CO2020 – Computer Aided Numerical Methods II
HomeWork 6
Hyperbolic PDEs

Ishaan Jain
CO21BTECH11006

Ans 1.

$$\frac{\partial u}{\partial t}+\frac{\partial F}{\partial x}=0$$

where $F=cu$ for $c>0$

(a) Explicit euler FTCS scheme:

Let's say, u is the numerical solution:

$$u_j^{n+1}=u_j^n-\frac{\Delta t}{2\Delta x}(F_{j+1}^n-F_{j-1}^n)$$

$$u_j^{n+1}=u_j^n-\frac{c\Delta t}{2\Delta x}(u_{j+1}^n-u_{j-1}^n)$$

$$u_j^{n+1}=u_j^n-\frac{CFL}{2}(u_{j+1}^n-u_{j-1}^n)$$

and v is the exact solution:

$$v_j^{n+1}\approx v_j^n-\frac{\Delta t}{2\Delta x}(F_{j+1}^n-F_{j-1}^n)$$

$$v_j^{n+1}\approx v_j^n-\frac{c\Delta t}{2\Delta x}(v_{j+1}^n-v_{j-1}^n)$$

$$v_j^{n+1}\approx v_j^n-\frac{CFL}{2}(v_{j+1}^n-v_{j-1}^n)$$

On subtracting these equations, we get

$$\varepsilon_j^{n+1}=\varepsilon_j^n-\frac{CFL}{2}(\varepsilon_{j+1}^n-\varepsilon_{j-1}^n)$$

The error is assumed to be of the form $\varepsilon_j^n=e^{\sigma t_n}e^{ik_m x_j}$, substituting this,

$$\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}=e^{\sigma\Delta t}=1-\frac{CFL}{2}(e^{ik_m\Delta x}-e^{-ik_m\Delta x})$$

$$\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}=e^{\sigma\Delta t}=1-\frac{CFL}{2}(2i\sin(k_m\Delta x))$$

We have to check for the values where

$$\left|\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}\right|\leq 1$$

$$\left(1-\frac{CFL}{2}(2i\sin(k_m\Delta x))\right)\left(1+\frac{CFL}{2}(2i\sin(k_m\Delta x))\right)\leq 1$$

$$\Rightarrow 1+CFL^2\sin^2(k_m\Delta x)\le 1 \Rightarrow CFL^2\sin^2(k_m\Delta x)\le 0$$

Here when the equality holds, $\Delta t=0$ which is can't be a solution.
So, the equation is unconditionally unstable.

(b) Lax method:

Let's say, u is the numerical solution:

$$u_j^{n+1}=\frac{u_{j+1}^n+u_{j-1}^n}{2}-\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n-F_{j-1}^n\right)$$

$$u_j^{n+1}=\frac{u_{j+1}^n+u_{j-1}^n}{2}-\frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n-u_{j-1}^n\right)$$

$$u_j^{n+1}=\frac{u_{j+1}^n+u_{j-1}^n}{2}-\frac{CFL}{2}\left(u_{j+1}^n-u_{j-1}^n\right)$$

and v is the exact solution:

$$v_j^{n+1}\approx\frac{v_{j+1}^n+v_{j-1}^n}{2}-\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n-F_{j-1}^n\right)$$

$$v_j^{n+1}\approx\frac{v_{j+1}^n+v_{j-1}^n}{2}-\frac{c\Delta t}{2\Delta x}\left(v_{j+1}^n-v_{j-1}^n\right)$$

$$v_j^{n+1}\approx\frac{v_{j+1}^n+v_{j-1}^n}{2}-\frac{CFL}{2}\left(v_{j+1}^n-v_{j-1}^n\right)$$

On subtracting these equations, we get

$$\varepsilon_j^{n+1}=\frac{\varepsilon_{j+1}^n+\varepsilon_{j-1}^n}{2}-\frac{CFL}{2}\left(\varepsilon_{j+1}^n-\varepsilon_{j-1}^n\right)$$

