Problem Set 3: Turing Machines

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1 Instructions for Submission

The **regular deadline** for this problem set is **Friday, November 22, at 11:59pm.** The **late deadline** for this problem set is **Sunday, December 1, at 11:59pm.** Please upload a PDF or image of your work to Canvas. Legible handwriting or typing is fine. Please submit **just your answers, and not the problem set document**.

2 How to solve/use this problem set

Some questions are graded on effort. These require a credible, reasonable attempt for credit. Any such attempt will receive full credit. Other questions are graded on correctness. **Questions that are graded on correctness are indicated as such in bold.**

3 Required Problems

1. (25 points) **(Graded on correctness)** Give an implementation-level description (not state diagrams) of Turing machine that recognizes the following language:

   $\{s\#t \mid s, t \in \{0,1\}^*$ and s is a substring of $t\}$.

   (Hint: try writing high-level pseudocode for an algorithm, then translate it into implementation-level language.)

The following machine M recognizes the language $\{s\#t \mid s,t \in \{0,1\}^*$ and s is a substring of $t\}$.
On input string w:

(a) Scan the tape from left to right to locate the first $\#$. If no $\#$ is found, reject. Mark the position of the $\#$.

(b) Identify s as the substring of w to the left of the $\#$ and t as the substring of w to the right of the $\#$. Place a marker at the beginning of s and t.

(c) For each character in t, compare s to the substring of t starting at the current position and repeat the following:

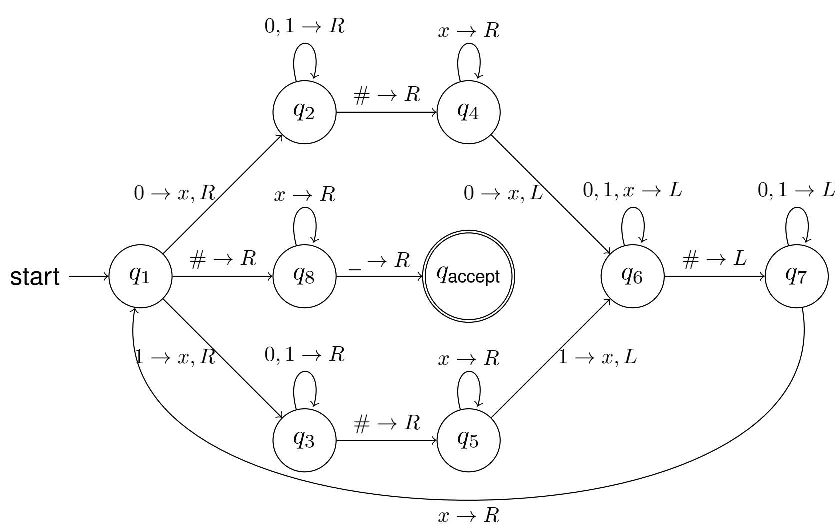
   i. Move s marker to the beginning of s.

   ii. Compare each character of s to the substring of t one character at a time. If characters match then move both markers to the right. If there is a mismatch, move the t marker to the right and repeat the comparison.

   iii. If all characters of s match the substring of t, accept.

(d) Move the t marker to the right and repeat comparison. If the end of t is reached and no match is found, reject.

2. (10 points) **(Graded on effort)** Consider the following TM state diagram from the first lecture on Turing machines (also from Sipser, Example 3.9, Figure 3.10, p. 173:



Draw the first 10 configurations that this machine goes through when reading the input $0010\#0010$. *(Hint: see bottom of p. 172 for examples of configurations from a different Turing machine).*

(a) $q_1 0010\#0010$

(b) $xq_2 010\#0010$

(c) $x0q_2 10\#0010$

(d) $x00q_2\#0010$

(e) $x00\#q_4 0010$

(f) $x00\#xq_6 010$

(g) $x00\#q_6 x010$

(h) $x00q_6\#x010$

(i) $x0q_6 0x010$

(j) $xq_6 00x010$

3. (15 points) **(Graded on correctness)** Give examples of languages that are...

