ESO 208A: Computational Methods in Engineering

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Non-linear equation

In computer, we have five approaches

- **Graphical method**
- **Bracketing methods:** Bisection, Regula-Falsi
- **Open methods:** Fixed point, Newton-Raphson, Secant
- **Special methods for polynomials:** Muller, Bairstow's
- **Hybrid methods:** Brent's

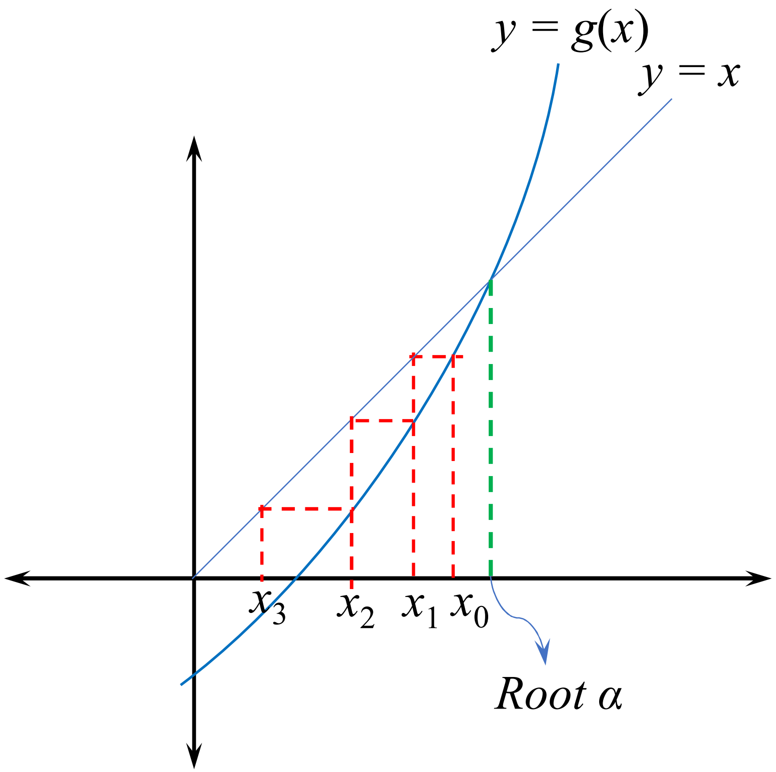
Open Methods

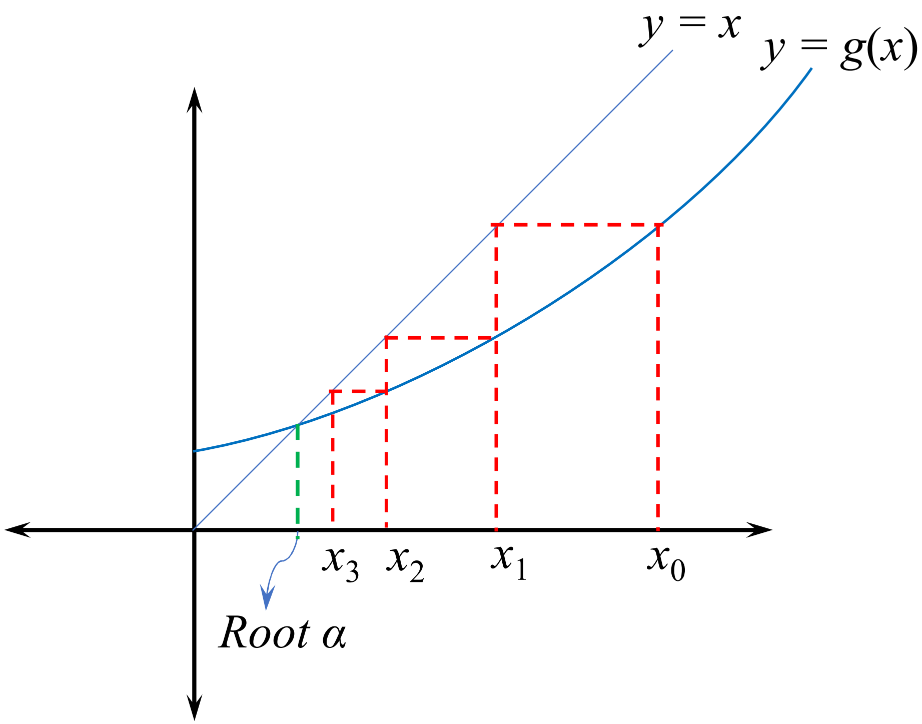
Distinguishing feature:

