COMBINATORIAL MATHEMATICS

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SYLLABUS

- Basic counting principles
- Permutations and combinations
- Pigeonhole principle
- Recurrence relations – Solving homogeneous and non-homogeneous recurrence relations
- Generating function.

BASIC COUNTING PRINCIPLES

- **Sum Rule Principle:**

  Suppose some event *E can occur in m ways and a second event F can occur in n ways, and suppose both* events cannot occur simultaneously.

  Then ***E or F*** *can occur in* ***m + n ways***

- If no two events can occur at the same time, then one of the events can occur in:

  n1 + n2 + n3 + · · · ways

- *Suppose A and B are disjoint sets.*

  Then $\boldsymbol{n(A \cup B) = n(A) + n(B)}$

BASIC COUNTING PRINCIPLES

- **Product Rule Principle:**

  Suppose there is an event *E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways.*

  Then combinations of ***E and F*** *can occur in*

  m.n ways

- If the events occur one after the other, then all the events can occur in the order indicated in:

  n1 · n2 · n3 · . . . ways.

- Let *A* × *B be the Cartesian product of sets A and B.*

  Then ***n(A × B) = n(A) · n(B)***

EXAMPLE:

- Suppose a college has 3 different history courses, 4 different literature courses, and 2 different sociology courses.

(a) The number *m of ways a student can choose one of each kind of courses is:*

$m = 3(4)(2) = 24$

(b) The number *n of ways a student can choose just one of the courses is:*

$n = 3 + 4 + 2 = 9$

Factorial function

- The product of the positive integers from 1 to n *inclusive is denoted by n!, read "n factorial." Namely:*
- $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)(n-1)n$

  $= n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- Accordingly, $1! = 1$ and $n! = n(n-1)!$.
- *It is also convenient to define* $0! = 1$.

EXAMPLE

- $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)(n-1)n$

  $= n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

- $3! = 3 \cdot 2 \cdot 1 = 6,$
- $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$
- $5! = 5 \cdot 4! = 5(24) = 120.$

EXERCISE

Compute:

(a) $4!$, $5!$;

(b) $6!$, $7!$, $8!$, $9!$;

(a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5(24) = 120.$

(b) Now use$(n + 1)! = (n + 1)n!$:

$6! = 5(5!) = 6(120) = 720,$

$7! = 7(6!) = 7(720) = 5\ 040,$

$8! = 8(7!) = 8(5040) = 40\ 320,$

$9! = 9(8!) = 9(40\ 320) = 362\ 880.$

EXERCISE

- Compute: (a) 13!/11!
  (b) 7!/ 10!

$$(a)\ \frac{13!}{11!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13\cdot 12 = 156.$$

Alternatively, this could be solved as follows:

$$\frac{13!}{11!} = \frac{13\cdot 12\cdot 11!}{11!} = 13\cdot 12 = 156.$$

$$(b)\ \frac{7!}{10!} = \frac{7!}{10\cdot 9\cdot 8\cdot 7!} = \frac{1}{10\cdot 9\cdot 8} = \frac{1}{720}.$$

EXERCISE

Simplify: (a) $\dfrac{n!}{(n-1)!}$; (b) $\dfrac{(n+2)!}{n!}$.

$$(a)\ \frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = n;\ \text{alternatively},\ \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n.$$

$$(b)\ \frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2+3n+2.$$

PERMUTATIONS

- Any arrangement of a set of *n objects in a given order is called a permutation of the object (taken all at a time).*

- The number of permutations of *n objects taken r at a time will be denoted by* ***P(n, r)***

- $P(n, r) = n(n-1)(n-2) \cdot \cdot \cdot (n-r+1)$
$$= n!\,/(n-r)!$$

EXAMPLE

- Find the number *m of permutations of six objects, say, A, B, C, D, E, F, taken three at a time.*

  Find the number of "three-letter words" using only the given six letters without repetition.

