M 361K: Real Analysis

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Fall 2022

Contents

0.1 August 25

0.1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1) $a + b = b + a$.
- (A2) $(a + b) + c = a + (b + c)$.
- (A3) $\exists$ an element $o \in \mathbb{R}$ such that $a + o = o + a = a$.
- (A4) For each element $a \in \mathbb{R}$, $\exists$ an element $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.
- (M1) $ab = ba$.
- (M2) $(ab)c = a(bc)$.
- (M3) $\exists$ an element $1 \in \mathbb{R}$ such that $a * 1 = 1 * a = a$.
- (M4) For each element $a \in \mathbb{R} \setminus 0$, $\exists$ an element $\frac{1}{a} \in \mathbb{R}$ such that $a * \frac{1}{a} = \frac{1}{a} * a = 1$.
- (D) $a * (b + c) = a * b + a * c$.

Note:-

If $a = b$ and $c = d$, then $a + c = b + d$ and $a * c = b * d$.

$\forall x, y, z \in \mathbb{R}$:

Theorem 0.1

If $x + z = y + z$ then $x = y$.

Proof.

$$\begin{aligned} x + z &= y + z \quad (A4) \\ (x + z) + (-z) &= (y + z) + (-z) \quad (A2) \\ x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\ x + 0 &= y + 0 \quad (A3) \\ x &= y \end{aligned}$$

😄

Theorem 0.2

For any $x \in \mathbb{R}$, $x * 0 = 0$.

Proof.

$$\begin{aligned} x * 0 &= x * (0 + 0) \\ x * 0 &= x * 0 + x * 0 \\ x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\ 0 &= x * 0 + (x * 0 + (-x * 0)) \\ &= x * 0 + 0 \\ &= x * 0 \end{aligned}$$

😄

Theorem 0.3
$-1 * x = -x$ i.e. $x + (-1) * x = 0$.

Proof.

$$\begin{aligned} x + (-1) * x &= x + x * (-1) \\ &= x * 1 + x * (-1) \\ &= x * (1 + (-1)) \\ &= x * 0 \\ &= 0 \end{aligned}$$

😄

Theorem 0.4 Zero-product property
$\forall x, y \in \mathbb{R},\ x * y = 0 \iff x = 0 \lor y = 0$.

Proof. Let $x, y \in \mathbb{R}$, if $x = 0$ or $y = 0$, then $x * y = 0$. Suppose $x \neq 0$, then we must show $y = 0$. Since $x \neq 0$, $\frac{1}{x}$ exists. Thus, if:

$$\begin{aligned} xy &= 0 \\ \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\ (\frac{1}{x} * (xy)) * y &= 0 \\ 1 * y &= 0 \\ y &= 0 \end{aligned}$$

😄

0.1.2 Order Axioms

$\forall x, y \in \mathbb{R}$:

- (O1) One of $x < y$, $x > y$ or $x = y$ is true.
- (O2) If $x < y$ and $y < z$, then $x < z$.
- (O3) If $x < y$ then $x + z < y + z$.
- (O4) If $x < y$ and $z > 0$ then $xz < yz$.

Theorem 0.5
If $x < y$ then $-y < -x$.

Proof.

$$\begin{aligned} x &< y \\ x + (-x + -y) &< y + (-x + -y) \\ (x + -x) + -y &< (y + -y) + -x \\ 0 + -y &< 0 + -x \\ -y &< -x \end{aligned}$$

😄

Theorem 0.6

If $x < y$ and $z > 0$ then $xz > yz$.

Proof. If $x < y$ and $z > 0$ then $-z > 0$. Thus, $x(-z) < y(-z)$. But,

$$\begin{aligned} x(-z) &= x(-1 * z) \\ &= (x * -1) * z \\ &= (-1 * x) * z \\ &= -1(x * z) \\ &= -x * z \end{aligned}$$

Similarly, $y(-z) = -y * z$. Thus, $-x * z < -y * z$, so $xz > yz$. 😄

Note:-

$\mathbb{R}$ is an ordered field. $\mathbb{R}$ is complete, while $\mathbb{Q}$ is not complete.

0.2 August 30

Theorem 0.7
$\sqrt{2}$ is irrational.

