(2)

For some normal distribution $N(0,1)$

$$M.G.F.\ (\phi) = e^{t^2/2}$$
$$= 1 + \frac{t^2}{2\ 1!} + \frac{t^4}{2^2\ 2!} + \frac{t^6}{2^3\ 3!} \cdots\cdots$$

From this we can deduce that

$$E(x^n) = \begin{cases} 0 & n \text{ is Odd by symmetry in probab} \\ \dfrac{\prod_{0}^{n/2-1}(n-k)}{2^{n/2}\ \cancel{\frac{n}{2}!}} & n \text{ is Even} \end{cases}$$

1a Given

$n \to \infty$

$p \to 0$

$np = \lambda$

$$P(X=x) = {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$$

$$= \frac{\lfloor n}{\lfloor x \, \lfloor n-x}\left(\frac{p}{1-p}\right)^{x}(1-p)^{n}$$

$$= \frac{(n)\times(n-1)\times(n-2)\cdots(n-(x-1))}{x!}\left(\frac{p}{1-p}\right)^{x}(1-p)^{n}$$

$$= \frac{\prod_{k=0}^{k=x-1}(n-k)}{x!}\cdot\left(\frac{p}{1-p}\right)^{x}(1-p)^{n}$$

$$\lim_{n\to\infty}\lim_{p\to 0} P(X=x) = \lim_{n\to\infty}\lim_{p\to 0}\frac{\prod_{k=0}^{x-1}(n-k)}{x!}\cdot\left(\frac{p}{1-p}\right)^{x}(1-p)^{n}$$

{ where $np = \lambda$ }

$$\Rightarrow P(X=x) = \lim_{n\to\infty}\prod_{k=0}^{x-1}(n-k)\lim_{p\to 0}\left(\frac{p}{1-p}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n}$$

Let ~~$\frac{\lambda}{n}$ = A~~

~~Now let $\lim_{p\to 0}\frac{p}{1-p}$~~

$$P(X=x) = \frac{1}{x!} \lim_{n\to\infty} \prod_{k=0}^{x-1} (n-k) \left(1-\frac{\lambda}{n}\right)^n \lim_{p\to 0} \left(\frac{p}{1-p}\right)^x$$

$$= \frac{1}{x!} \lim_{n\to\infty} \prod_{k=0}^{x-1} (n-k) \left(1-\frac{\lambda}{n}\right)^n p^x$$

$$= \frac{\cancel{1}}{x!} \cancel{p^x} \lim_{n\to\infty} \left[\prod_{k=0}^{x-1} x(n-k)\right] \left(1-\frac{\lambda}{n}\right)^n$$

let $\frac{-\lambda}{n} = \frac{1}{t}$

$\Rightarrow n = -t\lambda$ $\quad \lim_{n\to\infty} \Rightarrow \lim_{t\to\infty}$

$$\Rightarrow P(X=x) = \frac{1}{x!} \lim_{n\to\infty} \prod_{k=0}^{x-1} x(n-k) \left[\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t\right]^{-\lambda}$$

As we know $\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t = e$

$$\Rightarrow P(X=x) = \frac{1}{x!} \lim_{n\to\infty} \prod_{k=0}^{x-1} nx\, e^{-\lambda}$$

$k << n$

As $n \to \infty$

$$= \frac{1}{x!} \lim_{n\to\infty} \lambda^x e^{-\lambda}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

Thus If $n \to \infty$ $\quad p \to 0$ $\quad np = \lambda$

Binomial distribution tends to poission's distribution where its ~~P.D.~~ P.D.F. is given by

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

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Further $W = XY + X^2Y + XY^2$
$Z = XY^2 + X^2Y$

Now
$Cov(W,Z) = E(WZ) - E(W)E(Z)$
As X and Y are both normal distribution we can conclude that they are independent as no relation is yet defined.

$$E(W) = E(XY + X^2Y + XY^2)$$
$$= E(XY) + E(X^2Y) + E(XY^2)$$

Now As X and Y are independent
X^m, Y^n are also independent

Considering this fact

$$E(W) = E(X)E(Y) + E(X^2)E(Y) + E(X)E(Y^2)$$
$$= (0 \times 0) + (1 \times 0) + (0 \times 1)$$
$$= 0$$

