

# M 361K: Real Analysis

Ishan Shah

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## Contents

<b>1</b>	<b>August 25</b>	<b>3</b>
1.1	Algebraic Axioms . . . . .	3
1.2	Order Axioms . . . . .	4
<b>2</b>	<b>August 30</b>	<b>6</b>
2.1	Upper and Lower Bounds . . . . .	6
2.2	Completeness Axiom . . . . .	7
<b>3</b>	<b>September 6</b>	<b>8</b>
3.1	Cardinality . . . . .	8
3.2	Countability . . . . .	8
<b>4</b>	<b>September 8</b>	<b>10</b>
4.1	Limits of Sequences . . . . .	10
<b>5</b>	<b>September 13</b>	<b>12</b>
5.1	Monotone Sequences . . . . .	12
<b>6</b>	<b>September 15</b>	<b>13</b>
6.1	Cauchy Sequences . . . . .	13
<b>7</b>	<b>September 20</b>	<b>14</b>
7.1	Empty Set . . . . .	14
7.2	Topology of Real Numbers . . . . .	14
7.2.1	Neighborhoods . . . . .	14
7.2.2	Interior and Boundary Points . . . . .	15
7.3	Closure . . . . .	15
<b>8</b>	<b>September 22</b>	<b>16</b>
8.1	Set Covers . . . . .	16
8.2	Cauchy Convergence . . . . .	16

<b>9</b>	<b>September 27</b>	<b>18</b>
9.1	Limits of Functions . . . . .	18
<b>10</b>	<b>September 29</b>	<b>19</b>
10.1	Sums of Limits . . . . .	19
10.2	Continuity of Functions . . . . .	19
<b>11</b>	<b>October 6</b>	<b>21</b>
11.1	Derivatives . . . . .	21

# 1 August 25

## 1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1)  $a + b = b + a$ .
- (A2)  $(a + b) + c = a + (b + c)$ .
- (A3)  $\exists$  an element  $o \in \mathbb{R}$  such that  $a + o = o + a = a$ .
- (A4) For each element  $a \in \mathbb{R}$ ,  $\exists$  an element  $(-a) \in \mathbb{R}$  such that  $a + (-a) = 0$ .
- (M1)  $ab = ba$ .
- (M2)  $(ab)c = a(bc)$ .
- (M3)  $\exists$  an element  $1 \in \mathbb{R}$  such that  $a * 1 = 1 * a = a$ .
- (M4) For each element  $a \in \mathbb{R} \setminus 0$ ,  $\exists$  an element  $\frac{1}{a} \in \mathbb{R}$  such that  $a * \frac{1}{a} = \frac{1}{a} * a = 1$ .
- (D)  $a * (b + c) = a * b + a * c$ .

*Remark* (Equality property of  $\mathbb{R}$ ). If  $a = b$  and  $c = d$ , then  $a + c = b + d$  and  $a * c = b * d$ .

$\forall x, y, z \in \mathbb{R}$ :

**Theorem 1.1.** If  $x + z = y + z$  then  $x = y$ .

*Proof.*

$$\begin{aligned}x + z &= y + z \quad (A4) \\(x + z) + (-z) &= (y + z) + (-z) \quad (A2) \\x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\x + 0 &= y + 0 \quad (A3) \\x &= y\end{aligned}$$

□

**Theorem 1.2.** For any  $x \in \mathbb{R}$ ,  $x * 0 = 0$ .

*Proof.*

$$\begin{aligned}x * 0 &= x * (0 + 0) \\x * 0 &= x * 0 + x * 0 \\x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\0 &= x * 0 + (x * 0 + (-x * 0)) \\&= x * 0 + 0 \\&= x * 0\end{aligned}$$

□

**Theorem 1.3.**  $-1 * x = -x$  i.e.  $x + (-1) * x = 0$ .

*Proof.*

$$\begin{aligned}
 x + (-1) * x &= x + x * (-1) \\
 &= x * 1 + x * (-1) \\
 &= x * (1 + (-1)) \\
 &= x * 0 \\
 &= 0
 \end{aligned}$$

□

**Theorem 1.4** (Zero-product property).  $\forall x, y \in \mathbb{R}, x * y = 0 \iff x = 0 \vee y = 0$ .

*Proof.* Let  $x, y \in \mathbb{R}$ , if  $x = 0$  or  $y = 0$ , then  $x * y = 0$ . Suppose  $x \neq 0$ , then we must show  $y = 0$ . Since  $x \neq 0$ ,  $\frac{1}{x}$  exists. Thus, if:

$$\begin{aligned}
 xy &= 0 \\
 \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\
 \left(\frac{1}{x} * (xy)\right) * y &= 0 \\
 1 * y &= 0 \\
 y &= 0
 \end{aligned}$$

□

## 1.2 Order Axioms

$\forall x, y \in \mathbb{R}$ :

- (O1) One of  $x < y$ ,  $x > y$  or  $x = y$  is true.
- (O2) If  $x < y$  and  $y < z$ , then  $x < z$ .
- (O3) If  $x < y$  then  $x + z < y + z$ .
- (O4) If  $x < y$  and  $z > 0$  then  $xz < yz$ .

