Complex Analysis

MATH 463
HW #5

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Due February 10, 2026

Problem 1

Problem: Find the first few terms in the Laurent expansion of $\frac{1}{z^2(e^z - e^{-z})}$ valid for $0 < |z| < \pi$.

Solution.

Step 1: Identify singularities and region of validity.

The function is:

$$f(z) = \frac{1}{z^2(e^z - e^{-z})}$$

Singularities:

- $z = 0$ (from the z^2 factor, this is where we expand)
- Points where $e^z - e^{-z} = 0$, i.e., $e^z = e^{-z}$, giving $e^{2z} = 1$

From $e^{2z} = 1$, we get $2z = 2\pi ik$ for $k \in \mathbb{Z}$, so $z = \pi ik$.
The nearest non-zero singularities are at $z = \pm\pi i$ with $|z| = \pi$.
Therefore, the Laurent expansion is valid for $0 < |z| < \pi$.

Step 2: Express using hyperbolic sine.

Recall that $\sinh(z) = \frac{e^z - e^{-z}}{2}$, so:

$$e^z - e^{-z} = 2\sinh(z)$$

Therefore:

$$f(z) = \frac{1}{z^2 \cdot 2\sinh(z)} = \frac{1}{2z^2\sinh(z)}$$

Step 3: Find the Taylor series for $\sinh(z)$.

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$= z + \frac{z^3}{6} + \frac{z^5}{120} + \frac{z^7}{5040} + \cdots$$

Step 4: Factor out z from $\sinh(z)$.

$$\sinh(z) = z\left(1 + \frac{z^2}{6} + \frac{z^4}{120} + \frac{z^6}{5040} + \cdots\right)$$

Let $g(z) = 1 + \frac{z^2}{6} + \frac{z^4}{120} + \frac{z^6}{5040} + \cdots$
Then:

$$f(z) = \frac{1}{2z^2 \cdot z \cdot g(z)} = \frac{1}{2z^3 \cdot g(z)} = \frac{1}{2z^3} \cdot \frac{1}{g(z)}$$

Step 5: Expand $\frac{1}{g(z)}$ using geometric series.

Since $g(0) = 1$ and $g(z) = 1 + h(z)$ where $h(z) = \frac{z^2}{6} + \frac{z^4}{120} + \cdots$, we have for $|h(z)| < 1$:

$$\frac{1}{g(z)} = \frac{1}{1 + h(z)} = 1 - h(z) + h(z)^2 - h(z)^3 + \cdots$$

Computing term by term:
First order terms:

$$-h(z) = -\frac{z^2}{6} - \frac{z^4}{120} - \frac{z^6}{5040} - \cdots$$

Second order terms:

$$h(z)^2 = \left(\frac{z^2}{6} + \frac{z^4}{120} + \cdots\right)^2 = \frac{z^4}{36} + 2 \cdot \frac{z^2}{6} \cdot \frac{z^4}{120} + \cdots = \frac{z^4}{36} + O(z^6)$$

Coefficient of z^4:
From $-h(z)$: $-\frac{1}{120}$
From $h(z)^2$: $+\frac{1}{36}$
Total: $\frac{1}{36} - \frac{1}{120} = \frac{10}{360} - \frac{3}{360} = \frac{7}{360}$
Therefore:

$$\frac{1}{g(z)} = 1 - \frac{z^2}{6} + \frac{7z^4}{360} + O(z^6)$$

Step 6: Multiply by $\frac{1}{2z^3}$.

$$f(z) = \frac{1}{2z^3}\left(1 - \frac{z^2}{6} + \frac{7z^4}{360} + O(z^6)\right)$$

$$= \frac{1}{2z^3} - \frac{z^2}{12z^3} + \frac{7z^4}{720z^3} + O(z^3)$$

$$= \frac{1}{2z^3} - \frac{1}{12z} + \frac{7z}{720} + O(z^3)$$

Final Answer:

$$\boxed{\frac{1}{z^2(e^z - e^{-z})} = \frac{1}{2z^3} - \frac{1}{12z} + \frac{7z}{720} + O(z^3)}$$

valid for $0 < |z| < \pi$.

