Prepositional and Predicate Logic

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Prepositional and Predicate Logic: Syllabus

- Propositional logic,
- Truth tables,
- Normal forms (conjunctive and disjunctive),
- Validity of well-formed formula,
- Propositional inference rules,
- Predicate logic,
- Universal and existential quantifiers

PROPOSITION

- A *proposition (or statement) is a declarative statement which is true or false, but not both.*
- Each proposition will be represented by a propositional variable.
- Propositional variables are usually represented as lower-case letters, such as p, q, r, s, etc.
- Each variable can take one of two values:

  true or false.

PROPOSITION

(i) Ice floats in water.

(ii) China is in Europe.

(iii) $2 + 2 = 4$

(iv) $2 + 2 = 5$

(v) Where are you going?

(vi) Do your homework.

- The first four are propositions, the last two are not.
- Also, (i) and (iii) are true,

  but (ii) and (iv) are false.

Applications of propositional logic

- Translation of English sentences

- Inference and reasoning:
- New true propositions are inferred from existing ones
- Used in Artificial Intelligence:
  - Rule based (expert) systems
  - Automatic theorem provers

- Design of logic circuit

BASIC LOGICAL OPERATIONS

- The three basic logical operations are conjunction, disjunction, and negation which correspond, respectively, to "and," "or," and "not."

Logical NOT: ¬p Read "not p"

- ¬p is true if and only if p is false.
- Also called logical negation.

Logical AND: p ∧ q Read "p and q."

- p ∧ q is true if both p and q are true.
- Also called logical conjunction.

Logical OR: p ∨ q Read "p or q."

- p ∨ q is true if at least one of p or q are true (inclusive OR)
- Also called logical disjunction.

BASIC LOGICAL OPERATIONS

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(*b*) "p or q"

p	$\neg p$
T	F
F	T

(*c*) "not p"

Some other representations

AND $p \,\&\, q$, $p \cdot q$ or pq for $p \wedge q$

OR $p + q$ for $p \vee q$

NEGATION p' $\bar{p}$ or $\sim p$ for $\neg p$

Precedence of Operators

- Example: $\neg p \land q$ means $(\neg p) \land q$
  $p \land q \rightarrow r$ means $(p \land q) \rightarrow r$

Operator	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5

Truth Tables

- A truth table is a table showing the truth value of a propositional logic formula as a function of its inputs.
- **Implication**
  - The $\rightarrow$ connective is used to represent implications.
  - Its technical name is the material conditional operator

Truth Tables

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(*a*)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(*b*)

NOTE: $\neg p \wedge q$ ***means*** $(\neg p) \wedge q$ ***and not*** $\neg(p \wedge q)$.

TAUTOLOGIES AND CONTRADICTIONS

- **TAUTOLOGIES :** Some propositions P(p, q, . . .) contain only T (True) in the last column of their truth tables or, in other words, they are true for any truth values of their variables

- **CONTRADICTIONS :** if propositions P(p, q, . . .) contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables.
- **Note:** The negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

TAUTOLOGIES AND CONTRADICTIONS

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(a) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(b) $p \wedge \neg p$

Prove $[(A \to B) \land A] \to B[(A \to B) \land A] \to B$ is a tautology

A	B	$A \to B$	$(A \to B) \land A$	$[(A \to B) \land A] \to B$
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

Prove (A∨B)∧[(¬A)∧(¬B)](A∨B)∧[(¬A)∧(¬B)] is a contradiction

A	B	A ∨ B	¬ A	¬ B	(¬ A) ∧ (¬ B)	(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

CONTINGENCY

- A Contingency is a formula which has both some true and some false values for every value of its propositional variables.
- It is also said to be **Satisfiable.**
- **Example –** Prove $(A \vee B) \wedge (\neg A)$ is a contingency

A	B	A ∨ B	¬ A	(A ∨ B) ∧ (¬ A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

Inverse, Converse, and Contra-positive

- Implication / if-then (→) is also called a conditional statement. It has two parts −

  Hypothesis, p

  Conclusion, q

  As mentioned earlier, it is denoted as $p \rightarrow q$
- **Example of Conditional Statement −**

  "If you do your homework, you will not be punished."

  Here, "you do your homework" is the hypothesis, p, and

  "you will not be punished" is the conclusion, q.

