THE HARDY LITTLEWOOD CIRCLE METHOD

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1. Introduction

Additive number theory is the branch of number theory which studies subsets of the integers and how they behave under addition. The circle method is a tool which helps solve problems in additive number theory using complex analysis. The circle method was originally developed in 1918 by Ramanujan and G. H. Hardy as they were investigating the partition function $p(n)$, although Ramanujan had developed a more basic form of the circle method far before this. After Ramanujan's death, Hardy and Littlewood developed the method further and used to find to find the asymptotic formula's found in Waring's Problem. A little later, Ivan Vinogradov realized that the infinite series used in the circle method could be replaced with a Fourier series, transforming the method into one using exponential sums.

The circle method gives an integral representation for the coefficients of a generating function which represents the problem we are trying to solve. The integral can then be split up into main terms and error terms.

In this paper, we discuss the key ideas of the circle method, and then use it to walk the reader through a derivation of the asymptotic formula for the partitions. This solution is not Hardy and Ramanujan's original result, but instead is a simplified version devised by D. J. Newman. Finally, we briefly touch upon some other problems which the circle method can be applied to.

2. Partitions

We first begin by describing the problem of partitions. A partition of an integer n is a decomposition of n into a sum of integers where order does not matter. For example, if we consider the number 6, then some examples of partitions are

$$1 + 1 + 1 + 3$$
$$2 + 2 + 2$$
$$1 + 1 + 4$$

Note that $1 + 3 + 1 + 1$ is not considered a distinct partition and is the same as $1 + 1 + 1 + 3$. Therefore, we write a partition with the parts in ascending order. We define the function $p(n)$ to be the number of partitions of n. So what is $p(n)$? This proves to be a tricky problem. The first thing which we may think to do is to find a generating function for the partitions.

Definition 2.1 (Generating Function)**.** Let a_n be a sequence. Then the generating function for a_n is

$$F(z) = \sum_{z=0}^{\infty} a_n z^n$$

Date: November 17, 2025.

Essentially, we encode $p(n)$ as the coefficient of the x^n term in the generating function. This forces us to have $|z| < 1$

Theorem 2.2. *Let $P(x)$ denote the generating function for the partitions. Then,*

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

Proof. For any integer n, we can use it $0, 1, 2, 3 \cdots$ times in a partition. We can express this as the generating function

$$1 + x^n + x^{2n} + x^{3n} \cdots = \frac{1}{1-x^n}$$

When we multiply the generating functions over all integers n, we get the generating function for the partitions. Therefore,

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

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When treating this function in a purely formal sense, we do not have to worry about convergence. So, now that we have the generating function, what do we do with it? To continue, we will use the circle method.

3. The Circle Method

The main problem which the circle method seeks out to solve is problems of the form:

Question 3.1. *Given $A \subset \mathbb{N}$ and $s \in \mathbb{Z}_{>0}$, what is*

$$\{a_1 + a_2 \cdots + a_s : a_i \in A\} \cap \mathbb{N}$$

This question essentially asks what natural numbers can be written as a sum of elements from A.

Definition 3.2. Let $r(n, s, A)$ be the number of ways in which we can write n as the sum of s elements of A.

Notice that

$$p(n) = \sum_{s=1}^{\infty} r(n, s, \mathbb{N})$$

From here on, we assume that A is an infinite set, as otherwise the problem is not particularly interesting.

The circle method proceeds as follows. To solve our problem, we first construct a generating function which represents that problem. Then, we use a contour integral and the generating function to represent the solution to this problem. Next we split this integral up into "Major" and "Minor" arcs, which are essentially main terms and error terms. The integral over the major arcs are evaluated (often asymptotically), and the integral over the minor arcs is bounded so that its effect is negligible in comparison to the major arcs.

Theorem 3.3. *Consider Question 0.1, and let $F(z)$ be the generating function*

$$F(z) = \sum_{n \in A} z^n$$

Then, we know that for all $\rho \in (0, 1)$,

$$r(n, s, A) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{F(z)^s}{z^{N+1}} dz$$

Proof. We have that

$$\int_{|z|=\rho} \frac{F(z)^s}{z^{N+1}} dz = \int_{|z|=\rho} \left(\sum_{n \in A} z^n \right)^s z^{-N-1} dz = \int_{|z|=\rho} \left(\sum_{k=0}^{\infty} r(k, s, A) z^k \right) z^{-n-1} dz$$

$$= \int_{|z|=\rho} \left(\sum_{k=0}^{\infty} r(k, s, A) z^{k-n-1} \right) dz = \sum_{k=0}^{\infty} r(k, s, A) \int_{|z|=\rho} z^{k-n-1} dz = 2\pi i \cdot r(n, s, A)$$

This comes from the residue theorem, as the residues of z^{k-n-1} are 0 unless $k = n$. Therefore

$$\frac{1}{2\pi i} \int_{|z|=\rho} \frac{F(z)^s}{z^{N+1} dz} = \frac{1}{2\pi} \cdot 2\pi i \cdot r(n, s, A) = r(n, s, A)$$

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The next formulation of the circle method is due to Vinogradov. He replaced the generating function with a Fourier Series:

Theorem 3.4. *Consider Question 0.1, and let $G(z)$ be the generating function*

$$G(z) = \sum_{a \in A} e^{2\pi i a z}$$

Then,

$$r(n, s, A) = \int_0^1 \frac{G(z)^s}{e^{2\pi i n z}} dz$$

This proof is not relevant, so we omit it from our discussion.

