Thermoacoustic instability in a Rijke tube: Non-normality and nonlinearity

Koushik Balasubramanian and R. I. Sujith

Citation: Physics of Fluids **20**, 044103 (2008); doi: 10.1063/1.2895634
View online: http://dx.doi.org/10.1063/1.2895634
View Table of Contents: http://aip.scitation.org/toc/phf/20/4
Published by the American Institute of Physics

Thermoacoustic instability in a Rijke tube: Non-normality and nonlinearity

Koushik Balasubramanian and R. I. Sujith[a)]
Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai 600036, India

(Received 18 June 2007; accepted 7 January 2008; published online 10 April 2008)

The role of non-normality and nonlinearity in thermoacoustic interaction in a Rijke tube is investigated in this paper. The heat release rate of the heating element is modeled by a modified form of King's law. This fluctuating heat release from the heating element is treated as a compact source in the one-dimensional linear model of the acoustic field. The temporal evolution of the acoustic perturbations is studied using the Galerkin technique. It is shown that any thermoacoustic system is non-normal. Non-normality can cause algebraic growth of oscillations for a short time even though the eigenvectors of the system could be decaying exponentially with time. This feature of non-normality combined with the effect of nonlinearity causes the occurrence of triggering, i.e., the thermoacoustic oscillations decay for some initial conditions whereas they grow for some other initial conditions. If a system is non-normal, then there can be large amplification of oscillations even if the excited frequency is far from the natural frequency of the system. The dependence of transient growth on time lag and heater positions is studied. Such amplifications (pseudoresonance) can be studied using pseudospectra, as eigenvalues alone are not sufficient to predict the behavior of the system. The geometry of pseudospectra can be used to obtain upper and lower bounds on the growth factor, which provide both necessary and sufficient conditions for the stability of a thermoacoustic system. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2895634]

I. INTRODUCTION

The occurrence of combustion instabilities has been a plaguing problem in the development of combustors for rockets, jet engines, and power generating gas turbines.[1] Predicting and controlling combustion instability requires an understanding of the combustion-acoustic interactions.

Combustion instabilities are self-sustained large amplitude oscillations of pressure and velocity in combustors with the flame acting as an acoustic actuator and the combustion chamber as an acoustic resonator. If the heat released in the system depends on the pressure and velocity fluctuations in the system, a feedback loop that can destabilize the system is established. The occurrence of combustion instability depends on the phase between the heat release fluctuations and the pressure fluctuations at the flame.[2] Amplification of the pressure oscillations by the heat addition process will take place if the maximum and minimum of the heat addition occur during the compression and rarefaction phases of the pressure oscillation, respectively. In contrast, the pressure oscillations will be attenuated if the maximum and minimum of the heat addition occur during the rarefaction and compression phases of the pressure oscillation, respectively.

When pulsations start spontaneously, as they do in a Rijke tube with a heater at $\frac{1}{4}$ length from the bottom of the tube, the system is said to be linearly unstable, i.e., the system is unstable with respect to any small amplitude disturbance that may be present in the combustor. This scenario has been successfully modeled in various systems using classical linear stability analysis of the normal modes that model the system as a network model, in which each element is modeled using a linear transfer function. The stability of the system can then be determined easily by examining the eigenvalues of the system.

It is also possible that a linearly stable combustor (i.e., one that does not pulse spontaneously) could be "triggered" into pulsating operation by the introduction of a finite amplitude disturbance such as might be caused by a spark plug ignition or a small explosion. Such a system will be stable with respect to all disturbances whose amplitudes are below a certain threshold value, but transition into pulsating operation will occur when the amplitude of the disturbance exceeds this threshold value. Furthermore, there are instances of "bootstrapping" where a mode that decays initially can grow later and ultimately become unstable.[3] A linear stability analysis using normal modes would always indicate stability in the case of bootstrapping and miss the ultimate behavior. A comprehensive prediction of the conditions for the onset of instabilities is a difficult task, which is not yet mastered. In particular, predicting the conditions under which finite amplitude disturbances destabilize a linearly stable system and predicting the limit-cycle amplitude of the instability remain a key challenge, as little is known, even in a qualitative sense, about the key parameters controlling nonlinear flame dynamics even in simple laminar flames.[4] The pressing need to control this phenomenon in combustion chambers compounded by lack of understanding of combustion instability due to the complex nature of combustion-acoustic interaction in flames led to interest in simpler thermoacoustic systems such as Rijke tubes. The Rijke tube serves as a convenient prototypical system for the study of thermoacoustic phenomena.

Nonlinear effects in a Rijke tube were investigated in

[a)]Author to whom correspondence should be addressed. Electronic mail: sujith@iitm.ac.in.

detail by several authors. A Rijke tube is a relatively simple system: it is a duct with a heat source (often electrically heated wires) at quarter length from the bottom (if located vertically). Heckl[5] studied the nonlinear acoustic effects leading to limit cycles in unstable oscillations experimentally and theoretically for the case of a Rijke tube. She showed that the important nonlinear effects are (1) the reduction of the rate of heat transfer when the velocity amplitudes are of the same order as the mean velocity and (2) increased losses at the ends of the tube at very high amplitudes. Hantschk and Vortmeyer[6] showed that the limit-cycle amplitude in a Rijke tube is determined by nonlinearities in the heat flux from the heating element to the flow. Yoon *et al.*[3] proposed a nonlinear response model of a generalized Rijke tube. Their oscillatory heat release model was not derived from physical principles. They derived both closed form and numerical solutions for the acoustic field by an approximate modal analysis using a two-mode formulation. The two-mode nonlinear model is capable of predicting the bootstrapping effect which characterizes nonlinear velocity sensitive combustion response in rocket motors. However, they neglected nonlinear convection in their model. In order to explain the nonlinear effects of a Rijke tube, Matveev[7,8] constructed a simple theory using an energy approach. The equilibrium states of the system are found by balancing thermoacoustic energy input and acoustic losses. It is again confirmed that the nonlinearity of the unsteady heat transfer is a dominant factor in limit-cycle saturation. Further, he demonstrated the necessity of accurately modeling the effects of temperature gradient on the mode shapes to obtain accurate results for stability.[9]

Although considerable research has been performed on the nonlinear nature of thermoacoustic oscillations, their non-normal nature is an aspect that has not received any attention. Non-normality can lead to transient growth of a system even when the eigenvalues indicate linear stability. When the transient growth leads to high enough amplitudes, it can trigger nonlinearities in the system which can cause “nonlinear driving,” thereby driving a system which was thought to be linearly stable under the framework of classical linear stability. Further, non-normality leads to coupling between the modes, although this effect has been often attributed to nonlinearity. The role of non-normality in the context of thermoacoustic oscillations has been shown in the context of ducted diffusion flames by Balasubramanian and Sujith[10,11] using a Galerkin-type analysis. Nicoud *et al.*[12] have shown that the eigenvectors of a thermoacoustic system are nonorthogonal in the presence of heat release or in the presence of general complex impedance boundary conditions. Thus, there are two distinct aspects of combustion-acoustic interactions: non-normality of the modes and nonlinearities in the system.

