Tutorial - 11

speed & acceleration at 5 seconds

forward difference $(O(h^2))$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$$

$$f''(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$$

t	0	1	2	3	4	5	6	7	8	9
x	1	1.55	2.32	3.58	5.79	9.68	16.99	28.22	48.2	81.92
						i	$i+1$	$i+2$	$i+3$	

$h = 5 - 6 = 1$

$$f'(x_i)\Big|_{i=5} = \frac{-28.22 + 4\times(16.99) - 3\times(9.68)}{2}$$

$$f'(x_i)_{i=5} = 4.35 = \text{speed } (v)$$

$$f''(x_i) = \frac{-48.2 + 4\times(28.22) - 5\times(16.99) + 2\times 9.68}{(1)^2}$$

$$f''(x_i)_{i=5} = 1.59 = \text{acceleration } (a)$$

$$x = e^{0.5t} - 0.1t^2 \qquad \frac{dx}{dt} = 0.5e^{0.5t} - 0.2t$$

$$v\Big|_{t=5} = 0.5e^{0.5\times 5} - 0.2(5) = 5.091 \text{ m/s}$$

Backward diff.

(2)

$$\frac{d^2x}{dt^2} = a = 0.25e^{0.5t} - 0.2$$

$$a\Big|_{t=5} = 0.25e^{0.5\times5} - 0.2 = 2.846 \text{ m/s}^2$$

true error

$$e_{vel} = \left|\frac{5.091 - 4.350}{5.091}\right| \times 100$$

$$= 14.56$$

Similarly

$$e_{acc} = 44.12$$

Backward diff. $O(h^2)$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$$

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$$

t	2	3	4	5
x	2.32	3.58	5.79	9.68
	i-3	i-2	i-1	i

$$h = 5 - 4 = 1$$

$$f'(x_i) = \frac{3\times9.68 - 4\times5.79 + 3.58}{2\times(1)} = 4.730$$

$$f''(x_i) = 2.410$$

true Error

$E_{vel} = 7.10\ \%$

$E_{acc} = 15.31\ \%$

$\Rightarrow$ central diff $O(h^2)$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

t	4	5	6
x	5.79	9.68	16.99
	$i-1$	i	$i+1$

$$f'(x_i) = \frac{16.99 - 5.79}{2} = 5.35$$

$$f''(x_i) = \frac{16.99 - 2\times(9.68) + 5.79}{(1)^2} = 2.92$$

true Error

$E_{vel} = 5.08\ \%$

$E_{acc} = 2.61\ \%$

Richardson Extrapolation (velocity)

h	central diff $O(h^2)$ [itr $k=0$]	$O(h^4)$ [$k=1$]	$O(h^6)$ [$k=2$]
	$= \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$		$D_{h/2}^{k+1} = \frac{4^k D_{h/2} - D_h}{4^k - 1}$
4	$D_4^1 = \frac{f(x)\|_{t=9} - f(x)\|_{t=1}}{2\times 4}$	×	×
	$= \frac{81.92 - 1.55}{8} = (10.096)$		
2	$D_2^1 = \frac{f(x)\|_{t=7} - f(x)_{t=3}}{2\times 2}$	$D_2^2 = \frac{4\times 6.16 - 10.096}{3}$	×
	$= \frac{22.22 - 3.58}{4} = (6.16)$	$D_2^2 = (4.86)$	
1	$D_1^1 = \frac{f(x)\|_{t=6} - f(x)_{t=4}}{2\times(1)}$	$D_1^2 = \frac{4\times 5.35 - 6.16}{3}$	$D_1^3 = \frac{16\times 5.08 - 4.86}{15}$
	$= \frac{16.49 - 5.79}{2\times 1} = (5.35)$	$= (5.08)$	$= (5.094)$

Accelaration

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
4	4.007	X	X
2	3.110	2.811	X
1	2.920	2.857	2.860

corresponding Errors

$$= \left| \frac{\text{true Value} - \text{Approx Value}}{\text{true Value}} \right| \times 100$$

$$= \left| \frac{2.896 - 2.811}{2.896} \right| \times 100$$

$$= 1.22\ \%$$

$$= \left| \frac{2.896 - 2.857}{2.896} \right| \times 100$$

$$= 0.39\ \%$$

$$= \left| \frac{2.896 - 2.860}{2.896} \right| \times 100$$

$$= 0.49\ \%$$

Answer - 02

$$f(x) = \frac{\sin x}{x^3}$$

interval = $[1, 2\pi]$

20 equally spaced points $h = \left(\frac{2\pi - 1}{19}\right)$

$h = 0.278062$

$O(h) \Rightarrow$ backward diff.

