Tutorial - 09

$$f(x) = e^x$$

$x = 0, 0.5, 1.0, 1.5 \; \& \; 2$

$f(x)\Big|_{x=1.80} = ?$

Eqn of cubic spline.

$$q_i(x) = A_i(x - x_{i-1})^3 - B_i(x - x_i)^3 + C_i(x - x_{i-1}) - D_i(x - x_i) \longrightarrow (1)$$

where

$$A_i = \frac{\sigma_i}{6h_i} \qquad B = \frac{\sigma_{i-1}}{6h_i}$$

$\sigma_i = q_i''(x) \Rightarrow$ second order derivative

$h_i = x_i - x_{i-1} \Rightarrow$ difference

$$D_i = \frac{y_{i-1}}{h_i} - \frac{\sigma_{i-1}}{6} h_i$$

$$C_i = \frac{y_i}{h_i} - \frac{\sigma_i}{6} h_i$$

[y_i is function value at x_i]

Apply continuity Boundary condition in eqn (1)

$$\sigma_{i-1} h_i + 2\sigma_i (h_i + h_{i+1}) + \sigma_{i+1} h_{i+1} = 6(g_{i+1} - g_i) \longrightarrow (2)$$

$\hookrightarrow \dfrac{y_i - y_{i-1}}{h_i}$

Eq^n 2 can also be written as

$$\begin{bmatrix} B.C. & & & & \\ h_1 & 2(h_2+h_1) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-1} & 2(h_n+h_{n-1}) & h_n \\ & & & B.C. & \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \vdots \\ \vdots \\ \sigma_{n-1} \end{bmatrix} = 6 \begin{bmatrix} g_2-g_1 \\ \vdots \\ g_n-g_{n-1} \end{bmatrix}$$

i	x	$y = e^x$	$h_i = x_i - x_{i-1}$	$g = \frac{y_i - y_{i-1}}{h_i}$
0	0	1.000		
			0.5	1.297
1	0.5	1.649		
			0.5	2.139
2	1.0	2.718		
			0.5	3.527
3	1.5	4.482		
			0.5	5.815
4	2.0	7.389		

Natural Spline

$$\sigma_0 = 0$$

$$\sigma_n = \sigma_4 = 0$$

$$\begin{bmatrix} 2(h_1+h_2) & h_2 & 0 \\ h_2 & 2(h_2+h_3) & h_3 \\ 0 & h_3 & 2(h_3+h_4) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = 6 \begin{bmatrix} g_2 - g_1 \\ g_3 - g_2 \\ g_4 - g_3 \end{bmatrix}$$

$$\begin{bmatrix} 2.0 & 0.5 & 0 \\ 0.5 & 2.0 & 0.5 \\ 0 & 0.5 & 2.0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} 5.0501 \\ 8.3262 \\ 13.7275 \end{bmatrix}$$

use thomas algorithm or any other method of your convinience

$$\sigma_1 = 2.0062$$

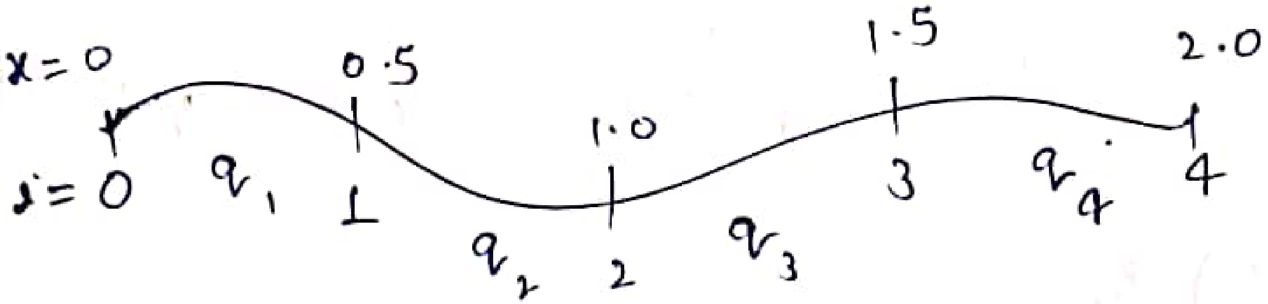
$$\sigma_2 = 2.0753$$

$$\sigma_3 = 6.3449$$

already know

$\sigma_0 = 0$

$\sigma_4 = 0$



$$q_1 = A_1(x_1 - x_0)^3 \pm B_1(x - x_1)^3 + C_1(x - x_0) - D_1(x - x_1)$$

$$q_1 = A_1(x - x_0)^3 - B_1(x - x_1)^3 + C_1(x - x_0) - D_1(x - x_1)$$

$x = 0$

$$A_1 = \frac{\sigma_1}{6h_1} = \frac{2.0062}{6 \times 0.5} = 0.6687$$

$$B_1 = \frac{\sigma_0}{6h_1} = \frac{0}{6 \times 0.5} = 0$$

$$C_1 = \frac{y_1}{h_1} - \frac{\sigma_1}{6} h_1 = \frac{1.649}{0.5} - \frac{2.0062}{6} 0.5$$

$$C_1 = 3.1303$$

$$D_1 = \frac{y_0}{h_1} - \frac{\sigma_0}{6} h_1 = \frac{1}{0.5} - \frac{0}{6} 0.5$$

$$D_1 = 2$$

$$q_1 = A_1(x-0)^3 - B_1(x-0.5)^3 + C_1(x-0) - D_1(x-0.5)$$

$$q_1 = 0.6687x^3 + 3.1303x - 2(x-0.5)$$

Similarly,

$$q_2 = 0.6918(x-0.5)^3 - 0.6687(x-1.0)^3$$
$$+ 5.2636(x-0.5) - 3.1303(x-1.0)$$

$$q_3 = 2.1150(x-1.0)^3 - 0.6918(x-1.5)^3$$
$$+ 8.4346(x-1.0) - 5.2636(x-1.5)$$

$$q_4 = 0(x-1.5)^3 - 2.1150(x-2.0)^3$$
$$+ 14.7781(x-1.5) - 8.4346(x-2.0)$$

$\tilde{y}(x=1.8) = ?$

$x = 1.8$ lies in q_4.

