CO2020- Computer Aided Numerical Methods
HomeWork 5 – Parabolic PDEs

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Ans1.
(a)

$$\vec{\nabla}=\frac{\partial}{\partial x}\hat{e}_x+\frac{\partial}{\partial y}\hat{e}_y+\frac{\partial}{\partial z}\hat{e}_z=\alpha\frac{\partial}{\partial r}\hat{e}_r+\beta\frac{\partial}{\partial\theta}\hat{e}_\theta+\frac{\partial}{\partial z}\hat{e}_z$$

We know that,

$$\frac{\partial}{\partial r}=x_r\frac{\partial}{\partial x}+y_r\frac{\partial}{\partial y} \qquad \text{eqn(i)}$$

$$\frac{\partial}{\partial\theta}=x_\theta\frac{\partial}{\partial x}+y_\theta\frac{\partial}{\partial y} \qquad \text{eqn(ii)}$$

where, $x=r\cos\theta$ and $y=r\sin\theta$,
On calculating, $x_r=\cos\theta$, $y_r=\sin\theta$, $x_\theta=-r\sin\theta$ and $y_\theta=r\cos\theta$

And the unit vectors are defined as:

$\hat{e}_r=\cos\theta\hat{e}_x+\sin\theta\hat{e}_y$ and $\hat{e}_\theta=\hat{e}_z\times\hat{e}_r=-\sin\theta\hat{e}_x+\cos\theta\hat{e}_y$

Substituting in the first equation,

$$\alpha\frac{\partial}{\partial r}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y} \qquad \text{eqn(iii)}$$

$$\beta\frac{\partial}{\partial\theta}=-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y} \qquad \text{eqn(iv)}$$

from i, ii, iii and iv,

$$\alpha\left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\Rightarrow\alpha=1$$

$$\beta\left(-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right)=-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}\Rightarrow\beta=\frac{1}{r}$$

i.e.

$$\vec{\nabla}=\frac{\partial}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta+\frac{\partial}{\partial z}\hat{e}_z$$

To solve the continuity equation,

$$\vec{\nabla}.\vec{v}=0$$

$$\vec{\nabla}.\vec{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\hat{e}_\theta+\frac{\partial v_z}{\partial z}\hat{e}_z+v_r(\nabla.\hat{e}_r)+v_\theta(\nabla.\hat{e}_\theta)+v_z(\nabla.\hat{e}_z)$$

$$\nabla.e_z=0$$
$$\nabla.e_r=\nabla.(\cos\theta e_x+\sin\theta e_y)=-\sin\theta\frac{\partial\theta}{\partial x}+\cos\theta\frac{\partial\theta}{\partial y}=-\sin\theta(\frac{-sin\theta}{r})+\cos\theta(\frac{\cos\theta}{r})=\frac{1}{r}$$

and

$$\nabla.e_\theta=\nabla.(-\sin\theta e_x+\cos\theta e_y)=-\cos\theta\frac{\partial\theta}{\partial x}-\sin\theta\frac{\partial\theta}{\partial y}=-\cos\theta(\frac{-sin\theta}{r})-\sin\theta(\frac{\cos\theta}{r})=0$$

Hence, $\nabla.\vec{v}=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}+\frac{u_r}{r}$

In 2D, we can neglect z-axis term;
So the final equation becomes:

$$\nabla.\vec{v}=\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}$$

(b) The Gradient operator in Cylindrical coordinates we already defined and proved in the last part, using that result:

$$\nabla^2 T=\vec{\nabla}.(\vec{\nabla}T)=(\frac{\partial}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{e}_\theta+\frac{\partial}{\partial z}\hat{e}_z).(\frac{\partial T}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial T}{\partial\theta}\hat{e}_\theta+\frac{\partial T}{\partial z}\hat{e}_z)$$

$$\nabla^2=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial T}{\partial\theta}\right)+\frac{\partial^2 T}{\partial z^2}+\frac{1}{r}\frac{\partial T}{\partial r}$$

$$\nabla^2 T=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{r}\frac{\partial T}{\partial r}$$

We can neglect the z axis for 2D form:

$$\nabla^2 T=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2}+\frac{1}{r}\frac{\partial T}{\partial r}$$

Ans 2.