The error is assumed to be of the form $\varepsilon_j^n=e^{\sigma t_n}e^{ik_m x_j}$, substituting this,

$$\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}=e^{\sigma\Delta t}=\frac{e^{ik_m\Delta x}+e^{-ik_m\Delta x}}{2}-\frac{CFL}{2}\left(e^{ik_m\Delta x}-e^{-ik_m\Delta x}\right)$$

$$\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}=e^{\sigma\Delta t}=\cos(k_m\Delta x)-\frac{CFL}{2}\left(2i\sin(k_m\Delta x)\right)$$

We have to check for the values where

$$\left|\frac{\varepsilon_j^{n+1}}{\varepsilon_j^n}\right|\le 1$$

$$\left(\cos(k_m\Delta x)-\frac{CFL}{2}(2i\sin(k_m\Delta x))\right)\left(\cos(k_m\Delta x)+\frac{CFL}{2}(2i\sin(k_m\Delta x))\right)\le 1$$

$$\Rightarrow\cos^2(k_m\Delta x)+CFL^2\sin^2(k_m\Delta x)\le 1$$

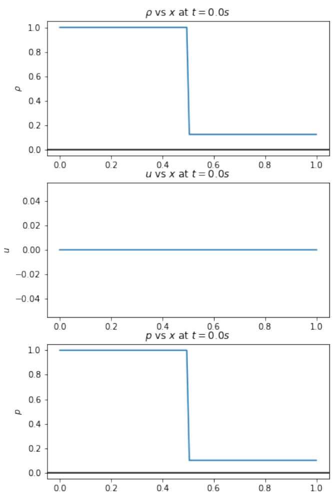
$$\Rightarrow CFL^2\sin^2(k_m\Delta x)\le\sin^2(k_m\Delta x)$$

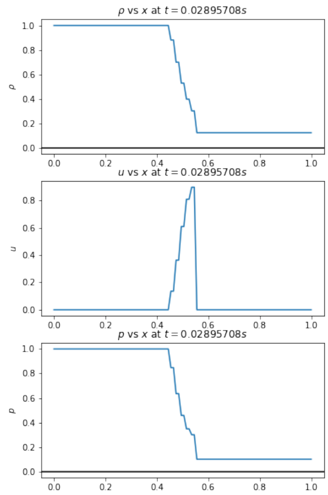
$$\Rightarrow CFL^2\le 1\Rightarrow CFL\le 1$$

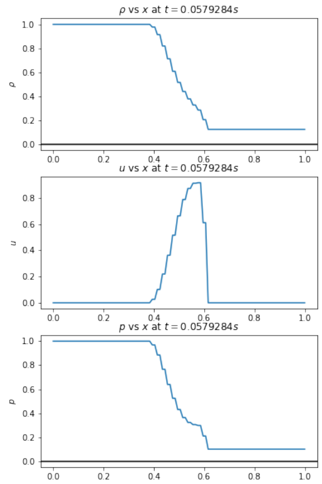
Hence, the Von Neumann stability analysis concludes that the Lax method is cconditionally stable with the condition being $CFL \leq 1$

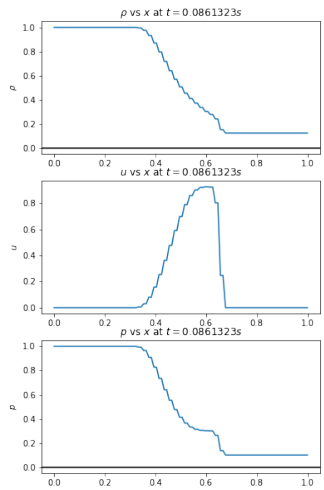
Ans 2.

