

# M 361K: Real Analysis

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# 1 August 25

## 1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1)  $a + b = b + a$ .
- (A2)  $(a + b) + c = a + (b + c)$ .
- (A3)  $\exists$  an element  $o \in \mathbb{R}$  such that  $a + o = o + a = a$ .
- (A4) For each element  $a \in \mathbb{R}$ ,  $\exists$  an element  $(-a) \in \mathbb{R}$  such that  $a + (-a) = 0$ .
- (M1)  $ab = ba$ .
- (M2)  $(ab)c = a(bc)$ .
- (M3)  $\exists$  an element  $1 \in \mathbb{R}$  such that  $a * 1 = 1 * a = a$ .
- (M4) For each element  $a \in \mathbb{R} \setminus 0$ ,  $\exists$  an element  $\frac{1}{a} \in \mathbb{R}$  such that  $a * \frac{1}{a} = \frac{1}{a} * a = 1$ .
- (D)  $a * (b + c) = a * b + a * c$ .

*Remark* (Equality property of  $\mathbb{R}$ ). If  $a = b$  and  $c = d$ , then  $a + c = b + d$  and  $a * c = b * d$ .

$\forall x, y, z \in \mathbb{R}$ :

**Theorem 1.1.** If  $x + z = y + z$  then  $x = y$ .

*Proof.*

$$\begin{aligned}x + z &= y + z \quad (A4) \\(x + z) + (-z) &= (y + z) + (-z) \quad (A2) \\x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\x + 0 &= y + 0 \quad (A3) \\x &= y\end{aligned}$$

□

**Theorem 1.2.** For any  $x \in \mathbb{R}$ ,  $x * 0 = 0$ .

*Proof.*

$$\begin{aligned}x * 0 &= x * (0 + 0) \\x * 0 &= x * 0 + x * 0 \\x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\0 &= x * 0 + (x * 0 + (-x * 0)) \\&= x * 0 + 0 \\&= x * 0\end{aligned}$$

□

**Theorem 1.3.**  $-1 * x = -x$  i.e.  $x + (-1) * x = 0$ .

*Proof.*

$$\begin{aligned}
 x + (-1) * x &= x + x * (-1) \\
 &= x * 1 + x * (-1) \\
 &= x * (1 + (-1)) \\
 &= x * 0 \\
 &= 0
 \end{aligned}$$

□

**Theorem 1.4** (Zero-product property).  $\forall x, y \in \mathbb{R}, x * y = 0 \iff x = 0 \vee y = 0$ .

*Proof.* Let  $x, y \in \mathbb{R}$ , if  $x = 0$  or  $y = 0$ , then  $x * y = 0$ . Suppose  $x \neq 0$ , then we must show  $y = 0$ . Since  $x \neq 0$ ,  $\frac{1}{x}$  exists. Thus, if:

$$\begin{aligned}
 xy &= 0 \\
 \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\
 \left(\frac{1}{x} * (xy)\right) * y &= 0 \\
 1 * y &= 0 \\
 y &= 0
 \end{aligned}$$

□

## 1.2 Order Axioms

$\forall x, y \in \mathbb{R}$ :

- (O1) One of  $x < y$ ,  $x > y$  or  $x = y$  is true.
- (O2) If  $x < y$  and  $y < z$ , then  $x < z$ .
- (O3) If  $x < y$  then  $x + z < y + z$ .
- (O4) If  $x < y$  and  $z > 0$  then  $xz < yz$ .

**Theorem 1.5.** If  $x < y$  then  $-y < -x$ .

*Proof.*

$$\begin{aligned}
 x &< y \\
 x + (-x + -y) &< y + (-x + -y) \\
 (x + -x) + -y &< (y + -y) + -x \\
 0 + -y &< 0 + -x \\
 -y &< -x
 \end{aligned}$$

□

**Theorem 1.6.** If  $x < y$  and  $z > 0$  then  $xz > yz$ .

*Proof.* If  $x < y$  and  $z > 0$  then  $-z < 0$ . Thus,  $x(-z) < y(-z)$ . But,

$$\begin{aligned}x(-z) &= x(-1 * z) \\&= (x * -1) * z \\&= (-1 * x) * z \\&= -1(x * z) \\&= -x * z\end{aligned}$$

Similarly,  $y(-z) = -y * z$ . Thus,  $-x * z < -y * z$ , so  $xz > yz$ . □

*Remark* (Completeness of  $\mathbb{R}$ ).  $\mathbb{R}$  is an ordered field.  $\mathbb{R}$  is complete, while  $\mathbb{Q}$  is not complete.

## 2 August 30

**Theorem 2.1.**  $\sqrt{2}$  is irrational.

*Proof.* Suppose not. Suppose that  $\sqrt{2}$  is rational. Then  $\exists m, n \in \mathbb{Z}$  such that  $\sqrt{2} = \frac{m}{n}, n \neq 0$  and  $m$  and  $n$  share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus,  $m^2$  is even and  $m$  is even. Then,  $m = 2k$  for some  $k \in \mathbb{Z}$ . But, by substituting  $m = 2k$  into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus,  $n^2$  is even, so  $n$  is even. So,  $n$  is a perfect square, which is a contradiction. Thus,  $\sqrt{2}$  is irrational.  $\square$

### 2.1 Upper and Lower Bounds

**Theorem:** Let  $S$  be a subset of  $\mathbb{R}$ . If there exists a real number  $m$  such that  $m \geq s \forall s \in S$ ,  $m$  is called an **upper bound** for  $S$ . If  $m \leq s \forall s \in S$ ,  $m$  is called a **lower bound** for  $S$ . **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for  $T$ .

**Definition 2.1** (Supremum). The least upper bound of a set is called the supremum of the set.

**Definition 2.2** (Infimum). The greatest lower bound of a set is called the infimum of the set.

## 2.2 Completeness Axiom

**Definition 2.3** (Completeness axiom). Every nonempty subset of  $\mathbb{R}$  that is bounded above has a least upper bound. That is,  $\sup S$  exists and is a real number.

**Theorem 2.2.** The set of natural numbers  $\mathbb{N}$  is unbounded above.

*Proof.* Suppose not. Suppose that  $\mathbb{N}$  is bounded above. If  $\mathbb{N}$  were bounded above, it must have a supremum  $m$ . Since  $\sup \mathbb{N} = m$ ,  $m - 1$  is not an upper bound. Thus,  $\exists n_0 \in \mathbb{N}$  such that  $n_0 > m - 1$ . But then,  $n_0 + 1 > m$ . This is a contradiction since  $n_0 + 1 \in \mathbb{N}$ . Thus,  $\mathbb{N}$  is unbounded above.  $\square$

**Theorem 2.3.** If  $A$  and  $B$  are nonempty subsets of  $\mathbb{R}$ , let  $C = \{x + y \mid x \in A, y \in B\}$ . If  $\sup A$  and  $\sup B$  exist, then  $\sup C = \sup A + \sup B$ .

