Tutorial - 08

$y = ae^{bx} \longrightarrow a \,\&\, b = ?$

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

$\left(y = ae^{bx} \right)$ take log

~~log~~ $\ln y = \ln a + bx$

$Y = a_0 + a_1 X$ [line]

compare $Y = \ln y$

$a_0 = \ln a$

$a = e^{a_0}$

$b = a_1$

$x = X$

from the concept of least square

$$\Sigma Y = \Sigma a_0 + \Sigma a_1 X$$

$$\Sigma Y = n a_0 + a_1 \Sigma X$$

$n \Rightarrow$ no of data points available

$$\sum XY = \sum a_0 X + \sum a_1 X^2$$

$$\sum XY = a_0 \sum X + a_1 \sum X^2$$

Required $\sum X$, $\sum Y$, $\sum XY$, $\sum X^2$

$X = x$	y	$Y = \ln(y)$	X^2	XY
2	4.077	1.4054	4.0000	2.8107
4	11.084	2.4055	16.0000	9.6220
6	30.128	3.4055	36.0000	20.4327
8	81.897	4.4055	64.0000	35.2437
10	222.620	5.4055	100.0000	54.0547
$\sum =$ 30.0000	not required	17.0272	220.0000	122.1638

$$\sum Y = n a_0 + a_1 \sum X$$

$$17.0272 = 5 a_0 + 30.0000 a_1$$

$$\sum XY = a_0 \sum X + a_1 \sum X^2$$

$$122.1638 = 30.0000 a_0 + 220.0000 a_1$$

Two equations two unknowns

$a_0 = 0.4054 = \ln a \Rightarrow a = e^{0.4054} = 1.4999$

$a_1 = 0.5000 = b$

$$\boxed{a = 1.4999 \ \& \ b = 0.5000}$$

Answer - 02

Orthogonal basis functions

Quadratic polynomial approximation of

$$f(x) = \frac{1}{1+x^2} \qquad x \in (-1, 1)$$

Approx func $= \hat{f}(x)$

$$\hat{f}(x) = a_0 \underbrace{P_0(x)}_{\text{given}} + a_1 \underbrace{P_1(x)}_{\text{given}} + a_2 \underbrace{P_2(x)}_{\text{given}} \quad -(1)$$

Legendre polynomials
$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x) \end{cases}$$

$$P_2(x) = \frac{3}{2} x \cdot x - \frac{1}{2}(1)$$
$[n=1]$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Now, a_0, a_1 & a_2

$$a_i = \frac{\langle P_i(x), f(x) \rangle}{\langle P_i(x), P_i(x) \rangle}$$

$$\langle f(x), g(x) \rangle = \int_{-L}^{L} f(x) g(x) \, dx \quad \left[\begin{array}{l} \text{formula for} \\ \text{inner product} \end{array} \right]$$

$$a_0 = \frac{\langle P_0(x), f(x) \rangle}{\langle P_0(x), P_0(x) \rangle}$$

$$= \frac{\int_{-1}^{1} 1 \left(\frac{1}{1+x^2} \right) dx}{\frac{2}{2 \times 0 + 1}} = \frac{1}{2} \tan^{-1}(x) \Big|_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$$

$$\boxed{a_0 = \frac{\pi}{4} = 0.7854}$$

$$a_1 = \frac{\langle P_1(x), f(x) \rangle}{\langle P_1(x), P_1(x) \rangle}$$

$$= \frac{\int_{-1}^{1} x \left(\frac{1}{1+x^2} \right) dx}{\frac{2}{2 \times 1 + 1}}$$

$$= \frac{\frac{1}{2}\int_{-1}^{1} \frac{2x}{1+x^2}\, dx}{2/3}$$

$$= \frac{3}{4}\left[\ln(x^2+1)\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\ln 2 - \ln 2\right]$$

$$a_1 = 0$$

$$a_2 = \frac{\langle P_2(x), f(x)\rangle}{\langle P_2(x), P_2(x)\rangle}$$

$$= \frac{\int_{-1}^{1} \frac{1}{2}(3x^2-1)\left(\frac{1}{1+x^2}\right) dx}{\frac{2}{2\times 2+1}}$$

$(+1 \& -1)$

$$= \frac{1}{2}\times\frac{5}{2}\left[\int_{-1}^{1} \frac{3x^2}{1+x^2}\, dx - \int_{-1}^{1}\frac{1}{1+x^2}\, dx\right]$$

$$= \frac{5}{4}\left[3\int_{-1}^{1} \frac{\cancel{x^2+1}}{\cancel{x^2+1}}\, dx - 3\int_{-1}^{1}\frac{1}{1+x^2}\, dx - \int_{-1}^{1}\frac{1}{1+x^2} dx\right]$$

$$= \frac{5}{4}\left[3\times 2 - 4\tan^{-1}(x)\Big|_{-1}^{1}\right]$$

$$a_2 = \frac{5}{4}\left[6 - 4\left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right]$$

$$a_2 = \frac{5}{4}\left[6 - 6.2831\right]$$

$$a_2 = \frac{5}{4}(-0.2831)$$

$$\boxed{a_2 = -0.3540}$$

a_0, a_1, a_2 & $P_0\ P_1\ P_2 \Rightarrow$ known

$$\hat{f}(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

$$= 0.7854(1) + 0(x) + \left[(-0.3540)\ \frac{1}{2}(3x^2-1)\right]$$

$$\boxed{\hat{f}(x) = 0.9624 - 0.5310x^2}$$

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(b.) Taylor's Series

$$\hat{f}(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) \quad \text{—} \; 2$$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = 1$$

$$f'(0) = \left. \frac{(1+x^2) \times 0 - 2x}{(1+x^2)^2} \right|_{x=0}$$

$$f'(0) = 0$$

$$f''(x) = \left. \frac{d}{dx}\left[\frac{-2x}{(1+x^2)^2}\right] \right|_{x=0}$$

$$= \left. \frac{\cancel{(1+x^2)^2}(-2)}{(1+x^2)^{\cancel{4}\,2}} - \frac{(-2x)\,2\,\cancel{(1+x^2)}\,(2x)}{(1+x^2)^{\cancel{4}\,3}} \right|_{x=0}$$

$$f''(x) = \left. \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3} \right|_{x=0}$$

$$f''(x) = -2$$

Replace the derivatives in equation 2

$$\hat{f}(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0)$$

$$= 1 + x(0) + \frac{x^2}{2}(-2)$$

$$\boxed{\hat{f}(x) = 1 - x^2}$$

(C) True relative error in $f(x)$ at

$x = -0.9, -0.5, 0, 0.5$

True relative error (E_T) $= \left| \dfrac{f(x) - \hat{f}(x)}{f(x)} \right| \times 100$

E_T for Legendre polynomial

x	$f(x) = 1/(1+x^2)$	$\hat{f}(x) = -0.5310x^2 + 0.9624$	E_T
-0.9	0.55	0.532	3.66
-0.5	0.80	0.830	3.71
0	1.00	0.962	3.76
0.5	0.80	0.830	3.71

E_T for Taylor's Series

x	$f(x) = \frac{1}{(1+x^2)}$	$\hat{f}(x) = 1-x^2$	$E_T = \left\lvert \frac{f(x)-\hat{f}(x)}{f(x)} \right\rvert \times 100$
-0.9	0.55	0.190	65.61
-0.5	0.80	0.750	6.25
0	1.00	1.000	0.00
0.5	0.80	0.750	6.25