ESO208A: Computational Methods in Engineering

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Recap

- Direct Methods:
  - Gauss Elimination,
  - Gauss-Jordon Elimination,
  - LU-Decomposition,
  - Thomas Algorithm (for tri-diagonal banded matrix)
  - Cholesky Decomposition

- Forward Error Analysis

Indirect Methods

Indirect or Iterative Methods

- Jacobi Iteration
- Gauss Seidal
- Relaxation Technique

All these methods are version of fixed-point iteration for linear system of equations

Fixed-Point Method

RECAP 1. $f(x) = 0$

Rearrange. $x_{i+1} = g(x_i)$

Convergence $|g'(s)| < 1$

Convergence rate : linear

2. $u(x, y) = 0 \Rightarrow x_{i+1} = g_1(x_i, y_i)$

$v(x, y) = 0 \Rightarrow y_{i+1} = g_2(x_{i+1}, y_i)$

Convergence

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$$

Fixed-Point Method

Example

a_{13}/a_{11} $\quad$ a_{13}/a_{11}

$E_1 \quad a_{11}x + a_{12}y + a_{13}z - b_1 = 0 \quad - ①$

$E_2 \quad a_{21}x + a_{22}y + a_{23}z - b_2 = 0$

$E_3 \quad a_{31}x + a_{32}y + a_{33}z - b_3 = 0$

$x = g_1(y,z) \qquad x = \dfrac{b_1 - a_{12}y - a_{13}z}{a_{11} = 1}$

$y = g_2(x,z) \qquad y = \dfrac{b_2 - a_{21}x - a_{23}z}{a_{22}}$

$z = g_3(x,y) \qquad z = \dfrac{b_3 - a_{31}x - a_{32}y}{a_{33}}$

Jacobi and Gauss Seidal

- Jacobi Iteration

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_i, z_i)$$
$$z_{i+1} = g_3(x_i, y_i)$$

- Gauss Seidal

$$x_{i+1} = g_1(y_i, z_i)$$
$$y_{i+1} = g_2(x_{i+1}, z_i)$$
$$z_{i+1} = g_3(x_{i+1}, y_{i+1})$$

- Gauss-Seidal method is faster than Jacobi method
- Relaxation Method: A way to improve convergence

Jacobi and Gauss Seidal

- Example

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$$

$$x = \frac{4 - y}{2}$$

$$y = \frac{8 - x - \cancel{y}}{2} \quad z$$

$$z = \frac{8 - y}{2}$$

- Jacobi Iteration

iter	x	y	z
0.	0	0	0
1	2	4	4
2.	0	1	2
3	1.5	3.0	3.5

Jacobi and Gauss Seidal

- Gauss Seidal

iter	x	y	z
0	0	0	0
1	2	3	2.5
2	0.5	2.5	2.75
3	0.75	2.25	2.875

Jacobi and Gauss Seidal

Comments

- Useful when dealing with large sparse systems
- To save computation time divide the equation by its diagonal.

  It saves computation, but can introduce round-off error.
- Convergence is not guaranteed [like FP methods]. If you get convergence, its linear convergence.

Jacobi and Gauss Seidal

Comments

- Convergence Criteria

$$\left|\cancel{\frac{\partial g_1}{\partial x}}\right| + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| < 1$$

$$x = \frac{b_1}{a_{11}} - \frac{a_{12}\,y}{a_{11}} - \frac{a_{13}}{a_{11}}z = g(y,z)$$

$$\frac{|a_{12}|}{|a_{11}|} + \frac{|a_{13}|}{|a_{11}|} < 1$$

$$\Rightarrow |a_{11}| > |a_{12}| + |a_{13}|$$

$$\Rightarrow |a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

Diagonally dominant system

The magnitude of diagonal element should be greater than sum of absolute values of all off diagonal terms

Jacobi and Gauss Seidal

- Convergence Criteria
- The magnitude of the diagonal element should be greater than the sum of absolute values of all off-diagonal elements. Such systems are called Diagonal dominant system
- The criteria for convergence is sufficient but not necessary i.e. the method may converge even if the criteria is not met.

Relaxation Techniques

Relaxation techniques

$$x'_{i+1} = g_1(y_i, z_i)$$

$$\boxed{\begin{aligned} x_{i+1} &= \lambda x'_{i+1} + (1-\lambda) x_i \\ \Rightarrow \quad x_{i+1} &= x_i + \lambda (x'_{i+1} - x_i) \end{aligned}}$$

λ is a weighing factor (Relaxation factor)

