ESO208A: Computational Methods in Engineering

Richa Ojha

Department of Civil Engineering

IIT Kanpur

Acknowledgements: Profs. Abhas Singh and Shivam Tripathi (CE)

Errors and Error Analysis

Significant digits

Significant digits of a number are those that can be used with confidence

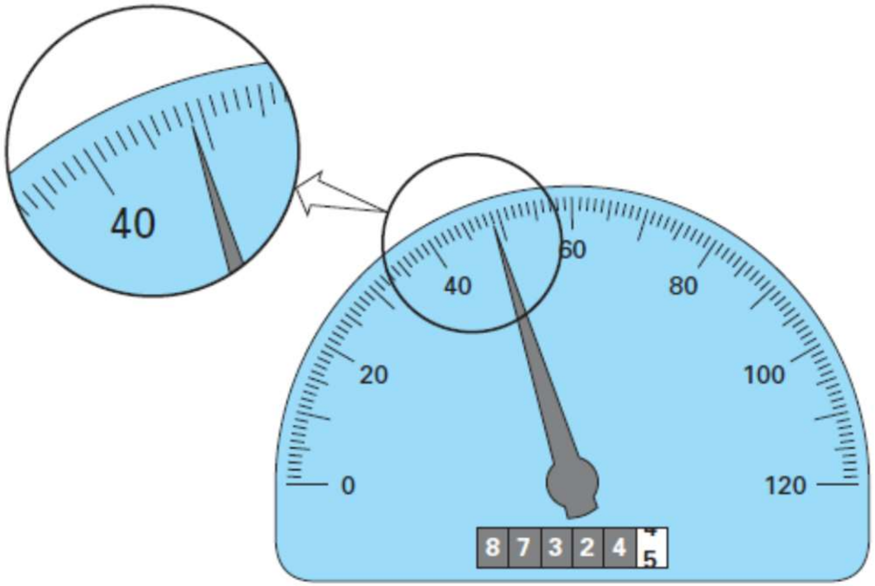


Fig: A speedometer (Source: Chapra and Canal)

Significant digits

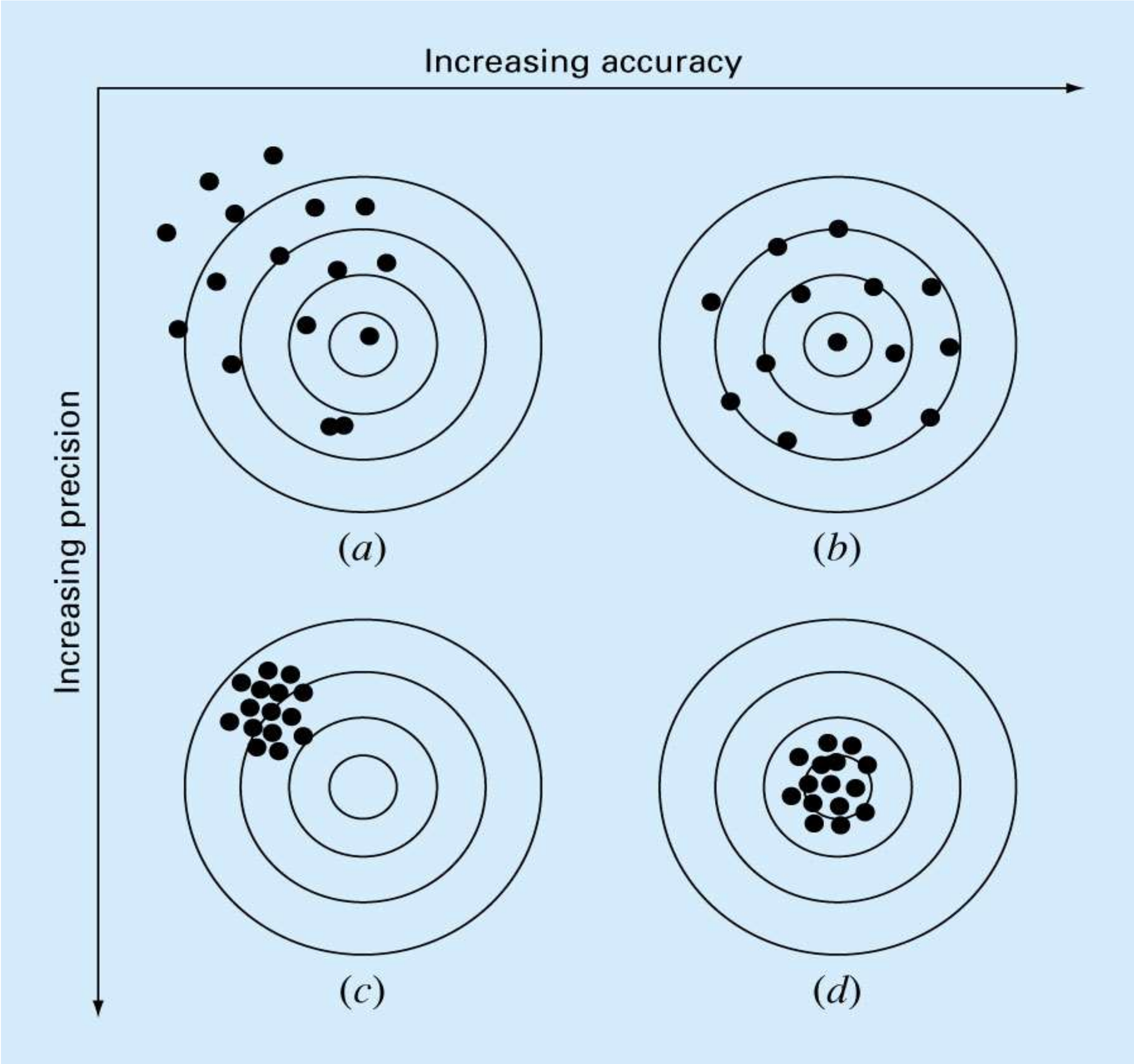
Number	Significant digits	Rule
228.18	5	All non-zero digits are significant
10.08	4	Zeros between non-zero digits are significant.
0034.5	3	Leading zeros are not significant.
34.500	5	In a decimal number trailing zeros are significant.
34500	3 or 4 or 5	In a non-decimal number trailing zeros may or may not be significant
3.450×10^4	4	No ambiguity in scientific notation