- **Inverse –** An inverse of the conditional statement is the negation of both the hypothesis and the conclusion.
- If the statement is “If p, then q”, the inverse will be “If not p, then not q”.
- Thus the inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

- **Example –** The inverse of “If you do your homework, you will not be punished” is

  “If you do not do your homework, you will be punished.”

- **Converse –** The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion.
- If the statement is "If p, then q", the converse will be "If q, then p". The converse of $p \rightarrow q$ is $q \rightarrow p$.

- **Example –** The converse of "If you do your homework, you will not be punished" is

  "If you will not be punished, you do your homework".

- **Contra-positive** – The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement.
- If the statement is “If p, then q”, the contra-positive will be “If not q, then not p”.
- The contra-positive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

(1 inverse $\neg p \rightarrow \neg q$. ; 2 converse : $\neg q \rightarrow \neg p$)

- **Example** – The Contra-positive of " If you do your homework, you will not be punished” is "If you are punished, you did not do your homework”.

Contrapositives

- The proposition $\neg q \rightarrow \neg p$ is called the Contrapositive of the proposition $p \rightarrow q$.
- They are logically equivalent. $p \rightarrow q \equiv \neg q \rightarrow \neg p$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\neg q \rightarrow \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

LOGICAL EQUIVALENCE

- Two propositions $P(p, q, \ldots)$ *and* $Q(p, q, \ldots)$ *are said to be logically equivalent, or simply equivalent or equal, denoted by*

  $$\boldsymbol{P(p, q, \ldots) \equiv Q(p, q, \ldots)}$$

  if they have identical truth tables.
- Example: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

LOGICAL EQUIVALENCE

- Let *p be "Roses are red" and q be "Violets are blue." Let S be the statement:

  "It is not true that roses are red and violets are blue." Then *S can be written in the form*

  $\neg(p \wedge q)$. *However,,* $\neg(p \wedge q) \equiv \neg p \vee \neg q$. *Accordingly, S* has the same meaning as the statement:

  "Roses are not red, or violets are not blue."

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

ALGEBRA OF PROPOSITIONS

Table 4-1 Laws of the algebra of propositions

Idempotent laws:	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws:	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	(5a) $p \vee F \equiv p$ (6a) $p \vee T \equiv T$	(5b) $p \wedge T \equiv p$ (6b) $p \wedge F \equiv F$
Involution law:	(7) $\neg\neg p \equiv p$	
Complement laws:	(8a) $p \vee \neg p \equiv T$ (9a) $\neg T \equiv F$	(8b) $p \wedge \neg p \equiv T$ (9b) $\neg F \equiv T$
DeMorgan's laws:	(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proof of a proposition

To prove: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by De Morgan's 2nd law
	$\equiv$	$\neg p \wedge (\neg(\neg p) \vee \neg q)$	by De Morgan's first law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the 2nd distributive law
	$\equiv$	$\mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv$	$(\neg p \wedge \neg q) \vee \mathbf{F}$	by commutativity of disj.
	$\equiv$	$\neg p \wedge \neg q$	by the identity law for $\mathbf{F}$

Prove $(p \land q) \rightarrow (p \lor q) \equiv T$.

$(p \land q) \rightarrow (p \lor q) \equiv \neg(p \land q) \lor (p \lor q)$

Substitution for $\rightarrow$

$\equiv (\neg p \lor \neg q) \lor (p \lor q)$ DeMorgan

$\equiv (\neg p \lor p) \lor (\neg q \lor q)$ Commutativity and Associativity

$\equiv T \lor T$ Because $\neg p \lor p \equiv T$

$\equiv T$

Prove $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

Note that, by "Substitution for $\to$", we have: RHS = $\neg p \vee (q \wedge r)$.

So, we start from the LHS and try to get this proposition:

$(p \to q) \wedge (p \to r) \equiv (\neg p \vee q) \wedge (\neg p \vee r)$

Substitution for $\to$, twice

$\equiv \neg p \vee (q \wedge r)$

Distribution law

$\equiv p \to (q \wedge r)$

Substitution for $\to$

CONDITIONAL AND BICONDITIONAL STATEMENTS

- **Conditional statements:** "If *p then q.*" $p \to q$
  - The conditional $p \to q$ *is frequently read*
  - *"p implies q" or "p only if q."*
  - The conditional $p \to q$ *is false only when the first part p is true and the second part q is false.*
  - *When p is false, the conditional* $p \to q$ *is true regardless of the truth value of q.*
- **Bi-conditional statements:** *"p if and only if q."*

$$p \leftrightarrow q$$

  - The bi-conditional $p \leftrightarrow q$ *is true whenever p and q have the same truth values and false otherwise.*
  - $A \leftrightarrow B$ is shorthand for $(A \to B) \wedge (B \to A)$.

