RELATIONS

PRESENTED BY:-

ER. HANIT KARWAL

ASSISTANT PROFESSOR

INFORMATION TECHNOLOGY DEPT.

GNDEC, LUDHIANA

hanitgndec@gmail.com

ORDERED PAIR

- An *ordered pair of elements a and b, where a is designated as the first element and b as the second element* , is denoted by (a, b).
- *In particular,* $(a, b) = (c, d)$

  if and only if $a = c$ *and* $b = d$.
- *Thus* $(a, b) = (b, a)$ *unless* $a = b$.
- *This contrasts with sets where the order of* elements is irrelevant; for example, $\{3, 5\} = \{5, 3\}$.

PRODUCT SET

- Consider two arbitrary sets *A and B. The set of all ordered pairs (a, b) where a* $\in$ *A and b* $\in$ *B is called* the *product, or Cartesian product, of A and B.*
- *A short designation of this product is* $A \times B$, *which is read "A cross B."*
- *By definition,* $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- *A^2 instead of* $A \times A$..

- *Example:*

  Let $A = \{1, 2\}$ *and* $B = \{a, b, c\}$.

  Then

  $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

  Similarly find $B \times A$, $A \times A$

Exercise

- Let $A = \{1, 2\}$ *and* $B = \{a, b, c\}$.

Then

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

RELATIONS

- Let *A and B be sets.*

  A binary relation (R) or, simply, relation (R) from A to B is a subset of $A \times B$.
- *R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair* $a \in A$ *and* $b \in B$*, exactly one of the following* is true:

(i) $(a, b) \in R$*; we then say "a is R-related to b", written aRb.*

(ii) $(a, b) / \in R$*; we then say "a is not R-related to b", written a/Rb.*

- *If A and B are sets with m' and 'n' elements respectively. Then total number of ordered pairs will be* $A \times B = mn$*. So the total number of relations from A to B is 2^(mn).*

DOMAIN – RANGE

- ***Domain of a relation R*** *is the set of all first elements of the ordered pairs which belong to R*
- ***Range*** *is the set of second elements.*

- Example:
- Let there are 2 sets $A = \{1, 2\}$ *and* $B = \{a, b, c\}$.

Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$R = \{(1, a), (2, b)\}$$

Here $1Ra$, $2Rb$

Domain: $(1, 2)$

Range : (a, b)

Exercise

- Find range & domain of the following

1. *$A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$*

2. $A=\{1,3,5,7\}$ and $B = \{2,4,6,8,\}$ and let $R= \{(1,8)\ (3,6)\ (5,2)\ (1,4)\}$

- 1Ry, 1Rz, 3Ry, but 1/Rx, 2/Rx, 2/Ry, 2/Rz, 3/Rx, 3/Rz
  The domain of R is {1, 3} and the range is {y, z}.
- The domain of R is {1, 3, 5} and
  the range is {2,4,6,8,}

INVERSE OF A RELATION

- Let *R be any relation from a set A to a set B.*
- *The inverse of R, denoted by* $R^{(-1)}$, is the relation from *B to A*
- which consists of those ordered pairs which, when reversed, belong to *R; that is,*
- $R^{(-1)} = \{(b, a) \mid (a, b) \in R\}$
- *Example:* $A = \{1, 2, 3\}$ *and* $B = \{x, y, z\}$*. Then the inverse of* $R = \{(1, y), (1, z), (3, y)\}$ *is*

  $$R^{(-1)} = \{(y, 1), (z, 1), (y, 3)\}$$
- *R is any relation, then* $(R^{(-1)(-1)}) = R$*. Also, the domain and range of* $R^{(-1)}$ are equal to that of R.

COMPLEMENT OF A RELATION

- Let *R be any relation from a set A to a set B.*
- *The complement of R, denoted by R^(c)*, is the relation from *A to B* which consists of those ordered pairs which don't belong to *R; that is,*
- $R^{(c)} = \{(a, b) \mid (a, b) \notin R\}$
- Let $A = \{1, 2\}$ *and* $B = \{a, b, c\}$.