The Laurent expansion has principal part $\frac{1}{2z^3} - \frac{1}{12z}$ and analytic part $\frac{7z}{720} + \cdots$ $\square$

Problem 2

Problem: Use the argument principle to find (geometrically) the number of zeros of $z^3 - z^2 + 3z + 5$ in the right half plane.

Solution.

Strategy:

By the argument principle, the number of zeros minus the number of poles of $p(z) = z^3 - z^2 + 3z + 5$ inside a contour equals:

$$N - P = \frac{1}{2\pi i} \int_\Gamma \frac{p'(z)}{p(z)} dz = \frac{1}{2\pi} \Delta_\Gamma \arg(p(z))$$

where $\Delta_\Gamma \arg(p(z))$ is the change in argument of $p(z)$ around Γ.

Since $p(z)$ is a polynomial, it has no poles, so $N = \frac{1}{2\pi} \Delta_\Gamma \arg(p(z))$.

Contour Selection:

We choose a contour that encloses the right half plane. Consider the semicircular contour:

- Γ_R: semicircle of radius R in the right half-plane from $-iR$ to iR (counterclockwise)
- Γ_I: the imaginary axis from iR to $-iR$ (downward)

As $R \to \infty$, this contour encloses the entire right half-plane.

Analysis on the semicircle Γ_R as $R \to \infty$:

For large $|z|$, the polynomial is dominated by its leading term:

$$p(z) = z^3 - z^2 + 3z + 5 = z^3 \left(1 - \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3}\right)$$

For $z = Re^{i\theta}$ with $\theta \in [-\pi/2, \pi/2]$ (right half-plane):

$$p(Re^{i\theta}) \approx R^3 e^{3i\theta} \text{ for large } R$$

As θ varies from $-\pi/2$ to $\pi/2$, the argument of $p(Re^{i\theta})$ changes by:

$$\Delta_{\Gamma_R} \arg = 3 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 3\pi$$

Analysis on the imaginary axis Γ_I:

We parametrize the imaginary axis as $z = it$ for $t \in \mathbb{R}$.

$$p(it) = (it)^3 - (it)^2 + 3(it) + 5 = -it^3 + t^2 + 3it + 5 = (t^2 + 5) + i(3t - t^3)$$

So:

- $\text{Re}(p(it)) = t^2 + 5 > 0$ for all t

- $\text{Im}(p(it)) = 3t - t^3 = t(3 - t^2)$

Since the real part is always positive, $p(it)$ never crosses the negative real axis, and the argument varies continuously.

Finding where Im$(p(it)) = 0$:

$$t(3 - t^2) = 0 \implies t = 0, \pm\sqrt{3}$$

Sign analysis:

- For $t > \sqrt{3}$: $\text{Im}(p(it)) < 0$ (fourth quadrant)
- For $0 < t < \sqrt{3}$: $\text{Im}(p(it)) > 0$ (first quadrant)
- For $-\sqrt{3} < t < 0$: $\text{Im}(p(it)) < 0$ (fourth quadrant)
- For $t < -\sqrt{3}$: $\text{Im}(p(it)) > 0$ (first quadrant)

Evaluating argument at key points:
As $t \to +\infty$: $p(it) \approx -it^3$, so $\arg(p(it)) \to -\pi/2$
As $t \to -\infty$: $p(it) \approx -it^3 = i|t|^3$, so $\arg(p(it)) \to \pi/2$
Argument change along Γ_I (from $t = +R$ to $t = -R$):
At $t = R$: $\arg(p(iR)) \approx -\pi/2$
At $t = -R$: $\arg(p(-iR)) \approx \pi/2$
Change: $\pi/2 - (-\pi/2) = \pi$
Since we traverse Γ_I downward (from iR to $-iR$), the argument change is:

$$\Delta_{\Gamma_I} \arg = -\pi$$

Total argument change:

$$\Delta_\Gamma \arg(p(z)) = \Delta_{\Gamma_R} + \Delta_{\Gamma_I} = 3\pi + (-\pi) = 2\pi$$

Number of zeros:

$$N = \frac{\Delta_\Gamma \arg(p(z))}{2\pi} = \frac{2\pi}{2\pi} = 1$$

Conclusion: There is exactly $\boxed{1}$ zero of $z^3 - z^2 + 3z + 5$ in the right half-plane. □

Problem 3

Problem: Evaluate $\int_\gamma \frac{\log z}{1+e^z} dz$ along the path γ indicated in Figure 4.2.2.