- Only one starting value
- Convergence is not always guaranteed
- If algorithm convergences, the rate of convergence may be faster

Open Methods

1. Fixed Point





$f(x) = 0 \Rightarrow x = g(x), \quad x_{i+1} = g(x_i)$

Fixed Point Method

- Problem: $f(x) = 0$, find a root $x = \alpha$ such that $f(\alpha) = 0$
  - Re-arrange the function: $f(x) = 0$ to $x = g(x)$
  - Iteration: $x_{k+1} = g(x_k)$
  - Stopping criteria: $\left|\frac{x_{k+1} - x_k}{x_{k+1}}\right| \leq \varepsilon$

Fixed Point Method

Example

$$f(x) = e^{-x} - x = 0$$

$S = 0.5671$

1 — $x = e^{-x}$

2 — $x = -\log(x)$

$x_0 = 0$

$x_1 = e^0 = 1$

$x_2 = e^{-1} = 0.3678$

$x_3 = e^{-0.36788} = 0.692$

$x_4 = e^{-x_3} = 0.5$

$x_0 = 0.50$

$x_1 = 0.69$

$x_2 = \vdots$

Fixed Point Method

Convergence of fixed point

$$x_{i+1} = g(x_i) \quad \text{—} (1)$$

$$S = g(s) \quad \text{—} (2)$$

2 − 1

$$S - x_{i+1} = \underbrace{g(s) - g(x_i)}_{MVT}$$

$$e_{i+1} = g'(\xi)(s - x_i)$$

$\xi \in (s, x_i)$

$$\Rightarrow e_{i+1} = g'(\xi)\, e_i$$

$$\Rightarrow \frac{|e_{i+1}|}{|e_i|} = |g'(\xi)|$$

as $i \to \infty$ $\left|\frac{e_{i+1}}{e_i}\right| = |g'(s)|$ ✓

If $|g'(s)| < 1$ algorithm converges

Linear convergence

$$\boxed{\frac{|e_{i+1}|}{|e_i|^p} = C}$$

If $|g'(s)| > 1$ – algorithm diverges

$g'(s) = +ive$

errors will reduce montonically

$g'(s) = -ive$

errors will oscillate

Fixed Point Method

