analytical photon number for the tc model

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1 Analytical Treatment of the Tavis-Cummings Model

The Tavis-Cummings (TC) model is a simple theoretical framework for describing the interaction between N two level systems (TLS) and an optical cavity mode. Its Hamiltonian is given by

$$H_{TC} = \frac{\omega_m}{2}\sum_{n=1}^{N}\sigma_z^n + \omega_c a^\dagger a + \frac{g}{\sqrt{N}}\sum_{n=1}^{N}(\sigma_+ a + \sigma_- a^\dagger) \tag{1}$$

where ω_m is the TLS transition frequency, ω_c is the cavity frequency, and $\frac{g}{\sqrt{N}}$ gives the collective coupling strength. Clearly, this model conserves excitations. Thus, we can solve the model analytically within an excitation subspace. Consider the second excitation manifold, whose symmetrized bright basis states are given by

$$|s_0, 2\rangle\,, \; |s_1, 1\rangle\,, \; |s_2, 0\rangle \tag{2}$$

1.1 Subspace Hamiltonian

We proceed by deriving Hamiltonian matrix in this subspace. The diagonal elements must be given by $2\omega_c$, $\omega_c + \omega_m$, and $2\omega_m$, respectively. Additionally, since the first and third basis states aren't coupled directly, we must have that the upper right and lower left entries are 0. Solving for the coupling terms benefits from working in the angular momentum basis, with resulting Hamiltonian

$$H_{TC} = \frac{\omega_m}{2}J_z + \omega_c a^\dagger a + \frac{g}{\sqrt{N}}\left(J_+ a + J_- a^\dagger\right) \tag{3}$$

We calculate the following action

$$J_+ a\,|s_1, 1\rangle = J_+\,|j = N/2, m = -j+1\rangle \otimes a\,|1\rangle \tag{4}$$

The coefficient attached to the TLS ensemble term is given by

$$\sqrt{j(j+1) - m(m+1)} = \sqrt{j(j+1) - (1-j)(2-j)} = \sqrt{4j-2} = \sqrt{2N-2}$$

So eq. (4) becomes

$$\sqrt{2(N-1)}\,|s_2, 0\rangle \tag{5}$$

Similarly

$$J_- a^\dagger\,|s_1, 1\rangle = \sqrt{2N}\,|s_0, 2\rangle \tag{6}$$

Using these results, we can calculate the matrix elements at (0, 1) and (1, 2) as follows

$$\langle s_2, 0|H_{INT}|s_1, 1\rangle = \frac{g}{\sqrt{N}}\sqrt{2(N-1)}\,\langle s_2, 0|s_2, 0\rangle = g\sqrt{\frac{2(N-1)}{N}} \tag{7}$$

$$\langle s_0, 2|H_{INT}|s_1, 1\rangle = \frac{g}{\sqrt{N}}\sqrt{2N}\,\langle s_0, 2|s_0, 2\rangle = g\sqrt{2} \tag{8}$$

Put $\Omega_k = g\sqrt{2K/N}$; then, our Hamiltonian matrix becomes

$$\begin{pmatrix} 2\omega_c & \Omega_N & 0 \\ \Omega_N & \omega_c + \omega_m & \Omega_{N-1} \\ 0 & \Omega_{N-1} & 2\omega_m \end{pmatrix} \tag{9}$$

1.2 Resonant Solution

We limit our focus to a small number of TLS–specifically, we do not take the thermodynamic limit in which $N \gg 1$, which would've allowed for the approximation $\Omega_{N-1} \approx \Omega_N$. This system can still be solved exactly, but doing so is *extremely* bashy and realistically requires the assistance of a symbolic calculator. However, given resonance ($\omega_m = \omega_c$), the computation becomes much more managable. Additionally, the resonance dynamics are of significant interest.

1.2.1 Eigenenergies & Eigenstates

We have

$$\begin{aligned} 0 = \det\left(\begin{pmatrix} 2\omega & \Omega_N & 0 \\ \Omega_N & 2\omega & \Omega_{N-1} \\ 0 & \Omega_{N-1} & 2\omega \end{pmatrix} - \lambda I\right) &= \begin{vmatrix} 2\omega - \lambda & \Omega_N & 0 \\ \Omega_N & 2\omega - \lambda & \Omega_{N-1} \\ 0 & \Omega_{N-1} & 2\omega - \lambda \end{vmatrix} \\ &= (2\omega - \lambda)((2\omega - \lambda)^2 - \Omega_{N-1}^2) - \Omega_N(\Omega_N(2\omega - \lambda)) \\ &= (2\omega - \lambda)[(2\omega - \lambda)^2 - \Omega_N^2 - \Omega_{N-1}^2] \end{aligned}$$

This leads to three polariton solutions

$$\lambda = 2\omega, \quad \lambda_\pm = 2\omega \pm \sqrt{\Omega_N^2 + \Omega_{N-1}^2} \tag{10}$$

