THE GROUP EXTENSION PROBLEM

ISHAN JOSHI

Abstract. In this paper we will discuss the group extension problem, and the classification of group extensions in the case of split extensions, and in the case where K is a normal abelian subgroup. When $G/K \cong H$, it can be noted that H acts upon K. In other words there exists a homomorphism $\theta : H \to \mathrm{Aut}(K)$. This homomorphism is determined by the congruence classes of the extensions of H by K. Thus, the congruence classes of extensions are able to be partitioned by the elements of $\mathrm{Hom}(H, \mathrm{Aut}(K))$ For the fixed homomorphism θ it turns out that the congruence classes of extensions which induce θ is described by the second cohomology group $H^2_\theta(H, K)$.

1. Introduction

The idea of a group began in the $19th$ century, when Évariste Galois was studying the roots of polynomials. Today, groups similar to those that Galois worked with are called permutation groups. Groups were abstracted first in the 1850s, in a paper by Arthur Cayley. After a century of growth, group theory began to branch out into representation theory, algebraic groups, and group extensions. The subject also began to see use in fields such as geometry and number theory. In the 1950s, mathematicians took up the monumental task of classifying all finite simple groups, an effort which culminated in the 1980s. Additionally, hundreds of classification theorems exist all throughout mathematics. It can be seen that we place great importance in classifying objects. A solution to the group extension problem would greatly expand our classification of groups, by allowing us to classify, as well as construct all finite groups. If the classification of finite simple groups is like a periodic table for groups, the solution to the group extension problem would tell us how to create all the other elements using that periodic table. It is for these reasons that we care about the group extension problem. In this paper, we review what is known about the classification of groups which have a normal subgroup. When a group G has $K \lhd G$, we can "factor" G as K and G/K. In this way, we are able to classify normal subgroups, by considering the group G. The natural question to ask is when can we go backwards? If we know G/K and K, can we determine G? Although the group group extension problem has not been solved in its entirety, solutions do exist in some cases. In this paper, we review the group extension problem in the cases of split extensions, and when K is a normal abelian subgroup. In the case of split extensions, we describe group extensions through semidirect products, and in the case when K is abelian, extensions are described through group cohomology. Intuitively, cohomology can describe how something can be extended, which is why we consider it a useful tool for the group extension problem.

Date: July 15, 2024.

2. Problem Statement and Preliminaries

We first begin by generalizing the idea of K being a normal subgroup of G.

Definition 1. A sequence of groups:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} H \xrightarrow{f_4} 1$$

Is called *short exact* if all f_i are homomorphisms and $\operatorname{im}(f_i) = \ker(f_{i+1})$.

When K is a normal subgroup of G, $G/K \cong H$. In general, $G/\operatorname{im}(f_2) \cong H$. This is a result of the first isomorphism theorem. We can also see that since $\operatorname{im}(f_1) = \ker(f_2)$, then the kernel of f_2 is trivial, meaning f_2 is injective. Similarly $\operatorname{im}(f_3) = \ker(f_4)$,so the image of $f_3 = H$ meaning f_3 is surjective.

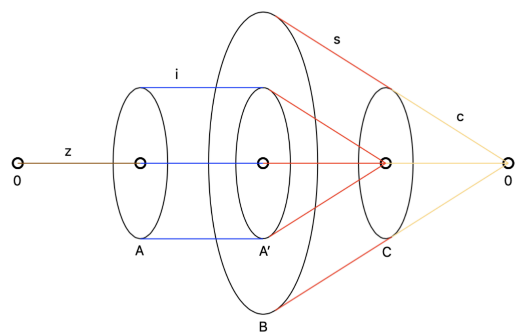

Figure 1. A visual description of the short exact sequence $1 \xrightarrow{z} A \xrightarrow{i} B \xrightarrow{s} C \xrightarrow{c} 1$

.

Example. If $K \lhd G$, then the sequence:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} G/K \xrightarrow{f_4} 1$$

is short exact. We let $f_2(k) = k$, and $f_3(g) = gK$. We can see that f_2 is injective, and f_3 is surjective, and that they are both homomorphisms. Now we only need to check:

$$f_3(f_2(k)) = f_3(k) = K.$$

Which shows that $\operatorname{im}(f_2) = \ker(f_3)$.

Using the short exact sequence, we can rephrase the definition of a group extension.

Definition 2. G is called an *extension* of H by K if G fits into the short exact sequence:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} H \xrightarrow{f_4} 1$$

.

Example. The direct product $K \times H$ is an example of an extension of H by K. In other words, there exits a short exact sequence:

$$1 \xrightarrow{f_1} K \xrightarrow{f_2} K \times H \xrightarrow{f_3} H \xrightarrow{f_4} 1.$$

We define $f_2(k) = (k, 1)$, and $f_3(k, h) = h$. We can indeed see that f_2 is injective, and f_3 is surjective. Additionally, $f_3(f_2(k)) = f_3(k, 1) = 1$, which shows that $\operatorname{im}(f_2) = \ker(f_3)$.