Convergence of fixed point

Example

(1) $x = e^{-x}$

$g'(x) = -e^{-x}$

$|g'(s)| < 1$

$s = 0.567$

(2) $x = -\log(x)$

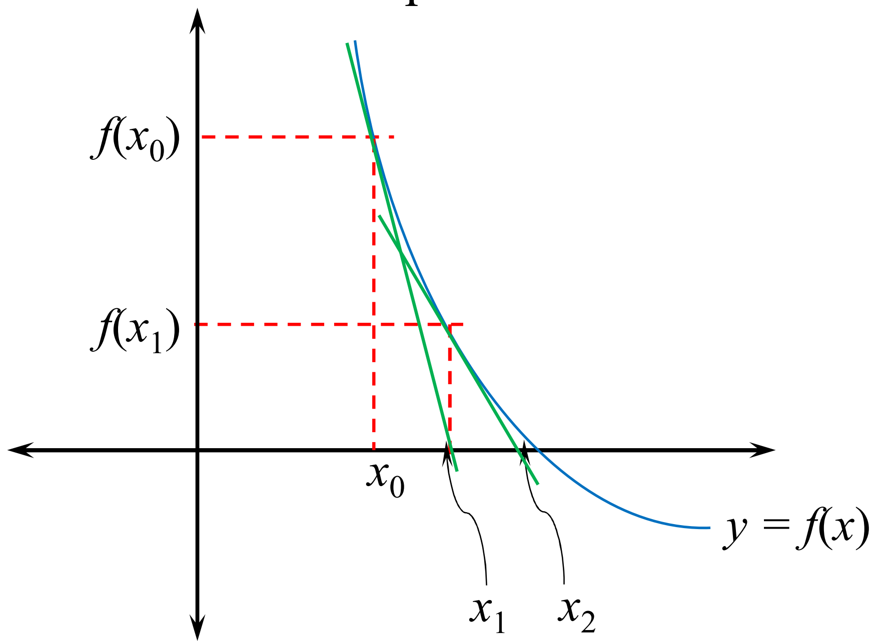
$g'(x) = -\frac{1}{x}$

$s = 0.567$ $\quad |g'(s)| > 1$

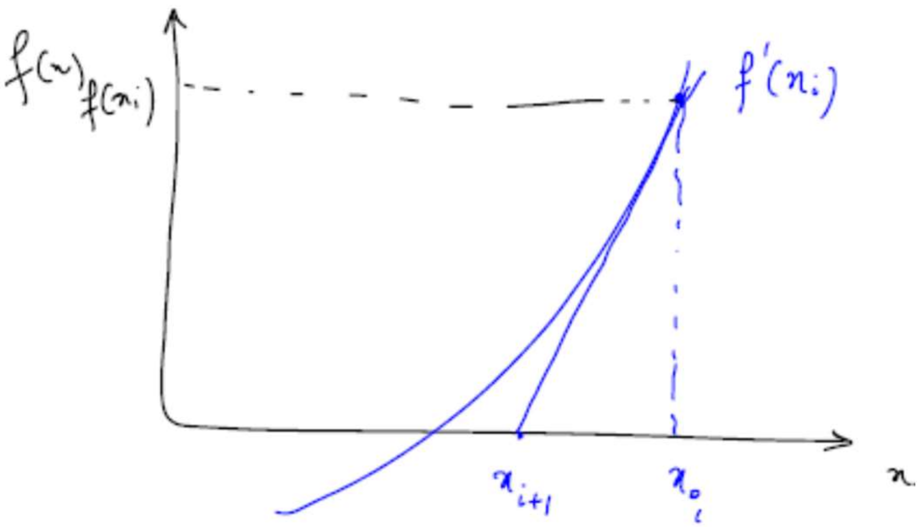
Open Methods

2. Newton Raphson Method

- Problem: $f(x) = 0$, find a root $x = \alpha$ such that $f(\alpha) = 0$
- Principle: Approximate the function as a straight line having same slope as the original function at the point of iteration.



Newton-Raphson Method



$$f'(x_i) = \frac{-f(x_i)}{x_{i+1} - x_i}$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson Method

Example

$$x = \sqrt{a}$$

$$f(x) = x^2 - a = 0$$

$$f'(x) = 2x$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - a}{2x_i}$$

$$= \frac{x_i^2 + a}{2x_i}$$

$$\boxed{x_{i+1} = \frac{1}{2}\left[x_i + \frac{a}{x_i}\right]}$$

Newton-Raphson Method

Convergence

Convergence of NR method

Taylor series

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

$$\Rightarrow \boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Let's assume that at i^{th} step we are just one step away from the true solution

$$f(s) = f(x_i) + (s - x_i) f'(x_i) + \frac{1}{2}(s-x_i)^2 f''(\xi_i)$$

$$\xi \in (x_i, s)$$

Newton-Raphson Method

Convergence

We know S is the solution

$$f(s) = 0$$

$$\Rightarrow \quad 0 = f(x_i) + (s - x_i) f'(x_i) + \frac{1}{2}(s - x_i)^2 f''(\xi)$$

