CO2020 : Computer-Aided Numerical Methods II

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Homework 6

Due Date: (optional) 3 May 2023

Question 1: Consider the 1D hyperbolic equation,

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0.$$

If the flux is $F = cu$ where $c > 0$ is a constant, then this is the 1D wave equation. The solution of this equation represents a wave moving to the right with speed c. Applying the explicit Euler time stepping scheme and 2nd order central difference scheme gives the update formula

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right).$$

The Lax method of discretization yields the update formula

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right).$$

Apply von Neumann stability analysis to determine stability conditions for the combination $c\Delta t/\Delta x$ which is called the CFL number.

Question 2: Study the code given to you along with this HW. This code implements the Lax method to solve the 1D Euler equations, given an initial state. The equation of state implemented and the initial conditions in the code have been specified so as to solve Sod's shock tube problem with the left state $\rho_L = 1, p_L = 1, u_L = 0$ and the right state $\rho_R = 0.125, p_R = 0.1, u_R = 0$. Note that the CFL number is defined for Euler equations based on the maximum wavespeed (or the maximum among the three eigenvalues of the Jacobian), $\max\{u, u+a, u-a\}$. Here, $a = \sqrt{\gamma p/\rho}$ is the speed of sound in the medium.

Run this code for $nx = 100$ and two different CFL numbers, 0.3 and 0.99, until time $t_{end} = 0.25$. Generate plots of the density, velocity and pressure for the two cases along with the exact solution (also provided to you).

Comment (no need to explain) on whether the features such as shock wave, contact discontinuity and expansion wave are captured better or worse by the numerical method on decreasing the CFL number. If the result seems counter-intuitive, you can read from our reference books on the effect of CFL number on numerical solution of hyperbolic systems.

Question 3: Implement the Lax-Wendroff method in the code. Note that the skeleton of the code is already provided to you. Run the code for the above shock tube problem and generate plots of the three primitive variables (density, velocity, pressure) for the Lax-Wendroff method at $t_{end} = 1.25$. You may use $nx = 100$, $CFL = 0.8$. Also include th exact solution and solution of the Lax method for the same numerical parameters on each plot. Comment on how Lax-Wendroff behaves as compared to the Lax method.

General instructions:

1. Your codes should be spread across two or more files (e.g. main function in one .c file and other functions in a different file). You should use Makefile to compile your code.

2. Use Matlab or any other postprocessing software to generate line plots. Ensure that the font size of the legend and labels is large enough to be easily visible. Ensure that the line thickness is appropriately large. Export your image as a png or eps file and include it in your report.

3. Prepare a short report documenting the results (mainly figures with captions) and brief comments as directed. To reduce your work, do not repeat the problem statement.

4. Please submit your report as one pdf file as per the directions above. In addition, submit a zip or tar file with all codes, any input files or Makefile you may have used.