*Proof.* Let  $\sup A = a$  and  $\sup B = b$ . Then if  $z \in C$ ,  $z = x + y$  for some  $x \in A, y \in B$ . Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom,  $\exists$  a least upper bound of  $C$ ,  $c = \sup C$ . It must be that  $c \leq a + b$ , so we must show  $c \geq a + b$ . Let  $\varepsilon > 0$ . Since  $a = \sup A$ ,  $a - \varepsilon$  is not an upper bound for  $A$ .  $\exists x \in A$  such that  $a - \varepsilon < x$ . Likewise,  $\exists y \in B$  such that  $b - \varepsilon < y$ . Then,

$$(a - \varepsilon) + (b - \varepsilon) = a + b - 2\varepsilon < x + y \leq c$$

Thus,  $a + b < c + 2\varepsilon \forall \varepsilon > 0$ . So,  $a + b \leq c \therefore c = a + b$ .  $\square$

## 3 September 6

### 3.1 Cardinality

**Definition 3.1** (Cardinality). The cardinality of a set  $A$  is the number of elements in  $A$ . We denote this as  $|A|$ . We say that two sets  $A$  and  $B$  have the same cardinality if and only if  $\exists$  a bijection  $f : A \rightarrow B$ , or  $|A| = |B|$ .

*Remark.* This bijection holds true because cardinality is reflexive (via the identity function), symmetric (via the inverse function), and transitive (via composition).

*Remark.* The following examples demonstrate how to prove whether two sets have the same cardinality.

- $|\text{even integers}| = |\text{odd integers}|$ :  $f(2n) = 2n + 1$ .
- $|\mathbb{Z}| = |\mathbb{Z}^+|$ :  $f(0) = 1, f(1) = 2, f(-1) = 3, f(2) = 4, \dots$
- $|\mathbb{Q}^+| = |\mathbb{Z}^+|$ : We can create a diagonal mapping by taking  $\frac{n}{m}$  for counting numbers on the rows and columns.
- $|\mathbb{Q}| = |\mathbb{Z}^+|$ :  $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ , so we can repeat the diagonal mapping for  $\mathbb{Q}^-$ . This is because any subset of a countable set is countable.
- $|\mathbb{Q}| \neq |\mathbb{R}|$ : For the real numbers, Cantor's Diagonal Argument proves the sets have different cardinality since no possible surjection exists.

In essence, if we show that there exists some one-to-one mapping between the two sets we can claim that  $|A| = |B|$ .

### 3.2 Countability

**Definition 3.2** (Countable). If a set is finite or has the same cardinality as  $\mathbb{N}$  (i.e.  $\mathbb{Z}^+$ ), we say that the set is countable.

**Theorem 3.1.** Any subset of a countable set is countable.

**Theorem 3.2.** Any set that contains an uncountable set is uncountable.

**Theorem 3.3.** If  $[a_n, b_n] \forall n \in \mathbb{N}$  is a nested sequence of closed bounded intervals,  $\exists \delta \in \mathbb{R}$  such that  $\delta \in I_n \forall n \in \mathbb{N}$ .

*Proof.*  $I_n \subseteq I_1 \forall n \in \mathbb{N}$ . Thus,  $a_n \subseteq b_1 \forall n \in \mathbb{N}$ . So,  $b_n$  is an upper bound for  $\{a_n \mid n \in \mathbb{N}\}$ . Let  $\delta$  be the supremum of  $\{a_n \mid n \in \mathbb{N}\}$ . Thus,  $a_n \leq \delta \forall n \in \mathbb{N}$ .

We have now shown that  $a_n \leq \delta \forall n \in \mathbb{N}$ , and we need to show that  $\delta \leq b_n \forall n \in \mathbb{N}$ . This is left as an exercise for the reader.  $\square$

*Remark.* A nested sequence means that successive subsets contain the previous subset. For example,  $[0, 1] \subseteq [0, 2] \subseteq [0, 3] \subseteq \dots$  is a nested sequence.



**Theorem 3.4.**  $[0, 1]$  is uncountable.

*Proof.* Assume  $[0, 1]$  is countable. That is,  $[0, 1] = I = \{x_1, x_2, x_3, \dots\}$ . Select a closed interval  $I_1 \subseteq I$  such that  $x_1 \notin I_1$ . Next, select a closed interval  $I_2 \subseteq I_1$  such that  $x_2 \notin I_2$ , and so on. Then, we have

$$I_n \subseteq \dots \subseteq I_2 \subseteq I_1 \subseteq I$$

and  $x_n \notin I_n \forall n \in \mathbb{N}$ . By **Theorem 3.3**,  $\exists \delta \in I$  such that  $\delta \in I_n \forall n \in \mathbb{N}$ . This implies that  $\delta \neq x_n \forall n \in \mathbb{N}$ . Thus,  $\delta \notin I$ , which is a contradiction. Therefore,  $[0, 1]$  is uncountable.  $\square$

## 4 September 8

### 4.1 Limits of Sequences

**Definition 4.1** (Limit of a sequence). A sequence  $a_n$  is said to converge to a real number  $s$ , if for any  $\varepsilon > 0$ ,  $\exists$  a real number  $k$  such that for all  $n \geq k$ , the terms  $a_n$  satisfy  $|a_n - s| < \varepsilon$ .

**Theorem 4.1.**  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ .

*Proof.* We need to find some  $N$  such that  $n > N \forall \varepsilon > 0$ .

$$\begin{aligned} \left| \frac{1}{\sqrt{n}} - 0 \right| &< \varepsilon \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \frac{1}{n} &< \varepsilon^2 \\ n &> \frac{1}{\varepsilon^2} \end{aligned}$$

Let  $\varepsilon > 0$  and  $N = \frac{1}{\varepsilon^2}$ . Then, if  $n > N$ , we have that

$$\begin{aligned} \left| \frac{1}{\sqrt{n}} - 0 \right| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} \\ &= \varepsilon \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ . □

**Theorem 4.2.**  $\lim_{n \rightarrow \infty} 1 + \frac{1}{2^n} = 1$ .

*Proof.* Let  $\varepsilon > 0$  and  $N = \frac{1}{\varepsilon}$ . Then, we have

$$\begin{aligned} \left| 1 + \frac{1}{2^n} - 1 \right| &< \varepsilon \\ \left| \frac{1}{2^n} \right| &= \frac{1}{2^n} < \frac{1}{n} < \frac{1}{\frac{1}{\varepsilon}} < \varepsilon \\ n &> \frac{1}{\varepsilon} \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} 1 + \frac{1}{2^n} = 1$ . □

**Theorem 4.3.** Every convergent sequence is bounded.

*Proof.* Let  $S_n$  be a convergent sequence with a limit  $s$  and  $\varepsilon = 1$ . Then, there exists some  $N$  such that  $|S_n - s| < 1$  when  $n > N$ . That is,  $|S_n| < |s| + 1$ .