**Theorem 1.5.** If  $x < y$  then  $-y < -x$ .

*Proof.*

$$\begin{aligned}
 x &< y \\
 x + (-x + -y) &< y + (-x + -y) \\
 (x + -x) + -y &< (y + -y) + -x \\
 0 + -y &< 0 + -x \\
 -y &< -x
 \end{aligned}$$

□

**Theorem 1.6.** If  $x < y$  and  $z > 0$  then  $xz > yz$ .

*Proof.* If  $x < y$  and  $z > 0$  then  $-z < 0$ . Thus,  $x(-z) < y(-z)$ . But,

$$\begin{aligned}x(-z) &= x(-1 * z) \\&= (x * -1) * z \\&= (-1 * x) * z \\&= -1(x * z) \\&= -x * z\end{aligned}$$

Similarly,  $y(-z) = -y * z$ . Thus,  $-x * z < -y * z$ , so  $xz > yz$ . □

*Remark* (Completeness of  $\mathbb{R}$ ).  $\mathbb{R}$  is an ordered field.  $\mathbb{R}$  is complete, while  $\mathbb{Q}$  is not complete.

## 2 August 30

**Theorem 2.1.**  $\sqrt{2}$  is irrational.

*Proof.* Suppose not. Suppose that  $\sqrt{2}$  is rational. Then  $\exists m, n \in \mathbb{Z}$  such that  $\sqrt{2} = \frac{m}{n}, n \neq 0$  and  $m$  and  $n$  share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus,  $m^2$  is even and  $m$  is even. Then,  $m = 2k$  for some  $k \in \mathbb{Z}$ . But, by substituting  $m = 2k$  into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus,  $n^2$  is even, so  $n$  is even. So,  $n$  is a perfect square, which is a contradiction. Thus,  $\sqrt{2}$  is irrational.  $\square$

### 2.1 Upper and Lower Bounds

**Theorem:** Let  $S$  be a subset of  $\mathbb{R}$ . If there exists a real number  $m$  such that  $m \geq s \forall s \in S$ ,  $m$  is called an **upper bound** for  $S$ . If  $m \leq s \forall s \in S$ ,  $m$  is called a **lower bound** for  $S$ . **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for  $T$ .

**Definition 2.1** (Supremum). The least upper bound of a set is called the supremum of the set.

**Definition 2.2** (Infimum). The greatest lower bound of a set is called the infimum of the set.

## 2.2 Completeness Axiom

**Definition 2.3** (Completeness axiom). Every nonempty subset of  $\mathbb{R}$  that is bounded above has a least upper bound. That is,  $\sup S$  exists and is a real number.

**Theorem 2.2.** The set of natural numbers  $\mathbb{N}$  is unbounded above.

*Proof.* Suppose not. Suppose that  $\mathbb{N}$  is bounded above. If  $\mathbb{N}$  were bounded above, it must have a supremum  $m$ . Since  $\sup \mathbb{N} = m$ ,  $m - 1$  is not an upper bound. Thus,  $\exists n_0 \in \mathbb{N}$  such that  $n_0 > m - 1$ . But then,  $n_0 + 1 > m$ . This is a contradiction since  $n_0 + 1 \in \mathbb{N}$ . Thus,  $\mathbb{N}$  is unbounded above.  $\square$

**Theorem 2.3.** If  $A$  and  $B$  are nonempty subsets of  $\mathbb{R}$ , let  $C = \{x + y \mid x \in A, y \in B\}$ . If  $\sup A$  and  $\sup B$  exist, then  $\sup C = \sup A + \sup B$ .

*Proof.* Let  $\sup A = a$  and  $\sup B = b$ . Then if  $z \in C$ ,  $z = x + y$  for some  $x \in A, y \in B$ . Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom,  $\exists$  a least upper bound of  $C$ ,  $c = \sup C$ . It must be that  $c \leq a + b$ , so we must show  $c \geq a + b$ . Let  $\epsilon > 0$ . Since  $a = \sup A$ ,  $a - \epsilon$  is not an upper bound for  $A$ .  $\exists x \in A$  such that  $a - \epsilon < x$ . Likewise,  $\exists y \in B$  such that  $b - \epsilon < y$ . Then,

$$(a - \epsilon) + (b - \epsilon) = a + b - 2 * \epsilon < x + y \leq c$$

Thus,  $a + b < c + 2 * \epsilon \forall \epsilon > 0$ . So,  $a + b \leq c \therefore c = a + b$ .  $\square$

## 3 September 6

### 3.1 Cardinality

**Definition 3.1** (Cardinality). The cardinality of a set  $A$  is the number of elements in  $A$ . We denote this as  $|A|$ . We say that two sets  $A$  and  $B$  have the same cardinality if and only if  $\exists$  a bijection  $f : A \rightarrow B$ , or  $|A| = |B|$ .

*Remark.* This bijection holds true because cardinality is reflexive (via the identity function), symmetric (via the inverse function), and transitive (via composition).

*Remark.* The following examples demonstrate how to prove whether two sets have the same cardinality.

- $|\text{even integers}| = |\text{odd integers}|$ :  $f(2n) = 2n + 1$ .
- $|\mathbb{Z}| = |\mathbb{Z}^+|$ :  $f(0) = 1, f(1) = 2, f(-1) = 3, f(2) = 4, \dots$
- $|\mathbb{Q}^+| = |\mathbb{Z}^+|$ : We can create a diagonal mapping by taking  $\frac{n}{m}$  for counting numbers on the rows and columns.
- $|\mathbb{Q}| = |\mathbb{Z}^+|$ :  $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ , so we can repeat the diagonal mapping for  $\mathbb{Q}^-$ . This is because any subset of a countable set is countable.
- $|\mathbb{Q}| \neq |\mathbb{R}|$ : For the real numbers, Cantor's Diagonal Argument proves the sets have different cardinality since no possible surjection exists.