Proof. Suppose not. Suppose that $\sqrt{2}$ is rational. Then $\exists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}, n \neq 0$ and m and n share no common factors. Then,

$$2 = \frac{m^2}{n^2}$$
$$2n^2 = m^2$$

Thus, m^2 is even and m is even. Then, $m = 2k$ for some $k \in \mathbb{Z}$. But, by substituting $m = 2k$ into the above equation, we get

$$2n^2 = (2k)^2$$
$$2n^2 = 4k^2$$
$$n^2 = 2k^2$$

Thus, n^2 is even, so n is even. So, n is a perfect square, which is a contradiction. Thus, $\sqrt{2}$ is irrational. ☺

0.2.1 Upper and Lower Bounds

Theorem: Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geqslant s \forall s \in S$, m is called an **upper bound** for S. If $m \leqslant s \forall s \in S$, m is called a **lower bound** for S. **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leqslant q \leqslant \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for T.

Definition 0.1: Supremum

The least upper bound of a set is called the supremum of the set.

Definition 0.2: Infimum

The greatest lower bound of a set is called the infimum of the set.

0.2.2 Completeness Axiom

Definition 0.3: Completeness axiom

Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 0.8
The set of natural numbers $\mathbb{N}$ is unbounded above.

Proof. Suppose not. Suppose that $\mathbb{N}$ is bounded above. If $\mathbb{N}$ were bounded above, it must have a supremum m. Since $\sup \mathbb{N} = m$, $m-1$ is not an upper bound. Thus, $\exists n_0 \in \mathbb{N}$ such that $n_0 > m-1$. But then, $n_0+1 > m$. This is a contradiction since $n_0+1 \in N$. Thus, N is unbounded above. 😃

Theorem 0.9

If A and B are nonempty subsets of $\mathbb{R}$, let $C = \{x+y \mid x \in A, y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C = \sup A + \sup B$.

Proof. Let $\sup A = a$ and $\sup B = b$. Then if $z \in C, z = x+y$ for some $x \in A, y \in B$. Then,

$$z = x + y \leqslant a + b = \sup A + \sup B$$

By the completeness axiom, $\exists$ a least upper bound of $C, c = \sup C$. It must be that $c \leqslant a+b$, so we must show $c \geqslant a+b$. Let $\epsilon > 0$. Since $a = \sup A$, $a - \epsilon$ is not an upper bound for A. $\exists x \in A$ such that $a - \epsilon < x$. Likewise, $\exists y \in B$ such that $b - \epsilon < y$. Then,

$$(a-\epsilon) + (b-\epsilon) = a + b - 2 * \epsilon < x + y \leqslant c$$

Thus, $a + b < c + 2 * \epsilon \forall \epsilon > 0$. So, $a + b \leqslant c \therefore c = a + b$. 😃

0.3 September 6

0.3.1 Cardinality

Definition 0.4: Cardinality

The cardinality of a set A is the number of elements in A. We denote this as $|A|$. We say that two sets A and B have the same cardinality if and only if $\exists$ a bijection $f : A \longrightarrow B$, or $|A| = |B|$.

Note:-

This bijection holds true because cardinality is reflexive (via the identity function), symmetric (via the inverse function), and transitive (via composition).

Note:-

The following examples demonstrate how to prove whether two sets have the same cardinality.

- $|\text{even integers}| = |\text{odd integers}|$: $f(2n) = 2n + 1$.
- $|\mathbb{Z}| = |\mathbb{Z}^+|$: $f(0) = 1$, $f(1) = 2$, $f(-1) = 3$, $f(2) = 4$, ...
- $|\mathbb{Q}^+| = |\mathbb{Z}^+|$: We can create a diagonal mapping by taking $\frac{n}{m}$ for counting numbers on the rows and columns.
- $|\mathbb{Q}| = |\mathbb{Z}^+|$: $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$, so we can repeat the diagonal mapping for $\mathbb{Q}^-$. This is because any subset of a countable set is countable.
- $|\mathbb{Q}| \neq |\mathbb{R}|$: For the real numbers, Cantor's Diagonal Argument proves the sets have different cardinality since no possible surjection exists.

In essence, if we show that there exists some one-to-one mapping between the two sets we can claim that $|A| = |B|$.

0.3.2 Countability

Definition 0.5: Countable

If a set is finite or has the same cardinality as $\mathbb{N}$ (i.e. $\mathbb{Z}^+$), we say that the set is countable.

Theorem 0.10

Any subset of a countable set is countable.

Theorem 0.11

Any set that contains an uncountable set is uncountable.

Theorem 0.12

If $[a_n, b_n] \forall n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, $\exists \delta \in \mathbb{R}$ such that $\delta \in I_n \forall n \in \mathbb{N}$.

