ESO208A: Computational Methods in Engineering

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Recap

- Direct Methods:
  - Gauss Elimination,
  - Gauss-Jordon Elimination,
  - LU-Decomposition,
  - Thomas Algorithm (for tri-diagonal banded matrix)
  - Cholesky Decomposition

LU decomposition

Error Analysis

Forward Error Analysis

Recall $f(x)$ $x+\Delta x$

$$C_p = \frac{|\Delta f / f|}{|\Delta x / x|}$$

$|x|$

$|\Delta x|$

$$\Rightarrow \frac{|\Delta f|}{|f|} = C_p \left|\frac{\Delta x}{x}\right|$$

Linear system

$$Ax = b$$

$$x = A^{-1} b$$

What would be the relative changes in x, ie $\frac{\Delta x}{x}$, due to small perturbations in A, $\frac{\Delta A}{A}$, or b, $\frac{\Delta b}{b}$.

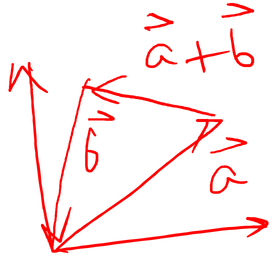
LU decomposition

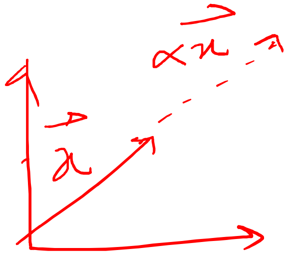
For Error Analysis, we need to first understand vector and matrix norms

Vector Norm

A vector norm is a measure (in some sense) of the size or "length" of a vector

- Properties of Vector Norm:
  - $\|\boldsymbol{x}\| > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$; $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$
  - $\|\alpha \boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$ for a scalar α
  - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$





- L_p-Norm of a vector x:

$$\|\boldsymbol{x}\|_p = \left(|x_1|^p + |x_2|^p \ldots + |x_n|^p\right)^{1/p}$$

- Example Norms:
  - $p = 1$: sum of the absolute values
  - $p = 2$: Euclidean norm
  - $p \rightarrow \infty$: maximum absolute value, $\|\boldsymbol{x}\|_\infty = \max\limits_{0 \leq i \leq n} |x_i|$

$$x = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$L_\infty = 4$$

LU decomposition

Matric Norm: A matrix norm is a measure of the size of a matrix

- Properties of Matrix norm:
  - $||\boldsymbol{A}|| > 0$ for $\boldsymbol{A} \neq \boldsymbol{0}$; $||\boldsymbol{A}|| = 0$ iff $\boldsymbol{A} = \boldsymbol{0}$
  - $||\alpha\boldsymbol{A}|| = |\alpha|||\boldsymbol{A}||$ for a scalar α
  - $||\boldsymbol{A} + \boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||$
  - $||\boldsymbol{AB}|| \leq ||\boldsymbol{A}||||\boldsymbol{B}||$
  - $||\boldsymbol{Ax}|| \leq ||\boldsymbol{A}||||\boldsymbol{x}||$ for consistent matrix and vector norms
- L_p Norm of a matrix $\boldsymbol{A}$:

$$||\boldsymbol{A}||_p = \max_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{||\boldsymbol{Ax}||_p}{||\boldsymbol{x}||_p}$$

LU decomposition

Matric Norm:

- *Column-Sum* norm: $\|\boldsymbol{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$
- *Spectral* norm: $\|\boldsymbol{A}\|_2 = \left(\max_{1\le j\le n} |\lambda_j|\right)^{1/2}$ where, λ_j are the eigenvalues of the square symmetric matrix $\boldsymbol{A}^T\boldsymbol{A}$.
- *Row-Sum* norm: $\|\boldsymbol{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$
- *Frobenius* norm: $\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\mathrm{trace}(\boldsymbol{A}^T\boldsymbol{A})}$

Trace of a matrix is the sum of elements on the main diagonal

LU decomposition

Matric Norm

- Spectral Radius: largest absolute eigenvalue of matrix $\boldsymbol{A}$ denoted by $\rho(A)$

  - If there are *m* distinct eigenvalues of A: $\rho(\boldsymbol{A}) = \max_{1\le i\le m} |\lambda_i|$

  - Lower bound of all matrix norms: $\rho(\boldsymbol{A}) \le \|\boldsymbol{A}\|$

$\boxed{A x_i = \lambda_i x_i}$ λ_i – eigen values
x_i – eigen vectors

Spectral radius provides a lower bound on the matrix norm

$\| A x_i \| = \| \lambda_i x_i \|$

$\| A x_i \| = |\lambda_i| \, \| x_i \|$

But $\| A x_i \| \le \| A \| \, \| x_i \|$

$\Rightarrow \| A \| \, \| x_i \| \ge |\lambda_i| \, \| x_i \|$

$\Rightarrow \boxed{\| A \| \ge |\lambda_i| = \rho(A)}$ $\max |\lambda_i|$

  - For any norm of matrix A: $\rho(\boldsymbol{A}) = \lim_{n\to\infty} \|\boldsymbol{A}^n\|^{1/n}$

LU decomposition

Matric Norm:

Example

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\|A\|_\infty = 9$$

$$\|A\|_1 = 8 \qquad \begin{bmatrix} 6 & 8 \end{bmatrix}$$

LU decomposition

Condition Number

Condition number $\qquad Ax = b$

(a) Perturb A

$$(A + \Delta A)(x + \Delta x) = b$$

$$\cancel{Ax} + \Delta A x + A \Delta x + \Delta A \Delta x = \cancel{b}$$

$$\Rightarrow \quad \Delta x = -A^{-1} \Delta A (x + \Delta x)$$

Norm

$$\|\Delta x\| = \|A^{-1} \Delta A (x + \Delta x)\|$$

$$\leq \|A^{-1}\| \, \|\Delta A (x + \Delta x)\|$$

$$\leq \|A^{-1}\| \, \|\Delta A x\| + \|A^{-1}\| \cancel{\|\Delta A \Delta x\|}$$