(a) The stencils for $i=0$ and $i=n_x-2$ are calculated the by the following taylor series expansion:

$$T_0=T_1-\Delta x T'_1+\left(\frac{\Delta x^2}{2}\right)T''_1-\left(\frac{\Delta x^3}{6}\right)T'''_1+\left(\frac{\Delta x^4}{24}\right)T''''_1+O(\Delta x^5)$$

$$T_2=T_1+\Delta x T'_1+\left(\frac{\Delta x^2}{2}\right)T''_1+\left(\frac{\Delta x^3}{6}\right)T'''_1+\left(\frac{\Delta x^4}{24}\right)T''''_1+O(\Delta x^5)$$

$$T_3 = T_1 + 2\Delta x T'_1 + \left(\frac{(2\Delta x)^2}{2}\right) T''_1 + \left(\frac{(2\Delta x)^3}{6}\right) T'''_1 + \left(\frac{(2\Delta x)^4}{24}\right) T''''_1 + O(\Delta x^5)$$

$$T_4 = T_1 + 3\Delta x T'_1 + \left(\frac{(3\Delta x)^2}{2}\right) T''_1 + \left(\frac{(3\Delta x)^3}{6}\right) T'''_1 + \left(\frac{(3\Delta x)^4}{24}\right) T''''_1 + O(\Delta x^5)$$

$$\begin{bmatrix} T_0 - T_1 \\ T_2 - T_1 \\ T_3 - T_1 \\ T_4 - T_1 \end{bmatrix} = \begin{bmatrix} -1 & 1/2 & -1/6 & 1/24 \\ 1 & 1/2 & 1/6 & 1/24 \\ 2 & 2 & 4/3 & 2/3 \\ 3 & 9/2 & 9/2 & 27/8 \end{bmatrix} \begin{bmatrix} \Delta x T'_1 \\ \Delta x^2 T''_1 \\ \Delta x^3 T'''_1 \\ \Delta x^4 T''''_1 \end{bmatrix}$$

$$b = Ax \Rightarrow A^{-1} b = x$$

Find the inverse and find the second row which will give the value of T''_1 .

Similarly find the right stencil at i = nx-2.

The values calculated are implemented in the code.

(b) Von-Neumann stability analysis:

$$r = k\frac{\Delta t}{\Delta x^2}$$

$T_j^n = z^n e^{ij\theta}$ where n in the time instance and j is the position.

$$T_j^{n+1} - T_j^n = r\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)$$

$$z^{n+1} e^{ij\theta} - z^n e^{ij\theta} = r\left(z^n e^{i(j+1)\theta} - 2z^n e^{i(j)\theta} + z^n e^{i(j-1)\theta}\right)$$

$$z - 1 = r\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$z = 1 - r(2 - 2\cos\theta)$$

For The analysis to be stable,

$$-1 \leqslant z \leqslant 1 \Rightarrow -1 \leqslant 1 - 2r(1 - \cos\theta) \leqslant 1$$

$$-1 \leqslant -r\left(2\sin^2(\theta/2)\right) \leqslant 0$$

$$0 \leqslant r \leqslant 1/2\sin^2(\theta/2)$$

In limiting case, r holds it’s maximum value as $r = 1/2$

at r = 0.5, $0.5 = k\frac{\Delta t}{\Delta x^2} \Rightarrow 0.5 = \frac{10^{-4}}{\Delta x^2}$