Example. Let $K = \mathbb{Z}$ and let $H = [0, 1)$ be a group with operation $\star : H \times H \longrightarrow H$, such that $h_1 \star h_2 = h_1 + h_2 - \lfloor h_1 + h_2 \rfloor$. In particular, H is the set or reals mod 1. Then an extension of H by K is $\mathbb{R}$.

Definition 3. A diagram of groups and homomorphisms *commutes* if every path of functions in the diagram leads to the same result. For example, the diagram:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "a"'] & H \arrow[d, "g"] \\ & K \end{tikzcd}$$

commutes if and only if $f(g(x)) = a(x)$ for $x \in G$.

In other words, all of the compositions of functions lead to the same result.

Definition 4. Two extensions G and G' are considered equivalent if there exists and isomorphism $\beta : G \longrightarrow G'$ and, the following diagram commutes:

$$\begin{tikzcd} & & G \arrow[dd, "\beta"] \arrow[dr, two heads, "f_3"] & & & \\ 1 \arrow[r, hook, "f_1"] & K \arrow[ur, hook, "f_2"] \arrow[dr, hook, "f_2'"'] & & H \arrow[r, two heads, "f_4"] & 1 \\ & & G' \arrow[ur, two heads, "f_3'"'] & & \end{tikzcd}$$

It is important to note that even if there exists an isomorphism $\pi : G \longrightarrow G'$, the extensions may not be equivalent.

Proposition 2.1. *The definition above creates an equivalence relation on the set of extensions.*

Proof. We can see reflexivity by simply setting β to be the identity map. We symmetry by considering the following:

$$\begin{tikzcd} & & G \arrow[d, "\alpha"] \arrow[dr, "f_3"] & & \\ 1 \arrow[r, "f_1"] & K \arrow[ur, "f_2"] \arrow[r, "f_2'"] \arrow[dr, "f_2"'] & G' \arrow[r, "f_3'"] \arrow[d, "\beta"] & H \arrow[r, "f_4"] & 1 \\ & & G \arrow[ur, "f_3"'] & & \end{tikzcd}$$

Similarly, we see transitivity by considering:

$$\begin{tikzcd} & & G \arrow[d, "\alpha"] \arrow[dr, "f_3"] & & \\ 1 \arrow[r, "f_1"] & K \arrow[ur, "f_2"] \arrow[r, "f_2'"] \arrow[dr, "f_2''"'] & G' \arrow[r, "f_3'"] \arrow[d, "\beta"] & H \arrow[r, "f_4"] & 1 \\ & & G'' \arrow[ur, "f_3''"'] & & \end{tikzcd}$$

Which shows that the definition of equivalent extensions is an equivalence relation. ∎

3. Classification of Split Extensions

Definition 5. A short exact sequence is called *split* is there exists a homomorphism $\pi : H \longrightarrow G$ such that $f_3 \circ \pi = id_H$.

This definition is very important, as it allows us to explicitly identify G. In this section we will see how we use semidirect products to classify split extensions.

Example. The sequence:

$$1 \xrightarrow{f_1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_3} \mathbb{Z}/2\mathbb{Z} \xrightarrow{f_4} 1$$

is split. Let $f_2(a) = (a, 0)$, and let $f_3(a, b) = b$, for all $a, b \in \mathbb{Z}/2\mathbb{Z}$. Now let $\pi : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be a function such that $\pi(b) = (0, b)$. It can easily be seen that π is a homomorphism. We need to check:

$$f_3(\pi(b)) = f_3(0, b) = b$$

which verifies that the extension splits.

Definition 6. For groups H and K, and with the action $\varphi : H \longrightarrow \mathrm{Aut}(K)$, the corresponding semidirect product $K \rtimes_\varphi H$ is the set $K \times H$, along with the multiplication:

$$(k_1, h_1)(k_2, h_2) = (k_1\varphi(h_1)(k_2), h_1h_2)$$

We still have to verify that the set $K \rtimes_\varphi H$ is a group.

Corollary 3.1. *The semidirect product $K \rtimes_\varphi H$ is a group.*

Proof. We first prove that there exists an identity element, which is $(1, 1)$.

$$(k, h)(1, 1) = (k\varphi(h)(1), h) = (k, h) = (1\varphi(1)(k), 1h) = (1, 1)(k, h)$$

Next we prove that inverses exist. To do this we will have to find the inverse element for (k, h). We are trying to find a (k', h') such that $(k, h)(k', h') = (1, 1)$ From the second element, we already know that $h' = h^{-1}$. Now we need a k', such that $k^{-1} = \varphi(h)(k')$ Since ϕ is an automorphism, then $\varphi(h)(k') = k^{-1}$ means that $k' = \varphi(h^{-1})(k^{-1}) = (\varphi(h^{-1})(k))^{-1}$ So, $(k, h)^{-1} = (\varphi(h^{-1})(k))^{-1}, h^{-1})$. Now we check that this is acutally an inverse:

$$(k, h)(\varphi(h^{-1})(k))^{-1}, h^{-1}) = (k\varphi(h)(\varphi(h^{-1})(k^{-1})), 1) = (kk^{-1}, 1) = (1, 1)$$

and,

$$(\varphi(h^{-1})(k))^{-1}, h^{-1})(k, h) = ((\varphi(h^{-1})(k))^{-1}\varphi(h^{-1})(k), 1) = (1, 1)$$

