M 361K Homework 3

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5.1

3. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in [b, c]$. Prove that h is continuous on $[a, c]$.

Proof. We have that h is continuous from $[a, b) \cup (b, c]$ from the definition of h and the continuity of f and g. We now need to show that h is continuous at b. Now, $\lim_{x \to b^-} = \lim_{x \to b^-} = f(b)$ and $\lim_{x \to b^+} = \lim_{x \to b^+} = g(b)$.

Because $f(b) = g(b)$, $\lim_{x \to b^-} h(x) = \lim_{x \to b^+} h(x) = h(b)$. Thus, $\lim_{x \to b} h(x) = h(b)$. Therefore, since h is continuous from $[a, b) \cup [b] \cup (b, c]$, h is continuous on $[a, c]$. $\square$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.

Proof. Let $\epsilon = \frac{f(c)}{2} > 0$. Then, let there be some $\delta > 0$ such that there exists some $V_\delta(c) = (c - \delta, c + \delta)$. Then, let $x \in V_\delta(c) = (c - \delta, c + \delta)$. Then,

$$\begin{aligned} x \in (c - \delta, c + \delta) &\implies |f(x) - f(c)| < \epsilon \\ &\implies -\epsilon < f(x) - f(c) < \epsilon \\ &\implies f(c) - \frac{f(c)}{2} < f(x) \\ &\implies f(x) > 0 \end{aligned}$$

Thus, $f(x) > 0$ for all $x \in V_\delta(c)$. $\square$

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find some sequence of rational numbers (x_n) that converges to some x in $\mathbb{R}$. We know that f is continuous at x, so we can say that $(f(x_n))$ converges to $f(x)$ and $f(x_n) = 0 \ \forall \ n \in \mathbb{N}$ because $f(r) = 0 \ \forall \ r \in \mathbb{Q}$. Thus, $f(x) = \lim f(x_n) = \lim(0) = 0$. $\square$

5.2

2. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on A.

Proof. We can do a proof by induction where the base case is $n = 1$. Then, $f^1(x) = f(x)$ which is trivially continuous on A by our original assumption.

Our inductive case is that $f^{n+1} = f^n f$ which is true by Theorem 5.2.1 which states that the product of two continuous functions is continuous. Thus, f^{n+1} is continuous on A. $\square$

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that the sum $f + g$ is continuous at c and the product $f \cdot g$ is continuous at c.

Proof. Let $f(x)$ and $g(x)$ be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x \neq c \end{cases}$$

Then, $f + g$ is continuous at c because $f(c) + g(c) = 1$ and $f \cdot g$ is continuous at c because $f(c) \cdot g(c) = 0$ for all values of x. $\square$

7. Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

Proof. The Density Theorem tells us that for every pair of rational numbers, there exists an irrational number between them, and vice versa. Thus, let f be defined as follows:

$$f(x) = \begin{cases} -1 & \text{if } x = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \\ 1 & \text{if } x \neq \frac{p}{q} \text{ for all } p, q \in \mathbb{N} \end{cases}$$

Then, $|f|$ is 1 everywhere. Thus, f is discontinuous at every point of $[0, 1]$ but $|f|$ is continuous on $[0, 1]$. $\square$

8. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

Proof. The Density Theorem states that there is a irrational number between every pair of rational numbers. Because $f(r) = g(r)$ for the rationals, $f(s) = g(s)$ for all irrational numbers s in order to maintain continuity for f and g using the Density Theorem. Thus, $f(x) = g(x)$ for all $x \in \mathbb{R}$ is true. $\square$

6.1

3. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Prove the following:

(a) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c.

Proof. We can use the definition of the derivative to show this.

$$\begin{aligned}(\alpha f)'(c) &= \lim_{x \to c} \frac{\alpha f(x) - \alpha f(c)}{x - c} \\ &= \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \alpha f'(c)\end{aligned}$$

□

(b) The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.

Proof.

$$\begin{aligned}(f+g)'(c) &= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c)\end{aligned}$$

□

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$.

Proof. Let $g(x) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$ so $g(x) := x$ for x rational and $g(x) := 0$ for x irrational. To show that f is differentiable at 0 we have to show that $\lim_{x \to 0} g(x) = f'(0)$ exists.

We know that $-|x| \leq g(x) \leq |x|$ for all $x \in \mathbb{R}$ and that $\lim_{x \to 0} -|x| = -0$ and $\lim_{x \to 0} |x| = 0$. Thus, $\lim_{x \to 0} g(x) = 0$ by Theorem 3.2.7 (Squeeze Theorem) so f is differentiable at $x = 0$ and $f'(0) = 0$. □

7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Proof. We want to show that if $g(x)$ is differentiable at c, then $f'(c) = 0$. Suppose not. Suppose that $g(x)$ is differentiable at c and $f'(c) \neq 0$. This means that the function $f(c)$ in some neighborhood around c must cross the x-axis at least once in order to satisfy the conditions that $f(c) = 0$ and $f'(c) \neq 0$ using the Intermediate Value Theorem. When we take the absolute value of $f(x)$ to produce $g(x)$ we must reflect the negative portion of $f(x)$ over the x-axis, creating a cusp at c. This means $g(x)$ is not differentiable at c, which is a contradiction. Thus, $f'(c) = 0$.