Non-normality has been found to play an important role in several fields such as turbulence,[13–15] instability of magnetic plasmas,[16] formation of cyclones,[17] etc. Recently, the authors have shown that the combined role of non-normality and nonlinearity can lead to the triggering of thermoacoustic oscillations in a linearly stable system in the context of ducted diffusion flames.[10,11] This is a general feature of all thermoacoustic systems and need not be restricted to diffusion flames. In the case of ducted diffusion flames, the combustion modes themselves are non-normal.[10,11]

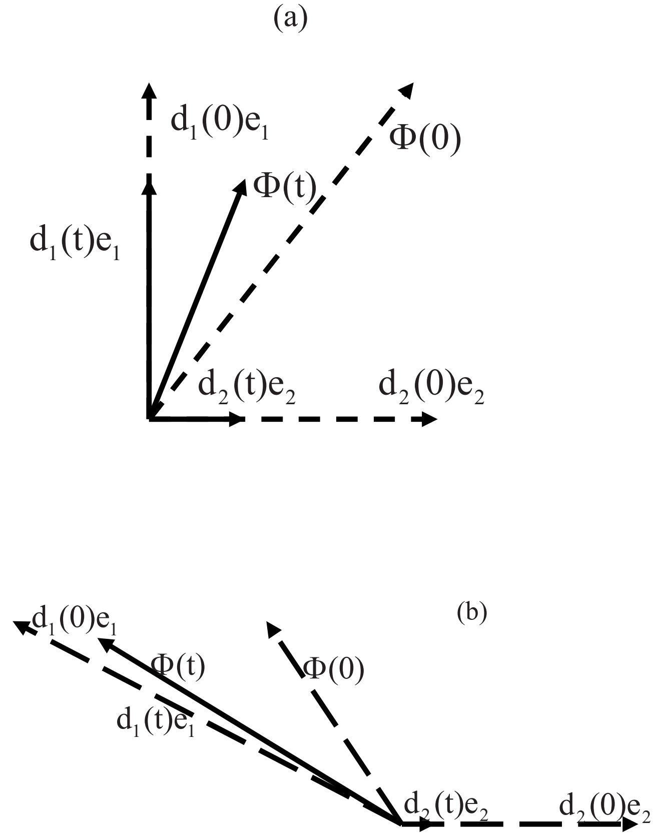

FIG. 1. (a) Monotonic decay of a normal system; (b) transient growth of a non-normal system. The initial state is $\Phi(0)=d_1(0)e_1+d_2(0)e_2$, and the final state is $\Phi(t)=d_1(t)e_1+d_2(t)e_2$. The dashed lines denote the vectors at time $t=0$ and the solid lines denote the vector at some time t.

The objective of this paper is to examine the role of non-normality in a simpler model for a thermoacoustic system, constructed for a Rijke tube. The rest of this paper is organized as follows. In Sec. II, the characteristics of a non-normal operator and its consequences are discussed in the context of thermoacoustic instabilities. The schematic of a horizontal Rijke tube considered in this study is described in Sec. III. The equations governing the evolution of the acoustic field in this horizontal Rijke tube are derived in Sec. IV. In Sec. V, examples highlighting the effects of non-normality and nonlinearity are discussed. The results and inferences are summarized in Sec. VI.

II. NON-NORMALITY AND TRANSIENT GROWTH OF THERMOACOUSTIC OSCILLATIONS

A system is said to be non-normal if its eigenvectors are not orthogonal. The evolution of such a system is governed by an operator which does not commute with its adjoint. The nonorthogonal nature of eigenvectors can lead to transient growth of oscillations before they eventually decay. This property is illustrated in Figs. 1(a) and 1(b). The vectors e_1 and e_2 represent the direction of the eigenvectors and Φ is a vector which is expressed as a linear combination of the eigenvectors. Figure 1(a) shows that for a normal system, Φ decreases monotonically if the amplitudes of the individual

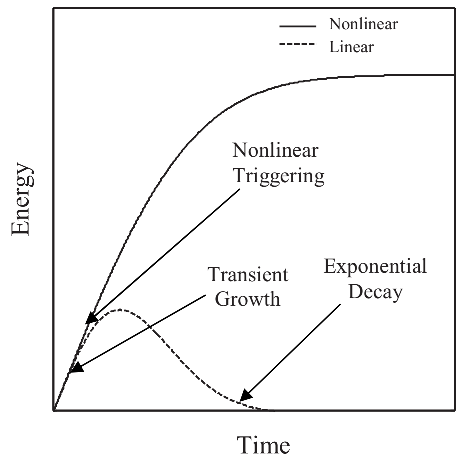

FIG. 2. Schematic of linear and nonlinear evolutions of the acoustic energy $[\frac{1}{2}p'^2+\frac{1}{2}(\gamma M u')^2]$.

eigenvectors themselves decay. On the contrary, for the non-normal system shown in Fig. 1(b), Φ increases even when the amplitudes of individual eigenvectors decay. However, Φ decays after a sufficiently long time if nonlinear effects do not become significant during the transient growth. There could be situations where the short-term growth of fluctuations can lead to significant amplitudes where nonlinear effects could cause nonlinear driving, as illustrated in Fig. 2. Such a scenario arises in the evolution of thermoacoustic oscillations.

The acoustic equations (the assumptions in deriving these equations and the details of nondimensionalization are given in Sec. IV) in the presence of a heat source can be written as

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{1}$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1)\gamma \frac{L_a}{c_0} \frac{\dot{\tilde{Q}}'}{\rho_0 c_0^2}. \tag{2}$$

The heat release rate in the above equation is calculated from a model for the heat source. The linearized oscillatory heat release rate can be written as $\dot{Q}'=R(x,\varepsilon_i)\gamma M u'+S(x,\mu_i)p'$, where R and S can be treated as continuous functions of x (which could even be sharply peaked at the flame location as in the case of a compact heat source), and ε_i and μ_i are parameters which affect heat release rate. The heat release rate could have an explicit dependence on time as well. Equations (1) and (2) can be recast in the matrix form as

$$\begin{bmatrix} \dfrac{\partial}{\partial t} & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} - \dfrac{R L_a(\gamma-1)}{\rho_0 c_0^3} & \dfrac{\partial}{\partial t} - \dfrac{S L_a \gamma(\gamma-1)}{\rho_0 c_0^3} \end{bmatrix} \begin{bmatrix} \gamma M u' \\ p' \end{bmatrix} = 0. \tag{3}$$

The matrix in Eq. (3) is a matrix of operators. The above operator does not commute with its adjoint for nonzero R and S. (The adjoint of a real matrix is simply the transpose of the matrix. The adjoint of a first derivative operator is its negative.) Therefore, it is clear that the thermoacoustic interaction is non-normal. In the absence of heat release, the matrix is symmetric and hence normal.

The non-normal behavior of the system causes large transient growth which can potentially trigger nonlinearities in the system. Classical linear stability analysis then becomes a poor indicator of system stability due to this algebraic growth.[14] This phenomenon has been studied in detail in the context of turbulence by Baggett *et al.*[13] They explained that in the non-normal evolution, the input and output structures (such as streamwise vortices, streaks, etc.) are different and nonlinearity closes the feedback loop by converting some of the output into input. In a similar manner, the interplay between transient linear growth resulting from non-normality and "nonlinear mixing" can indeed lead to the growth of the thermoacoustic oscillations.

Equation (3) is analyzed by reducing the infinite dimensional operator to a finite dimensional operator, such as a finite dimensional matrix. In this paper, this is achieved by reducing the partial differential equations governing the thermoacoustic interaction to a set of ordinary differential equations (ODEs) using the Galerkin technique. ODEs in time domain are obtained by decomposing the spatial variation using basis functions. This is similar to decomposing a vector along some basis. The basis functions used in this study are not the eigenmodes of the linearized system, but they are the eigenmodes of the self-adjoint part of the linearized system. Such an approach has been used in solving partial differential equations; see, e.g., Henningson and Schmid.[18] These evolution equations are solved numerically using the fourth order Runge–Kutta scheme. The complete evolution equations are linearized and the linearized equations are found to be non-normal. It must be emphasized that the eigenvalues of the linearized equations are not the wave numbers of the basis functions used in the Galerkin technique.