  Let us represent the general three-letter word by the following three positions:

  $$\_\_\_\_, \_\_\_\_, \_\_\_\_$$

  The first letter can be chosen in 6 ways; following this the second letter can be chosen in 5 ways; and, finally, the third letter can be chosen in 4 ways. Write each number in its appropriate position as follows:

  $$6\ ,\ 5\ ,\ 4$$

  By the Product Rule there are $m = 6 \cdot 5 \cdot 4 = 120$

  possible three-letter words without repetition from the six

  letters. Namely, there are 120 permutations of 6 objects taken 3 at a time.

  This agrees with the formula $P(6, 3) = 6 \cdot 5 \cdot 4 = 120$

PERMUTATIONS WITH REPETITIONS

- The number of permutations of a multi-set, that is, a set of objects some of which are alike.
- We will let *P(n; n1, n2, . . . , nr)* denote the number of permutations of *n objects of which n1 are alike, n2 are alike, . . ., nr are alike.*
- *The general* formula follows:

  P(n; n1, n2, . . . , nr) = n! / n1! n2! . . . nr !

- Accordingly, the number of different five-letter words that can be formed using the letters from the word "*BABBY*" *is:*

  P(5; 3) = 5! / 3! = 20

EXAMPLE

- Find the number *m of seven-letter words that can be formed using the letters of the word "BENZENE."*
- We seek the number of permutations of 7 objects of which 3 are alike (the three *E's), and 2 are alike (the two N's).*

$$m = P(7; 3, 2) = 7! \,/\, 3!2!$$
$$= (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) / (3 \cdot 2 \cdot 1 \cdot 2 \cdot 1)$$
$$= 420$$

Ordered Samples

- **Sampling with replacement**
- Here the element is replaced in the set S *before the next element is chosen. Thus, each time there are n ways* to choose an element (repetitions are allowed).
- The Product rule tells us that the number of such samples is:
- $n \cdot n \cdot n \cdot \cdot \cdot n \cdot n$ *(r factors)* $= nr$

ORDERED SAMPLES

- **Sampling without replacement**
- Here the element is not replaced in the set *S before the next element is chosen.*
- *Thus, there is no repetition* in the ordered sample. Such a sample is simply an *r-permutation*.
- *Thus the number of such samples is:*

$$P(n, r) = n(n-1)(n-2) \cdot \cdot \cdot (n-r+1)$$
$$= n! \,/\, (n-r)!$$

EXAMPLE

- Three cards are chosen one after the other from a 52-card deck. Find the number *m of ways*

  this can be done: (a) with replacement; (b) without replacement.

(a)Each card can be chosen in 52 ways.

Thus $m = 52(52)(52) = 140\ 608$.

(b) Here there is no replacement. Thus the first card can be chosen in 52 ways, the second in 51 ways, and the third in 50 ways.

Therefore: $m = P(52, 3) = 52(51)(50) = 132\ 600$

EXERCISE

Q1. Find: (*a) P(7, 3); (b) P(14, 2)*.

Q2. Find the number *m of ways that 7 people can arrange themselves:*

(*a) In a row of chairs; (b) Around a circular table.*

Q3. Find the number *n of distinct permutations that can be formed from all the letters of each word:*

THOSE; (b) UNUSUAL; (c) SOCIOLOGICAL

Q4. A class contains 8 students. Find the number *n of samples of size 3:*

(*a)With replacement; (b)Without replacement*

SOLUTIONS

- A1. *(a)* $P(7, 3) = 7 \cdot 6 \cdot 5 = 210$;
  (b) $P(14, 2) = 14 \cdot 13 = 182$.
- *A2*. Here $m = P(7, 7) = 7!$ *ways*.

(*b) One person can sit at any place at the table. The other 6 people can arrange themselves in 6! ways around the* table; that is $m = 6!$. This is an example of a *circular permutation*.

In general, n objects can be arranged in a circle in (n − 1)! ways.

- *A3*. (*a*) $n = 5! = 120$, *since there are 5 letters and no repetitions*.