Finally, we prove associativity:

$$\begin{aligned}
\Big((k_1, h_1)(k_2, h_2)\Big)(k_3, h_3) &= \big(k_1\varphi(h_1)(k_2), h_1h_2\big)(k_3, h_3) \\
&= \big(k_1\varphi(h_1)(k_2)\varphi(h_1h_2)(k_3), h_1h_2h_3\big) \\
&= \big(k_1\varphi(h_1)(k_2)\varphi(h_1)\big(\varphi(h_2)(k_3)\big), h_1h_2h_3\big) \\
&= \Big(k_1\varphi(h_1)\big(k_2\varphi(h_2)(k_3)\big), h_1h_2h_3\Big) \\
&= (k_1, h_2)(k_2\varphi(h_2)(k_3), h_2h_3) \\
&= (k_1, h_1)((k_2, h_2)(k_3, h_3))
\end{aligned}$$

Therefore, $K \rtimes_\varphi H$ is a group.

∎

Remark 3.2. The direct product $K \times H$ is a semidirect product, $K \rtimes_\varphi H$, when ϕ is the trivial action on K.

Example. We can create a semidirect product where $K = H = \mathbb{Z}$ The automorphism group for $\mathbb{Z}$ is $\{\pm 1\}$. We define the function $\varphi : \mathbb{Z} \longrightarrow \pm 1$ where $\varphi(n) = (-1)^n$ Then, we define the multiplication on $\mathbb{Z} \times \mathbb{Z}$ as $(a_1, b_1)(a_2, b_2) = (a_1 + (-1)^{b_1} a_2, b_1 + b_2)$ which is a semidirect product.

A natural question to ask is that when can we tell if a group has a semidirect product structure? Similarly to direct products, we can create a recognition theorem for semidirect products. We use many of the properties of the semidirect product for this theorem.

Lemma 3.3. *Let G be a group with subgroups K and H, where*

- $G = KH$
- $K \cap H = \{1\}$
- $K \lhd G$

Define $\varphi : H \longrightarrow \mathrm{Aut}(K)$ and let $\varphi(h)(k) = hkh^{-1}$. Then φ is a homomorphism and the function $f : K \rtimes_\varphi H \longrightarrow G$ where $f(k, h) = kh$ is an isomorphism.

Proof. We begin by showing f is an isomorphism. We need to show f is a homomorphism, and then that it is bijective.

$$\begin{aligned} f((k_1, h_1)(k_2, h_2)) &= f(k_1 \varphi(h_1)(k_2), h_1 h_2) \\ &= k_1 \phi(h_1)(k_2) h_1 h_2 \\ &= k_1 h_1 k_2 h_1^{-1} h_1 h_2 \\ &= k_1 h_1 k_2 h_2 \\ &= f(k_1, h_2) f(k_2, h_2). \end{aligned}$$

This proves that f is a homomorphism. Now, we prove bijectivity. From the first property, we can see that f is surjective, as its image is $KH = G$. We prove injectivity by showing that the kernel is trivial. Assume $f(k, h) = 1$ Then $kh = 1$, so $k = h^{-1}$. Since K is a subgroup of G, this means $h^{-1} \in K$. But h^{-1} is also in H. So $h = h^{-1} = 1$. Similarly, this logic also works for K. So $f(k, h) = f(1, 1)$. This proves that f is an isomorphism. To conclude our proof, we show that φ is a homomorphism.

$$\begin{aligned} \varphi(h_1 h_2)(k) &= h_1 h_2 k (h_1 h_2)^{-1} \\ &= h_1 h_2 k h_2^{-1} h_1^{-1} \\ &= h_1 \varphi(h_2)(k) h_1^{-1} \\ &= \varphi(h_1)[\varphi(h_2)(k)]. \end{aligned}$$

Which gives us that φ is a homomorphism, thereby completing our proof. ∎

Now, we have enough information and intuition to prove that all split extensions are semidirect products.

Theorem 3.4. *Let $1 \xrightarrow{f_1} K \xrightarrow{f_2} G \xrightarrow{f_3} H \xrightarrow{f_4} 1$ be a short exact sequence. Then, the sequence is split if and only if there exists a homomorphism $\varphi : H \longrightarrow \mathrm{Aut}(K)$ such that the extension of H by K: $K \rtimes_\varphi H$ is equivalent to the extension G of H by K.*

Proof. We begin by proving the forward direction. First we have to construct an action φ of H on K using the homomorphism $\pi : H \longrightarrow G$ and create an isomorphism of G and $K \rtimes_\varphi H$. To do this, we use conjugation in G to construct φ: We consider $\pi(h)f_2(k)\pi(h^{-1}) \in \ker(f_3)$, as seen by:

$$f_3(\pi(h)f_2(k)\pi(h^{-1})) = f_3(\pi(h))f_3(f_2(k))f_3(\pi(h^{-1}) = h \cdot 1 \cdot h^{-1} = 1.$$