Solution.

Step 1: Understand the contour.
Following the arrows in Figure 4.2.2, the path γ is clockwise and forms a rectangular annulus (dogbone shape):

- Outer rectangle: corners at $-5+10i$, $10+10i$, $10-2i$, $-5-2i$
- Inner rectangle (hole): corners at $-5+5i$, $5+5i$, $5-5i$, $-5-5i$

The contour encloses the region between these two rectangles.
Step 2: Identify singularities.
The integrand is $f(z) = \frac{\log z}{1+e^z}$ where we use the principal branch of the logarithm (branch cut along the negative real axis).
Poles from the denominator: $1+e^z = 0 \implies e^z = -1$
This gives $z = i\pi(2k+1)$ for $k \in \mathbb{Z}$.
Relevant poles:

- $k = 1$: $z_1 = 3\pi i \approx 9.42i$
- $k = 0$: $z_0 = \pi i \approx 3.14i$
- $k = -1$: $z_{-1} = -\pi i \approx -3.14i$

Step 3: Determine which poles are enclosed.
The annular region consists of points in the outer rectangle but not in the inner rectangle.
Upper region: $5 < \mathrm{Im}(z) < 10$

- $z_1 = 3\pi i \approx 9.42i$ is enclosed ✓

Inner hole: $-5 < \mathrm{Im}(z) < 5$ (with $-5 < \mathrm{Re}(z) < 5$)

- $z_0 = \pi i \approx 3.14i$ is in the hole, not enclosed

Lower region: $-5 < \mathrm{Im}(z) < -2$

- $z_{-1} = -\pi i \approx -3.14i$ is enclosed ✓

Enclosed poles: $3\pi i$ and $-\pi i$
Step 4: Compute residues.
For a simple pole at z_k where $e^{z_k} = -1$:

$$\mathrm{Res}\left(\frac{\log z}{1+e^z}, z_k\right) = \frac{\log z_k}{(1+e^z)'|_{z=z_k}} = \frac{\log z_k}{e^{z_k}} = \frac{\log z_k}{-1} = -\log z_k$$

At $z_1 = 3\pi i$:

$$\log(3\pi i) = \ln|3\pi i| + i\arg(3\pi i) = \ln(3\pi) + i\frac{\pi}{2}$$

$$\text{Res}(f, 3\pi i) = -\ln(3\pi) - i\frac{\pi}{2}$$

At $z_{-1} = -\pi i$:

$$\log(-\pi i) = \ln|-\pi i| + i\arg(-\pi i) = \ln(\pi) + i\left(-\frac{\pi}{2}\right) = \ln(\pi) - i\frac{\pi}{2}$$

$$\text{Res}(f, -\pi i) = -\ln(\pi) + i\frac{\pi}{2}$$

Step 5: Apply Residue Theorem.

Sum of residues:

$$\begin{aligned}\sum \text{Res} &= \left(-\ln(3\pi) - i\frac{\pi}{2}\right) + \left(-\ln(\pi) + i\frac{\pi}{2}\right) \\ &= -\ln(3\pi) - \ln(\pi) \\ &= -\ln(3\pi \cdot \pi) \\ &= -\ln(3\pi^2)\end{aligned}$$

Since γ is traversed clockwise (negative orientation):

$$\int_\gamma \frac{\log z}{1+e^z} dz = -2\pi i \cdot (-\ln(3\pi^2)) = \boxed{2\pi i \ln(3\pi^2)}$$

□

Problem 4

Problem: Let f be as in Theorem 4.3.3. Use the formula for f^{-1} derived therein to show that f^{-1} is analytic on $f(\Omega)$. (Show that f^{-1} is representable in $f(\Omega)$ by power series.)