Accuracy vs Precision

- Accuracy - How closely a measured/computed value agrees with the true value
  - opposite sense: Inaccuracy (or *bias*) A systematic deviation from the actual value

- Precision (or *reproducibility*)- How closely individual computed/measured values agree with each other
  - opposite sense: Imprecision (or *uncertainty*). Magnitude of scatter

Errors and Error Analysis



Source: Chapra and Canale

Define Error:

True Value (a) = Approximate Value $(\tilde{a})$ + Error (e)

Absolute Error: $e = (a - \tilde{a})$

Relative Error: $e_r = \frac{\varepsilon}{a} = \frac{(a-\tilde{a})}{a}$

- Relative error is often expressed as (%) by multiplying (e) with 100.
- Absolute error can have sign as well as $|\ .\ |$
- If the error is computed with respect to the true value (if known), a prefix 'True' is added.

Define Error:

- For an iterative process, the true value '*a*' is replaced with the current iteration value and a prefix 'approximate' is added. This is used for testing convergence of the iterative process.

Example – For iterative algorithms

$$\varepsilon = \text{Current approximation} - \text{Previous approx.}$$

$$\varepsilon_r = \frac{\text{Current approx} - \text{Previous approx.}}{\text{Current approx}}$$

Example

1. LHC to OAT

$$d = 800\ m$$
$$\tilde{d} = 1000\ m$$
$$e = -200\ m$$
$$e_r = -\frac{200}{800} = -\frac{1}{4}$$

2. Camplus to railway station

$$d = 14.7\ km$$
$$\tilde{d} = 15\ km$$
$$e = 0.3\ km \qquad e_r = \frac{0.3}{1.5}$$

We will never have the true value, but would like to have an idea about the error of the algorithm

- How to get an error bound?
- Error bound should be a tight bound

$$\boxed{E \geq e}$$

Sources of Error in computation?

- Model Error: physical processes are too complex or some of the processes cannot be characterized
- Data Error: initial and boundary conditions, measured values of the parameters and constants in the model
- Round-off Error: irrational numbers, product and division of two numbers, limited by the machine capability
- Truncation Error: truncation of an infinite series, often arises in the design of the numerical method through approximation of the mathematical problem.