Then

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

- *Let* $R = \{(1, a), (2, b), (2, c)\}$
- $R^{(c)} = \{ (1, b), (1, c), (2, a)\}$

COMPOSITION OF RELATIONS

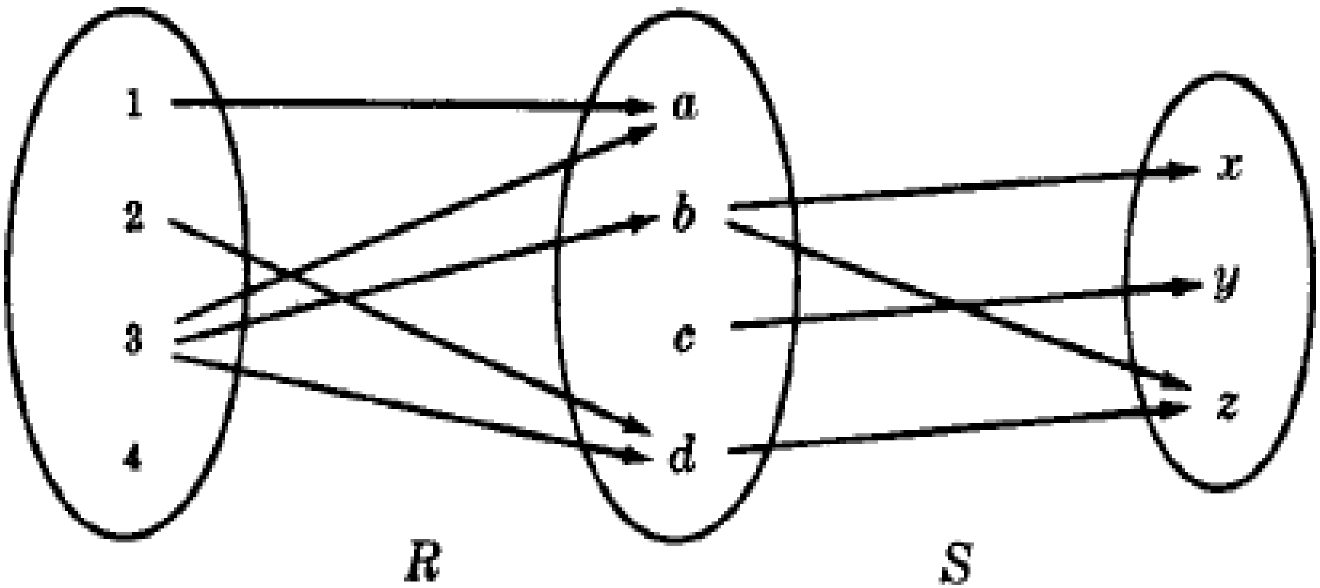
- Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by:
- $a(R \circ S)c$ if for some $b \in B$ we have aRb and bSc.
- $R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$
- The relation $R \circ S$ is called the composition of R and S; it is sometimes denoted simply by RS.

- Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ *and let*

$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ *and*

$S = \{(b, x), (b, z), (c, y), (d, z)\}$

$R \circ S = \{(2, z), (3, x), (3, z)\}$



EXERCISE

- Let $A = \{1, 2, 3, 4\}, B = \{a, b, c\}, C = \{x, y, z\}$. *Consider the relations R from A to B and S from B to C as follows:*

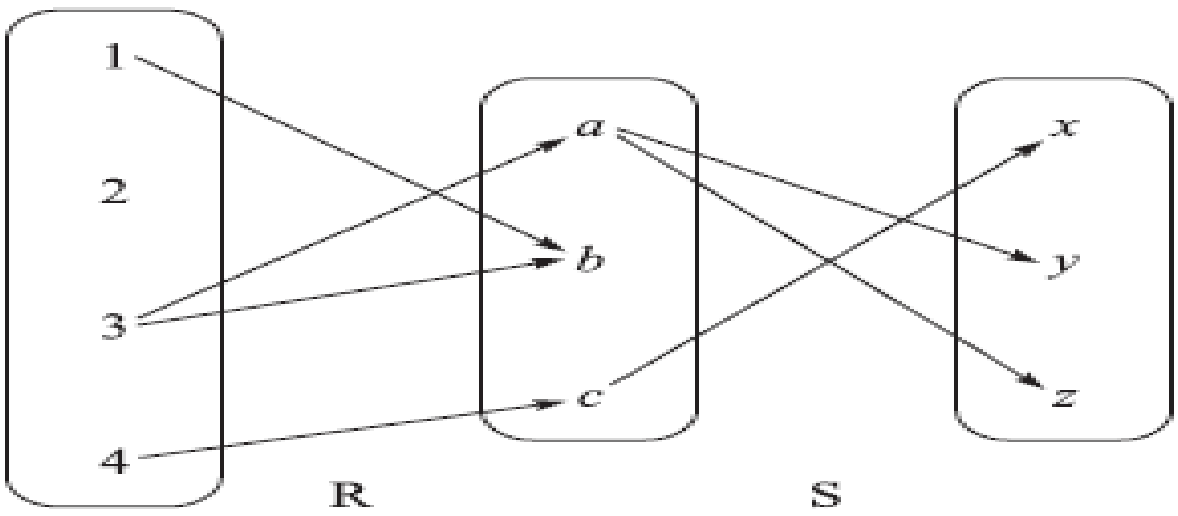
$R = \{(1, b), (3, a), (3, b), (4, c)\}$ *and*

