M 361K: Real Analysis

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1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1) $a + b = b + a$.

- (A2) $(a+b)+c = a+(b+c)$.
- (A3) $\exists$ an element $o \in \mathbb{R}$ such that $a+o = o+a = a$.
- (A4) For each element $a \in \mathbb{R}$, $\exists$ an element $(-a) \in \mathbb{R}$ such that $a+(-a) = 0$.
- (M1) $ab = ba$.
- (M2) $(ab)c = a(bc)$.
- (M3) $\exists$ an element $1 \in \mathbb{R}$ such that $a*1 = 1*a = a$.
- (M4) For each element $a \in \mathbb{R} \setminus 0$, $\exists$ an element $\frac{1}{a} \in \mathbb{R}$ such that $a * \frac{1}{a} = \frac{1}{a} * a = 1$.
- (D) $a*(b+c) = a*b + a*c$.

Remark (Equality property of $\mathbb{R}$). If $a = b$ and $c = d$, then $a+c = b+d$ and $a*c = b*d$.

$\forall x, y, z \in \mathbb{R}$:

Theorem 1.1. If $x+z = y+z$ then $x = y$.

Proof.

$$\begin{aligned} x+z &= y+z \ \ (A4) \\ (x+z)+(-z) &= (y+z)+(-z) \ \ (A2) \\ x+(z+(-z)) &= y+(z+(-z)) \ \ (A4) \\ x+0 &= y+0 \ \ (A3) \\ x &= y \end{aligned}$$

□

Theorem 1.2. For any $x \in \mathbb{R}$, $x*0 = 0$.

Proof.

$$\begin{aligned} x*0 &= x*(0+0) \\ x*0 &= x*0 + x*0 \\ x*0 + (-x*0) &= (x*0 + x*0) + (-x*0) \\ 0 &= x*0 + (x*0 + (-x*0)) \\ &= x*0 + 0 \\ &= x*0 \end{aligned}$$

□

Theorem 1.3. $-1*x = -x$ i.e. $x + (-1)*x = 0$.

Proof.

$$\begin{aligned} x + (-1) * x &= x + x * (-1) \\ &= x * 1 + x * (-1) \\ &= x * (1 + (-1)) \\ &= x * 0 \\ &= 0 \end{aligned}$$

□

Theorem 1.4 (Zero-product property)**.** $\forall x, y \in \mathbb{R}$, $x * y = 0 \iff x = 0 \lor y = 0$.

Proof. Let $x, y \in \mathbb{R}$, if $x = 0$ or $y = 0$, then $x * y = 0$. Suppose $x \neq 0$, then we must show $y = 0$. Since $x \neq 0$, $\frac{1}{x}$ exists. Thus, if:

$$\begin{aligned} xy &= 0 \\ \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\ (\frac{1}{x} * (xy)) * y &= 0 \\ 1 * y &= 0 \\ y &= 0 \end{aligned}$$

□

1.2 Order Axioms

$\forall x, y \in \mathbb{R}$:

- (O1) One of $x < y$, $x > y$ or $x = y$ is true.
- (O2) If $x < y$ and $y < z$, then $x < z$.
- (O3) If $x < y$ then $x + z < y + z$.
- (O4) If $x < y$ and $z > 0$ then $xz < yz$.

Theorem 1.5. If $x < y$ then $-y < -x$.

Proof.

$$\begin{gathered} x < y \\ x + (-x + -y) < y + (-x + -y) \\ (x + -x) + -y < (y + -y) + -x \\ 0 + -y < 0 + -x \\ -y < -x \end{gathered}$$

□

Theorem 1.6. If $x < y$ and $z > 0$ then $xz > yz$.

Proof. If $x < y$ and $z > 0$ then $-z > 0$. Thus, $x(-z) < y(-z)$. But,

$$\begin{aligned} x(-z) &= x(-1 * z) \\ &= (x * -1) * z \\ &= (-1 * x) * z \\ &= -1(x * z) \\ &= -x * z \end{aligned}$$

Similarly, $y(-z) = -y * z$. Thus, $-x * z < -y * z$, so $xz > yz$. $\square$

Remark (Completeness of $\mathbb{R}$). $\mathbb{R}$ is an ordered field. $\mathbb{R}$ is complete, while $\mathbb{Q}$ is not complete.

2 August 30

Theorem 2.1. $\sqrt{2}$ is irrational.

