ESO208A: Computational Methods in Engineering

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Non-linear equation

In computer, we have five approaches

- **Graphical method**
- **Bracketing methods:** Bisection, Regula-Falsi
- **Open methods:** Fixed point, Newton-Raphson, Secant
- **Special methods for polynomials:** Muller, Bairstow's
- **Hybrid methods:** Brent's

Open Methods

System of non-linear equations

1. Fixed point

$$\begin{cases} u(x,y) = 0 \\ v(x,y) = 0 \end{cases}$$

$$x_{i+1} = g_1(x_i, y_i)$$

$$y_{i+1} = g_2(x_{i+1}, y_i)$$

Convergence

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$$

2. Newton-Raphson

$$u(x_i, y_i) = u_i$$

$$v(x_i, y_i) = v_i$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x_i, y_i)} = \frac{\partial u_i}{\partial x}$$

Taylor's series

$$\begin{cases} u_{i+1} = u_i + (x_{i+1} - x_i)\dfrac{\partial u_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial u_i}{\partial y} \\ v_{i+1} = v_i + (x_{i+1} - x_i)\dfrac{\partial v_i}{\partial x} + (y_{i+1} - y_i)\dfrac{\partial v_i}{\partial y} \end{cases}$$

x_{i+1} $\qquad$ y_{i+1}

Hybrid Method

Combined Approach

- Bracketing method (when starting)
- Open method (when close to the solution)

Two popular methods:

- Dekker Method: Combines Bisection method and Secant Method
- Brent Algorithm: Combines Bisection method and Open Method (inverse quadratic)

In matlab fzero function is used, this function uses Brent Algorithm

Multiple Roots

What to do when your function has multiple roots?

Multiple roots

A function can have more than one roots of the same value

Example $f(x) = (x-2)^2 = 0$ - Double root

$f(x) = (x+3)^3 (x-4) = 0$ - 4 roots
Triple root
Single root

$$\left.\begin{aligned} f(x) &= (x-2)^2 = 0 \\ x=2 \quad f'(x) &= 2(x-2) = 0 \end{aligned}\right\}$$

Let s be a solution of the function $f(x)$ which can be factorized as

$$\boxed{f(x) = (x-s)^m h(x)}$$

with integer $m \geq 1$ and continuous function $h(x)$ for which $h(s) \neq 0$

Then, we say that s is a root of $f(x)$ of multiplicity m

If s is a root of multiplicity m, then

$$f(s) = f'(s) = f''(s) = \cdots = f^{m-1}(s) = 0$$

$$f^{m}(s) \neq 0$$

Multiple Roots

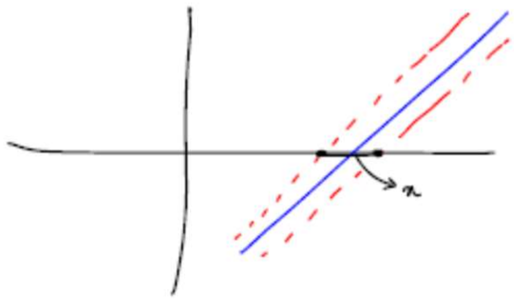
What to do when your function has multiple roots?

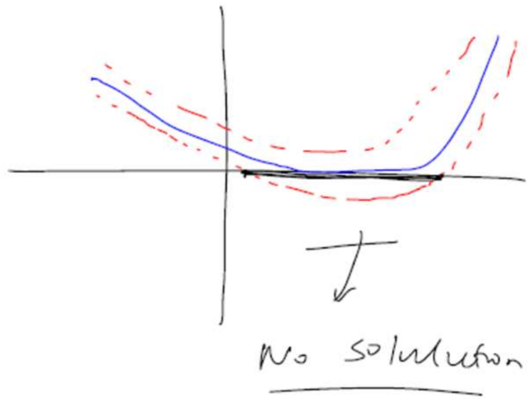
Problems with multiple roots

1) Bracketing method cannot be used when m is even

2) Newton-Raphson may not work as f'(x)=0

3) Large interval of uncertainty for solution of f(x)





