ESO 208A: Computational Methods in Engineering

Richa Ojha

Department of Civil Engineering

IIT Kanpur

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Non-linear equation

In computer, we have five approaches

- **Graphical method**
- **Bracketing methods:** Bisection, Regula-Falsi
- **Open methods:** Fixed point, Newton-Raphson, Secant
- **Special methods for polynomials:** Muller, Bairstow's
- **Hybrid methods:** Brent's

Bairstow's Method

1. Bairstow's method is an iterative approach loosely related to both Müller and Newton Raphson methods

2. It is based on dividing the given polynomial by a quadratic polynomial x^2-rx-s:

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \dots + a_n x^n$$
$$= \left(x^2 - rx - s\right) f_{n-2}(x) + R$$

where

$$f_{n-2}(x) = b_2 + b_3 x + \dots + b_{n-1} x^{n-3} + b_n x^{n-2}$$
$$R = b_1(x - r) + b_o$$

Bairstow's Method

3. The coefficients b's are obtained very easily by using recursive relation

$$b_n = a_n$$

$$b_{n\text{-}1} = a_{n\text{-}1} + rb_n$$

$$b_i = a_i + rb_{i+1} + sb_{i+2} \qquad i = n - 2 \text{ to } 0$$

4. Using Newton Raphson approach, r and s are adjusted so as to make both b_o *and* b_1 approach zero

$$b_1 = a_1 + rb_2 + sb_3 \Rightarrow u(r,s)$$

$$b_0 = a_0 + rb_1 + sb_2 \Rightarrow v(r,s)$$

Bairstow's Method

5. Obtain corrections in r and s by Newton-Raphson method

Changes Δs and Δr needed to improve guesses will be estimated by

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1$$

$$\frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s = -b_o$$

Bairstow's Method

6. Bairstow (1920) showed that the partial derivatives of b_0 and b_1 are obtained by the recursive relation

$$c_n = b_n$$

$$c_{n-1} = b_{n-1} + rc_n$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \qquad i = n-2 \; to \; 2$$

where

$$\frac{\partial b_o}{\partial r} = c_1 \qquad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \qquad \frac{\partial b_1}{\partial s} = c_3$$

7. Iterate the steps untill $(\Delta r/r)$ and $(\Delta s/s)$ drops below a specified threshold

Polynomial Methods: Single Root

$$p_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

If we divide by a factor (x - r) such that, $r = \alpha$ is a root of the polynomial, we will get an exact polynomial of order (n - 1), say $q_{n-1}(x)$.

$$q_{n-1}(x) = \sum_{k=0}^{n-1} b_{k+1} x^k = b_1 + b_2 x + b_3 x^2 + \cdots + b_n x^{n-1}$$

If $r \neq \alpha$, dividing by a factor (x - r) will have a remainder b_0.

Polynomial Methods: Single Root

$$p_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$= (x-r)q_{n-1}(x) + b_0 = (x-r)\sum_{k=0}^{n-1} b_{k+1} x^k + b_0$$

$$= b_0 + b_1(x-r) + b_2 x(x-r) + b_3 x^2(x-r) + \cdots + b_{n-2}x^{n-3}(x-r)$$
$$+ b_{n-1}x^{n-2}(x-r) + b_n x^{n-1}(x-r)$$

$$= (b_0 - rb_1) + x(b_1 - rb_2) + x^2(b_2 - rb_3) + \cdots + x^{n-2}(b_{n-2} - rb_{n-1})$$
$$+ x^{n-1}(b_{n-1} - rb_n) + b_n x^n$$

$$b_n = a_n;\;\; b_i - rb_{i+1} = a_i;\quad i = (n-1), (n-2), \cdots 2, 1, 0$$

$$b_n = a_n;\;\; b_i = a_i + rb_{i+1};\quad i = (n-1), (n-2), \cdots 2, 1, 0$$

For a given $p_n(x)$, a_i are known. For a choice of r, one can determine b_i from n+1 equations above having n+1 unknowns

Polynomial Methods: Single Root

Remainder b_0 is a function of $r \longrightarrow b_0(r)$, at $r = \alpha$, $b_0(r) = 0$

Problem: $f(x) = 0$, find a root $x = \alpha$ such that $f(\alpha) = 0$

Problem: $b_0(r) = 0$, find a root $r = \alpha$ such that $b_0(\alpha) = 0$

Apply Newton-Raphson:

Iteration Formula for Step k:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \text{or} \quad r_{k+1} = r_k - \frac{b_0(r_k)}{b_0'(r_k)}$$

$$b_0 = a_0 + r b_1 \longrightarrow b_0'(r) = b_1 \longrightarrow r_{k+1} = r_k - \frac{b_0(r_k)}{b_1(r_k)}$$

Assume a value of r, estimate b_0 and b_1, compute new r.