In essence, if we show that there exists some one-to-one mapping between the two sets we can claim that  $|A| = |B|$ .

### 3.2 Countability

**Definition 3.2** (Countable). If a set is finite or has the same cardinality as  $\mathbb{N}$  (i.e.  $\mathbb{Z}^+$ ), we say that the set is countable.

**Theorem 3.1.** Any subset of a countable set is countable.

**Theorem 3.2.** Any set that contains an uncountable set is uncountable.

**Theorem 3.3.** If  $[a_n, b_n] \forall n \in \mathbb{N}$  is a nested sequence of closed bounded intervals,  $\exists \delta \in \mathbb{R}$  such that  $\delta \in I_n \forall n \in \mathbb{N}$ .

*Proof.*  $I_n \subseteq I_1 \forall n \in \mathbb{N}$ . Thus,  $a_n \subseteq b_1 \forall n \in \mathbb{N}$ . So,  $b_n$  is an upper bound for  $\{a_n \mid n \in \mathbb{N}\}$ . Let  $\delta$  be the supremum of  $\{a_n \mid n \in \mathbb{N}\}$ . Thus,  $a_n \leq \delta \forall n \in \mathbb{N}$ .

We have now shown that  $a_n \leq \delta \forall n \in \mathbb{N}$ , and we need to show that  $\delta \leq b_n \forall n \in \mathbb{N}$ . This is left as an exercise for the reader.  $\square$

*Remark.* A nested sequence means that successive subsets contain the previous subset. For example,  $[0, 1] \subseteq [0, 2] \subseteq [0, 3] \subseteq \dots$  is a nested sequence.



**Theorem 3.4.**  $[0, 1]$  is uncountable.

*Proof.* Assume  $[0, 1]$  is countable. That is,  $[0, 1] = I = \{x_1, x_2, x_3, \dots\}$ . Select a closed interval  $I_1 \subseteq I$  such that  $x_1 \notin I_1$ . Next, select a closed interval  $I_2 \subseteq I_1$  such that  $x_2 \notin I_2$ , and so on. Then, we have

$$I_n \subseteq \dots \subseteq I_2 \subseteq I_1 \subseteq I$$

and  $x_n \notin I_n \forall n \in \mathbb{N}$ . By **Theorem 3.3**,  $\exists \delta \in I$  such that  $\delta \in I_n \forall n \in \mathbb{N}$ . This implies that  $\delta \neq x_n \forall n \in \mathbb{N}$ . Thus,  $\delta \notin I$ , which is a contradiction. Therefore,  $[0, 1]$  is uncountable.  $\square$

## 4 September 8

### 4.1 Limits of Sequences

**Definition 4.1** (Limit of a sequence). A sequence  $a_n$  is said to converge to a real number  $s$ , if for any  $\epsilon > 0$ ,  $\exists$  a real number  $k$  such that for all  $n \geq k$ , the terms  $a_n$  satisfy  $|a_n - s| < \epsilon$ .

**Theorem 4.1.**  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ .

*Proof.* We need to find some  $N$  such that  $n > N \forall \epsilon > 0$ .

$$\begin{aligned} \left| \frac{1}{\sqrt{n}} - 0 \right| &< \epsilon \\ \frac{1}{\sqrt{n}} &< \epsilon \\ \frac{1}{n} &< \epsilon^2 \\ n &> \frac{1}{\epsilon^2} \end{aligned}$$

Let  $\epsilon > 0$  and  $N = \frac{1}{\epsilon^2}$ . Then, if  $n > N$ , we have that

$$\begin{aligned} \left| \frac{1}{\sqrt{n}} - 0 \right| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} \\ &= \epsilon \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ . □

**Theorem 4.2.**  $\lim_{n \rightarrow \infty} 1 + \frac{1}{2^n} = 1$ .

*Proof.* Let  $\epsilon > 0$  and  $N = \frac{1}{\epsilon}$ . Then, we have

$$\begin{aligned} \left| 1 + \frac{1}{2^n} - 1 \right| &< \epsilon \\ \left| \frac{1}{2^n} \right| &= \frac{1}{2^n} < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} < \epsilon \\ n &> \frac{1}{\epsilon} \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} 1 + \frac{1}{2^n} = 1$ . □

**Theorem 4.3.** Every convergent sequence is bounded.

*Proof.* Let  $S_n$  be a convergent sequence with a limit  $s$  and  $\epsilon = 1$ . Then, there exists some  $N$  such that  $|S_n - s| < 1$  when  $n > N$ . That is,  $|S_n| < |s| + 1$ .

Let  $M = \max\{S_1, S_2, \dots, S_N, |s| + 1\}$ . Then,  $|S_n| \leq M$ , so  $S_n$  is bounded. □

**Theorem 4.4.** If a sequence converges, its limit is unique.

*Proof.* Suppose a sequence  $S_n$  converges to  $s$  and  $t$ . Let  $\epsilon > 0$ . Then,  $\exists N_1$  such that  $|S_n - s| < \frac{\epsilon}{2}$ . For  $n > N_1$ ,  $\exists N_2$  such that  $|S_n - t| < \frac{\epsilon}{2}$ . For  $n > N_2$ , let  $N = \max\{N_1, N_2\}$ . Then, for  $n > N$ , we have

$$\begin{aligned} |s - t| &= |s + S_n - S_n - t| \\ &= |s - S_n + S_n - t| \\ &\leq |s - S_n| + |S_n - t| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, the limit is unique. □

## 5 September 13

### 5.1 Monotone Sequences

**Definition 5.1** (Monotone sequence). A sequence  $S_n$  of real numbers is said to be increasing  $\iff S_n \leq S_{n+1} \forall n \in \mathbb{N}$  and decreasing  $\iff S_n \geq S_{n+1} \forall n \in \mathbb{N}$ .