Truncation error

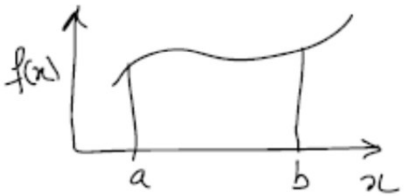
Errors

- Model error
- Data error
- ✓ Truncation error
- Round-off error

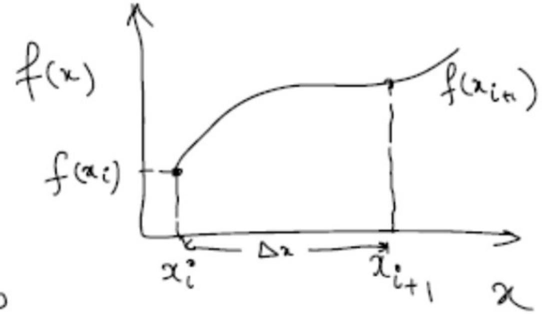
$\left.\begin{array}{l}\text{Truncation error}\\ \text{Round-off error}\end{array}\right\}$ Finite nature of computers

Truncation error – Error committed when a limiting process is truncated before one has reached the limiting value

$$\int_a^b f(x)\,dx \approx \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$



Function approximation



Taylor series

$$f(x_{i+1}) = f(x_i) + \Delta x\, f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) + \dots + \frac{\Delta x^n}{n!} f^{n}(x_i) + R_n$$

$$R_n = \frac{\Delta x^{n+1}}{(n+1)!} f^{n+1}(\xi) \qquad x_i \le \xi \le x_{i+1}$$

Truncation error

Example

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$x_i = 0 \quad f(x_i = 0) = 1.2$

$x_{i+1} = 1 \quad f(x_{i+1} = 1) = 0.2 \qquad \Delta x = 1$

1. Zero $\quad f(x_{i+1}) = f(x_i) = 1.2$

$$e = 0.2 - 1.2 = -1.0$$

2. First order $\quad f(x_{i+1}) = f(x_i) + \Delta x\, f'(x_i)$

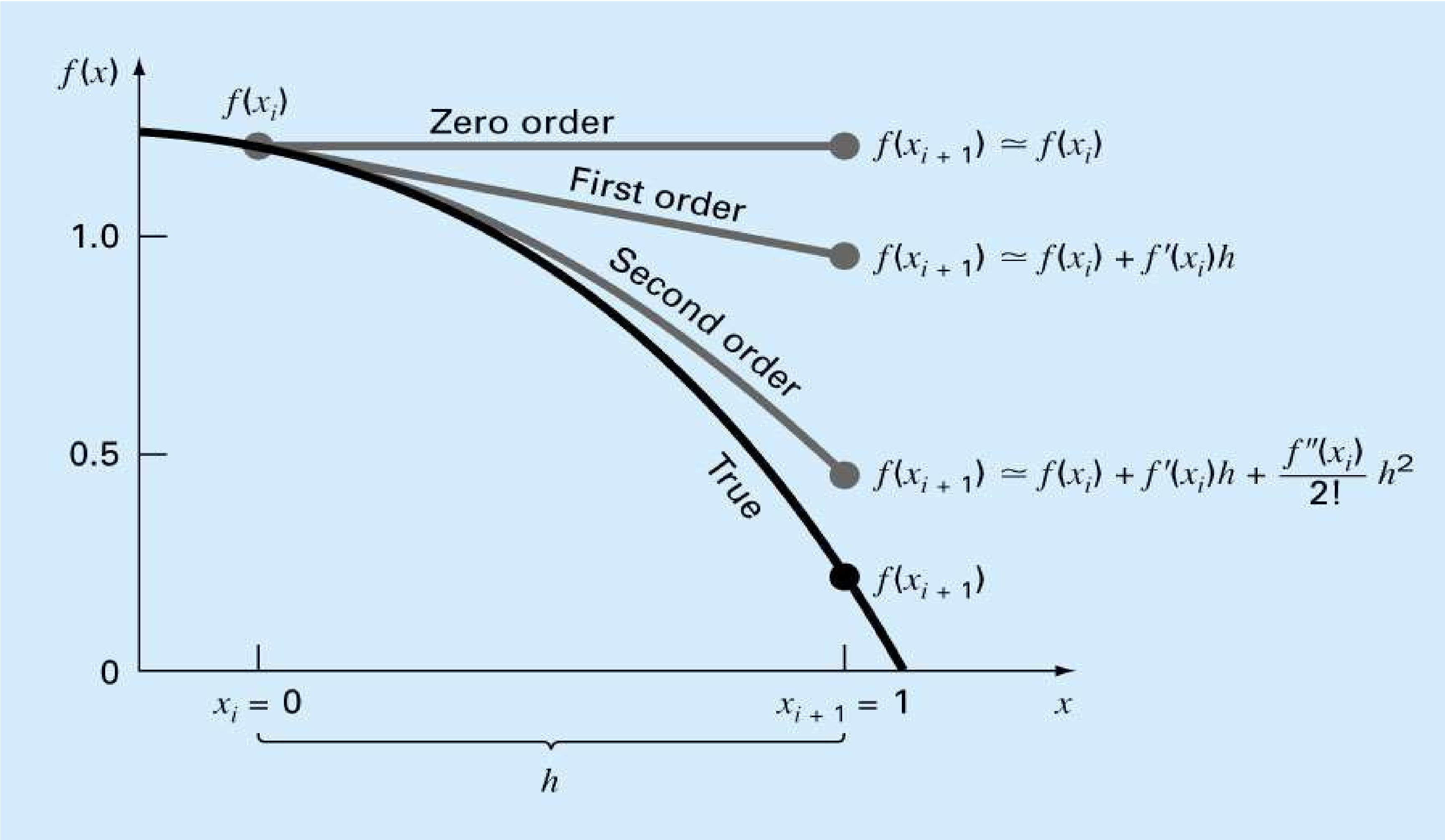
$$f'(x) = -0.4x^3 - 0.45x^2 - 0.1x - 0.25 \Big|_{x=0}$$