Solution.

Given Information:

Theorem 4.3.3 states: Let f and g be analytic on Ω and assume that f is one-to-one. Then for each $z_0 \in \Omega$ and each r such that $\overline{D}(z_0, r) \subseteq \Omega$, we have:

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{C(z_0,r)} z \frac{f'(z)}{f(z) - w} dz$$

for every $w \in f(D(z_0, r))$.

Proof that f^{-1} is analytic on $f(\Omega)$:

Let $w_0 \in f(\Omega)$ be arbitrary. We will show that f^{-1} is representable by a power series in a neighborhood of w_0.

Let $z_0 = f^{-1}(w_0) \in \Omega$. Since Ω is open, there exists $r > 0$ such that $\overline{D}(z_0, r) \subseteq \Omega$.

Step 1: Apply the formula from Theorem 4.3.3.

For any $w \in f(D(z_0, r))$:

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{C(z_0,r)} z \frac{f'(z)}{f(z) - w} dz$$

Step 2: Rewrite the integrand.

For $z \in C(z_0, r)$ and w near w_0, write:

$$\frac{1}{f(z) - w} = \frac{1}{f(z) - w_0 - (w - w_0)} = \frac{1}{f(z) - w_0} \cdot \frac{1}{1 - \frac{w - w_0}{f(z) - w_0}}$$

Step 3: Use geometric series expansion.

Let $\delta = \min_{z \in C(z_0,r)} |f(z) - w_0|$. Note that $\delta > 0$ because:

- $C(z_0, r)$ is compact
- $|f(z) - w_0|$ is continuous on $C(z_0, r)$
- $f(z) \neq w_0$ for all $z \in C(z_0, r)$ (since f is one-to-one and $f(z_0) = w_0$ but $z_0 \notin C(z_0, r)$)

For w satisfying $|w - w_0| < \delta$, we have $\left| \frac{w - w_0}{f(z) - w_0} \right| < 1$ for all $z \in C(z_0, r)$.

Therefore:

$$\frac{1}{f(z) - w} = \frac{1}{f(z) - w_0} \sum_{n=0}^{\infty} \left(\frac{w - w_0}{f(z) - w_0} \right)^n$$

This series converges uniformly in z on $C(z_0, r)$ for fixed w with $|w - w_0| < \delta$.

Step 4: Substitute into the integral.

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{C(z_0,r)} z f'(z) \cdot \frac{1}{f(z) - w_0} \sum_{n=0}^{\infty} \left(\frac{w - w_0}{f(z) - w_0} \right)^n dz$$

By uniform convergence, we can interchange the sum and integral:

$$f^{-1}(w) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (w - w_0)^n \int_{C(z_0,r)} \frac{z f'(z)}{(f(z) - w_0)^{n+1}} dz$$

Step 5: Define power series coefficients.
Let:

$$a_n = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{z f'(z)}{(f(z) - w_0)^{n+1}} dz$$

These coefficients are well-defined complex numbers since:

- $z f'(z)$ is analytic on Ω (product of analytic functions)
- $f(z) \neq w_0$ for $z \in C(z_0, r)$ (since f is one-to-one and $f(z_0) = w_0$)
- The integral exists for each fixed n

Step 6: Verify the power series representation.
We have shown that:

$$f^{-1}(w) = \sum_{n=0}^{\infty} a_n (w - w_0)^n$$

where this power series converges for $|w - w_0| < \delta$ with $\delta = \min_{z \in C(z_0,r)} |f(z) - w_0| > 0$.

Since the series converges in a neighborhood $D(w_0, \delta) \subseteq f(\Omega)$, the function f^{-1} is representable by a power series at w_0.

Step 7: Conclusion.

Since $w_0 \in f(\Omega)$ was arbitrary, f^{-1} is representable by a power series at every point in $f(\Omega)$.

Therefore, f^{-1} is analytic on $f(\Omega)$. □