# Tutorial - 10

(1.) $$I = \int_0^2 e^x dx$$

Analytical Sol$^n$ $= e^x \Big|_0^2$

$$I_{true} = e^2 - 1$$

$$I_{true} = 6.389$$

Numerical integration

divide the range of integration into 1, 2, 4, 8 segments.

1 segment

| i | x | y = f(x) |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 2 | 7.389 |

$h = 2$

Trapezoidal Rule

$$I = h\left[\frac{f_0}{2} + \frac{f_n}{2} + \sum_{i=1}^{n-1} f_i\right]$$

$$I = 2\left[\frac{1}{2} + \frac{7.389}{2}\right]$$

$$I = 8.389$$

$$E_r = \left|\frac{I_{true} - I}{I_{true}}\right| \times 100 = 100\left|\frac{6.389 - 8.389}{6.389}\right|$$

$$E_r = 31.3\ \%$$

Simpson's 1/3 rule for 1 point segment

⇒ for simpson's 1/3 rule we need ~~minimum~~ at least 3 no of data points. Hence for 1 point segment simpson 1/3 rule can not be used.

8 point segment.

| i | x | y = f(x) |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.25 | 1.284 |
| 2 | 0.5 | 1.648 |
| 3 | 0.75 | 2.117 |
| 4 | 1.0 | 2.718 |
| 5 | 1.25 | 3.49 |
| 6 | 1.5 | 4.481 |
| 7 | 1.75 | 5.75 |
| 8 | 2 | 7.389 |

[ $h = 0.25$ ]

Trapezoidal Rule. [n=8]

$$I = h\left[\frac{f_0}{2} + \frac{f_n}{2} + \sum_{i=1}^{n-1} f_i\right]$$

$$I = \frac{h}{2}\left[f_0 + f_n + 2\sum_{i=1}^{n-1} f_i\right]$$

$$I = 6.4222$$

$$E_r = \left|\frac{I_{true} - I}{I_{true}}\right| \times 100 = \left|\frac{6.389 - 6.42}{6.389}\right| \times 100$$

$$\boxed{E_r = 0.520}$$

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Simpson's 1/3 Rule

$$I = \frac{h}{3}\left[ f_0 + f_n + 4\sum_{i=1,3,5}^{n-1} f_i + 2\sum_{i=2,4,6}^{n-2} f_i \right]$$

$$I = 6.38919$$

$$E_r = \left| \frac{I_{true} - I}{I_{true}} \right| \times 100 = \left| \frac{6.389056 - 6.38919}{6.389056} \right| \times 100$$

$$\boxed{E_r = 0.00215\%}$$

Ratio of errors for successive interval sizes

$$\text{Ratio for 8 segment} = \frac{E_r \text{ for 4 segments}}{E_r \text{ for 8 segments}}$$

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for Simpson 1/3 Rule

| Segment | $h$ | I | $E_r$ | Ratio |
|---|---|---|---|---|
| 1 | 2 | — | | |
| 2 | 1 | 6.42073 | 0.4957 | |
| 4 | 0.5 | 6.39121 | 0.0337 | $= \frac{0.4957}{.0337} = 14.703$ |
| 8 | 0.25 | 6.38919 | 0.0021 | $= \frac{.0337}{.0021} = 15.6517$ |

## Answer-02 (a)

$O(h^8)$ Romberg integration

Trapezoidal Rule ⇒ global error $2^{nd}$ order

(can use other simpson's Rule as well)

Analytical integration

$$I_{true} = \int_{-2}^{2} x e^{-x} dx = -x e^{-x} - e^{-x} \Big|_{-2}^{2}$$

$$I_{true} = -3e^{-2} - e^{2}$$

$$I_{true} = -7.795$$

for $O(h^8)$ Romberg integration we need to evaluate integration upto 8 point segment.