(*b*) $n = 7!/\ 3! = 840$, since there are 7 letters of which 3 are *U and no other letter is repeated.*

(*c*) $n = 12!\ /3!2!2!2!$,

since there are 12 letters of which 3 are *O, 2 are C, 2 are I, and 2 are L. (We leave the answer*
using factorials, since the number is very large.)

- A4. (*a*) *Each student in the ordered sample can be chosen in 8 ways; hence, there are*

  $n = 8 \cdot 8 \cdot 8 = 83 = 512$ *samples of size 3 with replacement.*

(*b*) *The first student in the sample can be chosen in 8 ways, the second in 7 ways, and the last in 6 ways. Thus, there* are $n = 8 \cdot 7 \cdot 6 = 336$ *samples of size 3 without replacement*

COMBINATIONS

- Let *S be a set with n elements. A combination of these n elements taken r at a time is any selection of r of* the elements where order does not count. Such a selection is called an *r-combination; it is simply a subset of S* with *r elements.*
- *The number of such combinations will be denoted by C(n, r)*
- $\boldsymbol{C(n, r) = P(n, r) / r!}$

  $\boldsymbol{C(n, r) = n! / r! .(n - r)!}$

EXAMPLE

- A farmer buys 3 cows, 2 pigs, and 4 hens from a man who has 6 cows, 5 pigs, and 8 hens. Find the number *m of choices that the farmer has.*
- The farmer can choose the cows in *C(6, 3) ways, the pigs in C(5, 2) ways, and the hens in C(8, 4) ways.*
- Thus the number *m of choices follows:*

$$m = \binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}\cdot\frac{5\cdot 4}{2\cdot 1}\cdot\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = 20\cdot 10\cdot 70 = 14\,000$$

EXERCISE

Q1. A class contains 10 students with 6 men and 4 women. Find the number *n of ways to:*

(*a*) *Select a 4-member committee from the students.*

(*b*) *Select a 4-member committee with 2 men and 2 women.*

(*c*) *Elect a president, vice president, and treasurer.*

Q2. A box contains 8 blue socks and 6 red socks. Find the number of ways two socks can be drawn from the box if:

(*a*) *They can be any color. (b) They must be the same color.*

SOLUTION

A1.(a)This concerns combinations, not permutations, since order does not count in a committee. There are "10 choose 4" such committees.

That is:

$n = C(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 / 4 \cdot 3 \cdot 2 \cdot 1 = 210$

(*b*) *The 2 men can be chosen from the 6 men in C(6, 2) ways, and the 2 women can be chosen from the 4 women in C(4, 2) ways. Thus, by the Product Rule:*

$n = 6 \cdot 5 / 2 \cdot 1 \ . 4 \cdot 3 / 2 \cdot 1 = 15(6) = 90$

(*c*) *This concerns permutations, not combinations, since order does count. Thus,*

$n = P(6, 3) = 6 \cdot 5 \cdot 4 = 120$

SOLUTION

A2. (*a*) *There are "14 choose 2" ways to select 2 of the 14 socks.*

Thus: $n = C(14, 2) = (14 \cdot 13) / (2 \cdot 1) = 91$

(*b*) *There are* $C(8, 2) = 28$ *ways to choose 2 of the 8 blue socks, and*

$C(6, 2) = 15$ *ways to choose 2 of the 4 red* socks.

By the Sum Rule, $n = 28 + 15 = 43$.

- From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

We may have (3 men and 2 women) or (4 men and 1 woman) or (5 men only)

$\therefore$ Required number of ways $= ({}^7C_3 \times {}^6C_2) + ({}^7C_4 \times {}^6C_1) + ({}^7C_5)$

$$= \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{6 \times 5}{2 \times 1}\right) + ({}^7C_3 \times {}^6C_1) + ({}^7C_2)$$

$$= 525 + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 6\right) + \left(\frac{7 \times 6}{2 \times 1}\right)$$

$$= (525 + 210 + 21)$$

$$= 756.$$

THE PIGEONHOLE PRINCIPLE

- If *n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is* occupied by more than one pigeon.

