PROBLEM SET 5 – PHYS 0500

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Solution 7-5.
We choose to proceed with polar coordinates. Additionally, choose the horizontal axis of the vertical plane to be the 'ground' i.e. 0 gravitational potential. Then, recalling the conversions from rectangular to polar coordinates, we may write

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r})^2 + \frac{1}{2}m(r\dot{\theta})^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

And for potential energy, we have

$$U_f = -\int f\,dl = -\int_0^r -Al^{\alpha-1}\,dl = \frac{A}{\alpha}r^\alpha, \quad U_g = mgr\sin\theta$$
$$U = \frac{A}{\alpha}r^\alpha + mgr\sin\theta$$

Then the Lagrangian is

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{\alpha}r^\alpha - mgr\sin\theta$$

Finally, we apply the Euler-Lagrange equation for both r and θ

$$mr\dot{\theta}^2 - Ar^{\alpha-1} - mg\sin\theta - m\ddot{r} = 0$$
$$mgr\cos\theta + \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

Clearly, angular momentum is not conserved since $L = mvr = mr\dot{\theta}r = mr^2\dot{\theta}$ has a non-zero rate of change. This is due to the torque from the constant gravitational field.

Solution 7-7.
We denote the coordinates of the first bob as (x_1, y_1) and the coordinates of the second bob as (x_2, y_2). Additionally, we denote the angle between each pendulum and the vertical as θ_1 and θ_2. Then, we may write

$$x_1 = l\cos(\theta_1)$$
$$y_1 = l\sin(\theta_1)$$
$$x_2 = l\cos(\theta_1) + l\cos(\theta_2)$$
$$y_2 = l\sin(\theta_1) + l\sin(\theta_2)$$

Then the kinetic energy of the system is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2 + \dot{x_2}^2 + \dot{y_2}^2)$$

$$= \frac{ml^2}{2}(\dot{\theta_1}^2 \sin^2\theta_1 + \dot{\theta_1}^2 \cos^2\theta_1 + \dot{\theta_1}^2 \sin^2\theta_1 + 2\dot{\theta_1}\dot{\theta_2}\sin\theta_1\sin\theta_2 + \dot{\theta_2}^2\sin^2\theta_2 +$$

$$\dot{\theta_1}^2\cos^2\theta_1 + 2\dot{\theta_1}\dot{\theta_2}\cos\theta_1\cos\theta_2 + \dot{\theta_2}^2\cos^2\theta_2)$$

$$= \frac{ml^2}{2}(2\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2))$$

And the potential energy is given by

$$V = mg(-l\cos\theta_1) + mg(-l(\cos\theta_1 + \cos\theta_2))$$

$$V = -mgl(2\cos\theta_1 + \cos\theta_2)$$

Then our Lagrangian is

$$L = T - V = \frac{ml^2}{2}(2\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2)) + mgl(2\cos\theta_1 + \cos\theta_2)$$

We calculate the derivatives of L with respect to $\theta_1, \dot{\theta_1}, \theta_2$, and $\dot{\theta_2}$

$$\frac{\partial L}{\partial \theta_1} = -ml^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1 - \theta_2) - 2mgl\sin(\theta_1)$$

$$\frac{\partial L}{\partial \dot{\theta_1}} = 2ml^2\dot{\theta_1} + ml^2\dot{\theta_2}\cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = ml^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1 - \theta_2) - mgl\sin(\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta_2}} = ml^2\dot{\theta_2} + ml^2\dot{\theta_1}\cos(\theta_1 - \theta_2)$$

Then, the equation of motion for θ_1 becomes

$$ml^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1 - \theta_2) + 2mgl\sin(\theta_1) + 2ml^2\ddot{\theta_1} + ml^2\ddot{\theta_2}\cos(\theta_1 - \theta_2) - ml^2\dot{\theta_2}(\dot{\theta_1} - \dot{\theta_2})\sin(\theta_1 - \theta_2) = 0$$

$$ml^2\dot{\theta_2}^2\sin(\theta_1 - \theta_2) + 2mgl\sin\theta_1 + ml^2\ddot{\theta_2}\cos(\theta_1 - \theta_2) + 2ml^2\ddot{\theta_1} = 0$$

$$\dot{\theta_2}^2\sin(\theta_1 - \theta_2) + 2\frac{g}{l}\sin\theta_1 + \ddot{\theta_2}\cos(\theta_1 - \theta_2) + 2\ddot{\theta_1} = 0$$