$S = \{(a, y), (c, x), (a, z)\}$

1) Draw the diagrams of R *and* S.
2) Write $R(-1)$ and the composition $R \circ S$ *as sets of ordered pairs.*

SOLUTIONS

- $R\,(-1) = \{(b, 1), (a, 3), (b, 3), (c, 4)\}$,
- $R \circ S \quad = \{(3, y), (3, z), (4, x)\}$.



(b)

TYPES OF RELATIONS

- **Reflexive Relations**
- A relation *R on a set A is reflexive if aRa for every* $a \in A$*, that is, if* $(a, a) \in R$ *for every* $a \in A$*. Thus R is* not reflexive if there exists $a \in A$ *such that* $(a, a) \notin R$.
- Let $A = \{1, 2, 3, 4\}$

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$

$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R3 = \{(1, 3), (2, 1)\}$

$R4 = \emptyset$, *the empty relation*

$R5 = A \times A$, *the universal relation*

- Since *A contains the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs*
- *(1, 1), (2, 2), (3, 3), and (4, 4). Thus only R2 and the universal relation* $R5 = A \times A$ *are reflexive.*
- *R1,R3, and R4 are not reflexive since, for example, (2, 2) does not belong to any of them.*

TYPES OF RELATIONS

- **Symmetric Relation**
- A relation R *on a set* A *is symmetric if whenever* aRb *then* bRa*, that is, if whenever* $(a, b) \in R$ *then* $(b, a) \in R$.
- Thus R *is not symmetric if there exists* $a, b \in A$ *such that* $(a, b) \in R$ *but* $(b, a) / \in R$.

- Let $A = \{1, 2, 3, 4\}$

$R1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$

$R2 = \{(1, 1), (2, 2), (3, 3), (1,3)\}$

R1 is symmetric , R2 is not symmetric

TYPES OF RELATIONS

- **Antisymmetric Relation**
- A relation *R on a set A is antisymmetric if whenever aRb and bRa then a = b, that is, if a = b and aRb then bRa.*
- Thus *R is not antisymmetric if there exist distinct elements a*
- Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 1), (2, 2), (3, 3), (4,4)\}$

The identity relation is an antisymmetric relation.

TYPES OF RELATIONS

- **Transitive Relations**
- A relation *R on a set A is transitive if whenever aRb and bRc then aRc, that is, if whenever*
  $(a, b), (b, c) \in R$ then $(a, c) \in R$.
- *Thus R is not transitive if there exist a, b, c* $\in R$ *such that* $(a, b), (b, c) \in R$ *but* $(a, c) \notin R$.

- Let $A = \{1, 2, 3, 4\}$:
- $R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
- $R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
- $R3 = \{(1, 3), (2, 1)\}$
- $R4 = \emptyset$, *the empty relation*
- $R5 = A \times A$, *the universal relation*
- The relation *R3 is not transitive since*
  $(2, 1), (1, 3) \in R3$ *but* $(2, 3) \notin R3$.
  All the other relations are transitive.

EQUIVALENCE RELATIONS

- Consider a nonempty set S.

  A relation R on S is an equivalence relation if R is reflexive, symmetric, and transitive.

(1) For every $a \in S$, aRa : $a = a$ for every $a \in S$.

