ESO201A : THERMODYNAMICS
2021-22 Ist semester
IIT Kanpur

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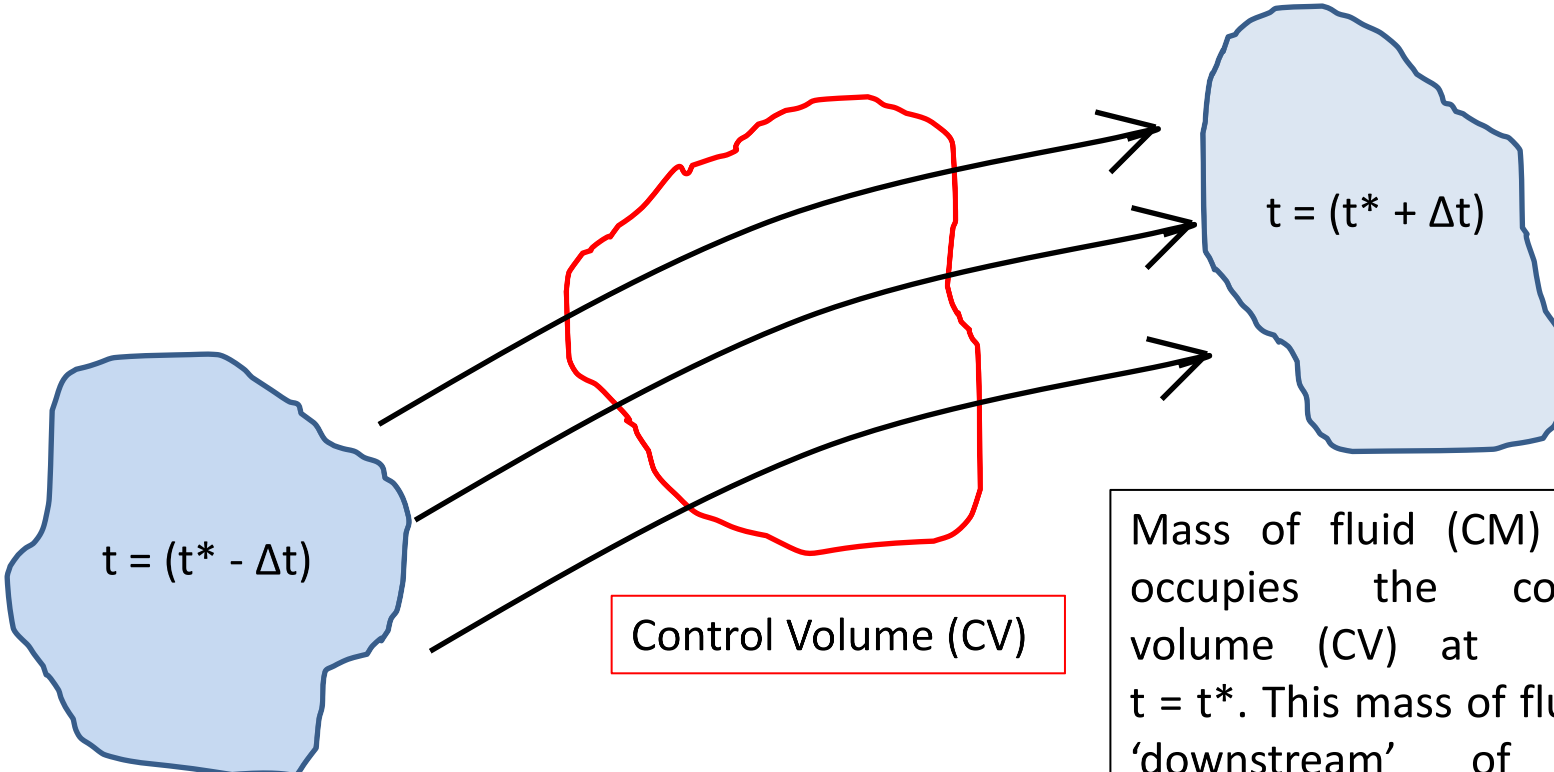
Lecture 9

Reynolds Transport Theorem:



Mass of fluid (CM) that occupies the control volume (CV) at time $t = t^*$. This mass of fluid is 'upstream' of the control volume at $t = (t^* - \Delta t)$, i.e., it has not yet entered the CV at $t = (t^* - \Delta t)$.

