Finite Difference Model for Joule Heating

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1 Introduction

The goal of this assignment is to model the heat equation in the 1D case using the Finite Difference Method. The governing equation is the first law of thermodynamics. In the general 3D case, it is written as:

$$\rho\dot{\omega} = \boldsymbol{\sigma} : \boldsymbol{\nabla}\dot{\boldsymbol{u}} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho z$$

***Note* :** Here and throughout this project report, vectors, matrices, and higher order tensors are in boldface, while scalars are not.

In the above equation, ρ is mass density, ω is stored energy per unit mass, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{q}$ is the heat flux, and ρz is a thermal source term given by:

$$\rho z = a(\boldsymbol{J} \cdot \boldsymbol{E})$$

where $\boldsymbol{J}$ is the current, $\boldsymbol{E}$ is the electric field, and a is an absorption constant with possible values $0 \leq a \leq 1$. We can use this equation in order to model Joule heating. For this assignment, we will treat the applied electric field as fixed. The electric field helps us find the current density J through Ohm's Law, also presented in the general case:

$$\boldsymbol{J} = \sigma^c \boldsymbol{E}$$

where σ^c is the electrical conductivity. In general, electrical conductivity is a 3x3 matrix, however since we are assuming isotropic or uniform conductivity, we can instead represent it as a scalar, which is why it isn't bolded in the above equation. Before proceeding further, we make a series of simplifying assumptions and discuss how they affect our general first law of thermodynamics equation:

- Stress power is negligible, or $\boldsymbol{\sigma} : \boldsymbol{\nabla}\dot{\boldsymbol{u}} \approx \boldsymbol{0}$
- Deformations are negligible, or $\frac{\partial\theta}{\partial t} \approx \dot{\theta}$, where we define $\dot{\theta} = \frac{d\theta}{dt}$
- Energy is only stored as heat, so $\dot{\omega} = C\dot{\theta}$, where C is heat capacity
- As mentioned earlier, we are modelling the heat equation in only 1 dimension. This is equivalent to assuming that the temperature is constant in the other two. For convenience, let the dimension that temperature varies in be the $\hat{x}_1$ direction, meaning it is constant in $\hat{x}_2$ and $\hat{x}_3$. This is equivalent to saying that $\theta = \theta(x_1, t)$
- All exchange of heat ie. heat flux takes place through conduction alone- convection and radiation terms are negligible; In other words, $\boldsymbol{q} = -(\kappa\nabla\theta)$

- Body density is constant in both time and space, or $\rho(x_1, T) = \rho_0$
- Similarly, thermal conductivity is assumed to be constant in both time and space, or $\kappa(x_1, T) = \kappa_0$

This simplifies the first law to

$$\rho_0 C\dot{\theta} = \rho_0 C\frac{d\theta}{dt} = \nabla \cdot (\kappa \nabla \theta) + a(\boldsymbol{J} \cdot \boldsymbol{E}) = \kappa_0 \nabla^2 \theta + a(\boldsymbol{J} \cdot \boldsymbol{E})$$

In the 1D case, the laplacian ∇^2 of the temperature is its second derivative with respect to the spatial dimension x. In the 1D case, both $\boldsymbol{J}$ and $\boldsymbol{E}$ reduce to scalars, allowing us to further simplify our equation to:

$$\rho_0 C\frac{d\theta}{dt} = \kappa_0 \frac{d^2\theta}{dx^2} + aJE$$

Rearranging the terms, that leaves us with:

$$\frac{d\theta}{dt} = \frac{1}{\rho_0 C}(\kappa_0 \frac{d^2\theta}{dx^2} + aJE)$$

We now have an expression for the rate of change of temperature, which we can use to step forward in time at a given node using a Forward Euler scheme, provided we find an actual value for this expression. Looking at our formula, we realise we must find an expression for $\frac{d^2\theta}{dx^2}$. Note that this is a partial differential equation, meaning it cannot be solved without boundary conditions. We make use of the Finite Difference Method, which allows us to solve for transient solutions. In the 1D case, the Finite Difference Method involves discretising a domain of finite length L into n evenly spaced "nodes". We use the central difference method to discretise in space, and the aforementioned Forward Euler scheme to disretise in time. The central difference method allows us to write the spatial derivative as:

$$\frac{d\theta_i^j}{dx^2} = \frac{\theta_{i-1}^j - 2\theta_i^j + \theta_{i+1}^j}{\Delta x^2}$$

where θ_i^j is the temperature at time t_j at node x_i. In our earlier notation, this would be expressed as $\theta(x_i, t_j)$. The equation we derive from the central difference scheme also shows us that for a given node at a given time t, the spatial derivative is a function of the temperature of that node itself, but also of the temperature of the neighbouring nodes, all at the same time t. Δx represents the finite difference here, and is the distance between two consecutive nodes. Since the nodes are equally distributed across the 1D domain, the distance between them is uniform and we do not need to specify nodes.