And for θ_2 it becomes

$$ml^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1 - \theta_2) - mgl\sin(\theta_2) - ml^2\ddot{\theta_2} - ml^2\ddot{\theta_1}\cos(\theta_1 - \theta_2) + ml^2\dot{\theta_1}\sin(\theta_1 - \theta_2)\cdot(\dot{\theta_1} - \dot{\theta_2}) = 0$$

$$ml^2\dot{\theta_1}^2\sin(\theta_1 - \theta_2) - mgl\sin\theta_2 - ml^2\ddot{\theta_2} - ml^2\ddot{\theta_1}\cos(\theta_1 - \theta_2) = 0$$

$$\ddot{\theta_2} + \frac{g}{l}\sin\theta_2 + \ddot{\theta_1}\cos(\theta_1 - \theta_2) - \dot{\theta_1}^2\sin(\theta_1 - \theta_2) = 0$$

And we are done.

Solution 7-9.

The coordinates for this one are a bit tricky. We will use the standard x and y axes, but, for the sake of simplicity, we will also represent the direction of the incline with w. The angle of incline with the horizontal will be α, and the angle between the pendulum and the vertical will be β. Then, we can specify the coordinates of the disk and the pendulum bob as follows

$$(x_d, y_d) = (w\cos\alpha, -w\sin\alpha)$$
$$(x_p, y_p) = (w\cos\alpha + l\sin\beta, -w\sin\alpha - l\cos\beta)$$

To find the system's kinetic energy, we consider the kinetic energies of the disk and the pendulum separately. The kinetic energy of the pendulum is given by

$$\begin{aligned} T_p &= \frac{1}{2}m(\dot{x_p}^2 + \dot{y_p}^2) \\ &= \frac{1}{2}m \cdot \left((l\cos\beta \cdot \dot{\beta} + \dot{w}\cos\alpha)^2 + (-\dot{w}\sin\alpha + l\sin\beta \cdot \dot{\beta})^2\right) \\ &= \frac{1}{2}m\left(l^2\dot{\beta}^2 + \dot{w}^2 + 2l\dot{w}\dot{\beta}\cos\beta\cos\alpha - 2l\dot{w}\dot{\beta}\sin\beta\sin\alpha\right) \\ &= \frac{1}{2}m\left(l^2\dot{\beta}^2 + \dot{w}^2 + 2l\dot{w}\dot{\beta}\cos(\alpha+\beta)\right) \end{aligned}$$

And for the kinetic energy of the disk, we recall the rotational kinetic energy formula as well as its translational kinetic energy

$$\begin{aligned} T_d &= \frac{1}{2}I\omega^2 + \frac{1}{2}M\dot{w}^2 \\ &= \frac{1}{2}I(\dot{\theta}^2) + \frac{1}{2}M\dot{w}^2 \end{aligned}$$

Using the fact that $I = \frac{1}{2}MR^2$ for a disk and the relationship $w = R\theta \to \dot{w} = R\dot{\theta}$, we can simplify this to

$$\begin{aligned} T_d &= \frac{1}{2} \cdot \frac{1}{2}MR^2 \cdot \frac{\dot{w}^2}{R^2} + \frac{1}{2}M\dot{w}^2 \\ &= \frac{3}{4}M\dot{w}^2 \end{aligned}$$

The combined potential energy of the system is simply

$$\begin{aligned} V &= Mg(-w\sin\alpha) + mg(-w\sin\alpha - l\cos\beta) \\ &= -(M+m)gw\sin\alpha - mgl\cos\beta \end{aligned}$$

Then, the Lagrangian is

$$L = \frac{1}{2}m\left(l^2\dot{\beta}^2 + \dot{w}^2 + 2l\dot{w}\dot{\beta}\cos(\alpha+\beta)\right) + \frac{3}{4}M\dot{w}^2 + (M+m)gw\sin\alpha + mgl\cos\beta$$

We calculate the partial derivatives with respect to both of our generalized coordinates

$$\begin{aligned} \frac{\partial L}{\partial w} &= (M+m)g\sin\alpha \\ \frac{\partial L}{\partial \dot{w}} &= ml\dot{\beta}\cos(\alpha+\beta) + \frac{3}{2}M\dot{w} + m\dot{w} \\ \frac{\partial L}{\partial \beta} &= -ml\dot{w}\dot{\beta}\sin(\alpha+\beta) - mgl\sin\beta \\ \frac{\partial L}{\partial \dot{\beta}} &= ml^2\dot{\beta} + ml\dot{w}\cos(\alpha+\beta) \end{aligned}$$