Let  $M = \max\{S_1, S_2, \dots, S_N, |s| + 1\}$ . Then,  $|S_n| \leq M$ , so  $S_n$  is bounded. □

**Theorem 4.4.** If a sequence converges, its limit is unique.

*Proof.* Suppose a sequence  $S_n$  converges to  $s$  and  $t$ . Let  $\varepsilon > 0$ . Then,  $\exists N_1$  such that  $|S_n - s| < \frac{\varepsilon}{2}$ . For  $n > N_1$ ,  $\exists N_2$  such that  $|S_n - t| < \frac{\varepsilon}{2}$ . For  $n > N_2$ , let  $N = \max\{N_1, N_2\}$ . Then, for  $n > N$ , we have

$$\begin{aligned} |s - t| &= |s + S_n - S_n - t| \\ &= |s - S_n + S_n - t| \\ &\leq |s - S_n| + |S_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, the limit is unique. □

## 5 September 13

### 5.1 Monotone Sequences

**Definition 5.1** (Monotone sequence). A sequence  $S_n$  of real numbers is said to be increasing  $\iff S_n \leq S_{n+1} \forall n \in \mathbb{N}$  and decreasing  $\iff S_n \geq S_{n+1} \forall n \in \mathbb{N}$ .

*Remark.* The Fibonacci sequence is an example of an increasing sequence.

**Definition 5.2** (Monotone convergence theorem). A monotone sequence is convergent if and only if it is bounded.

**Theorem 5.1.** An increasing bounded sequence is convergent.

*Proof.* Suppose  $S_n$  is a bounded increasing sequence. Let  $S$  be the set  $\{S_n \mid n \in \mathbb{N}\}$ . By the completeness axiom,  $\sup S$  exists. Let  $s = \sup S$ . We claim  $\lim_{n \rightarrow \infty} S_n = s$ . Given  $\varepsilon > 0$ ,  $s - \varepsilon$  is not an upper bound for  $S$ .

Thus,  $\exists N \in \mathbb{N}$  such that  $S_N > s - \varepsilon$ . Furthermore, since  $S_n$  is increasing and  $s$  is an upper bound for  $S$ , we have  $s - \varepsilon < S_N \leq S_n \leq s \forall n \geq N$ .  $\square$

*Remark.* This is an elementary proof because it only uses axioms to make the conclusion.

Ex.  $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$ .

**Theorem 5.2.** If  $S_n$  is an unbounded increasing sequence, then  $\lim_{n \rightarrow \infty} S_n = \infty$ .

*Proof.* Let  $S_n$  be an increasing unbounded sequence. Then,  $\{S_n \mid n \in \mathbb{N}\}$  is not bounded above, but  $S$  is bounded below by  $S_1$ . Thus, given  $M \in \mathbb{R}, \exists N \in \mathbb{N}$  such that  $S_N > M$ . But since  $S_n$  is increasing,  $S_n > M \forall n > N$ . Thus,  $\lim_{n \rightarrow \infty} S_n = \infty$ .  $\square$

## 6 September 15

### 6.1 Cauchy Sequences

**Definition 6.1** (Cauchy sequence). A sequence of real numbers  $S_n$  is called a Cauchy sequence if and only if for each  $\varepsilon > 0$ ,  $\exists N$  such that  $m, n > N \implies |S_m - S_n| < \varepsilon$ .

*Remark.* This means the elements of the sequence get closer to each other as  $N$  increases.

**Theorem 6.1.** Every convergent sequence is Cauchy.

*Proof.* Let  $S_n$  be a convergent sequence. Then  $\exists N$  such that  $n > N \implies |S_n - s| < \frac{\varepsilon}{2}$  for some  $s \in \mathbb{R}$ . Then, for  $n, m > N$ , we have

$$\begin{aligned} |S_n - S_m| &= |S_n - s + s - S_m| \\ &\leq |S_n - s| + |s - S_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus,  $S_n$  is Cauchy. □

**Theorem 6.2.** A sequence of real numbers is Cauchy if and only if it is convergent.

*Remark.* We cannot prove this yet.

## 7 September 20

### 7.1 Empty Set

**Theorem 7.1.** The empty set is a subset of any set.

*Proof.* Suppose not. That is, suppose  $\exists A$  such that  $\emptyset \not\subset A$ . Thus,  $\exists x \in \emptyset$  such that  $x \notin A$ . This is a contradiction because the empty set has no elements. Therefore,  $\emptyset \subset A$ .  $\square$

**Theorem 7.2.** There is only one set with no elements.

*Proof.* Suppose not. That is, suppose  $\exists$  two empty sets  $E_1, E_2$ . Then  $E_1 \subseteq E_2$  and  $E_2 \subseteq E_1$ . Thus,  $E_1 = E_2$ . This is a contradiction because  $E_1$  and  $E_2$  are two different sets. Therefore, there is only one empty set.  $\square$

*Remark* (Closedness of  $\emptyset$ ). The empty set is open and closed (vacuously true).

### 7.2 Topology

Let  $S \subseteq \mathbb{R}$  for the following definitions.

**Definition 7.1** (Neighborhood). A neighborhood of  $x$  in  $S$  can be thought of an varepsilon-sized ball around  $x$ , i.e.  $N(x, \varepsilon) = \{y \in \mathbb{R} \mid 0 \leq |x - y| < \varepsilon\}$ .

**Definition 7.2** (Deleted neighborhood). A deleted neighborhood is the same as a neighborhood except that  $x$  is not included, i.e.  $N^*(x, \varepsilon) = \{y \in \mathbb{R} \mid 0 < |x - y| < \varepsilon\}$ .

**Definition 7.3** (Accumulation point).  $x \in \mathbb{R}$  is an accumulation point of  $S$  if and only if every deleted neighborhood of  $x$  contains a point of  $S$ .

*Remark.*  $(0, \infty)$  has accumulation points  $[0, \infty)$ .  $(0, 1)$  does not contain all of its accumulation points since 0 and 1 are both accumulation points of the set.

**Theorem 7.3.**  $S \subseteq \mathbb{R}$  is closed if and only if  $S$  contains all of its accumulation points.

*Proof.* Suppose  $S$  is closed. Let  $x$  be an accumulation point of  $S$ . If  $x \notin S$ , then  $x \in S^c$ . Thus,  $\exists$  a neighborhood  $N$  of  $x$  such that  $N \subseteq S^c$ . But  $N \cap S = \emptyset$ , which contradicts  $x$  being an accumulation point of  $S$ .

Conversely, suppose  $S$  contains all of its accumulation points. Let  $x \in S^c$ , then  $x$  is not an accumulation point of  $S$ . Thus,  $\exists N^*(x, \varepsilon)$  that misses  $S$ . Since  $x \notin S$ ,  $N(x, \varepsilon)$  misses  $S$ . Therefore,  $S^c$  is open, which means  $S$  is closed.  $\square$

**Theorem 7.4.** If  $S$  is a nonempty closed bounded subset of  $\mathbb{R}$ , then  $S$  has a max.

*Proof.* Let  $s = \sup S$ . Then,  $s$  is an accumulation point of  $S$ . Since  $S$  is closed,  $s \in S$ . Thus,  $s$  is a max of  $S$ .  $\square$

**Definition 7.4** (Interior point).  $x \in S$  is an interior point of  $S$  if and only if  $\exists N(x, t)$  such that  $N(x, t) \subset S$ .

**Definition 7.5** (Boundary point).  $x \in S$  is a boundary point of  $S$  if and only if every neighborhood  $N$  of  $x$  has  $N \cap S \neq \emptyset$  and  $N \cap S^c \neq \emptyset$ .

