GA of WEEK 7-8

MATHS 2

STATS 2

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Week-7
Mathematics for Data Science - 2
Affine subspace, Equivalence and Similarity of the matrices,
Length of a vector, Inner products
Graded Assignment

1 Multiple Select Questions (MSQ)

1. Consider the vector spaces V and functions $\langle .,. \rangle : V \times V \to \mathbb{R}$ defined as follows:

   i) $V = \mathbb{R}^2$ and $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2$.

   ii) $V = M_{2\times2}(\mathbb{R})$ and $\langle A, B \rangle = Tr(AB)$,
   where $Tr(M)$ denotes the trace of a matrix M, i.e., the sum of the diagonal elements of M.

   iii) $V = M_{2\times1}(\mathbb{R})$ and $\langle A, B \rangle = Tr(AB^t)$,
   where $Tr(X)$ denotes the trace of a matrix X, i.e., the sum of the diagonal elements of X. Y^t denotes the transpose of matrix Y.

   iv) $V = \mathbb{R}^2$ and $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$.

   ○ **Option 1:** (i) is an inner product.
   ○ Option 2: (ii) is an inner product.
   ○ **Option 3:** (iii) is an inner product.
   ○ Option 4: (iv) is an inner product.

Soln. i) $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2$

* $\langle (x_1, x_2), (x_1, x_2) \rangle = x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2$
$= (x_1 - x_2)^2 + x_2^2 \geq 0$ $\quad \forall\, x_1, x_2 \in \mathbb{R}$

* If $\langle (x_1, x_2), (x_1, x_2) \rangle = (x_1 - x_2)^2 + x_2^2 = 0$
$\Rightarrow x_1 - x_2 = 0,\ x_2 = 0$
$\Rightarrow x_1 = 0.$

then $(x_1, x_2) = (0, 0)$.

$\langle (x_1, x_2), (x_1, x_2) \rangle = 0$ iff $(x_1, x_2) = (0,0)$.

✱ $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 2x_2 y_2$

$= y_1 x_1 - y_1 x_2 - y_2 x_1 + 2 y_2 x_2$

$= \langle (y_1, y_2), (x_1, x_2) \rangle$

✱ $\langle (x_1, x_2) + (z_1, z_2), (y_1, y_2) \rangle$

$= \langle (x_1 + z_1, x_2 + z_2), (y_1, y_2) \rangle$

$= (x_1 + z_1) y_1 - (x_2 + z_2) y_1 - (x_1 + z_1) y_2 + 2(x_2 + z_2) y_2$

$= x_1 y_1 + z_1 y_1 - x_2 y_1 - z_2 y_1 - x_1 y_2 - z_1 y_2 + 2x_2 y_2 + 2z_2 y_2$

$= (x_1 y_1 - x_2 y_1 - x_1 y_2 + 2x_2 y_2) + (z_1 y_1 - z_2 y_1 - z_1 y_2 + 2z_2 y_2)$

$= \langle (x_1, x_2), (y_1, y_2) \rangle + \langle (z_1, z_2), (y_1, y_2) \rangle$

✱ $\langle \alpha (x_1, x_2), (y_1, y_2) \rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2) \rangle$

$= \alpha x_1 y_1 - \alpha x_2 y_1 - \alpha x_1 y_2 + 2\alpha x_2 y_2$

$= \alpha \langle (x_1, x_2), (y_1, y_2) \rangle$

This is an inner product.

ii) $\langle A, B\rangle = Tr(AB)$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\langle A, A\rangle = Tr(A \cdot A) = Tr(A^2) = Tr\left(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\right)$

$= 0$

<u>But $A \neq 0$.</u>

Hence this not an inner product.

iii) $\langle A, B\rangle = Tr(AB^t)$

$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow B^t = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$

$AB^t = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$

$Tr(AB^t) = a_1 b_1 + a_2 b_2$

(*) $\langle A, A\rangle = Tr(AA^t) = a_1^2 + a_2^2 \geq 0 \quad \forall \; A \in M_{2\times 1}(\mathbb{R})$

$$\langle A, A\rangle = 0 \Rightarrow a_1^2 + a_2^2 = 0 \Rightarrow a_1 = 0 = a_2$$

$$\Rightarrow A = 0$$

$\langle A, A\rangle = 0$ iff $A = 0$.

$$\langle A, B\rangle = \mathrm{Tr}(AB^t) = a_1 b_1 + a_2 b_2$$

$$\langle B, A\rangle = \mathrm{Tr}(BA^t) = b_1 a_1 + b_2 a_2$$

$$\Rightarrow \langle A, B\rangle = \langle B, A\rangle$$

$$\langle A+C, B\rangle = \mathrm{Tr}((A+C)\cdot B^t)$$

$$= \mathrm{Tr}(AB^t + CB^t)$$

$$= \mathrm{Tr}(AB^t) + \mathrm{Tr}(CB^t)$$

$$= \langle A, B\rangle + \langle C, B\rangle$$

$$\langle \alpha A, B\rangle = \mathrm{Tr}(\alpha AB^t)$$

$$= \alpha\,\mathrm{Tr}(AB^t) = \alpha\langle A, B\rangle$$

This is an inner product.

iv) $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_2 + x_2 y_1$

$\langle (1, -1), (1, -1) \rangle = -1 - 1 = -2 < 0$.

It is not an inner product.

2. Consider two linear transformations T and S from $\mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x,y) = (2x+y, x+y)$ and $S(x,y) = (x+cy, x+2y)$. Let A and B be matrix representations of linear transformations T and S with respect to the standard bases of $\mathbb{R}^2$ respectively.
Consider the following statements:

- **P:** If $c = 1$, then A and B are similar matrices.
- **Q:** If $c = 2$, then A and B are similar matrices.
- **R:** If $c = 1$ and $P^{-1}AP = B$, then P can be the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.
- **S:** If $c = 1$ and $P^{-1}AP = B$, then P can be the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.
- **T:** If $c = 1$, then there are infinitely many P satisfying the equation $P^{-1}AP = B$.

Which of the following options are true?

- ○ **Option 1:** P is true but Q is false.
- ○ Option 2: Both P and Q are true.
- ○ Option 3: Both R and S are true.
- ○ **Option 4:** R is false but S is true.
- ○ **Option 5:** T is true.

Soln:

$T(x,y) = (2x+y, x+y)$ $\quad$ $S(x,y) = (x+cy, x+2y)$

$T(1,0) = (2,1)$ $\quad$ $S(1,0) = (1,1)$

$T(0,1) = (1,1)$ $\quad$ $S(0,1) = (c,2)$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 1 & c \\ 1 & 2 \end{bmatrix}$

If $c = 2$, then $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$\det(B) = 2 - 2 = 0$

$\det(A) = 2 - 1 = 1$

$\det(A) \neq \det(B)$

So, A and B are not similar matrices.