Mass of fluid (CM) that occupies the control volume (CV) at time $t = t^*$. This mass of fluid is 'downstream' of the control volume at $t = (t^* + \Delta t)$, i.e., it has now come out of the CV. Note that it has a different shape and volume than what it had at $t = (t^* - \Delta t)$

Reynolds Transport Theorem:

Consider a scalar quantity b which is a function of coordinates and time, i.e., b=b(x,y,z,t). As an example, this scalar quantity can be density of the fluid. Then we define two functions of time f(t) and g(t) as follows :

$$f(t) = \int_{CV} b(x,y,z,t)dV$$

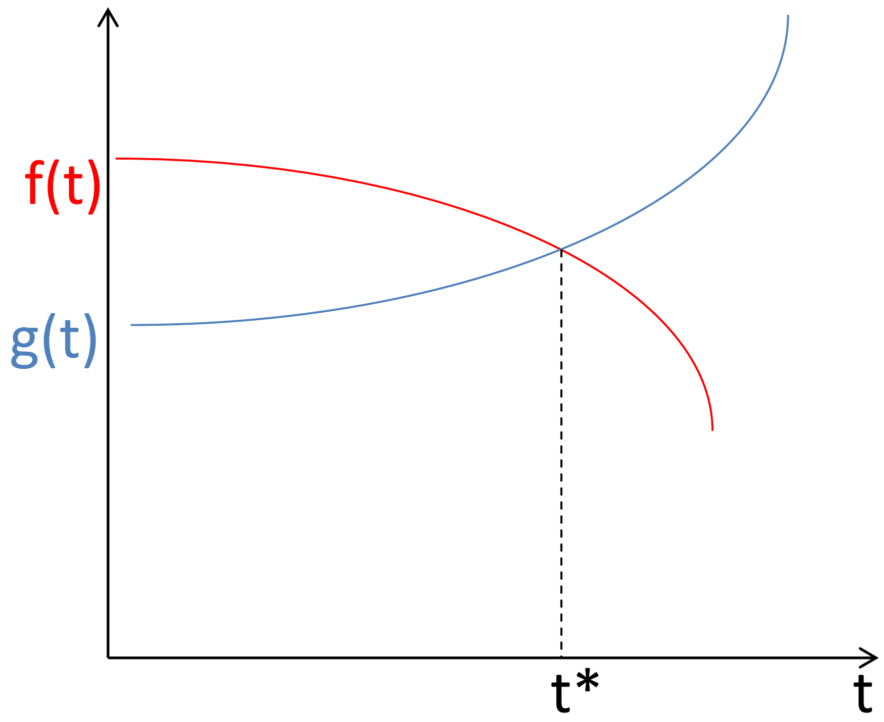
$$g(t) = \int_{CM} b(x,y,z,t)dV$$

In defining f(t), integration is taken over the control volume (CV) [see figure in the last slide]

In defining g(t), integration is taken over the mass of fluid (CM) that occupies the control volume at t = t* [see last slide]

Reynolds Transport Theorem:

The two functions of time can be plotted on a graph and the graph might look as depicted below.



Since the mass of the fluid (CM) occupies the control volume CV

At t = t*, the two functions of time will have the same value :

$$f(t^*) = g(t^*)$$

Thus the two curves interesect at t = t* as seen in above plot.

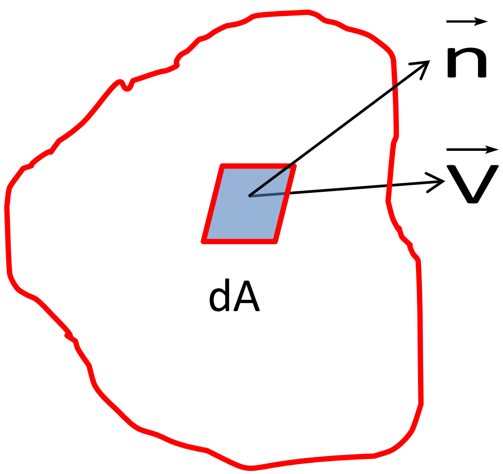
Reynolds Transport Theorem:

However the derivatives of the functions f(t) and g(t) at t = t* will not be equal. This can also be seen by the fact that slopes of the functions f(t) and g(t) plotted on last slide are not the same at t = t *. This condition is mathematically expressed as :

$$\left(\frac{dg}{dt}\right)_{t=t^*} \neq \left(\frac{df}{dt}\right)_{t=t^*}$$

The difference between the values of the two derivatives at t = t* is given by **Reynolds Transport Theorem** as follows :

$$\boxed{\left(\frac{dg}{dt}\right)_{t=t^*} = \left(\frac{df}{dt}\right)_{t=t^*} + \int_{CS} b\,(\vec{V}.\vec{n})\,dA}$$



The integral in the above equation is taken over the entire surface (CS) bounding the control volume (CV). The integration is performed at t = t*.