Proof. $I_n \subseteq I_1 \forall n \in \mathbb{N}$. Thus, $a_n \subseteq b_1 \forall n \in \mathbb{N}$. So, b_n is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Let δ be the supremum of $\{a_n \mid n \in \mathbb{N}\}$. Thus, $a_n \leqslant \delta \forall n \in \mathbb{N}$.

We have now shown that $a_n \leqslant \delta \forall n \in \mathbb{N}$, and we need to show that $\delta \leqslant b_n \forall n \in \mathbb{N}$. This is left as an exercise for the reader. 😆

Note:-

A nested sequence means that successive subsets contain the previous subset. For example, $[0, 1] \subseteq [0, 2] \subseteq [0, 3] \subseteq \ldots$ is a nested sequence.

Theorem 0.13

$[0, 1]$ is uncountable.

Proof. Assume $[0, 1]$ is countable. That is, $[0, 1] = I = \{x_1, x_2, x_3, \dots\}$. Select a closed interval $I_1 \subseteq I$ such that $x_1 \notin I_1$. Next, select a closed interval $I_2 \subseteq I_1$ such that $x_2 \notin I_2$, and so on. Then, we have

$$I_n \subseteq \dots \subseteq I_2 \subseteq I_1 \subseteq I$$

and $x_n \notin I_n \forall n \in \mathbb{N}$. By **Theorem 3.3**, $\exists \delta \in I$ such that $\delta \in I_n \forall n \in \mathbb{N}$. This implies that $\delta \neq x_n \forall n \in \mathbb{N}$. Thus, $\delta \notin I$, which is a contradiction. Therefore, $[0, 1]$ is uncountable. 😄

0.4 September 8

0.4.1 Limits of Sequences

Definition 0.6: Limit of a sequence

A sequence a_n is said to converge to a real number s, if for any $\epsilon > 0$, $\exists$ a real number k such that for all $n \geqslant k$, the terms a_n satisfy $|a_n - s| < \epsilon$.

Theorem 0.14

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof. We need to find some N such that $n > N \forall \epsilon > 0$.

$$\begin{aligned}
|\frac{1}{\sqrt{n}} - 0| &< \epsilon \\
\frac{1}{\sqrt{n}} &< \epsilon \\
\frac{1}{n} &< \epsilon^2 \\
n &> \frac{1}{\epsilon^2}
\end{aligned}$$

Let $\epsilon > 0$ and $N = \frac{1}{\epsilon^2}$. Then, if $n > N$, we have that

$$\begin{aligned}
|\frac{1}{\sqrt{n}} - 0| &= \frac{1}{\sqrt{n}} \\
&< \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} \\
&= \epsilon
\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. ☺

Theorem 0.15

$\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$.

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\epsilon}$. Then, we have

$$\begin{aligned}
|1 + \frac{1}{2^n} - 1| &< \epsilon \\
|\frac{1}{2^n}| = \frac{1}{2^n} < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} &< \epsilon \\
n &> \frac{1}{\epsilon}
\end{aligned}$$

Thus, $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$. ☺

Theorem 0.16

Every convergent sequence is bounded.

Proof. Let S_n be a convergent sequence with a limit s and $\epsilon = 1$. Then, there exists some N such that $|S_n - s| < 1$. That is, $|S_n| < |s| + 1$.

Let $M = \max\{S_1, S_2, \dots, S_n, |s| + 1\}$. Then, $|S_n| \leqslant M$, so S_n is bounded. ☺

Theorem 0.17

If a sequence converges, its limit is unique.

Proof. Suppose a sequence S_n converges to s and t. Let $\epsilon > 0$. Then, $\exists N_1$ such that $|S_n - s| < \frac{t}{2}$. For $n > N_1$, $\exists N_2$ such that $|S_n - t| < \frac{t}{2}$. For $n > N_2$, let $N = m + \{N_1, N_2\}$. Then, for $n > N$, we have

$$\begin{aligned}
|s - t| &= |s + S_n - S_n - t| \\
&= |s - S_n + S_n - t| \\
&\leqslant |s - S_n| + |S_n - t| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
|s - t| &= \epsilon
\end{aligned}$$

Thus, the limit is unique. 😄

0.5 September 13

0.5.1 Monotone Sequences

Definition 0.7: Monotone sequence

A sequence S_n of real numbers is said to be increasing $\iff$ $S_n \leqslant S_{n+1}\ \forall\ n \in \mathbb{N}$ and decreasing $\iff$ $S_n \geqslant S_{n+1}\ \forall\ n \in \mathbb{N}$.

The Fibonacci sequence is an example of an increasing sequence.

Definition 0.8: Monotone convergence theorem

A monotone sequence is convergent if and only if it is bounded.