III. THE HORIZONTAL RIJKE TUBE

A horizontal Rijke tube with an electric heat source is a system convenient for studying the fundamental principles of thermoacoustic instabilities both experimentally and theoretically. In such a setup, the mean flow is provided by a blower, which sucks air in the tube. This enables us to control the heater power and the mean flow independently. If the tube were oriented vertically, as in the classical Rijke tube, the effect of mean flow component caused by natural convection will have to be accounted for in the stability analysis. The horizontal orientation of the Rijke tube is implemented to exclude the influence of natural convention on the mean flow rate. Such a setup has been used by Matveev,[7,8] Heckl,[5] and Kopitz and Polifke.[19] A schematic of the horizontal Rijke tube setup is shown in Fig. 3. The current study models such a setup.

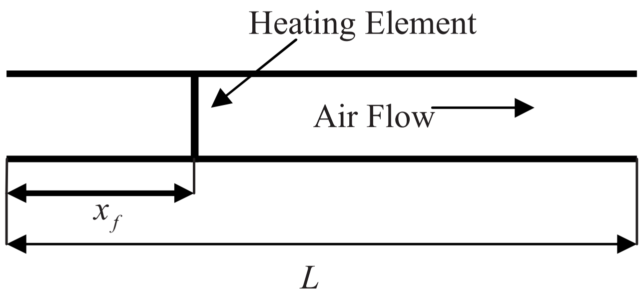

FIG. 3. Schematic of the horizontal Rijke tube setup.

IV. GOVERNING EQUATIONS

Neglecting the effect of mean flow and mean temperature gradient in the duct, the governing equations for the one-dimensional acoustic field are

$$\bar{\rho}\frac{\partial \tilde{u}'}{\partial \tilde{t}} + \frac{\partial \tilde{p}'}{\partial \tilde{x}} = 0 \quad \text{(acoustic momentum)}, \tag{4}$$

$$\frac{\partial \tilde{p}'}{\partial \tilde{t}} + \gamma \bar{p}\frac{\partial \tilde{u}'}{\partial \tilde{x}} = (\gamma - 1)\dot{\tilde{Q}}' \quad \text{(acoustic energy)}. \tag{5}$$

A modified form of King's law is used to model the heat release rate. Since King's law exhibits nonlinearity only for velocity perturbations greater than the mean fluctuations, Heckl[5] suggested the following empirical model:

$$\dot{\tilde{Q}}' = \frac{2L_w(T_w - \bar{T})}{S\sqrt{3}}\sqrt{\pi\lambda C_v\bar{\rho}\frac{d_w}{2}} \times \left[\sqrt{\left|\frac{u_0}{3} + u_f'(t-\tau)\right|} - \sqrt{\frac{u_0}{3}}\right]\delta(\tilde{x} - \tilde{x}_f). \tag{6}$$

In the above expression, L_w is the equivalent length of the wire, λ is the heat conductivity of air, C_v is the specific heat of air at constant volume, τ is the time lag, $\bar{\rho}$ is the mean density of air, d_w is the diameter of the wire, $(T_w - \bar{T})$ is the temperature difference, and S is the cross-sectional area of the duct. The above equations can be nondimensionalized as follows:

$$\tilde{x} = L_a x, \quad \tilde{t} = \frac{L_a}{c_0}t, \quad \tilde{u}' = u_0 u', \quad \tilde{p}' = \bar{p}p', \quad M = \frac{u_0}{c_0},$$

where c_0 is the speed of sound, L_a is the duct length, $\bar{p}$ is the pressure of the undisturbed medium, and u_0 is the mean flow velocity. The acoustic equations in the nondimensional form can be written as follows:

$$\gamma M\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{7}$$

$$\frac{\partial p'}{\partial t} + \gamma M\frac{\partial u'}{\partial x} = k\left[\sqrt{\left|\frac{1}{3} + u'(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f), \tag{8}$$

where

$$k = (\gamma - 1)\frac{2L_w(T_w - \bar{T})}{Sc_0\bar{p}\sqrt{3}}\sqrt{\pi\lambda C_v\bar{\rho}\frac{d_w}{2}u_0}. \tag{9}$$

The above set of partial differential equation can be reduced to ODEs using the Galerkin technique.[20] The velocity and pressure field can be written in terms of the duct's natural modes as follows:[21]

$$u' = \sum_{j=1}^{\infty}\eta_j\cos(j\pi x) \quad \text{and } p' = -\sum_{j=1}^{\infty}\frac{\gamma M}{j\pi}\dot{\eta}_j\sin(j\pi x). \tag{10}$$

The Galerkin technique makes use of the fact that any function in a domain can be expressed as a superposition of expansion functions which form a complete basis in that domain. The basis functions are chosen such that they satisfy the boundary conditions. However, the choice of the basis functions is not unique. The basis functions chosen here are just an arbitrary basis and not the eigenfunctions of the system. They are the eigenfunctions of the self-adjoint part of the linearized system. Clearly, the expansion functions chosen here satisfy the boundary conditions and they form a complete basis.

Substituting the above expansions into Eqs. (7) and (8) and projecting along the basis functions [the component or the projection of a function f along a basis function ψ_n is given by their inner product $\langle f|\psi_n\rangle$ which is defined as $\int_{\text{domain}} f(x)\psi_n(x)dx$], the following evolution equations are obtained:

$$\frac{d\eta_j}{dt} = \dot{\eta}_j, \tag{11}$$

$$\frac{d\dot{\eta}_j}{dt} + k_j^2\eta_j = -\frac{2k}{\gamma M}j\pi\left[\sqrt{\left|\frac{1}{3} + u_f'(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f). \tag{12}$$

Equations (11) and (12) can be expanded to second order for low amplitudes to yield the following matrix differential equation:

$$\frac{d\chi}{dt} + B_{\text{NL}}(\chi^T,\chi) + B_{\text{NN}}\chi = 0, \tag{13}$$

where $\chi = [\eta_1\ \dot{\eta}_1/\pi\ \eta_2\ \dot{\eta}_2/2\pi \cdots \eta_N\ \dot{\eta}_N/N\pi]^T$,

$$B_{\mathrm{NN}} = \begin{bmatrix} 0 & -\omega_1 & 0 & 0 & \cdots & 0 & 0 \\ \omega_1+\beta_1\cos(\pi x_f) & -\tau\beta_1\cos(\pi x_f) & \beta_1\cos(2\pi x_f) & -\tau\beta_1\cos(2\pi x_f) & \cdots & \beta_1\cos N\pi x_f & -\tau\beta_1\cos(N\pi x_f) \\ 0 & 0 & 0 & -\omega_2 & \cdots & 0 & 0 \\ \beta_2\cos(\pi x_f) & -\tau\beta_2\cos(\pi x_f) & \omega_2+\beta_2\cos(2\pi x_f) & -\tau\beta_2\cos(2\pi x_f) & \cdots & \beta_2\cos N\pi x_f & -\tau\beta_2\cos(N\pi x_f) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -\omega_1 \\ \beta_N\cos(\pi x_f) & -\tau\beta_N\cos(\pi x_f) & \beta_N\cos(2\pi x_f) & -\tau\beta_N\cos(2\pi x_f) & \cdots & \omega_N+\beta_N\cos(N\pi x_f) & -\tau\beta_N\cos(N\pi x_f) \end{bmatrix},$$

$$B_{\mathrm{NL}} = u'(t-\tau)\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_1\cos(\pi x_f) & -\alpha_1\tau\cos(\pi x_f) & \alpha_1\cos(2\pi x_f) & -\alpha_1\tau\cos(2\pi x_f) & \cdots & \alpha_1\cos(N\pi x_f) & -\alpha_1\tau\cos(2\pi x_f) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2\cos(\pi x_f) & -\alpha_2\tau\cos(\pi x_f) & \alpha_2\cos(2\pi x_f) & -\alpha_2\tau\cos(2\pi x_f) & \cdots & \alpha_2\cos(N\pi x_f) & -\alpha_2\tau\cos(N\pi x_f) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \alpha_N\cos(\pi x_f) & -\alpha_N\tau\cos(\pi x_f) & \alpha_N\cos(2\pi x_f) & -\alpha_N\tau\cos(2\pi x_f) & \cdots & \alpha_N\cos(N\pi x_f) & -\alpha_N\tau\cos(N\pi x_f) \end{bmatrix},$$

where $\beta_j=(k/\gamma M)\sqrt{3}\sin(j\pi x_f)$ and $\alpha_j=-3\beta_j/4$.