*Remark.* The Fibonacci sequence is an example of an increasing sequence.

**Definition 5.2** (Monotone convergence theorem). A monotone sequence is convergent if and only if it is bounded.

**Theorem 5.1.** An increasing bounded sequence is convergent.

*Proof.* Suppose  $S_n$  is a bounded increasing sequence. Let  $S$  be the set  $\{S_n \mid n \in \mathbb{N}\}$ . By the completeness axiom,  $\sup S$  exists. Let  $s = \sup S$ . We claim  $\lim_{n \rightarrow \infty} S_n = s$ . Given  $\epsilon > 0$ ,  $s - \epsilon$  is not an upper bound for  $S$ .

Thus,  $\exists N \in \mathbb{N}$  such that  $S_N > s - \epsilon$ . Furthermore, since  $S_n$  is increasing and  $s$  is an upper bound for  $S$ , we have  $s - \epsilon < S_N \leq S_n \leq s \forall n \geq N$ .  $\square$

*Remark.* This is an elementary proof because it only uses axioms to make the conclusion.

Ex.  $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$ .

**Theorem 5.2.** If  $S_n$  is an unbounded increasing sequence, then  $\lim_{n \rightarrow \infty} S_n = \infty$ .

*Proof.* Let  $S_n$  be an increasing unbounded sequence. Then,  $\{S_n \mid n \in \mathbb{N}\}$  is not bounded above, but  $S$  is bounded below by  $S_1$ . Thus, given  $M \in \mathbb{R}, \exists N \in \mathbb{N}$  such that  $S_N > M$ . But since  $S_n$  is increasing,  $S_n > M \forall n > N$ . Thus,  $\lim_{n \rightarrow \infty} S_n = \infty$ .  $\square$

## 6 September 15

### 6.1 Cauchy Sequences

**Definition 6.1** (Cauchy sequence). A sequence of real numbers  $S_n$  is called a Cauchy sequence if and only if for each  $\epsilon > 0$ ,  $\exists N$  such that  $m, n > N \implies |S_m - S_n| < \epsilon$ .

*Remark.* This means the elements of the sequence get closer to each other as  $N$  increases.

**Theorem 6.1.** Every convergent sequence is Cauchy.

*Proof.* Let  $S_n$  be a convergent sequence. Then  $\exists N$  such that  $n > N \implies |S_n - s| < \frac{\epsilon}{2}$  for some  $s \in \mathbb{R}$ . Then, for  $n, m > N$ , we have

$$\begin{aligned} |S_n - S_m| &= |S_n - s + s - S_m| \\ &\leq |S_n - s| + |s - S_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus,  $S_n$  is Cauchy. □

**Theorem 6.2.** A sequence of real numbers is Cauchy if and only if it is convergent.

*Remark.* We cannot prove this yet.

## 7 September 20

### 7.1 Empty Set

**Theorem 7.1.** The empty set is a subset of any set.

*Proof.* Suppose not. That is, suppose  $\exists A$  such that  $\emptyset \not\subset A$ . Thus,  $\exists x \in \emptyset$  such that  $x \notin A$ . This is a contradiction because the empty set has no elements. Therefore,  $\emptyset \subset A$ .  $\square$

**Theorem 7.2.** There is only one set with no elements.

*Proof.* Suppose not. That is, suppose  $\exists$  two empty sets  $E_1, E_2$ . Then  $E_1 \subseteq E_2$  and  $E_2 \subseteq E_1$ . Thus,  $E_1 = E_2$ . This is a contradiction because  $E_1$  and  $E_2$  are two different sets. Therefore, there is only one empty set.  $\square$

*Remark* (Closedness of  $\emptyset$ ). The empty set is open and closed (vacuously true).

### 7.2 Topology of Real Numbers

Let  $S \subseteq \mathbb{R}$  for the following definitions.

#### 7.2.1 Neighborhoods

**Definition 7.1** (Neighborhood). A neighborhood of  $x$  in  $S$  can be thought of an epsilon-sized ball around  $x$ , i.e.  $N(x, \epsilon) = \{y \in \mathbb{R} \mid 0 \leq |x - y| < \epsilon\}$ .

**Definition 7.2** (Deleted neighborhood). A deleted neighborhood is the same as a neighborhood except that  $x$  is not included, i.e.  $N^*(x, \epsilon) = \{y \in \mathbb{R} \mid 0 < |x - y| < \epsilon\}$ .

**Definition 7.3** (Accumulation point).  $x \in \mathbb{R}$  is an accumulation point of  $S$  if and only if every deleted neighborhood of  $x$  contains a point of  $S$ .

*Remark.*  $(0, \infty)$  has accumulation points  $[0, \infty)$ .  $(0, 1)$  does not contain all of its accumulation points since 0 and 1 are both accumulation points of the set.