Theorem 0.18

An increasing bounded sequence is convergent.

Proof. Suppose S_n is a bounded increasing sequence. Let S be the set $\{S_n \mid n \in \mathbb{N}\}$. By the completeness axiom, $\sup S$ exists. Let $s = \sup S$. We claim $\lim_{n\to\infty} S_n = s$. Given $\epsilon > 0, s - \epsilon$ is not an upper bound for S.
Thus, $\exists\ N \in \mathbb{N}$ such that $S_N > s - \epsilon$. Furthermore, since S_n is increasing and s is an upper bound for S, we have $s - \epsilon < S_N \leqslant S_n \leqslant s\ \forall\ n \geqslant N$. 😃

Note:-

This is an elementary proof because it only uses axioms to make the conclusion.

Ex. $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$.

Theorem 0.19

If S_n is an unbounded increasing sequence, then $\lim_{n\to\infty} S_n = \infty$.

Proof. Let S_n be an increasing unbounded sequence. Then, $\{S_n \mid n \in \mathbb{N}\}$ is not bounded above, but S is bounded below by S_1. Thus, given $M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $S_N > M$. But since S_n is increasing, $S_n > M\ \forall\ n > N$. Thus, $\lim_{n\to\infty} S_n = \infty$. 😃

0.6 September 15

0.6.1 Cauchy Sequences

Definition 0.9: Cauchy sequence

A sequence of real numbers S_n is called a Cauchy sequence if and only if for each $\epsilon > 0$, $\exists N$ such that $m, n > N \implies |S_m - S_n| < \epsilon$.

Note:-

This means the elements of the sequence get closer to each other as N increases.

Theorem 0.20

Every convergent sequence is Cauchy.

Proof. Let S_n be a convergent sequence. Then $\exists N$ such that $n > N \implies |S_n - s| < \frac{\epsilon}{2}$ for some $s \in \mathbb{R}$. Then, for $n, m > N$, we have

$$\begin{aligned} |S_n - S_m| &= |S_n - s + s - S_m| \\ &\leqslant |S_n - s| + |s - S_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, S_n is Cauchy. ☺

Theorem 0.21

A sequence of real numbers is Cauchy if and only if it is convergent.

Note:-

We cannot prove this yet.

0.7 September 20

0.7.1 Empty Set

Theorem 0.22

The empty set is a subset of any set.

Proof. Suppose not. That is, suppose $\exists A$ such that $\emptyset \not\subset A$. Thus, $\exists x \in \emptyset$ such that $x \notin A$. This is a contradiction because the empty set has no elements. Therefore, $\emptyset \subset A$. 😄

Theorem 0.23

There is only one set with no elements.

Proof. Suppose not. That is, suppose $\exists$ two empty sets E_1, E_2. Then $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$. Thus, $E_1 = E_2$. This is a contradiction because E_1 and E_2 are two different sets. Therefore, there is only one empty set. 😄

Note:-

The empty set is open and closed (vacuosly true).

0.7.2 Topology of Real Numbers

Let $S \subseteq \mathbb{R}$ for the following definitions.

Neighborhoods

Definition 0.10: Neighorhood

A neighborhood of x in S can be thought of an epsilon-sized ball around x, i.e. $N(x, \epsilon) = \{y \in R \mid 0 \leqslant |x - y| < \epsilon\}$.

Definition 0.11: Deleted neighborhood

A deleted neighorhood is the same as a neighorhood except that x is not included, i.e. $N^\star(x, \epsilon) = \{y \in R \mid 0 < |x - y| < \epsilon\}$.

Definition 0.12: Accumulation point

$x \in \mathbb{R}$ is an accumulation point of S if and only if every deleted neighborhood of x contains a point of S.

Note:-

$(0, \infty)$ has accumulation points $[0, \infty)$. $(0, 1)$ does not contain all of its accumulation points since 0 and 1 are both accumulation points of the set.

Theorem 0.24

$S \in \mathbb{R}$ is closed if and only if S contains all of its accumulation points.

Proof. Suppose S is closed. Let x be an accumulation point of S. If $x \notin S$, then $x \in S^\complement$. Thus, $\exists$ a neighborhood N of x such that $N \subseteq S^\complement$. But $N \cap S = \emptyset$, which contradicts x being an accumulation point of S.

Conversely, suppose S contains all of its accumulation points. Let $x \in S^\complement$, then x is not an accumulation point of S. Thus, $\exists N^\star(x, \epsilon)$ that misses S. Since $x \notin S$, $N(x, \epsilon)$ misses S. Therefore, $S^\complement$ is open, which means S is closed. 😄

Theorem 0.25

If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a max.