$$I^{k+1}(h) = \frac{4^k I(h/2) - I_k(h)}{4^k - 1}$$

| K | $O(h^2)$ [$=I$ from trapezoidal method] | $O(h^4)$ | $O(h^6)$ | $O(h^8)$ |
|---|---|---|---|---|
| 1 | $I_1 = -29.014$ (Segment 1) | | | |
| | | $I_1' = \frac{4}{3} I_2 - \frac{1}{3} I_1 = -9.6716$ | | |
| 2 | $I_2 = -14.507$ (Segment 2) | | $I_1'' = \frac{16}{15} I_2' - \frac{1}{15} I_1' = -7.856$ | |
| | | $I_2' = \frac{4}{3} I_3 - \frac{1}{3} I_2 = -7.969$ | | $I_1''' = \frac{64}{63} I_2'' - \frac{1}{63} I_1'' = -7.7955$ |
| 3 | $I_3 = -9.604$ (Segment 4) | | $I_2'' = \frac{16}{15} I_3' - \frac{1}{15} I_2' = -7.796$ | |
| | | $I_3' = \frac{4}{3} I_4 - \frac{1}{3} I_3 = -7.807$ | | |
| 4 | $I_4 = -8.2565$ (Segment 8) | | | |

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True Error

Order $\quad E_t$

$O(h^2)$ $= \left|\frac{I_{true} - I_1}{I_{true}}\right| \times 100 = 272.22$

$O(h^4)$ $= \left|\frac{I_{true} - I_1'}{I_{true}}\right| \times 100 = 24.073$

$O(h^6)$ $= \left|\frac{I_{true} - I_1''}{I_{true}}\right| \times 100 = 0.7845$

$O(h^8)$ $= \left|\frac{I_{true} - I_1'''}{I_{true}}\right| \times 100 = 0.006374$

Approximate Error

$$E_a = \left|\frac{I^{k+1}(h) - I^k(h/2)}{I^{k+1}(h)}\right| \times 100$$

Order $\quad E_a$

$O(h^2)$ — [ b/c $I^k(h/2)$ not available ]

$O(h^4)$ $= \left|\frac{I_1' - I_2}{I_1'}\right| \times 100$

$= 50.0$

$O(h^6)$ $= \left|\frac{I_1'' - I_2'}{I_1''}\right| \times 100 = \left|\frac{-7.856 + 7.969}{-7.856}\right| \times 100$

$= 1.4442$

$O(h^8)$ $= \left|\frac{I_1''' - I_2''}{I_1'''}\right| \times 100 = 0.0122$

Answer - 02 (b)

$$I = \int_{-2}^{2} x e^{-x} dx$$

use two & three point Gauss Legendre formulas

Analytical Integration

$$I = -7.795062$$

$$\int_{-1}^{1} f(y)dy = \sum_{i=0}^{n} c_i f(y_i)$$

Two point $(i=1)$

$$\int_{-1}^{1} f(y)dy = c_0 f(y_0) + c_1 f(y_1)$$

$$c_0 = c_1 = 1$$

$$y_0 = -1/\sqrt{3} \quad y_1 = 1/\sqrt{3}$$

$[i=2]$ Three point

$$\int_{-1}^{1} f(y)dy = c_0 f(y_0) + c_1 f(y_1) + c_2 f(y_2)$$

$$c_0 = c_2 = 5/9 \quad c_1 = 8/9$$

$$y_0 = -\sqrt{3/5} \quad y_1 = 0 \quad y_2 = \sqrt{3/5}$$

$$I = \int_{-2}^{2} f(x)\,dx$$

limits are not $[-1, 1]$

$$y = x/2$$

$$I = \int_{-1}^{1} f(y)\,dy$$

$$= \int_{-1}^{1} (2y)\exp(-2y)\,2\,dy$$

$$I = \int_{-1}^{1} (4y)\,e^{-2y}\,dy \qquad \left[f(y) = 4y\,e^{-2y}\right]$$

Two-point Gauss - Legendre formula

$$\tilde{I} = \int_{-1}^{1} f(y)\,dy = c_0 f(y_0) + c_1 f(y_1)$$

$$c_0 = c_1 = 1$$

$$y_0 = -1/\sqrt{3} \quad y_1 = 1/\sqrt{3}$$

$$\tilde{I} = f(-1/\sqrt{3}) + f(1/\sqrt{3}) \qquad \left[f(y) = 4y\,e^{-2y}\right]$$

$$= 4\left(\frac{-1}{\sqrt{3}}\right)e^{-2(-1/\sqrt{3})} + 4\left(\frac{1}{\sqrt{3}}\right)e^{-2(1/\sqrt{3})}$$