- **Example**: Suppose a department contains 13 employees , then two of the employees (pigeons) were born in the same month (pigeonholes).

Generalized Pigeonhole Principle

- If *n pigeonholes are occupied by* ***kn + 1*** *or more pigeons, where k is a* positive integer, then at least one pigeonhole is occupied by ***k + 1 or more pigeons.***
- ***Example:*** Find the minimum number of students in a class to be sure that three of them are born in the same month.

  Here the $n = 12$ *months are the pigeonholes, and* $k + 1 = 3$ *so* $k = 2$.

  Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

EXERCISE

Q1. Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

Q2. A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

SOLUTIONS

A1. Here the *n = 4 classes are the pigeonholes and k+1 = 5 so k = 4. Thus among any kn+1 = 17 students (pigeons),* five of them belong to the same class.

A2. Apply pigeonhole principle.
No. of colors (pigeonholes) n = 3
No. of marbles (pigeons) K+1 = 4
Therefore the minimum no. of marbles required = Kn+1
By simplifying we get Kn+1 = 10.
i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10

RECURRENCE RELATION

- Let S be a sequence of numbers. A recurrence relation on S is a formula that relates all, but a finite number of terms S, to the previous terms of S.
- It is also known as Difference Equation.
- Example: $13\ a_{r} + 2\ a_{r-1} = 0$

- **Degree of Recurrence Relation -** It is defined as the highest power of f(x).
- Example: $Y_K^{\,3} - 2\ Y_K^{\,2} + 2Y_K^{\,1} = 6$

  Here degree is 3

- **Order of Recurrence Relation:** It is defined as the difference between its highest and lowest subscripts of f(x).

- Example: $\mathbf{13\ a_{r} + 2\ a_{r-1} = 0}$

  Order = r – (r-1)=1 => It is first order recurrence relation

- Example : $\mathbf{8\ f(x) + 4\ f(x-1) + 6\ f(x-2) = 0}$

  Order = x - (x-2) =2 => It is second order recurrence relation

LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

- A *linear kth-order recurrence relation with constant coefficients is a recurrence relation* of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + f(n)$$

where *C1,C2, . . . , Ck are constants with Ck* = 0, and *f (n) is a function of n.*

- *The meanings of the names linear* and constant coefficients follow:
- *Linear: There are no powers or products of the aj 's.*
- *Constant coefficients: The C1C2, . . . ,Ck are constants (do not depend on n).*
- If *f (n) = 0, then the relation is also said to be homogeneous.*

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is – $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is –

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots –

Case 1 – If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 – If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1, then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 – If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then $F_n = r^n(acos(n\theta) + bsin(n\theta))$ is the solution.

Solve the RR

$F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Order: $H - L = (n) - (n-2) = n - n + 2$
$= 2$

Eqn: $a^2 - 5a + 6 = 0$ $\qquad P = 6$, Sum $= -5$
$-3, -2$

Solve and find roots: $a^2 - 3a - 2a + 6 = 0$
$a(a-3) - 2(a-3) = 0$
$(a-3)(a-2) = 0 \longrightarrow$ $a - 3 = 0$, $a = 3$; $a - 2 = 0$, $a = 2$

Roots $a_1 = 3$ and $a_2 = 2$

$\Rightarrow$ Roots are real and distinct.

Solution: $F_n = C_1 a_1^n + C_2 a_2^n$

$F_n = C_1(3)^n + C_2(2)^n \qquad \because a_1 = 3, a_2 = 2$

Now $F_0 = 1 \Rightarrow$ Put $n = 0$

$F_0 = \underbrace{C_1(3)^0 + C_2(2)^0 = 1}$ —— (I)

Now $F_1 = 4 \Rightarrow$ Put $n = 1$

$F_1 = \underbrace{C_1(3)^1 + C_2(2)^1 = 4}$ —— (II)

Solve (I) and (II) $\qquad C_1 + C_2 = 1 \qquad$ [(I) $\times 2$ and sub]
$3C_1 + 2C_2 = 4$