Option: change or reformulate f(x)=0 to u(x)=0, such that u(x) has a solution

Multiple Roots

Two modifications of Newton Raphson Method

a) First modification

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

b) Second modification

$$f(x) = 0$$

$$\Rightarrow \quad u(x) = \frac{f(x)}{f'(x)}$$

Multiple Roots

Instead of $f(x) = 0$

solve $u(x) = \frac{f(x)}{f'(x)}$

$$f(x) = (x-s)^m h_1(x)$$

$$f'(x) = m(x-s)^{m-1} h_1(x) + (x-s)^m h_1'(x)$$

$$= (x-s)^{m-1} h_2(x)$$

$h_1(s), h_2(s)$
$h_3(s) \neq 0$

$$u(x) = \frac{(x-s)^m h_1(x)}{(x-s)^{m-1} h_2(x)}$$

$$\boxed{u(x) = (x-s)\, h_3(x)}$$

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)}$$

$$= x_i - \frac{f(x_i) f'(x_i)}{f'(x_i)^2 - f(x_i) f''(x_i)}$$

$f(x_i)\ f'(x_i)\ f''(x_i)$

- Need to evaluate $f(x_i)$, $f'(x_i)$ and $f''(x_i)$ at every iteration
- Each iteration is more expensive, even though it converges rapidly

Polynomials

Consider an n^{th} order polynomial

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = 0$$

If a's are real,

- This polynomial will have n roots (real or complex)
- If n is odd, atleast one root will be real
- Complex roots occur in conjugate pairs

We are interested in finding the roots of the polynomials

Polynomials

Certain characteristics of polynomials:

1. Evaluation of polynomials by a computer

a. $f_3(x) = a_0 + a_1 . x + a_2 . x . x . + a_3 x . x . x$

$\underline{n^{th} \text{ order}}$ – n addition

$\frac{n(n+1)}{2}$ multiplications

b. $f_3(x) = a_0 + x . (a_1 + x . (a_2 + x . a_3))$

n – addition

n - multiplication

Polynomials

2. Division of polynomials

$$f_3(x) = x^3 - 13x - 12$$

Divide $x^2 - x - 1$

$$
\begin{array}{r|l}
 & x + 1 \\
\hline
x^2 - x - 1 & \cancel{x^3} + 0x^2 - 13x - 12 \\
 & \underline{-\cancel{x^3} \mp x^2 \mp x} \\
 & x^2 - 12x - 12 \\
 & \underline{-x^2 \mp x \mp 1} \\
 & -11x - 11
\end{array}
$$

$$f_n(x) = (x^2 - rx - s)\, f_{n-2}(x) + R$$

Polynomials

3. Deflation of Polynomials

Let's assume that we have determined 's' to be a root of

$f_n(x)$

$$f_n(x) = (x - s) f_{n-1}(x) = 0$$

Polynomials

4. Effective degree of Polynomials

$$f(x) = x^{12} - 6x^{8} + 4x^{4} + 1 = 0$$

In x it's 12^{th} order polynomial but in x^4 it's a cubic polynomial

$$= (x^4)^3 - (6x^4)^2 + 4x^4 + 1 = 0$$

So, try to reduce a polynomial to a lesser degree

Roots of Polynomials

Two methods that can be used to find roots of polynomials

a) Muller method

b) Bairstow method

Müller Method

Müller's method obtains a root estimate by projecting a parabola to the x axis through three function values.