If $c=1$, then $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ Let $P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$

$$AP = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} 2p_1 + p_3 & 2p_2 + p_4 \\ p_1 + p_3 & p_2 + p_4 \end{bmatrix}$$

$$PB = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} p_1 + p_2 & p_1 + 2p_2 \\ p_3 + p_4 & p_3 + 2p_4 \end{bmatrix}$$

If $P^{-1}AP = B$ i.e. $AP = PB$, then,

$2p_1 + p_3 = p_1 + p_2$ $\qquad$ $2p_2 + p_4 = p_1 + 2p_2$

$p_1 + p_3 = p_3 + p_4$ $\qquad$ $p_2 + p_4 = p_3 + 2p_4$

i.e., $p_1 + p_3 = p_2$, $p_1 = p_4$

$P = \begin{bmatrix} p_1 & p_2 \\ p_2 - p_1 & p_1 \end{bmatrix}$ Any such matreices will satisfy the condition, $P^{-1}AP = B$.

$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ is not of this form.

i.e., if $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ then, $P^{-1}AP \neq B$.

But $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is of this form

and in this case we have,

$$P^{-1}AP = B$$

where, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

As we have already seen

$P = \begin{bmatrix} p_1 & p_2 \\ p_2 - p_1 & p_1 \end{bmatrix}$ for any real numbers p_1 and p_2.

Hence there are infinitely many such matrices.

3. Let $\langle ., . \rangle$ denote the standard inner product on $\mathbb{R}^2$, i.e., $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2$. Which one of the following options is (are) true for the vector $\gamma \in \mathbb{R}^2$, such that $\langle \alpha, \gamma \rangle = 4$ and $\langle \beta, \gamma \rangle = 8$, where $\alpha = (3, 1)$ and $\beta = (6, 2)$.

- ○ Option 1: No such γ exists.
- ○ **Option 2:** There are infinitely many such vectors which satisfy the properties of γ.
- ○ Option 3: γ is unique in $\mathbb{R}^2$.
- ○ **Option 4:** Any vector in the set $\{(t, 4 - 3t) \mid t \in \mathbb{R}\}$ satisfies the properties of γ.
- ○ Option 5: $(1, 1)$ is the only vector which satisfies the properties of γ.

Soln:

Let $\gamma = (a, b)$

$\langle \alpha, \gamma \rangle = 4 \Rightarrow 3a + b = 4$

$\langle \beta, \gamma \rangle = 8 \Rightarrow 6a + 2b = 8$

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 6 & 2 & 8 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Solution set for $\gamma = \{(t, 4 - 3t) \mid t \in \mathbb{R}\}$

Hence, option 2 and 4 are correct.

4. Let $\langle .,. \rangle$ denote the standard inner product on $\mathbb{R}^2$, i.e., $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2$ and let $v \in \mathbb{R}^2$. Consider a linear transformation $T_v : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$T_v(u) = \langle u, v \rangle,$$

where $v \in \mathbb{R}^2$. Which of the following options is (are) true for T_v?

- ◯ Option 1: T_v is one-one for all $v \neq 0 \in \mathbb{R}^2$.
- ◯ **Option 2:** T_v is onto for all $v \neq 0 \in \mathbb{R}^2$.
- ◯ Option 3: T_v is onto for all $v \in \mathbb{R}^2$.
- ◯ **Option 4:** T_v is not one-one for every $v \in \mathbb{R}^2$.
- ◯ **Option 5:** There exists a $v \in \mathbb{R}^2$ such that T_v is not onto.

Soln:

$T_v : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$T_v(u) = \langle u, v \rangle$

If $v = 0$, then for any nonzero $u \in \mathbb{R}$,

$T_v(u) = \langle u, 0 \rangle = 0$

In this case, T_v is not one to one.

and also not onto.

If $v \neq 0$, then $T_v(v) = \langle v, v \rangle > 0$

Hence, T_v is nonzero linear transformation.

$\dim(\text{codomain}(T_v)) = \dim(\mathbb{R}) = 1$

$\text{rank}(T_v) = 1$. as, $T_v \neq 0$.

Hence, T_v is onto.

$$\text{Nullity}(T_\theta) + \text{rank}(T_\theta) = \dim(\mathbb{R}^2) = 2$$

$$\Rightarrow \text{Nullity}(T_\theta) + 1 = 2$$

$$\Rightarrow \text{Nullity}(T_\theta) = 1.$$

Hence, T_θ is not one to one.

5. Let L and L' be affine subspaces of $\mathbb{R}^3$, where $L = (0,1,1)+U$ and $L' = (0,1,0)+U'$, for some vector subspaces U and U' of $\mathbb{R}^3$. Let a basis for U be given by $\{(1,1,0),(1,0,1)\}$ and a basis for U' be given by $\{(1,0,0)\}$. Suppose there is a linear transformation $T : U \to U'$ such that $(1,0,1) \in \ker(T)$ and $T(1,1,0) = (1,0,0)$. An affine mapping $f : L \to L'$ is obtained by defining $f((0,1,1)+u) = (0,1,0)+T(u)$, for all $u \in U$. Which of the following options are true?

- ○ **Option 1:** $L = \{(x, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ○ Option 2: $L' = \{(x, y+1, 0) \mid x, y \in \mathbb{R}\}$.
- ○ Option 3: $L = \{(x-y, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ○ Option 4: $L = \{(x, x+1, y+1) \mid x, y \in \mathbb{R}\}$.
- ○ **Option 5:** $f(x, y+1, x-y+1) = (y, 1, 0)$
- ○ Option 6: $f(x-y, y+1, x-y+1) = (x, y+1, 0)$
- ○ Option 7: $f(x, x+1, y+1) = (y, 1, 0)$
- ○ Option 8: $f(x, y+1, x-y+1) = (0, 1, y)$

<u>Soln.</u>

$$U = \{a(1,1,0) + b(1,0,1) \mid a, b \in \mathbb{R}\}$$
$$= \{(a,a,0) + (b,0,b) \mid a, b \in \mathbb{R}\}$$
$$= \{(a+b, a, b) \mid a, b \in \mathbb{R}\}$$

If $x = a+b$, $y = a$ then, $b = x - y$

$$U = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$$

$$U' = \{x(1,0,0) \mid x \in \mathbb{R}\} = \{(x,0,0) \mid x \in \mathbb{R}\}$$

$$L = (0,1,1) + U = \{(x, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$$

$$L' = (0,1,0) + U' = \{(x, 1, 0) \mid x \in \mathbb{R}\}$$

$T : U \longrightarrow U'$

$T(1,0,1) = (0,0,0)$

$T(1,1,0) = (1,0,0)$

$(x, y, x-y) = a(1,0,1) + b(1,1,0)$

$= (a+b, b, a)$

$a+b = x \qquad b = y, \qquad a = x-y$

$(x, y, x-y) = (x-y)(1,0,1) + y(1,1,0)$

$T(x, y, x-y) = (x-y)T(1,0,1) + yT(1,1,0)$

$= (x-y)(0,0,0) + y(1,0,0)$

$= (y,0,0)$

$T : U \longrightarrow U'$

$T(x, y, x-y) = (y,0,0)$

$f : L \rightarrow L'$

$f((0,1,1) + (x, y, x-y)) = (0,1,0) + (y,0,0)$

$\Rightarrow f(x, y+1, x-y+1) = (y,1,0)$

2 Numerical Answer Type (NAT)

6. Let θ be the angle between the vectors $u = (4, 7, 3)$ and $v = (1, 2, -6)$, then what will be the value of $cos(\theta)$? [Answer: 0]

$$u \cdot v = \|u\| \|v\| \cos\theta$$

$$\Rightarrow \quad 4 + 14 - 18 = \|u\| \|v\| \cos\theta$$

$$\Rightarrow \quad \underline{\cos\theta = 0}.$$

7. Consider a basis

$$\{v_1 = (1, 2, 0), v_2 = (2, -1, 0), v_3 = (0, 0, 2)\}$$

of $\mathbb{R}^3$ with usual inner product. Suppose $v = (x, y, \frac{3x+y}{5}) \in V$ is written as $v = c_1 v_1 + c_2 v_2 + c_3 v_3$, such that $c_1 + c_2 = 4$. What will be the value of c_3? [Answer: 2]