It can be seen that B_{NN} does not commute with its adjoint, and hence it is non-normal. In the above equation, B_{NL} is the nonlinear matrix. In this paper, the role of damping is not considered as the authors would like to emphasize the role of non-normality. The role of damping in Rijke tube can be studied using the model suggested by Matveev.[8] The combined role of damping and non-normality in thermoacoustic systems was studied by the authors in the context of diffusion flames.[10,11] Various cases highlighting the non-normal and nonlinear nature of thermoacoustic oscillations and their consequences are discussed using examples in the following section.

V. RESULTS AND DISCUSSIONS

The duct acoustic modes are normal in the absence of heat addition. However, heat addition makes the system non-normal, as can be seen from the following physical arguments. The heat transfer is a function of the velocity, as given by Eq. (6). A small disturbance in the velocity can alter the heat release rate which in turn acts as the source of disturbance for the acoustic oscillations.

The acoustic pressure and the fluctuating heat release rate at the heater location can be expressed as

$$p'(x_f,t) = \sum_{k=1}^{\infty} P_k \psi_k(x_f)\exp(i\lambda_k t), \quad q' = \sum_{k=1}^{\infty} Q_k \exp(i\lambda'_k t),$$

where ψ_k is an eigenmode, and P_k and Q_k are some coefficients. The heat release drives the acoustic oscillations in the duct when it is in phase with acoustic pressure. The phase between heat release rate and pressure oscillations is then given by their correlation, i.e.,[10,11]

$$\phi(t) = \phi(0) + \int_0^t p'(x_f,t)q'(t)dt \Bigg/ \sqrt{\int_0^t p'^2(x_f,t)dt\int_0^t q'^2(t)dt}.$$

For a general system, the phase between heat release rate and pressure evolves with time. Hence, the phase difference between the heat release fluctuations and acoustic pressure oscillations at the heater at some instant depends on their phase difference at an earlier instant. The acoustic modes that are driven by the heat release oscillations at these two instants are different. Since the phase difference between the heat release oscillations and a particular mode of the acoustic field depends on the phase difference at an earlier time, the interaction would depend on that mode of the acoustic field which was in phase with the thermal process at an earlier time. Hence the mode which is driven at a particular instant of time depends on which mode got driven at an earlier time. Hence, the energy in a mode at some instant will depend on

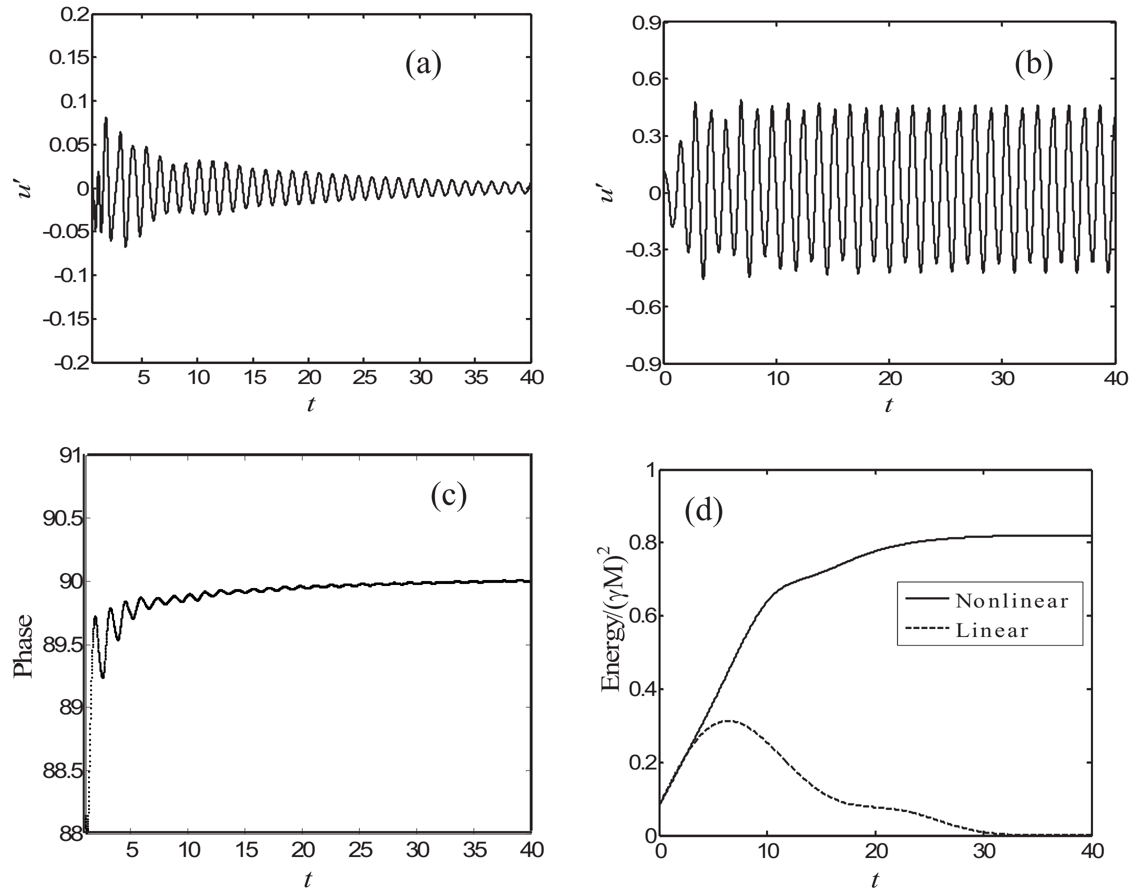

FIG. 4. Evolution of nondimensional acoustic velocity when the initial conditions are (a) $\eta_1(0)=0.15$ and (b) $\eta_1(0)=0.2$, $x_f=0.29$, $c_0=399.6$ m/s, $L_w=3.6$ m, $\bar{u}=0.5$ m/s, $\bar{\rho}=1.205$ kg/m^3, $T_w=1000$ K, $C_v=719$ J/kg K, and $\lambda=0.0328$ W/m K. (c) Evolution of phase and (d) linear and nonlinear evolutions of the acoustic energy when the initial condition is $\eta_1(0)=0.2$.

the energy of another mode which was driven at an earlier instant of time. This indicates that there is a nonzero projection of a mode along another mode, indicating non-normal behavior.

The role played by non-normality on the thermoacoustic interaction can be studied through the following examples. The numerical simulations were performed by keeping some of the parameters fixed and varying others. The parameters that were kept fixed are $\lambda=0.0328$ W/m K, $C_v=719$ J/kg K, and $\bar{\rho}=1.205$ kg/m^3. The set of first order ODEs (11) and (12) was integrated numerically using the fourth order adaptive Runge–Kutta technique. The numerical simulations were performed with ten acoustic modes so that the change in the solution with increase in number of modes is less than 5%.