Proof. Let $s = \sup S$. Then, s is an accumulation point of S. Since S is closed, $s \in S$. Thus, s is a max of S. 😄

Interior and Boundary Points

Definition 0.13: Interior point

$x \in S$ is an interior point of S if and only if $\exists N(x,t)$ such that $N(x,t) \subset S$.

Definition 0.14: Boundary point

$x \in S$ is a boundary point of S if and only if every neighborhood N of x has $N \cap S \neq \emptyset$ and $N \cap S^{\complement} \neq \emptyset$.

0.7.3 Closure

Definition 0.15: Open set

S is an open set if and only if every point in S is an interior point of S. $\forall x \in S, \exists$ a neighborhood $N(x,\epsilon)$ for some $\epsilon > 0$ such that $N(x,\epsilon) \subseteq S$.

Definition 0.16: Closed set

S is a closed set if and only S contains at least one of its boundary points. Additionally, $S^{\complement}$ must be an open set.

Note:-

$\mathbb{R}$ is open because all of its points are interior points. $\mathbb{R}$ is also closed because $\mathbb{R}$ has no boundary points, therefore implying that it contains at least one of its boundary points (vacuously true).

Theorem 0.26

The union of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $\exists$ a neighborhood N_1 of x such that $N_1 \subseteq A$. But then, $N_1 \subseteq A \cup B$. If $x \in B$, then $\exists$ a neighborhood N_2 of x such that $N_2 \subseteq B$. But then, $N_2 \subseteq A \cup B$.

Thus, in either case, $\exists$ a neighborhood N of x such that $N \subseteq A \cup B$. Therefore, $A \cup B$ is open. 😄

Theorem 0.27

An arbitrary union of open sets is open.

Proof. Let $A_1, A_2, \ldots, A_n$ be open sets. Let $x \in \bigcup_{i=1}^{n} A_i$. Then $x \in A_i$ for some i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcup_{i=1}^{n} A_i$. Therefore, $\bigcup_{i=1}^{n} N_i \subseteq \bigcup_{i=1}^{n} A_i$.

Thus, $\bigcup_{i=1}^{n} N_i$ is a neighborhood of x such that $\bigcup_{i=1}^{n} N_i \subseteq \bigcup_{i=1}^{n} A_i$. Therefore, $\bigcup_{i=1}^{n} A_i$ is open. 😄

Theorem 0.28

The intersection of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, $\exists$ neighborhoods $N_1(x,\epsilon_1)$ and $N_2(x,\epsilon_2)$. Let $\epsilon = min\{\epsilon_1, \epsilon_2\}$. Then $N_1(x,\epsilon) \subseteq A$ and $N_2(x,\epsilon) \subseteq B$.

Thus, $N(x,\epsilon) \subseteq A \cap B$. Therefore, $A \cap B$ is open. 😄

Theorem 0.29

A finite intersection of open sets is open.

Proof. Let $A_1, A_2, \ldots, A_n$ be open sets. Let $x \in \bigcap_{i=1}^{n} A_i$. Then $x \in A_i$ for all i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcap_{i=1}^{n} A_i$. Therefore, $\bigcap_{i=1}^{n} N_i \subseteq \bigcap_{i=1}^{n} A_i$.

Thus, $\bigcap_{i=1}^{n} N_i$ is a neighborhood of x such that $\bigcap_{i=1}^{n} N_i \subseteq \bigcap_{i=1}^{n} A_i$. Therefore, $\bigcap_{i=1}^{n} A_i$ is open. 😃

Theorem 0.30

An arbitrary intersection of open sets is open.

Note:-

$\bigcap_{i=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \emptyset$.

0.8 September 22

0.8.1 Set Covers

Definition 0.17: Open cover

An open cover F of some subset $S \in \mathbb{R}$ is a collection of open sets whose union contains S.

Note:-

If $E \subseteq F$ and E also covers S, we call E a **subcover**.

Definition 0.18: Compact

A set S is said to be compact is and only if whenever S is contained in the union of a family F of open sets, then it is contained in a finite number of the sets in F (every open cover has a finite subcover).

Note:-

It is hard to show that a set is compact since we have to consider *every* open cover.

Theorem 0.31 Heine-Borel

A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.