$\langle v_1, v_2 \rangle = 0$, $\langle v_2, v_3 \rangle = 0$, $\langle v_1, v_3 \rangle = 0$

Hence, $\{v_1, v_2, v_3\}$ is an orthogonal basis.

$$\langle v, v_1 \rangle = \langle c_1 v_1 + c_2 v_2 + c_3 v_3 , v_1 \rangle$$
$$= c_1 \langle v_1, v_1 \rangle + c_2 \langle v_2, v_1 \rangle + c_3 \langle v_3, v_1 \rangle$$
$$= c_1 \langle v_1, v_1 \rangle$$

$\Rightarrow \quad x + 2y = 5c_1$

$\langle v, v_2 \rangle = c_2 \langle v_2, v_2 \rangle$

$\Rightarrow \quad 2x - y = 5c_2$

$x + 2y + 2x - y = 5(c_1 + c_2)$

$\Rightarrow \quad 3x + y = 5(4) = 20$

$\langle v, v_3 \rangle = c_3 \langle v_3, v_3 \rangle$

$\Rightarrow \quad \frac{2(3x+y)}{5} = 4c_3$

$\Rightarrow \quad \frac{1}{10}(3x+y) = c_3$

$\Rightarrow \quad c_3 = 2$

8. Let $V = \mathbb{R}^2$ be a vector space. Consider two inner products $\langle .,. \rangle_1$ and $\langle .,. \rangle_2$ on V defined as

$$\langle (x_1, y_1), (x_2, y_2) \rangle_1 = x_1x_2 - x_1y_2 - x_2y_1 + 4y_1y_2$$

and

$$\langle (x_1, y_1), (x_2, y_2) \rangle_2 = 3x_1x_2 + 2y_1y_2.$$

If $\langle (a, b), (8, 9) \rangle_1 = 215$ and $\langle (a, b), (8, 9) \rangle_2 = 360$, then find the value $a + 2b$. [Ans: 25]

$$8a - 9a - 8b + 36b = 215$$

$$\Rightarrow \quad -a + 28b = 215$$

$$24a + 18b = 360$$

$$\Rightarrow \quad 4a + 3b = 60$$

$$-4a + 112b = 860$$

$$4a + 3b = 60$$

$$115b = 920$$

$$b = \frac{920}{115} = 8$$

$$4a + 24 = 60$$

$$\Rightarrow 4a = 36$$

$$\Rightarrow a = 9$$

$$a + 2b = 9 + 16 = \underline{25}$$

9. Let $V = \mathbb{R}^3$ be the inner product space with usual inner product. If θ is the angle between $(14, 11, 5)$ and (a, b, c) where $14a + 11b + 5c = 0$ and $(a^2 + b^2 + c^2) \neq 0$, then find the value of $\cos\theta$. [Ans 0]

$$(14, 11, 5) \cdot (a, b, c) = \|(14, 11, 5)\| \, \|(a, b, c)\| \cos\theta$$

$$\Rightarrow \quad 14a + 11b + 5c = \|(14, 11, 5)\| \, (a^2 + b^2 + c^2)^{1/2} \cos\theta$$

$$\Rightarrow \quad \underline{\cos\theta = 0} \; .$$

10. Consider a vector $v = (x - 11, 2, 1) \in \mathbb{R}^3$. Find the value of x so that the length of the vector v is minimum. [Ans: 11]

$$\|v\|^2 = (x-11)^2 + 4 + 1 = (x-11)^2 + 5 = f(x)$$

We have to find the minima of $f(x)$.

$$f'(x) = 2(x-11) = 0$$

$$\Rightarrow \underline{x = 11}$$

$$\underline{f''(x) = 2 > 0}.$$

Hence, $x = 11$ is minima of $f(x)$.

Week-8
Mathematics for Data Science - 2
Projection, Gram-Schmidt process, Orthogonal transformation
Graded Assignment-Solutions

1 Multiple Select Questions (MSQ)

1. An inner product on $\mathbb{R}^3$ is defined as:

$$\langle .,. \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$
$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

Match the sets of vectors in column A with their properties of orthogonality or orthonormality in column B with respect to the above inner product.

	Set of vectors (Column A)		Properties (Column B)
a)	$\{(2,3,4), (-1,2,-1)\}$	i)	Forms a basis but not orthogonal
b)	$\{\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(-1,0,-1)\}$	ii)	Forms an orthogonal basis
c)	$\{(2,3,4), (-1,2,-1), (0,4,-3)\}$	iii)	Orthogonal but not orthonormal, and does not form a basis of $\mathbb{R}^3$
d)	$\{(2,3,4), (-1,2,-1), (11,2,-7)\}$	iv)	Orthonormal, but does not form a basis of $\mathbb{R}^3$

Table : M2W6G1

Choose the correct options.

- ◯ Option 1: a $\to$ iv)
- ◯ **Option 2:** a $\to$ iii)
- ◯ **Option 3:** b $\to$ iv)
- ◯ Option 4: b $\to$ iii)
- ◯ Option 5: c $\to$ ii)
- ◯ **Option 6:** c $\to$ i)
- ◯ Option 7: d $\to$ i)
- ◯ **Option 8:** d $\to$ ii)

Solution: Note that if $x, y \in V$ are non-zero and orthogonal, then $\{x, y\}$ is a linearly independent set. Also, since dimension of $\mathbb{R}^3$ is 3, any set of 3 linearly independent vectors forms a basis.

a) $\langle(2,3,4),(-1,2,-1)\rangle = -2+6-4 = 0$ and $\langle(2,3,4),(2,3,4)\rangle \neq 1$. so the set is orthogonal but not orthonormal. Also, since there are only 2 vectors, it cannot form a basis for $\mathbb{R}^3$. This matches with iii).

b) $\langle\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = 0$. Also $\langle\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(1,0,-1)\rangle = 1$ and $\langle\frac{1}{\sqrt{2}}(-1,0,1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = 1$. Thus these vectors are orthonormal. But this does not form a basis for $\mathbb{R}^3$, since there are only two vectors. This matches with iv).

c) $\langle(-1,2,-1),(0,4,-3)\rangle = 11 \neq 0$, so the set is not orthogonal. If we show that the set is linearly independent, then it forms a basis for $\mathbb{R}^3$ (this is because a linearly independent set with number of elements equal to $\dim(V)$ is a basis for V). To show that the set is linearly independent. show that the system $\alpha(2,3,4) + \beta(-1,2,-1) + \gamma(0,4,-3) = 0$ has only one solution $\alpha = \beta = \gamma = 0$. This matches with i).

d) $\langle(2,3,4),(-1,2,-1)\rangle = \langle(2,3,4),(11,2,-7)\rangle = \langle(-1,2,-1),(11,2,-7)\rangle = 0$. So the set is orthogonal and hence is linearly independent. Thus it forms a basis (since the linearly independent set has 3(=$\dim(\mathbb{R}^3)$) elements). This matches with ii).

