PROBLEM SET 4 – APMA 0360

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Solution 1.
If u_1 and u_2 are solutions to

$$\begin{cases} u_t - u_{xx} = 0 & , \quad 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0 & , \quad t > 0 \end{cases}$$

we can consider the purported solution $a_1u_1 + a_2u_2$ by plugging it into the PDE directly. This gives

$$(a_1u_1 + a_2u_2)_t - (a_1u_1 + a_2u_2)_{xx} = 0$$

Since the derivative is a linear operator, we may expand this as

$$a_1(u_1)_t + a_2(u_2)_t - a_1(u_1)_{xx} - a_2(u_2)_{xx} = 0$$

Rearranging gives

$$a_1((u_1)_t - (u_1)_{xx}) + a_2((u_2)_t - (u_2)_{xx}) = 0$$

Since u_1 and u_2 are solutions to the equation $u_t - u_{xx} = 0$, this simplifies to

$$a_1(0) + a_2(0) = 0 \to 0 = 0$$

Which verifies solution to the PDE itself. As for the boundary conditions, we have

$$u_1(0,t) = u_1(\pi,t) = 0$$
$$u_2(0,t) = u_2(\pi,t) = 0$$
$$a_1u_1(0,t) + a_2u_2(0,t) = a_1(0) + a_2(0) = 0$$
$$a_1u_1(\pi,t) + a_2u_2(\pi,t) = a_1(0) + a_2(0) = 0$$

Verifying that a linear combination of solutions for this PDE with the given boundary conditions is also a valid solution. However, for the boundary condition $u(0,t) = u(\pi,t) = 1$, we have

$$a_1u_1(0,t) + a_2u_2(0,t) = a_1 + a_2$$

Which is not necessarily equal to 1, meaning that these boundary conditions don't allow for satisfication of the superposition principle.

Solution 2.
We assume a solution of the form $u(x,t) = X(x)T(t)$. Then, we have

$$\begin{aligned} u_t - u_{xx} &= 0 \\ XT' - X''T &= 0 \\ \frac{T'}{T} - \frac{X''}{X} &= 0 \\ \frac{T'}{T} = \frac{X''}{X} &= -k^2 \end{aligned}$$

We solve for general forms of X and T

$$\begin{aligned} T' = -k^2 T \to T = Ae^{-k^2 t} \\ X'' = -k^2 X \to X = B\cos(kx) + C\sin(kx) \\ u(x,t) = Ae^{-k^2 t}\left(B\cos(kx) + C\sin(kx)\right) \end{aligned}$$

We plug in boundary conditions

$$\begin{aligned} u_x(x,t) = Ae^{-k^2 t}\left(-Bk\sin(kx) + Ck\cos(kx)\right) \\ u_x(0,t) = Ae^{-k^2 t} \cdot Ck = 0 \to C = 0 \\ u_x(x,t) = Ae^{-k^2 t}(-Bk\sin(kx)) \\ u_x(\pi,t) = -Bk\sin(\pi k) = 0 \to k \in \mathbb{N} \end{aligned}$$

Then, by the principle of superposition, we can write a general solution to be of the form

$$u(x,t) = \sum_{n=0}^{\infty} D_n e^{-n^2 t}\cos(nx)$$

Finally, we use the last initial condition $u(x,0) = 1 + 2\cos(3x)$ to see

$$u(x,0) = \sum_{n=0}^{\infty} D_n \cos(nx) = 1 + 2\cos(3x)$$

Meaning $D_0 = 1$, $D_3 = 2$, and all other $D_n = 0$, giving the final answer

$$u(x,t) = 1 + 2e^{-9t}\cos(3x)$$

Solution 3.
Assuming a solution of the form $u(x,t) = X(x)T(t)$, we have

$$\begin{aligned} tXT' - TX'' &= 0 \\ \frac{tT'}{T} - \frac{X''}{X} &= 0 \\ \frac{tT'}{T} = \frac{X''}{X} &= -k^2 \end{aligned}$$

We start with the ODE in t

$$\begin{gathered}\frac{tT'}{T} = -k^2 \\ T' + k^2 t^{-1} T = 0 \\ t^{k^2} T' + k^2 t^{k^2-1} T = 0 \\ \frac{d}{dt}\left(t^{k^2} T\right) = 0 \\ t^{k^2} T = A \\ T = A t^{-k^2}\end{gathered}$$