$$\tilde{y}(x=1.8) = -2.1150(1.8-2.0)^3$$
$$+ 14.7781(1.8-1.5) - 8.4346(1.8-2.0)$$

$$\tilde{y}(x=1.8) = 6.13727$$

$$y(x=1.8) = e^{1.8} = 6.0496$$

$$E_a = \left| \frac{\tilde{y} - y}{y} \right| \times 100$$

$$= \left| \frac{6.13727 - 6.0496}{6.0496} \right| \times 100$$

$$E_a = 1.449\,\%$$

not a knot spline $[q_1'''(x_1) = q_2'''(x_1) \rightarrow d_1 = d_2]$

$[q_{n-1}'''(x_{n-1}) = q_n'''(x_{n-1}) \rightarrow d_{n-1} = d_n]$

$$\begin{bmatrix} h_2 & -(h_1+h_2) & h_1 & 0 & 0 \\ h_1 & 2(h_1+h_2) & h_2 & 0 & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & 0 \\ 0 & 0 & h_3 & 2(h_3+h_4) & h_4 \\ 0 & 0 & h_4 & -(h_3+h_4) & h_3 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} = 6 \begin{bmatrix} 0 \\ g_2 - g_1 \\ g_3 - g_2 \\ g_4 - g_3 \\ 0 \end{bmatrix}$$

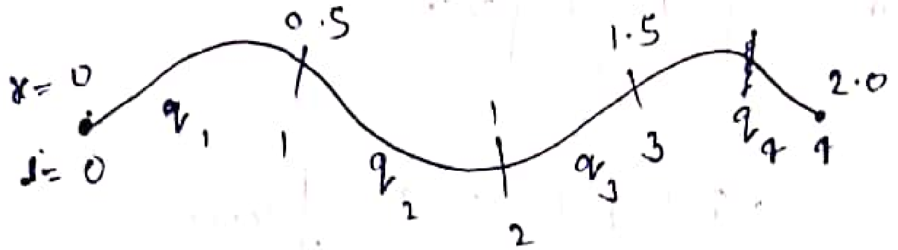
$$\begin{bmatrix} 0.5 & -1.0 & 0.5 & 0 & 0 \\ 0.5 & 2.0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 2.0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 2.0 & 0.5 \\ 0 & 0 & 0.5 & -1.0 & 0.5 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 5.0501 \\ 8.3262 \\ 13.725 \\ 0.0 \end{bmatrix}$$

$$A\sigma = b$$

$$\sigma = A^{-1}b$$ [can be directly calculated]

$$\sigma = \begin{bmatrix} 0.7684 \\ 1.6834 \\ 2.5983 \\ 4.5758 \\ 6.5534 \end{bmatrix}$$

Scanned with CamScanner



$$q_1 = A_1(x-x_0)^3 - B_1(x-x_1)^3 + C_1(x-x_0) - D_1(x-x_1)$$

$$A_1 = \frac{\sigma_1}{6h_1} = \frac{1.6834}{6 \times 0.5} = 0.5611$$

$$B_1 = \frac{\sigma_0}{6h_1} = \frac{0.7684}{6 \times 0.5} = = 0.2561$$

$$C_1 = \frac{y_1}{h_1} - \frac{\sigma_1}{6} h_1 = \frac{1.649}{0.5} - \frac{1.6834}{6} 0.5$$

$$C_1 = 3.1577$$

$$D_1 = \frac{y_0}{h_1} - \frac{\sigma_0}{6} h_1$$

$$D_1 = \frac{1}{0.5} - \frac{0.7684}{6} \times 0.5$$

$$D_1 = 1.936$$

$$q_1 = 0.5611(x)^3 - 0.2561(x-0.5)^3 + 3.1577x - 1.936(x-0.5)$$

Similarly,

$$q_2 = 0.8661(x-0.5)^3 - 0.5611(x-1.0)^3 + 5.22(x-0.5) - 3.1572(x-1.0)$$

$$q_3 = 1.5253(x-1.0)^3 - 0.8661(x-1.5)^3 + 8.5821(x-1.0) - 5.22(x-1.5)$$

$$q_4 = 2.1845(x-1.5)^3 - 1.5253(x-2.0)^3 + 14.2320(x-1.5) - 8.5821(x-2.0)$$

$x = 1.8$ will lie in q_4

$$\tilde{y}(x=1.8) = 2.1845(x-1.5)^3 - 1.5253(x-2.0)^3 + 14.2320(x-1.5) - 8.5821(x-2.0)$$

$$\tilde{y}(x=1.8) = 2.1845(.3)^3 - 1.5253(-.2)^3 + 14.2320(.3) - 8.5821(-.2)$$

$$\tilde{y}(x=1.8) = 6.0572$$

$$y(x=1.8) = 6.0496$$

$$E_a = \left|\frac{6.0496 - 6.0572}{6.0496}\right| \times 100$$

$$E_a = 0.1247\ \%$$

Reason:- Boundary conditions for not a knot spline are more realistic for exponential function compared to a natural spline.