CHAPTER 1 PROBLEMS – ANALYSIS

Ishaan Ganti
Principles of Mathematical Analysis

25 april 2024

Solution 1-1.
For the sake of contradiction, assume $r + x$ is rational. Then, we can express it in *lowest terms* as some p/q, where $p, q \in \mathbb{Z}$. Since r is rational, we may express it as some m/n, $m, n \in \mathbb{Z}$. But then

$$x = r + x - r = \frac{p}{q} - \frac{m}{n} = \frac{pn}{qn} - \frac{mq}{nq} = \frac{pn - mq}{nq}$$

Implying x is rational, a contradiction. Similarly, for the sake of contradiction, assume rx is rational. We use our previous definition of r and now define $rx = p/q$. We write

$$x = rx \cdot r^{-1} = \frac{p}{q} \cdot \left(\frac{m}{n}\right)^{-1} = \frac{p}{q} \cdot \frac{n}{m} = \frac{pn}{qm}$$

Implying x is rational, a contradiction.

Solution 1-2.
For the sake of contradiction, let $r \in \mathbb{Q}$ satisfy $r^2 = 12$. Then, we may write $r = p/q$ where $p, q \in \mathbb{Z}$, $p, q \neq 0$, and the fraction is in *lowest terms*. We write

$$r^2 = (p/q)^2 = 12$$
$$p^2 = 12q^2$$

So $2 \mid p$ and $3 \mid p$, which means $6 \mid p$. Let $p = 6h$. We may write

$$(6h/q)^2 = 12 \to 36h^2 = 12q^2 \to 3h^2 = q^2$$

So $3 \mid q$. But $3 \mid p$, so p/q cannot be in lowest terms, contradiction.

Solution 1-4.
Consider some $e \in E$. Since α bounds E from below, $\alpha \leq e$. And, since β bounds E from above, $\beta \geq e$. Then $\alpha \leq e \leq \beta \to \alpha \leq \beta$.

Solution 1-5.
Since A is bounded below, its infimum exists. Let its infimum be α. $-A$ is given by the set of numbers $-x$ for all $x \in A$. Since $\alpha \leq x$ for all $x \in A$, $-\alpha \geq -x$. This follows from

$$\alpha \leq x \to \alpha - x \leq 0 \to -(\alpha - x) \geq 0 \to -\alpha - (-x) \geq 0 \to -\alpha \geq -x$$

Then, $-\alpha$ must bound $-A$ from above. To determine whether $-\alpha$ is the supremum of $-A$, consider some $\beta < -\alpha$. Assume β is the supremum of $-A$. Then

$-\beta > \alpha$, so there exists some $x \in A$ for which $-\beta > x$. But then $\beta < -x \in -A$, meaning that β cannot be the supremum of $-A$, contradiction. Thus

$$-\alpha = -\inf A = \sup(-A) \to \inf A = -\sup(-A)$$

Solution 1-6.

a. Since $m/n = p/q$, we have $mq = np = v$. Then, we may use the integer exponentiation laws to write

$$((b^m)^{1/n})^v = ((b^m)^{1/n})^{np} = (b^m)^p = b^{mp}$$
$$((b^p)^{1/q})^v = ((b^p)^{1/q})^{mq} = (b^p)^m = b^{pm} = b^{mp}$$

Then, both $(b^m)^{1/n}$ and $(b^p)^{1/q}$ are solutions to the equation

$$y^v = b^{mp}$$

But the theorem states that the solution y to this equation is unique, so we must have $(b^m)^{1/n} = (b^p)^{1/q}$.

b. Since r and s are rational, we may express them as some p/q and m/n. Additionally, note that $r+s$ is then equal to $(pn+mq)/(nq)$. Then consider

$$(b^{r+s})^{nq} = (b^{(pn+mq)/(nq)})^{nq} = b^{pn+mq}$$
$$(b^r b^s)^{nq} = b^{rnq}b^{snq} = b^{(p/q)nq}b^{(m/n)nq} = b^{np}b^{mq} = b^{np+mq}$$

And, like in (a), Theorem 1.21 allows us to use this fact to conclude that $b^{r+s} = b^r b^s$.

c. Let $r = p/q$ and let $m/n = t < r$. Then

$$t < r \to \frac{m}{n} < \frac{p}{q} \to mq < np$$

Now, we consider $(b^r)^{nq} = (b^{p/q})^{nq} = b^{np}$ and $(b^t)^{nq} = (b^{(m/n)})^{nq} = b^{mq}$. We can use the fact that $b > 1$ and integer exponentiation to conclude that $b^{mq} < b^{np}$, so $(b^r)^{nq} > (b^t)^{nq}$. Taking the nqth root of both sides of this inequality yields $b^r > b^t$. Thus, b^r bounds $B(r)$ from above, and since $b^r \in B(r)$, we must have $b^r = \sup B(r)$.

It is important to note that we can only take the nqth root and not change the inequality because we may write

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$
$$\to (b^r)^{nq} - b^{mq} = (b^{rn})^q - (b^m)^q = (b^{rn} - b^m) \cdot f(b)$$

Where $f(b)$ is clearly non-negative since $b > 1$. We may rewrite $(b^{rn} - b^m)$ as

$$(b^{rn} - b^{m/n \cdot n}) = (b^r)^n - (b^t)^n = (b^r - b^t) \cdot g(b)$$

Where, again, $g(b)$ is clearly non-negative. But then we have

$$(b^r)^{nq} - (b^t)^{nq} = (b^{rn})^q - (b^m)^q = (b^{rn} - b^m) \cdot f(b) = (b^r - b^t) \cdot g(b) \cdot f(b)$$

So, the sign of $b^r - b^t$ must be the same as the sign of $(b^r)^{nq} - (b^t)^{nq}$, justifying taking the nqth root.

d. By definition, $b^{x+y} = \sup B(x+y)$. This is equal to $\sup\{b^p \mid p \in \mathbb{Q}, p \leq x+y\} = \sup\{b^{a+b} \mid a \in \mathbb{Q}, b \in \mathbb{Q}, a \leq x, b \leq y\} = \sup\{b^a b^b \mid a \in \mathbb{Q}, b \in \mathbb{Q}, a \leq x, b \leq y\} = \sup\{b^a \mid a \in \mathbb{Q}, a \leq x\} \sup\{b^b \mid b \in \mathbb{Q}, b \leq y\} = b^x b^y$.

   Note: I accidentally used b as an exponent. These bs should be different.

Solution 1-8.
Say we take i to be positive. Then, we can multiply both sides of an inequality by it and keep the sign orientation. But $i > 0 \to i^2 > 0 \cdot i \to -1 > 0 \to -1 \cdot i > 0 \cdot i \to -i > 0 \to i < 0$. Doing the same process in reverse shows that i cannot be negative either, and it obviously cannot be 0.

Solution 1-9.