We want to show that if $f'(c) = 0$, then $g(x)$ is differentiable at c. For a function to be differentiable at a point, the function must be continuous and the slope of the tangent line at the point must equal the limit of the function as x approaches the point. Because $f'(c) = 0$, the slope of the tangent line at c is 0. Because $g(x)$ is continuous at c, the limit of $g(x)$ as x approaches c must also be 0. Thus, $g(x)$ is differentiable at c.

Using both parts of this proof we can show that $g(x)$ is differentiable at c if and only if $f'(c) = 0$. □

9. Prove the following:

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable even function, then the derivative f' is an odd function.

Proof. Because f is even, $f(x) = f(-x)\ \forall\ x \in \mathbb{R}$.

$$\begin{aligned} f'(-c) &= \lim_{x \to -c} \frac{f(x) - f(-c)}{x - (-c)} \\ &= \lim_{x \to c} \frac{f(-x) - f(c)}{-x + c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{-(x - c)} \\ &= -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= -f'(c) \end{aligned}$$

Then, $f'(-c) = -f'(c)\ \forall\ c \in \mathbb{R}$, so f' is odd. □

(b) If $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then the derivative g' is an even function.

Proof. Because g is odd, $g(-x) = -g(x)\ \forall\ x \in \mathbb{R}$.

$$\begin{aligned} g'(-c) &= \lim_{x \to -c} \frac{g(x) - g(-c)}{x - (-c)} \\ &= \lim_{x \to c} \frac{g(-x) - (-g(c))}{-x + c} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to c} \frac{-g(x)+g(c)}{-x+c} \\
&= \lim_{x\to c} \frac{-(g(x)-g(c))}{-(x-c)} \\
&= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= g'(c)
\end{aligned}
$$

Then, $g'(-c) = g'(c)\ \forall\ c \in \mathbb{R}$, so g' is even. $\square$

6.2

5. Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$. *Hint: Show that $f(x) := x^{1/n} - (x-1)^{1/n}$ is decreasing for $x \geq 1$, and evaluate f at 1 and a/b.*

Proof. Let $f(x) := x^{1/n} - (x-1)^{1/n}$. Then, $f'(x) = \frac{1}{n}(x^{\frac{1}{n}-1} - (x-1)^{\frac{1}{n}-1})$. Since $n \geq 2$, $1 - \frac{1}{n} \geq \frac{1}{2}$ which means that $x^{1-\frac{1}{n}} > (x-1)^{1-\frac{1}{n}}$ when $x > 1$. Thus, $f'(x) < 0$ when $x > 1$, so $f(x)$ is decreasing when $x \geq 1$. For $a > b > 0$, we get $\frac{a}{b} > 1$ so that $f(\frac{a}{b}) = (\frac{a}{b})^{1/n} - (\frac{a}{b} - 1)^{1/n} < f(1) = 1$, i.e. $a^{1/n} - b^{1/n} < (a-b)^{1/n}$. $\square$

6. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbb{R}$.

Proof. The Mean Value Theorem states $f'(c) = \frac{f(b)-f(a)}{b-a} \implies f(b) - f(a) = f'(c)(b-a)$. We can apply the sin function here because it is continuous and differentiable. Then,

$$
\begin{aligned}
|\sin x - \sin y| &= |\cos c(x-y)| \\
\frac{|\sin x - \sin y|}{|x-y|} &= |\cos c|
\end{aligned}
$$

Since $\cos c$ is bounded by 1, we get that

$$
\frac{|\sin x - \sin y|}{|x-y|} \leq 1
$$

Therefore, $|\sin x - \sin y| \leq |x-y|$. $\square$

8. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x\to a} f'(x) = A$, then $f'(a)$ exists and equals A. *Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.*

Proof. $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ by the definition of the derivative. Using the Mean Value Theorem, we have that $f'(c) = \frac{f(x)-f(a)}{x-a}$. Then,

$$
\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{c\to a} f'(x) = A
$$

Thus, $f'(a)$ exists and equals A. $\square$

10. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of 0 the derivative $g'(x)$ takes on both positive and negative values. Thus g is not monotonic in any neighborhood of 0.

Proof. The derivative $g'(x) = 4x \sin(1/x) - 2\cos(1/x) + 1$. We can compute $g'(0)$ from the limit definition:

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x + 2x^2 \sin(1/x) - 0}{x - 0} \\ &= \lim_{x\to 0} 1 + 2x \sin(1/x) \\ &= 1 \end{aligned}$$

Then, we can let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(4n+1)\pi}$. This means $g'(x_n) = -1 < 0$ and $g'(y_n)\ 1 + \frac{4}{(4n+1)\frac{\pi}{2}} > 0$. Thus, in every neighborhood of 0 $g'(x)$ takes both positive and negative values so g is not monotonic in any neighborhood of 0. $\square$

13. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.

Proof. Let there be some $x, y \in I$ such that $x < y$. From the Mean Value Theorem we have that $f(y) - f(x) = f'(c)(y - x)$. Since f' is positive on I, $f'(c)(y - x) > 0$. Thus, $f(y) - f(x) > 0$ for all $x, y \in I$ so f is strictly increasing on I. $\square$