M 361K Homework 2

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3.3

5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}\ \forall\ n \in \mathbb{N}$. Show that (y_n) converges and find the limit.

Proof. We want to show that (y_n) is both monotonically increasing and bounded. First, we show that (y_n) is monotonically increasing. We have that $y_1 = \sqrt{p}$ and $y_2 = \sqrt{p + y_1} = \sqrt{p + \sqrt{p}}$. Thus, $y_2^2 - y_1^2 = p + \sqrt{p} - p = \sqrt{p} > 0$. Thus, $y_2 > y_1$. We can continue this argument to show that $(y_{n+1}) > (y_n)$ as $(y_{n+1}) - (y_n) = (y_n) + (y_{n-1}) > 0$. Thus, (y_n) is monotonically increasing.

Now, we show that (y_n) is bounded. We have that $y_1 = \sqrt{p} < 1 + 2\sqrt{p}$. Next, we have that $(y_n) < 1 + 2\sqrt{p}$. From here, $(y_{n+1}^2) < p + (y_n) < p + 1 + \sqrt{p} < (\sqrt{p} + 1)^2 < 1 + 2\sqrt{p}$. Thus, since $(y_n) < 1 + 2\sqrt{p}$, (y_n) is bounded, so (y_n) converges to some value c.

Now, we want to find this value c. We have that $\lim_{n\to\infty}(y_n) = \lim_{n\to\infty}\sqrt{p + y_{n-1}} = \sqrt{p + \lim_{n\to\infty}(y_{n-1})}$. Then, $c = \sqrt{p + c}$. Thus, $c^2 = p + c \implies c = \frac{1}{2}(1 + \sqrt{1 + 4p})$. □

8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n\ \forall\ n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$.

Proof. Since (b_n) is decreasing, b_1 is the upper bound of (b_n) and also the upper bound of (a_n) since $a_n \leq b_n\ \forall\ n \in \mathbb{N}$. Thus, (a_n) is bounded below by a_1 and above by b_1, and (a_n) is monotonic, so it must converge to some limit. Similarly, (b_n) is bounded below by a_1 and above by b_1, and (b_n) is monotonic, so it must converge to some limit.

Now, we can use Theorem 3.2.5 which states that for two convergent sequences (a_n) and (b_n), if $a_n \leq b_n\ \forall\ n \in \mathbb{N}$, then $\lim(a_n) \leq \lim(b_n)$. Thus, $\lim(a_n) \leq \lim(b_n)$. □

12. Establish the convergence and find the limits of the following sequences.

(a) $((1 + 1/n)^{n+1})$

$$\begin{aligned}\lim_{n\to\infty}(1 + 1/n)^{n+1} &= \lim_{n\to\infty}(1 + 1/n)^n \cdot (1 + 1/n)\\ &= \lim_{n\to\infty}(1 + 1/n)^n \cdot \lim_{n\to\infty}(1 + 1/n)\\ &= e \cdot 1\\ &= e\end{aligned}$$

(b) $((1+1/n)^{2n})$

$$\begin{aligned}\lim_{n\to\infty}(1+1/n)^{n+1} &= \lim_{n\to\infty}(1+1/n)^n \cdot (1+1/n)^n \\ &= \lim_{n\to\infty}(1+1/n)^n \cdot \lim_{n\to\infty}(1+1/n)^n \\ &= e \cdot e \\ &= e^2\end{aligned}$$

3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.

Proof. Let there be some sequence (x_n) where $(x_n) = 1$ if n is even and $(x_n) = n$ if n is odd. Then, (x_n) is an unbounded sequence, yet the susbequence (x_{2n}) is convergent as it is bounded and monotonic. Thus, (x_n) has a convergent subsequence. $\square$

4b. Show that the sequence $(\sin n\pi/4)$ is divergent.

Proof. Let $(x_n) = \sin n\pi/4$. We want to show that (x_n) has two convergent subsequences whose limits are not equal. Let $(y_n) = (x_{4n})$ and $(z_n) = (x_{8n+1})$ be subsequences of (x_n).