If we assume $\|\Delta A \Delta x\| \leq$

$$\|\Delta x\| \leq \|A^{-1}\| \, \|\Delta A\| \, \|x\|$$

$$\Rightarrow \quad \frac{\|\Delta x\|}{\|x\|} \leq \underbrace{\|A^{-1}\| \, \|A\|}_{C(A)} \frac{\|\Delta A\|}{\|A\|}$$

$$\boxed{C_A = \|A^{-1}\| \, \|A\|}$$

(b) Perturb b

$$A(x + \Delta x) = (b + \Delta b)$$

$$\Rightarrow \quad \Delta x = A^{-1} \Delta b$$

$$\Rightarrow \quad \|\Delta x\| \leq \|A^{-1}\| \, \|\Delta b\|$$

$$\leq \|A^{-1}\| \, \|b\| \frac{\|\Delta b\|}{\|b\|}$$

$$\leq \|A^{-1}\| \, \|A\| \, \|x\| \frac{\|\Delta b\|}{\|b\|}$$

$$\Rightarrow \quad \boxed{\frac{\|\Delta x\|}{\|x\|} \leq \|A^{-1}\| \, \|A\| \frac{\|\Delta b\|}{\|b\|}} \qquad C(A) = \|A^{-1}\| \, \|A\|$$

LU decomposition

Condition Number

Example

$x_1 + 2x_2 = 10$ (4 4 3)

$1.1x_1 + 2x_2 = 10.4$

1.05

8 4 1

$\|\Delta x\|_\infty = 4$

$$A = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix}$$

$$\|A\|_\infty = 3.1$$

$$A^{-1} = \begin{bmatrix} -10 & 10 \\ 5.5 & -5 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = 20$$

$$C(A) = \|A^{-1}\|_\infty \|A\|_\infty = 3.1 \times 20 = 62$$

ill-conditioned

Smallest condition number

$A = I \quad C(A) = 1$

LU decomposition

Condition Number

If we change 1.1 to 1.05, what would be the corresponding change in x

$$\frac{\|\Delta x\|}{\|x\|} \leq C_p \frac{\|\Delta A\|}{\|A\|}$$

$$\leq 62 \cdot \frac{0.05}{3.1}$$

$$x = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\frac{\|\Delta x\|}{\|x\|} \leq 1$$

$$\|\Delta x\| \leq \|x\|_\infty$$

$$\|\Delta x\| \leq 4$$

Condition Number

Condition Number

Recall: Determinant is not a good measure of the ill or well conditioning of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix} \qquad \det(A) = -0.2$$

$$A = \begin{bmatrix} 1000 & 2000 \\ 1100 & 2000 \end{bmatrix} \qquad \det(A) = 2\times 10^5$$

$C(A) = 62$

$$\|A\|_\infty = 3100$$

$$A^{-1} = \begin{bmatrix} -0.01 & 0.01 \\ 0.0055 & -0.005 \end{bmatrix} \qquad \|A^{-1}\|_\infty = 0.02$$

$$C(A) = \|A^{-1}\|_\infty \|A\|_\infty = \underline{\underline{62}}$$

Invariant

Measure of C(A) is independent of the scaling, which is a good thing

Condition Number

Question: It is always recommended that after estimating X, substitute it in the equation and see whether the equation is satisfied or not. Is the residual r= $b - \tilde{b}$ a good measure for e= $x - \tilde{x}$

$$Ax = b$$

Instead of x we estimated $\tilde{x}$

$$A\tilde{x} = \tilde{b}$$

$r = b - \tilde{b}$ is small, then the estimate $\tilde{x}$ is good?

Condition Number

Question: It is always recommended that after estimating X, substitute it in the equation and see whether the equation is satisfied or not.

$$e = x - \tilde{x}$$

$$r = b - \tilde{b}$$

$$Ax - A\tilde{x} = r$$

$$\Rightarrow A(x - \tilde{x}) = r$$

$$\Rightarrow e = A^{-1} r$$

$$\Rightarrow \|e\| \le \|A^{-1}\| \, \|r\| \quad \text{--- (1)}$$

$$Ax = b$$

$$\|b\| = \|Ax\|$$

$$\|b\| \le \|A\| \, \|x\|$$

$$\Rightarrow \|x\| \ge \frac{\|b\|}{\|A\|} \quad \text{--- (2)}$$

$$\frac{\|e\|}{\|x\|} \le \frac{\|A^{-1}\| \, \|r\|}{\|b\| / \|A\|}$$

$$\le \|A^{-1}\| \, \|A\| \frac{\|r\|}{\|b\|}$$

$$\Rightarrow \boxed{\frac{\|e\|}{\|x\|} \le C(A) \frac{\|r\|}{\|b\|}}$$

Iterative Refinement or Improvement

Iterative Refinement or Improvement

$$\boxed{Ax = b}$$

$$A\tilde{x} = \tilde{b}$$

$$Ax - A\tilde{x} = b - \tilde{b}$$

$$A(x - \tilde{x}) = r$$

$$\Rightarrow \boxed{Ae = r}$$

Unknow "e" can be estimated by $O(n^2)$

$$e = x - \tilde{x}$$

$$\Rightarrow \boxed{x = \tilde{x} + e}$$

Example

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

$$A\tilde{x} = \tilde{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$r = b - \tilde{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$Ae = r \longrightarrow e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$x = \tilde{x} + e$$

$$= \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Summary

- Forward error analysis
- Vector norm and matrix norm
- Condition number of a matrix