Since $\pi(h)f_2(k)\pi(h^{-1}) \in \ker(f_3)$, and $\mathrm{im}(f_2) = \ker(f_3)$, we can express $\pi(h)f_2(k)\pi(h^{-1}) \in \ker(f_3)$ as $f_2(k')$ for some $k' \in K$. Furthermore, only one such k' exists, because f_2 is injective. We define this k' to be $\varphi(h)(k)$. So,

$$\pi(h)f_2(k)\pi(h^{-1}) = f_2(\varphi(h)(k)).$$

Since $\varphi(h)(k) \in K$, we have that $\varphi(h) : K \longrightarrow K$. Now we need to show that $\varphi(h)(k)$ is an isomorphism and that $\varphi : H \longrightarrow \mathrm{Aut}(K)$ is a homomorphism. We start by checking that $\varphi(h)$ is a homomorphism.

$$\begin{aligned} f_2(\varphi(h)(k_1k_2)) &= \pi(h)f_2(k_1k_2)\pi(h)^{-1} \\ &= \pi(h)f_2(k_1)\pi(h)^{-1}\pi(h)f_(k_2)\pi(h)^{-1} \\ &= f_2(\varphi(h)(k_1))f_2(\varphi(h)(k_2)) \\ &= f_2(\varphi(h)(k_1)\varphi(h)(k_2)). \end{aligned}$$

Since f_2 is injective, we have that $\varphi(h)(k_1k_2) = \varphi(h)(k_1)\varphi(h)(k_2)$ Next we prove that φ is a homomorphism.

$$\begin{aligned} f_2(\varphi(h_1h_2)(k)) &= \pi(h_1h_2)f_2(k)\pi(h_1h_2)^{-1} \\ &= \pi(h_1)\pi(h_2)f_2(k)\pi(h_2)^{-1}\pi(h_1)^{-1} \\ &= \pi(h_1)f_2(\varphi(h_2)(k))\pi(h_1)^{-1} \\ &= f_2(\varphi(h_1)(\varphi(h_2)(k))). \end{aligned}$$

By injectivity, $\varphi(h_1h_2)(k) = \varphi(h_1)(\varphi(h_2)(k))$ This proves that φ is an action on K by H. Therefore, π can induce an action, meaning that we have the semidirect product $K \rtimes_\varphi H$. Now, we need to prove that the extension that is the semidirect product is equivalent to G. We want to find an isomorphism from G to $K \rtimes_\varphi H$. However, this is unnecessary, as it is easier to find an isomorphism from $K \rtimes_\varphi H$ to G. We call this function τ. We define τ by $\tau(k, h) = f_2(k)\pi(h)$. First, we check that τ is a homomorphism.

$$\begin{aligned} \tau((k_1, h_1)(k_2, h_2)) &= \tau(k_1\varphi(h_1)(k_2), h_1h_2) \\ &= f_2(k_1\varphi(h_1)(k_2))\pi(h_1h_2) \\ &= f_2(k_1)f_2(\varphi(h_1)(k_2))\pi(h_1)\pi(h_2) \\ &= f_2(k_1)\pi(h_1)f_2(k_2)\pi(h_1)^{-1}\pi(h_1)\pi(h_2) \\ &= f_2(k_1)\pi(h_1)f_2(k_2)\pi(h_2) \\ &= \tau(k_1, h_1)\tau(k_2, h_2). \end{aligned}$$

This gives us that τ is a homomorphism. Next, we show injectivity. Assume that $\tau(k, h) = 1$. Then, $f_3(\tau(k, h)) = f_3(f_2(k))f_3(\pi(h)) = \pi(1) = 1$ This means that $f_3(\pi(h)) = 1$, or that $h = 1$. Then, we get that $\tau(k, 1) = f_2(k) = 1$, which by the injectivity of f_3 implies that $k = 1$. Thus, the kernel of τ is trivial, proving injectivity. For surjectivty, let $g \in G$. Then we need to find k and h such that $\tau(k, h) = f_2(k)\pi(h) = g$ We consider $h = f_3(g)$. This comes from applying f_3 to both sides. We get $f_3(f_2(k))f_3(\pi(h)) = f_3(g)$ This means that $h = f_3(g)$ Now we search for a k such that $f_2(k) = g\pi(f_3(g))^{-1}$. Therefore, we need to check that $g\pi(f_3(g))^{-1} \in \ker(f_3)$.

$$f_3(g\pi(f_3(g))^{-1}) = f_3(g)f_3(\pi(f_3(g^{-1}))) = f_3(g)f_3(g)^{-1} = 1$$

Therefore, τ is an isomorphism. Now, we need to prove that this diagram commutes:

$$\begin{tikzcd}
 & & G \arrow[dr, two heads, "f_3"] & & \\
1 \arrow[r, "f_1"] & K \arrow[ur, hook, "f_2"] \arrow[dr, hook, "f_2'"'] & & H \arrow[r, hook, two heads, "f_4"] & 1 \\
 & & K \rtimes_\varphi H \arrow[uu, "\tau"] \arrow[ur, two heads, "f_3'"'] & &
\end{tikzcd}$$

