Tutorial - 06

$$d_1 x + by + cz = f_1$$
$$bx + d_2 y + az = f_2$$
$$cx + ay + d_3 z = f_3$$

$d_1 = 4.34 \pm 0.05$

$d_2 = 7.8 \pm 0.10$

$d_3 = 4.2 \pm 0.07$

$b = 2.1 \pm 0.02$

$a = 1.8 \pm 0.01$

$c = -2.4 \pm 0.11$

$f_1 = 87.65 \pm 0.56$

$f_2 = 121.76 \pm 1.80$

$f_3 = -2.0 \pm 0.03$

(1.a) obtain a LU decomposition of the coefficient matrix using cholesky's method by considering the mean values of constants.

$$A = \begin{bmatrix} 4.34 & 2.1 & -2.4 \\ 2.1 & 7.8 & 1.8 \\ -2.4 & 1.8 & 4.2 \end{bmatrix}$$

Coeff. Mat.

(+ve definite Matrix)

Cholesky's Method $A = LL^T$

So we do not need to calculate UTM

diagonal elements of LTM

$$l_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2}$$

off - diagonal elements of matrix

$$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}} \qquad \left[\begin{array}{l} j < i \\ j = 1 \ldots n \\ i = j+1 \ldots n \end{array}\right.$$

$$l_{11} = \sqrt{a_{11}} = \sqrt{4.34} = 2.083$$

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{2.1}{2.083} = 1.008$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{7.8 - (1.008)^2} = 2.604$$

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{-2.4}{2.083} = -1.152$$

$$l_{32} = \frac{a_{32} - l_{31} \times l_{21}}{l_{22}} = \frac{1.8 - (-1.152) \times 1.008}{2.604}$$

$$l_{32} = 1.137$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{4.2 - (-1.152)^2 - (1.137)^2}$$

$$l_{33} = 1.257$$

$$LTM = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$LTM = \begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.604 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix}$$

$$UTM = L^T$$

$$UTM = \begin{bmatrix} 2.083 & 1.008 & -1.152 \\ 0 & 2.604 & 1.137 \\ 0 & 0 & 1.257 \end{bmatrix}$$

(b.) obtain solution vector using the LU decomposition of part (a) & the mean values for constants in the right hand side vector

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

Coeff. Mat. — Solution vector

$\underline{\underline{A}} = LU$

$$LUx = b$$

$$Ux = y \longrightarrow (2.)$$

$$\boxed{Ly = b} \longrightarrow ~~A~~ (1.)$$

$$\begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.604 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 87.65 \\ 121.76 \\ -2.0 \end{bmatrix}$$

forward Substitution

$R_1 \longrightarrow \quad 2.083 y_1 = 87.65$

$y_1 = 42.073$

$R_2 \rightarrow \quad y_2 = 30.465$

$R_3 \rightarrow \quad y_3 = 9.413$

Now, y_1, y_2 & y_3 are known

we $\quad Ux = y \quad$ where $U = L^T$

$$\begin{bmatrix} 2.083 & 1.008 & -1.152 \\ 0 & 2.604 & 1.137 \\ 0 & 0 & 1.257 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 42.073 \\ 30.465 \\ 9.413 \end{bmatrix}$$

Backward substitution

$$R_3 \rightarrow \quad z = 9.413 / 1.257$$

$$z = 7.488$$

$$R_2 \rightarrow \quad y = 8.428$$

$$R_1 \rightarrow \quad x = 20.259$$

Solution Vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20.259 \\ 8.428 \\ 7.488 \end{bmatrix}$

(c.) Compute the inverse of coefficient matrix

$$AA^{-1} = I$$

↳ identity matrix

$$LUA^{-1} = I$$

Assume $UA^{-1} = B$

$$LB = I$$

$$\begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.605 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} (i.) & (ii) & (iii) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solve one by one with column vectors

(i.)

$$\Rightarrow \begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.605 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

forward Substitution $[R_1 \Rightarrow R_2 \Rightarrow R_3]$

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.480 \\ -0.185 \\ 0.607 \end{bmatrix}$$

