PROBLEM SET 3 – PHYS0500

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29 februari 2024

Solution 5-5.
Let R be the initial distance that the particle is from the earth. Then, we can write a consrvation of energy equation as follows

$$\frac{-GMm}{R} = \frac{1}{2}m\dot{x}^2 - \frac{GMm}{x}$$

We then solve for $t(x)$

$$\frac{1}{2}\dot{x}^2 = \frac{GM}{x} - \frac{GM}{R}$$

$$\frac{dx}{dt} = \sqrt{2\left(\frac{GM}{x} - \frac{GM}{R}\right)}$$

$$\frac{dx}{dt} = \sqrt{2\left(\frac{GMR - GMx}{Rx}\right)}$$

$$\sqrt{\frac{Rx}{GMR - GMx}}\,dx = \sqrt{2}\,dt$$

Letting $x = y^2$ and $dx = 2y\,dy$, we write

$$\sqrt{\frac{Ry^2}{GMR - GMy^2}} \cdot 2y\,dy = \sqrt{2}\,dt$$

$$\int \frac{2\sqrt{R}y^2}{\sqrt{GMR - GMy^2}}\,dy = \sqrt{2}t$$

$$2\sqrt{\frac{R}{GM}} \int \frac{y^2}{\sqrt{R - y^2}}\,dy = \sqrt{2}t$$

The solution to integrals of this form is given in Appendix E as

$$2\sqrt{\frac{R}{GM}}\left[-\frac{y}{2}\sqrt{R - y^2} + \frac{R}{2}\arcsin\frac{y}{\sqrt{R}}\right] = \sqrt{2}t$$

We switch back to x and then evaluate the integral from $x = R$ to $x = \frac{R}{2}$ as well as from $x = R$ to $x = 0$

$$2\sqrt{\frac{R}{GM}}\left[-\frac{\sqrt{x}}{2}\sqrt{R-x}+\frac{R}{2}\arcsin\frac{\sqrt{x}}{\sqrt{R}}\right] = \sqrt{2}t$$

$$\sqrt{2}T_1 = 2\sqrt{\frac{R}{GM}}\left[-\frac{\sqrt{x}}{2}\sqrt{R-x}+\frac{R}{2}\arcsin\frac{\sqrt{x}}{\sqrt{R}}\right]_R^0 = -2\sqrt{\frac{R}{GM}}\cdot R\frac{\pi}{4}$$

$$\sqrt{2}T_{\frac{1}{2}} = 2\sqrt{\frac{R}{GM}}\left[-\frac{\sqrt{x}}{2}\sqrt{R-x}+\frac{R}{2}\arcsin\frac{\sqrt{x}}{\sqrt{R}}\right]_R^{\frac{R}{2}} = 2\sqrt{\frac{R}{GM}}\left(\frac{R\pi-2R}{8}-\frac{2R\pi}{8}\right)$$

Taking their ratio gives

$$\frac{T_{\frac{1}{2}}}{T_1} = \frac{-\pi-2}{8}\cdot\frac{-4}{\pi} = 0.818 \approx \frac{9}{11}$$

Solution 5-7.

We note that the contribution the the gravitational potential by an infinitesimal line element dx will be

$$-\frac{G}{\sqrt{R^2+x^2}}\cdot \rho dx$$

where $\rho = \frac{M}{l}$. Integrating over the entire rod then gives

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} -\frac{M}{l}\frac{G}{\sqrt{R^2+x^2}}\,dx$$

$$-\frac{GM}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{dx}{\sqrt{R^2+x^2}}$$

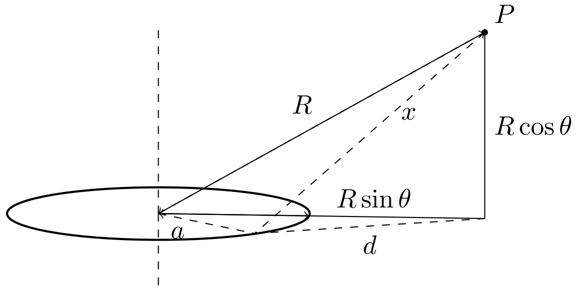
The solution to this integral is given in Appendix E as follows

$$-\frac{GM}{l}\left[\ln(x+\sqrt{x^2+R^2})\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$-\frac{GM}{l}\ln\left(\frac{0.5l+\sqrt{0.25l^2+R^2}}{-0.5l+\sqrt{0.25l^2+R^2}}\right)$$

And we are done.

Solution 5-10.