Define $f_2'(k) = (k, 1)$, and $f_3'(k, h) = h$. We need to show that $f_3(f_2(k)) = f_3(\tau(f_2'(k)))$.

$$f_3(\tau(f_2'(k))) = f_3(\tau(k, 1)) = f_3(f_2(k)\pi(1)) = f_3(f_2(k)).$$

This concludes the forward direction of the proof. Now we will prove the backwards direction. We define $\pi : H \longrightarrow G$ to be $\pi(h) = \beta^{-1}(1, h)$, where β is an isomorphism from $G \longrightarrow K \rtimes_\varphi H$ such that the corresponding diagram commutes. This is a homomorphism as β^{-1} is a homomorphism, as well as the map $h \mapsto (1, h)$. $f_3(\pi(h)) = f_3(\beta^{-1}(1, h)) = h$, because of the commutivity of the corresponding diagram. This proves that the extension is split. $\blacksquare$

4. Abelian Normal Subgroup

Before classifying the extensions of abelian normal subgroups, we find a way build up G using K as well as a transversal.

Definition 7. A *transversal* of a subgroup K in G is a set T, such that T contains exactly one element from every coset of K. A transversal can also be thought of as a function $\phi : H \longrightarrow G$, such that $\phi(H) = T$.

Example. Consider the group of integers $\mathbb{Z}$ with normal subgroup $3\mathbb{Z}$. The cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ are $0 + 3\mathbb{Z}, 1 + 3\mathbb{Z},$ and $2 + 3\mathbb{Z}$. Therefore, a valid transversal of $3\mathbb{Z}$ in $\mathbb{Z}$ is $\{0, 1, 2\}$.

The next lemma shows how G is built out of K and the transversal T.

Lemma 4.1. *Let T be a transversal of K in G. Then*

$$G = \bigcup_{t \in T} tK$$

Proof. Every element of the transversal T is of the form $t = gk$ for $k \in K$ and $g \in G$. Thus,

$$tK = gkK = gK.$$

Since the cosets of K partition G, this completes our proof. ■

As a result of this lemma, for any $g \in G$, there exists a unique $t \in T$ and a unique $k \in K$ such that $g = kt$. We also know that due to the definition of a transversal, there exists a $h \in H$, such that $\phi(h) = t$. We can therefore express any element g as a Cartesian pair, (k, h), with the property that

$$(k, h) = k\phi(h) = kt = g$$

We then perform the calculations:

$$\begin{aligned} g_1 g_2 &= (k_1, h_1)(k_2, h_2) \\ &= k_1\phi(h_1)k_2\phi(h_2) \\ &= k_1 t_1 k_2 t_2 \\ &= k_1 t_1 k_2 t_1^{-1} t_1 t_2 \\ &= k_1 k_2^{t_1^{-1}} t_1 t_2 \\ &= k_1 k_2^{t_1^{-1}} \phi(h_1)\phi(h_2) \\ &= (k_1 k_2^{t_1^{-1}} \phi(h_1)\phi(h_2)\phi(h_1h_2)^{-1})(\phi(h_1h_2)) \end{aligned}$$

From here, we can define a unique automorphism of K, corresponding to elements in H. When K is abelian, there exists a homomorphism $\theta : H \longrightarrow \mathrm{Aut}(K)$, where

$$\theta(h) = f_h \text{ and } f_h(k) = \phi(h) + k - \phi(h)$$

Now we will always assume K to be abelian, and therefore use additive notation for the group operation in K.

Note 1. From here on, we choose to suppress θ, and depict the action as a multiplication of elements of H and K.

Definition 8. An ordered triple (H, K, θ) is called *data* if K is abelian, H is a group, and $\theta : H \longrightarrow \mathrm{Aut}(K)$ is a homomorphism. If G is an extension of H by K, then G *realizes* the data if for every transversal $\phi : H \longrightarrow K$:

$$xa = \phi(x) + a - \phi(x).$$

For $a \in K$ and $x \in H$

Intuitively, θ describes how K is a normal subgroup of G. We rephrase the group extension problem as find all extensions G such that G realizes the data (H, K, θ)

Definition 9. A factor set for the group G is a function $f : H \times H \longrightarrow K$ such that for all $x, y \in H$ and some transversal ϕ,

$$\phi(x) + \phi(y) = f(x, y) + \phi(xy)$$

A factor set describes how close ϕ is to being a homomorphism.

Remark 4.2. ϕ is a homomorphism if and only if the factor set f is the zero map.

A factor set therefore also is a measure of how far away G is from being a semidirect product, as if ϕ is a homomorphism, then G is a semidirect product.