(ii)
$$\begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.605 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

use forward substitution

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.383 \\ -0.347 \end{bmatrix}$$

(iii)
$$\begin{bmatrix} 2.083 & 0 & 0 \\ 1.008 & 2.605 & 0 \\ -1.152 & 1.137 & 1.257 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

use forward substitution

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.795 \end{bmatrix}$$

Now all the b's are known hence

$$B = \begin{bmatrix} 0.480 & 0 & 0 \\ -0.185 & 0.383 & 0 \\ 0.607 & -0.347 & 0.795 \end{bmatrix}$$

$$AA^{-1} = I$$

$$\underbrace{LUA^{-1}}_{B} = I$$

$$LB = I$$

$\hookrightarrow$ known

$$\boxed{UA^{-1} = B}$$

$$\begin{bmatrix} 2.083 & 1.008 & -1.152 \\ 0 & 2.605 & 1.137 \\ 0 & 0 & 1.257 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{array}{ccc} (i) & (ii) & (iii) \\ \end{array} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

known

Solve one by one with 1 column vector

(i)

$$\begin{bmatrix} 2.083 & 1.008 & -1.152 \\ 0 & 2.605 & 1.137 \\ 0 & 0 & 1.257 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.480 \\ -0.180 \\ 0.607 \end{bmatrix}$$

Backward Substitution

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 0.634 \\ -0.282 \\ 0.483 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 2.083 & 1.008 & -1.152 \\ 0 & 2.605 & 1.132 \\ 0 & 0 & 1.257 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.383 \\ -0.347 \end{bmatrix}$$

Backward Substitution

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -0.282 \\ 0.268 \\ -0.276 \end{bmatrix}$$

Similary for 3rd column

$$\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0.484 \\ -0.276 \\ 0.633 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.635 & -0.282 & 0.484 \\ -0.282 & 0.268 & -0.276 \\ 0.484 & -0.276 & 0.633 \end{bmatrix}$$

(d) derive an analytical expression for the maximun norm of relative error in the solution vector for small perturbations in both, coeff matrix & the right hand side vectors.

$$Ax = b$$

with small perturbations

$$(A + \delta A)(x + \delta x) = (b + \delta b)$$

$$\cancel{Ax} + \delta A x + A\delta x + \delta A \delta x = \cancel{b} + \delta b$$

$$A\delta x = \delta b - \delta A \delta x - \delta A x$$

$$\underbrace{\delta x}_{\text{error}} = -A^{-1}\delta A x + A^{-1}\delta b - A^{-1}\delta A \delta x$$

take the norm of error

$$\|\delta x\| \leq \|A^{-1}\| \, \|\delta A\| \, \|x\| + \|A^{-1}\| \, \|\delta b\|$$

$$+ \|A^{-1}\| \, \|\delta A\| \, \|\delta x\|$$

for relative error in norm

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\| \, \|\delta x\|}{\|A\| \, \|x\|} \right]$$

At this point one can ignore the product of two errors w.r.t. other two terms

$$\|A\| \cdot \|A^{-1}\| = C_p(A)$$

↳ defined as condition Number

$$\frac{\|\delta x\|}{\|x\|} = C_p(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right]$$

(1.e.) Obtain the maximum norm of the relative error in the solution vector for one standard deviation perturbations in all the constants of the set of equations.

⇒ use column sum norm for matrices & L_∞ norm for vectors

column sum norm ⇒

$$\|A\| = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$A = \begin{bmatrix} 4.34 & 2.1 & -2.4 \\ 2.1 & 7.8 & 1.8 \\ -2.4 & 1.8 & 4.2 \end{bmatrix}$$

column sum $\Rightarrow$ 8.84   11.7   8.4

maximum value = norm

$$\boxed{\|A\| = 11.7}$$

$$A^{-1} = \begin{bmatrix} 0.635 & -0.282 & 0.484 \\ -0.282 & 0.268 & -0.276 \\ 0.484 & -0.276 & 0.633 \end{bmatrix}$$