   (a) Context free but not regular
   (b) Decidable but not context free
   (c) TM-recognizable but not decidable
   (d) Not TM-recognizable

(a) $L = \{a^n b^n \mid n \geq 0\}$ is context-free but not regular.
(b) $L = \{a^n b^n c^n \mid n \geq 0\}$ is decidable but not context-free.
(c) $EQ_{CFG} = \{\langle G, H\rangle \mid$ CFGs G, H generate the same language $\}$ is TM-recognizable but not decidable.
(d) $\overline{A_{TM}} = \{\langle M, w\rangle \mid M$ is a TM and M does not accept $w\}$ is not TM-recognizable.

4. (25 points) **(Graded on correctness)** (Sipser Exercise 4.3) Show that the following language is decidable:

   $ALL_{DFA} = \{\langle A\rangle \mid A$ is a DFA and $L(A) = \Sigma^*\}$

   (Hint: Use one of the languages that we have seen in class is decidable, along with a closure property of regular languages.)

We know that $E_{DFA} = \{\langle A \rangle \mid A$ is a DFA and $L(A) = \emptyset\}$ is decidable. We also know that decidable languages are closed under complement. Since $ALL_{DFA} = \overline{E_{DFA}}$, because $\forall w \in \Sigma^*$, we know that $w \notin E_{DFA}$. So since E_{DFA} is decidable and decidable languages are closed under complement, ALL_{DFA} is decidable.

5. (25 points) **(Graded on correctness)** (Sipser Exercises 5.1-5.2) Consider the languages

$$ALL_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$$

and

$$EQ_{CFG} = \{\langle G, H \rangle \mid \text{ CFGs } G, H \text{ generate the same language }\}$$

ALL_{CFG} is undecidable (the proof is in the Sipser textbook).

Prove that EQ_{CFG} is...

(a) Undecidable (use the fact that ALL_{CFG} is undecidable)

(b) Co-TM-recognizable

(c) Not TM-recognizable

(a) Suppose EQ_{CFG} is decidable. Then there exists a TM Q that decides EQ_{CFG}. Construct the following TM, D, that decides $ALL_{CFG} = \{\langle G \rangle \mid G$ is a CFG and $L(G) = \Sigma^*\}$. High-level description of D:
On input $\langle G \rangle$:

(a) Construct a CFG H such that $L(H) = \Sigma^*$.

(b) Run Q on input $\langle G, H \rangle$.

(c) If Q accepts, accept; else reject.

D accepts $\langle G \rangle$ if $L(G) = L(H) = \Sigma^*$, else D rejects $\langle G \rangle$. $\therefore$ D decides ALL_{CFG}. (contradiction)

(b) Construct the following TM M that recognizes $\overline{EQ_{CFG}} = \{\langle G, H\rangle \mid L(G) \neq L(H)\}$.
High-level description of M:
On input $\langle G, H\rangle$:

(a) Construct a CFG I such that $L(I) = \Sigma^*$.

(b) Run Q on input $\langle G, I\rangle$.

(c) If there exists a string $w \in \Sigma^*$ such that $w \in L(G)$ and $w \notin L(H)$ or $w \notin L(G)$ and $w \in L(H)$, accept; else reject.

$\therefore M$ recognizes $\overline{EQ_{CFG}}$.

(c) Suppose for a contradiction that EQ_{CFG} is TM-recognizable. Then there exists a TM M that recognizes EQ_{CFG}. If EQ_{CFG} is TM-recognizable, and $\overline{EQ_{CFG}}$ is co-TM-recognizable, then both EQ_{CFG} and $\overline{EQ_{CFG}}$ would be TM-recognizable. If both a language and its complement are TM-recognizable, then the language is decidable. However, we know that EQ_{CFG} is undecidable. (contradiction)