# ESO 208A: Computational Methods in Engineering

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# Recap

- What is a system of linear equations?
- Different kind of matrices
- Direct method-Gauss Elimination Method

# Today's lecture

- Situations under which Gauss Elimination method will not work
- Gauss Jordan Method
- How to find algorithm complexity?
- LU decomposition method

# Gauss Elimination Algorithm

***Forward Elimination:***

For $k = 1, 2, \ldots. (n - 1)$

Define multiplication factors: $l_{ik} = \dfrac{a_{ik}}{a_{kk}}$

Compute: $a_{ij} = a_{ij} - l_{ik}\, a_{kj}$; $\quad b_i = \; b_i - l_{ik}\;\, b_k$ for

$i = k+1, k+2, \ldots.n$ and $j = k+1, k+2, \ldots.n$

Resulting System of equation is upper triangular. Solve it using the ***Back-Substitution algorithm:***

$$x_n = \frac{b_n}{a_{nn}}; x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}; \;\; i = (n-1), (n-2), \ldots 3, 2, 1$$

# Gauss Elimination

## Difficult Cases

### a) Division by zero

$E_1$ : ✓ $3x_2 - x_3 = 5$

$E_2$ : $4x_1 + 4x_2 - 3x_3 = 3$

$E_3$ : $-2x_1 + 3x_2 - x_3 = 1$

$$A \quad \left[\begin{array}{ccc|c} 0 & 2 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$$

$$l_{21} = \frac{a_{21}}{a_{11}}$$

- $l_{21}$ can not be calculated, exchange the rows, which one to pick
- Does it matter? Yes, in terms of round off error.
- When we switch the rows, it is called as pivoting or row pivoting
- When we switch columns, it is column pivoting. In this case, we need to reorder the unknowns.
- When we switch both, it is total pivoting

Partial Pivoting

# Gauss Elimination

## Difficult Cases

### b) Ill-conditioned

$x_1+2x_2=10$
$1.1x_1+2x_2=10.4$

Solution: $x_1=4,\ x_2=3$

Now if I slightly change the coefficients
$x_1+2x_2=10$
$1.05x_1+2x_2=10.4$

Solution: $x_1=8,\ x_2=1$

- By just changing the coefficient slightly, the solution changes significantly
- This is very costly
- Can we without solving find if the system is ill conditioned

# Gauss Elimination

## Difficult Cases

### c) Round-off Error

$$0.0004\,x_1 + 1.402\,x_2 = 1.406$$

$$0.4003\,x_1 - 1.502\,x_2 = 2.501$$

True Solution: $x_1=10$, $x_2=1$

What if you are using a computer that has four significant digits

# Gauss Elimination

## Difficult Cases

### c) Round-off Error

$$\begin{bmatrix} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{bmatrix}$$

$$R_2 = R_2 - l_{21} R_1 \qquad l_{21} = \frac{a_{21}}{a_{11}}$$

$$= \frac{0.4003}{0.0004} = 0.1001 \times 10^4$$

$$-1405 x_2 = 1404$$

$$x_2 = 0.9993$$

$$x_1 = \frac{(1.406 - 1.402 \times 0.9993)}{0.0004}$$

$$= 12.5$$

- The solution is very different from the actual solution
- Before solving the problem, can we know our system will have round-off problem

# Gauss Elimination

## Difficult Cases: Options for handling

### a) Ill-Conditioned

$$x_1 + 2x_2 = 10$$

$$1.1x_1 + 2x_2 = 10.4$$

$$1.05 \quad — \quad \left\{ \begin{array}{l} x_1 = 8 \\ x_2 = 1 \end{array} \right.$$

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_2 = -\frac{a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + \frac{b_2}{b_1}$$

$$\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}} \quad - 1$$

$$\boxed{a_{11}a_{22} - a_{12}a_{21} \approx 0}$$

- If determinant is close to zero- ill conditioned
- If determinant is zero- singular

# Gauss Elimination

## Difficult Cases: Options for handling

### a) Ill-Conditioned

Can we use determinant as a measure of ill-conditioning?

$$\begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix} \Rightarrow D = 2 - 2.2 = -0.2$$

Suppose in the example we multiply the equations by 10

$$\begin{bmatrix} 10 & 20 \\ 11 & 20 \end{bmatrix} \Rightarrow D = \underline{\underline{-20}}$$

Now the determinant is significantly different from 0. On its own D is not a good measure of ill-conditioning

# Gauss Elimination

## Difficult Cases: Options for handling

The three issues mentioned earlier can be avoided by:

- Use of more significant digits
- Pivoting: Row or partial pivoting-exchange row of the augmented matrix
  - Exchange rows which will result in largest magnitude of pivot element

# Gauss Elimination

## Difficult Cases: Options for handling

Example:

$$0.0004\, x_1 + 1.402\, x_2 = 1.406$$

$$0.4003\, x_1 - 1.502\, x_2 = 2.501$$

Exact — $x_1 = 10$
$x_2 = 1$

If we solved the problem by a 4-digit

— $x_1 = 0.9993$

— $x_2 = 12.5$

# Gauss Elimination

## Difficult Cases: Options for handling

Example:

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{array}\right]$$

Pivoting

$$\left[\begin{array}{ccc} 0.4003 & -1.502 & 2.501 \\ 0.0004 & 1.402 & 1.406 \end{array}\right]$$

$$l_{21} = \frac{0.0004}{0.4003} = 0.9993 \times 10^{-3}$$

$$\left[\begin{array}{cc|c} 0.4003 & -1.502 & 2.501 \\ 0 & 1.404 & 1.404 \end{array}\right]$$

Back sub.

$$x_2 = 1.0$$

$$x_1 = \frac{2.501 - (-1.502 \times x_2)}{0.4003} = \frac{4.003}{0.4003} = \underline{\underline{10}}$$

We did row pivoting

# Gauss Elimination

## Difficult Cases: Options for handling

Example:

$$\left[\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{array}\right]$$

Total pivoting

$$\left[\begin{array}{cc|c} -1.502 & 0.4003 & 2.502 \\ 1.402 & 0.0004 & 1.406 \end{array}\right]$$

$$\ell_{21} = -\frac{1.402}{1.502} = -0.9334$$

We did total pivoting

$x_1$=10, and $x_2$=1

# Gauss Elimination

## Difficult Cases: Options for handling

Why pivoting has worked?

$$10^m \times 0.0004\, x_1 + 10^m \times 1.402\, x_2 = 10^m \times 1.406$$

$$0.4003\, x_1 - 1.502\, x_2 = 2.501$$

$$l_{21} = \frac{0.4003}{0.0004 \times 10^m} = 1001 \times 10^{-m}$$

$$a_{22} = -1.502 - \left(1.402 \times \cancel{10^m} \times 1001 \times \cancel{10^{-m}}\right)$$

$$= -1405$$

$$b_2 = 1404$$

$$x_1 = \underline{\underline{0.9993}}$$

$$x_2 = 12.5$$

So, it is not the magnitude of pivot element that avoided round-off errors

$$|a_{12}| > |a_{11}|$$

# Gauss Elimination

## Difficult Cases: Options for handling

Why pivoting has worked?

- Even by making the pivot large still we get round-off error.
- It is not the magnitude of the pivot element but relative magnitude of elements that leads to round-off error
- Scaling of elements of ‘A’ governs the round-off errors

# Gauss Elimination

## Difficult Cases: Options for handling

### Scaling

$$\left[\begin{array}{cc|c} \boxed{2 \quad 10^5} & & 10^5 \\ 1 & 1 & 2 \end{array}\right]$$

Exact

$x_1 = 1.00002$

$x_2 = 0.99998$

If we use 3-digit

$l_{21} = 1/2 = 0.5$

$x_2 = 1.0$

$x_1 = 0.0$

Scaling - Divide the $\left|\frac{a_{ij}}{A_i}\right|$

$A_j$ is the maximum value in a row

# Gauss Elimination

## Difficult Cases: Options for handling

**Scaling**

$$\left[\begin{array}{cc|c} 2\times10^{-5} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

partial pivoting

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2\times10^{-5} & 1 & 1 \end{array}\right]$$

Perform pivoting by using scaled coefficients but perform computations (GE) using original coefficients

# Gauss Elimination

## Difficult Cases: Options for handling

### Scaling

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 10^5 & 10^5 \end{array}\right]$$

$$\underline{x_1 = x_2 = 1.0}$$

Perform pivoting by using scaled coefficients but perform computations (GE) using original coefficients

# Gauss Elimination

## Difficult Cases: Options for handling

Most common implementations of GE:

- Use scaled values of the coefficients as a criterion to decide pivoting
- Retain the original coefficients for actual elimination and substitution
- "No general pivoting strategy that will work for all linear systems"
  - Example: If coefficient matrix is a positive definite matrix, the BEST strategy is no interchange
- If you know, any special characteristics of the system use it to decide the pivoting strategy

# Direct Methods: Gauss Jordon

In this method, the coefficient matrix is reduced to an Identity matrix

- Requires a minor modification in GE algorithm
  - At each step, first the pivot element is made unity by dividing pivot equation by the pivot element
  - In addition to sub-diagonal elements, the above diagonal elements are also made 0.

# Direct Methods: Gauss Jordon

Example

<u>Example</u>

① $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right]$

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$

$R_1 = \frac{R_1}{a_{11}}$

$$\left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 2.5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1.5 & -0.5 & 2.5 \\ 0 & -\underline{\underline{2}} & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right]$$

$R_2 = R_2 / a_{22}$

$$\left[\begin{array}{cccc} 1 & 1.5 & -0.5 & 2.5 \\ 0 & 1 & 0.5 & 3.5 \\ 0 & 6 & -2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 0 & -1.25 & -2.75 \\ 0 & 1 & 0.5 & 3.5 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

# Summary

- Under what situations Gauss Elimination will not work
- Gauss Jordan Method