Tutorial-07

Eigen Value Problem ⇒ Power Method

find max & min Eigen Values.

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.3750 & -0.2500 & 0.1250 \\ -0.2500 & 0.0000 & 0.2500 \\ 0.1250 & 0.2500 & -0.2917 \end{bmatrix}$$

formula $\quad x_{i+1} = \dfrac{Ax_i}{\lambda_{i+1}} = \dfrac{y_i}{\lambda_{i+1}}$

where λ_{i+1} = dominant component of Ax_i

Part-01 Maximum Eigen Value

Iteration-01

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_i = Ax_i$$

$$y_0 = Ax_0$$

$$y_0 = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y_0 = \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$

λ_1 = dominant component of y_0

$$\lambda_1 = 14$$

$$X_1 = \frac{y_0}{\lambda_1}$$

$$= \frac{\begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}}{14}$$

$$X_1 = \begin{bmatrix} 0.5714 \\ 1.0000 \\ 0.8571 \end{bmatrix}$$

Iteration - 02

$$\lambda_1 = 14 \qquad X_1 = \begin{bmatrix} 0.5714 \\ 1.0000 \\ 0.8571 \end{bmatrix}$$

$$X_2 = \frac{AX_1}{\lambda_2} = \frac{y_1}{\lambda_2}$$

$$y_1 = AX_1$$

$$y_1 = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \begin{bmatrix} 0.5714 \\ 1.0000 \\ 0.8571 \end{bmatrix}$$

$$y = \begin{bmatrix} 6.2857 \\ 12.2857 \\ 10.2857 \end{bmatrix}$$

λ_2 = dominant component of y_1

$$\lambda_2 = 12.2857$$ Eigen Valul

$$X_2 = \begin{bmatrix} 6.2857 \\ 12.2857 \\ 10.2857 \end{bmatrix} / 12.2857$$

$$X_2 = \begin{bmatrix} 0.5116 \\ 1.0000 \\ 0.8372 \end{bmatrix}$$ Eigen Vector

$$\cancel{E_a} = \left| \frac{\cancel{y_2 - y_1}}{} \right.$$

Normalised Eigen Vector

$$= \frac{X_2}{\|X_2\|} \rightarrow \frac{X_2}{\sqrt{(0.5116)^2 + (1.0000)^2 + (0.8372)^2}}$$

$$= \begin{bmatrix} 0.3586 \\ 0.7171 \\ 0.5976 \end{bmatrix}$$

$$E_a = \left| \frac{\lambda_2 - \lambda_1}{\lambda_2} \right| \times 100$$

$$= \left| \frac{12.2857 - 14}{12.2857} \right| \times 100$$

$$E_a = 13.95\,\%$$

Part-02 Minimum Eigen Value.

$$AX = \lambda X$$

$$A^{-1}AX = A^{-1}\lambda X$$

$$X = \lambda A^{-1}X$$

$$\left(\frac{1}{\lambda}\right) X = A^{-1}X$$

Eigen values for A^{-1} are $\left(\frac{1}{\lambda}\right)$

So maximum eigen value for A^{-1} will be $\left(\frac{1}{\text{minimum eigen value}}\right)$ of A

$$B = A^{-1} = \begin{bmatrix} 0.3750 & -0.2500 & 0.1250 \\ -0.2500 & 0.0000 & 0.2500 \\ 0.1250 & 0.2500 & -0.2917 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

<u>Iteration-01</u>

$$y_i = B x_i$$

$$y_0 = B X_0$$

$$y_0 = \begin{bmatrix} 0.25 \\ 0 \\ 0.0833 \end{bmatrix}$$

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$\lambda_1 =$ dominant component in y_0

$\lambda_1 = 0.25$

$$x_1 = \frac{Bx_0}{\lambda_1}$$

$$= \begin{bmatrix} 0.25 \\ 0 \\ 0.0833 \end{bmatrix} / 0.25$$

$$x_1 = \begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.3333 \end{bmatrix}$$

Iteration - 02

$\lambda_1 = 0.25 \qquad x_1 = \begin{bmatrix} 1.0000 \\ 0.0000 \\ 0.3333 \end{bmatrix}$