column sum $\Rightarrow$ 1.400   0.826   1.39

maximum value = norm

$$\|A^{-1}\| = 1.4$$

$$\delta A = \begin{bmatrix} 0.05 & 0.02 & 0.11 \\ 0.02 & 0.1 & 0.01 \\ 0.11 & 0.01 & 0.07 \end{bmatrix}$$

colum sum   0.18   0.13   0.19

maximum value = norm

$$\|\delta A\| = 0.19$$

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$$b = \begin{bmatrix} 87.65 \\ 121.76 \\ -2 \end{bmatrix} \qquad \delta b = \begin{bmatrix} 0.56 \\ 1.8 \\ 0.03 \end{bmatrix}$$

L_∞ norm of a vector = maximum absolute value in the vector

$\|b\| = 121.76$ $\qquad$ $\|\delta b\| = 1.8$

$\|A\| = 11.7$

$\|A^{-1}\| = 1.4$

$\|\delta A\| = 0.19$

$$\varphi(A) = \|A\|\,\|A^{-1}\|$$

$$= 11.7 \times 1.4$$

$$= 16.38$$

$$\frac{\|\delta x\|}{\|x\|} \leq \varphi(A)\left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right]$$

$$\frac{\|\delta x\|}{\|x\|} \leq 16.38\left(\frac{0.19}{11.7} + \frac{1.8}{121.76}\right)$$

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq 0.508}$$

Answer - 02

Use Thomas Algorithm to solve the given system of equation.

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -2 \end{bmatrix}$$

i	l	d	u	b	λ	α	β	x
index	elements below diagonal	diagonal element	upper diagonal elements	right side column vector				

$\alpha_1 = d_1 \qquad \beta_1 = b_1$

$\cancel{\beta_i = d_i - \lambda_i u_{i-1}}$

$\alpha_i = d_i - \lambda_i u_{i-1}$

$\beta_i = b_i - \lambda_i \beta_{i-1}$

$\lambda_i = \dfrac{l_i}{\alpha_{i-1}}$

↓

multiplying factor

$$x_i = \frac{\beta_i - u_i x_{i+1}}{\alpha_i}$$

$$x_n = \beta_n / \alpha_n$$

i	l	d	u	b	α
1		-2	1	3	$\alpha_1 = d_1 = -2$
2	1	-4	1	1	$\alpha_2 = d_2 - \left(\frac{l_2}{\alpha_1}\right)u_1 = -3.5$
3	1	-4	1	2	$\alpha_3 = d_3 - \left(\frac{l_3}{\alpha_2}\right)u_2 = -4 - \left(\frac{1}{-3.5}\right)1 = -3.71$
4	1	-2		-2	$\alpha_4 = d_4 - \left(\frac{l_4}{\alpha_3}\right)u_3 = -2 - \left(\frac{1}{-3.71}\right)1 = -1.730$

$\Rightarrow$ calculate β

index	β
1	$\beta_1 = b_1 = 3$
2	$b_2 - \left(\frac{l_2}{\alpha_1}\right)\beta_1 = 1 - \left(\frac{1}{-2}\right)3 = 2.5 = \beta_2$
3	$b_3 - \left(\frac{l_3}{\alpha_2}\right)\beta_2 = 2 - \left(\frac{1}{-3.5}\right)2.5 = 2.71 = \beta_3$
4	$b_4 - \left(\frac{l_4}{\alpha_3}\right)\beta_3 = -2 - \left(\frac{1}{-3.71}\right)2.71 = -1.269 = \beta_4$

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index	x (start from last row)
1	$\frac{\beta_1 - u_1 x_2}{\alpha_1} = -1.933 = x_1$
2	$\frac{\beta_2 - u_2 x_3}{\alpha_2} = -0.866 = x_2$
3	$\frac{\beta_3 - u_3 x_4}{\alpha_3} = -0.533 = x_3$
4	$\frac{\beta_4}{\alpha_4} = \frac{-1.269}{1.73} = 0.733 = x_4$

i	l	d	u	b	α	β	x
1		-2	1	3	-2	3	-1.9
2	1	-4	1	1	-3.5	2.5	-0.86
3	1	-4	1	2	-3.71	2.71	-0.53
4	1	-2		-2	-1.73	-1.269	0.73