CONDITIONAL AND BICONDITIONAL STATEMENTS

- ***Conditional Statement*** *: $p \rightarrow q$ is logically equivalent to $\neg p \vee q$; that is, $p \rightarrow q \equiv \neg p \vee q$*
- "If *p then q" is logically equivalent to the statement "Not p or q"*

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $p \leftrightarrow q$

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(c) $\neg p \vee q$

Logical Equivalences

Logical Equivalences Involving Conditional Statements

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing the truth table

- Example: Construct the truth table for

$$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$$

p	q	¬p			$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Rows: all possible combinations of values for elementary propositions: 2^n values

Constructing the truth table

- Example: Construct the truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Constructing the truth table

- Examples: Construct a truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Translating logical formulas to English sentences

- Using the above logic operators, we can construct more complicated logical formulas. (They are called compound propositions.)
- **Example;**

  Proposition p: Alice is smart.

  Proposition q: Alice is honest
- $\neg p \land q$: Alice is not smart but honest.
- $p \lor (\neg p \land q)$: Either Alice is smart, or she is not smart but honest.
- $p \rightarrow \neg q$: If Alice is smart, then she is not honest.

Translating logical formulas from English sentences

- We can also go in the other direction, translating English sentences to logical formulas:

  Proposition p: Alice is smart.

  Proposition q: Alice is honest

- Alice is either smart or honest, but Alice is not honest if she is smart:

  $(p \lor q) \land (p \rightarrow \neg q)$.

- Alice is smart is necessary and sufficient for Alice to be honest:

  $(p \rightarrow q) \land (q \rightarrow p)$.

  (This is often written as $p \leftrightarrow q$).

TRANSLATION

- Assume a sentence: If you are older than 13 or you are with your parents then you can attend a PG-13 movie.
- If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie)
- Propositions: –

  A= you are older than 13

  B= you are with your parents

  C=you can attend a PG-13 movie
- Translation: $A \vee B \rightarrow C$

TRANSLATION :GENERAL RULE

- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.
- **Step 1:** find logical connectives : if, and

  You can have free coffee **if** you are senior citizen **and** it is a Tuesday
- **Step 2:** break the sentence into elementary propositions

  You can have free coffee (a) **if** you are senior citizen (b) **and** it is a Tuesday (c)
- **Step 3:** rewrite the sentence in propositional logic

$$b \wedge c \rightarrow a$$

- Assume two elementary statements:

  p: you drive over 65 mph ;

  q: you get a speeding ticket

- **Translate each of these sentences to logic**
  - you do not drive over 65 mph. $(\neg p)$
  - you drive over 65 mph, but you don't get a speeding ticket. $(p \wedge \neg q)$
  - you will get a speeding ticket if you drive over 65 mph. $(p \rightarrow q)$
  - if you do not drive over 65 mph then you will not get a speeding ticket. $(\neg p \rightarrow \neg q)$
  - driving over 65 mph is sufficient for getting a speeding ticket. $(p \rightarrow q)$
  - you get a speeding ticket, but you do not drive over 65 mph. $\mathbf{(q \wedge \neg p)}$

ARGUMENTS

- An *argument is an assertion that a given set of propositions P1, P2, . . . , Pn, called* ***premises****, yields (has a* consequence) another proposition *Q, called the* ***conclusion***.
- *Such an argument is denoted by P1, P2, . . . , Pn |- Q*

- An argument *P1, P2, . . . , Pn |- Q is said to be valid if Q is true whenever all the premises*
- *P1, P2, . . . , Pn are true.*
- An argument which is not valid is called ***fallacy***.

- The argument *P1, P2, . . . , Pn* |- *Q is valid if and only if the proposition* $(P1 \wedge P2 \ldots \wedge Pn) \rightarrow Q$ is a tautology.

Example: "If *p implies q and q implies r, then p implies r*"

- That is, the following argument is valid:
- $p \rightarrow q, q \rightarrow r$ |- $p \rightarrow r$ ***(Law of Syllogism)***

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

EXAMPLE:

S1 : If a man is a bachelor, he is unhappy.