Example. We continue with our previous example. Let $G = \mathbb{Z}$ and let $K = 3\mathbb{Z}$. Finally, let H be a group of order 3, which is given by the following multiplication table:

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

We let $\phi(e) = 0, \phi(a) = 2$, and $\phi(b) = 1$. We define the corresponding factor set to be $f(x, y) = \phi(x) + \phi(y) - \phi(xy)$. Then $f(e, e) = f(a, e) = f(e, a) = f(b, e) = f(e, b) = 0$, and $f(a, b) = f(b, a) = 3$.

We now introduce a theorem which motivates an important construction for the extensions of H by K.

Theorem 4.3. *Given data (H, K, θ), A function $f : H \times H \longrightarrow K$ is a factor set if and only if f satisfies the cocycle identity:*

$$f(x, 1) = 0 = f(1, y)$$

and

$$xf(y, z) + f(x, yz) = f(x, y) + f(x, yz)$$

for all $x, y, z \in H$. In particular, there exists an extension G which realizes the data (H, K, θ) and a transversal ϕ where f is the resulting factor set.

Proof. We begin by proving the first claim. We have previously assumed that $\phi(1) = 0$. Then we get:

$$f(x, 1) = \phi(x) + \phi(1) - \phi(x1) = 0 = \phi(1) + \phi(y) - \phi(y1) = f(1, y).$$

Now, we prove the forward direction of the second claim. We begin by considering the equation:

$$\begin{aligned}(\phi(x)+\phi(y))+\phi(z) &= \phi(x)+(\phi(y)+\phi(z))\\ f(x,y)+\phi(xy)+\phi(z) &= \phi(x)+f(y,z)+\phi(yz)\\ f(x,y)+f(xy,z)+\phi(xyz) &= f(y,z)+f(x,yz)+\phi(xyz)\\ f(x,y)+f(xy,z) &= \phi(x)+f(y,z)-\phi(x)+f(x,yz)\\ f(x,y)+f(xy,z) &= xf(y,z)+f(x,yz).\end{aligned}$$

This proves the forward direction of the second claim. To prove the other direction, we construct an extension G which realizes the data and has ϕ as transversal, such that f is the factor set corresponding to ϕ. We consider the set $K\times H$, with the group operation defined as:

$$(k,x)+(l,y)=(k+xl+f(x,y),xy).$$

Inverses are defined as: $-(k,x)=(-x^{-1}k-x^{-1}f(x,x^{-1}),x^{-1})$, and the identity is $(0,1)$. Next, we show that G is a extension of H by K and that G realizes the data. Let f_3 be a function for which $f_3(k,x)=x$ for all $k\in K$. This function is surjective, and has kernel $(k,1)$. We then construct the function $f_2 : K \longrightarrow G$ where $f_2(k)=(k,1)$. The image of this function is precisely $\ker(f_3)$. Therefore G is an extension of H by K. Next, we check that G realizes the data. We need to show that $xa=\phi(x)+a-\phi(x)$. We have $\phi(x)$ as (b,x) for $b\in K$. So,

$$\begin{aligned}\phi(x)+a-\phi(x) &= (b,x)+(a,1)-(b,x)\\ &= (b+xa+f(x,1),x)+(-x^{-1}b-x^{-1}f(x,x^{-1}),x^{-1})\\ &= (b+xa+x[x^{-1}b-x^{-1}f(x,x^{1})]+f(x,x^{-1}),1)\\ &= (xa,1)\end{aligned}$$

This proves that G realizes the data. Finally, we prove that f is a factor set for G. Let F be the factor set corresponding to ϕ. Then:

$$\begin{aligned}F(x,y) &= \phi(x)+\phi(y)-\phi(xy)\\ &= (b,x)+(b,y)-(b,xy)\\ &= (b+xb+f(x,y),xy)+(-(xy)^{-1}b-(xy)^{-1}f(xy,(xy)^{-1}),(xy)^{-1})\\ &= (b+xb+f(x,y)+xy[-(xy)^{-1}b-(xy)^{-1}f(xy,(xy)^{-1})]+f(xy,(xy)^{-1},xy(xy)^{-1})\\ &= (f(x,y),1).\end{aligned}$$

Which concludes our proof. ∎

We denote the construction of G with factor set f, as G_f. This is very important, as we define a way to construct G using H and K without using the multiplication in G

Definition 10. Let $Z^2_\theta(H,K)$ be the set of all factor sets for the group G which realizes the data (H,K,θ).

Remark 4.4. $Z^2_\theta(H,K)$ is an abelian group under pointwise addition.

The following lemma describes how two factor sets of the same group, G are different.

Lemma 4.5. *Let G be an extension realizing the data (H, K, θ), with transversals ϕ and ϕ', which have factor sets f and f' respectively. There is a function $\ell : H \longrightarrow K$ where $\ell(1) = 0$ and*

$$f'(x, y) - f(x, y) = x\ell(x) - \ell(xy) + \ell(x)$$

for all $x, y \in H$.

