M 361K: Real Analysis

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1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1) $a + b = b + a$.
- (A2) $(a + b) + c = a + (b + c)$.
- (A3) $\exists$ an element $o \in \mathbb{R}$ such that $a + o = o + a = a$.
- (A4) For each element $a \in \mathbb{R}$, $\exists$ an element $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.
- (M1) $ab = ba$.
- (M2) $(ab)c = a(bc)$.
- (M3) $\exists$ an element $1 \in \mathbb{R}$ such that $a * 1 = 1 * a = a$.
- (M4) For each element $a \in \mathbb{R} \setminus 0$, $\exists$ an element $\frac{1}{a} \in \mathbb{R}$ such that $a * \frac{1}{a} = \frac{1}{a} * a = 1$.

- (D) $a * (b + c) = a * b + a * c$.

Remark (Equality property of $\mathbb{R}$). If $a = b$ and $c = d$, then $a + c = b + d$ and $a * c = b * d$.

$\forall x, y, z \in \mathbb{R}$:

Theorem 1.1. If $x + z = y + z$ then $x = y$.

Proof.

$$\begin{aligned} x + z &= y + z \quad (A4) \\ (x + z) + (-z) &= (y + z) + (-z) \quad (A2) \\ x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\ x + 0 &= y + 0 \quad (A3) \\ x &= y \end{aligned}$$

$\square$

Theorem 1.2. For any $x \in \mathbb{R}$, $x * 0 = 0$.

Proof.

$$\begin{aligned} x * 0 &= x * (0 + 0) \\ x * 0 &= x * 0 + x * 0 \\ x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\ 0 &= x * 0 + (x * 0 + (-x * 0)) \\ &= x * 0 + 0 \\ &= x * 0 \end{aligned}$$

$\square$

Theorem 1.3. $-1 * x = -x$ i.e. $x + (-1) * x = 0$.

Proof.

$$\begin{aligned} x + (-1) * x &= x + x * (-1) \\ &= x * 1 + x * (-1) \\ &= x * (1 + (-1)) \\ &= x * 0 \\ &= 0 \end{aligned}$$

$\square$

Theorem 1.4 (Zero-product property)**.** $\forall x, y \in \mathbb{R}$, $x * y = 0 \iff x = 0 \lor y = 0$.

Proof. Let $x, y \in \mathbb{R}$, if $x = 0$ or $y = 0$, then $x * y = 0$. Suppose $x \neq 0$, then we must show $y = 0$. Since $x \neq 0$, $\frac{1}{x}$ exists. Thus, if:

$$\begin{aligned} xy &= 0 \\ \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\ (\frac{1}{x} * (xy)) * y &= 0 \\ 1 * y &= 0 \\ y &= 0 \end{aligned}$$

□

1.2 Order Axioms

$\forall x, y \in \mathbb{R}$:

- (O1) One of $x < y$, $x > y$ or $x = y$ is true.
- (O2) If $x < y$ and $y < z$, then $x < z$.
- (O3) If $x < y$ then $x + z < y + z$.
- (O4) If $x < y$ and $z > 0$ then $xz < yz$.

Theorem 1.5. If $x < y$ then $-y < -x$.

Proof.

$$\begin{gathered} x < y \\ x + (-x + -y) < y + (-x + -y) \\ (x + -x) + -y < (y + -y) + -x \\ 0 + -y < 0 + -x \\ -y < -x \end{gathered}$$

□

Theorem 1.6. If $x < y$ and $z > 0$ then $xz > yz$.

Proof. If $x < y$ and $z > 0$ then $-z > 0$. Thus, $x(-z) < y(-z)$. But,

$$\begin{aligned} x(-z) &= x(-1 * z) \\ &= (x * -1) * z \\ &= (-1 * x) * z \\ &= -1(x * z) \\ &= -x * z \end{aligned}$$

Similarly, $y(-z) = -y * z$. Thus, $-x * z < -y * z$, so $xz > yz$. □

Remark (Completeness of $\mathbb{R}$). $\mathbb{R}$ is an ordered field. $\mathbb{R}$ is complete, while $\mathbb{Q}$ is not complete.

2 August 30

Theorem 2.1. $\sqrt{2}$ is irrational.

Proof. Suppose not. Suppose that $\sqrt{2}$ is rational. Then $\exists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}, n \neq 0$ and m and n share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus, m^2 is even and m is even. Then, $m = 2k$ for some $k \in \mathbb{Z}$. But, by substituting $m = 2k$ into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus, n^2 is even, so n is even. So, n is a perfect square, which is a contradiction. Thus, $\sqrt{2}$ is irrational. $\square$

2.1 Upper and Lower Bounds

Theorem: Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s \forall s \in S$, m is called an **upper bound** for S. If $m \leq s \forall s \in S$, m is called a **lower bound** for S. **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for T.

Definition 2.1 (Supremum)**.** The least upper bound of a set is called the supremum of the set.

Definition 2.2 (Infimum)**.** The greatest lower bound of a set is called the infimum of the set.

2.2 Completeness Axiom

Definition 2.3 (Completeness axiom)**.** Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 2.2. The set of natural numbers $\mathbb{N}$ is unbounded above.

