ESO 208A: Computational Methods in Engineering

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Recap

- What is a system of linear equations?
- Different kind of matrices
- Direct method-Gauss Elimination Method

Today's lecture

- Situations under which Gauss Elimination method will not work
- Gauss Jordan Method
- How to find algorithm complexity?
- LU decomposition method



Gauss Elimination Algorithm

Forward Elimination:

For
$$k = 1, 2, (n - 1)$$

Define multiplication factors: $l_{ik} = \frac{a_{ik}}{a_{kk}}$

Compute: $a_{ij} = a_{ij}$ - l_{ik} a_{kj} ; $b_i = b_i - l_{ik}$ b_k for i = k+1, k+2,n and j = k+1, k+2,n

Resulting System of equation is upper triangular. Solve it using the *Back-Substitution algorithm*:

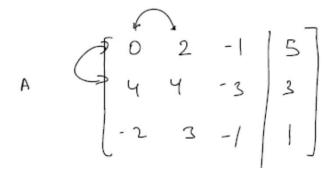
$$x_n = \frac{b_n}{a_{nn}}; x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij} x_j}{a_{ii}}; i = (n-1), (n-2), \dots 3, 2, 1$$



Difficult Cases

a) Division by zero

$$E_1$$
: $\sqrt{3n_2 - n_3} = 5$
 E_2 : $4n_1 + 4n_2 - 3n_3 = 3$
 E_3 : $-2n_1 + 3n_2 - n_3 - 1$



- l_{21} can not be calculated, exchange the rows, which one to pick
- Does it matter? Yes, in terms of round off error.
- When we switch the rows, it is called as pivoting or row pivoting
- When we switch columns, it is column pivoting. In this case, we need to reorder the unknowns.
- When we switch both, it is total pivoting

Partial Pivoting



Difficult Cases

b) Ill-conditioned

$$x_1 + 2x_2 = 10$$

1.1 $x_1 + 2x_2 = 10.4$

Solution: $x_1=4$, $x_2=3$

Now if I slightly change the coefficients $x_1+2x_2=10$ 1.05 $x_1+2x_2=10.4$

Solution: $x_1 = 8$, $x_2 = 1$

- By just changing the coefficient slightly, the solution changes significantly
- This is very costly
- Can we without solving find if the system is ill conditioned

Difficult Cases

c) Round-off Error

$$0.0004 \, n_1 + 1.402 \, n_2 = 1.406$$
 $0.4003 \, n_1 - 1.502 \, n_1 = 2.501$

True Solution: $x_1 = 10$, $x_2 = 1$

What if you are using a computer that has four significant digits

Difficult Cases

c) Round-off Error

$$R_{2} = R_{2} - l_{21}R_{1}$$

$$= \frac{q_{21}}{q_{1}}$$

$$= \frac{0.0003}{0.0009} = 0.1001 \times 10^{10}$$

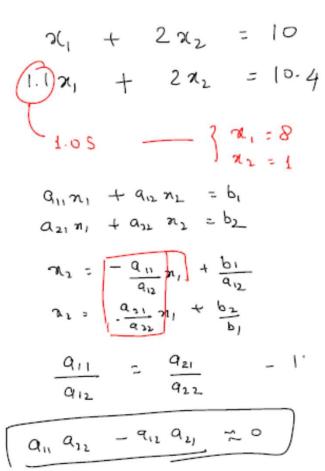
$$71 = \frac{1405 \times 1 = 1404}{1.406 - 1.402 \times 0.9913}$$

$$= \frac{12.5}{0.0004}$$

- The solution is very different from the actual solution
- Before solving the problem, can we know our system will have roundoff problem

Difficult Cases: Options for handling

a) Ill-Conditioned



- If determinant is close to zero- ill conditioned
- If determinant is zero- singular

Difficult Cases: Options for handling

a) Ill-Conditioned

Can we use determinant as a measure of ill-conditioning?

$$\begin{bmatrix} 1 & 2 \\ 1 \cdot 1 & 2 \end{bmatrix} \Rightarrow \sum = 2 - 2 \cdot 2 = -02$$

Suppose in the example we multiply the equations by 10

$$\begin{bmatrix} 10 & 20 \\ 11 & 20 \end{bmatrix} \Rightarrow D = -\frac{20}{20}$$