$\boxed{\lambda \in (0,2)}$

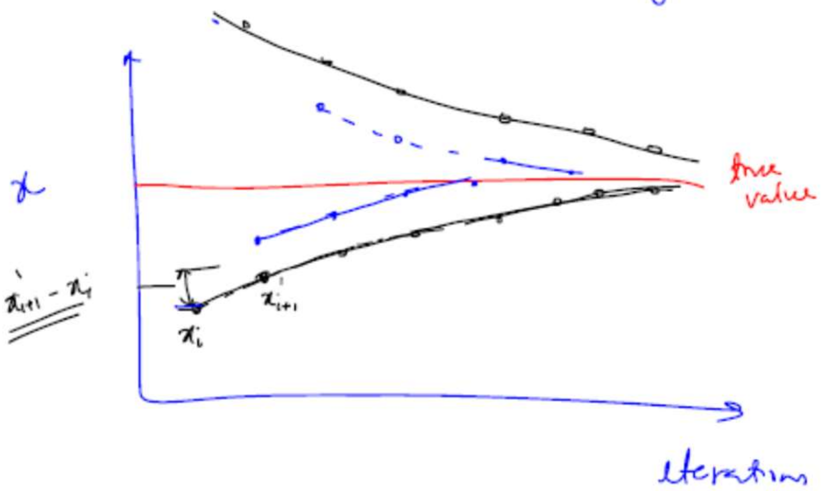
If $\lambda = 1$ $\quad x_{i+1} = x'_{i+1}$

Gauss-Seidel method

Relaxation Techniques

if $\lambda > 1$ - more weightage to the present term x'_{i+1}

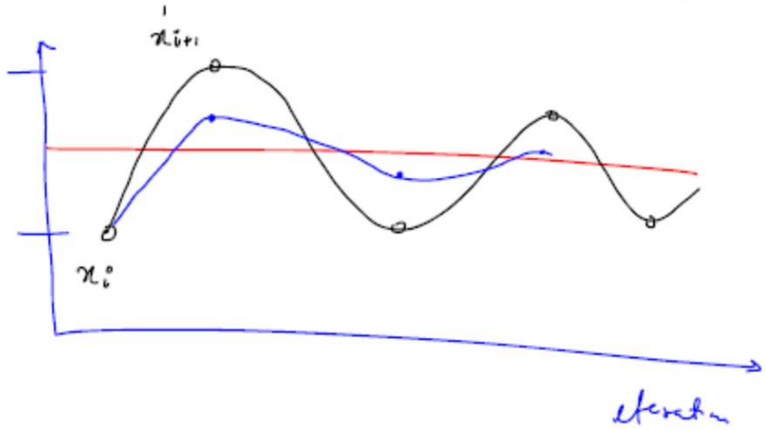
Over-relaxation - Improves convergence



Successive over relaxation (SOR)

if $\lambda < 1$

Under relaxation - dampens the oscillation



Relaxation Techniques

- Example

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

Gauss-Seidel

$$A = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$x = 2 - 0.5y$$

$$y = 4 - 0.5x - 0.5z$$

$$z = 4 - 0.5y$$

$$x_{i+1} = g_1(y_i, z_i)$$

Relaxation Techniques

- Example

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

Gauss-Seidel

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$$x = 2 - 0.5y$$
$$y = 4 - 0.5x - 0.5z$$
$$z = 4 - 0.5y$$

$$x_{i+1} = g_1(y_i, z_i)$$

iteration	x	y	z	e_a
0	0	0	0	
1	2	3	2.5	100 %
2.	0.5	2.5	2.75	

e_a is the maximum of relative error in x, y and z

Relaxation Techniques

- Example

Example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}$$

Gauss-Seidel

$$A = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$x = 2 - 0.5y$$
$$y = 4 - 0.5x - 0.5z$$
$$z = 4 - 0.5y$$

$$x_{i+1} = g_1(y_i, z_i)$$

Suppose it is given $e_a < 0.1\%$. If any of the variable, exceeds your threshold go to next iteration.

Step 1:

$$x'_{i+1} = 2 - 0.5y_i$$
$$x_{i+1} = x_i + \lambda(x'_{i+1} - x_i)$$

$$y'_{i+1} = 4 - 0.5x_{i+1} - 0.5\, z_i$$
$$y_{i+1} = y_i + \lambda(y'_{i+1} - y_i)$$

Over relaxation $\lambda = 1.2$

iteration	x'	x	y'	y	z'	z
0		0		0		0
1	2	2.4	2.8	3.36	2.32	2.784
2	0.32	-0.096				

After 7 iterations, the solution converges. Try it!

Relaxation Techniques

How do we get the optimal value of λ?

- Problem Specific
- The usual procedure is to do empirical evaluation
  - Useful when the system has to be solved a number of times
- Can use this λ for solving x for different values of b

GAPS

- Why GS is faster than Jacobi?
- The convergence criteria is sufficient (not necessary)
- Why the relaxation techniques work?
- $\lambda \in (0,2)$, Why this range works?

To answer these, we need to study Eigen Values and Eigen Vectors

Summary

- Gauss-Seidal
- Jacobi Iteration
- Successive Over Relaxation Technique