**Theorem 7.3.**  $S \subseteq \mathbb{R}$  is closed if and only if  $S$  contains all of its accumulation points.

*Proof.* Suppose  $S$  is closed. Let  $x$  be an accumulation point of  $S$ . If  $x \notin S$ , then  $x \in S^c$ . Thus,  $\exists$  a neighborhood  $N$  of  $x$  such that  $N \subseteq S^c$ . But  $N \cap S = \emptyset$ , which contradicts  $x$  being an accumulation point of  $S$ .

Conversely, suppose  $S$  contains all of its accumulation points. Let  $x \in S^c$ , then  $x$  is not an accumulation point of  $S$ . Thus,  $\exists N^*(x, \epsilon)$  that misses  $S$ . Since  $x \notin S$ ,  $N(x, \epsilon)$  misses  $S$ . Therefore,  $S^c$  is open, which means  $S$  is closed.  $\square$

**Theorem 7.4.** If  $S$  is a nonempty closed bounded subset of  $\mathbb{R}$ , then  $S$  has a max.

*Proof.* Let  $s = \sup S$ . Then,  $s$  is an accumulation point of  $S$ . Since  $S$  is closed,  $s \in S$ . Thus,  $s$  is a max of  $S$ .  $\square$

### 7.2.2 Interior and Boundary Points

**Definition 7.4** (Interior point).  $x \in S$  is an interior point of  $S$  if and only if  $\exists N(x, t)$  such that  $N(x, t) \subseteq S$ .

**Definition 7.5** (Boundary point).  $x \in S$  is a boundary point of  $S$  if and only if every neighborhood  $N$  of  $x$  has  $N \cap S \neq \emptyset$  and  $N \cap S^c \neq \emptyset$ .

### 7.3 Closure

**Definition 7.6** (Open set).  $S$  is an open set if and only if every point in  $S$  is an interior point of  $S$ .  $\forall x \in S, \exists$  a neighborhood  $N(x, \epsilon)$  for some  $\epsilon > 0$  such that  $N(x, \epsilon) \subseteq S$ .

**Definition 7.7** (Closed set).  $S$  is a closed set if and only if  $S$  contains at least one of its boundary points. Additionally,  $S^c$  must be an open set.

*Remark* (Closure of  $\mathbb{R}$ ).  $\mathbb{R}$  is open because all of its points are interior points.  $\mathbb{R}$  is also closed because  $\mathbb{R}$  has no boundary points, therefore implying that it contains at least one of its boundary points (vacuously true).

**Theorem 7.5.** The union of two open sets is open.

*Proof.* Let  $A$  and  $B$  be open sets. Let  $x \in A \cup B$ . Then  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $\exists$  a neighborhood  $N_1$  of  $x$  such that  $N_1 \subseteq A$ . But then,  $N_1 \subseteq A \cup B$ . If  $x \in B$ , then  $\exists$  a neighborhood  $N_2$  of  $x$  such that  $N_2 \subseteq B$ . But then,  $N_2 \subseteq A \cup B$ .

Thus, in either case,  $\exists$  a neighborhood  $N$  of  $x$  such that  $N \subseteq A \cup B$ . Therefore,  $A \cup B$  is open.  $\square$

**Theorem 7.6.** An arbitrary union of open sets is open.

*Proof.* Let  $A_1, A_2, \dots, A_n$  be open sets. Let  $x \in \bigcup_{i=1}^n A_i$ . Then  $x \in A_i$  for some  $i$ . Let  $N_i$  be a neighborhood of  $x$  such that  $N_i \subseteq A_i$ . Then  $N_i \subseteq A_i \subseteq \bigcup_{i=1}^n A_i$ . Therefore,  $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$ .

Thus,  $\bigcup_{i=1}^n N_i$  is a neighborhood of  $x$  such that  $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$ . Therefore,  $\bigcup_{i=1}^n A_i$  is open.  $\square$

**Theorem 7.7.** The intersection of two open sets is open.

*Proof.* Let  $A$  and  $B$  be open sets. Let  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B$ . Thus,  $\exists$  neighborhoods  $N_1(x, \epsilon_1)$  and  $N_2(x, \epsilon_2)$ . Let  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ . Then  $N_1(x, \epsilon) \subseteq A$  and  $N_2(x, \epsilon) \subseteq B$ .

Thus,  $N(x, \epsilon) \subseteq A \cap B$ . Therefore,  $A \cap B$  is open.  $\square$

**Theorem 7.8.** A finite intersection of open sets is open.

*Proof.* Let  $A_1, A_2, \dots, A_n$  be open sets. Let  $x \in \bigcap_{i=1}^n A_i$ . Then  $x \in A_i$  for all  $i$ . Let  $N_i$  be a neighborhood of  $x$  such that  $N_i \subseteq A_i$ . Then  $N_i \subseteq A_i \subseteq \bigcap_{i=1}^n A_i$ . Therefore,  $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$ .

Thus,  $\bigcap_{i=1}^n N_i$  is a neighborhood of  $x$  such that  $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$ . Therefore,  $\bigcap_{i=1}^n A_i$  is open.  $\square$

**Theorem 7.9.** An arbitrary intersection of open sets is open.

*Remark* (Counterexample).  $\bigcap_{i=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \emptyset$ .

## 8 September 22

### 8.1 Set Covers

**Definition 8.1** (Open cover). An open cover  $F$  of some subset  $S \subseteq \mathbb{R}$  is a collection of open sets whose union contains  $S$ .

*Remark.* If  $E \subseteq F$  and  $E$  also covers  $S$ , we call  $E$  a **subcover**.

