

CHAPTER 1 PROBLEMS – ANALYSIS

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Solution 1-1.

For the sake of contradiction, assume $r + x$ is rational. Then, we can express it in *lowest terms* as some p/q , where $p, q \in \mathbb{Z}$. Since r is rational, we may express it as some m/n , $m, n \in \mathbb{Z}$. But then

$$x = r + x - r = \frac{p}{q} - \frac{m}{n} = \frac{pn}{qn} - \frac{mq}{nq} = \frac{pn - mq}{nq}$$

Implying x is rational, a contradiction. Similarly, for the sake of contradiction, assume rx is rational. We use our previous definition of r and now define $rx = p/q$. We write

$$x = rx \cdot r^{-1} = \frac{p}{q} \cdot \left(\frac{m}{n}\right)^{-1} = \frac{p}{q} \cdot \frac{n}{m} = \frac{pn}{qm}$$

Implying x is rational, a contradiction.

Solution 1-2.

For the sake of contradiction, let $r \in \mathbb{Q}$ satisfy $r^2 = 12$. Then, we may write $r = p/q$ where $p, q \in \mathbb{Z}$, $p, q \neq 0$, and the fraction is in *lowest terms*. We write

$$\begin{aligned} r^2 &= (p/q)^2 = 12 \\ p^2 &= 12q^2 \end{aligned}$$

So $2 \mid p$ and $3 \mid p$, which means $6 \mid p$. Let $p = 6h$. We may write

$$(6h/q)^2 = 12 \rightarrow 36h^2 = 12q^2 \rightarrow 3h^2 = q^2$$

So $3 \mid q$. But $3 \mid p$, so p/q cannot be in lowest terms, contradiction.

Solution 1-4.

Consider some $e \in E$. Since α bounds E from below, $\alpha \leq e$. And, since β bounds E from above, $\beta \geq e$. Then $\alpha \leq e \leq \beta \rightarrow \alpha \leq \beta$.

Solution 1-5.

Since A is bounded below, its infimum exists. Let its infimum be α . $-A$ is given by the set of numbers $-x$ for all $x \in A$. Since $\alpha \leq x$ for all $x \in A$, $-\alpha \geq -x$. This follows from

$$\alpha \leq x \rightarrow \alpha - x \leq 0 \rightarrow -(\alpha - x) \geq 0 \rightarrow -\alpha - (-x) \geq 0 \rightarrow -\alpha \geq -x$$

Then, $-\alpha$ must bound $-A$ from above. To determine whether $-\alpha$ is the supremum of $-A$, consider some $\beta < -\alpha$. Assume β is the supremum of $-A$. Then

$-\beta > \alpha$, so there exists some $x \in A$ for which $-\beta > x$. But then $\beta < -x \in -A$, meaning that β cannot be the supremum of $-A$, contradiction. Thus

$$-\alpha = -\inf A = \sup(-A) \rightarrow \inf A = -\sup(-A)$$

Solution 1-6.

- a. Since $m/n = p/q$, we have $mq = np = v$. Then, we may use the integer exponentiation laws to write

$$\begin{aligned} ((b^m)^{1/n})^v &= ((b^m)^{1/n})^{np} = (b^m)^p = b^{mp} \\ ((b^p)^{1/q})^v &= ((b^p)^{1/q})^{mq} = (b^p)^m = b^{pm} = b^{mp} \end{aligned}$$

Then, both $(b^m)^{1/n}$ and $(b^p)^{1/q}$ are solutions to the equation

$$y^v = b^{mp}$$

But the theorem states that the solution y to this equation is unique, so we must have $(b^m)^{1/n} = (b^p)^{1/q}$.

- b. Since r and s are rational, we may express them as some p/q and m/n . Additionally, note that $r+s$ is then equal to $(pn+mq)/(nq)$. Then consider

$$\begin{aligned} (b^{r+s})^{nq} &= (b^{(pn+mq)/(nq)})^{nq} = b^{pn+mq} \\ (b^r b^s)^{nq} &= b^{r nq} b^{s nq} = b^{(p/q)nq} b^{(m/n)nq} = b^{np} b^{mq} = b^{np+mq} \end{aligned}$$

And, like in (a), Theorem 1.21 allows us to use this fact to conclude that $b^{r+s} = b^r b^s$.