# PROBLEM SET 6 – PHYS 0500

Ishaan Ganti
Brown University
Advanced Mechanics

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**Solution 8-3.**
In a circular orbit, recall that

$$\frac{v^2}{r} = \frac{k}{\mu r^2} \to v^2 = \frac{k}{r\mu} \tag{1}$$

And the potential energy $V$ is given by $\int k/r^2 \, dr = -k/r$. The kinetic energy is given by

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu \cdot \frac{k}{r\mu} = -\frac{1}{2}V \tag{2}$$

But then if $k$ is cut into half, the potential energy gets cut in half, but the kinetic energy remains the same. If the new potential energy is $V'$ and the new kinetic energy is $T'$, we have

$$V' = \frac{1}{2}V, \quad T = -\frac{1}{2}V \to E' = T' + V' = 0 \tag{3}$$

Now, we may use the equation of Kepler's problem to model our trajectory as the force is inversely proprtional to the square of the distance between the particle and the force center

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta \tag{4}$$

where $\alpha = \frac{l^2}{\mu k}$ and $\epsilon = \sqrt{1 + \frac{2E'l^2}{\mu k^2}}$. But if $E' = 0$, $\epsilon = 1$, giving

$$r = \frac{\alpha}{1 + \cos\theta} \tag{5}$$

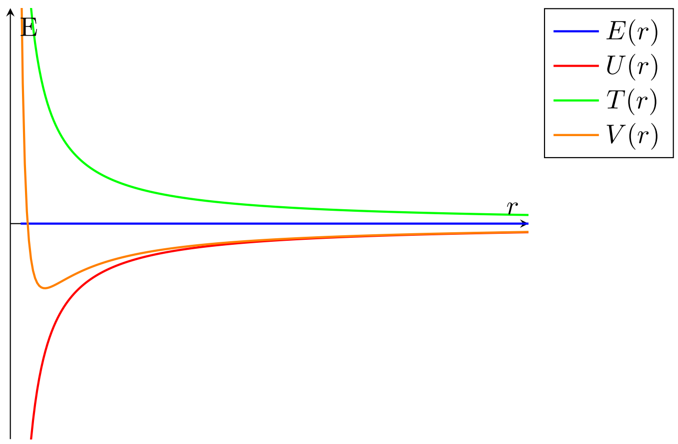
The equation of a parabola.
**Solution 8-9.**

a. As we saw in 8-3, the energy of a particle in a circular orbit is given by $T + V = -\frac{1}{2}V + V = \frac{1}{2}V$. With the new radial velocity, the kinetic energy doubles and the potential energy stays the same, giving the new total energy

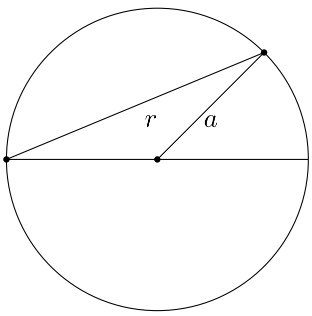
$$E' = T' + V' = 2 \cdot -\frac{1}{2}V + V = 0 \tag{6}$$

So $E'/E = 0$. Since angular momentum is given $r \times p$ and the firing of the rocket is done in the radial direction, the new component of the angular momentum is 0. So, $l'/l = 1$.

b. Since $E' = 0$, the motion will be parabolic. Additionally, $U(r) = -T(r)$, and both of these go to 0 as $r \to \infty$ like $\frac{1}{r}$. Finally, $V(r)$ will behave something like $-\frac{1}{r} + \frac{l^2}{2\mu r^2}$, giving the graphs




**Solution 8-11.**




The motion of the particle is given by $r = 2a\cos\theta$, where $\theta$ is between the chord and the diameter and varies from 0 to $\pi$ for one orbit. Then, we can use the equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\mu\frac{r^2}{l^2}F(r) \tag{7}$$

We will set constants to $q$ by dividing both sides of the equation by $2a$ and defining $q = -\frac{\mu}{2al^2}$ for the sake of simplification; this will not affect anything besides constants in our solution for $F(r)$. We write

$$\frac{d^2}{d\theta^2}\sec\theta + \sec\theta = -q(\cos^2\theta)F(r) \tag{8}$$

$$\frac{d}{d\theta}(\sec\theta\tan\theta) + \sec\theta = -q\cos^2\theta\cdot F(r) \tag{9}$$

$$\sec\theta\tan^2\theta + \sec^3\theta + \sec\theta = -q\cos^2\theta\cdot F(r) \tag{10}$$

$$\tan^2\theta + \sec^2\theta + 1 = -q\cos^3\theta\cdot F(r) \tag{11}$$

$$2\sec^2\theta = -q\cos^3\theta\cdot F(r) \tag{12}$$

$$\frac{2}{q}\sec^5\theta = F(r) \tag{13}$$

$$F(r) = \frac{k}{r^5} \tag{14}$$

Where we have defined $k = \frac{2}{q}\cdot 32a^5$.

**Solution 8-14.**

We start with the same equation as in 8-11. This gives

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\mu\frac{r^2}{l^2}F(r) \tag{15}$$

Since $r = k\theta^2$, we write

$$\frac{d^2}{d\theta^2}\left(\frac{1}{k\theta^2}\right) + \frac{1}{k\theta^2} = -\mu\frac{k^2\theta^4}{l^2}F(r) \tag{16}$$

$$6k^{-1}\theta^{-4} + k^{-1}\theta^{-2} = -\mu\frac{k^2\theta^4}{l^2}\cdot F(r) \tag{17}$$

$$-\frac{l^2}{\mu k^3}(6\theta^{-8} + \theta^{-6}) = F(r) \tag{18}$$

Since $\theta^{-2} = r^{-1}k$, we can plug this in to get

$$F(r) = -\frac{l^2}{\mu k^3}(6r^{-4}k^4 + r^{-3}k^3) \tag{19}$$

$$F(r) = -\frac{l^2}{\mu}(6r^{-4}k + r^{-3}) \tag{20}$$