Q3 use Jacobi Method

$$6x - 2y + z = 11$$
$$-2x + 7y + 2z = 5$$
$$x + 2y - 5z = -1$$

convergence criterion satisfied

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$$

Now write the equation in $x = g(y,z)$ form

$$x_{k+1} = \frac{1}{6}(11 + 2y_k - z_k) = g_1(y_k, z_k)$$

$$y_{k+1} = \frac{1}{7}(5 + 2x_k - 2z_k) = g_2(x_k, z_k)$$

$$z_{k+1} = \frac{1}{-5}(-1 - x_k - 2y_k) = g_3(x_k, y_k)$$

first iteration $k = 0$ $[x_0, y_0, z_0 = 0]$

$$x_1 = \frac{1}{6}[11 + 2\times 0 - 0] = 1.833$$

$$y_1 = \frac{1}{7}[5 + 2\times 0 - 2\times 0] = 0.714$$

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$$z_1 = \frac{1}{-5}[-1]$$

$$z_1 = 0.2$$

Second iteration $k = 1$

$$x_2 = g_1(y_1, z_1)$$
$$= 2.038$$

$$y_2 = g_2(x_1, z_1)$$
$$= 1.181$$

$$z_2 = g_3(x_1, y_1)$$
$$= 0.852$$

Error [After first iteration]

$$= \text{Max}\left[\frac{|x_2 - x_1|}{x_2}, \frac{|y_2 - y_1|}{y_2}, \frac{|z_2 - z_1|}{z_2}\right] \times 100$$

$$= \text{Max}\left[\frac{.2}{.2}, \frac{2.038}{2.038}, \frac{0.852}{0.852}\right] \times 100$$

$$\boxed{e_a = 100\%}$$

Similarly calculate error after all the iterations for $k = 8$

$x_9 = 2.0001$

$y_9 = 1.0001$

$z_9 = 1.0003$

$$\boxed{e_a = 0.04\%}$$

(b.) Gauss Seidel

Convergence criteria already satisfied [from Jacobi part]

$$x_{k+1} = \frac{1}{6}\left[11 + 2y_k - 2z_k\right] = g_1(y_k, z_k)$$

$$y_{k+1} = \frac{1}{7}\left[5 + 2x_{k+1} - 2z_k\right] = g_2(\underline{\underline{x_{k+1}}}, z_k)$$

$$z_{k+1} = \frac{1}{-5}\left[-1 - x_{k+1} - 2y_{k+1}\right] = g_3(\underline{\underline{x_{k+1}}}, \underline{\underline{y_{k+1}}})$$

first iteration

$$x_1 = \frac{1}{6}\left[11 + 2\times 0 - 0\right] = \frac{11}{6} = 1.833$$

$$y_1 = \frac{1}{7}\left[5 + 2\times 1.833 - 2\right] = \frac{8.666}{7}$$

$$= 1.238$$

$$z_1 = \frac{1}{-5}\left[-1 - 1.833 - 2\times 1.238\right]$$

$$= \frac{-5.309}{-5}$$

$$z_1 = 1.061$$

$$\text{Error} = \text{Max}\left[\frac{|x_2 - x_0|}{x_1}, \frac{|y_1 - y_0|}{y_1}, \frac{|z_1 - z_0|}{z_1}\right] \times 100$$

$$= 100\%$$

Second iteration

$$x_2 = g_1(y_1, z_1) = 2.069$$

$$y_2 = g_2(x_2, z_1) = 1.002$$

$$z_2 = g_3(x_2, y_2) = 1.014$$

After 5^{th} iteration

$$x_5 = 2.0001$$
$$y_5 = 1.0001$$
$$z_5 = 1.0001$$

$$\boxed{e_a = 0.07\%}$$