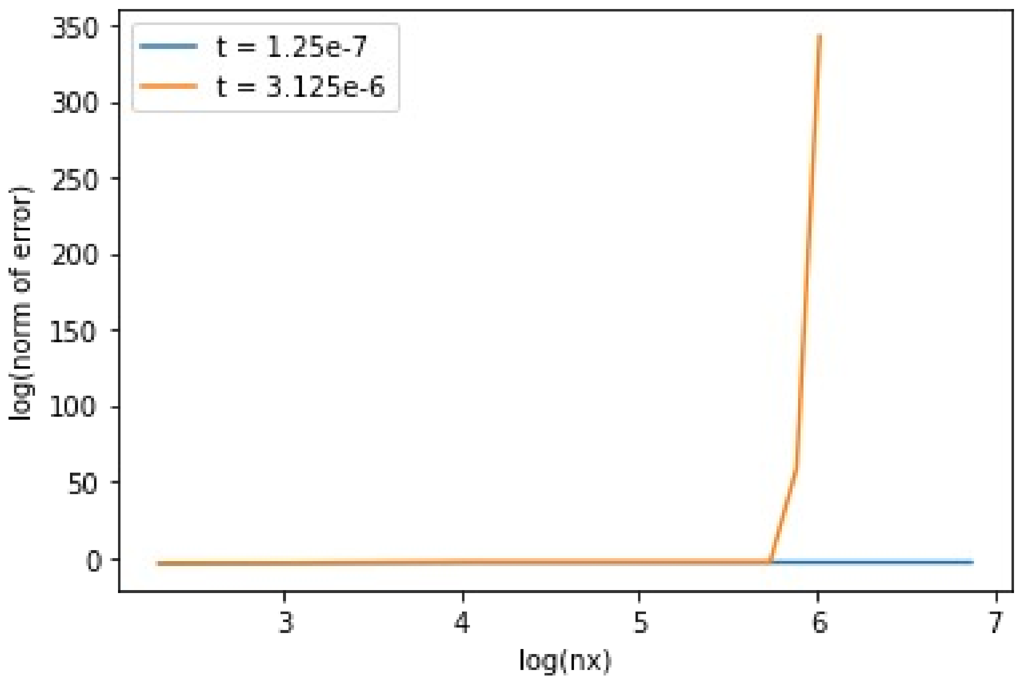
$$\Delta x = \sqrt{2\times 10^{-3}} = 0.0141421$$

number of partitions $= \frac{1-0}{\Delta x} = 70$ at which the system is stable. After this the error starts growing.

There may be an error in calculations because for me, after n = 62, error is starting to grow.

```
ishaan@insane-insaan:~/Desktop/Codes/CO2020/PDE/HW5/hw5_template$ gcc parabolic_pde.c -lm
ishaan@insane-insaan:~/Desktop/Codes/CO2020/PDE/HW5/hw5_template$ ./a.out
Inputs are: 62 0.000000 1.000000 0.000000 0.100000
Done writing solution for time step = 0
Done writing solution for time step = 100
Done writing solution for time step = 200
Done writing solution for time step = 300
Done writing solution for time step = 400
Done writing solution for time step = 500
Done writing solution for time step = 600
Done writing solution for time step = 700
Done writing solution for time step = 800
Done writing solution for time step = 900
Done writing solution for time step = 1000
Norm for 62 steps = 0.130123
ishaan@insane-insaan:~/Desktop/Codes/CO2020/PDE/HW5/hw5_template$ ./a.out
Inputs are: 63 0.000000 1.000000 0.000000 0.100000
Done writing solution for time step = 0
Done writing solution for time step = 100
Done writing solution for time step = 200
Done writing solution for time step = 300
Done writing solution for time step = 400
Done writing solution for time step = 500
Done writing solution for time step = 600
Done writing solution for time step = 700
Done writing solution for time step = 800
Done writing solution for time step = 900
Done writing solution for time step = 1000
Norm for 63 steps = 238280708129.695496
```

(c)



The graph above is the error vs nx plot for two different time intervals.

Initially, the log-log values are parallel to the straight line with slope = -4.

for second case, after nx = 350 (approx), the error starts to grow and we see a asymptotic curve in our plot.