ESO208A: Computational Methods in Engineering

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System of linear equations

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System of Linear Equations

n- equations , n - unknowns

$$E_1 : a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$E_2 : a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$E_n : a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

Coefficient matrix $A_{n\times n}$

Augmented matrix $\tilde{A}_{n\times n+1} = [A \;\; b]$

Homogeneous $\underline{b} = 0$

Non-homogeneous $\underline{b} \neq 0$

Preliminaries

Important square matrices:

1. Symmetric Matrix

$$a_{ij} = a_{ji}$$

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

2. Diagonal Matrix

$$A = \begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & a_{33} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

Preliminaries

3. Identity Matrix

$$I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

4. Upper Triangular Matrix

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

All elements below the main diagonal are zero

Preliminaries

5. Lower Triangular Matrix

$$L = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ \vdots & & & \ddots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

All elements above the main diagonal are zero

6. Banded Matrix

$$\begin{array}{cc} & \leftarrow \; a \; \rightarrow \\ \begin{array}{c} \uparrow \\ \\ b \\ \downarrow \\ \\ \\ \\ \end{array} & \begin{bmatrix} \times & \times & \times & 0 & 0 & 0\ldots & 0 \\ \times & \times & \times & \times & 0 & 0\ldots & 0 \\ \times & \times & \times & \times & \times & 0\ldots & 0 \\ \times & \times & \times & \times & \times & \times\ldots & 0 \\ 0 & \times & \times & \times & \times & \times\ldots & 0 \\ 0 & 0 & \times & \times & \times & \times\ldots & 0 \\ . & . & . & . & . & . & . \end{bmatrix} \end{array}$$

All elements are zero except for a band centered on the main diagonal

Band Width $= a + b - 1$

Preliminaries

7. Sparse Matrix

Most of the elements are zero

8. Dense Matrix

Most of the elements are non-zero

9. Positive Definite Matrix

A symmetric matrix, such that x^TAx is positive for every non-zero column vector x of n real number

$x^T A x \rightarrow$ Scalar

$1{\times}n \quad n{\times}n \quad n{\times}1$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x = \begin{bmatrix} a \\ b \end{bmatrix}$

$x^T I x = a^2 + b^2$

Solution of system of linear equations

- Direct Methods:
  - One obtains the exact solution (ignoring the round-off errors) in a finite number of steps.
  - These group of methods are more efficient for dense and banded matrices.
  - Gauss Elimination; Gauss-Jordon Elimination, LU-Decomposition, Thomas Algorithm (for tri-diagonal banded matrix)
- Iterative Methods:
  - Solution is obtained through successive approximation.
  - Number of computations is a function of desired accuracy/precision of the solution and are not known apriori.
  - More efficient for sparse matrices.
  - Jacobi Iterations, Gauss Seidal Iterations with Successive Over/Under Relaxation