Divide $f'(x_i)$

$$-\frac{f(x_i)}{f'(x_i)} = (s - x_i) + \frac{1}{2}(s - x_i)^2 \frac{f''(\xi)}{f'(x_i)}$$

$$x_{i+1} - \cancel{x_i} - s + \cancel{x_i} = \frac{1}{2}(s - x_i)^2 \frac{f''(\xi_i)}{f'(x_i)}$$

$$e_{i+1} = -\frac{1}{2} e_i^2 \frac{f''(\xi_i)}{f'(x_i)}$$

$$\Rightarrow \quad \frac{|e_{i+1}|}{|e_i|^2} = \left| \frac{1}{2} \frac{f''(\xi)}{f'(x_i)} \right|$$

$i \to \infty$

$$\boxed{\frac{|e_{i+1}|}{|e_i|^2} = \left| \frac{1}{2} \frac{f''(s)}{f'(s)} \right|}$$

- Quadratic convergence

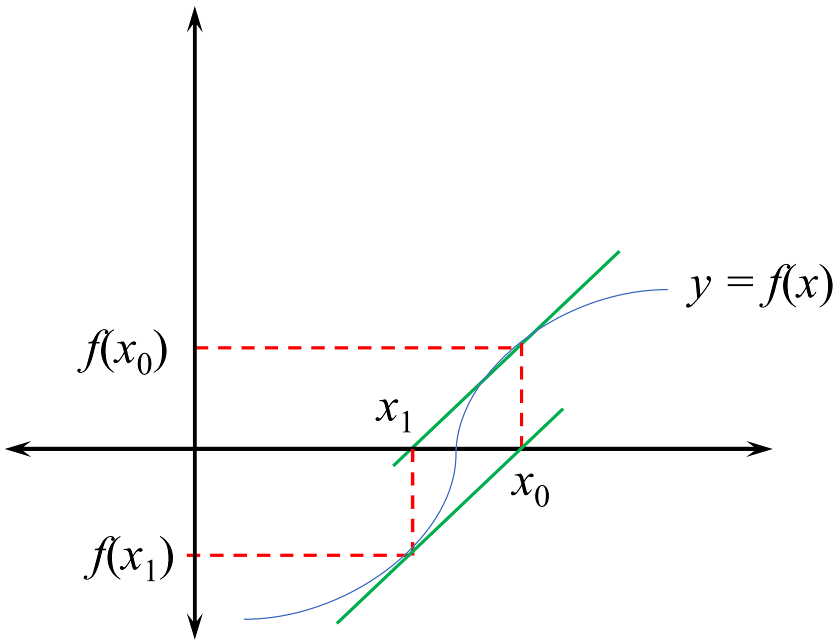
Newton-Raphson Method

Advantages:

Faster convergence (quadratic)

Disadvantages:

Need to calculate derivate



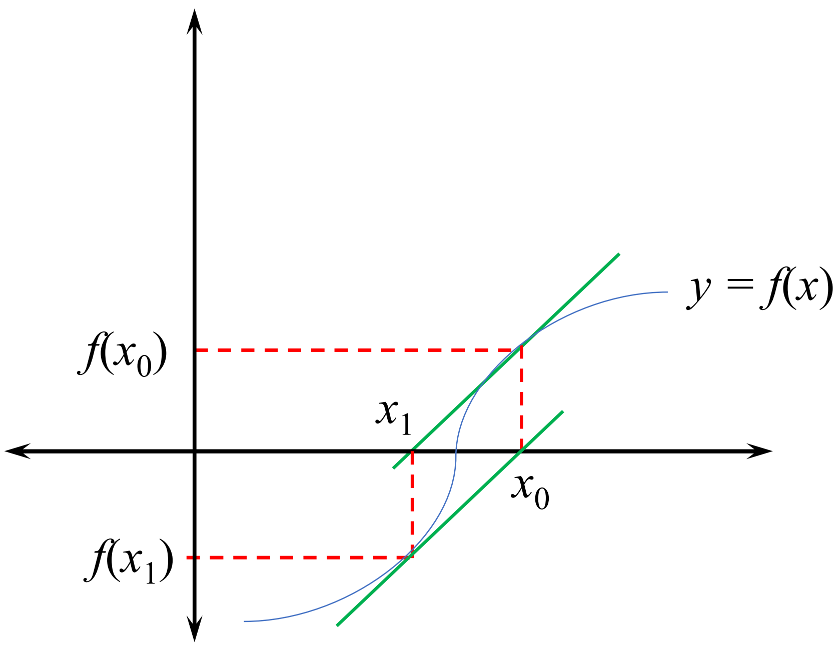
Newton-Raphson method may get stuck!