Proof. Let S be a compact set. Observe the open cover $(-n, n) \forall n \in \mathbb{N}$. Since S is compact, $\exists$ a finite subcover $(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)$. $\exists$ one of these sets such that $\bigcup_{i=1}^{k}(-n_i, n_i) = (-n_m, n_m)$ for some $m = 1, 2, \ldots k$. Thus, $S \subseteq (-n_m, n_m)$, so S is bounded.

Let S be a compact set. Suppose S is not closed. Let p be a boundary point of S, and Let $U_n = \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}] \forall n \in \mathbb{N}$. $S \subseteq \bigcup U_n = \mathbb{R}\ p$. $\exists$ a finite subcover $n_1, n_2, \ldots, n_k$ such that $S \subseteq \bigcup_{i=1}^{k} U_{n_i}$. $\exists k$ such that $S \subseteq U_{n_k}$. But, this is a contradiction with p being a boundary point. Therefore, S is closed.

The proof in the other direction is similar, yet non-trivial. 😊

Theorem 0.32 Bolzano-Weierstrass

If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then $\exists$ at least one accumulation point of S.

Proof. Let S be a boudned infinite subset of $\mathbb{R}$. Suppose S has no accumulation points, then S is closed. By Heine-Borel, S must be compact. Define neighorhoods N_x such that $N_x(x) \cap S = x \forall x \in S$. Clearly, $S \subseteq \bigcup_x N_x$. But, the collection of all N_x must contain a finite subcover. That is,

$$S \subseteq N_{x_1} \cup N_{x_2} \cup \ldots \cup N_{x_k}$$

for some $k \in \mathbb{N}$. This contradicts that S is infinite. Therefore, S has an accumulation point. 😊

0.8.2 Cauchy Convergence

Theorem 0.33

Every Cauchy sequence is convergent.

Proof. S_n is Cauchy, so $S = \{S_n \mid n \in \mathbb{N}\}$. By Bolzano-Weierstrass, $\exists$ an accumulation point s of S. We claim that $S_n \longrightarrow s$. Given $\epsilon > 0$, $\exists$ N such that $m, n > N$. Then $|S_m - S_n| < \frac{\epsilon}{2}$. $(S - \frac{\epsilon}{2}, S + \frac{\epsilon}{2})$ contains an infinite number of points.

$\exists m > N$ such that $S_m \in N(s, \frac{\epsilon}{2})$. But then, $|S_n - s| = |S_n - S_m + S_m - s| \leqslant |S_n - S_m| + |S_m - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $S_n \longrightarrow s$. 😊

Theorem 0.34

Let x_n be a sequence of non-negative real numbers. $\sum x_n$ converges if S_k, the sequence of partial sums is bounded.

Proof. $\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k$. S_k is increasing and bounded, it is convergent by the monotone convergence theorem. 😄

0.9 September 27

0.9.1 Limits of Functions

Definition 0.19: Limit of a function

Let $f : D \to \mathbb{R}$ and let c be an accumulation point of the function. Then, $\lim_{x \to c} f(x) = L$ if and only if given $\epsilon > 0, \exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Note:-

Suppose we want to show that $\lim_{x \to 2} S_x + 1 = 11$. We are looking for some $\delta > 0$ such that $0 \leqslant |x - 2| < \delta$ and $|S_x + 1 - 11| < \epsilon$. This is structured similarly to proofs of limits of sequences.
Additionally, the limit must go to an accumulation point of the function because we cannot find the limit of a value outside the function's domain.

Theorem 0.35

$\lim_{x \to 5} 10x + 2 = 52$.

Proof. We need to find some $\delta > 0$ such that whenever $0 < |x - 5| < \delta$, $|10x + 2 - 52| < \epsilon$.

$$|10x - 50| < \epsilon$$
$$10|x - 5| < \epsilon$$
$$|x - 5| < \frac{\epsilon}{10}$$

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{10}$. Then, whenever $0 < |x - 5| < \delta$, we have $|10x + 2 - 52| = |10x - 50| = 10|x - 5| < 10 * \frac{\epsilon}{10} = \epsilon$. 😃

Theorem 0.36

$\lim_{x \to 3} x^2 + 2x + 6 = 21$.

Proof. We need to find some $\delta > 0$ such that whenever $0 < |x - 3| < \delta$, $|(x^2 + 2x + 6) - 21| < \epsilon$.

$$|x^2 + 2x + 6 - 21| < \epsilon$$
$$|x^2 + 2x - 15| < \epsilon$$
$$|x + 5||x - 3| < \epsilon$$

If $\delta < 1 \implies |x + 5||x - 3| < 9|x - 3| < \epsilon$. Thus $|x - 3| < \frac{\epsilon}{9}$. We let $\delta = \min\{1, \frac{\epsilon}{9}\}$.