$y_1 = Bx_1$

$$y_1 = \begin{bmatrix} 0.4167 \\ -0.167 \\ 0.0278 \end{bmatrix}$$

$\lambda_2 =$ dominant component in y_1

$\lambda_2 = 0.4167$

$$X_2 = \frac{BX_1}{\lambda_2}$$

$$= \begin{bmatrix} 0.4167 \\ -0.167 \\ 0.0278 \end{bmatrix} \Big/ 0.4167$$

$$X_2 = \begin{bmatrix} 1.0000 \\ -0.4000 \\ 0.0667 \end{bmatrix}$$

Eigen Vector

$$E_a = \left| \frac{\lambda_2 - \lambda_1}{\lambda_2} \right| \times 100 = \left| \frac{0.4167 - 0.25}{0.4167} \right| \times 100$$

$$\boxed{E_a = 40.00\ \%}$$

Normalised Eigen Vector $= \dfrac{X_2}{\|X_2\|}$

$$= \frac{\begin{bmatrix} 1.0000 \\ -0.4999 \\ 0.0001 \end{bmatrix}}{\sqrt{(1)^2 + (-0.4999)^2 + (0.0001)^2}}$$

Normalised Eigen Vector $= \begin{bmatrix} 0.8945 \\ -0.4472 \\ 0.0000 \end{bmatrix}$

After 6 iterations $\lambda_6 = 0.4999$

$E_a = 0.08\% < 0.1\%$

Eigen Value for $A = \lambda$

Eigen Value for $A^{-1} = 1/\lambda$

λ_{max} for $A^{-1} = \dfrac{1}{\lambda_{min} \text{ for } A}$

λ_{min} for $A = \dfrac{1}{0.4999}$

$$\boxed{\lambda_{min} = 2.000}$$

Q-02 QR method

$$A = \begin{bmatrix} 40 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix}$$

$$\underline{A} = QR$$

using gram schmidt process

$$\begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \underline{a}_1 & \underline{q}_1^T \underline{a}_2 & \underline{q}_1^T \underline{a}_3 \\ 0 & \underline{q}_2^T \underline{a}_2 & \underline{q}_2^T \underline{a}_3 \\ 0 & 0 & \underline{q}_3^T \underline{a}_3 \end{bmatrix}$$

$$a_j' = a_j - (q_1^T a_j) q_1 - (q_2^T a_j) q_2 \ldots\ldots$$

$$q_j = \frac{a_j'}{\|a_j'\|}$$

$$a_1' = a_1$$

$$a_1' = \begin{bmatrix} 40.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$$

$$q_1 = \frac{a_1'}{\|a_1'\|}$$

$$= \frac{\begin{bmatrix} 40.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}}{\sqrt{(40)^2 + (1)^2 + (1)^2}}$$

$$q_1 = \begin{bmatrix} 0.9994 \\ 0.0250 \\ 0.0250 \end{bmatrix}$$

$$\left[\|a_1'\| = 40.0250\right]$$

$$a_2' = a_2 - \underbrace{(q_1^T a_2)}_{\Downarrow} q_1$$

$$q_1^T a_2 = \begin{bmatrix} 0.9994 & 0.0250 & 0.0250 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$$q_1^T a_2 = 1.1244$$

$$a_2' = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} - 1.1244 \begin{bmatrix} 0.9994 \\ 0.0250 \\ 0.0250 \end{bmatrix}$$

$$a_2' = \begin{bmatrix} -0.1236 \\ 4.9719 \\ -0.0281 \end{bmatrix}$$

$$q_2 = \frac{a_2'}{\|a_2\|'} = \begin{bmatrix} -0.0249 \\ 0.9997 \\ -0.0056 \end{bmatrix}$$

$$[\|a_2'\|] = 4.9735$$

$$a_3' = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$q_1^T a_3 = \begin{bmatrix} 0.9994 & 0.025 & 0.025 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1.0244$$

$$q_2^T a_3 = \begin{bmatrix} -0.0249 & 0.9997 & -0.0056 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= -0.0305$$

$$a_3' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 1.0244 \begin{bmatrix} 0.9994 \\ 0.0250 \\ 0.0250 \end{bmatrix} - (-0.0305) \begin{bmatrix} -0.0249 \\ 0.9997 \\ -0.0056 \end{bmatrix}$$