**Definition 8.2** (Compact). A set  $S$  is said to be compact if and only if whenever  $S$  is contained in the union of a family  $F$  of open sets, then it is contained in a finite number of the sets in  $F$  (every open cover has a finite subcover).

*Remark.* It is hard to show that a set is compact since we have to consider *every* open cover.

**Theorem 8.1** (Heine-Borel). A subset  $S$  of  $\mathbb{R}$  is compact if and only if  $S$  is closed and bounded.

*Proof.* Let  $S$  be a compact set. Observe the open cover  $(-n, n) \forall n \in \mathbb{N}$ . Since  $S$  is compact,  $\exists$  a finite subcover  $(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k)$ .  $\exists$  one of these sets such that  $\bigcup_{i=1}^k (-n_i, n_i) = (-n_m, n_m)$  for some  $m = 1, 2, \dots, k$ . Thus,  $S \subseteq (-n_m, n_m)$ , so  $S$  is bounded.

Let  $S$  be a compact set. Suppose  $S$  is not closed. Let  $p$  be a boundary point of  $S$ , and let  $U_n = \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}] \forall n \in \mathbb{N}$ .  $S \subseteq \bigcup U_n = \mathbb{R} \setminus p$ .  $\exists$  a finite subcover  $U_{n_1}, U_{n_2}, \dots, U_{n_k}$  such that  $S \subseteq \bigcup_{i=1}^k U_{n_i}$ .  $\exists k$  such that  $S \subseteq U_{n_k}$ . But, this is a contradiction with  $p$  being a boundary point. Therefore,  $S$  is closed.

The proof in the other direction is similar, yet non-trivial.  $\square$

**Theorem 8.2** (Bolzano-Weierstrass). If a bounded subset  $S$  of  $\mathbb{R}$  contains infinitely many points, then  $\exists$  at least one accumulation point of  $S$ .

*Proof.* Let  $S$  be a bounded infinite subset of  $\mathbb{R}$ . Suppose  $S$  has no accumulation points, then  $S$  is closed. By Heine-Borel,  $S$  must be compact. Define neighborhoods  $N_x$  such that  $N_x(x) \cap S = \{x\} \forall x \in S$ . Clearly,  $S \subseteq \bigcup_x N_x$ . But, the collection of all  $N_x$  must contain a finite subcover. That is,

$$S \subseteq N_{x_1} \cup N_{x_2} \cup \dots \cup N_{x_k}$$

for some  $k \in \mathbb{N}$ . This contradicts that  $S$  is infinite. Therefore,  $S$  has an accumulation point.  $\square$

### 8.2 Cauchy Convergence

**Theorem 8.3.** Every Cauchy sequence is convergent.

*Proof.*  $S_n$  is Cauchy, so  $S = \{S_n \mid n \in \mathbb{N}\}$ . By Bolzano-Weierstrass,  $\exists$  an accumulation point  $s$  of  $S$ . We claim that  $S_n \rightarrow s$ . Given  $\epsilon > 0$ ,  $\exists N$  such that  $m, n > N$ . Then  $|S_m - S_n| < \frac{\epsilon}{2}$ .  $(S - \frac{\epsilon}{2}, S + \frac{\epsilon}{2})$  contains an infinite number of points.

$\exists m > N$  such that  $S_m \in N(s, \frac{\epsilon}{2})$ . But then,  $|S_n - s| = |S_n - S_m + S_m - s| \leq |S_n - S_m| + |S_m - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ . Therefore,  $S_n \rightarrow s$ .  $\square$



**Theorem 8.4.** Let  $x_n$  be a sequence of non-negative real numbers.  $\sum x_n$  converges if  $S_k$ , the sequence of partial sums is bounded.

*Proof.*  $\sum_{n=1}^{\infty} x_n = \lim_{k \rightarrow \infty} S_k$ .  $S_k$  is increasing and bounded, it is convergent by the monotone convergence theorem.  $\square$

## 9 September 27

### 9.1 Limits of Functions

**Definition 9.1** (Limit of a function). Let  $f : D \rightarrow \mathbb{R}$  and let  $c$  be an accumulation point of the function. Then,  $\lim_{x \rightarrow c} f(x) = L$  if and only if given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that if  $|x - c| < \delta$ , then  $|f(x) - L| < \epsilon$ .

*Remark.* Suppose we want to show that  $\lim_{x \rightarrow 2} S_x + 1 = 11$ . We are looking for some  $\delta > 0$  such that  $0 \leq |x - 2| < \delta$  and  $|S_x + 1 - 11| < \epsilon$ . This is structured similarly to proofs of limits of sequences.

Additionally, the limit must go to an accumulation point of the function because we cannot find the limit of a value outside the function's domain.