$\vec{n}$= normal vector (directed outwards) to the differential area 'dA' on the surface of the control volume

$\vec{V}$= Velocity vector at the center of the differential area 'dA'

Mass balance:

The mass of fluid is obtained by integrating the density ρ(x,y,z,t) over the volume :

$$M_{CM}(t) = \int_{CM} \rho(x,y,z,t)\,dV$$

$$M_{CV}(t) = \int_{CV} \rho(x,y,z,t)\,dV$$

Here $M_{CM}(t)$ is the mass of the specific body of the fluid that occupies control volume (CV) at a certain time t*. We are tracing that body with respect to time as it moves and $M_{CM}(t)$ is the mass of that body of fluid at time t. $M_{CV}(t)$ is the mass of the body of fluid that occupies the control volume at a given time t. Since for a given body of fluid, its mass is fixed (we are not considering nuclear reactions here !). Hence we have the condition :

$$\frac{dM_{CM}}{dt} = 0$$

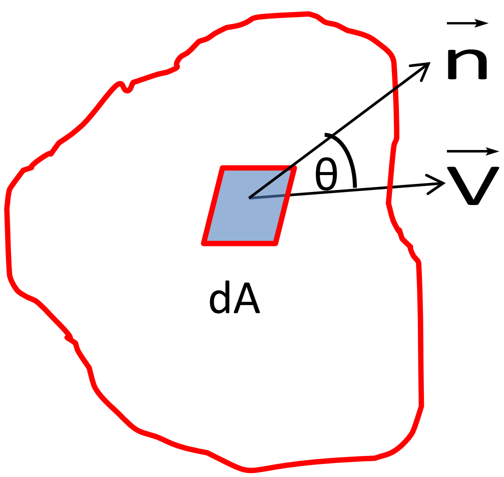
Mass balance:

Applying Reynolds transport theorem to the left hand side of the last equation :

$$\frac{dM_{CV}}{dt} + \int_{CS} \rho \, (\vec{V}.\vec{n}) \, dA = 0$$

Note that $(\vec{V}.\vec{n}) = V \cos\theta$

Here θ is the angle between the velocity vector and the normal to the differential Area dA (see figure)



Control Volume (CV)

If $\theta = 0^0$, $(\vec{V}.\vec{n}) = V \cos(0^0) = V$

Then the flow is directed **outwards** and along the same direction as that $\vec{n}$

If $\theta = 180^0$, $(\vec{V}.\vec{n}) = V \cos(180^0) = -V$

Then the flow is directed **inwards** and opposite to the direction of $\vec{n}$

How to choose a control volume (CV) ?

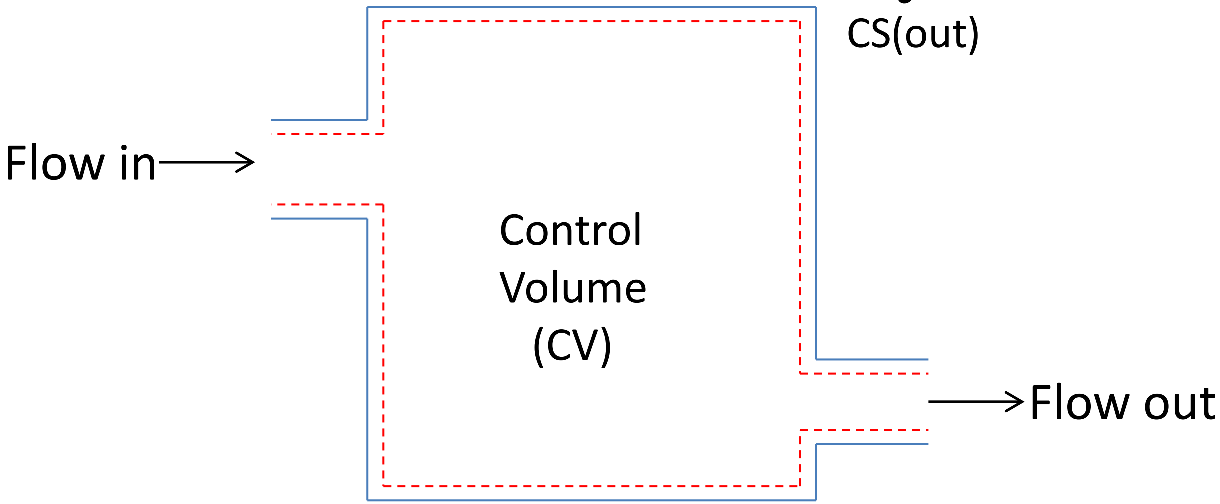
CV is chosen such that a given problem can be simplified as much as possible. As a general rule, CV is chosen such that surface (CS) bounding the control volume is perpendicular to the flow direction along inlets and outlets (see figure below).