2. Choose the set of correct options.

   ◯ **Option 1:** Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis of an inner product space V. If there exists some $v \in V$, such that $\langle v, v_i \rangle = 0$ for all $i = 1, 2, \ldots, n$, then $v = 0$.

   ◯ **Option 2:** There exists an orthonormal basis for $\mathbb{R}^n$ with the standard inner product.

   ◯ **Option 3:** If P_W denotes the linear transformation which projects the vectors of an inner product space V to a subspace W of V, then range$(P_W) \cap$ null space$(P_W) = \{0\}$, where 0 denotes the zero vector of V.

   ◯ **Option 4:** $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ cannot represent a matrix corresponding to some projection.

Solution: A linear transformation P is a projection if $P^2 = P$.

- **Option 1:** Since β is a basis, $v \in V$ can be written as $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$. Now $\langle v, v_i \rangle = \alpha_i \langle v_i, v_i \rangle$ for all $i = 1, 2, ..., n$. Since it is given that v is such that $\langle v, v_i \rangle = 0$ for all i, we have $\alpha_i \|v_i\|^2 = 0$ for all i. But $\|v_i\| \neq 0$ (since $v_i \neq 0$). Thus $\alpha_i = 0$ for all i and hence $v = 0$.
- **Option 2:** $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis for V.
- **Option 3:** Since P_W is a projection, $P_W^2 = P_W$. Let $x \in$ range$(P_W) \cap$ null space(P_W). Since $x \in$ range(P_W), $x = P_W y$ for some $y \in V$. But $x \in$ null space(P_W) implies that $P_W x = 0$, i.e., $P_W^2 y = 0$. But $P_W^2 = P_W$ and hence $x = P_W y = 0$. Thus range$(P_W) \cap$ null space$(P_W) = \{0\}$.
- **Option 4:** Note that $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So the matrix cannot represent a projection.

2 Numerical Answer Type (NAT)

3. If A is an orthogonal matrix of order 5, then find nullity of the matrix A. [Ans: 0]

   Solution: A is orthogonal of order 5 $\implies$ A is invertible. Therefore nullity of A is 0.

4. Let $v \in \mathbb{R}^3$ be a vector such that $||v|| = 5$. If u is the vector obtained from v after the anti- clock wise rotation of XY-plane with angle $70°$ about the Z- axis, then find the length of the vector u. [Ans: 5]

   Solution: Rotation matrix is an orthogonal matrix. It preserves the length of a vector, that is, $||Av|| = ||v||$ for all v. Therefore $||v|| = ||u|| = 5$.

5. Let $v = (1, 2, 2)$ be a vector in $\mathbb{R}^3$. If (a, b, c) is the vector obtained from v after the anti-clock wise rotation of YZ-plane with angle $60°$ about the X- axis, then find the value of $a + b + c$. [Ans: 3]

Solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60) & \sin(60) \\ 0 & \sin(60) & \cos(60) \end{bmatrix}$ represents the rotation matrix of XY-plane with angle $60°$ about the X-axis.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60) & -\sin(60) \\ 0 & \sin(60) & \cos(60) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - \sqrt{3} \\ 1 + \sqrt{3} \end{bmatrix}.$$

Hence $a + b + c = 3$.

6. Consider a vector space $M_{2\times2}(\mathbb{R})$ and a norm on the vector space defined as $\|A\| = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\}$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times2}(\mathbb{R})$.

Let $B = \begin{bmatrix} x & \sqrt{2}\, x \\ -\sqrt{2}\, y & y \end{bmatrix}$ be an orthogonal matrix i.e. $BB^T = B^T B = I$ and assume $x, y > 0$.

Then find the norm of the matrix $C = \begin{bmatrix} \sqrt{3}\, x & \sqrt{3}\, x \\ \sqrt{3}\, y & \sqrt{3}\, y \end{bmatrix}$ (i.e., $||C||$) [Ans: 2]

Solution: Note that $BB^T = \begin{bmatrix} x & \sqrt{2}\, x \\ -\sqrt{2}\, y & y \end{bmatrix} \begin{bmatrix} x & -\sqrt{2}\, y \\ \sqrt{2}\, x & y \end{bmatrix} = \begin{bmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{bmatrix}$. But since B is an orthogonal matrix, $BB^T = I$. So $3x^2 = 3y^2 = 1$. Thus $\sqrt{3}x = \sqrt{3}y = \pm 1$. This gives $\|C\| = \max\{2, 2\} = 2$.

7. Let $V = \mathbb{R}^2$ be the inner product space with usual inner product and a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (\frac{a}{\sqrt{a^2+4}}x + \frac{1}{\sqrt{b^2+1}}y, \frac{2}{\sqrt{a^2+4}}x + \frac{b}{\sqrt{b^2+1}}y)$. If T is an orthogonal linear transformation, then find the value of $a + 2b$. [Ans: 0]

Solution: If T is an orthogonal transformation, then $\langle Tu, Tv\rangle = \langle u, v\rangle$, for all $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Thus we have

$$\begin{aligned} \langle Tu, Tv\rangle &= \langle (u_1, u_2), (v_1, v_2)\rangle \\ (\frac{a}{\sqrt{a^2+4}}u_1 + \frac{1}{\sqrt{b^2+1}}u_2)(\frac{a}{\sqrt{a^2+4}}v_1 + \frac{1}{\sqrt{b^2+1}}v_2) & \\ + (\frac{2}{\sqrt{a^2+4}}u_1 + \frac{b}{\sqrt{b^2+1}}u_2)(\frac{2}{\sqrt{a^2+4}}v_1 + \frac{b}{\sqrt{b^2+1}}v_2) &= u_1v_1 + u_2v_2 \end{aligned}$$

Thus we get $\dfrac{a+2b}{\sqrt{(a^2+4)}\sqrt{(b^2+1)}}(u_1v_2+u_2v_1)=0$. Since $\langle Tu, Tv\rangle = \langle u, v\rangle$ is true for all u, v, we can choose $u = v = (1,1)$. For this choice of u, v, we have $\dfrac{a+2b}{\sqrt{(a^2+4)}\sqrt{(b^2+1)}} = 0$. Thus $a+2b=0$.

3 Comprehension Type Question:

With a particular frame of reference (in $\mathbb{R}^3$), position of a target is given as the vector $(3,4,5)$. Three shooters S_1, S_2, and S_3 are moving along the lines $x=y$, $x=-y$, and $x=2y$, on the XY-plane (i.e., $z=0$) to shoot the target. Suppose that, there is another shooter S_4, who is moving on the plane $x+y+z=0$. Suppose all of them shoot the target so that the target is at the closest distance from the respective path or plane on which they are travelling.
Answer questions 8,9 and 10 using the given information.