### 7.3 Closure

**Definition 7.6** (Open set).  $S$  is an open set if and only if every point in  $S$  is an interior point of  $S$ .  $\forall x \in S, \exists$  a neighborhood  $N(x, \varepsilon)$  for some  $\varepsilon > 0$  such that  $N(x, \varepsilon) \subseteq S$ .

**Definition 7.7** (Closed set).  $S$  is a closed set if and only if  $S$  contains at least one of its boundary points. Additionally,  $S^c$  must be an open set.

*Remark* (Closure of  $\mathbb{R}$ ).  $\mathbb{R}$  is open because all of its points are interior points.  $\mathbb{R}$  is also closed because  $\mathbb{R}$  has no boundary points, therefore implying that it contains at least one of its boundary points (vacuously true).

**Theorem 7.5.** The union of two open sets is open.

*Proof.* Let  $A$  and  $B$  be open sets. Let  $x \in A \cup B$ . Then  $x \in A$  or  $x \in B$ . If  $x \in A$ , then  $\exists$  a neighborhood  $N_1$  of  $x$  such that  $N_1 \subseteq A$ . But then,  $N_1 \subseteq A \cup B$ . If  $x \in B$ , then  $\exists$  a neighborhood  $N_2$  of  $x$  such that  $N_2 \subseteq B$ . But then,  $N_2 \subseteq A \cup B$ .

Thus, in either case,  $\exists$  a neighborhood  $N$  of  $x$  such that  $N \subseteq A \cup B$ . Therefore,  $A \cup B$  is open.  $\square$

**Theorem 7.6.** An arbitrary union of open sets is open.

*Proof.* Let  $A_1, A_2, \dots, A_n$  be open sets. Let  $x \in \bigcup_{i=1}^n A_i$ . Then  $x \in A_i$  for some  $i$ . Let  $N_i$  be a neighborhood of  $x$  such that  $N_i \subseteq A_i$ . Then  $N_i \subseteq A_i \subseteq \bigcup_{i=1}^n A_i$ . Therefore,  $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$ .

Thus,  $\bigcup_{i=1}^n N_i$  is a neighborhood of  $x$  such that  $\bigcup_{i=1}^n N_i \subseteq \bigcup_{i=1}^n A_i$ . Therefore,  $\bigcup_{i=1}^n A_i$  is open.  $\square$

**Theorem 7.7.** The intersection of two open sets is open.

*Proof.* Let  $A$  and  $B$  be open sets. Let  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B$ . Thus,  $\exists$  neighborhoods  $N_1(x, \varepsilon_1)$  and  $N_2(x, \varepsilon_2)$ . Let  $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ . Then  $N_1(x, \varepsilon) \subseteq A$  and  $N_2(x, \varepsilon) \subseteq B$ .

Thus,  $N(x, \varepsilon) \subseteq A \cap B$ . Therefore,  $A \cap B$  is open.  $\square$

**Theorem 7.8.** A finite intersection of open sets is open.

*Proof.* Let  $A_1, A_2, \dots, A_n$  be open sets. Let  $x \in \bigcap_{i=1}^n A_i$ . Then  $x \in A_i$  for all  $i$ . Let  $N_i$  be a neighborhood of  $x$  such that  $N_i \subseteq A_i$ . Then  $N_i \subseteq A_i \subseteq \bigcap_{i=1}^n A_i$ . Therefore,  $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$ .

Thus,  $\bigcap_{i=1}^n N_i$  is a neighborhood of  $x$  such that  $\bigcap_{i=1}^n N_i \subseteq \bigcap_{i=1}^n A_i$ . Therefore,  $\bigcap_{i=1}^n A_i$  is open.  $\square$

**Theorem 7.9.** An arbitrary intersection of open sets is open.

*Remark* (Counterexample).  $\bigcap_{i=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \emptyset$ .

## 8 September 22

### 8.1 Set Covers

**Definition 8.1** (Open cover). An open cover  $F$  of some subset  $S \subseteq \mathbb{R}$  is a collection of open sets whose union contains  $S$ .

*Remark.* If  $E \subseteq F$  and  $E$  also covers  $S$ , we call  $E$  a **subcover**.

**Definition 8.2** (Compact). A set  $S$  is said to be compact if and only if whenever  $S$  is contained in the union of a family  $F$  of open sets, then it is contained in a finite number of the sets in  $F$  (every open cover has a finite subcover).

*Remark.* It is hard to show that a set is compact since we have to consider *every* open cover.

**Theorem 8.1** (Heine-Borel). A subset  $S$  of  $\mathbb{R}$  is compact if and only if  $S$  is closed and bounded.

*Proof.* Let  $S$  be a compact set. Observe the open cover  $(-n, n) \forall n \in \mathbb{N}$ . Since  $S$  is compact,  $\exists$  a finite subcover  $(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k)$ .  $\exists$  one of these sets such that  $\bigcup_{i=1}^k (-n_i, n_i) = (-n_m, n_m)$  for some  $m = 1, 2, \dots, k$ . Thus,  $S \subseteq (-n_m, n_m)$ , so  $S$  is bounded.

Let  $S$  be a compact set. Suppose  $S$  is not closed. Let  $p$  be a boundary point of  $S$ , and let  $U_n = \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}] \forall n \in \mathbb{N}$ .  $S \subseteq \bigcup U_n = \mathbb{R} \setminus p$ .  $\exists$  a finite subcover  $U_{n_1}, U_{n_2}, \dots, U_{n_k}$  such that  $S \subseteq \bigcup_{i=1}^k U_{n_i}$ .  $\exists k$  such that  $S \subseteq U_{n_k}$ . But, this is a contradiction with  $p$  being a boundary point. Therefore,  $S$  is closed.

The proof in the other direction is similar, yet non-trivial.  $\square$

**Theorem 8.2** (Bolzano-Weierstrass). If a bounded subset  $S$  of  $\mathbb{R}$  contains infinitely many points, then  $\exists$  at least one accumulation point of  $S$ .

*Proof.* Let  $S$  be a bounded infinite subset of  $\mathbb{R}$ . Suppose  $S$  has no accumulation points, then  $S$  is closed. By Heine-Borel,  $S$  must be compact. Define neighborhoods  $N_x$  such that  $N_x(x) \cap S = \{x\} \forall x \in S$ . Clearly,  $S \subseteq \bigcup_x N_x$ . But, the collection of all  $N_x$  must contain a finite subcover. That is,

$$S \subseteq N_{x_1} \cup N_{x_2} \cup \dots \cup N_{x_k}$$

for some  $k \in \mathbb{N}$ . This contradicts that  $S$  is infinite. Therefore,  $S$  has an accumulation point.  $\square$

### 8.2 Cauchy Convergence

**Theorem 8.3.** Every Cauchy sequence is convergent.

*Proof.*  $S_n$  is Cauchy, so  $S = \{S_n \mid n \in \mathbb{N}\}$ . By Bolzano-Weierstrass,  $\exists$  an accumulation point  $s$  of  $S$ . We claim that  $S_n \rightarrow s$ . Given  $\varepsilon > 0$ ,  $\exists N$  such that  $m, n > N$ . Then  $|S_m - S_n| < \frac{\varepsilon}{2}$ .  $(S - \frac{\varepsilon}{2}, S + \frac{\varepsilon}{2})$  contains an infinite number of points.

