ESO208A: Computational Methods in Engineering

Richa Ojha

Department of Civil Engineering

IIT Kanpur

Acknowledgements: Profs. Abhas Singh and Shivam Tripathi (CE)

LU decomposition

Consider the system

$$\boldsymbol{A}x = \boldsymbol{b}$$

- In most engineering problems, the matrix $\boldsymbol{A}$ remains constant while the vector $\boldsymbol{b}$ changes with time.
- The matrix $\boldsymbol{A}$ describes the system and the vector $\boldsymbol{b}$ describes the external forcing. e.g., all network problems (pipes, electrical, canal, road, reactors, etc.); structural frames; many financial analyses.
- If all $\boldsymbol{b}$'s are available together, one can solve the system by augmented matrix but in practice, they are not!

LU decomposition

For the system,

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

- Perform a decomposition of the form $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, where $\boldsymbol{L}$ is a *lower-triangular* and $\boldsymbol{U}$ is an *upper-triangular* matrix!
- For any given $\boldsymbol{b}$, solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{L}\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b}$
- This is equivalent to solving two triangular systems:
  - Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ using *forward substitution* to obtain $\boldsymbol{y}$
  - Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ using *back substitution* to obtain $\boldsymbol{x}$
- Most frequently used method for engineering applications!

LU decomposition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$a_{11} = l_{11}u_{11} \qquad a_{12} = l_{11}u_{12} \qquad a_{13} = l_{11}u_{13}$$

$$a_{21} = l_{21}u_{11} \qquad a_{22} = l_{21}u_{12} + l_{22}u_{22}$$

$$a_{23} = l_{21}u_{13} + l_{22}u_{23}$$

$$a_{31} = l_{21}u_{11} \qquad a_{32} = l_{31}u_{12} + l_{32}u_{22}$$

$$a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33}$$

12 Unknowns and 9 equations! 3 free entries!

In general, n^2 equations and $n^2 + n$ unknows! n free entries!

LU decomposition

In general, n^2 equations and $n^2 + n$ unknown, n free entries!

It means we cannot have a unique solution for l_{ij} and u_{ij}. However, if we fix 'n' terms, we will get a unique solution

LU decomposition Theorem

If A is a square matrix of size n × n and if $\det(A) \neq 0$. Then there exists a lower triangular matrix (L) and an upper triangular matrix (U) such that A=LU.

Further, if the diagonal elements of either L or U are unity, i.e l_{ii} or u_{ii} =1 for i=1,2,….n, then both L and U are unique

LU decomposition

How to get elements of both L and U

1. Gauss Elimination gives both L and U
2. Dolittle Method
3. Crout Method
4. Thomas Algorithm- Tri-diagonal matrix
5. Cholesky Algorithm- Positive definite matrix

$l_{ii}=1$

$u_{ii}=1$

LU decomposition

1. Gauss Elimination Method for L and U

$$\text{GE}\begin{cases} A \longrightarrow U \\ \text{multiplication factors} \quad \underline{\underline{l_{ij}}} = \dfrac{a_{ij}}{a_{ii}} \end{cases}$$

Example

$$\begin{bmatrix} 2 & 3 \\ 8 & 5 \end{bmatrix}$$

$$l_{21} = \frac{8}{2} = 4$$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -7 \end{bmatrix}}_{U}$$

LU decomposition

Gauss Elimination Method for L and U

Previous Example

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

multiplication factors

$$l_{21} = 4/2 = 2$$

$$l_{31} = -2/2 = -1$$

$$l_{32} = -3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -5 \end{bmatrix}$$

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 3 - 10 = -7 \\ 1-(-1\times 5)-(-3\times -7) = -15 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \\ -15 \end{bmatrix}$$

$$Ux = y$$

LU decomposition

Gauss Elimination Method for L and U

$$y = \begin{bmatrix} 5 \\ -7 \\ -15 \end{bmatrix}$$

$$Ux = y$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ -15 \end{bmatrix}$$

$$x_3 = 3$$

$$x_2 = 2$$

$$x_1 = 1$$

'L' is like a "recorder" of the operations to be applied on 'b'

LU decomposition

Comparison of GE and LU

Forward elimination   Backward substitution

GE → $O(n^3)$ $O(n^2)$

n^2+n^2

LU → $O(n^3)$ $O(n^2)$

r - equations

GE $O(r n^3)$

LU $O(n^3 + r n^2) \sim O(n^3)$

LU decomposition

Comparison of GE and LU

Example – Inverse of a matrix

$$A\underline{x}_1 = \underline{b}_1 \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$A x_2 = b_2 \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}} = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$$

$$A A^{-1} = I$$

GE	$O(n^4)$	MATLAB for estimation inverse of a matrix
LU	$O(n^3 + n^3) \sim O(n^3)$	

Example

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 0.4 & 0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

$D = 10$

$$\overset{L}{\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 3 & 1 \\ 0 & 10/3 \end{bmatrix}}$$

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$$

$$Ux = y$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 10/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$$

LU decomposition

2. Crout's Method

$$u_{ii} = 1$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \cdots & u_{1n} \\ 0 & 1 & u_{23} \cdots & u_{2n} \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$a_{11} = l_{11} \qquad \Rightarrow \qquad l_{i1} = a_{i1} \qquad i = 1, 2, \ldots n$$

$$a_{21} = l_{21}$$

$$a_{12} = l_{11} u_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}}$$

$$a_{13} = l_{11} u_{13} \Rightarrow u_{13} = \frac{a_{13}}{l_{11}} \qquad \Rightarrow \qquad u_{1j} = \frac{a_{1j}}{l_{11}} \quad j = 2, \ldots n$$

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ l_{21} & - & - & \cdot & u_{2n} \\ \vdots & & & & \\ l_{n1} & - & - & & l_{nn} \end{bmatrix}$$

LU decomposition

2. Crout's Method

$$a_{22} = l_{21} u_{12} + l_{22}$$

$$\Rightarrow l_{22} = a_{22} - l_{21} u_{12}$$

For $j = 2, 3, \ldots n-1$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad i = j, j+1, \ldots n$$

$$u_{jk} = \frac{a_{jk} - \sum_{i=1}^{j-1} l_{ji} u_{ik}}{l_{jj}} \qquad k = j+1, \ldots n$$

end

$$l_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk} u_{kn}$$

LU decomposition

3. Dolittle Method

$$l_{ii} = 1$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

Summary

- What is LU decomposition
- Crout's method
- Dolittle method