$$\tilde{I} = -6.60009$$

$$E_r = \left|\frac{I_{true} - \tilde{I}}{I_{true}}\right| \times 100 = \left|\frac{-7.795 + 6.6001}{-7.795}\right| \times 100$$

$$\boxed{E_r = 15.33\%}$$

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Three point Gauss - Legendre Quadrature

$$\int_{-1}^{1} f(y)dy = c_0 f(y_0) + c_1 f(y_1) + c_2 f(y_2)$$

$$c_0 = c_2 = 5/9 \qquad c_1 = 8/9$$

$$y_0 = -\sqrt{3/5} \qquad y_1 = 0 \qquad y_2 = \sqrt{3/5}$$

$$\tilde{I} = \frac{5}{9} f(-\sqrt{3/5}) + 8/9\, f(0) + 5/9\, f(\sqrt{3/5})$$

$$\tilde{I} = \frac{5}{9}\left[4\left(-\frac{\sqrt{3}}{\sqrt{5}}\right)e^{2\sqrt{3/5}}\right] + 0 + \frac{5}{9}\left[4\sqrt{\frac{3}{5}}\, e^{-2\sqrt{3/5}}\right]$$

$$\tilde{I} = -7.7378$$

$$E_r = \left|\frac{-7.795 + 7.7378}{-7.795}\right| \times 100$$

$$\boxed{E_r = 0.7346\ \%}$$

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Answer - 03

$$I = \int_{-2}^{\infty} x e^{-x} \, dx \quad \text{[improper integral]}$$

$$I = \underbrace{\int_{-2}^{2} x e^{-x} \, dx}_{I_A} + \underbrace{\int_{2}^{\infty} x e^{-x} \, dx}_{I_B}$$

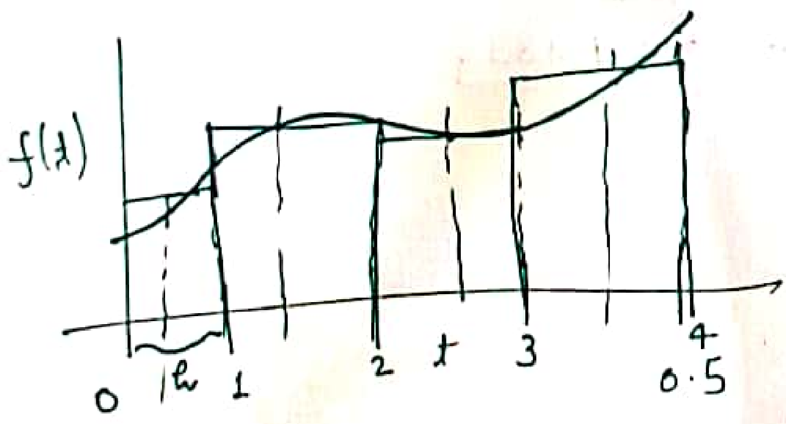
$I_A \rightarrow$ known from Q2 (a)

$= -7.7955$

$$I_B = \int_{2}^{\infty} x e^{-x} \, dx$$

[substitute $x = 1/t$ ]

$$I_B = \int_{0}^{1/2} \left(\frac{1}{t^3}\right) e^{-1/t} \, dt$$




$$h = \frac{0.5}{4}$$

| $i$ | $t$ | $f(t) = \frac{1}{t^3} e^{-1/t}$ |
|---|---|---|
| 0 | $h/2 = 0.0625$ | 0.000461 |
| 1 | $h/2 + h/4 = 0.1875$ | 0.732918 |
| 2 | $h + h/4 = 0.3125$ | 1.33569 |
| 3 | $\frac{3h}{2} + h/4 = 0.4375$ | 1.21448 |
| | | $\Sigma f(t) = 3.2830$ |

$$I_B = h \times \Sigma f(t)$$ [ if $h$ is constant throughout intervals ]

$$= 0.125 \times 3.283$$

$$I_B = 0.41038$$

$$I = I_A + I_B$$

$$I = -7.7955 + 0.41038$$

$$\boxed{I = -7.389}$$

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