ESO208A: Computational Methods in Engineering

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Consider the system

$$Ax = b$$

- In most engineering problems, the matrix *A* remains constant while the vector *b* changes with time.
- The matrix *A* describes the system and the vector *b* describes the external forcing. e.g., all network problems (pipes, electrical, canal, road, reactors, etc.); structural frames; many financial analyses.
- If all *b*'s are available together, one can solve the system by augmented matrix but in practice, they are not!



For the system,

$$Ax = b$$

- Perform a decomposition of the form A = LU, where L is a lower-triangular and U is an upper-triangular matrix!
- For any given b, solve Ax = LUx = b
- This is equivalent to solving two triangular systems:
 - Solve Ly = b using forward substitution to obtain y
 - Solve Ux = y using back substitution to obtain x
- Most frequently used method for engineering applications!



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$a_{11} = l_{11}u_{11}$$
 $a_{12} = l_{11}u_{12}$ $a_{13} = l_{11}u_{13}$

$$a_{21} = l_{21}u_{11} \qquad a_{22} = l_{21}u_{12} + l_{22}u_{22}$$

$$a_{23} = l_{21}u_{13} + l_{22}u_{23}$$

$$a_{31} = l_{21}u_{11}$$
 $a_{32} = l_{31}u_{12} + l_{32}u_{22}$
 $a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33}$

12 Unknowns and 9 equations! 3 free entries!

In general, n^2 equations and $n^2 + n$ unknows! n free entries!



In general, n^2 equations and $n^2 + n$ unknown, n free entries!

It means we cannot have a unique solution for l_{ij} and u_{ij} . However, if we fix 'n' terms, we will get a unique solution

LU decomposition Theorem

If A is a square matrix of size $n \times n$ and if $det(A) \neq 0$. Then there exists a lower triangular matrix (L) and an upper triangular matrix (U) such that A=LU.

Further, if the diagonal elements of either L or U are unity, i.e l_{ii} or u_{ii} = 1 for i=1,2,...n, then both L and U are unique



How to get elements of both L and U

- 1. Gauss Elimination gives both L and U
- 2. Dolittle Method
- 3. Crout Method $u_{ii}=1$
- 4. Thomas Algorithm- Tri-diagonal matrix
- 5. Cholesky Algorithm-Positive definite matrix



1. Gauss Elimination Method for L and U

GE A
$$\longrightarrow$$

multiplicate factors $\lim_{x \to a_{ij}} \frac{a_{ij}}{a_{ij}}$

Comple $\begin{bmatrix} 2 & 3 \\ 8 & 5 \end{bmatrix}$
 $\lim_{x \to a_{ij}} \frac{a_{ij}}{a_{ij}}$
 $\lim_{x \to a_{ij}} \frac{a_{ij}}{a_{ij}}$



Gauss Elimination Method for L and U

A:
$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$
multiplication factors
$$l_{21} = \frac{4}{2} = 2$$

$$l_{31} = \frac{-2}{2} = 1$$

$$l_{32} = -3$$

$$-1 - 3 \cdot 1$$

$$y_{2} = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 \cdot 1 \end{bmatrix} \begin{bmatrix} y_{1} \\ y_{2} \\ -1 & -3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$y_{3} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$y_{3} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

$$y_{4} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

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Gauss Elimination Method for L and U

$$\begin{bmatrix}
2 & 3 & -1 \\
0 & -2 & -1 \\
0 & 0 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
5 \\
-7 \\
-15
\end{bmatrix}$$



Comparison of GE and LU

Forward element. Broker and substitution of
$$GE \rightarrow O(n^3)$$
 $O(n^2)$
 $S \rightarrow O(n^3)$ $O(n^3)$
 $S \rightarrow O(n^3)$ $O(n^3)$
 $S \rightarrow O(n^3)$ $O(n^3)$



Comparison of GE and LU

Example - Invari Ja matrix
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

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$$A$$

Three Ja matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.4 & 0.1 \\ -0.2/0.3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2/3 & 1 \end{bmatrix}$$

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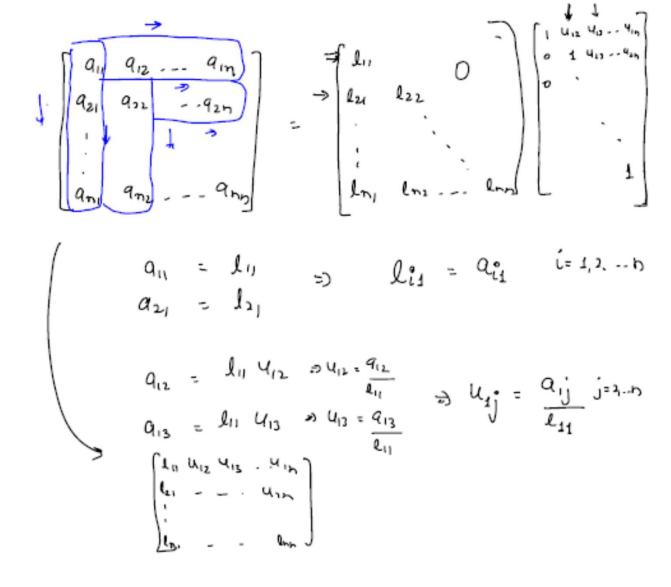
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2. Crout's Method



2. Crout's Method

$$a_{22} = l_{2}, u_{12} + l_{22}$$

$$\Rightarrow l_{22} = a_{22} - l_{21} u_{12}$$

$$for j = 2, 3, ... n-1$$

$$lij = a_{ij} - \sum_{k=1}^{j-1} lik u_{kj} \quad i=j,j+1,...n$$

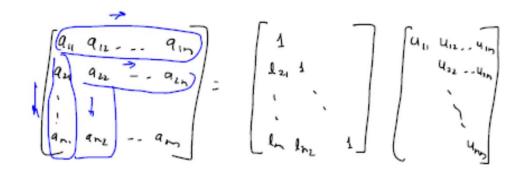
$$u_{jk} = a_{jk} - \sum_{i=1}^{j-1} lii u_{ik} \quad k=j+1,...n$$

$$lij = a_{nn} - \sum_{k=1}^{j-1} ln_k u_{kn}$$

$$lnn = a_{nn} - \sum_{k=1}^{j} ln_k u_{kn}$$



3. Dolittle Method





Summary

- What is LU decomposition
- Crout's method
- Dolittle method