(2) If aRb, then bRa : If $a = b$, then $b = a$

(3) If aRb and bRc, then aRc : If $a = b$, $b = c$, then $a = c$

PARTIAL ORDERING RELATIONS

- A relation *R on a set S is called a partial ordering or a partial order of S if R is reflexive, antisymmetric, and transitive.*
- A set *S together with a partial ordering R is called a partially ordered set or poset.*

(1) For every $a \in S$, aRa : $a = a$ *for every* $a \in S$.

(2) If aRb and bRa , a=b : Identity relation

(3) If aRb and bRc, then aRc : If a = b, b = c, then a = c

CLOSURES

- Consider the relation $R = \{(a, a), (a, b), (b, c), (c, c)\}$ *on the set* $A = \{a, b, c\}$. *Find: (a) reflexive(R); (b) symmetric(R); (c) transitive(R).*
- *The* ***reflexive closure*** *on R is obtained by adding all diagonal pairs of* $A \times A$ *to R which are not currently in R.*

$$\text{reflexive}(R) = R \cup \{(b, b)\}$$
$$= \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$$

- *The* ***symmetric closure*** *on R is obtained by adding all the pairs in* $R^{(-1)}$ to *R which are not currently in R. Hence,*

$$\text{symmetric}(R) = R \cup \{(b, a), (c, b)\}$$
$$= \{(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)\}$$

CLOSURES

- *The **transitive closure** on R, since A has three elements, is obtained by taking the union of R with* $R^2 = R \circ R$ *and* $R^3 = R \circ R \circ R$.

  Note that

  $R^2 = R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$

  $R^3 = R \circ R \circ R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$

  $$\text{transitive}(R) = R \cup R^2 \cup R^3$$
  $$= \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$$

EXERCISE

Que1) Consider the following five relations on the set

$A = \{1, 2, 3\}$:

$R = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$,

$\emptyset = \text{empty relation}$

$S = \{(1, 1)(1, 2), (2, 1)(2, 2), (3, 3)\}$,

$A \times A = \text{universal relation}$

$T = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$

- **Determine whether or not each of the above relations on *A is:***

(a) reflexive;

(b) symmetric;

(c) transitive;

(d) antisymmetric.

SOLUTIONS

- *(a) R is not reflexive since* $2 \in A$ *but* $(2, 2) \notin R$*. T is not reflexive since* $(3, 3) \notin T$ *and, similarly,* $\emptyset$ *is not reflexive. S and* $A \times A$ *are reflexive.*
- *(b) R is not symmetric since* $(1, 2) \in R$ *but* $(2, 1) \notin R$*, and similarly T is not symmetric. S,* $\emptyset$*, and* $A \times A$ *are symmetric.*
- *(c) T is not transitive since (1, 2) and (2, 3) belong to T, but (1, 3) does not belong to T. The other four relations are* transitive.
- *(d) S is not antisymmetric since* $1 = 2$*, and (1, 2) and (2, 1) both belong to S. Similarly,* $A \times A$ *is not antisymmetric*. The other three relations are antisymmetric.

Que 2) Let R *and* S *be the following relations on*

$B = \{a, b, c, d\}$:

$R = \{(a, a), (a, c), (c, b), (c, d), (d, b)\}$ *and*

$S = \{(b, a), (c, c), (c, d), (d, a)\}$

Find the following composition relations:

(a) $R \circ S$; (b) $S \circ R$; (c) $R \circ R$; (d) $S \circ S$.

(a) $R \circ S = \{(a, c), (a, d), (c, a), (d, a)\}$

(b) $S \circ R = \{(b, a), (b, c), (c, b), (c, d), (d, a), (d, c)\}$

(c) $R \circ R = \{(a, a), (a, b), (a, c), (a, d), (c, b)\}$

(d) $S \circ S = \{(c, c), (c, a), (c, d)\}$

Que 3) Let R *be the relation on* $\mathbf{N}$ ***defined by***

$\boldsymbol{x + 3y = 12}$***, i.e.*** $\boldsymbol{R = \{(x, y) \mid x + 3y = 12\}}$**.**

(a) Write R *as a set of ordered pairs*

(b) Find the domain , range of R *and* $R(-1)$

(d) Find the composition relation $R \circ R$.

(a) $\{(9, 1), (6, 2), (3, 3)\}$;

(b) (i) $\{9, 6, 3\}$,

(ii) $\{1, 2, 3\}$,

(iii) $\{(1, 9), (2, 6), (3, 3)\}$;

(c) $\{(3, 3)\}$.