Proof. For all $x \in H$, $\phi(x)$ and $\phi'(x)$ are part of the same coset of K. So, there is an element, $\ell(x) \in K$ such that $\phi'(x) = \ell(x) + \phi(x)$. So,

$$\begin{aligned}
\phi'(x) + \phi'(y) &= (\ell(x) + \phi(x)) + (\ell(y) + \phi(y)) \\
&= \ell(x) + \phi(x) + \ell(y) - \phi(x) + \phi(x) + \phi(y) \\
&= \ell(x) + x\ell(y) + \phi(x) + \phi(y) \\
&= \ell(x) + x\ell(y) + f(x, y) + \phi(xy) \\
&= \ell(x) + x\ell(y) + f(x, y) + \phi'(xy) - \ell(xy).
\end{aligned}$$

From here, we can easily obtain $f'(x, y) - f(x, y) = x\ell(y) - \ell(xy) + \ell(x)$, as desired. ■

Definition 11. Given the data, (H, K, θ), a coboundary is a function $g : H \times H \longrightarrow K$, such that for all $x, y \in H$,

$$g(x, y) = xe(y) - e(xy) + e(x)$$

Where $e : H \longrightarrow K$ and $e(1) = 0$. The set of all coboundaries is an is denoted $B^2_\theta(H, K)$.

The definition of a coboundary as well as lemma 4.5 lead us to an important realization.

Corollary 4.6. *If two extensions are equivalent, their factor sets differ by a coboundary.*

Proof. G and G' are isomorphic, so if we identify G' with G, its isomorphic image, then we get that $f' - f \in B^2_\theta(H, K)$ ■

This is motivation to uncover some other properties about coboundaries.

Corollary 4.7. *A coboundary is also a factor set.*

Proof. We begin by proving the first part of the cocycle identity.

$$g(x, 1) = xe(1) - e(x) + e(x) = 0 = 1e(y) - e(y) + e(1) = g(1, y).$$

Next we prove the second part:

$$\begin{aligned}
xg(y, z) + g(x, yz) &= x[ye(z) - e(yz) + e(y)] + xe(yz) - e(xyz) + e(x) \\
&= xye(z) + xe(y) - e(xyz) + e(x) \\
&= xye(z) - e(xyz) + e(xy) - e(xy) + xe(y) + e(x) \\
&= g(xy, z) - e(xy) + xe(y) + e(x) \\
&= g(xy, z) + g(x, y).
\end{aligned}$$

Which proves that a coboundary satisfies the cocycle identity, and is therefore a factor set. ■

Corollary 4.8. *$B^2_\theta(H, K)$ is closed under addition.*

Proof. Let f, g be coboundaries with corresponding functions h and e respectively. Then:

$$\begin{aligned} f+g = f(x,y)+g(x,y) &= xh(y)-h(xy)+h(x)+xe(y)-e(xy)+e(x) \\ &= x(h+e)(y)-(h+e)(xy)+(h+e)(x) \end{aligned}$$

Which gives us closure, as desired. ■

The previous two corollaries give us that the set of coboundaries is an abelian subgroup of the set of factor sets.

Definition 12. Let G be a group. A G-module is an abelian group Q along with an action $\psi : G \times Q \longrightarrow Q$ such that

$$g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2.$$

Example. Every group G has $\mathbb{Z}$ as a G-module using the trivial action.

Definition 13. Let ${A^i}_{i\in\mathbb{N}}$ be a sequence of abelian H modules, and let $d^i : A^i \longrightarrow A^{i+1}$ be a collection of homomorphisms, with the property that $d^{i+1} \circ d^i = 0$ We define the cochain complex (A^*, d^*) as:

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \xrightarrow{d^{i+2}} \cdots$$

Definition 14. We define the i^{th} cohomology group H^i to be $\ker(d^i)/\operatorname{im}(d^{i-1})$.

We know that this definition is well defined as due to the property that $d^{i+1} \circ d^i = 0$, this $\operatorname{im}(d^{i-1}) \subset \ker(d^i)$. This is a subgroup, and additionally it is normal, as we are considering abelian groups.

Intuitively, the group cohomology describes how we can extend a group action.

Definition 15. We denote the set of all functions from H^i to K as F^i.

Corollary 4.9. *F^i is a group under pointwise addition.*

Proof. F^i is defined over pointwise addition, and therefore inherits the group structure from K. ■

Definition 16. We define $d^i : F^i \longrightarrow F^{i+1}$ as:

$$d^i(f)(x_0, \ldots, x_i) = x_0 f(x_1, \ldots x_i) + \sum_{j=1}^{i} (-1)^j f(x_0, \ldots x_{j-1}x_j \ldots x_i) + (-1)^{i+1} f(x_0, \ldots, x_{i-1})$$

Lemma 4.10. $d^{i+1} \circ d^i = 0$

Proof.