8. Choose the set of correct options.
   - ◯ Option 1: S_1 will shoot the target from the point $(\frac{7}{2}, -\frac{7}{2}, 0)$.
   - ◯ **Option 2:** S_1 will shoot the target from the point $(\frac{7}{2}, \frac{7}{2}, 0)$.
   - ◯ Option 3: S_1 will shoot the target from the point $(1,1,0)$.
   - ◯ Option 4: S_2 will shoot the target from the point $(-\frac{1}{2}, -\frac{1}{2}, 0)$.
   - ◯ Option 5: S_2 will shoot the target from the point $(1,-1,0)$.
   - ◯ **Option 6:** S_2 will shoot the target from the point $(-\frac{1}{2}, \frac{1}{2}, 0)$.
   - ◯ **Option 7:** S_3 will shoot the target from the point $(4,2,0)$.
   - ◯ Option 8: S_3 will shoot the target from the point $(2,1,0)$.
   - ◯ Option 9: S_3 will shoot the target from the point $(0,0,0)$.

   Solution: Note that the closest point to shoot the target can be obtained by finding the orthogonal projection of (3,4,5) onto the subspace on which S_i's are moving.

   Equation of line of motion of S_1 is $x=y$ $(x-y=0)$. The subspace $W_1 = \{(x,y,z) | x = y, z = 0\}$ is spanned by the vector $(1,1,0)$. The projection of $(3,4,5)$ onto W_1 is $\frac{1}{\langle(1,1,0),(1,1,0)\rangle}\langle(3,4,5),(1,1,0)\rangle(1,1,0) = (\frac{7}{2}, \frac{7}{2}, 0)$.

   Equation of line of motion of S_2 is $x=-y$ $(x+y=0)$. The subspace $W_2 = \{(x,y,z) | x = -y, z = 0\}$ is spanned by the vector $(1,-1,0)$. The projection of $(3,4,5)$ onto W_2 is $\frac{1}{\langle(1,-1,0),(1,-1,0)\rangle}\langle(3,4,5),(1,-1,0)\rangle(1,-1,0) = (-\frac{1}{2}, \frac{1}{2}, 0)$.

Equation of line of motion of S_3 is $x = 2y$ $(x-2y=0)$. The subspace $W_3 = \{(x,y,z)|x = 2y, z = 0\}$ is spanned by the vector $(2,1,0)$. The projection of $(3,4,5)$ onto W_3 is $\frac{1}{\langle(2,1,0),(2,1,0)\rangle}\langle(3,4,5),(2,1,0)\rangle(2,1,0) = (4,2,0)$.

9. If (a,b,c) is the point from which the shooter S_4 will shoot the target, then find the value of $a+2b+3c$. [Ans: 2]

   Solution: Equation of line of motion of S_4 is $x+y+z=0$. The subspace $W_4 = \{(x,y,z)|x+y+z=0\}$ is spanned by the vectors $(1,0,-1)$ and $(0,1,-1)$. Using Gram Schmidt orthogonalization, we can get an orthonormal (orthogonal is sufficient) basis for W_4. $\{(1,0,-1),(-\frac{1}{2},1,-\frac{1}{2})\}$ is an orthogonal basis for W_4. The projection of $(3,4,5)$ onto W_4 is
   $\frac{1}{\langle(1,0,-1),(1,0,-1)\rangle}\langle(3,4,5),(1,0,-1)\rangle(1,0,-1)+\frac{1}{\langle(-\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},1,-\frac{1}{2})\rangle}\langle(3,4,5),(-\frac{1}{2},1,-\frac{1}{2})\rangle(-\frac{1}{2},1,-\frac{1}{2})$
   $= (-1,0,1)+(0,0,0) = (-1,0,1)$.

10. Let d_i be the distance of the target from the point where the shooter S_i shoots the target, for i = 1,2,3,4 and let d be the minimum amongst the d_i. Find the value of d^2. [Ans: 25.5]

    Solution: Distance between the points can be calculated using $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$. Thus we have $d_1 = \sqrt{\frac{51}{2}}$, $d_2 = \sqrt{\frac{99}{2}}$, $d_3 = \sqrt{30}$ and $d_4 = 4\sqrt{3}$.

Statistics for Data Science - 2

Week 7 Graded assignment

1. Let X_1, X_2, X_3 are three independent and identically distributed random variables with mean μ and variance σ^2. Given below are 3 different formulations of sample mean. (Observe that $E[A] = E[B] = E[C]$).

$$\begin{aligned} A =& \frac{X_1 + X_2 + X_3}{3} \\ B =& 0.1X_1 + 0.3X_2 + 0.6X_3 \\ C =& 0.2X_1 + 0.3X_2 + 0.5X_3 \end{aligned}$$

Choose the correct option from the following:

(a) $\text{Var}(A) = \text{Var}(B) = \text{Var}(C)$

(b) $\text{Var}(A) \geq \text{Var}(B) \geq \text{Var}(C)$

(c) $\text{Var}(A) \leq \text{Var}(B) \leq \text{Var}(C)$

(d) $\text{Var}(A) \leq \text{Var}(C) \leq \text{Var}(B)$

Solution:

Let $X_1, X_2, X_3 \sim$ i.i.d.X, where $E[X] = \mu, \text{Var}(X) = \sigma^2$

$$\begin{aligned} \text{Var}(A) =& \text{Var}\left(\frac{X_1 + X_2 + X_3}{3}\right) \\ =& \frac{1}{9}\left(\text{Var}[X_1] + \text{Var}[X_2] + \text{Var}[X_3]\right) \\ =& \frac{1}{9}(3\sigma^2) = \frac{\sigma^2}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}(B) =& \text{Var}\left(0.1X_1 + 0.3X_2 + 0.6X_3\right) \\ =& 0.01\text{Var}[X_1] + 0.09\text{Var}[X_2] + 0.36\text{Var}[X_3] \\ =& 0.46\sigma^2 \end{aligned}$$

$$\begin{aligned} \text{Var}(C) =& \text{Var}\left(0.2X_1 + 0.3X_2 + 0.5X_3\right) \\ =& 0.04\text{Var}[X_1] + 0.09\text{Var}[X_2] + 0.25\text{Var}[X_3] \\ =& 0.38\sigma^2 \end{aligned}$$

Therefore, $\text{Var}(B) \geq \text{Var}(C) \geq \text{Var}(A)$.

2. A random sample of size 25 is collected from a normal population with mean of 50 and standard deviation of 5. Find the variance of the sample mean.

Solution:
We know that variance of the sample mean $\overline{X}$ is given by

$$\begin{aligned}\text{Var}[\overline{X}] &= \frac{\sigma^2}{n} \\ &= \frac{5^2}{25} = 1\end{aligned}$$

---

3. A fair die is rolled 100 times. Let X denote the number of times six is obtained. Find a bound for the probability that $\frac{X}{100}$ differs from $\frac{1}{6}$ by less than 0.1 using weak law of large numbers.