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Then $Cov(\omega, z) = E(\omega z) - E(\omega)E(z)$

$= E(\omega z) - E(z) \times 0$

$= E(\omega z)$

$= E([x^2y^2][1+x+y][x^2+y^2])$

$= E(x^4y^2 + x^2y^4 + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5)$

Now As each term with a odd power component will have Expected value as 0

$\Rightarrow Cov(\omega, z) = E(x^4)E(y^2) + E(x^2)E(y^4)$

$= \left(\frac{4\times3}{4}\right)\left(\frac{2}{2^1}\right) + \left(\frac{4\times3}{4}\right)\left(\frac{2}{2^1}\right)$

$= 3+3$

$= 6$

$\Rightarrow \boxed{Cov(\omega, z) = 6}$

3 By Chebyshev's inequality

$$P(|\overline{X}-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Now for $P(|\overline{X}-\mu| \geq \varepsilon)$

ε being some constant with $\varepsilon > 0$

Now $\varepsilon = k\sigma/\sqrt{n}$

$$\Rightarrow \frac{\sigma}{\sqrt{n}\varepsilon} = \frac{1}{k}$$

$$\Rightarrow \frac{\sigma^2}{n\varepsilon^2} = \frac{1}{k^2}$$

Now $P(|\overline{X}-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$ By Chebyshev's inequality

$$\Rightarrow P(|\overline{X}-\mu| \geq \varepsilon) \leq \frac{\sum_{i=0}^{n}(x_i-\mu)^2}{Nn\varepsilon^2}$$

$$\Rightarrow P(|\overline{X}-\mu| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\left[\sum_{i=0}^{n}(x_i-\mu)^2 / Nn\right]$$

Now

$$\lim_{n\to\infty} P(|\overline{X}-\mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{1}{\varepsilon^2}\left[\frac{\sum_0^n (x_i-\mu)^2}{Nn}\right]$$

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X}-\mu| \geq \varepsilon) \leq \sigma^2 \lim_{n\to\infty} \frac{1}{\varepsilon^2 n}$$

As σ is not dependent on n but data itself that is population

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X}-\mu| \geq \varepsilon) \leq \sigma^2 . 0$$

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X}-\mu| \geq \varepsilon) \leq 0$$

Hence W.L.L.N is proved

N is size of population

_/_/_

5)

$$f(x,y) = \begin{cases} 1 & 0 \le x < \infty,\ 0 \le y \le e^{-x} \\ 0 & \text{Else} \end{cases}$$

$$f_x(x) = \int_0^{e^{-x}} 1 \cdot dy$$

$$= e^{-x}$$ //

f_y $\quad y \le e^{-x}$

$\Rightarrow x \le -\log y$ — (i)

Also $\infty > x \ge 0$ — (ii)

If $y > 1$

$x \le$ $x < 0$

Contradicting (ii)

Now

$$f_Y(y) = \int_0^{-\log y} 1 \cdot dx$$

$$= -\log y$$

$$f_y(y) \begin{cases} -\log y & y < 1 \\ 0 & y \ge 1 \end{cases}$$

Now $f_y(y) \cdot f_x(x) = -\log y \, e^{-x} \ne 1$ For $0 \le x < \infty$, $0 \le y \le e^{-x}$

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i.e. $f_X(x) \cdot f_Y(y) \neq f_{X,Y}(x,y)$

Hence X and Y are not independent.