$\Rightarrow \quad 2C_1 + \cancel{2C_2} = 2$
$\quad\; 3C_1 + \cancel{2C_2} = 4$
$(-) \quad (-) \quad (-)$
$\cancel{-}C_1 = \cancel{-}2$

$\Rightarrow \boxed{C_1 = 2}$

Now $C_1 + C_2 = 1$
$2 + C_2 = 1$

$\Rightarrow \boxed{C_2 = -1}$

Substitute value of C_1 and C_2 in Solution

$$F_n = 2(3)^n + (-1)\,2^n$$

$$F_n = 2.3^n - 2^n$$ is the required solution

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 5x + 6 = 0,$$

So, $(x-3)(x-2) = 0$

Hence, the roots are −

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is −

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n \ (As\ x_1 = 3\ and\ x_2 = 2)$

Therefore,

$1 = F_0 = a3^0 + b2^0 = a + b$

$4 = F_1 = a3^1 + b2^1 = 3a + 2b$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is −

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

Solve the recurrence relation − $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 10x - 25 = 0$$

So $(x-5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

$$F_n = ax_1^n + bnx_1^n$$

$$3 = F_0 = a.5^0 + (b)(0.5)^0 = a$$

$$17 = F_1 = a.5^1 + b.1.5^1 = 5a + 5b$$

Solving these two equations, we get $a = 3$ and $b = 2/5$

Hence, the final solution is − $F_n = 3.5^n + (2/5).n.2^n$

EXERCISE

Q1. Solve the recurrence relation:

$t_n = 0$; if n=0

$t_n = 5$; if n=1

$t_n = 3t_{n-1} + 4t_{n-2}$; otherwise

Q2. Solve the recurrence relation:

$t_n = 4(t_{n-1} - t_{n-2})$ where $t_n = 1$;

if n=0 and n=1

Q3. Solve the recurrence relation:

$t_n = 5t_{n-1} - 6t_{n-2}$ where $t_n = 7$; if n=0

and $t_n = 16$; if n=1

EXERCISE

Q4. Solve the recurrence relation:

$t_n = 7t_{n-1} - 10\, t_{n-2}$; where

$t_n = 5$; if $n=0$ and $t_n = 16$; if $n=1$

Q5. It is given that white tiger population is 30 at time n=0 and 32 at time n=1. Also the increase from time n-1 to time n is twice the increase from time n-2 to time n-1. Write the recurrence relation for growth rate of tiger and solve it.

SOLUTION

A1. Equation $x^2 - 3x - 4 = 0$

Roots -1, 4 ; Constants C1= -1 and C2=1

Solution $t_n = 4^n - (-1)^n$

A2. Equation $x^2 - 4x + 4 = 0$

Roots 2,2 ; Constants C1= 1 and C2= -1/2

Solution $t_n = 2n - n\,(2)^{n-1}$

A3. Equation $x^2 - 5x + 6 = 0$

Roots 2,3 ; Constants C1= 5 and C2= 2

Solution $t_n = 5\,(2)^n + 2\,(3)^n$

A4. Equation $x^2 - 7x + 10 = 0$

Roots 2,5 ; Constants C1= 3 and C2= 2

Solution $t_n = 3\,(2)^n + 2\,(5)^n$

SOLUTION

A5. It is given that white tiger population is 30 at time n=0 and 32 at time n=1. Also the increase from time n-1 to time n is twice the increase from time n-2 to time n-1. Write the recurrence relation for growth rate of tiger and solve it

increase from time n-1 to time n : $t_n - t_{n-1}$

increase from time n-2 to time n-1: $t_{n-1} - t_{n-2}$

Recurrence relation: $t_n - t_{n-1} = 2(t_{n-1} - t_{n-2})$

Initial conditions: $t_n = 30$; if n=0 and

$t_n = 32$; if n=1

Equation: $x^2 - 3x + 2 = 0$

Roots: 1,2 ; Constants: C1= 28 and C2= 2

Solution: $t_n = 28 + 2(2)^n$