**Theorem 9.1.**  $\lim_{x \rightarrow 5} 10x + 2 = 52$ .

*Proof.* We need to find some  $\delta > 0$  such that whenever  $0 < |x - 5| < \delta$ ,  $|10x + 2 - 52| < \epsilon$ .

$$\begin{aligned} |10x - 50| &< \epsilon \\ 10|x - 5| &< \epsilon \\ |x - 5| &< \frac{\epsilon}{10} \end{aligned}$$

Given  $\epsilon > 0$ , let  $\delta = \frac{\epsilon}{10}$ . Then, whenever  $0 < |x - 5| < \delta$ , we have  $|10x + 2 - 52| = |10x - 50| = 10|x - 5| < 10 * \frac{\epsilon}{10} = \epsilon$ .  $\square$

**Theorem 9.2.**  $\lim_{x \rightarrow 3} x^2 + 2x + 6 = 21$ .

*Proof.* We need to find some  $\delta > 0$  such that whenever  $0 < |x - 3| < \delta$ ,  $|(x^2 + 2x + 6) - 21| < \epsilon$ .

$$\begin{aligned} |x^2 + 2x + 6 - 21| &< \epsilon \\ |x^2 + 2x - 15| &< \epsilon \\ |x + 5||x - 3| &< \epsilon \end{aligned}$$

If  $\delta < 1 \implies |x + 5||x - 3| < 9|x - 3| < \epsilon$ . Thus  $|x - 3| < \frac{\epsilon}{9}$ . We let  $\delta = \min\{1, \frac{\epsilon}{9}\}$ .

Given  $\epsilon > 0$ , let  $\delta = \min\{1, \frac{\epsilon}{9}\}$ . Then, whenever  $0 < |x - 3| < \delta$ , we have that  $|x + 5| < 9$ , thus,  $|(x^2 + 2x + 6) - 21| = |x^2 + 2x - 15| = |x + 5||x - 3| < \min\{1, \frac{\epsilon}{9}\} * \frac{\epsilon}{9} = \epsilon$ .  $\square$

*Remark.* These proofs have two phases. First, we determine some  $\delta$  as an upper bound. Then, we show how this choice of  $\delta$  implies the limit is bounded by some  $\epsilon$ .

**Theorem 9.3.** Let  $f : D \rightarrow \mathbb{R}$  and  $c$  is an accumulation point of  $D$ . Then,  $\lim_{x \rightarrow c} f(x) = L$  if and only if for every sequence  $S_n \in D$  such that  $S_n \rightarrow c$ ,  $S_n \neq c \forall n$ , then  $f(S_n)$  converges to  $L$ .

*Proof.*  $\lim_{x \rightarrow c} f(x) = L$  and  $S_n \rightarrow L \implies f(S_n) \rightarrow L$ . We need to find  $N$  such that  $n > N$  and  $|f(S_n) - L| < \epsilon$ . We know that  $\exists \delta$  such that  $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$  and  $\exists N$  such that  $n > N \implies |S_n - c| < \delta$ . Thus, for  $n > N$  we have  $|f(S_n) - L| < \epsilon$ .

Suppose  $L$  is not the limit of  $f$  as  $x$  approaches  $c$ . We must find  $(S_n)$  that converges to  $c$ , but  $f(S_n)$  does not converge to  $L$  (contrapositive).  $\exists \epsilon > 0$  such that  $\forall \delta > 0, 0 < |x - c| < \delta \implies |f(x) - L| \geq \epsilon$ . For each  $n \in \mathbb{N}$ ,  $\exists S_n \in D$  such that  $0 < |S_n - c| < \frac{1}{n}$  and  $|f(S_n) - L| \geq \epsilon$ . Then,  $S_n \rightarrow c$ , but  $f(S_n) \not\rightarrow L$ . This is a contradiction.  $\square$

## 10 September 29

### 10.1 Sums of Limits

**Theorem 10.1.** Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $\lim_{x \rightarrow c} g(x) = M$ . Then,  $\lim_{x \rightarrow c} (f + g)(x) = L + M$ .

*Proof (Definition 9.1).* Given  $\epsilon > 0$ , let  $\delta_1 > 0$  be such that  $0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\epsilon}{2}$ . Let  $\delta_2 > 0$  be such that  $0 < |x - c| < \delta_2 \implies |g(x) - M| < \frac{\epsilon}{2}$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then, for  $0 < |x - c| < \delta$ , we have

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

*Proof (Theorem 9.3).* Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $\lim_{x \rightarrow c} g(x) = M$ , and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} (f + g)(S_n) = \lim_{n \rightarrow \infty} f(S_n) + g(S_n) = \lim_{n \rightarrow \infty} f(S_n) + \lim_{n \rightarrow \infty} g(S_n) = L + M$$

Thus,  $\lim_{x \rightarrow c} (f + g)(x) = L + M$ .

□

*Remark.* This is true for  $-$ ,  $\times$ , and  $\div$  as well.

**Definition 10.1** (Sequential criterion for functional limits).  $\lim_{x \rightarrow c} f(x) = L$  if and only if whenever  $S_n \rightarrow c$ ,  $\lim_{n \rightarrow \infty} f(S_n) = L$ .

**Theorem 10.2.** Let  $k \in \mathbb{R}$ . If  $\lim_{x \rightarrow c} f(x) = L$ , then  $\lim_{x \rightarrow c} kf(x) = kL$ .

*Proof.* Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $k \in \mathbb{R}$ , and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} kf(S_n) = k \lim_{n \rightarrow \infty} f(S_n) = kL$$

Thus,  $\lim_{x \rightarrow c} kf(x) = kL$ .

□

### 10.2 Continuity of Functions

**Definition 10.2** (Continuous Function). A function  $f$  is continuous at  $x = c$  if and only if  $\lim_{x \rightarrow c} f(x) = f(c)$ . Let  $s$  be an accumulation point of the domain  $f : D \rightarrow \mathbb{R}$ . Then,  $f$  is continuous at  $s$  if and only if for each  $\epsilon > 0$ ,  $\exists \delta > 0$  such that whenever  $0 < |x - s| < \delta$ ,  $|f(x) - f(s)| < \epsilon$ .