Proof. Suppose not. Suppose that $\sqrt{2}$ is rational. Then $\exists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}, n \neq 0$ and m and n share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus, m^2 is even and m is even. Then, $m = 2k$ for some $k \in \mathbb{Z}$. But, by substituting $m = 2k$ into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus, n^2 is even, so n is even. So, n is a perfect square, which is a contradiction. Thus, $\sqrt{2}$ is irrational. $\square$

2.1 Upper and Lower Bounds

Theorem: Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s \forall s \in S$, m is called an **upper bound** for S. If $m \leq s \forall s \in S$, m is called a **lower bound** for S. **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2

- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for T.

Definition 2.1 (Supremum)**.** The least upper bound of a set is called the supremum of the set.

Definition 2.2 (Infimum)**.** The greatest lower bound of a set is called the infimum of the set.

2.2 Completeness Axiom

Definition 2.3 (Completeness axiom)**.** Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 2.2. The set of natural numbers $\mathbb{N}$ is unbounded above.

Proof. Suppose not. Suppose that $\mathbb{N}$ is bounded above. If $\mathbb{N}$ were bounded above, it must have a supremum m. Since $\sup \mathbb{N} = m$, $m - 1$ is not an upper bound. Thus, $\exists n_0 \in \mathbb{N}$ such that $n_0 > m - 1$. But then, $n_0 + 1 > m$. This is a contradiction since $n_0 + 1 \in N$. Thus, N is unbounded above. □

Theorem 2.3. If A and B are nonempty subsets of $\mathbb{R}$, let $C = \{x + y \mid x \in A, y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C = \sup A + \sup B$.

Proof. Let $\sup A = a$ and $\sup B = b$. Then if $z \in C, z = x + y$ for some $x \in A, y \in B$. Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom, $\exists$ a least upper bound of $C, c = \sup C$. It must be that $c \leq a + b$, so we must show $c \geq a + b$. Let $\epsilon > 0$. Since $a = \sup A$, $a - \epsilon$ is not an upper bound for A. $\exists x \in A$ such that $a - \epsilon < x$. Likewise, $\exists y \in B$ such that $b - \epsilon < y$. Then,

$$(a - \epsilon) + (b - \epsilon) = a + b - 2 * \epsilon < x + y \leq c$$

Thus, $a + b < c + 2 * \epsilon \forall \epsilon > 0$. So, $a + b \leq c \therefore c = a + b$. □

3 September 8

3.1 Limits of Sequences

Definition 3.1 (Limit of a sequence)**.** A sequence a_n is said to converge to a real number s, if for any $\epsilon > 0$, $\exists$ a real number k such that for all $n \geq k$, the terms a_n satisfy $|a_n - s| < \epsilon$.

Theorem 3.1. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof. We need to find some N such that $n > N \forall \epsilon > 0$.

$$|\frac{1}{\sqrt{n}} - 0| < \epsilon$$
$$\frac{1}{\sqrt{n}} < \epsilon$$
$$\frac{1}{n} < \epsilon^2$$
$$n > \frac{1}{\epsilon^2}$$

Let $\epsilon > 0$ and $N = \frac{1}{\epsilon^2}$. Then, if $n > N$, we have that

$$\begin{aligned} |\frac{1}{\sqrt{n}} - 0| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. □

Theorem 3.2. $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$.

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\epsilon}$. Then, we have

$$|1 + \frac{1}{2^n} - 1| < \epsilon$$
$$|\frac{1}{2^n}| = \frac{1}{2^n} < \frac{1}{n} < \frac{1}{\frac{1}{\epsilon}} < \epsilon$$
$$n > \frac{1}{\epsilon}$$

Thus, $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$. □

Theorem 3.3. Every convergent sequence is bounded.

Proof. Let S_n be a convergent sequence with a limit s and $\epsilon = 1$. Then, there exists some N such that $|S_n - s| < 1$. That is, $|S_n| < |s| + 1$.

Let $M = \max\{S_1, S_2, \ldots, S_n, |s| + 1\}$. Then, $|S_n| \leq M$, so S_n is bounded. □

Theorem 3.4. If a sequence converges, its limit is unique.

Proof. Suppose a sequence S_n converges to s and t. Let $\epsilon > 0$. Then, $\exists N_1$ such that $|S_n - s| < \frac{t}{2}$. For $n > N_1$, $\exists N_2$ such that $|S_n - t| < \frac{t}{2}$. For $n > N_2$, let $N = m + \{N_1, N_2\}$. Then, for $n > N$, we have

$$|s - t| = |s + S_n - S_n - t|$$

$$
\begin{aligned}
&= |s - S_n + S_n - t| \\
&\leq |s - S_n| + |S_n - t| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
|s - t| &= \epsilon
\end{aligned}
$$

Thus, the limit is unique. □

4 September 13

4.1 Monotone Sequences

Definition 4.1 (Monotone sequence)**.** A sequence S_n of real numbers is said to be increasing $\iff S_n \leq S_{n+1} \; \forall \; n \in \mathbb{N}$ and decreasing $\iff S_n \geq S_{n+1} \; \forall \; n \in \mathbb{N}$.