4. Back to Partitions

Now we will apply the circle method the integer partitions. We aim to find an approximation for the generating function of the partitions and bound the difference using the circle method. Remember our generating function for the partitions:

$$P(z) = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}$$

In this instance, we do care that this generating function converges. Since we already used a summation of geometric series while finding the generating function, we require $|z| < 1$.

Lemma 4.1.

$$P(z) < e^{\frac{1}{|1-z|} + \frac{1}{1-|z|}}$$

Proof. Let $z \in \mathbb{C}$ where $|z| < 1$. This means that $1 - z^n$ is in the open right half plane, meaning that $(1 - z^n)^{-1}$ is also in the open right half plane. This means that logarithms are well defined.

Taking the logarithm, we get:

$$\log(P(z)) = \sum_{n=1}^{\infty} -\log(1 - z^n) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{z^{nk}}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} z^{nk} = \sum_{k=1}^{\infty} \frac{z^k}{(1 - z^k)k}$$

These steps are justified, since the series converges absolutely when $|z| < 1$. Therefore, the sum can be rearranged.

The next step is the bound the logarithm. By the triangle inequality:

$$|\log(P(z))| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{|1 - z^k|k} = \frac{|z|}{|1 - z|} + \sum_{k=2}^{\infty} \frac{|z|^k}{|1 - z^k|k} \leq \frac{|z|}{|1 - z|} + \sum_{k=2}^{\infty} \frac{|z|^k}{(1 - |z|^k)k}$$

$$= \frac{|z|}{|1 - z|} + \frac{1}{1 - |z|} \sum_{k=2}^{\infty} \frac{|z|^k}{(1 + |z| + |z|^2 \cdots |z|^{k-1})k}$$

$$= \frac{|z|}{|1 - z|} + \frac{1}{1 - |z|} \sum_{k=2}^{\infty} \frac{1}{(|z|^{-k} + |z|^{-k+1} + |z|^{-k+2} \cdots |z|^{-1})k}$$

$$\leq \frac{|z|}{|1 - z|} + \frac{1}{1 - |z|} \sum_{k=2}^{\infty} \frac{2}{(k)(k+1)k} \leq \frac{|z|}{|1 - z|} + \frac{1}{1 - |z|} \sum_{k=2}^{\infty} \frac{1}{k^2}$$

$$\leq \frac{1}{|1 - z|} + \frac{1}{1 - |z|}$$

Finally, from here, we get that

$$|P(z)| = |e^{\log(P(z))}| = |e^{\Re(\log(P(z)))}| = e^{\Re(\log(P(z)))} \leq e^{|\log(P(z))|} < e^{\frac{1}{|1-z|} + \frac{1}{1-|z|}}$$

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In addition to giving us the estimate we want, this additionally shows that $|z| \leq 1$ is sufficient for this to converge.

The next thing we want to do is to approximate $P(z)$ when z is close to 1.

Definition 4.2.

$$\phi(z) = \left(\frac{1 - z}{2\pi}\right)^{-1/2} e^{\frac{\pi^2}{12}(-1 + \frac{2}{1-z})}$$

Lemma 4.3. *Whenever* $|z| < 1$ *and* $|1 - z| \leq 2(1 - |z|)$

$$P(z) = \phi(z)(1 + O(1 - z))$$

Proof. Recall the identity

$$\log(P(z)) = \sum_{k=1}^{\infty} \frac{z^k}{(1 - z^k)k}$$

We proceed by setting $z = e^{-w}$ where $|\Im w| \leq \pi$. Then, we have that

$$\log(P(e^{-w})) = \sum_{k=1}^{\infty} \frac{e^{-kw}}{(1 - e^{-kw})k} = \sum_{k=1}^{\infty} \frac{1}{(e^{kw} - 1)k}$$

