PROBLEM SET 5 – APMA 0360

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Solution 1.

a. We have

$$\begin{aligned}
&\int_0^\pi \cos(nx)\cos(mx)\,dx \\
&= \frac{1}{2}\int_0^\pi \cos(nx+mx)+\cos(nx-mx)\,dx \\
&= \frac{1}{2}\left[\frac{\sin(nx+mx)}{n+m}+\frac{\sin(nx-mx)}{n-m}\right]_0^\pi \\
&= \frac{1}{2}\left[\frac{\sin((n+m)x)}{n+m}+\frac{\sin((n-m)x)}{n-m}\right]_0^\pi
\end{aligned}$$

And since $n, m \in \mathbb{N}$, $n+m \in Z$ and $n-m \in \mathbb{Z}$, meaning that every term becomes 0 when we plug in any integer multiple of π for x. Thus, this is equal to 0. Note that $n \neq m$ is vital since it means $n-m \neq 0$.

b. We have

$$\begin{aligned}
&\int_{-\pi}^\pi \sin(nx)\cos(mx)\,dx \\
&= \frac{1}{2}\int_{-\pi}^\pi \sin(nx+mx)+\sin(nx-mx)\,dx \\
&= \frac{1}{2}\left[-\frac{\cos(nx+mx)}{n+m}+\frac{\cos(nx-mx)}{m-n}\right]_{-\pi}^\pi \\
&= \frac{1}{2}\left(-\frac{\cos((n+m)\pi)}{n+m}+\frac{\cos((n-m)\pi)}{m-n}+\frac{\cos(-(n+m)\pi)}{n+m}-\frac{\cos(-(n-m)\pi)}{m-n}\right) \\
&= \frac{1}{2}\left(-\frac{\cos((n+m)\pi)}{n+m}+\frac{\cos((n-m)\pi)}{m-n}+\frac{\cos((n+m)\pi)}{n+m}-\frac{\cos((n-m)\pi)}{m-n}\right) \\
&= 0
\end{aligned}$$

And we are done.

Solution 2.

Assuming a solution of the form $u(x,t) = X(x)T(t)$, we write

$$u_t - u_{xx} = 0$$
$$XT' - X''T = 0$$
$$\frac{T'}{T} = \frac{X''}{X} = -k^2$$
$$T = Ae^{-k^2 t}, \quad X = B\cos(kx) + C\sin(kx)$$
$$u(x,t) = Ae^{-k^2 t}(B\cos(kx) + C\sin(kx))$$

We plug in the u_x boundary conditions

$$u_x(x,t) = Ae^{-k^2 t}(-Bk\sin(kx) + Ck\cos(kx))$$
$$u_x(0,t) = Ae^{-k^2 t}(Ck) = 0 \to C = 0$$
$$u_x(\pi,t) = Ae^{-k^2 t}(-Bk\sin(\pi k)) = 0 \to k \in \mathbb{N}$$

Then, we have

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-n^2 t}\cos(nx)$$

Since $u(x,0) = x^2$, we have

$$\sum_{n=0}^{\infty} A_n \cos(nx) = x^2$$
$$A_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx$$
$$= -\frac{4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx$$
$$= \frac{4}{n^2\pi}\left[x\cos(nx)\right]_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos(nx)\,dx$$
$$= \frac{4}{n^2}\cdot(-1)^n$$

We find A_0

$$A_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

Then

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2 t}\cos(nx)$$
$$u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\cdot\frac{4}{n^2}\cos(nx)e^{-n^2 t}$$

Solution 3.

We have

$$f(x) = \begin{cases} -1, & 0 < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

$$A_n = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} -\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin(nx)\,dx\right)$$

$$A_n = \frac{2}{n\pi}\left(\cos(0.5\pi n) - 1 + \cos(0.5\pi n) - \cos(n\pi)\right)$$

If $4 \mid n$, then the expression inside the parentheses becomes 0. If $n \equiv 1 \pmod 4$, the expression becomes 0 as well. If $n \equiv 2 \pmod 4$, the expression becomes -4, and if $n \equiv 3 \pmod 4$, the expression becomes 0 again. This gives

$$\begin{aligned} f(x) &= \frac{-8}{\pi}\left(\frac{\sin 2x}{2} + \frac{\sin 6x}{6} + \frac{\sin 10x}{10} + \dots\right) \\ &= \sum_{n=0}^{\infty} -\frac{8}{(4n+2)\pi}\sin((4n+2)x), \end{aligned}$$

To find the value of $A_n \sin(n\pi/2)$ for $n \in \mathbb{N}$, we take cases with $n \equiv 0, 1, 2, 3 \pmod 4$:

$$\begin{aligned} n \equiv 0 &\to A_n = 0 \to A_n \sin(n\pi/2) = 0 \\ n \equiv 1 &\to A_n = 0 \to A_n \sin(n\pi/2) = 0 \\ n \equiv 2 &\to A_n = -\frac{8}{n\pi} \to -\frac{8}{n\pi}\sin(k\pi) = 0 \\ n \equiv 3 &\to A_n = 0 \to A_n \sin(n\pi/2) = 0 \end{aligned}$$

So it is always 0. Also, I am aware my sine series for the jump isn't in the form $\sum_{n=1}^{\infty} A_n \sin(nx)$, but I didn't think writing it in that form was convenient.

Solution 4.

We assume a solution of the form $u(x,t) = X(x)T(t)$, yielding

$$XT'' - X''T = 0$$

$$\frac{T''}{T} - \frac{X''}{X} = 0$$

$$\frac{T''}{T} = \frac{X''}{X} = -k^2$$

$$X = A\cos(kx) + B\sin(kx), \quad T = C\cos(kt) + D\sin(kt)$$

We plug in the boundary conditions

$$A \cdot T = 0 \to A = 0$$

$$T \cdot B\sin(k\pi) = 0 \to k \in \mathbb{N}$$

Giving a general solution

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(nt) + B_n \sin(nt)) \sin(nx)$$

Note that

$$u_t(x,0) = \sin(2x) = \sum_{n=1}^{\infty} n B_n \sin(nx)$$

So $B_2 = \frac{1}{2}$ and all other $B_n = 0$. And

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = x^2$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin(nx)\, dx$$

$$\begin{aligned} A_n &= -\frac{2}{n\pi} \left[x^2 \cos(nx)\right]_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} 2x \cos(nx)\, dx \\ &= -\frac{2\pi}{n} \cdot (-1)^n - \frac{4}{n^2\pi} \int_0^{\pi} \sin(nx)\, dx \\ &= -\frac{2\pi}{n} \cdot (-1)^n + \frac{4}{n^3\pi} [\cos(n\pi) - 1] \\ &= -\frac{2\pi}{n} \cdot (-1)^n - \frac{4}{n^3\pi} + \frac{4}{n^3\pi} \cdot (-1)^n \\ &= -\frac{2\pi}{n} \cdot (-1)^n + \frac{4 \cdot (-1)^n - 4}{n^3\pi} \end{aligned}$$

Putting everything together gives

$$u(x,t) = \frac{1}{2} \sin(2t) \sin(2x) + \sum_{n=1}^{\infty} \left(-\frac{2\pi}{n} \cdot (-1)^n + \frac{4((-1)^n - 1)}{n^3\pi} \right) \cos(nt) \sin(nx)$$