Now the determinant is significantly different from 0. On its own D is not a good measure of ill-conditioning

Difficult Cases: Options for handling

The three issues mentioned earlier can be avoided by:

- Use of more significant digits
- Pivoting: Row or partial pivoting-exchange row of the augmented matrix
 - Exchange rows which will result in largest magnitude of pivot element

Difficult Cases: Options for handling

Example:

$$0.0004 \, n_1 + 1.402 \, n_2 = 1.406$$
 $0.4003 \, n_1 - 1.502 \, n_2 = 2.50$

Exact $- n_1 = 10$
 $n_2 = 1$

If we solved the problem by a 4 -digit
 $- n_1 = 0.9993$
 $- n_2 = 12.5$

Difficult Cases: Options for handling

Example:
$$\begin{vmatrix}
0.0004 & 1.402 & 1.406 \\
0.4003 & -1.502 & 2.501
\end{vmatrix}$$

$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0.4003 & -1.502 & 2.501
\end{vmatrix}$$

$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0.4003 & -1.502 & 2.501
\end{vmatrix}$$

$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0 & 1.404 & 1.404
\end{vmatrix}$$

$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0 & 1.404 & 1.404
\end{vmatrix}$$

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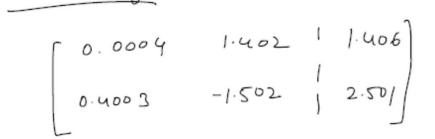
$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0 & 1.404 & 1.404
\end{vmatrix}$$

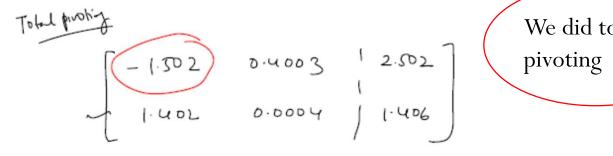
$$\begin{vmatrix}
0.4003 & -1.502 & 2.501 \\
0 & 1.404 & 1.404
\end{vmatrix}$$

We did row pivoting

Difficult Cases: Options for handling







We did total

$$x_1 = 10$$
, and $x_2 = 1$

Difficult Cases: Options for handling

Why pivoting has worked?

$$10 \times 0.0004 \, n_1 + 10 \times 1.402 \, n_2 = 1.406$$
 $0.4003 \, n_1 - 1.502 \, n_2 = 2.50$
 $121 = \frac{0.4003}{0.0004 \times 10^{m}} = 1001 \times 10^{m}$
 $122 = -1.502 - (1.402 \times 10^{m} \times 1001 \times 10^{m})$
 $121 = 1405$
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Difficult Cases: Options for handling

Why pivoting has worked?

- Even by making the pivot large still we get round-off error.
- It is not the magnitude of the pivot element but relative magnitude of elements that leads to round-off error
- Scaling of elements of 'A' governs the round-off errors

Difficult Cases: Options for handling

Scaling

Difficult Cases: Options for handling

Scaling
$$\begin{bmatrix} 2 \times 10^{5} & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 \times 10^{5} & 1 & 1 & 1 \end{bmatrix}$$
Parhal putty
$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 \times 10^{5} & 1 & 1 & 1 \end{bmatrix}$$

Perform pivoting by using scaled coefficients but perform computations (GE) using original coefficients

Difficult Cases: Options for handling

Scaling

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 10^{5} & 10^{5} \\ 31 & = 31.0 \end{bmatrix}$$

Perform pivoung by using scaled coefficients but perform computations (GE) using original coefficients

Difficult Cases: Options for handling

Most common implementations of GE:

- Use scaled values of the coefficients as a criterion to decide pivoting
- Retain the original coefficients for actual elimination and substitution
- "No general pivoting strategy that will work for all linear systems"
 - Example: If coefficient matrix is a positive definite matrix, the BEST strategy is no interchange
- If you know, any special characteristics of the system use it to decide the pivoting strategy

Direct Methods: Gauss Jordon

In this method, the coefficient matrix is reduced to an Identity matrix

- Requires a minor modification in GE algorithm
 - At each step, first the pivot element is made unity by dividing
 pivot equation by the pivot element
 - In addition to sub-diagonal elements, the above diagonal elements are also made 0.



Direct Methods: Gauss Jordon

Example



Summary

• Under what situations Gauss Elimination will not work

• Gauss Jordan Method