Inlet : If the flow direction is perpendicular for an inlet and ρ is taken to be constant over the inlet area, then

$$\int_{CS(in)} \rho\,(\vec{V}.\vec{n})\,dA = \rho \int_{CS(in)} (-V)\,dA = -\dot{m}_{in}$$

Outlet : If the flow direction is perpendicular to an outlet and ρ is taken to be constant over the outlet area, then

$$\int_{CS(out)} \rho\,(\vec{V}.\vec{n})\,dA = \rho \int_{CS(out)} V\,dA = \dot{m}_{out}$$



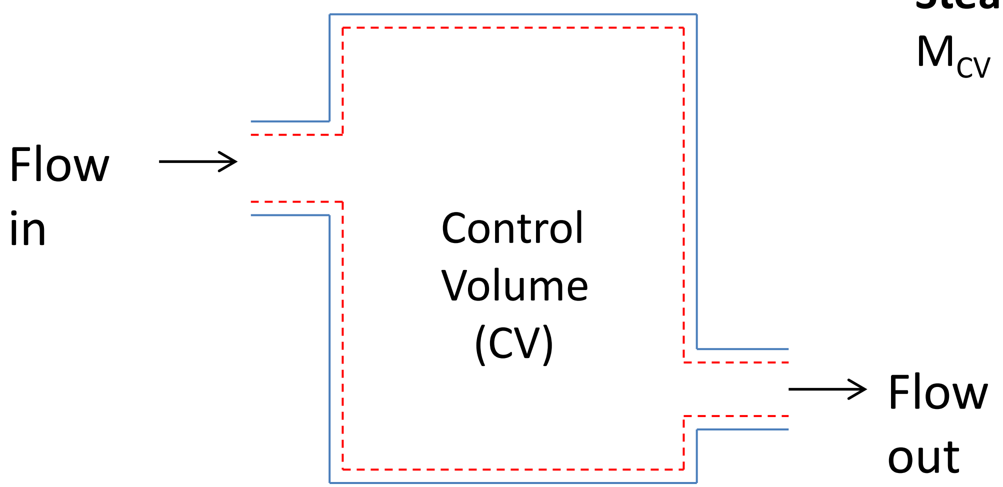
Mass balance :

If the CS is chosen such that direction of flow is perpendicular to inlets and outlets, (see last slide) then

$$\int_{CS} \rho\,(\vec{V}.\vec{n})\,dA = \sum_{out} \dot{m}_{out} - \sum_{in} \dot{m}_{in}$$

Then the mass balance takes a simpler form :

$$\frac{dM_{CV}}{dt} + \sum_{out} \dot{m}_{out} - \sum_{in} \dot{m}_{in} = 0$$

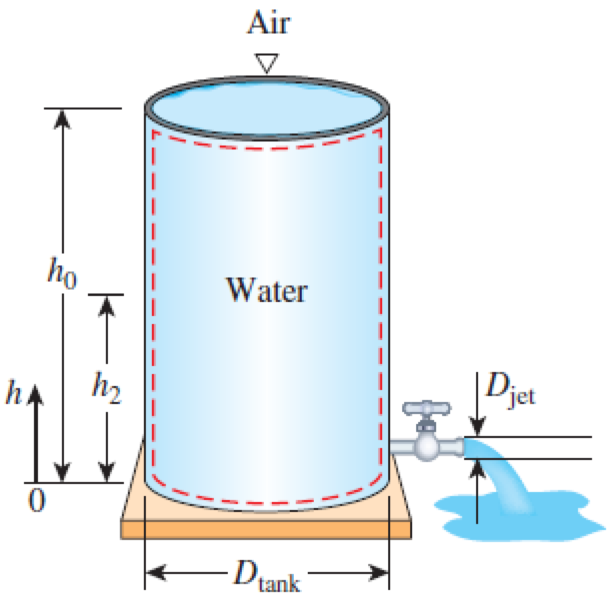


Steady flow problems :
M_{CV} does not change with time.