$\exists m > N$  such that  $S_m \in N(s, \frac{\varepsilon}{2})$ . But then,  $|S_n - s| = |S_n - S_m + S_m - s| \leq |S_n - S_m| + |S_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ . Therefore,  $S_n \rightarrow s$ .  $\square$



**Theorem 8.4.** Let  $x_n$  be a sequence of non-negative real numbers.  $\sum x_n$  converges if  $S_k$ , the sequence of partial sums is bounded.

*Proof.*  $\sum_{n=1}^{\infty} x_n = \lim_{k \rightarrow \infty} S_k$ .  $S_k$  is increasing and bounded, it is convergent by the monotone convergence theorem.  $\square$

## 9 September 27

### 9.1 Limits of Functions

**Definition 9.1** (Limit of a function). Let  $f : D \rightarrow \mathbb{R}$  and let  $c$  be an accumulation point of the function. Then,  $\lim_{x \rightarrow c} f(x) = L$  if and only if given  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that if  $|x - c| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

*Remark.* Suppose we want to show that  $\lim_{x \rightarrow 2} S_x + 1 = 11$ . We are looking for some  $\delta > 0$  such that  $0 \leq |x - 2| < \delta$  and  $|S_x + 1 - 11| < \varepsilon$ . This is structured similarly to proofs of limits of sequences.

Additionally, the limit must go to an accumulation point of the function because we cannot find the limit of a value outside the function's domain.

**Theorem 9.1.**  $\lim_{x \rightarrow 5} 10x + 2 = 52$ .

*Proof.* We need to find some  $\delta > 0$  such that whenever  $0 < |x - 5| < \delta$ ,  $|10x + 2 - 52| < \varepsilon$ .

$$\begin{aligned} |10x - 50| &< \varepsilon \\ 10|x - 5| &< \varepsilon \\ |x - 5| &< \frac{\varepsilon}{10} \end{aligned}$$

Given  $\varepsilon > 0$ , let  $\delta = \frac{\varepsilon}{10}$ . Then, whenever  $0 < |x - 5| < \delta$ , we have  $|10x + 2 - 52| = |10x - 50| = 10|x - 5| < 10 * \frac{\varepsilon}{10} = \varepsilon$ .  $\square$

**Theorem 9.2.**  $\lim_{x \rightarrow 3} x^2 + 2x + 6 = 21$ .

*Proof.* We need to find some  $\delta > 0$  such that whenever  $0 < |x - 3| < \delta$ ,  $|(x^2 + 2x + 6) - 21| < \varepsilon$ .

$$\begin{aligned} |x^2 + 2x + 6 - 21| &< \varepsilon \\ |x^2 + 2x - 15| &< \varepsilon \\ |x + 5||x - 3| &< \varepsilon \end{aligned}$$

If  $\delta < 1 \implies |x + 5||x - 3| < 9|x - 3| < \varepsilon$ . Thus  $|x - 3| < \frac{\varepsilon}{9}$ . We let  $\delta = \min\{1, \frac{\varepsilon}{9}\}$ .

Given  $\varepsilon > 0$ , let  $\delta = \min\{1, \frac{\varepsilon}{9}\}$ . Then, whenever  $0 < |x - 3| < \delta$ , we have that  $|x + 5| < 9$ , thus,  $|(x^2 + 2x + 6) - 21| = |x^2 + 2x - 15| = |x + 5||x - 3| < \min\{1, \frac{\varepsilon}{9}\} * \frac{\varepsilon}{9} = \varepsilon$ .  $\square$

*Remark.* These proofs have two phases. First, we determine some  $\delta$  as an upper bound. Then, we show how this choice of  $\delta$  implies the limit is bounded by some  $\varepsilon$ .

**Theorem 9.3.** Let  $f : D \rightarrow \mathbb{R}$  and  $c$  is an accumulation point of  $D$ . Then,  $\lim_{x \rightarrow c} f(x) = L$  if and only if for every sequence  $S_n \in D$  such that  $S_n \rightarrow c$ ,  $S_n \neq c \forall n$ , then  $f(S_n)$  converges to  $L$ .

*Proof.*  $\lim_{x \rightarrow c} f(x) = L$  and  $S_n \rightarrow L \implies f(S_n) \rightarrow L$ . We need to find  $N$  such that  $n > N$  and  $|f(S_n) - L| < \varepsilon$ . We know that  $\exists \delta$  such that  $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$  and  $\exists N$  such that  $n > N \implies |S_n - c| < \delta$ . Thus, for  $n > N$  we have  $|f(S_n) - L| < \varepsilon$ .

Suppose  $L$  is not the limit of  $f$  as  $x$  approaches  $c$ . We must find  $(S_n)$  that converges to  $c$ , but  $f(S_n)$  does not converge to  $L$  (contrapositive).  $\exists \varepsilon > 0$  such that  $\forall \delta > 0, 0 < |x - c| < \delta \implies |f(x) - L| \geq \varepsilon$ . For each  $n \in \mathbb{N}$ ,  $\exists S_n \in D$  such that  $0 < |S_n - c| < \frac{1}{n}$  and  $|f(S_n) - L| \geq \varepsilon$ . Then,  $S_n \rightarrow c$ , but  $f(S_n) \not\rightarrow L$ . This is a contradiction.  $\square$

## 10 September 29

### 10.1 Sums of Limits

**Theorem 10.1.** Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $\lim_{x \rightarrow c} g(x) = M$ . Then,  $\lim_{x \rightarrow c} (f + g)(x) = L + M$ .

*Proof (Definition 9.1).* Given  $\varepsilon > 0$ , let  $\delta_1 > 0$  be such that  $0 < |x - c| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$ . Let  $\delta_2 > 0$  be such that  $0 < |x - c| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$ .

Let  $\delta = \min\{\delta_1, \delta_2\}$ . Then, for  $0 < |x - c| < \delta$ , we have

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

*Proof (Theorem 9.3).* Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $\lim_{x \rightarrow c} g(x) = M$ , and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} (f + g)(S_n) = \lim_{n \rightarrow \infty} f(S_n) + g(S_n) = \lim_{n \rightarrow \infty} f(S_n) + \lim_{n \rightarrow \infty} g(S_n) = L + M$$

Thus,  $\lim_{x \rightarrow c} (f + g)(x) = L + M$ .

□

*Remark.* This is true for  $-$ ,  $\times$ , and  $\div$  as well.

**Definition 10.1** (Sequential criterion for functional limits).  $\lim_{x \rightarrow c} f(x) = L$  if and only if whenever  $S_n \rightarrow c$ ,  $\lim_{n \rightarrow \infty} f(S_n) = L$ .

**Theorem 10.2.** Let  $k \in \mathbb{R}$ . If  $\lim_{x \rightarrow c} f(x) = L$ , then  $\lim_{x \rightarrow c} kf(x) = kL$ .

*Proof.* Let  $\lim_{x \rightarrow c} f(x) = L$ ,  $k \in \mathbb{R}$ , and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} kf(S_n) = k \lim_{n \rightarrow \infty} f(S_n) = kL$$

Thus,  $\lim_{x \rightarrow c} kf(x) = kL$ .