If we add and subtract $\frac{1}{2}\log(1-e^{-w})+\frac{\pi^2}{6w}$ on one side, we get

$$\log(P(z)) = \frac{1}{2}\log(1-e^{-w}) + \frac{\pi^2}{6w} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{-kw}}{k} - \frac{1}{w}\sum_{k=0}^{\infty}\frac{1}{k^2} + \sum_{k=1}^{\infty}\frac{1}{(e^{kw}-1)k}$$

$$= \frac{1}{2}\log(1-e^{-w}) + \frac{\pi^2}{6w} + w\sum_{k=1}^{\infty}\frac{1}{(e^{kw}-1)wk} - \frac{1}{(wk)^2} + \frac{e^{-kw}}{2wk}$$

For convenience let $g(u) = \frac{1}{(e^u-1)u} - \frac{1}{u^2} + \frac{e^{-u}}{2u}$. Then, we get

$$\log(P(z)) = \frac{1}{2}\log(1-e^{-w}) + \frac{\pi^2}{6w} + w\sum_{k=0}^{\infty} g(wk)$$

g is chosen so that it is continuous at 0. Note that the sum is a Riemann sum for the integral from 0 to infinity of $g(u)$. In fact, it can be shown that

$$w\sum_{k=0}^{\infty} g(wk) = \int_0^{\infty} g(u)du + O(w) = -\frac{1}{2}\log(2\pi) + O(w)$$

The proof for this is quite involved and is not relevant to our discussion, so in order to see it look at [AC17]

So now, we have

$$\log(f(e^{-w})) = \frac{\pi^2}{6w} + \frac{1}{2}\log(1-e^{-w}) - \frac{1}{2}\log(2\pi) + O(w)$$

Then, replacing $z = e^{-w}$, we get

$$\log(f(z)) = \frac{\pi^2}{6w} + \frac{1}{2}\log(\frac{1-z}{2\pi}) + O(w)$$

Since we have that $|z| < 1$, and $w = -\log(z) = \log(1-(1-z)) = \sum_{k=0}^{\infty}(1-z)^k/k$. This means that $O(w) = O(1-z)$ as z approaches 1. In a similar fashion, we can also show that

$$\frac{1}{w} = \frac{1}{1-z} + -\frac{1}{2} + O(1-z)$$

Therefore,

$$\log(f(z)) = \frac{\pi^2}{6(1-z)} - \frac{\pi^2}{12} + \frac{1}{2}\log(\frac{1-z}{2\pi}) + O(1-z)$$

Finally, we get that

$$f(z) = \sqrt{\frac{1-z}{2\pi}}e^{\frac{\pi^2}{6(1-z)} - \frac{\pi^2}{12} + O(1-z)} = \phi(z)e^{O(1-z)}$$
$$= \phi(z)(1+O(1-z))$$

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Now, we are finally ready to use the circle method. Using the circle method, we will bound the difference between $P(z)$ and our approximation ϕ

Lemma 4.4. *Let $\phi(z) = \sum_{k=0}^{\infty} q(n)z^n$. We aim to show that*

$$p(n) = q(n) + O(n^{-5/4}e^{\pi\sqrt{2n/3}})$$

Proof. By the circle method, we know that

$$p(n) - q(n) = \frac{1}{2\pi i}\int_{|z|=1-\pi/\sqrt{6n}} \frac{f(z)-\phi(z)}{z^{n+1}}dz$$

We split the contour into major and minor arcs, letting the major arc

$$\mathcal{M} = \{z \in \mathbb{C} : |1-z| < \pi\sqrt{2/3n}\}$$

and the minor arc being everything else.

We can first bound the minor arc.

$$\int_m \frac{f(z)-\phi(z)}{z^{n+1}}dz \leq \int_m \frac{|f(z)|+|\phi(z)|}{|z^{n+1}|}|dz| \leq \int_m \frac{e^{\frac{1}{|1-z|}+\frac{1}{1-|z|}} - |\phi(z)|}{|z|^{n+1}}|dz|$$
$$= O\left(\int_m (e^{\frac{1}{|1-z|}+\frac{1}{1-|z|}} + e^{\frac{\pi^2}{6|1-z|}})|z|^{-(n+1)}|dz|\right)$$
$$= O\left(\int_m (e^{\frac{1}{\pi}(\sqrt{3n/2}+\sqrt{6n})} + e^{\frac{\pi^2}{6}\sqrt{3n/2}})e^{\pi\sqrt{n/6}}|dz|\right) = O(e^{a\sqrt{n}})$$

where $a < \pi\sqrt{2/3}$

Next, we bound the major arc with the knowledge that the length of $\mathcal{M}$ is $O(n^{-1/2})$

$$\int_{\mathcal{M}} \frac{f(z)-\phi(z)}{z^{n+1}}dz = O(\int_{\mathcal{M}} |z|^{-n}|1-z|^{3/2}e^{\frac{\pi}{6}\cdot\frac{1}{1-|z|}}|dz|) = O(e^{2\pi\sqrt{n/6}}\cdot n^{-3/4})\cdot O(n^{-1/2})$$
$$= O(n^{-5/4}e^{\pi\sqrt{2n/3}})$$

