PROBLEM SET 6 – PHYS 0500

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Solution 8-3.
In a circular orbit, recall that

$$\frac{v^2}{r} = \frac{k}{\mu r^2} \rightarrow v^2 = \frac{k}{r\mu} \tag{1}$$

And the potential energy V is given by $\int k/r^2 \, dr = -k/r$. The kinetic energy is given by

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu \cdot \frac{k}{r\mu} = -\frac{1}{2}V \tag{2}$$

But then if k is cut into half, the potential energy gets cut in half, but the kinetic energy remains the same. If the new potential energy is V' and the new kinetic energy is T', we have

$$V' = \frac{1}{2}V, \quad T = -\frac{1}{2}V \rightarrow E' = T' + V' = 0 \tag{3}$$

Now, we may use the equation of Kepler's problem to model our trajectory as the force is inversely proprtional to the square of the distance between the particle and the force center

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta \tag{4}$$

where $\alpha = \frac{l^2}{\mu k}$ and $\epsilon = \sqrt{1 + \frac{2E'l^2}{\mu k^2}}$. But if $E' = 0$, $\epsilon = 1$, giving

$$r = \frac{\alpha}{1 + \cos\theta} \tag{5}$$

The equation of a parabola.

Solution 8-9.

   a. As we saw in 8-3, the energy of a particle in a circular orbit is given by $T + V = -\frac{1}{2}V + V = \frac{1}{2}V$. With the new radial velocity, the kinetic energy doubles and the potential energy stays the same, giving the new total energy
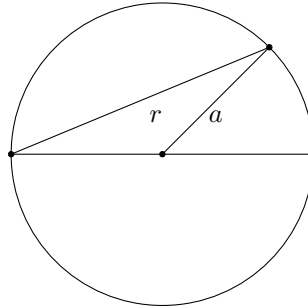
$$E' = T' + V' = 2 \cdot -\frac{1}{2}V + V = 0 \tag{6}$$

So $E'/E = 0$. Since angular momentum is given $r \times p$ and the firing of the rocket is done in the radial direction, the new component of the angular momentum is 0. So, $l'/l = 1$.

b. Since $E' = 0$, the motion will be parabolic. Additionally, $U(r) = -T(r)$, and both of these go to 0 as $r \to \infty$ like $\frac{1}{r}$. Finally, $V(r)$ will behave something like $-\frac{1}{r} + \frac{l^2}{2\mu r^2}$, giving the graphs
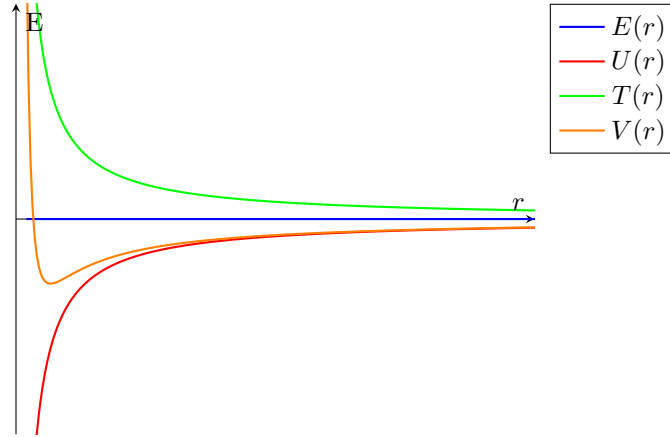
Solution 8-11.

The motion of the particle is given by $r = 2a\cos\theta$, where θ is between the chord and the diameter and varies from 0 to π for one orbit. Then, we can use the equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\mu \frac{r^2}{l^2} F(r) \tag{7}$$

We will set constants to q by multiplying both sides of the equation by $2a$ and defining $q = \frac{8a^3\mu}{l^2}$ for the sake of simplification; this will not affect anything

2

besides constants in our solution for $F(r)$. We write

$$\frac{d^2}{d\theta^2}\sec\theta + \sec\theta = -q(\cos^2\theta)F(r) \tag{8}$$

$$\frac{d}{d\theta}(\sec\theta\tan\theta) + \sec\theta = -q\cos^2\theta \cdot F(r) \tag{9}$$

$$\sec\theta\tan^2\theta + \sec^3\theta + \sec\theta = -q\cos^2\theta \cdot F(r) \tag{10}$$

$$\tan^2\theta + \sec^2\theta + 1 = -q\cos^3\theta \cdot F(r) \tag{11}$$

$$2\sec^2\theta = -q\cos^3\theta \cdot F(r) \tag{12}$$

$$-\frac{2}{q}\sec^5\theta = F(r) \tag{13}$$

$$F(r) = \frac{k}{r^5} \tag{14}$$

Where we have defined $k = -\frac{2}{q}\cdot 32a^5$.

Solution 8-14.

We start with the same equation as in 8-11. This gives

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\mu\frac{r^2}{l^2}F(r) \tag{15}$$

Since $r = k\theta^2$, we write

$$\frac{d^2}{d\theta^2}\left(\frac{1}{k\theta^2}\right) + \frac{1}{k\theta^2} = -\mu\frac{k^2\theta^4}{l^2}F(r) \tag{16}$$

$$6k^{-1}\theta^{-4} + k^{-1}\theta^{-2} = -\mu\frac{k^2\theta^4}{l^2}\cdot F(r) \tag{17}$$

$$-\frac{l^2}{\mu k^3}(6\theta^{-8} + \theta^{-6}) = F(r) \tag{18}$$

Since $\theta^{-2} = r^{-1}k$, we can plug this in to get

$$F(r) = -\frac{l^2}{\mu k^3}(6r^{-4}k^4 + r^{-3}k^3) \tag{19}$$

$$F(r) = -\frac{l^2}{\mu}(6r^{-4}k + r^{-3}) \tag{20}$$