Example

Discharge of water from a tank (unsteady flow problem)

A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5 in streams out (Fig. 5–11). The average velocity of the jet is approximated as $V = \sqrt{2gh}$, where h is the height of water in the tank measured from the center of the hole (a variable) and g is the gravitational acceleration. Determine how long it takes for the water level in the tank to drop to 2 ft from the bottom.



Ref. Cengel and Boles, 8th Edition (2015)

SOLUTION The plug near the bottom of a water tank is pulled out. The time it takes for half of the water in the tank to empty is to be determined.

Assumptions **1** Water is a nearly incompressible substance. **2** The distance between the bottom of the tank and the center of the hole is negligible compared to the total water height. **3** The gravitational acceleration is 32.2 ft/s^2.

Analysis We take the volume occupied by water as the control volume. The size of the control volume decreases in this case as the water level drops, and thus this is a variable control volume. (We could also treat this as a fixed control volume that consists of the interior volume of the tank by disregarding the air that replaces the space vacated by the water.) This is obviously an unsteady-flow problem since the properties (such as the amount of mass) within the control volume change with time.

The conservation of mass relation for a control volume undergoing any process is given in rate form as

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{dm_{\text{CV}}}{dt} \qquad (1)$$

Example (contd.)

Ref. Cengel and Boles, 8th Edition (2015)

During this process no mass enters the control volume ($\dot{m}_{\text{in}} = 0$), and the mass flow rate of discharged water is

$$\dot{m}_{\text{out}} = (\rho VA)_{\text{out}} = \rho\sqrt{2gh}A_{\text{jet}} \tag{2}$$

where $A_{\text{jet}} = \pi D_{\text{jet}}^2/4$ is the cross-sectional area of the jet, which is constant. Noting that the density of water is constant, the mass of water in the tank at any time is

$$m_{\text{CV}} = \rho V = \rho A_{\text{tank}} h \tag{3}$$

where $A_{\text{tank}} = D_{\text{tank}}^2/4$ is the base area of the cylindrical tank. Substituting Eqs. 2 and 3 into the mass balance relation (Eq. 1) gives

$$-\rho\sqrt{2gh}A_{\text{jet}} = \frac{d(\rho A_{\text{tank}} h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{\text{jet}}^2/4) = \frac{\rho(\pi D_{\text{tank}}^2/4)dh}{dt}$$

Canceling the densities and other common terms and separating the variables give

$$dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2}\frac{dh}{\sqrt{2gh}}$$

Example (contd.)

Ref. Cengel and Boles, 8th Edition (2015)

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{\text{tank}}^2}{D_{\text{jet}}^2\sqrt{2g}} \int_{h_0}^{h_2} \frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}} \left(\frac{D_{\text{tank}}}{D_{\text{jet}}}\right)^2$$

Substituting, the time of discharge is determined to be

$$t = \frac{\sqrt{4\text{ ft}} - \sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}} \left(\frac{3 \times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \mathbf{12.6\ min}$$

Therefore, it takes 12.6 min after the discharge hole is unplugged for half of the tank to be emptied.

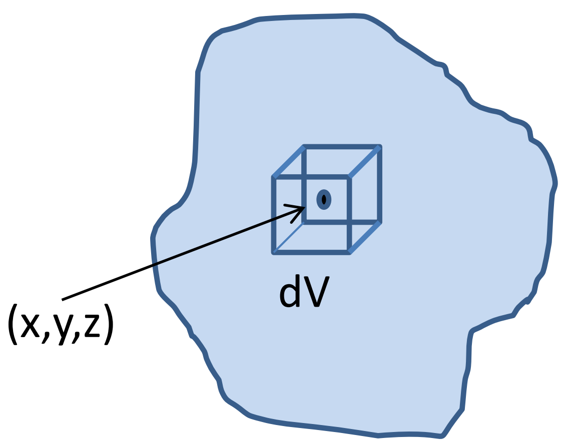
Discussion Using the same relation with $h_2 = 0$ gives $t = 43.1$ min for the discharge of the entire amount of water in the tank. Therefore, emptying the bottom half of the tank takes much longer than emptying the top half. This is due to the decrease in the average discharge velocity of water with decreasing h.