Newton-Raphson Method

Places where Newton-Raphson may not work

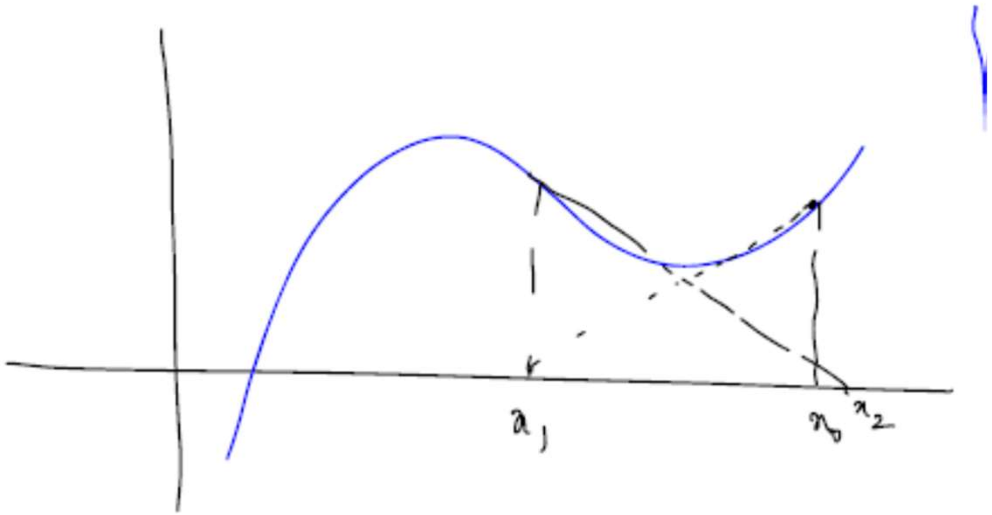
a) Inflection point

Double Derivative =0; Solution is diverging



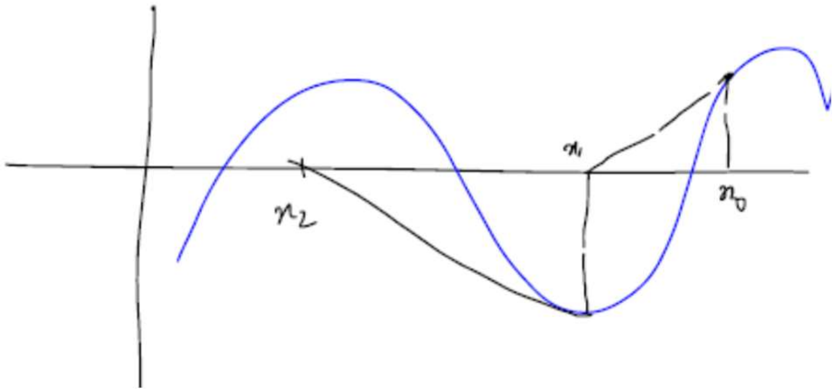
Newton-Raphson Method

b) If you have a local minima, it will trap your solution



Newton-Raphson Method

c) Multiple Solutions



- Convergence depends on the function
- Guess is close to the solution

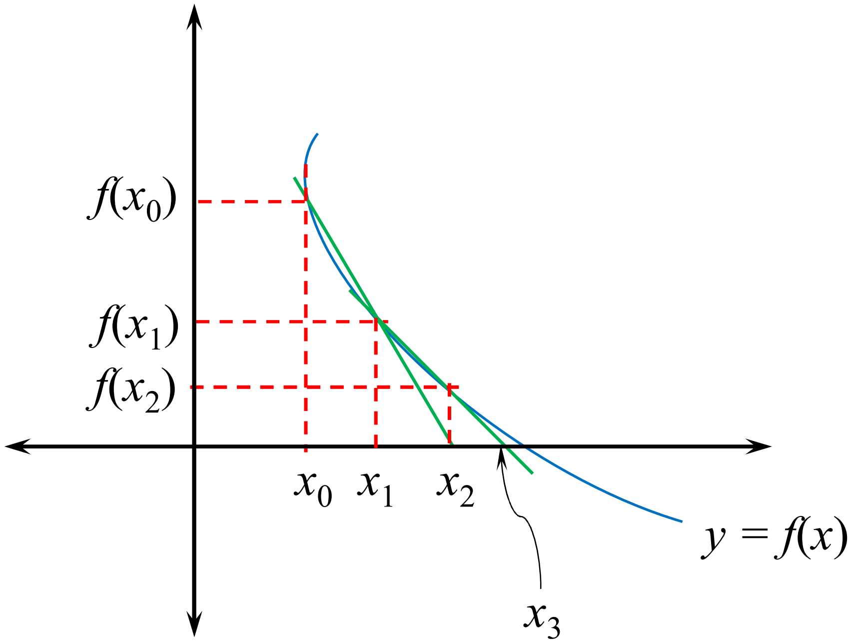
“ No substitute for understanding the problem”

Open Methods

3. Secant Method

- Principle: Use a difference approximation for the slope or derivative in the Newton-Raphson method. This is equivalent to approximating the tangent with a secant.

- Problem: $f(x) = 0$, find a root $x = \alpha$ such that $f(\alpha) = 0$



Secant Method

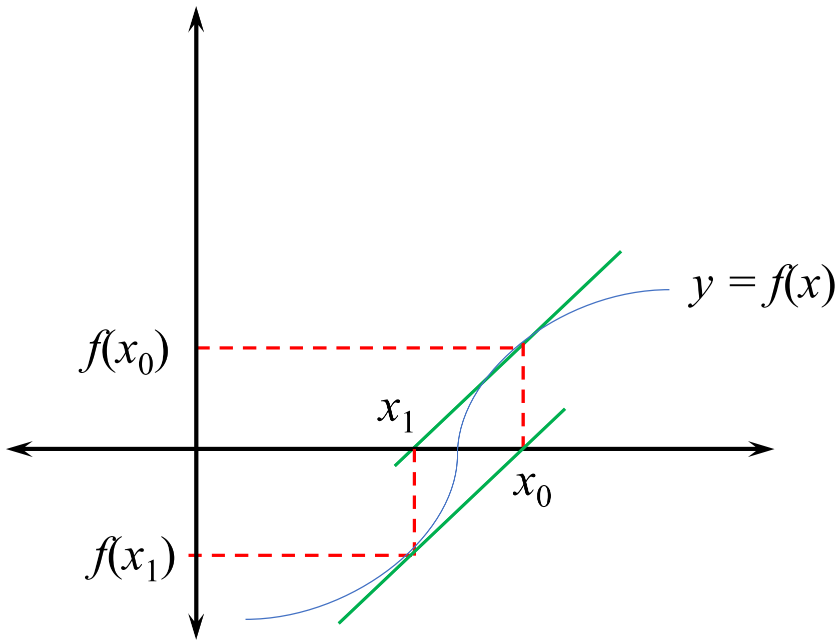
- Problem: $f(x) = 0$, find a root $x = \alpha$ such that $f(\alpha) = 0$

  - Initialize: choose two points x_0 and x_1 and evaluate $f(x_0)$ and $f(x_1)$
  - Approximation: $f'(x_k) \approx \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$, replace in Newton-Raphson
  - Iteration Formula: $x_{k+1} = x_k - f(x_k)\frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})}$
  - Stopping criteria: $\left|\frac{x_{k+1}-x_k}{x_{k+1}}\right| \leq \varepsilon$

Secant Method

Advantages:

- Fast convergence (slightly less than quadratic)
- Overcomes the disadvantage of having to calculate derivate



Secant method may also get stuck!

Look for Modified Secant Method!

Summary

What are fixed-point, Newton-Raphson, and Secant method?

Under what conditions these methods will not work.

What is the convergence rate of these methods?