CO2020 : Computer-Aided Numerical Methods II

Course Instructors: Niranjan S. Ghaisas and Raja Banerjee

Homework 3

Due Date: 6 March 2023

Topic: PDE Basics

Note: In this document, subscript denotes a differentiation, e.g. u_x denotes $\partial u/\partial x$. Variables such as x, y, z, t are generally independent variables. The unknown u is a function of the independent variables as per context. Any other symbol (e.g. c, ν) is a constant unless otherwise stated.

1. State the order of the following PDEs and whether they are homogeneous/heterogeneous and linear/non-linear.
   (a) $u_{xx} + u_{yy} = 0$
   (b) $u\ x\ u_x + u^2\ x\ y\ u_{xy} = 0$
   (c) $(u_t)^2 + f(x,y)\ u = g(x,y)$
   (d) $u_t + u\ u_x = \nu\ u_{xx}$
   (e) $u_t + c\ u_x = 0$
2. Identify whether the 2nd order PDEs given below are elliptic, parabolic or hyperbolic. If an equation is parabolic or hyperbolic, state the number of real characteristics and find their expressions.
   (a) $x^2\ u_{xx} - 2xy\ u_{xy} + y^2\ u_{yy} = 0$
   (b) $u_{xx} + (1+y^2)^2\ u_{yy} - 2y(1+y^2)\ u_y = 0$
   (c) $x^2\ u_{xx} - 2xy\ u_{xy} - 3y^2\ u_{yy} = u_y - u_x + f(x,y)\ u$
3. Using a change of variable, transform the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x)\frac{\partial u}{\partial x} = f(x,y)$$

into a Poisson equation

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = A(x) f(x, y)$$

where $a(x) = dA(x)/dx$.

4. Solve analytically the Laplace equation $T_{xx} + T_{yy} = 0$ for the temperature on a rectangular domain of size $a \times b$. The horizontal boundaries ($y = 0$ and $y = b$) are insulated, and the left ($x = 0$) and right ($x = a$) boundaries are held at T_0 and ($T_0[1 + 0.2(y/b - 0.5)]$) respectively. For $a = 1$ m, $b = 0.5$ m and $T_0 = 300$ K, plot (a) contours in the 2D domain; (b) profiles of T vs y for $x/a = 0.1, 0.3, 0.5, 0.7$ and 0.9; (c) profiles of T vs x for $y = 0.1, 0.5$ and 0.9. Generate one figure each for parts (a), (b) and (c).

General instructions:

1. For plotting figures, use Matlab or Python any other postprocessing software of your choice. Ensure that the font size of the legend and labels is large enough to be easily visible. Ensure that the line thickness is appropriately large. Export each image as a png or eps file and include them in a pdf document.

2. Print out this pdf and submit it along with your handwritten part for the rest of the questions.

3. Alternatively, you may scan your entire handwritten part to pdf, combine it with the pdf containing the figures for Q4, and submit the entire homework as a single pdf on google classroom.