□

### 10.2 Continuity of Functions

**Definition 10.2** (Continuous function). A function  $f$  is continuous at  $x = c$  if and only if  $\lim_{x \rightarrow c} f(x) = f(c)$ . Let  $s$  be an accumulation point of the domain  $f : D \rightarrow \mathbb{R}$ . Then,  $f$  is continuous at  $s$  if and only if for each  $\varepsilon > 0$ ,  $\exists \delta > 0$  such that whenever  $0 < |x - s| < \delta$ ,  $|f(x) - f(s)| < \varepsilon$ .

*Remark.* Let  $f(x) = x \sin(\frac{1}{x})$  where  $x \neq 0$ ,  $f(0) = 0$ . If we want to show that this function is continuous, we need to find some  $\delta > 0$  such that  $|x| < \delta \implies |f(x) - f(0)| < \varepsilon$ . Let  $\delta = \varepsilon$ , then when  $|x| < \delta$ ,  $|f(x) - f(0)| = |x \sin(\frac{1}{x}) - 0| = |x \sin(\frac{1}{x})| \leq |x| < \varepsilon$ .

**Theorem 10.3.** If  $f$  and  $g$  are continuous at  $x = c$ , then  $f + g$  is also continuous at  $x = c$ .

*Proof.* Let  $f$  and  $g$  be continuous at  $c$  and  $S_n$  be a sequence of real numbers such that  $S_n \rightarrow c$ . Then,

$$\lim_{n \rightarrow \infty} (f + g)(S_n) = \lim_{n \rightarrow \infty} f(S_n) + \lim_{n \rightarrow \infty} g(S_n) = f(c) + g(c)$$

Thus,  $\lim_{x \rightarrow c} (f + g)(x) = (f + g)(c)$ . □

**Theorem 10.4.** Let  $f : D \rightarrow E$  be continuous at  $x = c$  and let  $g : E \rightarrow R$  be continuous at  $x = f(c)$ . Then, the composition  $g \circ f$  is continuous at  $x = c$ .

*Proof.* This is left as an exercise for the reader. □

# 11 October 6

## 11.1 Derivatives

**Definition 11.1** (Derivative). Let  $f$  be a real-valued function defined on an open interval containing  $c$ . We say  $f$  is differentiable at  $c$  if  $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$  exists. We call this limit  $f'(c)$ .

**Theorem 11.1.** If  $f$  is differentiable at  $c$ , then  $f$  is continuous at  $c$ .

*Proof.* Let  $f$  be defined on some interval  $I$  containing  $c$ . Then if  $f$  is differentiable at  $c$ , if and only if for  $x \neq c$ ,

$$f(x) = (x - c) \frac{f(x) - f(c)}{x - c} + f(c)$$

Then,  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - c) \frac{f(x) - f(c)}{x - c} + f(c) = \lim_{x \rightarrow c} (x - c) f'(c) + f(c) = f(c)$ . Therefore,  $f$  is continuous at  $c$ .  $\square$

### Derivative Rules

- $\frac{d}{dx} kf = k \frac{df}{dx}$
- $\frac{d}{dx} f + g = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx} f \cdot g = \frac{df}{dx} g + \frac{dg}{dx} f$
- $\frac{d}{dx} \frac{f}{g} = \frac{\frac{df}{dx} g - \frac{dg}{dx} f}{g^2}$

**Theorem 11.2** (Product rule).

$$(fg)' = f'g + fg'$$

*Proof.* Suppose  $f$  and  $g$  are differentiable at  $c$ . Then,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)(g(x) - g(c))}{x - c} + \frac{g(c)(f(x) - f(c))}{x - c} \\ &= f(c)g'(c) + g(c)f'(c) \end{aligned}$$

$\square$

**Theorem 11.3** (Quotient rule).

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

*Proof.* Let  $f$  and  $g$  be differentiable at  $c$ . Then,

$$\begin{aligned}
\lim_{x \rightarrow c} \frac{\frac{f}{g}(x) - \frac{f}{g}(c)}{x - c} &= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x \rightarrow c} \frac{\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}}{x - c} \\
&= \lim_{x \rightarrow c} \frac{f(x)g(c) - g(c)f(c) + g(c)f(c) - f(c)g(x)}{(x - c)g(x)g(c)} \\
&= \lim_{x \rightarrow c} \frac{g(c)\frac{f(x)-f(c)}{(x-c)} + f(c)\frac{g(x)-g(c)}{(x-c)}}{g(c)g(x)} \\
&= \lim_{x \rightarrow c} \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)} Aa
\end{aligned}$$

□

**Theorem 11.4** (Power rule).

$$(x^n)' = nx^{n-1}f' \quad \forall n \in \mathbb{N}$$

*Proof by induction.*  $p(n) = (x^n)' = nx^{n-1}f'$ .

$$p(1): f(x) = x. \quad \lim_{x \rightarrow c} \frac{x-c}{x-c} = 1 = 1 \cdot x^0.$$

$$p(k) \rightarrow p(k+1):$$

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\
&= \left(\frac{d}{dx}x^k\right) \cdot x + x^k \left(\frac{d}{dx}x\right) \\
&= kx^{k-1} \cdot x + x^k \cdot 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

□

**Theorem 11.5** (Chain rule).

$$g(f(x))' = g'(f(x)) \cdot f'(x)$$

*Proof.*

$$\begin{aligned}
\lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \frac{f(x) - f(c)}{x - c} \\
&= g'(f(x))f'(x)
\end{aligned}$$

□

*Remark.* This will not hold if  $f(x) = f(c)$ . This is not the full proof.

## 12 October 13

### 12.1 Differentiability and Continuity

**Theorem 12.1.** Let  $f$  be defined on an interval  $I$  containing  $c$ . Then,  $f$  is differentiable at  $c$  if and only if  $\exists$  a function  $\varphi$  on  $I$  such that  $\varphi$  is continuous at  $c$  and

$$f(x) - f(c) = \varphi(x)(x - c) \forall x \neq c$$

In this case, we have  $\varphi(c) = f'(c)$ .

*Remark.* Let  $f(x) = x^3$ . Then,  $f(x) - f(c) = x^3 - c^3 = (x^2 + xc + c^2)(x - c)$ .  $\phi(c) = c^2 + c \cdot c + c^2 = 3c^2 = f'(c)$ .

*Proof.* If  $f'(c)$  exists, we can define  $\varphi$  as

$$\varphi(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

Then,  $\varphi$  is continuous. Since  $\lim_{x \rightarrow c} \varphi(x) = f'(c) = \varphi(c)$ . Thus, the function is differentiable. If  $x = c$ , the equation from the theorem holds as  $0 = 0$ .