Now

$$p(y|x) = \frac{p(x,y)}{p(x)}$$

$$= \frac{p(x,y)}{e^{-x}}$$

$$\Rightarrow p(y|x) \begin{cases} e^{x} & 0 \leq y \leq e^{-x} \\ 0 & e^{-x} < y \text{ or } y < 0 \\ \text{Undefined} & \text{otherwise} \end{cases}$$

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(7) $$E(XY) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} x \, e^{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)} \, dx \, dy$$

$$\Rightarrow E(XY) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} x \, e^{-\frac{1}{2(1-\rho^2)}((x-\rho y)^2 + (1-\rho^2)y^2)} \, dx \, dy$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} y \, e^{-y^2/2} \left[\int_{-\infty}^{\infty} t \, e^{-\frac{t^2}{2(1-\rho^2)}} dt + \int_{-\infty}^{\infty} \rho y \, e^{-\frac{t^2}{2(1-\rho^2)}} dt \right] dy$$

Let $t = x - \rho y$
$dt = dx$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} \rho y^2 e^{-y^2/2} \sqrt{1-\rho^2} \int_{-\infty}^{\infty} e^{-k^2/2} \, dk \, dy$$

$$k = \frac{t}{\sqrt{1-\rho^2}}$$

$$= \frac{\rho(1-\rho^2)^{1/2}}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} \cancel{\theta}\, y^2 e^{-y^2/2} \sqrt{\pi} \, dy$$

$$= \frac{\rho\sqrt{\pi}}{2\pi} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} \, dy$$

$$= \frac{2\sqrt{2}\,\rho\,\cancel{\theta}}{\cancel{\theta} \times 2\sqrt{\pi}} \int_{-\infty}^{\infty} \cancel{l} \left[d\left(\frac{-1}{2} e^{-l^2}\right) \right] dl$$

$$= \frac{\rho\sqrt{2}}{\sqrt{\pi}} \left[\left[\frac{-1}{2} e^{-l^2} \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-l^2} dl \right]$$

$$= \rho \sqrt{\frac{2}{\pi}} \, \frac{1}{2} \sqrt{\pi} \quad \Rightarrow \frac{\cancel{\sqrt{2}\,\rho}}{\cancel{\sqrt{2}}}\; \rho$$

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$$E(x) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dx\, dy$$

$$E(x) = \frac{\rho}{2\sqrt{\pi}} \int_{-\infty}^{\infty} y\, e^{-y^2/2}\, dy$$

$$\Rightarrow E(x) = 0$$

Similarly $E(y) = 0$

$$E(x^2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)}\, dx\, dy$$

$$\Rightarrow E(x^2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (t+\rho y)^2\, e^{-\frac{1}{2(1-\rho^2)}(t^2+(1-\rho^2)y^2)}\, dt\, dy$$

$$t = (x - \rho y)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-y^2/2} \int_{-\infty}^{\infty} t^2\, e^{\frac{-t^2}{2(1-\rho^2)}} + \rho^2 y^2\, e^{\frac{-t^2}{2(1-\rho^2)}}\, dt\, dy$$

$$= \frac{\cancel{(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-y^2/2} \left[\frac{(1-\rho^2)(\sqrt{1-\rho^2})(2\sqrt{2})}{2}\sqrt{\pi} + \rho^2\sqrt{1-\rho^2}\sqrt{2}\sqrt{\pi}\, y^2\right]$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\left[(1-\rho^2) + \rho^2 y^2\right] dy$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\left[(1-\rho^2)\sqrt{2\pi} + \rho^2 \sqrt{2\pi}\right]$$

$$\Rightarrow 2(\cancel{1}+\cancel{\rho^2}) \quad 2$$

$$E(y^2) = 2$$

$$\sigma_x = \sigma_y = \sqrt{2}$$

$$\text{Corr}(X,Y) = \frac{2\rho^2}{2}$$
$$= \rho^2$$

$$E(X^2Y^2) = \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{\infty} y^2 \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)} \, dx \, dy$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2}(1-\rho^2) + y^4 \rho^2 e^{-y^2/2} \, dy$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-\rho^2)\left(\frac{2\sqrt{2}\sqrt{\pi}}{2}\right) + \rho^2 6\sqrt{2\pi} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[1-\rho^2+6\rho^2 \right] \sqrt{2\pi}$$

$$= 2\left[1+5\rho^2\right]$$

$$\text{Cov}(X^2,Y^2) = 2\left[1+5\rho^2\right] - 4$$
$$= 2\left[5\rho^2-2\right]$$