S2 : If a man is unhappy, he dies young.

S : Bachelors die young

- $p \rightarrow q, q \rightarrow r \mid\!- \; p \rightarrow r$
- where *p is "He is a bachelor," q is "He is unhappy" and r is "He dies young;"*
- By Law of Syllogism the above argument is valid.

PROPOSITIONAL FUNCTIONS, QUANTIFIERS

- *A propositional function (or an open sentence or condition) defined on A is an expression*

  p(x) which has the property that *p(a) is true or false for each a* $\in$ *A. That is, p(x) becomes a statement (with a truth* value) whenever any element *a* $\in$ *A is substituted for the variable x.*
- *The set A is called the domain of p(x), and* the set *Tp of all elements of A for which p(a) is true is called the truth set of p(x).*
- *In other words,*

  $Tp = \{x \mid x \in A, p(x) \text{ is true}\}$ or $Tp = \{x \mid p(x)\}$

Universal Quantifier

- Let *p(x) be a propositional function defined on a set A. Consider the expression*

  $(\forall x \in A)p(x)$ *or* $\forall x\, p(x)$ which reads

  "For every *x in A, p(x) is a true statement" or, simply, "For all x, p(x)."*

- *The symbol* ∀ which reads "for all" or "for every" is called the *universal quantifier*.

Existential Quantifier

- Let *p(x) be a propositional function defined on a set A. Consider the expression*

  $(\exists x \in A)p(x)$ *or* $\exists x,\ p(x)$ which reads

"There exists an *x in A such that p(x) is a true statement" or, simply, "For some x, p(x)."*

- *The symbol* ∃ which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*

Duality Principle

- Duality principle states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true.
- If dual of any statement is the statement itself, it is said **self-dual** statement.
- Example – The dual of $(A \cap B) \cup C$ is $(A \cup B) \cap C$

Normal forms

- Due to the problem of finding whether a given statement is a tautology or contradiction or satisfiable in a number of steps, is called **Decision Making.**
- For decision problems, the construction of truth tables may not be practical always. So we consider alternate way known as **reduction to normal forms.**
- **2 normal forms are:**

-- Disjunctive Normal Form (DNF)

-- Conjunctive Normal Form (CNF)

Disjunctive Normal Form (DNF)

- A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs.
- In terms of set operations, it is a compound statement obtained by Union among variables connected with Intersections.
- DNF is an $\vee$ of $\wedge$s; an $\wedge$ of literals is called a term.
- Example: $(A \wedge B) \vee (A \wedge C) \vee (B \wedge C \wedge D)$

  $(P \cap Q) \cup (Q \cap R)$

Conjunctive Normal Form (CNF)

- A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs.
- In terms of set operations, it is a compound statement obtained by Intersection among variables connected with Unions.
- CNF is an ∧ of ∨s, where ∨ is over variables or their negations (literals); an ∨ of literals is also called a clause.
- Example: $(A \lor B) \land (A \lor C) \land (B \lor C \lor D)$

  $(P \cup Q) \cap (Q \cup R)$

From truth table to DNF and CNF

- A minterm is a conjunction of literals in which each variable is represented exactly once
- If a Boolean function (truth table) has the variables (p,q,r)then $p \wedge \neg q \wedge r$ is a minterm but $p \wedge \neg q$ is not.
- Each minterm is true for exactly one assignment. $p \wedge \neg q \wedge r$ is true if p is true (1) , q is false (0) and r is true (1).
- Any deviation from this assignment would make this particular minterm false.
- A disjunction of minterms is true only if at least one of its constituents minterms is true.

FROM TRUTH TABLE TO DNF

- If a function, e.g. F, is given by a truth table, we know exactly for which assignments it is true.
- Consequently, we can select the minterms that make the function true and form the disjunction of these minterms.
- F is true for three assignments:
- p, q, r are all true, $(p \wedge q \wedge r)$
- p, ¬q, r are all true, $(p \wedge \neg q \wedge r)$
- ¬p, ¬q, r are all true, $(\neg p \wedge \neg q \wedge r)$
- DNF of F:
- $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

p	q	r	F
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

FROM TRUTH TABLE TO CNF

- Complementation can be used to obtain conjunctive normal forms from truth tables.
- If A is a formula containing only the connectives $\neg$, $\vee$ and $\wedge$, then its complement is formed by

  replacing all $\vee$ by $\wedge$

  replacing all $\wedge$ by $\vee$

  replacing all atoms by their complements.
- The complement of q is $\neg$ q
- The complement of $\neg$ q is q
- Example: Find the complement of the formula
  - $(p \wedge q) \vee \neg r$ $(\neg p \vee \neg q) \wedge r$