Assume  $\varphi$  is continuous at  $c$  and satisfies the equation. Then, continuity of  $\varphi$  implies  $\varphi(c) = \lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x)-f(c)}{x-c} \implies \varphi(c) = f'(c)$  since  $f$  is differentiable.  $\square$

**Theorem 12.2** (Chain rule).

$$g(f(c))' = g'(f(c)) \cdot f'(c)$$

*Proof.* Let  $c \in I$ .  $f$  is continuous at  $c$ . Define

$$\varphi(x) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$$

Thus,  $\varphi$  is continuous at  $c$ . Then,

$$\begin{aligned} \lim_{x \rightarrow c} \varphi(f(x)) &= \varphi(f(c)) = g'(f(c)) \\ g(y) - g(f(c)) &= \varphi(y)(y - f(c)) \\ g(f(x)) - g(f(c)) &= \varphi(f(x))(f(x) - f(c)) \\ \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} &= \lim_{x \rightarrow c} \frac{\varphi(f(x))(f(x) - f(c))}{x - c} \\ g'(f(c)) &= \lim_{x \rightarrow c} \varphi(f(x)) \cdot \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \\ g'(f(c)) &= g'(f(c)) \cdot f'(c) \end{aligned}$$

Thus, the chain rule holds.  $\square$

**Theorem 12.3.** If  $S$  is a nonempty compact subset of  $\mathbb{R}$ ,  $S$  has a max and a min.

*Proof.* Let  $m = \sup S$  exist by the completeness axiom. Given  $t > 0$ ,  $\exists x$  such that  $m - t < x < m$ . Then,  $m$  is an accumulation point of  $S$ . But  $S$  is closed by Heine-Borel. Thus,  $m \in S$ .

The same proof holds for the min. □

**Theorem 12.4.** If  $f$  is continuous and  $D$  is compact, then  $f(D)$  is compact. (Note: this will be on the final).

*Proof.* We know that the inverse of a continuous function is continuous (final exam proof) and that if an open set is continuous its inverse is also continuous (exam 2 proof).

Take an open cover  $U = \{u_i\}$  of  $f(D)$ . Then,  $f^{-1}(u_i)$  is an open cover for  $D$ . But, only a finite number are needed  $(\{u_1, u_2, \dots, u_n\})$ . Then,  $(\{f(u_1), f(u_2), \dots, f(u_n)\})$  is a finite subcover of  $u_i$  for  $f(D)$ . □

**Theorem 12.5.** Let  $D$  be compact and suppose  $f : D \rightarrow \mathbb{R}$  is continuous, then  $f$  assumes a min and a max.

*Proof.* Since  $D$  is compact,  $f(D)$  is compact. Thus,  $f(D)$  has a min  $y_1$  and a max  $y_2$ . Since  $y_1, y_2 \in f(D)$ ,  $\exists x_1, x_2 \in D$  such that  $f(x_1) = y_1$  and  $f(x_2) = y_2$ . Thus,  $f(x_1) \leq f(x) \leq f(x_2) \forall x \in D$ . □

**Theorem 12.6.** If  $f$  is differentiable on an  $(a, b)$  and  $f$  assumes a max or min for some  $c \in (a, b)$ , then  $f'(c) = 0$ .

*Proof.* Suppose  $f$  assumes its max is at  $c$ . That is to say  $f(x) \leq f(c) \forall x \in (a, b)$ . Let  $x_n$  be a sequence converging to  $c$  such that  $a < x_n < c$ . Then,

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges to  $f'(c)$ . But, each term is nonnegative. Therefore, the derivative is nonnegative  $\implies f'(c) \geq 0$ . Now, define  $y_n$  as a sequence converging to  $c$  such that  $c < y_n < b$ .

If we look at the sequence  $\frac{f(y_n) - f(c)}{y_n - c}$ , we see that it converges to  $f'(c)$ . But, each term is nonpositive. Therefore, the derivative is nonpositive, so  $f'(c) \leq 0 \therefore 0 \leq f'(c) \leq 0$ , so we must have that  $f'(c) = 0$ . □



## 13 October 20

### 13.1 Mean Value Theorem

**Theorem 13.1** (Rolle's theorem). Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and let  $f(a) = f(b)$ . Then  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .

*Proof.* Since  $f$  is continuous and  $[a, b]$  is compact,  $\exists x_1, x_2 \in [a, b]$  such that  $f(x_1) \leq f(x) \leq f(x_2) \forall x \in [a, b]$ . If  $x_1$  and  $x_2$  are the endpoints of the interval, then  $f$  is a compact function, thus  $f'(c) = 0 \forall c \in (a, b)$ . Otherwise,  $f$  contains a max at  $x_2 \therefore f'(x_2) = 0$ . Thus  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .  $\square$

**Theorem 13.2** (Mean value theorem). Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then  $\exists c \in (a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .

*Proof.* Let  $g(x)$  be defined as  $g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ . Let  $h(x)$  be the distance from the graph of  $f \circ g$ . That is,  $h = f - g$ . Then,  $h$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Furthermore,  $h(a) = h(b) = 0$ .

By Rolle's Theorem,  $\exists c \in (a, b)$  such that  $h'(c) = 0$ . Thus,

$$0 = h'(c) = f'(c) - g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Therefore,  $f'(c) = \frac{f(b)-f(a)}{b-a}$ .  $\square$

**Theorem 13.3.** Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then if  $f'(x) = 0 \forall x \in (a, b)$ , then  $f$  is constant on  $[a, b]$ .

*Proof.* Suppose  $f$  is not constant. Then,  $\exists x_1, x_2$  such that  $a \leq x_1 < x_2 \leq b$  and  $f(x_1) \neq f(x_2)$ . By the Mean Value Theorem,  $\exists c \in (x_1, x_2)$  such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

But, this is a contradiction. Therefore,  $f$  is constant on  $[a, b]$ .  $\square$

**Theorem 13.4.** Let  $f$  be differentiable on an interval  $I$ . If  $f'(x) > 0 \forall x \in I$ , then  $f$  is strictly increasing on  $I$ .

*Proof.* Suppose  $f'(x) > 0 \forall x \in I$  and  $x_1, x_2 \in I$  such that  $x_1 < x_2$ . Mean Value Theorem implies that  $\exists c \in (x_1, x_2)$  such that  $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ . Which is to say that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Thus,  $f(x_2) - f(x_1)$  is positive since  $f'(c)$  and  $(x_2 - x_1)$  are both positive. Therefore,  $f$  is increasing.  $\square$

## 13.2 Intermediate Value Theorem

**Theorem 13.5** (Intermediate value theorem). Let  $f$  be continuous on  $[a, b]$  and suppose  $f(a) < 0 < f(b)$ . Then  $\exists c \in (a, b)$  such that  $f(c) = 0$ .

*Proof.* Let  $c$  be the largest value for which  $f(x) \leq 0$ . Let  $S = \{x \in [a, b] \mid f(x) \leq 0\}$ . Since  $a \in S$ ,  $S$  is nonempty. Thus,  $\sup S = c$  exists.

We claim that  $f(c) = 0$ . Suppose  $f(c) < 0$ , then  $\exists$  a neighborhood  $U$  of  $c$  such that  $f(x) < 0 \forall x \in U \cap [a, b]$ . Now,  $c \neq b$  since  $f(a) < 0 < f(b)$ . Thus,  $U$  contains a point  $p$  such that  $c < p < b$  where  $f(p) < 0$ . But, this is a contradiction since  $p \in S$  and  $p > c$ . Therefore,  $f(c) \not< 0$ .

Similarly, suppose  $f(c) > 0$ . We can follow this proof in the other direction to show that  $f(c) = 0$ . □

*Remark.* This is the baby version of the intermediate value theorem. The full version will be asked on exam 2.