The ODE in x is the same as in Problem 2, giving

$$X = B\cos(kx) + C\sin(kx)$$

Then, we have

$$u(x,t) = At^{-k^2}\left(B\cos(kx) + C\sin(kx)\right)$$

We plug in initial conditions

$$\begin{gathered}u(0,t) = At^{-k^2}(B) = 0 \to B = 0 \\ u(\pi,t) = At^{-k^2}(C\sin(k\pi)) = 0 \to k \in \mathbb{N}\end{gathered}$$

Then, by the principle of superposition, we may write our general solution as

$$u(x,t) = \sum_{n=1}^{\infty} D_n t^{-n^2} \sin(nx)$$

And we are done.

Solution 4.

To go through all cases, we consider a positive, negative, and 0 constant for the separated PDE. The constants reflecting these cases will be $k^2, -k^2$, and 0, respectively. We start by assuming a solution of the form $u(x,t) = X(x)T(t)$, separating variables, and then considering the $k = 0$ case. We have

$$\begin{gathered}XT' - X''T = 0 \\ \frac{T'}{T} - \frac{X''}{X} = 0 \\ \frac{T'}{T} = \frac{X''}{X} = 0\end{gathered}$$

We must have $X, T \neq 0$, so we have

$$\begin{gathered}X'' = 0 \to X = Ax + B \\ T' = 0 \to T = C \\ u(x,t) = C(Ax + B) \\ u(\pi,t) = CA\pi + CB = 0 \\ u_x(x,t) = CA \\ u_x(0,t) = CA = 0\end{gathered}$$

But if $CA = 0$, then we must have $CB = 0$, implying $u(x,t) = 0$, a trivial solution. We proceed with the positive case, which gives

$$\begin{aligned}
\frac{T'}{T} &= \frac{X''}{X} = k^2 \\
\frac{T'}{T} &= k^2 \to T = T = Ae^{k^2 t} \\
\frac{X''}{X} &= k^2 \to X = Be^{kx} + Ce^{-kx} \\
u(x,t) &= Ae^{k^2 t}\left(Be^{kx} + Ce^{-kx}\right) \\
u_x(x,t) &= Ae^{k^2 t}\left(Bke^{kx} - Cke^{-kx}\right) \\
u_x(0,t) &= Ae^{k^2 t}(Bk - Ck) = 0 \to B = C \\
u(\pi,t) &= Ae^{k^2 t}\left(Be^{k\pi} + Be^{-k\pi}\right) = 0
\end{aligned}$$

If $B = 0$, we have the trivial solution again. Otherwise, we have

$$e^{k\pi} + e^{-k\pi} = 0$$

A contradiction. Thus, for $\lambda = k^2 \geq 0$, the only possible solution is the trivial solution. Finally, we pick $\lambda = -k^2 < 0$, giving

$$\begin{aligned}
\frac{T'}{T} &= \frac{X''}{X} = -k^2 \\
\frac{T'}{T} &= -k^2 \to T = Ae^{-k^2 t} \\
\frac{X''}{X} &= -k^2 \to X = B\cos(kx) + C\sin(kx) \\
u(x,t) &= Ae^{-k^2 t}\left(B\cos(kx) + C\sin(kx)\right) \\
u_x(x,t) &= Ae^{-k^2 t}\left(-Bk\sin(kx) + Ck\cos(kx)\right) \\
u_x(0,t) &= Ae^{-k^2 t}(Ck) = 0 \to C = 0 \\
u(\pi,t) &= Ae^{-k^2 t} \cdot B\cos(k\pi) = 0 \\
k\pi &= \frac{\pi}{2} + n\pi, \quad n \in \mathbb{N} \\
k &= \frac{1}{2} + n, \quad n \in \mathbb{N}
\end{aligned}$$

Then, by the principle of superposition, we can take a linear combination of solutions to write a general solution in the following form

$$u(x,t) = \sum_{n=0}^{\infty} D_n \exp\left(-\left(\frac{1}{2} + n\right)^2 t\right)\cos\left(\frac{x}{2} + xn\right)$$

And we are done.