$$
\begin{aligned}
&d^{i+1}d^i(f)\\
&= x_0 d^i(f)(x_1 \dots x_{i+1}) + \sum_{j=1}^{i+1}(-1)^j d^i f(x_0 \dots x_{j-1}x_j \dots x_{i+1}) + (-1)^{i+2} d^i f(x_0 \dots x_i)\\
&= x_0(x_1 f(x_2 \dots x_{i+1})) + \sum_{j=2}^{i+1}(-1)^j f(x_1 \dots x_{j-1}x_j \dots x_{i+1}) + (-1)^{i+1} f(x_1 \dots x_i)\\
&\quad + \sum_{k=1}^{i+1}(-1)^k x_0[f(x_1 \dots x_{k-1}x_k \dots x_{i+1}) + \sum_{j=1}^{k-1}(-1)^j f(x_1 \dots x_{j-1}x_j \dots x_{k-1}x_k \dots x_{i+1})\\
&\quad + \sum_{j=k+1}^{i+1}(-1)^j f(x_1 \dots x_{k-1}x_k \dots x_{j-1}x_j \dots x_{i+1}) + (-1)^{i+1} f(x_1 \dots x_{k-1}x_k \dots x_i)]\\
&\quad + (-1)^{i+2}[x_0 f(x_1 \dots x_1) + \sum_{j=1}^{k}(-1)^j f(x_1 \dots x_{j-1}x_j \dots x_{i+1}) + (-1)^{i+1} x_0 f(x_1 \dots x_{i-1})]\\
&= x_0 x_1 f(x_2 \dots x_{i+1}) + \sum_{j=2}^{i+1}(-1)^j f(x_1 \dots x_{j-1}x_j \dots x_{i+1}) + (-1)^{i+1} x_0 f(x_1 \dots x_i)\\
&\quad + \sum_{k=1}^{i+1}\sum_{j=1}^{k-1}(-1)^{k+j} f(x_1 \dots x_{j-1}x_j \dots x_{k-1}x_k \dots x_{i+1})\\
&\quad + \sum_{k=1}^{i+1}\sum_{j=k+1}^{i+1}(-1)^{k+j} f(x_1 \dots x_{k-1}x_k \dots x_{j-1}x_j \dots x_{i+1})\\
&\quad + (-1)^{i+2} x_0 f(x_1 \dots x_i) + \sum_{j=1}^{k}(-1)^{j+i+2} f(x_0 \dots x_{j-1}x_j \dots x_i) + f(x_0 \dots x_{i-1})\\
&= 0
\end{aligned}
$$

■

Corollary 4.11. $\ker(d_2) = Z^2_\theta(H,K)$ *and* $\operatorname{im}(d_1) = B^2_\theta(H,K)$

Theorem 4.12. *There exists a bijection between $H^2_\theta(H,K)$ and the set of equivalence classes, E of extensions which realize the data (H,K,θ).*

Proof. We call the equivalence class of the extension G_f as $[G_f]$. We then define the function $\rho : H^2_\theta(H,K) \longrightarrow E$, such that $\rho(f + B^2_\theta(H,K)) = [G_f]$. First, we prove ρ is well defined. Let $f + B^2_\theta(H,K) = g + B^2(H,K)$. Then, f and g are in the same coset, meaning $G_f = G_g$, meaning $[G_f] = [G_g]$. Now, we prove that ϕ is injective. Let $\rho(f + B^2_\theta(H,K)) = \rho(g + B^2_\theta(H,K))$. Then $[G_f] = [G_g]$. This means that g and f differ by a coboundary, so in the quotient group $f + B^2_\theta(H,K) = g + B^2_\theta(H,K)$, proving injectivity. Next, we prove surjectivity. Let $[G] \in E$, Let f be a factor set for G. Then $[G] = [G_f] = \rho(f + B^2_\theta(H,K))$. This proves that ρ is a bijection, as desired. ■

This finally gives us the description for the set of equivalence classes of the extensions of H by K, which is:

$$\bigcup_{\theta \in \mathrm{Hom}(H,\mathrm{Aut}(K))} H^2_\theta(H,K).$$

5. Further Questions

Although this paper provides a overview of the group extension problem in a few cases, there is much more to the group extension problem than what is covered in this paper. There are many more results in relation to the problem which cover the solution in more generality. Given any two arbitrary groups K and H, we still cannot generate all possible group extensions of H by K. Unlike the well studied and understood subject of field extensions, group extensions are not nearly as well classified. Solving the group extension problem would give us a huge amount of new insight into the classification of groups, and thus continues to be an active area of research.

Acknowledgements

The author would like to thank Dr. Simon Rubinstein-Salzedo and Emma Cardwell for their support and help in making this paper possible.

References

[Ada18] Zachary W Adams. The group extensions problem and its resolution in cohomology for the case of an elementary abelian normal sub-group. Master's thesis, Colorado State University, 2018.

[Cona] Keith Conrad. Semidirect products. *URL: https://kconrad.math.uconn.edu/blurbs/grouptheory/splittinggp.pdf.*

[Conb] Keith Conrad. Splitting of short exact sequences for groups. *URL: https://kconrad. math. uconn. edu/blurbs/grouptheory/splittinggp. pdf.*

[Igu] Kiyoshi Igusa. Factor sets.

[Rot95] Joseph J. Rotman. *An introduction to the theory of groups.*, volume 148 of *Grad. Texts Math.* New York, NY: Springer-Verlag, 4th ed. edition, 1995.

[Rot95] [Conb] [Ada18] [Cona] [Igu]