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Now, all that is left to do is to compute $q(n)$ We begin with the identity

$$\pi\sqrt{2}e^{\pi^2/12}\phi(z) = (1-z)\int_{-\infty}^{\infty} e^{\pi t\sqrt{2/3}-(1-z)t^2}dt$$

We therefore have that

$$\pi\sqrt{2}e^{\pi^2/12}q(n) = \int_{-\infty}^{\infty} e^{\pi t\sqrt{2/3}-t^2}e^{t^2 n}(1-n)dt = \int_{-\infty}^{\infty} e^{\pi t\sqrt{2/3}-t^2}\left(\frac{t^{2n}}{n!} - \frac{t^{2n-2}}{(n-1)!}\right)dt$$

Letting $t = s+\sqrt{n}$ and then using Stirling's formula for $n!$, we get

$$\sim \frac{e^{\pi\sqrt{2n/3}}}{\sqrt{2\pi n}}\int_{-\infty}^{\infty} se^{\pi\sqrt{2/3}s-s^2-2\sqrt{n}s}\left(1+\frac{s}{\sqrt{n}}\right)^{2n-2}\left(2+\frac{s}{\sqrt{n}}\right)ds$$

Taking a limit inside the integral as n approaches infinity yields that

$$\pi\sqrt{2}e^{\pi^2/12}q(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{\sqrt{2\pi n}}\int_{-\infty}^{\infty} 2se^{\pi\sqrt{2/3}s-2s^2}ds = \frac{\pi}{2\sqrt{6}n}e^{\frac{\pi^2}{12}+\pi\sqrt{2n/3}}$$

This means that

$$q(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

Then, by lemma 4.4, we have that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} + O(n^{-5/4}e^{\pi\sqrt{2n/3}})$$

This completes the proof of the Hardy-Ramanujan Formula for $p(n)$

5. Other Problems and Conclusion

We will now look at some other examples where the circle method could be used, although we will only briefly look at each one.

Question 5.1. *For every $k \in \mathbb{N}$ does there exist an $s \in \mathbb{Z}_{>0}$ such that for every $n \in \mathbb{N}$, $a_1^k + a_2^k \cdots a_s^k = n$ for some $a_k \in \mathbb{N}$*

This problem is called Waring's problem, and it can be solved using the circle method. For this example Vinogradov's adaptation proves to be more useful. We have that A is the set of powers of k. For notational convenience, let Let $r(n, s) = \{a_i \in \mathbb{N} : a_1^k + a_2^k \cdots + a_s^k = n\}$. Note that this means that every $a_i \leq n^{1/k}$. We can now define

$$G(z) = \sum_{a \leq n^{1/k}} e^{2\pi i a z}$$

This allows us to say that

$$r(n, s) = \int_0^1 \frac{G(z)^s}{e^{2\pi i n z}} dz = \int_M \frac{G(z)^s}{e^{2\pi i n z}} dz + \int_m \frac{G(z)^s}{e^{2\pi i n z}} dz$$

Now we must define the major and minor arcs. First we define the major arc. Let $M_q = \{x \in \mathbb{R}/\mathbb{Z} : |x - \frac{a}{k}| \leq \frac{1}{qn}\}$, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}$ and $\delta > 0$. The major arc is essentially a small interval around every fraction a/k. We define

$$M = \bigcup_{1 \leq q \leq n^{1-1/k}} M_q$$

We define everything else to be the minor arc. We can show that the integral over the minor arcs

$$\int_m \frac{G(z)^s}{e^{2\pi i n z}} dz = o(n^{s/k-1})$$

Bounding the major arcs is also more tricky. For more information see [ASS22]

Now that we have outlined the broad strokes in the solution to Waring's Problem, we introduce an open problem:

Question 5.2. *Consider Question 0.1 with $A = P$ and $s = 2$, where P is the set of primes. This asks whether every even number is the sum of two primes.*

This is known as Goldbach's conjecture, which is a famous unsolved problem. Some progress can be made on it using the circle method.

In conclusion, the circle method is a powerful tool which can be used to analyze additive number theory problems. As seen in this paper, it can be adapted in a variety of ways to make it useful in different problems.

References

[AC17] Lambert A Campo. The circle method, applications to the partition function and beyond. 2017.

[ASS22] EDGAR ASSING. The circle method and waring's problem. 2022.

[MTB04] Steven J Miller and Ramin Takloo-Bighash. The circle method, 2004.

[New62] D. J. Newman. A simplified proof of the partition formula. *Michigan Mathematical Journal*, 9(3):283 – 287, 1962.

[RS] Simon Rubinstein-Salzedo. *Complex Analysis*.

[New62] [AC17] [MTB04] [RS] [ASS22]