A. Triggering

If the system is non-normal as well as nonlinear, oscillations can grow even when the individual eigenvalues indicate linear stability. For such systems, there exists some initial condition for which the oscillations decay and some other initial conditions for which they grow. This feature is captured by a heater located at $x_f=0.29$, $\bar{u}=0.5$ m/s, $c_0=399.6$ m/s, and $L_w=3.6$ m. Figures 4(a) and 4(b) show triggering in the absence of damping. Figure 4(a) shows that for initial conditions of $\eta_1(0)=0.15$ and $\eta_{i\neq 1}(0)=0$, the oscillations decay, whereas Fig. 4(b) shows that the oscillations grow for a different initial condition, i.e., for $\eta_1=0.2$ and $\eta_{i\neq 1}(0)=0$. Further, Fig. 4(b) shows that the amplitude of the oscillations saturates. Hence, this example shows that saturation can occur in the absence of damping. This interesting feature was discussed by Balasubramanian and Sujith[10,11] in the context of diffusion flames. The authors have explained that saturation can occur in the absence of damping if the phase difference (the phase between pressure and heat release rate is the correlation between the acoustic pressure and heat release rate.[10,11]) between the acoustic oscillations and the heat release oscillations evolves to 90°.[22] This is confirmed from the evolution of the phase to 90° as the system approaches limit cycle, as seen in Fig. 4(c). The evolution of the nondimensional acoustic energy $[\frac{1}{2}p'^2+\frac{1}{2}(\gamma M u')^2]$ after locally weighted regression smoothing calculated from both linear [integration of Eqs. (11) and (12) without nonlinear terms] and nonlinear simulations (with nonlinear terms included) are presented in Fig. 4(d). The linear simulation shows that the acoustic energy grows initially and eventually decays. The nonlinear simulation is almost identical to the linear simulation initially. After sufficient transient growth, the nonlinearity "picks up," which can be seen from the de-

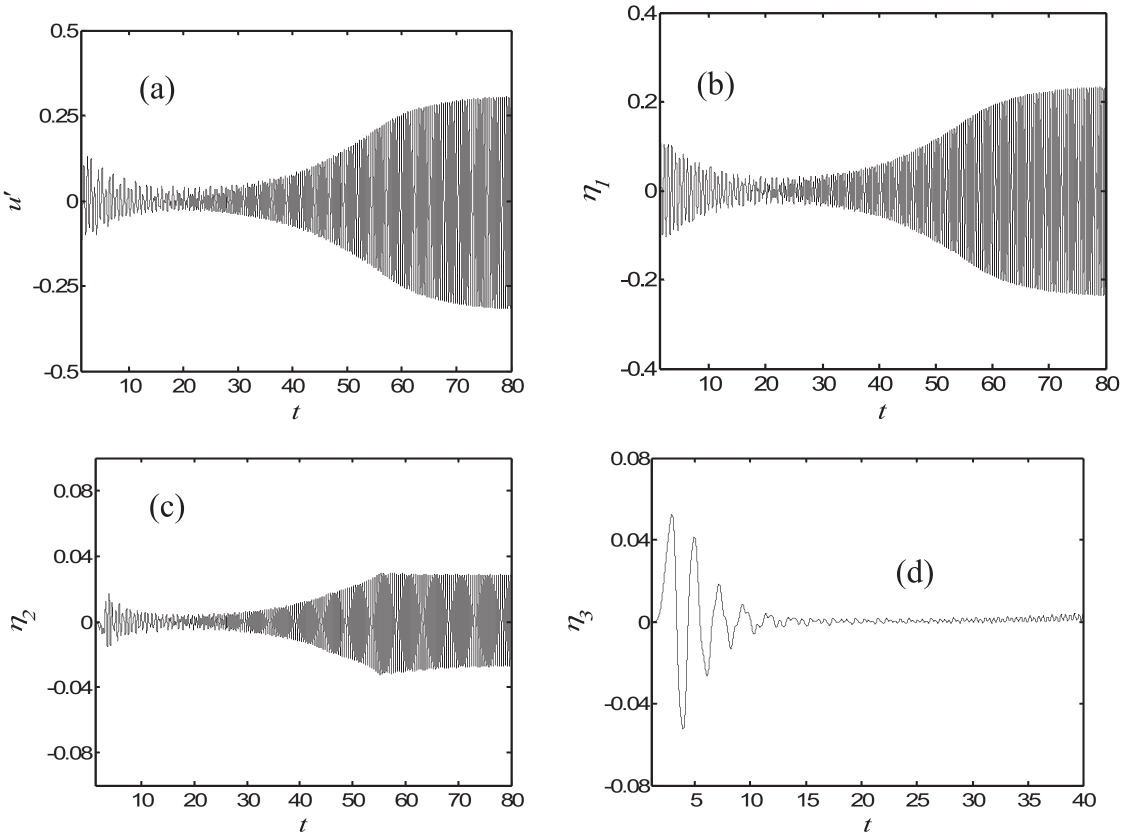

FIG. 5. (a) Nondimensional evolution of acoustic velocity when the initial condition is $\eta_1(0)=0.18$; [(b)–(d)] evolution of the acoustic velocity projected onto the various Galerkin modes. $L_w=3.6$ m, $\lambda=0.0328$ W/m K, $C_v=719$ J/kg K, $\bar{\rho}=1.205$ kg/m^3, $\bar{u}=0.5$ m/s, $T_w=1000$ K, $\tau=0.45\pi$, $x_f=0.29$, and $c_0=399.6$ m/s.

viation of the nonlinear evolution from the linear evolution. The nonlinear simulation also shows that, eventually, the acoustic energy grows and saturates.

The phenomenon of triggering has been observed in experiments with Rijke tube[9] and also in other thermoacoustic devices such as solid rocket motors.[23,24] This is usually attributed to nonlinearities. However, as shown in this example, this is not the complete picture. This example shows that the non-normal nature of the evolution equations is indeed responsible for raising the amplitudes to a level where the nonlinearity in the system is triggered.

Our view on non-normality does not diminish the importance of nonlinear effects. We are highlighting the role played by the linear (non-normal) evolution in increasing the amplitude to a level where nonlinearity becomes significant. The nonlinear terms in the equation become significant beyond a certain value of the amplitude (say, A_n). Non-normality causes an amplitude lesser than A_n to increase up to A_n. Hence, the threshold value beyond which triggering occurs is less than A_n.

B. Growth of oscillations in an initially decaying system

This section discusses bootstrapping in a thermoacoustic system which is stable according to classical linear stability analysis based on eigenvalues. In this example, the heater is located at $\frac{1}{4}$ duct length. The initial conditions chosen are $\eta_1(0)=0.18$, $\eta_{i\neq 1}(0)=0$, and $\dot{\eta}_i(0)=0$. L_w and T_w were chosen as 3.6 m and 1000 K, respectively. Other parameters are maintained to be the same as those in the previous example.

Figure 5(a) shows the evolution of acoustic velocity at the heater location. It can be seen that low frequency oscillations that are initially present in the system decay and high frequency oscillations set in after some time. Further, it can be seen that the oscillations eventually saturate after nonlinear growth. It must be emphasized that classical linear stability analysis based on the eigenvalues shows all eigenmodes of this coupled system to be stable.

Figures 5(b)–5(d) show the evolution of the acoustic velocity projected on the first three Galerkin expansion functions. It can be seen that while the acoustic velocity projected to the first expansion function decays, the projections on the second and third expansion functions grow. After sufficient energy is projected onto the second and third expansion functions, they project the energy back, causing the energy projected on the first expansion function to grow. This bootstrapping results in a shift in frequency during the evolution. The net effect of all these energy transfer causes the acoustic velocity to grow and eventually saturate. This feature has been discussed in the context of turbulence and it is known as bootstrapping.[14,15] Yoon *et al.*[3] have discussed bootstrapping in the context of Rijke tube using an *ad hoc* nonlinear model for the heat release rate.