   (a) at least $\frac{5}{36}$
   (b) at least $\frac{31}{36}$
   (c) at most $\frac{5}{36}$
   (d) at most $\frac{31}{36}$

Solution:
X denotes the number of times six is obtained on rolling a fair die 100 times.
Let $X_1, X_2, \ldots, X_{100}$ be 100 i.i.d. samples such that

$$X_i = \begin{cases} 1 & \text{if six appears on rolling a fair die} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i] = \mu = \frac{1}{6}$ and
$\text{Var}(X_i) = \sigma^2 = \frac{5}{36}$
Notice that $X = X_1 + X_2 + X_3 + \ldots + X_{100}$

To find: Bound on $P\left(\left|\frac{X}{100} - \frac{1}{6}\right| < 0.1\right)$.

By weak law of large numbers, we have

$$
\begin{aligned}
P(|\overline{X} - \mu| < \delta) &\geq 1 - \frac{\sigma^2}{n\delta^2} \\
\Rightarrow P\left(\left|\frac{X}{100} - \frac{1}{6}\right| < 0.1\right) &\geq 1 - \frac{5}{36 \times 100 \times 0.01} \\
\Rightarrow P\left(\left|\frac{X}{100} - \frac{1}{6}\right| < 0.1\right) &\geq 1 - \frac{5}{36} = \frac{31}{36}
\end{aligned}
$$

---

4. Let $X_1, X_2, \ldots, X_5$ be i.i.d. samples whose distribution has mean 20 and variance 4. Suppose the sample variance is defined as

$$S^2 = \frac{(X_1 - \overline{X})^2 + \cdots + (X_5 - \overline{X})^2}{5}$$

, where $\overline{X} = \dfrac{X_1 + X_2 + \cdots + X_5}{5}$. Find the expected value of S^2.

Solution:

$E[\bar{X}] = \mu = 20$ and $\text{Var}[\bar{X}] = \dfrac{\sigma^2}{n} = \dfrac{4}{5} = 0.8$.

$$
\begin{aligned}
E[S^2] =& E\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}\right] \\
=& \frac{1}{n}E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \\
=& \frac{1}{n}E\left[\sum_{i=1}^{n}(X_i^2 + \bar{X}^2 - 2X_i\bar{X})\right] \\
=& \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 + n\bar{X}^2 - 2n\bar{X}\sum_{i=1}^{n}X_i\right] \\
=& \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 + n\bar{X}^2 - 2n\bar{X}^2\right] \\
=& \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right] \\
=& \frac{1}{n}\left[\sum_{i=1}^{n}E[X_i^2] - nE[\bar{X}^2]\right] \\
=& \frac{1}{n}\left[\sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] \\
=& \frac{1}{n}\left[(n\sigma^2 + n\mu^2) - (\sigma^2 + n\mu^2)\right] \\
=& \frac{(n-1)\sigma^2}{n}
\end{aligned}
$$

Here, $n = 5$, therefore, $E[S^2] = \frac{4}{5} \times 4 = 3.2$

5. Suppose $X_i \sim$ Normal $\left(0, \frac{1}{i^2}\right)$, where $i = 1, 2, \ldots, 9$ and $X_1, X_2, \ldots, X_9$ are independent to each other. Let Y be a random variable defined as $Y = \sum_{i=1}^{9} iX_i$. Find the variance of Y.

   Answer: 9

Solution:

$$\begin{aligned}
\text{Var}(Y) =& \text{Var}\left(\sum_{i=1}^{9} iX_i\right) \\
=& \text{Var}(X_1 + 2X_2 + 3X_3 + \ldots + 9X_9) \\
=& \text{Var}(X_1) + \text{Var}(2X_2) + \ldots + \text{Var}(9X_9) \\
=& \text{Var}(X_1) + 4\text{Var}(X_2) + \ldots + 81\text{Var}(X_9) \\
=& \frac{1}{1^2} + 4\left(\frac{1}{2^2}\right) + \ldots + 81\left(\frac{1}{9^2}\right) \\
=& 9
\end{aligned}$$

6. A random sample of size 50 is collected from a population P, where $P \sim$ Uniform[0,12]. Find a lower bound on the probability that the sample mean will be at most 3 units away from the actual mean using the weak law of large numbers.
   Answer: 0.733

   Solution:

   $P \sim \text{Uniform}[0, 12]$
   $E[P] = \mu = \dfrac{0+12}{2} = 6, \text{Var}(P) = \sigma^2 = \dfrac{(12-0)^2}{12} = \dfrac{144}{12} = 12$

   By weak law of large numbers, we have

$$\begin{aligned}
&P(|\overline{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2} \\
\Rightarrow &P\left(|\overline{X} - \mu| < 3\right) \geq 1 - \frac{12}{50 \times 9} = \frac{73}{75} = 0.9733
\end{aligned}$$

7. Suppose a random sample is used to estimate the proportion of voters in a city. If the sample proportion is roughly 0.45, what sample size is necessary so that the standard deviation of the sample proportion is 0.02?
   Answer: 619

   Solution:

   Let the random variable X represents that the selected candidate is a voter.

   Let X_i be defined as

$$X_i = \begin{cases} 1, & \text{if the selected candidate is a voter} \\ 0, & \text{otherwise} \end{cases}$$

Define an event A as $A : X = 1$.

It is given that $P(A) = 0.45$.

We know that $\text{Var}(S(A)) = \dfrac{P(A)(1 - P(A))}{n}$

$$\sqrt{\frac{p(1-p)}{n}} = 0.02 \implies \sqrt{\frac{(0.45)(0.55)}{n}} = 0.02 \implies n = 618.75 \approx 619$$

8. The average life(in years) of an electronic watch follows an exponential distribution with parameter $\frac{1}{2}$. Find the lower bound on the probability that the mean life of a random sample of 50 such watches falls between 1 and 3 years. Enter your answer correct to two decimals.
Hint: Use weak law of large numbers.
Answer: 0.92

Solution:

Let the random variable X represents the life of an electronic watch.
It is given that $X \sim \text{Exp}(1/2)$ and 50 such samples are taken.
$E[X] = \mu = 2, \text{Var}(X) = \sigma^2 = 4$

To find: a lower bound on $P(1 < \overline{X} < 3)$.

By weak law of large numbers, we have

$$P(|\overline{X} - \mu| < \delta) \geq 1 - \frac{\sigma^2}{n\delta^2}$$
$$\Rightarrow P\left(|\overline{X} - 2| < 1\right) \geq 1 - \frac{4}{50 \times 1} = \frac{23}{25} = 0.92$$

Statistics for Data Science - 2

Week 8 Graded assignment

1. Let $X_1, X_2, \ldots, X_{50} \sim$ i.i.d. Poisson(0.04) and let $Y = \sum_{i=1}^{50} X_i$. Use Central Limit theorem to find $P(Y > 3)$. Enter the answer correct to 2 decimal places.