$$a_3' = \begin{bmatrix} -0.0245 \\ +0.0049 \\ 0.9742 \end{bmatrix}$$

$$q_3 = \frac{a_3'}{\|a_3\|'} = \frac{a_3'}{\|a_3'\|}$$

$$\|a_3'\| = 0.9746$$

$$q_3 = \begin{bmatrix} -0.0251 \\ 0.0050 \\ 0.9997 \end{bmatrix}$$

$$Q_1 = [q_1 \quad q_2 \quad q_3]$$

$$= \begin{bmatrix} 0.9994 & -0.0249 & -0.0251 \\ 0.0250 & 0.9997 & 0.0050 \\ 0.0250 & -0.0056 & 0.9997 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} \|a_1'\| & q_1^T a_2 & q_1^T a_3 \\ 0 & \|a_2'\| & q_2^T a_3 \\ 0 & 0 & \|a_3'\| \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 40.0250 & 1.1243 & 1.0244 \\ 0.0000 & 4.9735 & -0.0305 \\ 0.0000 & 0.0000 & 0.9746 \end{bmatrix}$$

$$A_2 = R_1 Q_1$$

$$A_2 = \begin{bmatrix} \overset{d_1}{40.0537} & 0.1235 & 0.0243 \\ 0.1235 & 4.9721 \; d_2 & -0.0055 \\ 0.0243 & -0.0055 & \underset{d_3}{0.9742} \end{bmatrix}$$

$$E_a = Max\left[\left|\frac{40.0537 - 40.0000}{40.0537}\right| \times 100, \left|\frac{4.9721 - 5.0000}{4.9721}\right| \times 100, \left|\frac{0.9742 - 1.0000}{0.9742}\right| \times 100\right]$$

$$\boxed{E_a = 2.6497\,\%}$$

<u>Iteration - 02</u>

$$A_2 = Q_2 R_2$$

$$[a_1 \; a_2 \; a_3] = [q_1 \; q_2 \; q_3] \begin{bmatrix} - & - & - \\ 0 & - & - \\ 0 & 0 & - \end{bmatrix}$$

$$a_1' = a_1$$

$$a_1' = \begin{bmatrix} 40.0537 \\ 0.1235 \\ 0.0243 \end{bmatrix}$$

$q_1' =$

$$q_1 = \frac{a_1'}{\|a_1'\|} \qquad \|a_1'\| = 40.0539$$

$$q_1 = \begin{bmatrix} 1.0000 \\ 0.0031 \\ 0.0006 \end{bmatrix}.$$

$$a_2' = a_2 - \underbrace{(q_1' a_2)}_{= 0.1388} q_1$$

$$a_2' = \begin{bmatrix} 0.1235 \\ 4.9721 \\ -0.0055 \end{bmatrix} - 0.1388 \begin{bmatrix} 1.0000 \\ 0.0031 \\ 0.0006 \end{bmatrix}$$

$$a_2' = \begin{bmatrix} -0.0153 \\ 4.9717 \\ -0.0056 \end{bmatrix}.$$

$$\|a_2'\| = 4.9717$$

$$q_2 = \frac{a_2'}{\|a_2'\|}$$

$$q_2 = \begin{bmatrix} -0.0031 \\ 1.0000 \\ -0.0011 \end{bmatrix}$$

$$a_3' = a_3 - \underbrace{(q_1^T a_3)}_{0.0249} q_1 - \underbrace{(q_2^T a_3)}_{-0.0067} q_2$$

$$a_3' = \begin{bmatrix} -0.0006 \\ 0.0011 \\ 0.9742 \end{bmatrix}$$

$$\|a_3'\| = 0.9742$$

$$q_3 = \begin{bmatrix} -0.0006 \\ 0.0011 \\ 1.0000 \end{bmatrix}$$

$$Q_2 = [q_1 \quad q_2 \quad q_3]$$

$$Q_2 = \begin{bmatrix} 1.0000 & -0.0031 & -0.0006 \\ 0.0031 & 1.0000 & 0.0011 \\ 0.0006 & -0.0011 & 1.0000 \end{bmatrix}$$

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$$R_2 = \begin{bmatrix} 40.0539 & 0.1388 & 0.0249 \\ 0.0000 & 4.9717 & -0.0067 \\ 0.0000 & 0.0000 & 0.9742 \end{bmatrix}$$

$$A_3 = R_2 Q_2$$

$$A_3 = \begin{bmatrix} 40.0541 & 0.0153 & 0.0006 \\ 0.0153 & 4.9717 & -0.0011 \\ 0.0006 & -0.0011 & 0.9742 \end{bmatrix}$$

$$E_a = \max\left[\left|\frac{40.0541 - 40.0537}{40.0541}\right| \times 100, \left|\frac{4.9717 - 4.9721}{4.9717}\right| \times 100, \left|\frac{0.9742 - 0.9742}{0.9742}\right| \times 100\right]$$

$$\boxed{E_a = 0.0085\ \%}$$