Solution of system of linear equations

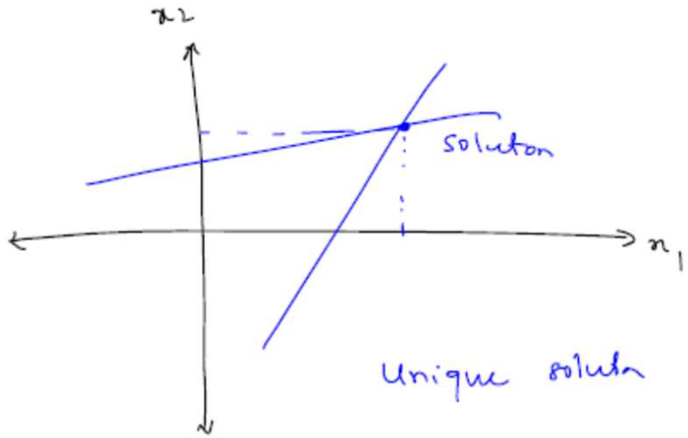
Graphical Interpretation

Let us take two linear equations with two unknowns

$a_1x_1+b_1x_2=c_1$

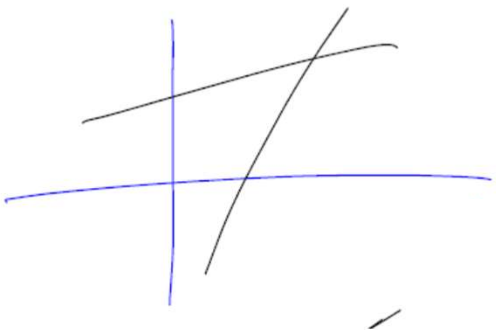
$a_2x_1+b_2x_2=c_2$

a) Unique Solution

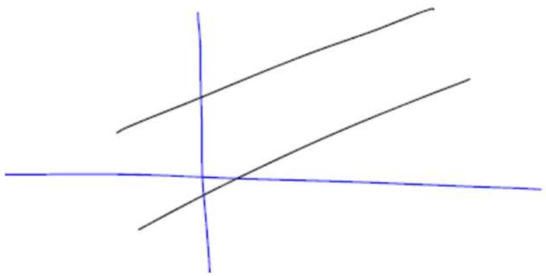


Solution of system of linear equations

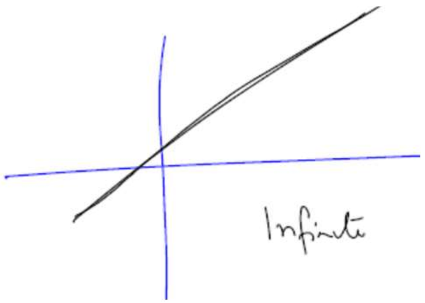
b) Unique Well Conditioned Solution



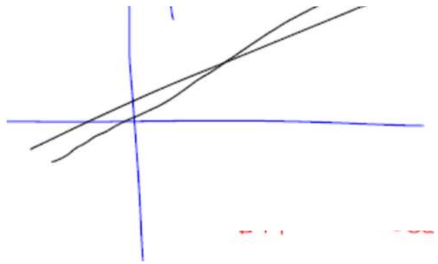
c) No Solution (Singular)



d) Infinite Solution (Singular)



e) Unique Ill conditioned



Solution of system of linear equations

Direct Methods

1) If A is I (Identity Matrix)

$$\begin{bmatrix} 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & \ddots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = b_1$$
$$\vdots$$
$$x_n = b_n$$

2) If A is a diagonal Matrix

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$
$$\vdots$$
$$x_n = \frac{b_n}{a_{nn}}$$

Solution of system of linear equations

Direct Methods

3) If A is an upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \\ 0 & 0 & 0 & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Back substitution

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

4) If A is a lower triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & & 0 \\ \vdots & & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j}{a_{ii}}$$

Solution of system of linear equations

Direct Methods

If the coefficient matrix A is "full". Can use

- Gauss Elimination
- Gauss-Jordon Elimination
- LU-Decomposition
- Thomas Algorithm (for tri-diagonal banded matrix)

All these methods belong to family of Gauss Elimination. Gauss Elimination is one of the ubiquitous algorithm

E: ax+by+cz=d

If we multiply, divide, add or subtract both sides nothing is going to change!

Direct Methods: Gauss Elimination

Gauss Elimination for the matrix equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

Approach in two steps:

a) Operating on rows of matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, transform the matrix $\boldsymbol{A}$ to an upper triangular matrix.

b) Solve the system using *Back substitution algorithm*.

Indices:

- i: Row index
- j: Column index
- k: Step index

Gauss Elimination

Gauss Elimination

objective – To convert A to U

$$E_1 : 2x_1 + 3x_2 - x_3 = 5$$

$$E_2 : 4x_1 + 4x_2 - 3x_3 = 3$$

$$E_3 : -2x_1 + 3x_2 - x_3 = 1$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$$

Step 1

multiplying factors

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{4}{2} = 2$$

Pivot

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{-2}{2} = -1$$

$$R_2 : R_2 - l_{21} R_1 \quad ; \quad R_3 = R_3 - l_{31} R_1$$

Pivot equation

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{bmatrix}$$

Step 2

$$R_3 = R_3 - l_{32} R_2$$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{6}{-2} = -3$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

U

Back substitution

$$x_3 = -15/-5 = 3$$

$$x_2 = \frac{-7 - (-1 \times 3)}{-2} = 2$$

$$x_1 = 1$$

Gauss Elimination Algorithm

Forward Elimination:

For $k = 1, 2, \ldots. (n - 1)$

Define multiplication factors: $l_{ik} = \frac{a_{ik}}{a_{kk}}$

Compute: $a_{ij} = a_{ij} - l_{ik}\, a_{kj}$; $b_i = b_i - l_{ik}\, b_k$ for

$i = k+1, k+2, \ldots.n$ and $j = k+1, k+2, \ldots.n$

Resulting System of equation is upper triangular. Solve it using the ***Back-Substitution algorithm:***

$$x_n = \frac{b_n}{a_{nn}}; x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}; \quad i = (n-1), (n-2), \ldots 3, 2, 1$$

Summary

- Different types of matrices
- Graphical interpretation of solution of system of linear equations
- Gauss-Elimination Method