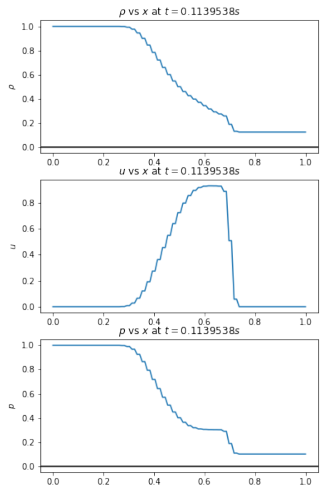
While using the lax method,
(a) when CFL number is **0.99**, we get the follwing profiles of density, velocity and pressure when plotted against x at different time steps.

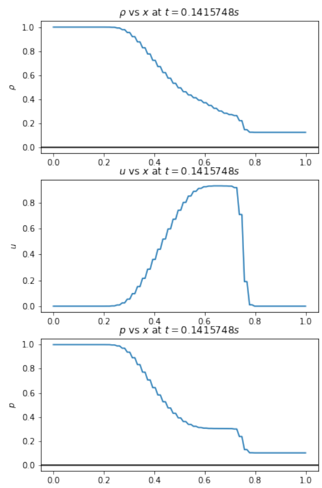


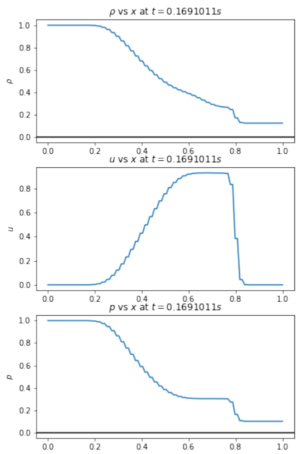


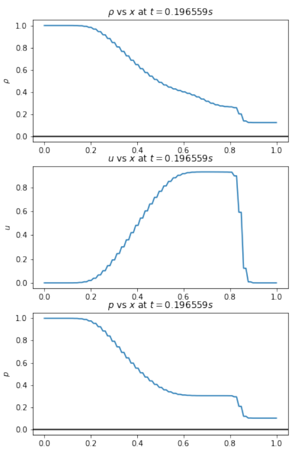