Continue until b_0 becomes zero. (with acceptable relative error)

Polynomial Methods: Bairstow's

$$p_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Let us divide by a factor $(x^2 - rx - s)$. If the factor is exact, the resulting polynomial will be of order $(n - 2)$. Two roots of the polynomial can be estimated simultaneously as the roots of the quadratic factor. For the complex roots, they will be the complex conjugates.

$$q_{n-2}(x) = \sum_{k=0}^{n-2} b_{k+2} x^k = b_2 + b_3 x + b_4 x^2 + \cdots + b_n x^{n-2}$$

If the factor $(x^2 - rx - s)$ is not exact, there will be two remainder terms, one function of x and another constant.

Let us express the remainder term as $b_1(x - r) + b_0$. This form instead of the standard $b_1 x + b_0$ is chosen to device a convenient iteration formula!

Polynomial Methods: Bairstow's

$$p_n(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$= (x^2 - rx - s)q_{n-2}(x) + b_1(x - r) + b_0$$

$$= (x^2 - rx - s)\sum_{k=0}^{n-2} b_{k+2}x^k + b_1(x - r) + b_0$$

$$= b_0 + b_1(x - r) + b_2(x^2 - rx - s) + b_3 x(x^2 - rx - s) + \cdots$$
$$+ b_{n-2}x^{n-4}(x^2 - rx - s) + b_{n-1}x^{n-3}(x^2 - rx - s) + b_n x^{n-2}(x^2 - rx - s)$$

$$= (b_0 - rb_1 - sb_2) + x(b_1 - rb_2 - sb_3) + x^2(b_2 - rb_3 - sb_4) + \cdots$$
$$+ x^{n-2}(b_{n-2} - rb_{n-1} - sb_n) + x^{n-1}(b_{n-1} - rb_n) + b_n x^n$$

$$b_n = a_n;\; b_{n-1} = a_{n-1} + rb_n\,;\; b_i = a_i + rb_{i+1} + sb_{i+2};\quad i = (n-2), \cdots 2, 1, 0$$

For a given $p_n(x)$, a_i are known. For a choice of r and s, one can determine b_i from n+1 equations above having n+1 unknowns

Polynomial Methods: Bairstow's

b_0 and b_1 are functions of r and s $\longrightarrow$ $b_0(r, s)$ and $b_1(r, s)$

Expand in Taylor's series: Apply 2-d Newton-Raphson

$$0 = b_0(r + \Delta r, s + \Delta s) = b_0 + \frac{\partial b_0}{\partial r}\Delta r + \frac{\partial b_0}{\partial s}\Delta s + HOT$$

$$0 = b_1(r + \Delta r, s + \Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s + HOT$$

$$\begin{bmatrix} \frac{\partial b_0}{\partial r} & \frac{\partial b_0}{\partial s} \\ \frac{\partial b_1}{\partial r} & \frac{\partial b_1}{\partial s} \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s \end{bmatrix} = \begin{bmatrix} -b_0 \\ -b_1 \end{bmatrix}$$

Need to evaluate: $\frac{\partial b_0}{\partial r}$, $\frac{\partial b_0}{\partial s}$, $\frac{\partial b_1}{\partial r}$ and $\frac{\partial b_1}{\partial s}$

Polynomial Methods: Bairstow's

$$b_n = a_n;\ b_{n-1} = a_{n-1} + rb_n\ ;\ b_i = a_i + rb_{i+1} + sb_{i+2};\quad i = (n-2), \cdots 2, 1, 0$$

Partial differentials with respect to r:

$$b_n = a_n \quad \rightarrow \frac{\partial b_n}{\partial r} = 0;$$

$$b_{n-1} = a_{n-1} + rb_n \ \rightarrow \frac{\partial b_{n-1}}{\partial r} = b_n = c_n$$

$$b_{n-2} = a_{n-2} + rb_{n-1} + sb_n \rightarrow \frac{\partial b_{n-2}}{\partial r} = b_{n-1} + r\frac{\partial b_{n-1}}{\partial r} + s\underbrace{\cancel{\frac{\partial b_n}{\partial r}}}_{= 0} = b_{n-1} + rc_n$$

$$= c_{n-1}$$

$$b_{n-3} = a_{n-3} + rb_{n-2} + sb_{n-1} \ \rightarrow \ \frac{\partial b_{n-3}}{\partial r} = b_{n-2} + r\frac{\partial b_{n-2}}{\partial r} + s\frac{\partial b_{n-1}}{\partial r}$$

$$= b_{n-2} + rc_{n-1} + sc_n = c_{n-2}$$

$$c_n = b_n;\ c_{n-1} = b_{n-1} + rc_n\ ;\ c_i = b_i + rc_{i+1} + sc_{i+2};\quad i = (n-2), \cdots 2, 1, 0$$

$$\frac{\partial b_i}{\partial r} = c_{i+1};\qquad i = (n-1), \cdots 2,1,0$$

Polynomial Methods: Bairstow's

$$b_n = a_n;\ b_{n-1} = a_{n-1} + rb_n\ ;\ b_i = a_i + rb_{i+1} + sb_{i+2};\quad i = (n-2), \cdots 2, 1, 0$$