The Fibonacci sequence is an example of an increasing sequence.

Definition 4.2 (Monotone convergence theorem)**.** A monotone sequence is convergent if and only if it is bounded.

Theorem 4.1. An increasing bounded sequence is convergent.

Proof. Suppose S_n is a bounded increasing sequence. Let S be the set $\{S_n \mid n \in \mathbb{N}\}$. By the completeness axiom, $\sup S$ exists. Let $s = \sup S$. We claim $\lim_{n\to\infty} S_n = s$. Given $\epsilon > 0, s - \epsilon$ is not an upper bound for S.
Thus, $\exists \; N \in \mathbb{N}$ such that $S_N > s - \epsilon$. Furthermore, since S_n is increasing and s is an upper bound for S, we have $s - \epsilon < S_N \leq S_n \leq s \; \forall \; n \geq N$. □

Remark. This is an elementary proof because it only uses axioms to make the conclusion.

Ex. $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$.

Theorem 4.2. If S_n is an unbounded increasing sequence, then $\lim_{n\to\infty} S_n = \infty$.

Proof. Let S_n be an increasing unbounded sequence. Then, $\{S_n \mid n \in \mathbb{N}\}$ is not bounded above, but S is bounded below by S_1. Thus, given $M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $S_N > M$. But since S_n is increasing, $S_n > M \; \forall \; n > N$. Thus, $\lim_{n\to\infty} S_n = \infty$. □

5 September 15

5.1 Cauchy Sequences

Definition 5.1 (Cauchy sequence)**.** A sequence of real numbers S_n is called a Cauchy sequence if and only if for each $\epsilon > 0$, $\exists N$ such that $m, n > N \implies |S_m - S_n| < \epsilon$.

Remark. This means the elements of the sequence get closer to each other as N increases.

Theorem 5.1. Every convergent sequence is Cauchy.

Proof. Let S_n be a convergent sequence. Then $\exists N$ such that $n > N \implies |S_n - s| < \frac{\epsilon}{2}$ for some $s \in \mathbb{R}$. Then, for $n, m > N$, we have

$$\begin{aligned} |S_n - S_m| &= |S_n - s + s - S_m| \\ &\leq |S_n - s| + |s - S_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, S_n is Cauchy. □

Theorem 5.2. A sequence of real numbers is Cauchy if and only if it is convergent.

Remark. We cannot prove this yet.

6 September 20

6.1 Empty Set

Theorem 6.1. The empty set is a subset of any set.

Proof. Suppose not. That is, suppose $\exists A$ such that $\emptyset \not\subset A$. Thus, $\exists x \in \emptyset$ such that $x \notin A$. This is a contradiction because the empty set has no elements. Therefore, $\emptyset \subset A$. □

Theorem 6.2. There is only one set with no elements.

Proof. Suppose not. That is, suppose $\exists$ two empty sets E_1, E_2. Then $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$. Thus, $E_1 = E_2$. This is a contradiction because E_1 and E_2 are two different sets. Therefore, there is only one empty set. □

Remark (Closedness of $\emptyset$)*.* The empty set is open and closed (vacuosly true).

6.2 Topology of $\mathbb{R}$

Let $S \subseteq \mathbb{R}$ for the following definitions.

6.2.1 Neighborhoods

Definition 6.1 (Neighorhood)**.** A neighorhood of x in S can be thought of an epsilon-sized ball around x, i.e. $N(x, \epsilon) = \{y \in R \mid 0 \leq |x - y| < \epsilon\}$.

Definition 6.2 (Deleted neighborhood)**.** A deleted neighorhood is the same as a neighorhood except that x is not included, i.e. $N^\star(x, \epsilon) = \{y \in R \mid 0 < |x - y| < \epsilon\}$.

Definition 6.3 (Accumulation point)**.** $x \in \mathbb{R}$ is an accumulation point of S if and only if every deleted neighborhood of x contains a point of S.

Remark. $(0, \infty)$ has accumulation points $[0, \infty)$. $(0, 1)$ does not contain all of its accumulation points since 0 and 1 are both accumulation points of the set.

