_{k \text{ odd}} \frac{(-iMt)^k}{k!} && (43)\\
&= I \sum_{k \text{ even}} \frac{(-i\beta^k t)^k}{k!} + \frac{M}{\beta} \sum_{k \text{ odd}} \frac{(-i\beta^k t)^k}{k!} && (44)\\
&= I\left(1 - \frac{\beta^2 t^2}{2!} + \frac{\beta^4 t^4}{4!} - \dots\right) + M\left(\frac{-i\beta t}{1!} + \frac{i\beta^3 t^3}{3!} - \dots\right) && (45)\\
&= I\cos(\beta t) - \frac{M}{\beta} i\sin(\beta t) && (46)\\
&= \begin{pmatrix} \cos(\beta_n t) - \frac{\Delta}{2\beta_n} i\sin(\beta_n t) & -\frac{j\sqrt{n+1}}{\beta_n} i\sin(\beta_n t) \\ -\frac{j\sqrt{n+1}}{\beta_n} i\sin(\beta_n t) & \cos(\beta_n t) + \frac{\Delta}{2\beta_n} i\sin(\beta_n t) \end{pmatrix} && (47)
\end{aligned}
$$

Then, our wave function coefficients are given by

$$
\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} A_0\left(\cos(\beta_n t) - \frac{\Delta}{2\beta_n} i\sin(\beta_n t)\right) - B_0\left(\frac{j\sqrt{n+1}}{\beta_n} i\sin(\beta_n t)\right) \\ -A_0\left(\frac{j\sqrt{n+1}}{\beta_n} i\sin(\beta_n t)\right) + B_0\left(\cos(\beta_n t) + \frac{\Delta}{2\beta_n} i\sin(\beta_n t)\right) \end{pmatrix} \quad (48)
$$