Then, $(y_n) = \sin(4n\pi/4) = \sin(n\pi) = 0$ and $(z_n) = \sin((8n+1)\pi/4) = \sin(2n\pi + \pi/4) = \sin(\pi/4) = \sqrt{2}/2$. Thus, (y_n) and (z_n) are both convergent subsequences of (x_n), yet their limits are not equal. Therefore, (x_n) is divergent. $\square$

10. Let (x_n) be a bounded subsequence and for each $n \in \mathbb{N}$, let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

Proof. For $\epsilon > 0$, there exists some $n \in \mathbb{N}$ such that $s_n < S + \epsilon$. We can choose $\epsilon = 1$ and m_1 such that $s_{m_1} - 1 < S + 1$ and $k_1 \geq m_1$ such that $s_{m_1} - 1 < x_{k_1} < s_{m_1}$ since $s_{m_1} = \sup\{x_n : k \geq m_1\}$.

Then, we can choose some $m_n > m_{n-1}$ such that $S \leq s_{m_n} < S + \frac{1}{n}$ and $k_n \geq m_n$ and $k_n > k_{n-1}$ such that $s_{m_n} - \frac{1}{n} < x_{k_n} < s_{m_n}$. Now, we have a subsequence (x_{k_n}) of (x_n) where $|x_{k_n} - S| \leq \frac{1}{n}$. Finally, we know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, $\lim_{n\to\infty} x_{k_n} = S$. $\square$

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty}(1/x_{n_k}) = 0$.

Proof. Let (x_n) be unbounded. Then, there exists some $n_1 \in \mathbb{N}$ such that $|x_{n_1}| \geq 1$. There also exists some $n_2 > n_1 \in \mathbb{N}$ such that $|x_{n_2}| \geq 2$. We can continue this with some arbitrary sequence $n_i \in \mathbb{N}$ such that $|x_{n_k}| \geq k$ for all $k \in \mathbb{N}$ because this sequence is unbounded. Then,

$$0 \leq \frac{1}{|x_{n_k}|} \leq \frac{1}{k}$$

We know that $\lim_{k\to\infty} \frac{1}{k} = 0$, so $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$ by Theorem 3.2.7 (Squeeze Theorem). $\square$

3.5

2. Show directly from the definition that the following are Cauchy sequences.

(a) $(\frac{n+1}{n})$

We want to show that $\exists N \in \mathbb{N}$ such that $m, n > N$ and $|S_m - S_n| < \epsilon$. Let there be some arbitrary N such that $\frac{1}{N} = \frac{\epsilon}{2}$ and some $m > n \geq N$. Then,

$$\begin{aligned}\left|\frac{m+1}{m} - \frac{n+1}{n}\right| &= \left|\frac{1}{m} - \frac{1}{n}\right| \\ &\leq \frac{1}{m} + \frac{1}{n} \\ &\leq \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \epsilon\end{aligned}$$

Thus, $(\frac{n+1}{n})$ is a Cauchy sequence.

(b) $(1 + \frac{1}{2!} + \cdots + \frac{1}{n!})$ We want to show that $\exists N \in \mathbb{N}$ such that $m, n > N$ and $|S_m - S_n| < \epsilon$. Let there be some arbitrary N such that $\frac{1}{2^N} < \epsilon$ and some $m > n \geq N$. Then,

$$\begin{aligned}\left|\left(1 + \frac{1}{2!} + \cdots + \frac{1}{m!}\right) - \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right| &= \frac{1}{(n+1)!} + \cdots + \frac{1}{m!} \\ &\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} \\ &= \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}}\right) \\ &\leq \frac{1}{2^n}\left(\frac{1}{2} + \frac{1}{4} + \cdots\right) \\ &= \frac{1}{2^n} * 1 \\ &\leq \frac{1}{2^N} \\ &< \epsilon\end{aligned}$$

Thus, $(1 + \frac{1}{2!} + \cdots + \frac{1}{n!})$ is a Cauchy sequence.

7. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

Proof. Let $\epsilon = 1$. Then, there must exist some N such that $m, n > N$ and $|x_m - x_n| < \epsilon$. However, since x_m and x_n are integers, $x_m = x_n$ in order to satisfy the Cauchy condition. Thus, (x_n) is ultimately constant. □

8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.

Proof. Let there be some bounded, monotone increasing sequence (x_n) that has some supremum M. We know that there exists some n_0 and $\epsilon > 0$ such that $M - \epsilon < x_{n_0} < M$. Since (x_n) is increasing, we also know that there exists some n_1 and n_2 such that $x_{n_0} \leq x_{n_1} \leq x_{n_2}$. We can combine these inequalities to get that $M - \epsilon < x_{n_0} \leq x_{n_1} \leq x_{n_2} < M$.

Now, we want to show that $|x_{n_1} - x_{n_2}| < \epsilon$ to satisfy the definition of a Cauchy sequence. Since we know that $M - \epsilon < x_{n_1} < M$ and $M - \epsilon < x_{n_2} < M$, we know that $-M < -x_{n_2} < \epsilon - M$. Then, we can combine our equations like so:

$$\begin{aligned} M - \epsilon - M < x_{n_1} - x_{n_2} &< M + \epsilon - M \\ \implies \epsilon < x_{n_1} - x_{n_2} &< \epsilon \\ \implies |x_{n_1} - x_{n_2}| &< \epsilon \end{aligned}$$

Thus, (x_n) is a Cauchy sequence. $\square$

4.1

2. Determine a condition on $|x - 4|$ to assure the following inequalities.

We can break down our original equation to yield a more useful form assuming that $x \geq 0$:

$$\begin{aligned} x - 4 &= (\sqrt{x} + 2)(\sqrt{x} - 2) \\ |\sqrt{x} - 2| &= \frac{|x - 4|}{\sqrt{x} + 2} \\ |\sqrt{x} - 2| &\leq \frac{|x - 4|}{2} \end{aligned}$$

We can use this new form to easily solve these inequalities.

(a) $|\sqrt{x} - 2| < \frac{1}{2}$

Proof. Let $|x - 4| < 1$. Then, $|\sqrt{x} - 2| < \frac{1}{2}$. $\square$

(b) $|\sqrt{x} - 2| < 10^{-2}$

Proof. Let $|x - 4| < 2 \cdot 10^{-2}$. Then, $|\sqrt{x} - 2| < 10^{-2}$. $\square$

6. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose that there exists constants K and L such that $|f(x) - L| \leq K|x - c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

Proof. We want to show that $\epsilon > 0$ as we can find some $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \epsilon$. Because $|f(x) - L| \leq K|x - c|$, we have $\delta = \frac{\epsilon}{K}$. If $|x - c| < \frac{\epsilon}{K}$, then $|f(x) - L| \leq K|x - c| < \epsilon$. Thus, $|f(x) - L| < \epsilon$, so $\lim_{x \to c} f(x) = L$. $\square$

9a. Use either the $\epsilon-\delta$ definition of a limit or the Sequential Criterion for limits to establish that $\lim_{x\to 2} \frac{1}{1-x} = -1$.

Proof. We assume we have some $(s_n) \to 2$. Then,

$$\lim_{n\to\infty} \frac{1}{1-x} = \frac{1}{1-2} = -1$$

Then, by the Sequential Criterion for limits, we have that $\lim_{x\to 2} \frac{1}{1-x} = -1$. $\square$

10a. Use the definition of the limit to show that $\lim_{x\to 2}(x^2 + 4x) = 12$.