Theorem 6.3. $S \in \mathbb{R}$ is closed if and only if S contains all of its accumulation points.

Proof. Suppose S is closed. Let x be an accumulation point of S. If $x \notin S$, then $x \in S^c$. Thus, $\exists$ a neighborhood N of x such that $N \subseteq S^c$. But $N \cap S = \emptyset$, which contradicts x being an accumulation point of S.

Conversely, suppose S contains all of its accumulation points. Let $x \in S^c$, then x is not an accumulation point of S. Thus, $\exists N^\star(x, \epsilon)$ that misses S. Since $x \notin S$, $N(x, \epsilon)$ misses S. Therefore, S^c is open, which means S is closed. □

Theorem 6.4. If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a max.

Proof. Let $s = \sup S$. Then, s is an accumulation point of S. Since S is closed, $s \in S$. Thus, s is a max of S. □

6.2.2 Interior and Boundary Points

Definition 6.4 (Interior point)**.** $x \in S$ is an interior point of S if and only if $\exists N(x, t)$ such that $N(x, t) \subset S$.

Definition 6.5 (Boundary point)**.** $x \in S$ is a boundary point of S if and only if every neighborhood N of x has $N \cap S \neq \emptyset$ and $N \cap S^c \neq \emptyset$.

6.3 Closure

Definition 6.6 (Open set)**.** S is an open set if and only if every point in S is an interior point of S. $\forall x \in S, \exists$ a neighborhood $N(x, \epsilon)$ for some $\epsilon > 0$ such that $N(x, \epsilon) \subseteq S$.

Definition 6.7 (Closed set)**.** S is a closed set if and only S contains at least one of its boundary points. Additionally, S^c must be an open set.

Remark (Closure of $\mathbb{R}$). $\mathbb{R}$ is open because all of its points are interior points. $\mathbb{R}$ is also closed because $\mathbb{R}$ has no boundary points, therefore implying that it contains at least one of its boundary points (vacuously true).

Theorem 6.5. The union of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $\exists$ a neighborhood N_1 of x such that $N_1 \subseteq A$. But then, $N_1 \subseteq A \cup B$. If $x \in B$, then $\exists$ a neighborhood N_2 of x such that $N_2 \subseteq B$. But then, $N_2 \subseteq A \cup B$.

Thus, in either case, $\exists$ a neighborhood N of x such that $N \subseteq A \cup B$. Therefore, $A \cup B$ is open. □

Theorem 6.6. An arbitrary union of open sets is open.

Proof. Let $A_1, A_2, \ldots, A_n$ be open sets. Let $x \in \bigcup_{i=1}^n A_i$. Then $x \in A_i$ for some i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcup_{i=1}^n A_i$. Therefore, $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$.

Thus, $\bigcup_{i=1}^n N_i$ is a neighborhood of x such that $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$. Therefore, $\bigcup_{i=1}^n A_i$ is open. □

Theorem 6.7. The intersection of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, $\exists$ neighborhoods $N_1(x, \epsilon_1)$ and $N_2(x, \epsilon_2)$. Let $\epsilon = min\{\epsilon_1, \epsilon_2\}$. Then $N_1(x, \epsilon) \subseteq A$ and $N_2(x, \epsilon) \subseteq B$.

Thus, $N(x, \epsilon) \subseteq A \cap B$. Therefore, $A \cap B$ is open. $\square$

Theorem 6.8. A finite intersection of open sets is open.

Proof. Let $A_1, A_2, \ldots, A_n$ be open sets. Let $x \in \bigcap_{i=1}^n A_i$. Then $x \in A_i$ for all i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcap_{i=1}^n A_i$. Therefore, $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$.

Thus, $\bigcap_{i=1}^n N_i$ is a neighborhood of x such that $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$. Therefore, $\bigcap_{i=1}^n A_i$ is open. $\square$

Theorem 6.9. An arbitrary intersection of open sets is open.

Remark (Counterexample). $\bigcap_{i=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \emptyset$.

6.4 Set Covers

Definition 6.8 (Open cover)**.** An open cover F of some subset $S \in \mathbb{R}$ is a collection of open sets whose union contains S.

Remark. If $E \subseteq F$ and E also covers S, we call E a **subcover**.

Definition 6.9 (Compact)**.** A set S is said to be compact is and only if whenever S is contained in the union of a family F of open sets, then it is contained in a finite number of the sets in F (every open cover has a finite subcover).

Remark. It is hard to show that a set is compact since we have to consider *every* open cover.

Theorem 6.10 (Heine-Borel)**.** A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.