The equation of motion for w is then

$$(M+m)g\sin\alpha - ml\ddot{\beta}\cos(\alpha+\beta) - (-ml\dot{\beta}\sin(\alpha+\beta)\cdot\dot{\beta}) - \frac{3}{2}M\ddot{w} - m\ddot{w} = 0$$

$$\boxed{(M+m)g\sin\alpha - ml(\ddot{\beta}\cos(\alpha+\beta) - \dot{\beta}^2\sin(\alpha+\beta)) - \ddot{w}\left(\tfrac{3}{2}M+m\right) = 0}$$

The equation of motion for β is

$$-ml\dot{w}\dot{\beta}\sin(\alpha+\beta) - mgl\sin\beta - ml^2\ddot{\beta} - ml\ddot{w}\cos(\alpha+\beta) + ml\dot{w}\sin(\alpha+\beta)\cdot\dot{\beta} = 0$$

$$\boxed{\ddot{\beta} + \tfrac{g}{l}\sin\beta + \tfrac{\ddot{w}}{l}\cos(\alpha+\beta) = 0}$$

Solution 7-10.

a. The system's Lagrangian is straightforward

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - Mgy$$

Where we've set the $y = 0$ level to be the horizontal of the block on the surface. Then, noting that we must have $\dot{x} = -\dot{y}$ and applying Euler's equation we can write

$$Mg + 2M\ddot{y} = 0$$
$$\ddot{y} = -\frac{g}{2}$$

Assuming initial position and velocity are 0, the solution is then

$$y(t) = -\frac{g}{4}t^2$$

And we are done.

b. If the string has mass m, we need to account for its kinetic and potential energy. Since the system moves together, the kinetic energy is

$$T = \frac{1}{2}m\dot{y}^2$$

The potential energy, however, should only take into account the part of the string hanging off the table, given by y. Additionally, we can reference the string's position (the portion hanging off the horizontal surface) by its center of mass. Assuming a constant density, this gives

$$V = \frac{m}{l}y\cdot g\cdot\frac{y}{2} = \frac{mg}{2l}y^2$$

It is *extremely* important to note that since the y coordinate is squared, the potential energy expression doesn't actually take into account that y is negative. Thus, we must insert the sign in ourselves

$$V = -\frac{mg}{2l}y^2$$

The new Lagrangian is then

$$L = M\dot{y}^2 - Mgy + \frac{mg}{2l}y^2 + \frac{1}{2}m\dot{y}^2$$

We apply Euler's equation

$$\begin{aligned}
-2M\ddot{y} - m\ddot{y} - Mg + \frac{mg}{l}y &= 0 \\
(2M+m)\ddot{y} &= \frac{mg}{l}\left(y - \frac{Ml}{m}\right) \\
\ddot{y} &= \frac{mg}{(2M+m)l}\left(y - \frac{Ml}{m}\right) \\
\frac{d^2}{dt^2}\left(y - \frac{Ml}{m}\right) &= \frac{mg}{(2M+m)l}\left(y - \frac{Ml}{m}\right)
\end{aligned}$$

For simplicity, we define $\alpha = \sqrt{\frac{mg}{(2M+m)l}}$, giving

$$y - \frac{Ml}{m} = Ae^{-\alpha t} + Be^{\alpha t}$$

This would be a final general solution. If we want to go further, however, we can impose implied initial conditions. Assuming the initial conditions $y(0) = \dot{y}(0) = 0$, we can solve for A and B

$$\begin{aligned}
-\frac{Ml}{m} &= A + B \\
A - B &= 0 \\
A = B &= -\frac{Ml}{2m}
\end{aligned}$$

Then

$$\begin{aligned}
y &= \frac{Ml}{m} - \frac{Ml}{m}\cosh(\alpha t) \\
y &= \frac{Ml}{m}\left(1 - \cosh(\alpha t)\right) \\
y &= \frac{Ml}{m}\left(1 - \cosh\left(\sqrt{\frac{mg}{(2M+m)l}} \cdot t\right)\right)
\end{aligned}$$

And we are done.