The above simulation was repeated in the absence of

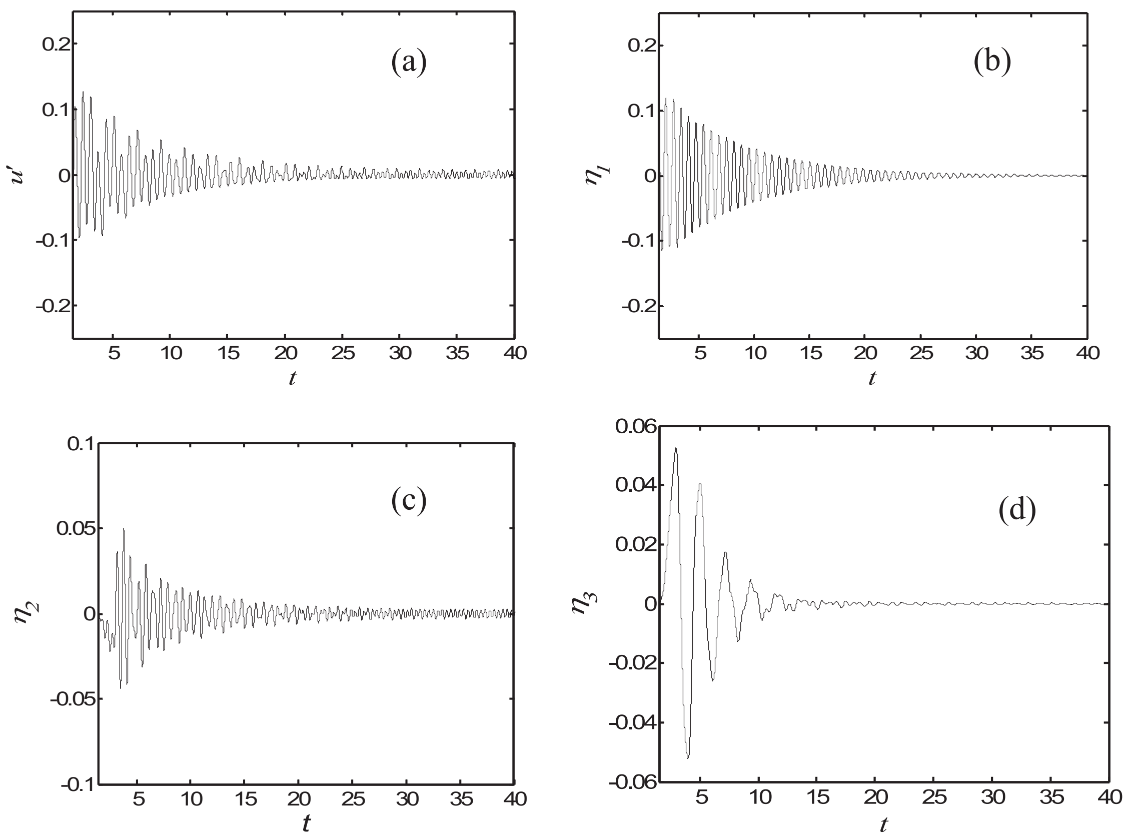

FIG. 6. (a) Linear evolution of acoustic velocity when the initial condition is $\eta_1(0)=0.18$; [(b)–(d)] evolution of the acoustic velocity projected onto the various Galerkin modes. The conditions are the same as those of Fig. 5.

nonlinear terms. Figure 6(a) shows the evolution of acoustic velocity obtained from linear simulations. Figures 6(b)–6(d) clearly shows energy exchange between the modes even in the absence of nonlinear terms. The occurrence of energy exchange between modes can be explained as follows.

There are two mechanisms by which energy can get redistributed in the system. The first mechanism is a nonlinear mechanism where two individual eigenmodes interact directly, causing exchange of energy between these two modes. Another mechanism which causes redistribution of energy is the interaction of various modes with the base flow. At low amplitudes when the nonlinear effects are not significant, the redistribution of energy mainly occurs due to the interaction of various modes with the base flow. When the disturbance is caused by exciting only one eigenmode, it is expected that the oscillations will decay if the system is linearly stable. However, if the system is non-normal, then the oscillations can grow if there is a small amount of energy in the unexcited modes. This could be due to noise in the system or due to a mild nonlinearity. The authors would like to emphasize that the mild nonlinearity just transfers some energy from the excited modes to other modes. Though the individual modes can decay, there can be an overall growth of amplitude due to non-normality. This overall growth can cause nonlinear effects to become significant and more energy can get exchanged between the various modes. Such a situation cannot occur in a linearly stable normal system (classical linear stability) if the nonlinearity is initially mild. This is because the amplitudes of a normal system decay if the individual eigenmodes decay, and hence nonlinearity becomes milder and milder. This explains how non-normality plays an important role in the exchange of energy between various eigenmodes.

C. Transient growth

As discussed earlier, the non-normal nature of a system can cause transient growth of oscillations which can trigger nonlinearities in the system. In this section, a method to analyze transient growth is discussed. Schmid and Henningson[25] gave a detailed discussion on the analysis of transient growth in the context of transition to turbulence in shear flows. They analyzed the stability of shear flows by studying the energy growth of the system. This analysis is general and can be applied to thermoacoustic systems as well. Balasubramanian and Sujith[10] performed such an analysis in the context of diffusion flames. The evolution of the acoustic oscillations in a Rijke tube is also non-normal as $B_{NN}B_{NN}^{\dagger} \neq B_{NN}^{\dagger}B_{NN}$. The symbol $\dagger$ denotes the adjoint (conjugate transpose) of an operator. The solution of the linearized system of evolution equations [Eq. (13) without the nonlinear term] can be written in the operator form as[25]

$$\tilde{\eta}(t) = \exp(Lt)\,\tilde{\eta}(0) = S^{-1}\exp(L_D t)S\,\tilde{\eta}(0), \qquad (14)$$

where $L=-B_{NN}$ is the stability operator, S is the similarity transformation that diagonalizes L, and L_D is the diagonal form of L. Since L is non-normal, S is nonunitary, indicating

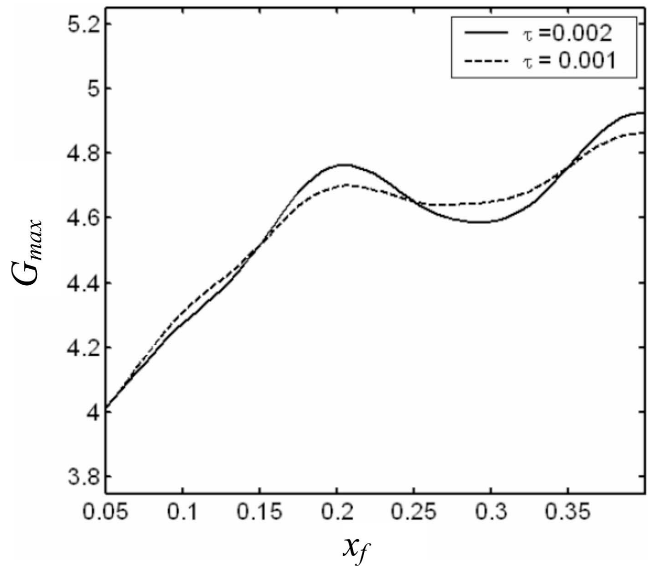

FIG. 7. Variation of maximum growth factor with heating element location for different values of time lag. $L_w=2.0$ m, $\lambda=0.0328$ W/m K, $C_v=719$ J/kg K, $\bar{\rho}=1.205$ kg/m^3, $c_0=401.2$ m/s, $\bar{u}=0.5$ m/s, and $T_w=1000$ K.