$O(h^2) \Rightarrow$ central diff.

$O(h^4) \Rightarrow$ central diff

$O(h)$ backward $f' = \frac{f(x_i) - f(x_{i-1})}{h}$

$O(h^2)$ central $f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$

$O(h^4)$ central $f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$

data required ~~$i = 1 \longrightarrow i = 22$~~

$i = -1 \longrightarrow i = 22$

for $x = 1$

x	$f(x) = \frac{\sin x}{x^3}$	
$1-2h = 0.9438$	4.910	$i-2$
$1-h = 0.7219$	1.756	$i-1$
1	0.8414	i
$1+h = 1.278$	0.458	$i+1$
$1+2h = 1.556$	0.2653	$i+2$

$O(h)$ backward $f'(x) = \frac{f(x_i) - f(x_{i-1})}{h}$

$$= \frac{0.8414 - 1.756}{0.2780}$$

$$= -3.2899$$

$O(h^2)$ central $f'(x) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$

$$= \frac{0.458 - 1.756}{2 \times 0.2780}$$

$$= -2.334$$

$O(h^4)$ central $f'(x) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$

$$= -\frac{0.2653 + 8 \times 0.458 - 8 \times 1.756 + 4.91}{12 \times 0.2780}$$

$$= -1.445$$

$$f'(\text{true}) = \frac{d}{dx}\frac{\sin x}{x^3}$$

$$= \frac{\cos x}{x^3} - \frac{3\sin x}{x^4}$$

$$f'(x)\Big|_{x=1} = \frac{\cos(1)}{(1)} - \frac{3\sin(1)}{(1)}$$ → radians.

$$\boxed{f'(x)\Big|_{x=1} = -1.98411}$$

Point/Node	x	f	Backward	Centralh^2	Centralh^4	True
	0.4438752	4.9104599				
	0.7219376	1.7562948				
1	1	0.841471	-3.289994705	-2.333403231	-1.445494924	-1.98411
2	1.2780624	0.4586314	-1.376811758	-1.035955339	-0.758288507	-0.93832
3	1.5561248	0.2653506	-0.695098921	-0.543336483	-0.420017191	-0.50767
4	1.8341872	0.1564685	-0.391574046	-0.313628884	-0.248639207	-0.29811
5	2.1122495	0.0909338	-0.235683722	-0.191485145	-0.153856817	-0.18384
6	2.3903119	0.0499789	-0.147286567	-0.120379491	-0.097441716	-0.11624
7	2.6683743	0.0239878	-0.093472414	-0.07623802	-0.061935159	-0.07382
8	2.9464367	0.0075811	-0.059003625	-0.047575282	-0.038672218	-0.04607
9	3.2244991	-0.00247	-0.036146938	-0.028411715	-0.023026999	-0.02743
10	3.5025615	-0.0082194	-0.020676491	-0.015405174	-0.012368718	-0.01473
11	3.7806238	-0.0110372	-0.010133857	-0.006570484	-0.005114629	-0.0061
12	4.0586862	-0.0118734	-0.00300711	-0.000662349	-0.000261948	-0.00032
13	4.3367486	-0.0114056	0.001682413	0.003140869	0.002856334	0.003392
14	4.614811	-0.0101267	0.004599325	0.00540742	0.004704657	0.005592
15	4.8928734	-0.0083984	0.006215515	0.006547571	0.005620261	0.006682
16	5.1709358	-0.0064854	0.006879627	0.006868779	0.005857081	0.006964
17	5.4489982	-0.0045785	0.00685793	0.006608414	0.005611933	0.006673
18	5.7270605	-0.0028103	0.006358898	0.005953586	0.005040429	0.005993
19	6.0051229	-0.0012676	0.005548275	0.005053395	0.004266852	0.005074
20	6.2831853	6.173E-18	0.004558514	0.004026687	0.003390418	0.004031
	6.5612477	0.0009718				
	6.8393101	0.0016501