Proof. Let $\delta = \min\{1, \frac{epsilon}{9}\}$ and $\epsilon > 0$. We also have x such that $|x - 2| < \delta$. We want to show that the difference at any two arbitrary points is less than ϵ. Then,

$$\begin{aligned}
|x^2 + 4x - 12| &= |(x-2)(x+6)| \\
&\leq |x-2||x+6| \\
&\leq \delta|x+6| \\
&= \delta|x-2+8| \\
&\leq \delta|\delta+8| \\
&\leq \delta(1+8) \\
&= \delta(9) \\
&\leq \epsilon
\end{aligned}$$

Thus, $\lim_{x\to 2}(x^2 + 4x) = 12$. $\square$

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

Proof. For some $\epsilon > 0$, choose $\delta = \epsilon$ such that $|x - 0| = |x| < \delta$. Then, $|f(x) - 0| = |f(x)| \leq |x| \leq \delta = \epsilon$. Thus, $f(x)$ has a limit at $x = 0$. $\square$

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

Proof. Let there be some $(x_n) \in \mathbb{Q}$ and $(y_n) \in \mathbb{R} \setminus \mathbb{Q}$ such that they both converge to c where $c \neq 0$. We know that $f(x_n) = x_n$ and $f(y_n) = 0$ from the definition of the function. Thus, we have two convergent subsequences that do not converge to the same limit. Therefore, f does not have a limit at c. $\square$

4.2

1. Apply Theorem 4.2.4 to determine the following limits:

(a) $\lim_{x\to 1}(x+1)(2x+3)$ where $(x \in \mathbb{R})$

Proof.

$$\begin{aligned}\lim_{x\to 1}(x+1)(2x+3) &= \lim_{x\to 1}(x+1)\cdot\lim_{x\to 1}(2x+3)\\ &= 2\cdot 5\\ &= 10\end{aligned}$$

Thus, $\lim_{x\to 1}(x+1)(2x+3) = 10$. □

(b) $\lim_{x\to 1}\frac{x^2+2}{x^2-2}$ where $(x > 0)$

Proof.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+2}{x^2-2} &= \frac{\lim_{x\to 1}(x^2+2)}{\lim_{x\to 1}(x^2-2)}\\ &= \frac{3}{-1}\\ &= -3\end{aligned}$$

Thus, $\lim_{x\to 1}\frac{x^2+2}{x^2-2} = -3$. □

(c) $\lim_{x\to 2}(\frac{1}{x+1} - \frac{1}{2x})$ where $(x > 0)$

Proof.

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{x+1} - \frac{1}{2x}\right) &= \lim_{x\to 2}\left(\frac{1}{x+1}\right) - \lim_{x\to 2}\left(\frac{1}{2x}\right)\\ &= \frac{1}{3} - \frac{1}{4}\\ &= \frac{1}{12}\end{aligned}$$

Thus, $\lim_{x\to 2}(\frac{1}{x+1} - \frac{1}{2x}) = \frac{1}{12}$. □

(d) $\lim_{x\to 0}\frac{x+1}{x^2+2}$ where $(x \in \mathbb{R})$

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{x+1}{x^2+2} &= \frac{\lim_{x\to 0}(x+1)}{\lim_{x\to 0}(x^2+2)}\\ &= \frac{1}{2}\end{aligned}$$

Thus, $\lim_{x\to 0}\frac{x+1}{x^2+2} = \frac{1}{2}$. □

4. Prove that $\lim_{x\to 0}\cos(1/x)$ does not exist but that $\lim_{x\to 0} x\cos(1/x) = 0$.

Proof. First, we will show that $\lim_{x\to 0}\cos(1/x)$ does not exist. Let $(x_n) = \frac{1}{n+\pi/2}$ and $(y_n) = \frac{1}{n+2\pi}$. Then, $\lim_{n\to\infty}(x_n) = 0$ and $\lim_{n\to\infty}(y_n) = 0$. $\forall n \in \mathbb{N}$, $\cos(1/x_n) = \cos(n+\pi/2) = 0$ and $\cos(1/y_n) = \cos(n+2\pi) = 1$. Thus, we have two convergent subsequences that do not converge to the same limit, so $\lim_{x\to 0}\cos(1/x)$ does not exist.