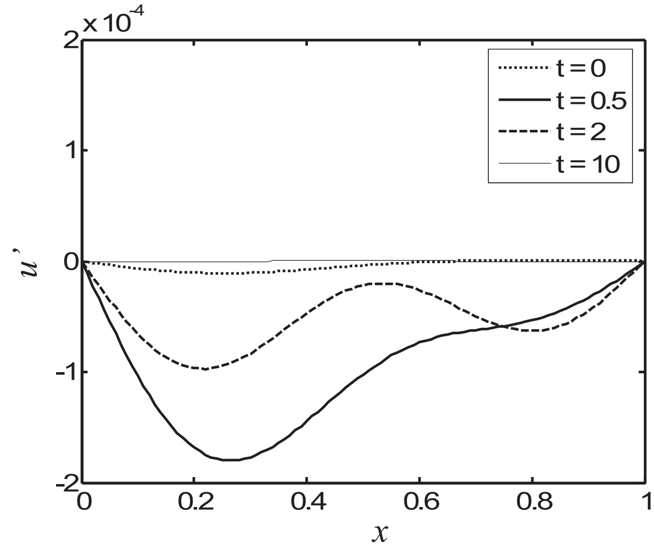

FIG. 8. Spatial variation of the acoustic velocity at various instants of time. The initial condition is chosen to be the optimum initial condition which yields the maximum growth factor. $L_w=2.0$ m, $\tau=0.1\pi$, $\lambda=0.0328$ W/m K, $C_v=719$ J/kg K, $\bar{\rho}=1.205$ kg/m^3, $c_0=401.2$ m/s, $\bar{u}=0.6$ m/s, and $T_w=900$ K.

that it is not possible to make the eigenvectors perpendicular by a simple rotation. Transient growth is quantified by maximum growth factor which is defined as[25]

$$G(t) = \max_{\eta}(\|\tilde{\eta}(t)\|^2/\|\tilde{\eta}(0)\|^2) = \|\exp(Lt)\|^2, \tag{15}$$

where "max" indicates that the ratio of the norms is maximized over all initial conditions. Growth factor is a measure of maximum amplification of energy density at an instant of time. The expression in Eq. (15) is maximized for various instants of times over all possible initial conditions. The maximum growth factor and the optimum initial condition were computed using singular value decomposition. The stability of a system can be studied using the maximum growth factor in a particular time interval $[0,t]$ which is defined as $G_{\max}=\max_t G(t)$. This maximum is obtained after smoothening $G(t)$.

The above maximum value is infinite if L has an eigenvalue with a positive real part. This corresponds to a linearly unstable system. In the present paper, $G_{\max}$ values for various parameters such as time lag and flame location are calculated and the regions with large transient growth are identified. Transient growth cannot occur for those parameters which have $G_{\max}=1$. When $G_{\max}=1$, the energy at any instant is less than the initial energy of the system, indicating that the energy of the system decays. When $G_{\max}>1$, the system will exhibit transient growth. Hence, a system which is linear initially will behave like a linear system throughout its evolution as there is no amplification of the oscillations to trigger nonlinear effects when $G_{\max}=1$. This fact is used in the next section to obtain necessary and sufficiency conditions for the stability of the system.

Figure 7 shows the variation of the maximum growth factor with heater location for different values of time lags. The parameters are given in the figure caption. The growth factor is a nonmonotonic function of the heater location. The dependence of growth factor with x_f is oscillatory in nature. Further, the growth factor is of the order of 10. The authors would like to point out that in the present analysis, a simple model for the heating element is used. This simplified approach was used to focus on the non-normal nature of the acoustic equations alone in the presence of heat source. However, the energy released at the heating element is governed by an advection-diffusion equation. It has been shown that the advection-diffusion equation is non-normal.[26,27] This may cause the growth factor to be much larger as the number of eigenmodes of the coupled system will be much larger.

Figure 8 shows the spatial variation of acoustic velocity in the Rijke tube at various instants of times. The figure describes how acoustic velocity varies with space as the system evolves from the optimal initial condition. This result was obtained by solving the linearized equations. The parameters are given in the figure caption. It is clear that the spatial variation is a superposition of several modes whose phase vary with time.

D. Pseudospectra

The degree of resonant amplification that may occur in a normal system in response to an input frequency is inversely proportional to the distance in the complex plane between the input frequency and nearest eigenvalues. However, in the case of a non-normal operator, the resonant amplifications may be orders of magnitude larger[14] and cannot be determined by the eigenvalues alone. Such a resonance of a non-normal system is known as pseudoresonance.

The concept of ε pseudoeigenvalues can be used to analyze the behavior of evolution governed by such non-normal operators.[26–28] z is an ε pseudoeigenvalue of A if it satisfies $\|(zI-A)^{-1}\|\geq\varepsilon^{-1}$. There are other equivalent definitions of pseudoeigenvalues and they have been discussed in great detail by Trefthen and Embree.[26–28] In the above expression, ε

is a measure of the distance of the forcing frequency from the eigenvalues of the system. This can be also interpreted as the noise level in a system when there is no external forcing. It should be noted that the system discussed in this paper is a self-excited system. In a linearly stable self-excited system, if the initial condition is along one of the eigenmodes, then there is no growth and the oscillations will decay throughout the evolution. However, in most situations, the initial conditions do not correspond to any particular eigenmode, and hence transient growth can occur. In these situations, ε is a measure of the deviation of the initially excited mode from the eigenmodes of the system.

Transient growth and the non-normal nature of the operator can be studied using pseudospectra. The pseudospectra of normal operators are closed circles. When a contour corresponding to some ε value does not lie entirely in the left half plane, the system exhibits transient growth, causing the amplitudes to increase to high values.[14] Further, it is possible to obtain necessary and sufficiency conditions for an oscillation to be stable based on the geometry of pseudospectra.

Trefethen *et al.*[14] have used the relation between the geometry of the pseudospectra and the lower bound on the transient growth factor to analyze hydrodynamic instability in Couette and Poiseuille flows. The lower bound on the transient growth factor is given by

$$\max_t \|e^{tA}\| \geq \max_\varepsilon \frac{\max_{z\in\Gamma(\varepsilon)} \mathrm{Re}(z)}{\varepsilon}. \tag{16}$$

This inequality serves as a necessary condition. The pseudospectra contours should protrude far into the right half plane for the system to exhibit large transient growth. Hence, for a system not to grow (either transient or exponential), pseudospectra must lie entirely on the left half plane.

Similarly, a sufficiency condition for a system to be stable can be obtained by relating the upper bound on transient growth to the geometry of pseudospectra. The upper bound on the transient growth factor of a non-normal operator can be obtained from the pseudospectra as follows.[28] The exponential of an operator A, e^{tA}, can be defined by the Dunford–Taylor integral (operator analog of Cauchy integral),

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma (z-A)^{-1} e^{tz} dz, \tag{17}$$

where $\Gamma(\varepsilon)$ is the boundary of pseudospectra corresponding to some ε. Hence, the norm of the evolution operator is bounded by the Cauchy integral of $|e^{tz}|\|(z-A)^{-1}\|$. When Γ encloses the ε pseudospectra, then the upper bound for the transient growth factor can be written as

$$\|\exp(tA)\| \leq \left[\frac{L_\varepsilon}{2\pi\varepsilon} \max_{z\in\Gamma(\varepsilon)} |\exp(tz)|\right], \tag{18}$$

where L_ε is the length of the contour (or convux hull) $\Gamma(\varepsilon)$.

The following sufficiency condition for a thermoacoustic system to be stable can be obtained by choosing A as the stability operator L. The upper bound on the maximum growth rate is given by

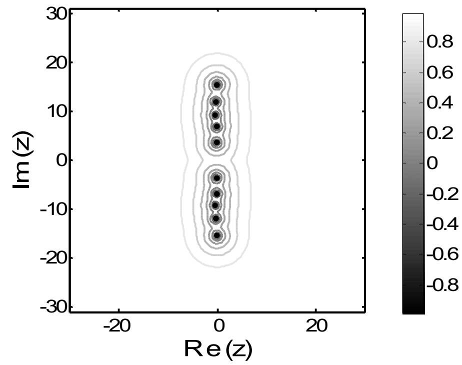

FIG. 9. Pseudospectra of a Rijke tube in which the heater is located at $x_f=0.18$. The time lag was chosen as $\tau=0.1\pi$. $\lambda=0.0328$ W/m K, $C_v=719$ J/kg K, $c_0=399.6$ m/s, $\bar{\rho}=1.205$ kg/m^3, $L_w=2.0$ m, $T_w=1000$ K, and $\bar{u}=0.5$ m/s.

$$G_{\max} = \max_t \|\exp(tL)\|^2 \leq \max_t \left[\frac{L_\varepsilon}{2\pi\varepsilon} \max_{z\in\Gamma(\varepsilon)} |\exp(tz)|\right]^2. \tag{19}$$

Hence, a system is stable if the real parts of all the eigenvalues are negative and if the right hand side of the above expression is 1. Hence, if the pseudospectra lie entirely in the left half plane and if $G_{\max}=1$, then the system will not grow for any initial condition.