Notably, the upper and lower polariton energies average to exactly the middle polariton energy. One may solve for the eigenstates analytically as well; the results are (in ascending order)

$$\begin{pmatrix} \sqrt{\frac{N}{N-1}} \\ -\sqrt{\frac{2N-1}{N-1}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{\frac{N-1}{N}} \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{\frac{N}{N-1}} \\ \sqrt{\frac{2N-1}{N-1}} \\ 1 \end{pmatrix} \tag{11}$$

1.2.2 Cavity Photon Expectation

we aim to calculate the dynamics of the cavity photon expectation value. it is clear that in this basis, the cavity number operator is given by

$$\hat{n} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

additionally, any state in this basis can be written like

$$\psi(t) = c_0 e^{-i\lambda_- t} |\lambda_-\rangle + c_1 e^{-i\lambda t} |\lambda\rangle + c_2 e^{-i\lambda_+ t} |\lambda_+\rangle \tag{13}$$

we need the action of the number operator acting on each eigenstate. for simplicity, we put $\alpha = \sqrt{N/(N-1)}$, $\beta = \sqrt{(2N-1)/(N-1)}$ as well as normalization constants γ_1 and γ_2.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma_1 \begin{pmatrix} \alpha \\ -\beta \\ 1 \end{pmatrix} = \gamma_1 \begin{pmatrix} 2\alpha \\ -\beta \\ 0 \end{pmatrix}$$

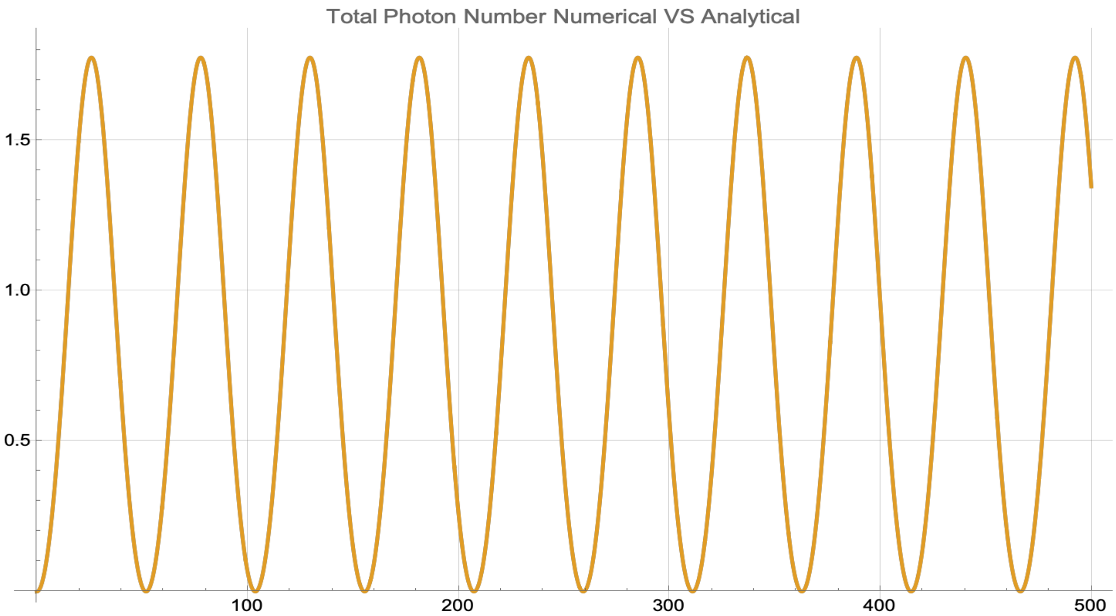
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma_2 \begin{pmatrix} -1/\alpha \\ 0 \\ 1 \end{pmatrix} = \gamma_2 \begin{pmatrix} -2/\alpha \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma_1 \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} = \gamma_1 \begin{pmatrix} 2\alpha \\ \beta \\ 0 \end{pmatrix}$$

consider an initial state with all excitations in the tls ensemble i.e. $|s_2, 0\rangle$. the projection of this initial state onto the eigenbasis yields projection coefficients $c_0 = \gamma_1$, $c_1 = \gamma_2$, and $c_2 = \gamma_1$. with this, we may expand $\langle \psi(t)|\hat{n}|\psi(t)\rangle$ to solve for $\langle \hat{n} \rangle (t)$. The calculation is also quite bashy; hence, omitted, but it yields the following:

$$\langle n \rangle (t) = -\frac{2(N-1)(1-4N+\cos(\Delta\lambda\, t))}{(2N-1)^2} \sin^2\left(\frac{\Delta\lambda\, t}{2}\right) \tag{14}$$

Where we've put $\Delta\lambda = \sqrt{\Omega_N^2 + \Omega_{N-1}^2}$. We can confirm this result numerically; consider $N = 2$ with coupling $g = 0.07$; this gives $\Delta\lambda \approx 0.12$. The resulting plot is



Figuur 1: Analytical expression and numerical results plotted on the same graph.

Essentially an exact agreement.