Proof. Suppose not. Suppose that $\mathbb{N}$ is bounded above. If $\mathbb{N}$ were bounded above, it must have a supremum m. Since $\sup \mathbb{N} = m$, $m - 1$ is not an upper bound. Thus, $\exists n_0 \in \mathbb{N}$ such that $n_0 > m - 1$. But then, $n_0 + 1 > m$. This is a contradiction since $n_0 + 1 \in N$. Thus, N is unbounded above. □

Theorem 2.3. If A and B are nonempty subsets of $\mathbb{R}$, let $C = \{x + y \mid x \in A, y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C = \sup A + \sup B$.

Proof. Let $\sup A = a$ and $\sup B = b$. Then if $z \in C, z = x + y$ for some $x \in A, y \in B$. Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom, $\exists$ a least upper bound of $C, c = \sup C$. It must be that $c \leq a+b$, so we must show $c \geq a + b$. Let $\epsilon > 0$. Since $a = \sup A$, $a - \epsilon$ is not an upper bound for A. $\exists x \in A$ such that $a - \epsilon < x$. Likewise, $\exists y \in B$ such that $b - \epsilon < y$. Then,

$$(a - \epsilon) + (b - \epsilon) = a + b - 2 * \epsilon < x + y \leq c$$

Thus, $a + b < c + 2 * \epsilon \forall \epsilon > 0$. So, $a + b \leq c \therefore c = a + b$. □

3 September 8

3.1 Limits of Sequences

Definition 3.1 (Limit of a sequence)**.** A sequence a_n is said to converge to a real number s, if for any $\epsilon > 0$, $\exists$ a real number k such that for all $n \geq k$, the terms a_n satisfy $|a_n - s| < \epsilon$.

Theorem 3.1. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof. We need to find some N such that $n > N \forall \epsilon > 0$.

$$\begin{aligned} |\frac{1}{\sqrt{n}} - 0| &< \epsilon \\ \frac{1}{\sqrt{n}} &< \epsilon \\ \frac{1}{n} &< \epsilon^2 \\ n &> \frac{1}{\epsilon^2} \end{aligned}$$

Let $\epsilon > 0$ and $N = \frac{1}{\epsilon^2}$. Then, if $n > N$, we have that

$$|\frac{1}{\sqrt{n}} - 0| = \frac{1}{\sqrt{n}}$$

$$< \frac{1}{\sqrt{\frac{1}{\epsilon^2}}}$$
$$= \epsilon$$

Thus, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. □

Theorem 3.2. $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$.

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\epsilon}$. Then, we have

$$|1 + \frac{1}{2^n} - 1| < \epsilon$$
$$|\frac{1}{2^n}| = \frac{1}{2^n} < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} < \epsilon$$
$$n > \frac{1}{\epsilon}$$

Thus, $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$. □

Theorem 3.3. Every convergent sequence is bounded.

Proof. Let S_n be a convergent sequence with a limit s and $\epsilon = 1$. Then, there exists some N such that $|S_n - s| < 1$. That is, $|S_n| < |s| + 1$.

Let $M = \max\{S_1, S_2, \ldots, S_n, |s| + 1\}$. Then, $|S_n| \leq M$, so S_n is bounded. □

Theorem 3.4. If a sequence converges, its limit is unique.

Proof. Suppose a sequence S_n converges to s and t. Let $\epsilon > 0$. Then, $\exists N_1$ such that $|S_n - s| < \frac{t}{2}$. For $n > N_1$, $\exists N_2$ such that $|S_n - t| < \frac{t}{2}$. For $n > N_2$, let $N = m + \{N_1, N_2\}$. Then, for $n > N$, we have

$$\begin{aligned}
|s - t| &= |s + S_n - S_n - t| \\
&= |s - S_n + S_n - t| \\
&\leq |s - S_n| + |S_n - t| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
|s - t| &= \epsilon
\end{aligned}$$

Thus, the limit is unique. □

4 September 13

4.1 Monotone Sequences

Definition 4.1 (Monotone sequence)**.** A sequence S_n of real numbers is said to be increasing $\iff S_n \leq S_{n+1} \; \forall \, n \in \mathbb{N}$ and decreasing $\iff S_n \geq S_{n+1} \; \forall \, n \in \mathbb{N}$.

The Fibonacci sequence is an example of an increasing sequence.

Definition 4.2 (Monotone convergence theorem)**.** A monotone sequence is convergent if and only if it is bounded.

Theorem 4.1. An increasing bounded sequence is convergent.

Proof. Suppose S_n is a bounded increasing sequence. Let S be the set $\{S_n \mid n \in \mathbb{N}\}$. By the completeness axiom, $\sup S$ exists. Let $s = \sup S$. We claim $\lim_{n\to\infty} S_n = s$. Given $\epsilon > 0, s - \epsilon$ is not an upper bound for S.
Thus, $\exists\ N \in \mathbb{N}$ such that $S_N > s - \epsilon$. Furthermore, since S_n is increasing and s is an upper bound for S, we have $s - \epsilon < S_N \leq S_n \leq s\ \forall\ n \geq N$. $\square$

Remark. This is an elementary proof because it only uses axioms to make the conclusion.

Ex. $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$.

Theorem 4.2. If S_n is an unbounded increasing sequence, then $\lim_{n\to\infty} S_n = \infty$.

Proof. Let S_n be an increasing unbounded sequence. Then, $\{S_n \mid n \in \mathbb{N}\}$ is not bounded above, but S is bounded below by S_1. Thus, given $M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $S \mid N > M$. But since S_n is increasing, $S_n > M\ \forall\ n > N$. Thus $\lim_{n\to\infty} S_n = \infty$. $\square$