Given $\epsilon > 0$, let $\delta = \min\{1, \frac{\epsilon}{9}\}$. Then, whenever $0 < |x - 3| < \delta$, we have that $|x + 5| < 9$, thus, $|(x^2 + 2x + 6) - 21| = |x^2 + 2x - 15| = |x + 5||x - 3| < \min\{1, \frac{\epsilon}{9}\} * \frac{\epsilon}{9} = \epsilon$. 😃

Note:-

These proofs have two phases. First, we determine some δ as an upper bound. Then, we show how this choice of δ implies the limit is bounded by some ϵ.

Theorem 0.37

Let $f : D \to \mathbb{R}$ and c is an accumulation point of D. Then, $\lim_{x \to c} f(x) = L$ if and only if for every sequence $S_n \in D$ such that $S_n \to c$, $S_n \neq c \forall n$, then $f(S_n)$ converges to L.

Proof. $\lim_{x \to c} f(x) = L$ and $S_n \to L \implies f(S_n) \to L$. We need to find N such that $n > N$ and $|f(S_n) - L| < \epsilon$. We know that $\exists \delta$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$ and $\exists N$ such that $n > N \implies |S_n - c| < \delta$. Thus, for $n > N$ we have $|f(S_n) - L| \in \epsilon$.

Suppose L is not the limit of f as x approaches c. We must find (S_n) that converges to c, but $f(S_n)$ does not converge to L (contrapositive). $\exists \epsilon > 0$ such that $\forall \delta > 0, 0 < |x - c| < \delta \implies |f(x) - L| \geqslant \epsilon$. For each $n \in N, \exists S_n \in D$ such that $0 < |S_n - c| < \frac{1}{n}$ and $|f(S_n) - L| \geqslant \epsilon$. Then, $S_n \to c$, but $f(S_n) \not\to L$. This is a contradiction. 😃

0.10 September 29

0.10.1 Sums of Limits

Theorem 0.38

Let $lim_{x \to c} f(x) = L$, $lim_{x \to c} g(x) = M$. Then, $lim_{x \to c}(f+g)(x) = L + M$.

Proof (Definition 9.1). Given $\epsilon > 0$, let $\delta_1 > 0$ be such that $0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2}$. Let $\delta_2 > 0$ be such that $0 < |x - c| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2}$.

Let $\delta = min\{\delta_1, \delta_2\}$. Then, for $0 < |x - c| < \delta$, we have

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leqslant |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

😊

Proof (Theorem 9.3). Let $lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, and S_n be a sequence of real numbers such that $S_n \longrightarrow c$. Then,

$$\lim_{n \to \infty} (f+g)(S_n) = \lim_{n \to \infty} f(S_n) + g(S_n) = \lim_{n \to \infty} f(S_n) + \lim_{n \to \infty} g(S_n) = L + M$$

Thus, $\lim_{x \to c}(f+g)(x) = L + M$. 😊

Note:-

This is true for $-$, $\times$, and $\div$ as well.

Theorem 0.39

Let $k \in \mathbb{R}$. If $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} kf(x) = kL$.

Proof. Let $\lim_{x \to c} f(x) = L$, $k \in \mathbb{R}$, and S_n be a sequence of real numbers such that $S_n \longrightarrow c$. Then,

$$\lim_{n \to \infty} kf(S_n) = k \lim_{n \to \infty} f(S_n) = kL$$

Thus, $\lim_{x \to c} kf(x) = kL$. 😊

0.10.2 Continuity of Functions

Definition 0.20: Continuous Function

A function f is continuous at $x = c$ if and only if $\lim_{x \to c} f(x) = f(c)$. Let s be an accumulation point of the domain $f : D \longrightarrow \mathbb{R}$. Then, f is continuous at s if and only if for each $\epsilon > 0$, $\exists \delta > 0$ such that whenever $0 < |x - s| < \delta$, $|f(x) - f(s)| < \epsilon$.

Note:-

Let $f(x) = x\sin(\frac{1}{x})$ where $x \neq 0$, $f(0) = 0$. If we want to show that this function is continuous, we need to find some $\delta > 0$ such that $|x| < \delta \implies |f(x) - f(0)| < \epsilon$. Let $\delta = \epsilon$, then when $|x| < \delta$, $|f(x) - f(0)| = |x\sin(\frac{1}{x}) - 0| = |x\sin(\frac{1}{x})| \leqslant |x| < \epsilon$.