Partial differentials with respect to *s*:

$$b_n = a_n \ \rightarrow \frac{\partial b_n}{\partial s} = 0;$$

$$b_{n-1} = a_{n-1} + rb_n \ \rightarrow \frac{\partial b_{n-1}}{\partial s} = 0$$

$$b_{n-2} = a_{n-2} + rb_{n-1} + sb_n \rightarrow \frac{\partial b_{n-2}}{\partial s} = b_n + r\overset{=0}{\cancel{\frac{\partial b_{n-1}}{\partial s}}} + s\overset{=0}{\cancel{\frac{\partial b_n}{\partial s}}} = b_n = c_n \text{ (say)}$$

$$b_{n-3} = a_{n-3} + rb_{n-2} + sb_{n-1} \ \rightarrow \frac{\partial b_{n-3}}{\partial s} = b_{n-1} + r\frac{\partial b_{n-2}}{\partial s} + s\frac{\partial b_{n-1}}{\partial s} = b_{n-1} + rc_n$$
$$= c_{n-1}$$

$$b_{n-4} = a_{n-4} + rb_{n-3} + sb_{n-2} \ \rightarrow \frac{\partial b_{n-4}}{\partial s} = b_{n-2} + r\frac{\partial b_{n-3}}{\partial s} + s\frac{\partial b_{n-2}}{\partial s}$$
$$= b_{n-2} + rc_{n-1} + sb_n = c_{n-2}$$

$$c_n = b_n;\ c_{n-1} = b_{n-1} + rc_n\ ;\ c_i = b_i + rc_{i+1} + sc_{i+2};\quad i = (n-2), \cdots 2, 1, 0$$

$$\frac{\partial b_i}{\partial s} = c_{i+2};\qquad i = (n-2), \cdots 2,1,0$$

Polynomial Methods: Bairstow's

$$\frac{\partial b_i}{\partial r} = c_{i+1};\ i = (n-1), \cdots 2,1,0 \ \text{ and } \ \frac{\partial b_i}{\partial s} = c_{i+2};\ i = (n-2), \cdots 2,1,0$$

$$\frac{\partial b_0}{\partial r} = c_1;\ \ \frac{\partial b_1}{\partial r} = c_2;\ \ \frac{\partial b_0}{\partial s} = c_2 \ \text{ and } \ \frac{\partial b_1}{\partial s} = c_3$$

$$\begin{bmatrix} \dfrac{\partial b_0}{\partial r} & \dfrac{\partial b_0}{\partial s} \\ \dfrac{\partial b_1}{\partial r} & \dfrac{\partial b_1}{\partial s} \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s \end{bmatrix} = \begin{bmatrix} -b_0 \\ -b_1 \end{bmatrix}$$

For any given polynomial, we know $\{a_0, a_1, \ldots a_n\}$. Assume r and s. Compute $\{b_0, b_1, \ldots b_n\}$ and $\{c_0, c_1, \ldots c_n\}$. Compute Δr and Δs.

Polynomial Methods: Bairstow's

- ✓ Step 1: input $a_0, a_1, \ldots a_n$ and initialize r and s.
- ✓ Step 2: compute $b_0, b_1, \ldots b_n$
- $b_n = a_n; b_{n-1} = a_{n-1} + rb_n$; $b_i = a_i + rb_{i+1} + sb_{i+2}$; $i = (n-2), \cdots 2, 1, 0$
- ✓ Step 3: compute $c_0, c_1, \ldots c_n$
- $c_n = b_n$; $c_{n-1} = b_{n-1} + rc_n$; $c_i = b_i + rc_{i+1} + sc_{i+2}$; $i = (n-2), \cdots 2, 1, 0$
- ✓ Step 4: compute Δr and Δs from $\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta s \end{bmatrix} = \begin{bmatrix} -b_0 \\ -b_1 \end{bmatrix}$
- ✓ Step 5: compute $r_{new} = r + \Delta r$, $s_{new} = s + \Delta s$
- ✓ Step 6: check for convergence, $\left|\frac{r_{new} - r}{r_{new}}\right|, \left|\frac{s_{new} - s}{s_{new}}\right| \leq \varepsilon$ and $b_0, b_1 \leq \varepsilon'$
- ✓ Step 7: Stop if all convergence checks are satisfied. Else, set $r = r_{new}$, $s = s_{new}$ and go to step 2.

Bairstow's Method

Step 8. The roots quadratic polynomial x^2-rx-s are obtained as

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2}$$

Step 9. At this point three possibilities exist:

1. *The quotient is a third-order polynomial or greater*. The previous values of r and s serve as initial guesses and Bairstow's method is applied to the quotient to evaluate new r and s values.

2. *The quotient is quadratic*. The remaining two roots are evaluated directly, using the above eqn.

3. *The quotient is a 1st order polynomial*. The remaining single root can be evaluated simply as x=-s/r.

Summary

- Bairstow method
- Derivation of Bairstow method