Moving on to time discretisation, we remember that as per the Forward Euler scheme, our update rule for a node x_i at time t_{j+1} is:

$$\theta_i^{j+1} = \theta_i^j + \Delta t \frac{d\theta_i^j}{dt}$$

We rewrite this in terms our previous equations in order to obtain a fully discretised update step for the temperature at a single node x_i:

$$\theta_i^{j+1} = \theta_i^j + \frac{\Delta t}{\rho_0 C}(\kappa_0 \frac{\theta_{i-1}^j - 2\theta_i^j + \theta_{i+1}^j}{\Delta x^2} + aJE)$$

Provided that we have an initial temperature for each node, we are now able solve for the temperature at each node for the next time step, by using the temperature of that node and its immediate neighbours at the current time step. Note that our expression for the spatial derivative doesn't make sense for either of the two terminal nodes, but we handle that by the imposition of boundary conditions. We employ Dirichlet boundary conditions, meaning the temperature is fixed at both terminal nodes, with the heat source term being applied to every other node in the system.

We have two different sets of boundary conditions. In either case, the temperature at the initial node is fixed at $\theta_0 = 300K$ or $\theta(0,t) = \theta_0 = 300K$. Additionally, in both cases, the initial temperature at all non terminal nodes is also θ_0, or equivalently: $\theta(x_1, 0) = \theta_0 = 300K$. The two sets of boundary conditions differ regarding their treatment of the last node, corresponding to $x_1 = L$, where L is the length of our domain. In the first set, the terminal nodes are symmetric: $\theta(0,t) = \theta(L,t) = \theta_0 = 300K$. This is a matched Dirichlet boundary condition. In the second set, $\theta(L,t) = \theta_0 + 200 = 500K$, which is a mismatched Dirichlet boundary condition.

2 Deliverables

We start by expressing our Joule heating term differently. In general, joule heating term $\boldsymbol{H}(x) = \boldsymbol{a}(x)\boldsymbol{J}(x)\boldsymbol{E}(x)$. In the 1D case, all of these simplify to scalars, giving us our earlier equation of $H = aJE$. We have also defined $\boldsymbol{J} = \sigma^c \boldsymbol{E}$

Again, in the 1D case, we rewrite this as $E = \frac{J}{\sigma^c}$, which gives us $H = \frac{aJJ}{\sigma^c} = \frac{aJ^2}{\sigma^c}$

We now find an analytical solution to the heat equation in the 1D case with our earlier assumptions, especially that thermal conductivity is isotropic and not a function of position. For steady state, $\frac{d\theta}{dt} = 0$ or $\kappa_0 \frac{d^2\theta}{dx^2} + aJE = 0$, where we denote aJE with H. This gives us $\kappa_0 \frac{d^2\theta}{dx^2} = -H$. Integrating with respect to x gives us

$$\kappa_0 \frac{d\theta}{dx} = -Hx + c_1$$

We integrate again to represent $\theta(x)$ as

$$\theta(x) = \theta(x=0) - \int_0^x \frac{Hx}{\kappa_0} dx + \int_0^x \frac{c_1}{\kappa_0} dx + c_2$$

We impose the first boundary condition that $\theta(x=0) = \theta_0$, yielding that $c_2 = 0$. The second boundary condition is $\theta(x=L) = \theta_L$, meaning

$$\theta_L - \theta_0 = -\int_0^L \frac{Hx}{\kappa_0} dx + \int_0^L \frac{c_1}{\kappa_0} dx$$

Noting that κ_0, H, c_1 are all constants, we integrate with respect to x again, yielding

$$\theta_L - \theta_0 = \frac{-HL^2}{2\kappa_0} + \frac{c_1 L}{\kappa_0}$$

Solving for the other constant, we get $c_1 = (\theta_L - \theta_0 + \dfrac{HL^2}{2\kappa_0})\dfrac{\kappa_0}{L}$

bringing us to $\theta(x) = \dfrac{-Hx^2}{2\kappa_0} + \dfrac{c_1 x}{\kappa_0} = \theta_0 - \dfrac{Hx^2}{2\kappa_0} + \dfrac{(\theta_L - \theta_0)x}{L} + \dfrac{HLx}{2\kappa_0}$

To confirm dimensional consistency, we look at each term in our equation, expecting each to be expressed in terms of K. Starting with θ_0, we have defined its value in K. $H = aJE$, so its units are $\frac{JV}{m^3}$ (recall that a represents efficiency and is hence unitless) x^2 has units m^2, and κ_0 has units $\frac{W}{m-K}$, thus the expression $\frac{-Hx^2}{2\kappa_0}$ has units K since the units of length in the numerator and denominator cancel out and JV = W in terms of units used. For the third term, we see that $\theta_L - \theta_0$ will be expressed in K, and $\frac{x}{L}$ will be dimensionless, hence the entire term has units in K. We see that the last term will have the same units as the second term since x and L are both measured in m. We have already shown that the second term has units K, so the fourth term must as well. Hence, we have confirmed the dimensional consistency of our formula for $\theta(x)$

We now plot the steady state solution, along with plots of the temperature profile

