PROBLEM SET 3 – APMA 0360

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Solution 1.
The solution formula states that for the initial condition $u(x,0) = f(x)$ with $f(x)$ in the Schwartz class, $u(x,t)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4Dt}\right) f(y)\, dy$$

Given $u(x,0) = e^x$, we write

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4Dt}\right) e^y\, dy$$

We focus on just the integral for now. We write

$$\int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4Dt} + y\right)\, dy$$

$$= \int_{-\infty}^{\infty} \exp\left(\frac{1}{4Dt}(-x^2 + 2xy - y^2 + 4Dty)\right)\, dy$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{4Dt}(-y^2 + 2(x+2Dt)y - x^2\right)\, dy$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{4Dt}(-y^2 + 2(x+2Dt)y - (x+2Dt)^2 + (x+2Dt)^2 - x^2\right)\, dy$$

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{4Dt}(-(y-(x+2Dt))^2 + (x+2Dt)^2 - x^2)\right)\, dy$$

$$\exp\left(\frac{(x+2Dt)^2 - x^2}{4Dt}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-(y-(x+2Dt))^2}{4Dt}\right)\, dy$$

Now, we can evaluate the integral as the integrand is a standard Gaussian function

$$\exp\left(\frac{(x+2Dt)^2 - x^2}{4Dt}\right) \cdot \sqrt{4Dt\pi}$$

Plugging this back into the original expression for $u(x,t)$ gives

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \cdot \exp\left(\frac{(x+2Dt)^2 - x^2}{4Dt}\right) \cdot \sqrt{4Dt\pi}$$
$$u(x,t) = \exp\left(\frac{(x+2Dt)^2 - x^2}{4Dt}\right)$$
$$u(x,t) = \exp\left(\frac{x^2 + 4Dtx + 4D^2t^2 - x^2}{4Dt}\right)$$
$$u(x,t) = \exp(x + Dt)$$
$$u(x,t) = e^{x+Dt}$$

Solution 2.
We have

$$u_{tt} = c^2 u_{xx}$$
$$\int_{-\infty}^{\infty} e^{ikx} u_{tt}\,dx = c^2 \int_{-\infty}^{\infty} e^{ikx} u_{xx}\,dx$$

We integrate by parts twice on the RHS and note that we can take the derivative out of the integral on the LHS to get

$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) = c^2(ik)^2\hat{u}(k,t)$$
$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) + c^2k^2\hat{u}(k,t) = 0$$
$$\hat{u}(k,t) = f(k)e^{ikct} + g(k)e^{-ikct}$$

Note that instead of having constants in front of the exponential functions, we have functions of k as the ODE was only with respect to t. We take the inverse fourier transform of both sides to get

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{ikct}e^{-ikx} + g(k)e^{-ikct}e^{-ikx}\,dk$$
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ik(x-ct)} + g(k)e^{-ik(x+ct)}\,dk$$

And using the shift property of Fourier transforms, we can create new functions to get the solution

$$u(x,t) = p(x-ct) + q(x+ct)$$

And we are done.

Solution 3.
We use the Fourier transform method again, assuming $u(x,t)$ is in the Schwartz class, which gives

$$u_{tt} - 3u_{xt} - 4u_{xx} = 0$$
$$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} e^{ikx}u\,dx - 3\int_{-\infty}^{\infty} e^{ikx}u_{xt}\,dx - 4\int_{-\infty}^{\infty} e^{ikx}u_{xx}\,dx = 0$$
$$\frac{\partial^2}{\partial t^2}\hat{u} + 3ik\frac{\partial}{\partial t}\hat{u} + 4k^2\hat{u} = 0$$

We solve this as a standard ODE

$$r^2 + 3ikr + 4k^2 = 0$$
$$r = \frac{-3ki \pm \sqrt{-9k^2 - 16k^2}}{2}$$
$$r = \frac{-3ki \pm 5ki}{2}$$
$$r = -4ki, \quad r = ki$$
$$\hat{u}(k,t) = f(k)e^{-4kit} + g(k)e^{kit}$$

And now we take the inverse Fourier transform to proceed, using the shift property in doing so

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ikx-4kit} + g(k)e^{-ikx+kit}\,dk$$
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ik(x+4t)} + g(k)e^{-ik(x-t)}\,dk$$
$$u(x,t) = F(x+4t) + G(x-t)$$

Now, we consider initial conditions. We have

$$u(x,0) = F(x) + G(x) = 2x^2$$
$$u_t(x,0) = \left[4F'(x-4t) - G'(x+t)\right]_{t=0} = 4F'(x) - G'(x) = e^{4x}$$

Then

$$\int_0^x 4F'(s) - G'(s)\,ds = \int_0^x e^{4s}\,ds$$
$$4F(x) - G(x) - 4F(0) + G(0) = \frac{e^{4x}}{4} - \frac{1}{4}$$
$$-G(x) + 4F(x) = \frac{e^{4x}}{4} + C_1$$
$$5F(x) = \frac{e^{4x}}{4} + 2x^2 + C_1$$
$$F(x) = \frac{2}{5}x^2 + \frac{e^{4x}}{20} + \frac{C_1}{5}$$
$$-5G(x) = \frac{e^{4x}}{4} - 8x^2 + C_1$$
$$G(x) = -\frac{e^{4x}}{20} + \frac{8}{5}x^2 - \frac{C_1}{5}$$

Then, putting everything together gives

$$u(x,t) = F(x+4t) + G(x-t)$$
$$u(x,t) = \frac{2}{5}(x+4t)^2 + \frac{e^{4(x+4t)}}{20} - \frac{e^{4(x-t)}}{20} + \frac{8}{5}(x-t)^2$$
$$2x^2 + 8t^2 + \frac{1}{20}e^{4x}\left(e^{16t} - e^{-4t}\right)$$

Solution 4.
Given the damped wave equation $u_{tt} + \alpha u_t = u_{xx}$ and its energy given by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t(x,t))^2 + (u_x(x,t))^2 \, dx$$

We start by differentiating $E(t)$, which gives

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} (u_t(x,t))^2 + (u_x(x,t))^2 \, dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} 2u_t(x,t) \cdot u_{tt}(x,t) + 2u_x(x,t) \cdot u_{xt}(x,t) \, dx \end{aligned}$$

We consider the potential energy integral and integrate by parts to get

$$\begin{aligned} &\int_{-\infty}^{\infty} u_x(x,t) \cdot u_{xt}(x,t) \, dx \\ &= -\int_{-\infty}^{\infty} u_{xx}(x,t) \cdot u_t(x,t) \, dx \end{aligned}$$

We can put this back into the expression for the derivative of $E(t)$, which gives

$$= \int_{-\infty}^{\infty} u_t(x,t) \cdot u_{tt}(x,t) - u_{xx}(x,t) \cdot u_t(x,t) \, dx$$

Using the fact that $u_{tt} = u_{xx} - \alpha u_t$ gives

$$\begin{aligned} &= \int_{-\infty}^{\infty} u_t(x,t) \cdot (u_{xx}(x,t) - \alpha u_t(x,t)) - u_{xx}(x,t) \cdot u_t(x,t) \, dx \\ &= \int_{-\infty}^{\infty} u_t(x,t) \cdot u_{xx}(x,t) - \alpha u_t(x,t) \cdot u_t(x,t) - u_{xx}(x,t) \cdot u_t(x,t) \, dx \\ &= \int_{-\infty}^{\infty} -\alpha (u_t(x,t))^2 \, dx \leq 0 \end{aligned}$$

This final integral has to be less than or equal to 0 as it is the product of a negative number ($\alpha > 0$ by selection) and a square, which must be non-negative.