$$= -0.25$$

$$f(x_{i+1}) = 1.2 - 0.25 = 0.95 \qquad e = -0.75$$

This series can go upto fourth order, but if a series can go upto infinite and we can do only upto certain order. The error is truncation error!

Truncation error



Source: Chapra and Canale

Truncation error-Error bound

Truncation error

Taylor Series

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f'(x_i)$$

$$+ \frac{(x_{i+1} - x_i)^2}{2!} f''(x_i) +$$

$$+ \ldots + \frac{(x_{i+1} - x_i)^n}{n!} f^n(x_i)$$

$$+ R$$

$$R = \frac{(x_{i+1} - x_i)^{n+1}}{(n+1)!} f^{n+1}(\xi)$$

$$x_i \leq \xi \leq x_{i+1}$$

Example

$x_0 = 0 \quad f(x_0) = 1.2$

$x = 1 \quad f(x) = 0.2$

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Zeroth order

$$\tilde{f}(x_{i+1}) = f(x_i) = 1.2$$

$$e = 0.2 - 1.2 = -1.0$$

$$R = \frac{(x_{i+1} - x_i)}{1!} f'(\xi)$$

$$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$E \geq |R|$$

$$E = 2.1$$

Truncation error-Error bound

First order

$$f(x_{i+1}) = 0.95$$

$$e = 0.2 - 0.95 = -0.75$$

$$R = \frac{(x_{i+1} - x_i)^2}{2!} f''(\xi)$$

$$f''(x) = -1.2x^2 - 0.9x - 1$$

$$E = 1.55$$

$$e_a = 0.95 - 1.2 = -0.25$$

Order	$\tilde{f}(x_{i+1})$	$\lvert e \rvert$	E	$\lvert e_a \rvert$
0	1.2	1.0	2.1	—
1st	0.95	0.75	1.55	0.25
2nd	0.45	0.25	0.56	0.50
3rd	0.30	0.10	0.10	0.15
4th	0.20	0.0	0.0	0.10

- E becomes closer to the true error as the no. of terms increases
- We try to use e and E to make decisions. Try to select a problem with min e and min E

Data error

Data error

$$y = f(x)$$

$$\tilde{x} = x - e$$

$$\tilde{y} = f(\tilde{x})$$

For some reason I can't get true *x* values. If there is an error in *x* what will be the error in *y*