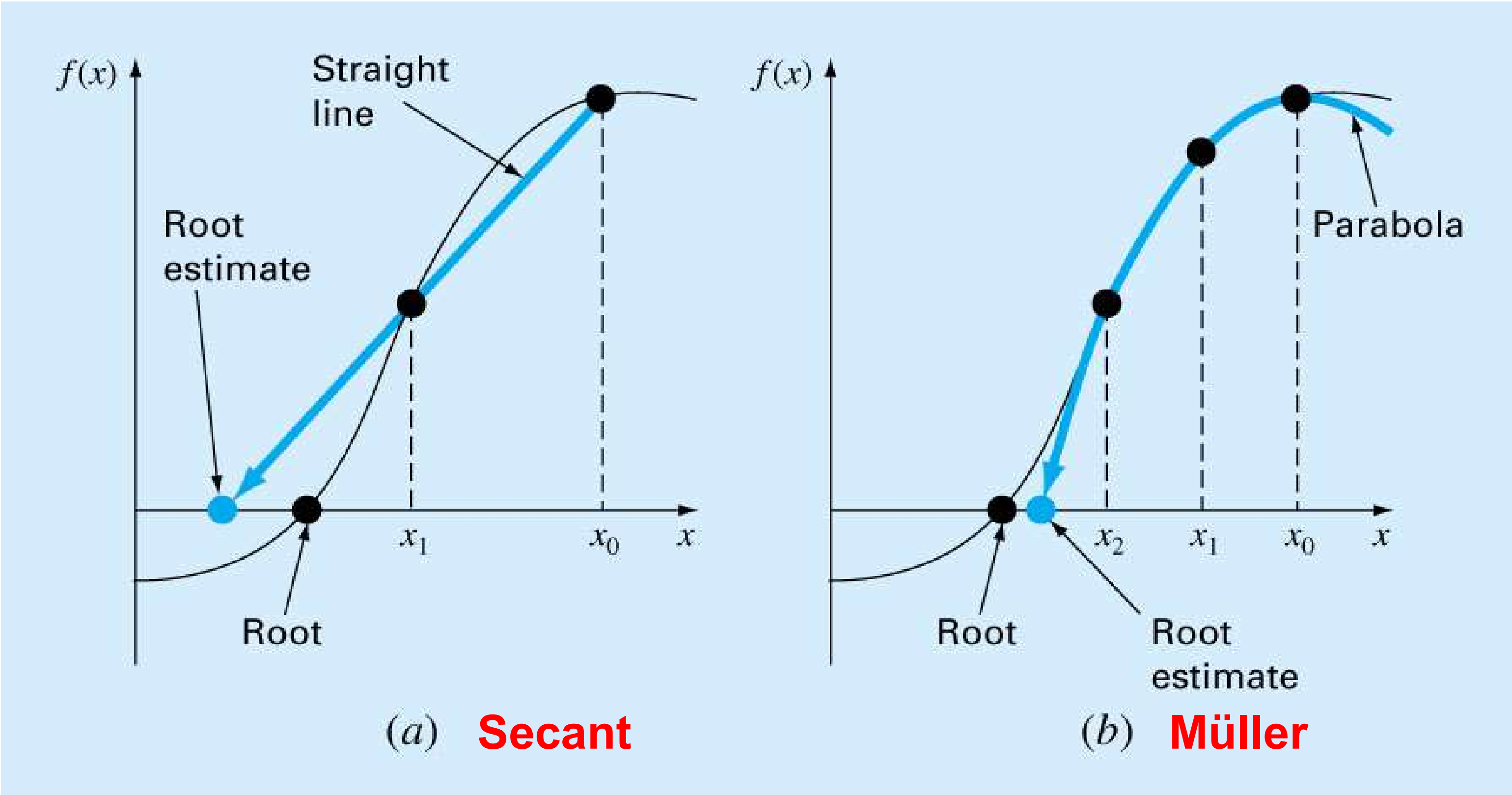


Figure 7.3 of C&C

Müller Method

1. Write the equation of a parabola in a convenient form:

$$f_2(x) = a(x - x_2)^2 + b(x - x_2) + c$$

2. The parabola should intersect the three points $[x_o, f(x_o)]$, $[x_1, f(x_1)]$, $[x_2, f(x_2)]$.

$$f(x_o) = a(x_o - x_2)^2 + b(x_o - x_2) + c$$
$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$
$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

Müller Method

3. The three equations can be solved to estimate a, b, and c

Define

$$h_o = x_1 - x_o \qquad h_1 = x_2 - x_1$$

$$\delta_o = \frac{f(x_1) - f(x_o)}{x_1 - x_o} \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

then,

$$a = \frac{\delta_1 - \delta_o}{h_1 + h_o} \qquad b = ah_1 + \delta_1 \qquad c = f(x_2)$$

Müller Method

4. Roots can be found by applying quadratic formula:

$$x_3 - x_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

5. $\pm$term yields two roots; the sign is chosen to agree with b. This will result in a large denominator, and will give root estimate that is closest to x_2.

Müller Method

6. Once x_3 is determined, the process is repeated by employing a sequential approach just like in secant method, x_1, x_2, and x_3 to replace x_0, x_1, and x_2.

Summary

- We looked at open methods for solving system of non-linear equations
- How to modify Newton-Raphson in case of multiple roots?
- Characteristics of a polynomial
- Muller method for solving a polynomial