Rayleigh criterion gives the condition for acoustic driving to occur. However, the prediction of transient growth by Rayleigh criterion requires the precise knowledge of initial conditions. The ambiguity of initial conditions due to noise makes the identification of transient growth using Rayleigh criterion difficult. However, the necessary and sufficiency conditions obtained in this paper are conditions on the evolution operator, and hence do not depend on the initial conditions.

Figure 9 shows the pseudospectra for a Rijke tube with length of 1 m and time lag $\tau=0.1\pi$ and $\bar{u}=0.5$ m/s. It is clear from the noncircular behavior that the system is nonnormal. It is clear that the contour is not entirely on the left half plane. Hence, it can be inferred from Eq. (24) that this system shows transient growth. This indicates that there is an "unstable" pseudoeigenvalue for some ε. Even if a system behaves linearly, the transient growth can cause the amplitudes to reach high values and trigger nonlinearities which can cause the oscillation to grow further.

VI. CONCLUSION

In this paper, the nature of thermoacoustic interaction is studied in the context of a simple model for a horizontal Rijke tube. The coupled thermoacoustic system is nonnormal, and hence the eigenvectors are nonorthogonal. Nonnormality can lead to short time growth of oscillations before they eventually decay even when the system is linearly stable. Transient growth can cause the amplitudes of the oscillations to trigger nonlinear driving. It has been observed that the various eigenmodes of the coupled thermoacoustic system interact, resulting in the growth of oscillations even when the eigenvalues indicate stability. The current methodology to study the onset of thermoacoustic oscillations involves looking for exponentially growing or decaying modes by calculating the individual eigenvalues of the linearized system. The role of non-normality has not been considered in predicting the stability margins of a system.

The stability of a system can be studied by calculating the transient growth factor. The growth factor shows a nonmonotonic variation with the heater location. For a nonnormal system, "pseudoresonance" can occur at frequencies far from the spectrum and pseudospectra can be used to analyze such systems. The pseudospectra of normal operators are disjoint circles. The pseudospectra of the thermoacoustic system considered in this study are noncircular, implying a highly non-normal nature of the system. Estimates of transient growth factors can be obtained by studying the geometry of the pseudospectra. The relation between the bounds of transient growth factor and the geometry of the pseudospectra provide necessary and sufficient conditions for the stability of a system. If the pseudospectra protrude into the right half plane, then the system can show transient growth.

ACKNOWLEDGMENTS

This work was funded by the Department of Science and Technology. The authors wish to thank Professor W. Polifke (Technishe Universität München), S. Muralidharan (Princeton University), and Professor R. Govindarajan (Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore) for their critical comments and suggestions.

[1]K. R. McManus, T. Poinsot, and S. M. Candel, "A review of active control of combustion instabilities," Prog. Energy Combust. Sci. **19**, 1 (1993).
[2]J. W. S. Rayleigh, "The explanation of certain acoustic phenomena," Nature (London) **18**, 319 (1878).
[3]H. G. Yoon, J. Peddieson, and K. R. Purdy, "Nonlinear response of a generalized Rijke tube," Int. J. Eng. Sci. **39**, 1707 (2001).
[4]B. T. Zinn and T. C. Lieuwen, "Combustion instabilities: Basic concepts," in *Combustion Instabilities in Gas Turbine Engines: Operational Experience, Fundamental Mechanisms, and Modeling*, edited by T. C. Lieuwen and V. Yang, Progress in Astronautics and Aeronautics Vol. 210 (AIAA, Reston, VA, 2005).
[5]M. A. Heckl, "Nonlinear acoustic effects in the Rijke tube," Acustica **72**, 63 (1990).
[6]C. C. Hantschk and D. Vortmeyer, "Numerical simulation of self-excited thermo-acoustic instabilities in a Rijke tube," J. Sound Vib. **277**, 511 (1999).
[7]K. I. Matveev, "Energy consideration of the nonlinear effects in a Rijke tube," J. Fluids Struct. **18**, 783 (2003).
[8]K. I. Matveev, "Thermo-acoustic instabilities in the Rijke tube: Experiments and modeling," Ph.D. thesis, California Institute of Technology, 2003.
[9]K. I. Matveev and F. E. C. Culick, "A study of the transition to the instability in a Rijke tube with axial temperature gradient," J. Sound Vib. **264**, 689 (2003).
[10]K. Balasubramanian and R. I. Sujith, "The role of non-normality in combustion-acoustic interactions in diffusion flames," AIAA Paper No. 2007-0567, 45th Aerospace Sciences Meeting and Exhibit, Reno, NV, 8–11 January 2007.
[11]K. Balasubramanian and R. I. Sujith, "Non-normality and nonlinearity in combustion-acoustic interactions in diffusion flames," J. Fluid Mech. **594**, 29 (2007).
[12]F. Nicoud, L. Benoit, C. Sensiau, and T. Poinsot, "Acoustic modes in combustors with complex impedances and multidimensional active flames," AIAA J. **45**, 426 (2007).
[13]J. S. Baggett, T. A. Driscoll, and L. N. Trefethen, "A mostly linear model of transition to turbulence," Phys. Fluids **7**, 833 (1995).
[14]L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, "Hydrodynamic stability without eigenvalues," Science **261**, 578 (1993).
[15]T. Gebgart and S. Grossmann, "Chaos transition despite linear stability," Phys. Rev. E **50**, 3705 (1994).
[16]W. Kerner, "Large scale complex eigenvalue problems," J. Comput. Phys. **85**, 1 (1989).
[17]B. F. Farrell, "Optimal excitation of baroclinic waves," J. Atmos. Sci. **46**, 1193 (1989).
[18]D. S. Henningson and P. J. Schmid, "Vector eigenfunctions for plane channel flows," Stud. Appl. Math. **87**, 15 (1992).
[19]J. Kopitz and W. Polifke, "CFD based analysis of thermoacoustic instabilities by determination of open-loop-gain," 12th International Congress on Sound and Vibration, The International Institute of Acoustics and Vibration, Lisbon (2005), Paper No. 389.
[20]L. Meirovitch, *Analytical Methods in Vibrations* (Macmillan, New York, 1967), Chap. 6.
[21]A. P. Dowling, "The calculation of thermoacoustic oscillations," J. Sound Vib. **180**, 557 (1995).
[22]W. Polifke, personal communication (2006).
[23]N. Ananthkrishnan, S. Deo, and F. E. C. Culick, "Reduced-order modeling and dynamics of nonlinear acoustic waves in a combustion chamber," Combust. Sci. Technol. **177**, 221 (2005).
[24]J. M. Wicker, W. D. Greene, S. I. Kim, and V. Yang, "Triggering of longitudinal combustion instabilities in rocket motors: Nonlinear combustion response," J. Propul. Power **12**, 1148 (1996).
[25]P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows* (Springer, New York, 2001).
[26]S. C. Reddy and L. N. Trefethen, "Pseudospectra of the convection-diffusion operator," SIAM J. Appl. Math. **54**, 1634 (1994).
[27]L. N. Trefethen, "Pseudospectra of linear operators," SIAM Rev. **39**, 383 (1997).
[28]L. N. Trefethen and M. Embree, *Spectra and Pseudospectra* (Princeton University Press, Princeton, 2005).