Data error

Data error

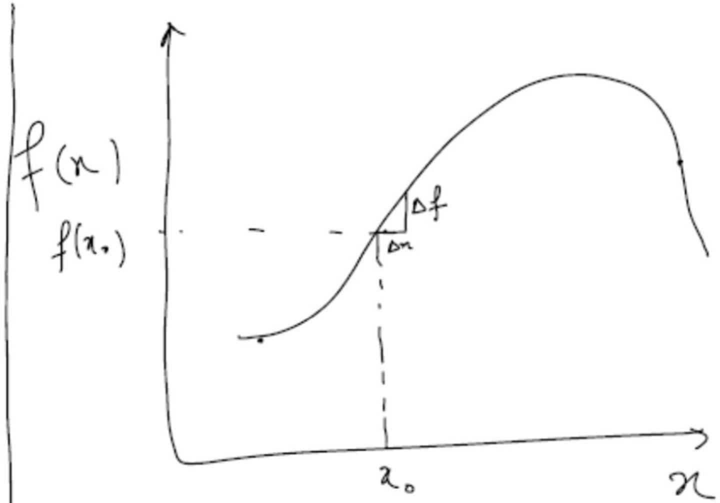
$$f(x+\Delta x) - f(x)$$

$$= \Delta x\, f'(x) + \frac{\Delta x^2}{2!} f''(x) + \ldots$$

propagation

First order error

$$f(x+\Delta x) - f(x) = \Delta x\, f'(x)$$



$$\frac{\Delta f}{\Delta x} = f'(x)$$

$$\Delta f = \Delta x\, f'(x)$$

Data error

Taylor series

$$f(x_0 + \Delta x) = f(x_0) + \Delta x \left.\frac{df}{dx}\right|_{x_0} + \dots$$

$$f(x_0 + \Delta x, z_0 + \Delta z) = f(x_0, z_0) + \Delta x \left.\frac{\partial f}{\partial x}\right|_{x_0} + \Delta z \left.\frac{\partial f}{\partial z}\right|_{z_0} + \dots$$

$$f(x_0 + \Delta x, z_0 + \Delta z) - f(x_0, z_0)$$

$$\Delta f = \Delta x \left.\frac{\partial f}{\partial x}\right|_{x_0} + \Delta z \left.\frac{\partial f}{\partial z}\right|_{z_0}$$

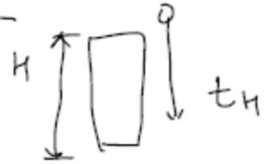
$$\Delta f(x_1, x_2, \dots x_m) = \left(\sum \left| \Delta x_i \frac{\partial f}{\partial x_i} \right| \right)$$

Quadrature sum

$$\Delta f(x_i, x_2, \dots x_m) = \sqrt{\sum_{i=1}^{m} \left(\Delta x_i \frac{\partial f}{\partial x_i} \right)^2}$$

Data error

Example



$$t_H = \sqrt{\frac{2H}{g}}$$

$$\Rightarrow g = \frac{2H}{t_H^2} = f(H, t)$$

$$\tilde{t} = t \pm \Delta t$$

$$\tilde{H} = H \pm \Delta H$$

$$\tilde{g} = g \pm \Delta g$$

$H = 660 \pm 0.01\ m$

$t = 11.65 \pm 0.01\ s$

Data error

$$\Delta g = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial H}\Delta H$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left(\frac{2H}{t^2}\right) = \frac{-4H}{t^3}$$

$$\frac{\partial f}{\partial H} = \frac{\partial}{\partial H}\left(\frac{2H}{t^2}\right) = \frac{2}{t^2}$$

$$\boxed{\Delta g = \frac{-4H}{t^3}\Delta t + \frac{2}{t^2}\Delta H}$$

Data error

$$\Delta g \approx \left|\Delta t \frac{4H}{t^3}\right| + \left|\Delta H \frac{2}{H^2}\right|$$

$H = 660 \pm 0.01$ m

$t = 11.65 \pm 0.01$ s

$$\Delta g = 0.01 \times 1.67 + 0.01 \times 0.00147$$

$$= 0.016998 \text{ m/s}^2$$

Summary

- What are significant digits?
- What are the sources of error in the computation?
- What is Truncation Error?
- What is Data Error?