PROBLEM SET 6 – APMA 0360

Ishaan Ganti
Brown University
Applied PDEs

7 april 2024

Solution 1.
We first calculate h_τ. We write

$$\frac{\partial h}{\partial \tau} = \frac{\partial h}{\partial t} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{N} \frac{\partial H}{\partial t} \cdot \frac{\partial t}{\partial \tau} = -\frac{bHI}{N\gamma} \tag{1}$$

And then for i_τ we have

$$\frac{\partial i}{\partial \tau} = \frac{\partial i}{\partial t} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{N} \frac{\partial I}{\partial t} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{N\gamma} \cdot (bHI - \gamma I + DI_{xx}) \tag{2}$$

In both of these equations, we use the conversion $h = H/N$ and $i = I/N$, which gives

$$h_\tau = -\frac{bHI}{N\gamma} = -\frac{bN^2 ih}{N\gamma} = -\frac{bNih}{\gamma} = -R_0 ih \tag{3}$$

$$i_\tau = \frac{1}{N\gamma}(bHI - \gamma I + DI_{xx}) = \frac{1}{N\gamma}(bN^2 hi - \gamma Ni + DNi_{xx}) = R_0 hi - i + di_{xx} \tag{4}$$

And we are done.

Solution 2.
We calculate the Laplacian, using the polar form of the expression

$$\nabla^2 \log(x^2 + y^2) \tag{5}$$

$$= \nabla^2 \log(r^2) \tag{6}$$

$$= \frac{\partial^2}{\partial r^2} \log(r^2) + \frac{1}{r} \frac{\partial}{\partial r} \log(r^2) + \frac{1}{r^2} \frac{\partial}{\partial \theta^2} \log(r^2) \tag{7}$$

$$= \frac{\partial}{\partial r} \frac{2r}{r^2} + \frac{2}{r^2} + 0 \tag{8}$$

$$= -\frac{2}{r^2} + \frac{2}{r^2} \tag{9}$$

$$= 0$$

Since the Laplacian is 0, u satisfies Laplaces equation on the given domain.

Solution 3.
We assume a solution of the form $u = X(x)Y(y)$. Then we have

$$\begin{gathered} X''Y + XY'' = 0 \\ \frac{X''}{X} + \frac{Y''}{Y} = 0 \\ \frac{X''}{X} = -\frac{Y''}{Y} = k^2 \\ Y = A\cos(ky) + B\sin(ky) \\ X = Pe^{kx} + Qe^{-kx} = C\cosh(kx) + D\sinh(kx) \end{gathered}$$

For $k > 0$. We attempt to apply some of the boundary conditions

$$u(x,y) = (A\cos(ky) + B\sin(ky))(C\cosh(kx) + D\sinh(kx)) \tag{10}$$
$$u_y(x,y) = (-Ak\sin(ky) + Bk\cos(ky))(C\cosh(kx) + D\sinh(kx)) \tag{11}$$
$$u_y(x,0) = (Bk)(C\cosh(kx) + D\sinh(kx)) = 0 \to B = 0 \tag{12}$$
$$u_y(x,\pi) = (-Ak\sin(k\pi))(C\cosh(kx) + D\sinh(kx)) = 0 \to k \in \mathbb{N} \tag{13}$$

We will consider the homogenous solution, first. We write

$$u(x,y) = \sum_{n=1}^{\infty} \cos(ny)(A_n\cosh(nx) + B_n\sinh(nx)) \tag{14}$$
$$u(0,y) = \sum_{n=1}^{\infty} A_n\cos(ny) = 0 \to A_n = 0 \tag{15}$$
$$u(x,y) = \sum_{n=1}^{\infty} B_n\sinh(nx)\cos(ny) \tag{16}$$
$$u(\pi,y) = \sum_{n=1}^{\infty} C_n\cos(ny) = 3\cos(2y) \tag{17}$$

So, $C_2 = 3$ and all other $C_l = 0$. Then, the same goes for all B_l, and $B_2 = \frac{3}{\sinh(2\pi)}$. This gives the homogenous solution

$$u(x,y) = \frac{3\cos(2y)\sinh(2x)}{\sinh(2\pi)} \tag{18}$$

For the particular solution, we consider the boundary conditions

$$\begin{gathered} u_y(x,0) = 0 \\ u_y(x,\pi) = 0 \\ u(0,y) = 0 \\ u(\pi,y) = 1 \end{gathered}$$

The last condition implies no dependence on y, meaning we have a solution just of the form $X(x)$. But then our solution is linear i.e. $X = Ax + B$. Using the conditions, we see that $B = 0$ and $A = \frac{1}{\pi}$. Putting both solutions together yields the final solution

$$u(x,t) = \frac{x}{\pi} + \frac{3\cos(2y)\sinh(2x)}{\sinh(2\pi)} \tag{19}$$

Solution 4.
As in the previous problem, we split this problem into two different boundary value problems. First, we place the condition that $u(x, \pi) = 0$ as opposed to 100. But then our initial conditions are

$$\begin{aligned} u(x,0) &= 0 \\ u(x,\pi) &= 0 \\ u(0,y) &= 0 \\ u(\pi,y) &= 100 \end{aligned}$$

We separate variables, giving

$$\begin{aligned} \frac{X''}{X} &= -\frac{Y''}{Y} = k^2 \\ X &= Ae^{kx} + Be^{-kx} \\ Y &= C\cos(ky) + D\sin(ky) \end{aligned}$$

We consider boundary conditions

$$u(x,0) \to C = 0 \to C = 0 \tag{20}$$

$$u(x,\pi) = D\sin(k\pi) = 0 \to k \in \mathbb{N} \tag{21}$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin(ny)(A_n \sinh(nx) + B_n \cosh(nx)) \tag{22}$$

$$u(0,y) = B_n \sin(ny) = 0 \to B_n = 0 \tag{23}$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh(nx)\sin(ny) \tag{24}$$

$$u(\pi,y) = \sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin(ny) = 100 \tag{25}$$

We solve for the coefficients, letting $P_n = A_n \sinh(n\pi)$

$$P_n = \frac{2}{\pi}\int_0^{\pi} 100\sin(ny)\,dy = -\frac{200}{n\pi}\left[\cos(ny)\right]_0^{\pi} = -\frac{200}{n\pi}(-1 + (-1)^n) \tag{26}$$

So then

$$A_n = -\frac{200}{n\pi\sinh(n\pi)}(-1 + (-1)^n) \tag{27}$$

And

$$u(x,y) = \sum_{n=1}^{\infty} -\frac{200}{n\pi\sinh(n\pi)}(-1 + (-1)^n)\sinh(nx)\sin(ny) \tag{28}$$

And we can see that for the second set of boundary conditions, the solution we get will be the same with the x and y terms switched, yielding the final solution

$$u(x,y) = \sum_{n=1}^{\infty} -\frac{200}{n\pi\sinh(n\pi)}(-1 + (-1)^n)(\sinh(nx)\sin(ny) + \sinh(ny)\sin(nx)) \tag{29}$$

And we are done.