From truth table to CNF

- Solution: $\neg G$ is true for the following assignments.
- $p = 1; q = 0; r = 1$
- $p = 1; q = 0; r = 0$
- $p = 0; q = 0; r = 1$
- The DNF of $\neg G$ is therefore:

  $(p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r$
- The formula has the complement:

  $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r)$

  It is the desired CNF of G

p	q	r	G
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	1

Transformation into Conjunctive Normal Form

- A literal is either a propositional variable, or the negation of one.

  Examples: p, $\neg p$.
- A clause is a disjunction of literals.

  Example: $p \vee \neg q \vee r$.
- A formula in conjunctive normal form (CNF) is a conjunction of clauses.

  Example: $(p \vee \neg q \vee r) \wedge (\neg p \vee \neg r)$.
- Similarly, one defines formulae in disjunctive normal form (DNF) by swapping

Transform the following formula into CNF. $\neg(P \rightarrow Q) \vee (R \rightarrow P)$

- Express implication by disjunction and negation.

  $\neg(\neg p \vee q) \vee (\neg r \vee p)$
- Push negation inwards by De Morgan's laws and double negation.

  $(p \wedge \neg q) \vee (\neg r \vee p)$
- Convert to CNF by associative and distributive laws.

  $(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$
- Optionally simplify by commutative and idempotent laws.

  $(p \vee \neg r) \wedge (\neg q \vee \neg r \vee p)$
- By commutative and absorbtion laws

  $(p \vee \neg r)$

EXERCISE:

Q1. Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements:

(a) $\neg p$ (b) $p \wedge q$ (c) $p \vee q$ (d) $q \vee \neg p$.

Q2. Verify that the proposition $p \vee \neg (p \wedge q)$ is a tautology.

Q3. Show that the propositions $\neg(p \wedge q)$ and

$\neg p \vee \neg q$ are logically equivalent.

Q4. Check if its true or not

$\neg(p \wedge q) \vee (\neg p \wedge q) \equiv \neg p$.

Q5. Consider the conditional proposition $p \rightarrow q$. The simple propositions $q \rightarrow p$, $\neg p \rightarrow \neg q$ and $\neg q \rightarrow \neg p$

are called, respectively, the converse, inverse, and contrapositive of the conditional $p \rightarrow q$.

Which if any of these propositions are logically equivalent to $p \rightarrow q$?

EXERCISE:

Q6. Test the validity of the following argument:

If I study, then I will not fail mathematics.

If I do not play basketball, then I will study.

But I failed mathematics.

SO,

Therefore I must have played basketball

Q7. Prove the following argument is valid: $p \rightarrow \neg q$, $r \rightarrow q$, r $\vdash$ $\neg p$.

SOLUTION

A1. (a) It is not cold. (c) It is cold or it is raining.

(b) It is cold and raining. (d) It is raining or it is not cold.

A2. YES

A3. YES (F T T T)

A4. False

A5. Only the contra-positive $\neg q \rightarrow \neg p$ is logically equivalent to the original conditional proposition $p \rightarrow q$.

p	q	$\neg p$	$\neg q$	Conditional $p \rightarrow q$	Converse $q \rightarrow p$	Inverse $\neg p \rightarrow \neg q$	Contrapositive $\neg q \rightarrow \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

A6. Let p be “I study,” q be “I failed mathematics,” and r be “I play basketball.” The argument has the form: $p \to \neg q$, $\neg r \to p$, $q \mid\!- r$.

Construct the truth, where the premises $p \to \neg q$, $\neg r \to p$, and q are true simultaneously only in the fifth line of the table, and in that case the conclusion r is also true. Hence the argument is valid.

A7. Here, *$p \to \neg q$, $r \to q$, and r are*

true simultaneously only in the fifth row of the table, where *$\neg p$ is also true. Hence the argument is valid.*

p	q	r	$\neg q$	$p \to \neg q$	$\neg r$	$\neg r \to p$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	F
F	F	T	T	T	F	T
F	F	F	T	T	T	F

	p	q	r	$p \to \neg q$	$r \to q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T