Rate form of First law for a control mass :

Recall that the rate form of the first law as applied to a control mass is

$$\frac{dE_{CM}}{dt} = \dot{Q}_{in} - \dot{W}_{out,\,t}$$

Here : E_{CM} is the total energy of a given control mass system and it can be expressed as $E_{CM} = \int_{CM} e\,\rho\,dV$



Control mass (CM)

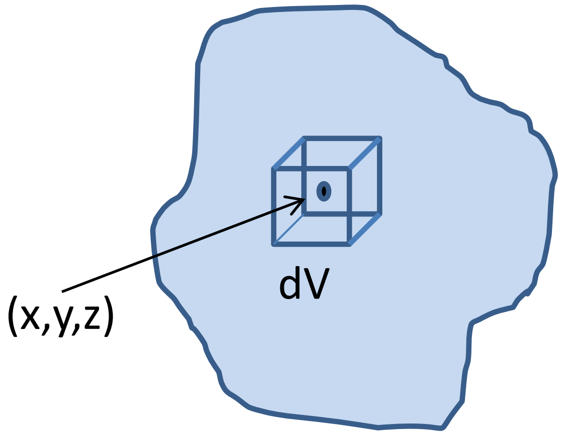
$\dot{Q}_{in}$ = Rate of energy transfer to the control mass as heat

$\dot{W}_{out,\,t}$ = Rate of energy transfer to the control mass as work

First law as applied to a control mass :

The total energy per unit mass is given by $$e = u + \frac{1}{2}V^2 + gZ$$

Here : u(x,y,z) is the specific internal energy of the fluid at (x,y,z), V is the magnitude of the velocity of the fluid at (x,y,z), and Z is the height of the volume element dV about the ground level. The second and third terms in the equation for E_{CM} (see last slide) are the kinetic and potential energies of the fluid per unit mass at (x,y,z).



Control mass (CM)

This integration in the expression for E_{CM} (see last slide) can be rationalized as follows : We consider a volume element dV around a point (x,y,z) within the body of the fluid (control mass). Here e(x,y,z,t) is the energy per unit mass and ρ(x,y,z,t) is the mass density of the fluid. Then, (e ρ dV) = total energy of the fluid inside the volume element dV. The entire control mass is divided into a large number of such volume elements. The energies (e ρ dV) are summed over all the volume elements, this summation yields E_{CM} in the limit as dV → 0 (i.e., integration is equal to the summation in the limit as the size of the volume elements approaches zero).

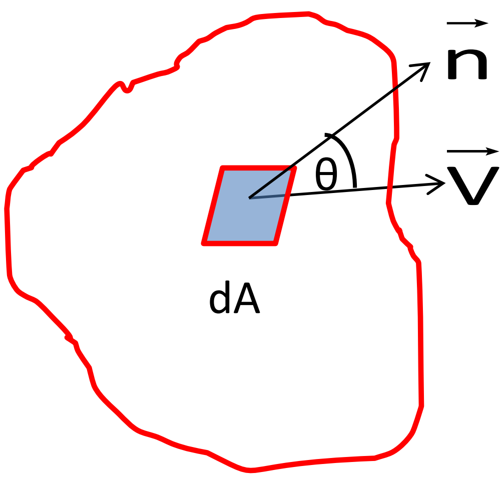
First law as applied to a control volume :

Applying Reynolds transport theorem to the left hand side of the rate form of first law (on previous slide) and splitting work term in two parts, we get

$$\frac{dE_{CV}}{dt} + \int_{CS} e\,\rho\,(\vec{V}.\vec{n})\,dA = \dot{Q}_{in} - \dot{W}_{out}^{non-flow} - \dot{W}_{out}^{flow}$$

Here : E_{CV} is the total energy of a the fluid inside the control volume at a given time t, expressed as

$$E_{CV} = \int_{CV} e\,\rho\,dV$$



Control Volume (CV)

Note that when we apply the first law to a control volume, the work term is split into two parts : **flow work** and **non-flow work**.

The non-flow work includes work done due to moving parts inside the control volume such as shaft work and other non-mechanical forms of work such as electrical work. We will consider flow work in detail in the next lecture.