Now, we will show that $\lim_{x\to 0} x\cos(1/x) = 0$. Let $\epsilon > 0$. We know $\exists\delta = \epsilon$ such that $|x-0| = |x| < \delta$. Then,

$$\begin{aligned} |x\cos(1/x) - 0| &= |x\cos(1/x)| \\ &\le |x| \\ &\le \delta \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{x\to 0} x\cos(1/x) = 0$. $\square$

6. Use the definition of the limit to prove that if $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then $\lim_{x\to c}(f+g) = L+M$.

Proof. Let $\epsilon > 0$. Then, we know there exists some $\delta_f, \delta_g > 0$ such that $|x-c| < \delta_f$ and $|x-c| < \delta_g \implies |f(x)-L| < \frac{\epsilon}{2}$ and $|g(x)-M| < \frac{\epsilon}{2}$.

We choose $\delta = \max\{\delta_f, \delta + g\}$. Then, for $|x-c| < \delta$, we have

$$\begin{aligned} |f(x)+g(x)-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\ &\le |f(x)-L| + |g(x)-M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{x\to c}(f+g) = L+M$. $\square$

10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f+g$ and fg have limits at c.

Proof. Let $f(x) = \operatorname{sgn}(x)$ and $g(x) = -\operatorname{sgn}(x)$ and $c = 0$. Then, f and g do not have limits at c, but $f+g = 0$ and $fg = -1$. Thus, $f+g$ and fg have limits at c. $\square$

11. Determine whether the following limits exist at $\mathbb{R}$.

(a) $\lim_{x \to 0} \sin(1/x^2)$ where $(x \neq 0)$.

Proof. Let $(x_n) = \frac{1}{\sqrt{n\pi}}$ and $(y_n) = \frac{1}{\sqrt{2\pi n + \pi/2}}$. Then, $\lim_{n\to\infty}(x_n) = 0$ and $\lim_{n\to\infty}(y_n) = 0$. $\forall n \in \mathbb{N}$, $\sin(1/x_n^2) = \sin(n\pi) = 0$ and $\sin(1/y_n^2) = \sin(2\pi n + \pi/2) = 1$. Thus, we have two convergent subsequences that do not converge to the same limit, so $\lim_{x\to 0} \sin(1/x^2)$ does not exist. $\square$

(b) $\lim_{x \to 0} x \sin(1/x^2)$ where $(x \neq 0)$.

Proof. We have some $\epsilon > 0$ and $\delta = \epsilon$ such that $|x - 0| = \delta$. Then,

$$\begin{aligned} |x\sin(1/x^2) - 0| &= |x \sin(1/x^2)| \\ &\leq |x| \\ &\leq \delta \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{x\to 0} x \sin(1/x^2) = 0$. $\square$

(c) $\lim_{x \to 0} \operatorname{sgn} \sin(1/x)$ where $(x \neq 0)$.

Proof. Let $(x_n) = \frac{1}{n\pi}$ and $(y_n) = \frac{1}{2\pi n + \pi/2}$. Then, $\lim_{n\to\infty}(x_n) = 0$ and $\lim_{n\to\infty}(y_n) = 0$. $\forall n \in \mathbb{N}$, $\operatorname{sgn} \sin(1/x_n) = \operatorname{sgn} \sin(n\pi) = 0$ and $\operatorname{sgn} \sin(1/y_n) = \operatorname{sgn} \sin(2\pi n + \pi/2) = 1$. Thus, we have two convergent subsequences that do not converge to the same limit, so $\lim_{x\to 0} \operatorname{sgn} \sin(1/x)$ does not exist. $\square$

(d) $\lim_{x \to 0} \sqrt{x} \sin(1/x^2)$ where $(x > 0)$.

Proof. We have some $\epsilon > 0$ and $\delta = \epsilon$ such that $|\sqrt{x} - 0| = \delta$. Then,

$$\begin{aligned} |\sqrt{x}\sin(1/x^2) - 0| &= |\sqrt{x} \sin(1/x^2)| \\ &\leq |\sqrt{x}| \\ &\leq \delta \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{x\to 0} \sqrt{x} \sin(1/x^2) = 0$. $\square$