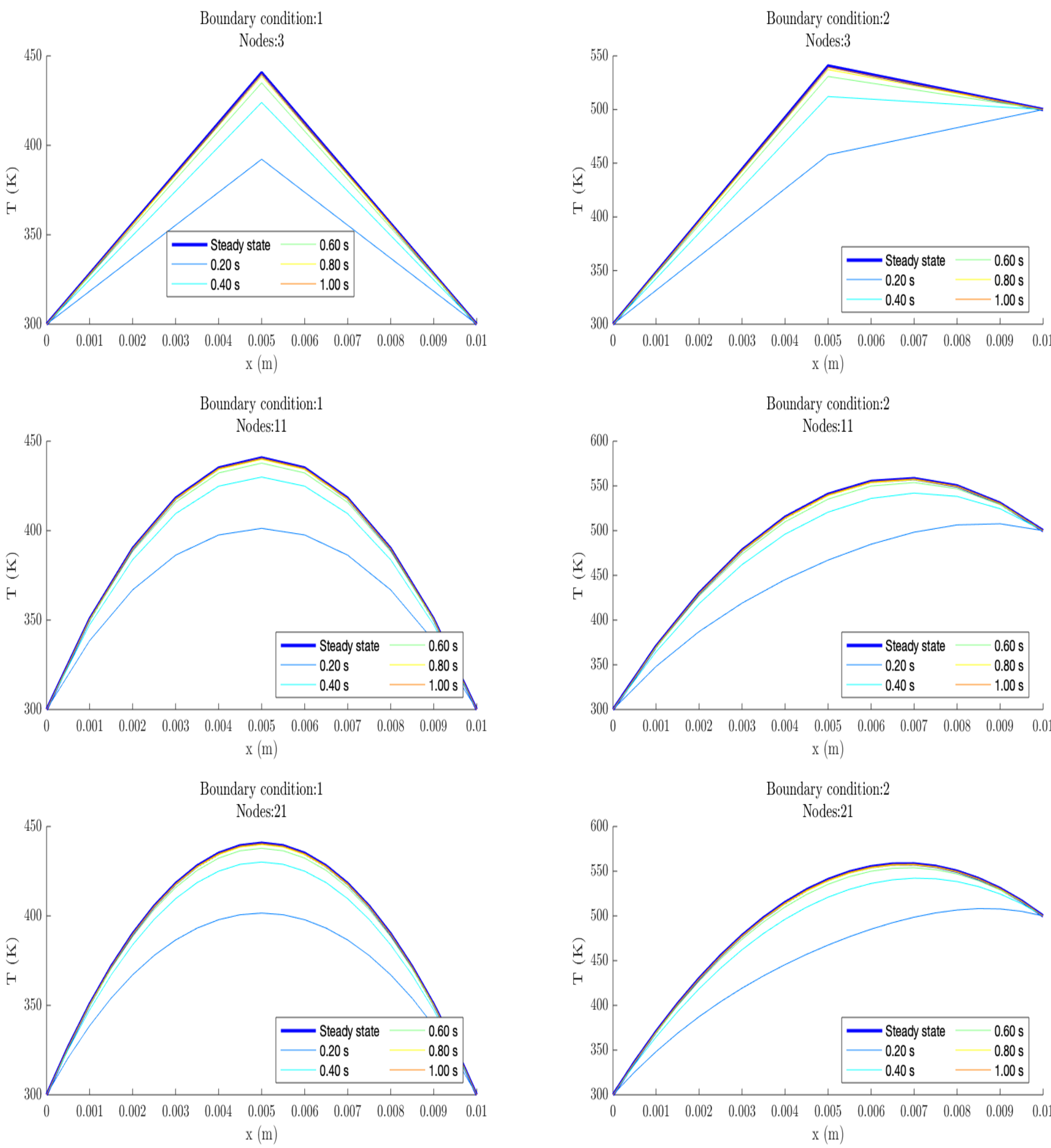

Figure 1: Temperature Profiles for Different Boundary Conditions and Number of Nodes

In the above plots, boundary condition 1 refers to the matched Dirichlet condition, while boundary condition 2 refers to the mismatched Dirichlet condition. Nodes refers to the number of nodes we discretised our spatial dimension into, including the two terminal nodes. The reason that both plots in the 3-nodal case look like piece-wise linear functions, while both plots in the 21-nodal case look quadratic is because MATLAB linearly interpolates between data points. With coarse mesh grid resolution (few nodes), the plots look linear, while finer mesh grid resolutions, the quadratic nature of the plots becomes readily apparent. In the 11-nodal case, the overall trend is quadratic for both plots, but we can still see the linear interpolation between consecutive nodes.

In all 6 plots, we rapidly converge to (but do not perfectly reach) the steady state solution. For all the plots, the orange plot corresponding temperature profile at $t = 1.00$ s is almost indistinguishable from the steady state analytical solution, denoted by the bolded dark blue line. Because of the aforementioned interpolation, the mesh grid resolution affects the final temperature profile in that a finer meshgrid resolution ie. greater number of nodes results in a higher or equivalent temperature profile, which can be visually seen in the case of the mismatched boundary conditions, where the maximum value is greatest in the 21-nodal case, while in the case of the matched boundary condition, the 21-nodal temperature profile is always equal to or greater than that corresponding to the 3-nodal case