Solution 7-15.
Let l be the length of the spring and let θ be the angle between the spring and vertical. Then, the kinetic energy of the system is given by

$$T = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2 \cdot \dot{\theta}^2 = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2)$$

The potential energy is given by

$$V = V_y + V_s = -mgl\cos\theta + \frac{1}{2}k(b-l)^2$$

Then

$$L = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) + mgl\cos\theta - \frac{1}{2}k(b-l)^2$$

The EoM for l becomes

$$ml\dot{\theta}^2 + mg\cos\theta - kl + kb - m\ddot{l} = 0$$

$$\boxed{\ddot{l} - l\dot{\theta}^2 - g\cos\theta + \tfrac{k}{m}(l-b) = 0}$$

And for θ we get

$$-mgl\sin\theta - ml^2\ddot{\theta} - 2ml\dot{l}\dot{\theta} = 0$$

$$\boxed{\ddot{\theta} + \tfrac{2}{l}\dot{\theta}\dot{l} + \tfrac{g}{l}\sin\theta = 0}$$

Solution 7-22.
The potential energy function is given by

$$U = -\int F\,dx = -\int \frac{k}{x^2}e^{-\frac{t}{\tau}}\,dx = \frac{k}{x}e^{-\frac{t}{\tau}}$$

So the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{k}{x}e^{-\frac{t}{\tau}}$$

The Hamiltonian is given by

$$H = \sum_j p_j\dot{q}_j - L = \frac{\partial L}{\partial \dot{x}}\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{k}{x}e^{-\frac{t}{\tau}} = \frac{1}{2}m\dot{x}^2 + \frac{k}{x}e^{-\frac{t}{\tau}}$$

So the Hamiltonian is the total energy of the system-specifically, $H = T + V$. Energy is not conserved as the Hamiltonian has a time dependence.

Solution 7-23.
First, we calculate the general Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

Using the definition of the Hamiltonian, we get

$$\begin{aligned}
H = \sum_j p_j\dot{q}_j - L &= m\dot{x} + m\dot{y} + m\dot{z} - L \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U \\
&= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + U
\end{aligned}$$

Applying Hamilton's equation $-\dot{p}_k = \frac{\partial H}{\partial q_k}$ gives

$$
\begin{aligned}
-m\ddot{x} = -\dot{p}_x = \frac{\partial H}{\partial x} = \frac{\partial U}{\partial x} \rightarrow m\ddot{x} = -\frac{\partial U}{\partial x} = F_x \\
-m\ddot{y} = -\dot{p}_y = \frac{\partial H}{\partial y} = \frac{\partial U}{\partial y} \rightarrow m\ddot{y} = -\frac{\partial U}{\partial y} = F_y \\
-m\ddot{z} = -\dot{p}_z = \frac{\partial H}{\partial z} = \frac{\partial U}{\partial z} \rightarrow m\ddot{z} = -\frac{\partial U}{\partial z} = F_z
\end{aligned}
$$

Which are Newton's equations.

Solution 7-24.

The Lagrangian is essentially the same as other pendulum Lagrangians we've done

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) + mgl\cos\theta \\
&= \frac{1}{2}m\left(\alpha^2 + l^2\dot{\theta}^2\right) + mgl\cos\theta
\end{aligned}
$$

The Hamiltonian is

$$
\begin{aligned}
H &= \sum_i p_i\dot{q}_i - L = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L \\
&= ml^2\dot{\theta}^2 - L \\
&= \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}m\alpha^2 - mgl\cos\theta
\end{aligned}
$$

So the Hamiltonian is not equal to the total energy. Energy is clearly not conserved since the shortening of the string is work being done on the system.

Solution 7-25.

The kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

And the potential energy is

$$V = mgz$$

Since there aren't any time-dependent forces (and all forces are conservative), we notice that $H = T + V$, giving

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz$$

We plug in the given substitutions

$$
\begin{aligned}
H &= \frac{1}{2}m(r^2\dot{\theta}^2 + k^2\dot{\theta}^2) + mgk\theta \\
&=
\end{aligned}
$$

Additionally, we note that the momentum is

$$\frac{\partial L}{\partial \dot{\theta}} = m(r^2 + k^2)\dot{\theta}$$

So the Hamiltonian can be rewritten as

$$\begin{aligned} H &= \frac{1}{2}m(r^2 + k^2) \cdot \left(\frac{p_\theta}{m(r^2 + k^2)}\right)^2 + mgk\theta \\ &= \frac{(p_\theta)^2}{2m(r^2 + k^2)} + mgk\theta \end{aligned}$$

Then via Hamilton's equations we may write

$$\begin{aligned} \dot{p_\theta} &= -\frac{\partial H}{\partial \theta} = -mgk \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m(r^2 + k^2)} \end{aligned}$$

We solve the system to get

$$\ddot{\theta} = -\frac{gk}{r^2 + k^2}$$

And since $z = k\theta$, we can express our answer in terms of z as

$$\ddot{z} = -\frac{gk^2}{r^2 + k^2}$$

And we are done.