Let the angle between a and $R\sin\theta$ be ϕ. Then, we see that

$$d^2 = a^2 + R^2\sin^2\theta - 2aR\sin\theta\cos\phi$$
$$x^2 = d^2 + R^2\cos^2\theta$$

We solve for x

$$x^2 = a^2 + R^2\sin^2\theta - 2aR\sin\theta\cos\phi + R^2\cos^2\theta$$
$$x^2 = a^2 + R^2 - 2aR\sin\theta\cos\phi$$

We also note that

$$dV = -\frac{G}{x}\cdot\rho\, dl$$
$$dV = -\frac{G}{x}\cdot\frac{M}{2\pi a}\cdot a\, d\phi$$

Putting everything together gives

$$V = -\frac{GM}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{a^2 + R^2 - 2aR\sin\theta\cos\phi}}$$
$$V = -\frac{GM}{2\pi R}\int_0^{2\pi}\left(1 + \frac{a^2}{R^2} - \frac{2a}{R}\sin\theta\cos\phi\right)^{-0.5} d\phi$$
$$V = -\frac{GM}{2\pi R}\int_0^{2\pi}\left(1 - \left(2\frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{R^2}\right)\right)^{-0.5} d\phi$$

At this point, we recall the Taylor Series for $(1-x)^{-0.5}$ about $x = 0$

$$(1-x)^{-0.5} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + O(x^3)$$

We plug in $x = 2\frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{R^2}$

$$\left(1 - \left(2\frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{R^2}\right)\right)^{-0.5}$$
$$\approx 1 + \frac{1}{2}\left(2\frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{R^2}\right) + \frac{3}{8}\left(2\frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{R^2}\right)^2$$
$$\approx 1 + \frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{2R^2} + \frac{3}{2}\frac{a^2}{R^2}\sin^2\theta\cos^2\phi$$

And now we use this approximated integrand in our original integral

$$V = -\frac{GM}{2\pi R}\int_0^{2\pi} 1 + \frac{a}{R}\sin\theta\cos\phi - \frac{a^2}{2R^2} + \frac{3}{2}\frac{a^2}{R^2}\sin^2\theta\cos^2\phi\, d\phi$$

We calculate the integral of $\cos^2\phi$ from 0 to 2π

$$\int_0^{2\pi}\cos^2\phi\, d\phi = \frac{1}{2}\int_0^{2\pi}\cos(2\phi) + 1\, d\phi = \pi$$

Which gives

$$V = -\frac{GM}{2\pi R}\left(2\pi - \frac{\pi a^2}{R^2} + \frac{3}{2}\frac{a^2}{R^2}\sin^2\theta\pi\right)$$
$$V = -\frac{GM}{R}\left(1 - \frac{1}{2}\frac{a^2}{R^2}\left(1 - \frac{3}{2}\sin^2\theta\right)\right)$$

Solution 5-15.
When the particle is a distance r away from the center of the earth, the force acting upon it is given by

$$F = \frac{Gm}{r^2}\cdot\left(\frac{4}{3}\pi r^3\rho\right)$$

We pretty much ignore the mass that is farther away from the Earth's center than we are due to Newton's Shell law. Then, we have

$$a = \frac{G}{r^2}\cdot\left(\frac{4}{3}\pi r^3\rho\right)$$
$$\omega^2 r = \frac{G}{r^2}\cdot\left(\frac{4}{3}\pi r^3\rho\right)$$
$$\omega^2 = \frac{4G}{3}\pi\rho$$
$$\omega = \sqrt{\frac{4G}{3}\pi\rho}$$
$$T = 2\pi\cdot\left(\frac{4G}{3}\pi\rho\right)^{-\frac{1}{2}}$$

Assuming $\rho \approx 5441.41\ \mathrm{kg/m^3}$, we get

$$T = 5094.33\ \text{sec}$$

And dividing by 60 to convert to minutes, we get $T = 84.9$ minutes, which is roughly correct.

Solution 5-16.
Noting that only the z component of the force will not cancel, we have the integral

$$F = \int_0^\infty \frac{GM}{r^2}\cos(\theta)\,dm$$

where θ is the angle between the vertical and the radial line. We note

$$dm = 2\pi r\rho\,dr$$

Which gives

$$F = \int_0^\infty \frac{GM}{(r^2+h^2)}\cdot\frac{h}{\sqrt{r^2+h^2}}\cdot 2\pi r\rho\,dr$$
$$F = GMh2\pi\rho\int_0^\infty \frac{r}{(r^2+h^2)^{1.5}}\,dr$$
$$F = GMh2\pi\rho\left[-(r^2+h^2)^{-0.5}\right]_0^\infty$$
$$F = GM\rho\cdot 2\pi$$

And we are done.