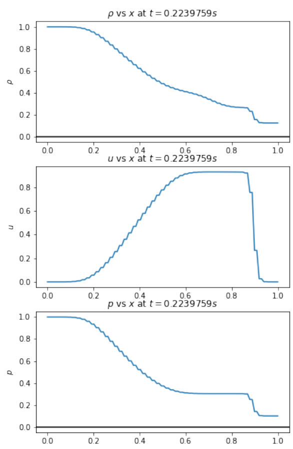


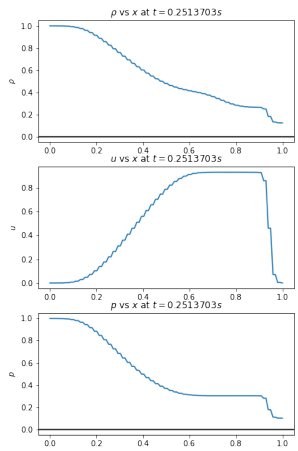




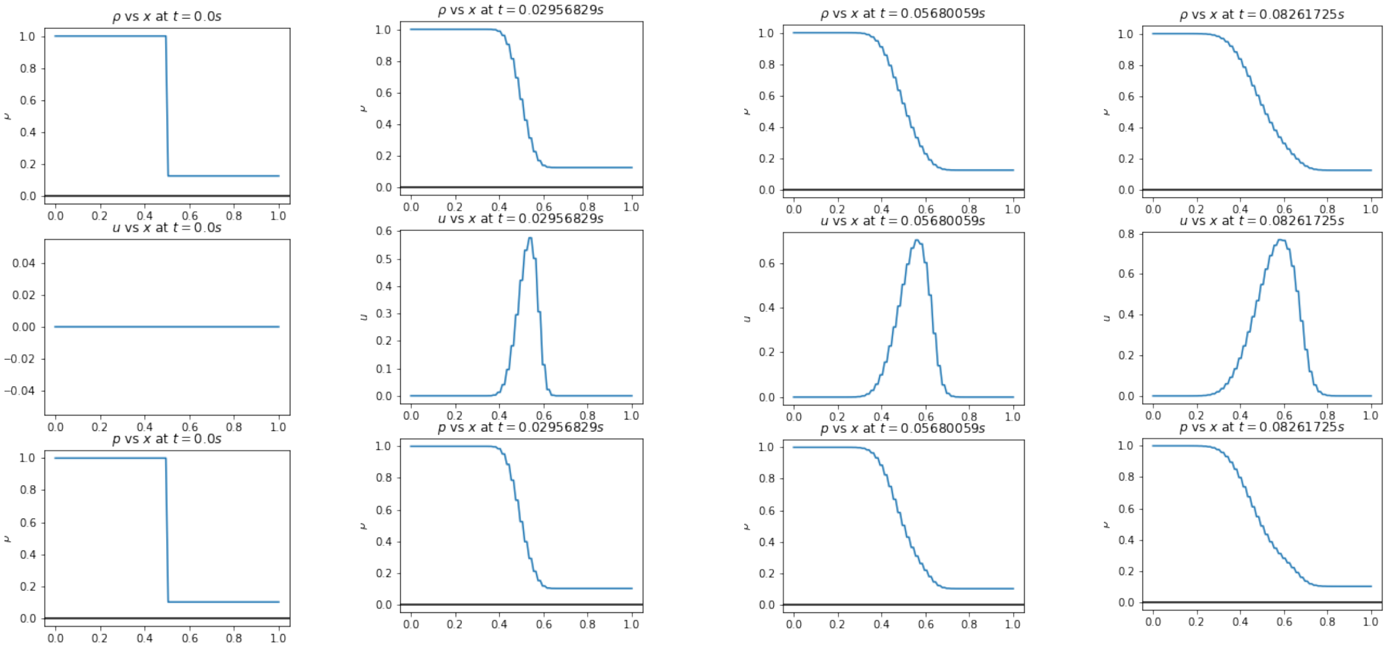


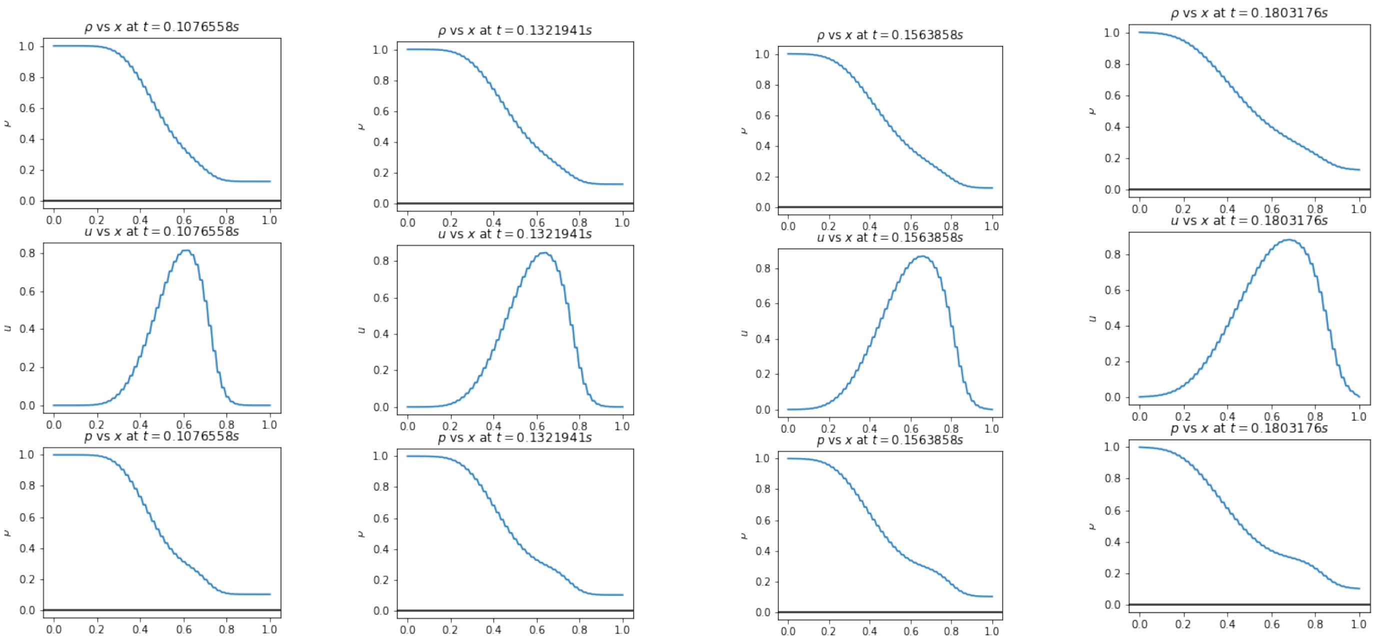


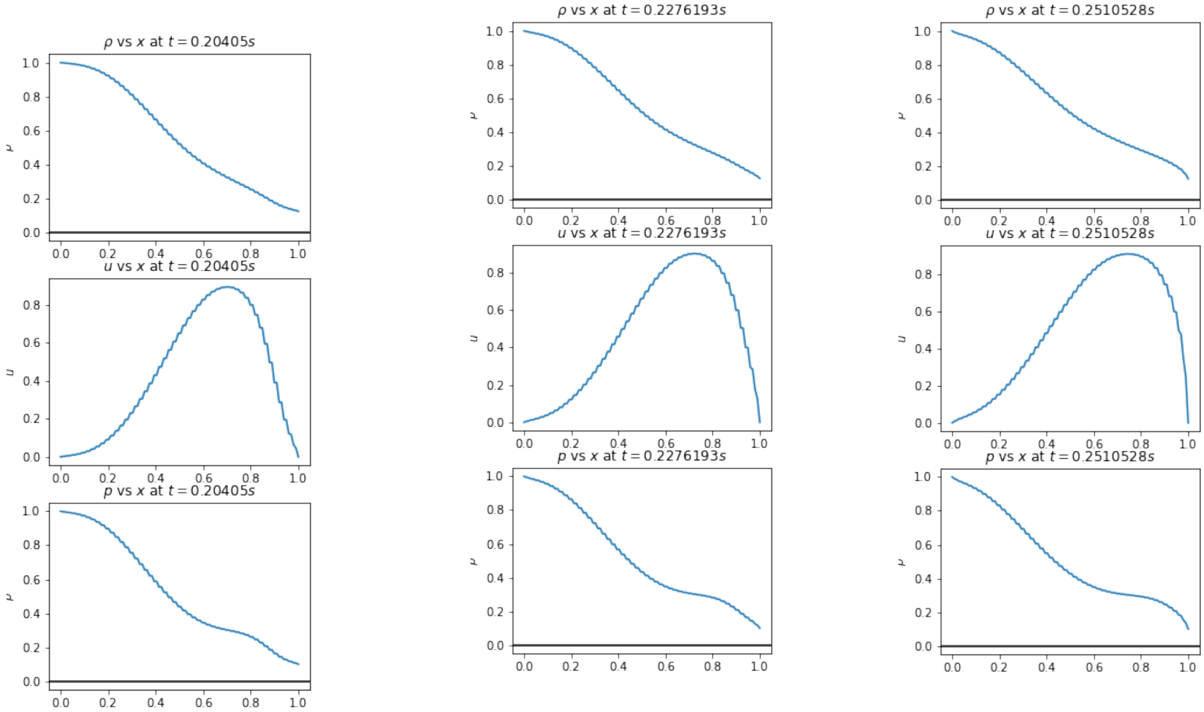




(b) When **CFL = 0.3,** we get the following profiles:







We can observe that the plots for CFL = 0.99 are less stable than CFL = 0.3. The reason is that in the first question itself, we derived that the CFL should be less than or equal to 1. As 0.99 is much closer to 0.3, so the solution when CFL = 0.99 should have more discontinuity, which is evident from the plots. So, decreasing CFL number results in more stability.