*Remark.* Let  $f(x) = x \sin(\frac{1}{x})$  where  $x \neq 0$ ,  $f(0) = 0$ . If we want to show that this function is continuous, we need to find some  $\delta > 0$  such that  $|x| < \delta \implies |f(x) - f(0)| < \epsilon$ . Let  $\delta = \epsilon$ , then when  $|x| < \delta$ ,  $|f(x) - f(0)| = |x \sin(\frac{1}{x}) - 0| = |x \sin(\frac{1}{x})| \leq |x| < \epsilon$ .

**Theorem 10.3.** If  $f$  and  $g$  are continuous at  $x = c$ , then  $f + g$  is also continuous at  $x = c$ .

*Proof.* Let  $f$  and  $g$  be continuous at  $c$  and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} (f + g)(S_n) = \lim_{n \rightarrow \infty} f(S_n) + \lim_{n \rightarrow \infty} g(S_n) = f(c) + g(c)$$

Thus,  $\lim_{x \rightarrow c} (f + g)(x) = (f + g)(c)$ . □

**Theorem 10.4.** Let  $f : D \rightarrow E$  be continuous at  $x = c$  and let  $g : E \rightarrow R$  be continuous at  $x = f(c)$ . Then, the composition  $g \circ f$  is continuous at  $x = c$ .

*Proof.* This is left as an exercise for the reader. □

# 11 October 6

## 11.1 Derivatives

**Definition 11.1** (Derivative). Let  $f$  be a real-valued function defined on an open interval containing  $c$ . We say  $f$  is differentiable at  $c$  if  $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$  exists. We call this limit  $f'(c)$ .

**Theorem 11.1.** If  $f$  is differentiable at  $c$ , then  $f$  is continuous at  $c$ .

*Proof.* Let  $f$  be defined on some interval  $I$  containing  $c$ . Then if  $f$  is differentiable at  $c$ , if and only if for  $x \neq c$ ,

$$f(x) = (x - c) \frac{f(x) - f(c)}{x - c} + f(c)$$

Then,  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - c) \frac{f(x) - f(c)}{x - c} + f(c) = \lim_{x \rightarrow c} (x - c) f'(c) + f(c) = f(c)$ . Therefore,  $f$  is continuous at  $c$ .  $\square$

### Derivative Rules

- $\frac{d}{dx} kf = k \frac{df}{dx}$
- $\frac{d}{dx} f + g = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx} f \cdot g = \frac{df}{dx} g + \frac{dg}{dx} f$
- $\frac{d}{dx} \frac{f}{g} = \frac{\frac{df}{dx} g - \frac{dg}{dx} f}{g^2}$

**Theorem 11.2** (Product rule).

$$(fg)' = f'g + fg'$$

*Proof.* Suppose  $f$  and  $g$  are differentiable at  $c$ . Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)(g(x) - g(c))}{x - c} + \frac{g(c)(f(x) - f(c))}{x - c} \\ &= f(c)g'(c) + g(c)f'(c) \end{aligned}$$

$\square$

**Theorem 11.3** (Quotient rule).

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

*Proof.* Let  $f$  and  $g$  be differentiable at  $c$ . Then,

$$\begin{aligned}
\lim_{x \rightarrow c} \frac{\frac{f}{g}(x) - \frac{f}{g}(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x \rightarrow c} \frac{\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}}{x - c} \\
&= \lim_{x \rightarrow c} \frac{f(x)g(c) - g(c)f(c) + g(c)f(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\
&= \lim_{x \rightarrow c} \frac{g(c)\frac{f(x)-f(c)}{(x-c)} + f(c)\frac{g(x)-g(c)}{(x-c)}}{g(c)g(x)} \\
&= \lim_{x \rightarrow c} \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)} Aa
\end{aligned}$$

□

**Theorem 11.4** (Power rule).

$$(x^n)' = nx^{n-1}f' \quad \forall n \in \mathbb{N}$$

*Proof by induction.*  $p(n) = (x^n)' = nx^{n-1}f'$ .

$$p(1): f(x) = x. \quad \lim_{x \rightarrow c} \frac{x-c}{x-c} = 1 = 1 \cdot x^0.$$

$p(k) \rightarrow p(k+1):$

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\
&= \left(\frac{d}{dx}x^k\right) \cdot x + x^k \left(\frac{d}{dx}x\right) \\
&= kx^{k-1} \cdot x + x^k \cdot 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

□

**Theorem 11.5** (Chain rule).

$$g(f(x))' = g'(f(x))f'(x)$$

*Proof.*

$$\begin{aligned}
\lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \frac{f(x) - f(c)}{x - c} \\
&= g'(f(x))f'(x)
\end{aligned}$$

□

*Remark.* This will not hold if  $f(x) = f(c)$ .

**Theorem 11.6.** Let  $f$  be defined on an interval  $I$  containing  $c$ . Then,  $f$  is differentiable at  $c$  if and only if  $\exists$  a function  $\varphi$  on  $I$  such that  $\varphi$  is continuous at  $c$  and

$$f(x) - f(c) = \varphi(x)(x - c) \forall x \neq c$$

In this case, we have  $\varphi(c) = f'(c)$ .

*Remark.* Let  $f(x) = x^3$ . Then,  $f(x) - f(c) = x^3 - c^3 = (x^2 + xc + c^2)(x - c)$ .  $\phi(c) = c^2 + c \cdot c + c^2 = 3c^2 = f'(c)$ .