Chapter 1 Reading Notes

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1 Introduction

We start by discussing an 'incompleteness' of the rationals that motivates the study of the reals. Put bluntly, there exists no such $p \in \mathbb{Q}$ that satisfies $p^2 = 2$. The proof of this is pretty straightforward and is widely found, hence omitted. Instead, I will prove a related result.

Theorem 1.1

Let A be the set of all $a \in Q$ such that $a^2 < 2$. Similarly, let B be the set of all $b \in Q$ such that $b^2 > 2$. Then A has no largest element and B has no smallest element.

Proof

For any $p \in Q$, consider the value given by

$$\gamma = p - \frac{p^2 - 2}{p + 2}$$

If $p \in A$, then either $p^2 - 2 < 0$ and $p + 2 > 0$ or $p^2 - 2 > 0$ and $p + 2 < 0$, so $\gamma > p$. Then, we calculate $\gamma^2 - 2$

$$\gamma = \frac{p^2 + 2p - p^2 + 2}{p + 2} = \frac{2p + 2}{p + 2}$$
$$\gamma^2 - 2 = \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p + 2)^2} = \frac{2p^2 - 4}{(p + 2)^2} = \frac{2(p^2 - 2)}{(p + 2)^2}$$

But then $p^2 - 2$ in the numerator is negative, so $\gamma^2 - 2 < 0$ and $\gamma \in A$. Practically the same logic applies for showing that B has no smallest element.

The bit of this proof that was tricky for me was defining the expression that explicitly shows that A has no largest element and B has no smallest element. The proof statement itself, however, felt pretty intuitive.

Theorem 1.2: Least upper bound property implies greatest lower bound

Suppose S is an ordered set with the least upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L$$

exists in S and $\alpha = \inf B$.

Proof

First, we show that L is bounded above. Since for all $b \in B$, $l \in L$, we know $l \leq b$, all b serve as upper bounds for L. Then, since S possesses the least upper-bound property, there exists some $\alpha = \sup L$ in S. This proves the first part of the theorem.

Now, consider some other γ in S. Assume $\gamma > \alpha$. Then, since α is the supremum of L, $\gamma \notin L$. By the definition of L, γ is not a lower bound of B. However, if $\gamma < \alpha$, then it cannot be the infimum of B as α bounds B from below as $\alpha \in L$ and $\alpha > \gamma$. Thus, α must be the infimum of B.

Note: paying attention to definitions is important. B being bounded below as a subset of S implies that all values that bound B below *are within* S by the definition of bounded below. So, we must have $L \subset S$.

2 Fields

The text goes on to discuss fields in some length. I will skip over the introductory propositions as I have experience in algebra, but I will prove one set of propositions just as a sort of checkpoint.

Proposition 2.1

The following statements are true in every ordered field.

1. If $x > 0$ then $-x < 0$ and vice versa.
2. If $x > 0$ and $y < z$ then $xy < xz$.
3. If $x < 0$ and $y < z$ then $xy > xz$.
4. If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
5. If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof

1. Since $x > 0$, $-x + x > -x$, but this is just $0 > -x$ or $-x < 0$.
2. $xz = x(z - y) + xy > 0 + xy = xy$.
3. $xy = x(y - z) + xz > 0 + xz > xz$.

I got too lazy to do the rest; they follow from the definitions and some of the earlier propositions.

3 The Real Field

The core theorem of this section is the following

Theorem 3.1: Existence Theorem

There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

The proof is in an Appendix in the text; I just skimmed through it, but I'll go through it in more detail later. As an exercise, I will try to prove the Archemedian property

Theorem 3.2: Archemedian Property of $\mathbb{R}$

If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$.

Proof

We proceed via contradiction. Let L be the set of all nx, where n is a positive integer. If no such n exists such that $nx > y$, y must bound L from above. Then, L must have a least upper bound. We denote this by $p = \sup L$. Since $x > 0$, $p - x < p$. So, $p - x$ does not bound L from above. Then, $\exists mx \in L$ such that $mx > p - x$. But then $p < mx + x = (m+1)x \in L$, a contradiction.

The approach for this proof was first recognizing that the statement of the proof seems fairly obvious. In a sense, it's like saying "whatever number you can think of, I can think of one that's bigger." So, I immediately thought of trying to come up with some sort of contradiction. Also, the fact that the existence theorem is presented as the 'core' theorem of this section made me think to use it.

Lemma 3.1

Given $x, y \in R$ that satisfy $x - y > 1$, there exists an integer between x and y.

Proof

First, we prove that given $x, y \in \mathbb{R}$ that satisfy $x - y > 1$, there exists an integer between x and y. Pick y' to be the smallest integer greater than or equal to y. Then, consider all $n \in \mathbb{Z}$ with $n > y'$. By the Archimedean principle, this subset is nonempty. Then, by the well-ordering principle, this subset has a minimal element, p. So, $p - 1$ cannot be in the subset and then $p - 1 < y'$. But $p - 1$ is an integer, and since $y' = \lceil y \rceil$, we must have $p - 1 \leq y \to p \leq y + 1$. We also know that $p > y' \geq y \to y < p$. We can combine these inequalities with the original inequality $x - y > 1$ to get

$$y < p \leq y + 1 < x$$
$$y < p < x$$

So p is the integer we desire.

Theorem 3.3: $\mathbb{Q}$ is dense in $\mathbb{R}$

For any $x, y \in \mathbb{R}$ where $x < y$, there exists some $p \in \mathbb{Q}$ such that $x < p < y$.

Proof

If $x < y$, then $y - x > 0$. By the Archimedean principle, there exists an integer n such that $n(y - x) > 1$ or $ny - nx > 1$. But then by Lemma 3.1 there exists an integer m between ny and nx, which means

$$nx < m < ny$$

Since n must be non-zero, we can divide by n

$$x < \frac{m}{n} < y$$

And m/n is the desired p.

Nice. This proof differs slightly from the one in Rudin, but it was what made sense to me.

Theorem 3.4

For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.