Solution:

Let $X \sim$ Poisson(0.04).
Consider the samples $X_1, X_2, \ldots, X_{50}$ from X.
$E[X] = \text{Var}[X] = 0.04$
$E[Y] = E\left[\sum_{i=1}^{50} X_i\right] = 50 \times 0.04 = 2$, $\text{Var}[Y] = \text{Var}\left[\sum_{i=1}^{50} X_i\right] = 50 \times 0.04 = 2$
To find: $P(Y > 3)$.
By CLT, we know that

$$\frac{Y - n\mu}{\sigma\sqrt{n}} \sim \text{Normal}(0, 1)$$
$$\Rightarrow \left(\frac{Y-2}{\sqrt{2}}\right) \sim \text{Normal}(0, 1)$$

Now,

$$\begin{aligned}
P(Y > 3) &= P(Y - 2 > 1) \\
&= P\left(\frac{Y-2}{\sqrt{2}} > \frac{3-2}{\sqrt{2}}\right) \\
&= P(Z > 0.707) \\
&= 1 - F_Z(0.707) = 1 - 0.76 = 0.24
\end{aligned}$$

---

2. Let the moment generating function of a random variable X be given by

$$M_X(\lambda) = \left(\frac{1}{4}\right)e^{-2\lambda} + \left(\frac{1}{40}\right) + \left(\frac{3}{10}\right)e^{-\lambda} + \left(\frac{3}{40}\right)e^{2\lambda} + \left(\frac{7}{20}\right)e^{\lambda}$$

Find the distribution of X.

X	-2	-1	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{3}{40}$	$\frac{3}{10}$	$\frac{1}{40}$	$\frac{7}{20}$

(a)

X	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{40}$	$\frac{3}{10}$	$\frac{3}{40}$	$\frac{7}{20}$

(b)

X	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{1}{40}$	$\frac{7}{20}$	$\frac{3}{40}$

(c)

X	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{40}$	$\frac{1}{40}$	$\frac{7}{20}$	$\frac{3}{10}$

(d)

Solution:

The MGF of a discrete random variable X with the PMF $f_X(x) = P(X = x)$, $x \in T_X$ is given by

$$\begin{aligned} M_X(\lambda) &= E[e^{\lambda X}] \\ &= \sum_{x \in T_X} P(X = x)e^{\lambda x} \end{aligned}$$

Now, MGF of a random variable X is given as

$$M_X(\lambda) = \left(\frac{1}{4}\right)e^{-2\lambda} + \left(\frac{1}{40}\right) + \left(\frac{3}{10}\right)e^{-\lambda} + \left(\frac{3}{40}\right)e^{2\lambda} + \left(\frac{7}{20}\right)e^{\lambda}$$

Therefore, distribution of X is given by

X	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{1}{40}$	$\frac{7}{20}$	$\frac{3}{40}$

---

3. A fair coin is tossed 1000 times. Use CLT to compute the probability that head appears at most 520 times. Enter the answer correct to 3 decimal places.

Solution:

Define a random variable X such that

$$X = \begin{cases} 1 & \text{if head appears on tossing a fair coin} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $E[X] = \mu = \dfrac{1}{2}$ and
$\text{Var}(X) = \sigma^2 = \dfrac{1}{2}.\dfrac{1}{2} = \dfrac{1}{4}$

Let $X_1, X_2, \ldots, X_{1000}$ be outcomes on tossing the fair coin 1000 times.
Notice that $X_1 + X_2 + \ldots + X_{1000}$ will denote the number of times head appears in 1000 tosses.

Let $S = X_1 + X_2 + \ldots + X_{1000}$

To find: $P(S \leq 520)$

By CLT, we know that

$$\frac{S - 1000\mu}{\sigma\sqrt{n}} \sim \text{Normal}(0, 1)$$
$$\Rightarrow \frac{S - 500}{5\sqrt{10}} \sim \text{Normal}(0, 1)$$

Now,

$$\begin{aligned} P(S \leq 520) &= P(S - 500 \leq 20) \\ &= P\left(\frac{S - 500}{5\sqrt{10}} \leq \frac{20}{5\sqrt{10}}\right) \\ &= P(Z \leq 1.26) \\ &= 0.896 \end{aligned}$$

---

4. Let $X_1, X_2, \ldots, X_{500} \sim$ i.i.d Normal$(0,1)$. Evaluate $P(X_1^2 + X_2^2 + \ldots + X_{500}^2 > 550)$ using Central Limit theorem. Enter the answer correct to 2 decimal places.
Hint: $(X_1^2 + X_2^2 + \ldots + X_{500}^2) \sim$ Gamma $(250, 0.5)$.

Solution:

Given $X_1, \ldots, X_{500} \sim$ i.i.d. Normal$(0, 1)$.

We know that if $X \sim \text{Normal}(0, 1) \implies X^2 \sim \text{Gamma}\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

Also, Sum of n independent Gamma(α, β) is Gamma$(n\alpha, \beta)$.

Therefore, $X_i^2 \sim \text{Gamma}\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, for all i.

and $(X_1^2 + X_2^2 + \ldots + X_{500}^2) \sim$ Gamma $(250, 0.5)$

Let $Y = Y_1 + Y_2 + \ldots + Y_{500}$, where $Y_i = X_i^2$ for all $i : 1 \to 500$

$E[Y_i] = \dfrac{0.5}{0.5} = 1$ and $\text{Var}[Y_i] = \dfrac{0.5}{0.25} = 2$, for $i : 1 \to 500$

$E[Y] = \frac{250}{0.5} = 500$ and $\text{Var}[Y] = \frac{250}{0.5^2} = 1000$

To find: $P(Y > 550)$

By CLT, we know that

$$\frac{Y - 500\mu}{\sigma\sqrt{n}} \sim \text{Normal}(0, 1)$$
$$\Rightarrow \frac{Y - 500}{10\sqrt{10}} \sim \text{Normal}(0, 1)$$

Now,

$$\begin{aligned} P(Y > 550) &= P(Y - 500 > 50) \\ &= P\left(\frac{Y - 550}{10\sqrt{10}} > \frac{5}{\sqrt{10}}\right) \\ &= P(Z > 1.58) \\ &= 1 - F_Z(1.58) = 1 - 0.94 = 0.06 \end{aligned}$$

Use the below information to answer questions 5 and 6.

Let X be a random variable having the gamma distribution with the parameters $\alpha = 2n$ and $\beta = 1$.
Hint:

- If $X \sim \text{Gamma}(\alpha, \beta), E[X] = \frac{\alpha}{\beta}$ and $\text{Var}[X] = \frac{\alpha}{\beta^2}$
- Sum of n independent $\text{Gamma}(\alpha, \beta)$ is $\text{Gamma}(n\alpha, \beta)$

5. Use the Weak Law of Large number to find the value of n such that

$$P\left(\left|\frac{X}{2n} - 1\right| > 0.01\right) < 0.01$$

(a) 505000

(b) 470000

(c) 498000

(d) 482000

Solution:

Given $X \sim \text{Gamma}(2n, 1)$
Let $X = X_1 + X_2 + X_3 + \ldots + X_{2n}$, where $X_i \sim \text{Gamma}(1, 1)$.