Theorem 0.40

If f and g are continuous at $x = c$, then $f + g$ is also continuous at $x = c$.

Proof. Let f and g be continuous at c and S_n be a sequence of real numbers such that $S_n \longrightarrow c$. Then,

$$\lim_{n \to \infty} (f+g)(S_n) = \lim_{n \to \infty} f(S_n) + \lim_{n \to \infty} g(S_n) = f(c) + g(c)$$

Thus, $\lim_{x \to c}(f+g)(x) = (f+g)(c)$. 😊

Theorem 0.41

Let $f : D \longrightarrow E$ be continuous at $x = c$ and let $g : E \longrightarrow R$ be continuous at $x = f(c)$. Then, the composition $g \circ f$ is continuous at $x = c$.

Proof. This is left as an exercise for the reader. ☺

0.11 October 6

0.11.1 Derivatives

Definition 0.21: Derivative

Let f be a real-valued function defined on an open interval containing c. We say f is differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. We call this limit $f'(c)$.

Theorem 0.42

If f is differentiable at c, then f is continuous at c.

Proof. Let f be defined on some interval I containing c. Then if f is differentiable at c, if and only if for $x \neq c$,

$$f(x) = (x-c)\frac{f(x)-f(c)}{x-c} + f(c)$$

Then, $\lim_{x\to c} f(x) = \lim_{x\to c}(x-c)\frac{f(x)-f(c)}{x-c} + f(c) = \lim_{x\to c}(x-c)f'(c) + f(c) = f(c)$. Therefore, f is continuous at c. 😆

Derivative Rules

- $\frac{d}{dx}kf = k\frac{df}{dx}$
- $\frac{d}{dx}f + g = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx}f \cdot g = \frac{df}{dx}g + \frac{dg}{dx}f$
- $\frac{d}{dx}\frac{f}{g} = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}$

Theorem 0.43 Product rule

$$(fg)' = f'g + fg'$$

Proof. Suppose f and g are differentiable at c. Then,

$$\begin{aligned}
\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} &= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)(g(x)-g(c))}{x-c} + \frac{g(x)(f(x)-f(c))}{x-c}. \\
&= f(c)g'(c) + g(c)f'(c)
\end{aligned}$$

😆

Theorem 0.44 Quotient rule

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

Proof. Let f and g be differentiable at c. Then,

$$\begin{aligned}
\lim_{x\to c}\frac{\frac{f}{g}(x)-\frac{f}{g}(c)}{x-c} &= \lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c} \\
&= \lim_{x\to c}\frac{\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)}}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(c)-g(c)f(c)+g(c)f(c)-f(c)g(x)}{(x-c)g(x)g(c)} \\
&= \lim_{x\to c}\frac{g(c)\frac{f(x)-f(c)}{(x-c)}+f(c)\frac{g(x)-g(c)}{(x-c)}}{g(c)g(x)} \\
&= \lim_{x\to c}\frac{g(c)f'(c)-f(c)g'(c)}{g^2(c)}Aa
\end{aligned}$$

😄

Theorem 0.45 Power rule

$$(x^n)' = nx^{n-1}f' \ \forall\ n \in \mathbb{N}$$

Proof by induction. $p(n) = (x^n)' = nx^{n-1}f'$.

$p(1)$: $f(x) = x$. $\lim_{x\to c}\frac{x-c}{x-c} = 1 = 1\cdot x^0$.

$p(k) \longrightarrow p(k+1)$:

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k\cdot x \\
&= (\frac{d}{dx}x^k)\cdot x + x^k(\frac{d}{dx}x) \\
&= kx^{k-1}\cdot x + x^k\cdot 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

😄

Theorem 0.46 Chain rule

$$g(f(x))' = g'(f(x))f'(x)$$

Proof.

$$\begin{aligned}
\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\frac{f(x)-f(c)}{x-c} \\
&= g'(f(x))f'(x)
\end{aligned}$$

😄

Note:-

This will not hold if $f(x) = f(c)$.

Theorem 0.47

Let f be defined on an interval I containing c. Then, f is differentiable at c if and only if $\exists$ a function φ

on I such that φ is continuous at c and

$$f(x) - f(c) = \varphi(x)(x - c) \forall x \neq c$$

In this case, we have $\varphi(c) = f'(c)$.

Note:-

Let $f(x) = x^3$. Then, $f(x) - f(c) = x^3 - c^3 = (x^2 + xc + c^2)(x - c)$. $\phi(c) = c^2 + c \cdot c + c^2 = 3c^2 = f'(c)$.