$E[X] = \mu = 1$ and $\text{Var}(X) = \sigma^2 = 1$
$E[\bar{X}] = 1$ and $\text{Var}[\bar{X}] = \dfrac{1}{2n}$

To find: The value of n such that $P\left(\left|\dfrac{X}{2n} - 1\right| > 0.01\right) < 0.01.$

By weak law of large numbers, we have

$$P(|\overline{X} - \mu| > \delta) \leq \frac{\sigma^2}{n\delta^2}$$
$$\Rightarrow P\left(\left|\frac{X}{2n} - 1\right| > 0.01\right) \leq \frac{1}{2n \times 0.01^2}$$

Therefore, $\dfrac{1}{2n \times 0.01^2} < 0.01 \implies 2n > \dfrac{1}{0.01^3} \implies n > 500000.$

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6. Use CLT to find the value of n such that

$$P\left(\left|\frac{X}{2n} - 1\right| > 0.01\right) < 0.01$$

Hint: Use $F_Z(2.58) = 0.995, F_Z(1.96) = 0.975$ if needed.

(a) 34570
(b) 33500
(c) 32500
(d) 30000

Solution:
$E[X_1 + \ldots + X_{2n}] = 2n$ and $\text{Var}[X_1 + \ldots + X_{2n}] = 2n$

To find: The value of n such that $P\left(\left|\dfrac{X}{2n} - 1\right| > 0.01\right) < 0.01.$

By CLT, we know that

$$\frac{X - 2n\mu}{\sigma\sqrt{n}} \sim \text{Normal}(0, 1)$$
$$\Rightarrow \frac{X - 2n}{\sqrt{2n}} \sim \text{Normal}(0, 1)$$

Now,

$$
\begin{aligned}
P\left(\left|\frac{X}{2n}-1\right|>0.01\right) &< 0.01\\
\implies P\left(\left|\frac{X_1+\ldots+X_n}{2n}-1\right|>0.01\right) &< 0.01\\
\implies P\left(\left|\frac{X_1+\ldots+X_n-2n}{\sqrt{2n}}\right|>0.01\sqrt{2n}\right) &< 0.01\\
\implies P(\mid Z\mid>0.01\sqrt{2n}) &< 0.01\\
\implies 2P(Z>0.01\sqrt{2n}) &< 0.01\\
\implies 1-F_Z(0.01\sqrt{2n}) &< \frac{0.01}{2}\\
\implies F_Z(0.01\sqrt{2n}) &> 0.995\\
\implies F_Z(0.01\sqrt{2n}) &> F_Z(2.58)\\
\implies n &> 33282
\end{aligned}
$$

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7. Let the time taken (in hours) for failure of an electric bulb follow the exponential distribution with the parameter 0.05. Suppose that 100 such light bulbs say $L_1, L_2, \ldots, L_{100}$ are used in the following manner: For every i, as soon as the light L_i fails, L_{i+1} becomes operative, where $i : 1 \to 99$ (i.e. If L_1 fails, L_2 becomes operative, if L_2 fails, L_3 becomes operative, and so on). Let the total time of operation of 100 bulbs be denoted by T. Using CLT, compute the probability that T exceeds 2500 hours.

   (a) $F_Z(1.5)$
   (b) $1 - F_Z(1.5)$
   (c) $F_Z(2.5)$
   (d) $1 - F_Z(2.5)$

   Solution:

   Given, time to failure (in hours) of an electric bulb has the exponential distribution with the parameter $\lambda = 0.05$.
   Since, the bulbs are used in such a way, that as soon as light L_1 fails, L_2 becomes operative, L_2 fails, L_3 becomes operative, and so on.

   We know that if $X \sim \text{Gamma}(\alpha, \beta)$ with parameter $\alpha = 1$, then $X \sim \text{Exp}(\beta)$.
   Also, sum of n i.i.d. $\text{Exp}(\lambda)$ is $\text{Gamma}(n, \lambda)$.

   Since each of the L_i's are exponentially distributed with parameter $= 0.05$, therefore

$$L_1 + \ldots + L_{100} \sim \text{Gamma}(n\alpha, \beta) = \text{Gamma}(100, 0.05)$$

Let $T = L_1 + \ldots + L_{100}$

$$E[L_i] = \mu = \frac{1}{0.05} = 20 \text{ and } \mathrm{SD}[L_i] = \sigma = \frac{1}{0.05} = 20$$

To find: $P(T \geq 2500)$

By CLT, we know that

$$\frac{T - 100\mu}{\sigma\sqrt{n}} \sim \text{Normal}(0, 1)$$
$$\Rightarrow \frac{T - 2000}{20\sqrt{100}} \sim \text{Normal}(0, 1)$$

Now,

$$\begin{aligned} P(T \geq 2500) &= P(T - 2000 \geq 500) \\ &= P\left(\frac{T - 2000}{200} \geq \frac{500}{200}\right) \\ &= P(Z \geq 2.5) \\ &= 1 - F_Z(2.5) \end{aligned}$$

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8. Suppose speeds of vehicles on a particular road are normally distributed with mean 36 mph and standard deviation 2 mph. Find the probability that the mean speed $\overline{X}$ of 20 randomly selected vehicles is between 35 and 38 mph.

   (a) $F_Z(\sqrt{5}) - F_Z(-\sqrt{5})$
   (b) $F_Z(\sqrt{20}) - F_Z(-\sqrt{20})$
   (c) $F_Z(\sqrt{38}) - F_Z(-\sqrt{35})$
   (d) $F_Z(\sqrt{20}) - F_Z(-\sqrt{5})$

   Solution:

   Let X denote the speed of a vehicle on a particular road.
   Given that $X \sim \text{Normal}(36, 2^2)$.
   Therefore, $\mu = 36$ and $\sigma = 2$
   Select $X_1, X_2, \ldots X_{20}$ samples such that $X_1, X_2, \ldots X_{20} \sim$ iid X

   Let $\overline{X} = \dfrac{X_1 + X_2 + \ldots + X_{20}}{20}$ and $S = X_1 + X_2 + \ldots + X_{20}$

   To find: $P(35 < \overline{X} < 38)$ From CLT, we know that

$$\begin{aligned}
&\frac{X_1 + X_2 \ldots + X_n - nE[X]}{\sqrt{n}\sigma} \sim \text{Normal}(0,1)\\
\Rightarrow &\frac{S - n\mu}{\sqrt{n}\sigma} \sim \text{Normal}(0,1)\\
\Rightarrow &\frac{(S - 36(20))}{(2\sqrt{20})} \sim \text{Normal}(0,1)
\end{aligned}$$

Now,

$$\begin{aligned}
P(35 < \overline{X} < 38) &= P(35 < \frac{S}{20} < 38)\\
&= P(-1 < \frac{S}{20} - 36 < 2)\\
&= P(-1 < \frac{S - 36(20)}{20} < 2)\\
&= P(\frac{-\sqrt{20}}{2} < \frac{S - 36(20)}{2\sqrt{20}} < \sqrt{20})\\
&= P(-\sqrt{5} < \frac{S - 36(20)}{2\sqrt{20}} < \sqrt{20})\\
&= F_Z(\sqrt{20}) - F_Z(-\sqrt{5})
\end{aligned}$$