## 14 October 25

### 14.1 Cauchy Mean Value Theorem

**Theorem 14.1** (Cauchy mean value theorem). Let  $f$  and  $g$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then,  $\exists$  at least one  $c \in (a, b)$  such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

*Proof.* Let  $h(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x) \forall x \in [a, b]$ .

Note that

$$\begin{aligned} h(a) &= (f(b) - f(a))g'(a) - (g(b) - g(a))f'(a) = 0 \\ &= f(b)g'(a) - f(a)g'(b) \end{aligned}$$

and

$$\begin{aligned} h(b) &= (f(b) - f(a))g'(b) - (g(b) - g(a))f'(b) = 0 \\ &= f(b)g'(b) - f(a)g'(b) \end{aligned}$$

Thus,  $h$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $h(a) = h(b)$ . Therefore, by Rolle's theorem,  $\exists c \in (a, b)$  such that  $h'(c) = 0$ . That is to say,

$$h'(c) = (f(b) - f(a))g''(c) - (g(b) - g(a))f''(c) = 0$$

which implies the desired equality. □

### 14.2 L'Hospital's Rule

**Theorem 14.2** (L'Hospital's rule). Let  $f$  and  $g$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  and  $f(c) = g(c) = 0$ . Also suppose that  $g'(c) \neq 0$  in some neighborhood of  $c$ .

If

$$\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L$$

*Proof.* Let  $x_n$  be a sequence that converges to  $c$ . By the Cauchy mean value theorem  $\exists$  a sequence  $c_n$  such that  $c_n$  is between  $x_n$  and  $c$  for each  $n$  and

$$(f(x_n) - f(c))g'(c_n) = (g(x_n) - g(c))f'(c_n)$$

Thus,

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)}$$

Furthermore, since  $x_n \rightarrow c$  and  $c_n \rightarrow c$ , we have that if  $\lim_{n \rightarrow \infty} \frac{f'(c_n)}{g'(c_n)} = L$ , then  $\lim_{n \rightarrow \infty} \frac{f(x_n)}{g(x_n)} = \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L$ . □

### 14.3 Taylor's Theorem

**Theorem 14.3** (Taylor's theorem). Let  $f$  and its first  $n$  derivatives be continuous on  $[a, b]$  (implying that they are also differentiable). Let  $x_0 \in [a, b]$ . Then, for each  $x \in [a, b]$  with  $x \neq x_0$ ,  $\exists$  a  $c$  between  $x$  and  $x_0$  such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

*Proof.* Let  $x_0$  and  $x$  be given and let  $J = [x_0, x]$  or  $[x, x_0]$ . We will define  $F$  on  $J$  as follows:

$$F(t) = f(x) - f(t) - (x - t)f'(t) - \frac{(x - t)^2}{2!}f''(t) - \cdots - \frac{(x - t)^n}{(n)!}f^{(n)}(t)$$

Note that

$$F'(t) = \frac{-(x - t)^n}{n!}f^{(n+1)}(t)$$

and define  $G$  by

$$G(t) = F(t) - \left(\frac{x - t}{x - x_0}\right)^{n+1} F(x_0)$$

Note that  $G(x_0) = 0 = G(x)$ . Then, by Rolle's Theorem,  $\exists c$  between  $x$  and  $x_0$  such that  $G'(c) = 0$ . That is,

$$0 = G'(c) = F'(c) + (n+1)\frac{(x - c)^n}{(x - x_0)^{n+1}}F(x_0)$$

Hence,

$$\begin{aligned} F(x_0) &= -\left(\frac{1}{n+1}\right)\left(\frac{(x - x_0)^{n+1}}{(x - c)^n}\right)F'(c) \\ &= \left(\frac{1}{n+1}\right)\left(\frac{(x - x_0)^{n+1}}{(x - c)^n}\right)\left(\frac{(x - c)^n}{n!}\right)f^{(n+1)}(c) \\ &= \left(\frac{(x - x_0)^{n+1}}{(n+1)!}\right)f^{(n+1)}(c) \end{aligned}$$

which implies the desired equality. □

## 15 October 27

### 15.1 Applications of Taylor's Theorem

**Definition 15.1** (Taylor polynomial). We denote a Taylor polynomial  $\mathcal{P}_n(x)$  as

$$\mathcal{P}_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and a remainder term  $R_n(x)$  with some  $c \in \mathbb{R}$  where  $x_0 \leq c \leq x$  as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

**Example 15.1.** Estimate  $e^6$  on  $[-1, 1]$  using a Taylor polynomial. Let  $f(x) = e^x$ ,  $x_0 = 0$  and  $n = 5$ .

$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \end{aligned}$$

You can place an upper bound on the remainder term on the interval  $[-1, 1]$  ( $c = 1$  maxes out  $f'(c)$  and  $x = 1$  maxes out  $x^6$ ).

$$|R_5(x)| = \left| \frac{f^{(6)}(c)}{6!}x^6 \right| = \frac{|f^{(6)}(c)|}{6!} |x^6| \leq \frac{e \cdot 1}{6!}$$

**Example 15.2.** Estimate  $\cos(1)$  to within  $1/1000$  using a Taylor polynomial. Take  $x_0 = 0$ , on  $[-1, 1]$ . We need

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \right| \leq \frac{1}{1000}$$

If you find the Taylor polynomial of cosine to the 6th degree,

$$\begin{aligned} \cos(0) &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ \therefore \left| -\frac{x^6}{6!} \right| &\leq \frac{1}{1000} \text{ on } [-1, 1] \end{aligned}$$

Hence, this is a good enough approximation that estimates  $\cos(x)$  on  $[-1, 1]$  within an error of  $1/1000$ .

*Remark.* You can estimate  $\pi$  with  $\tan^{-1}(1)$  because it equals  $\pi/4$ .

**Theorem 15.1.**  $e$  is irrational.

*Proof.* We know that  $e < 3$ . Then, we have that

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) < \frac{3}{(n+1)!} < \frac{3}{(n+1)!}$$

We can assume  $e = \frac{a}{b}$  where  $b \neq 0$  and  $a, b \in \mathbb{Z}$ . Then,

$$0 < \frac{a}{b} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}$$

Let  $M$  be the middle term. Then, take  $n > \max\{b, 3\}$ . Finally, we have

$$0 < M < a - n! \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) < \frac{3}{n+1} < \frac{3}{4}$$

This is a contradiction because there is no integer between 0 and  $\frac{3}{4}$  even though  $M$  is an integer.  $\square$

## 15.2 Riemann Integrability

**Definition 15.2** (Partition). Let  $[a, b]$  be an interval in  $\mathbb{R}$ . A partition  $\mathcal{P}$  of  $[a, b]$  is a finite set of points  $\{x_0, x_1, \dots, x_n\}$  such that  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ .

**Definition 15.3** (Upper and lower sums). Let  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  be a partition of  $[a, b]$ , and let

$$\begin{aligned} M_i(f) &= \sup \{f(x) : [x_{i-1}, x_i]\} \\ m_i(f) &= \inf \{f(x) : [x_{i-1}, x_i]\} \end{aligned}$$

For example,  $f(x) = x + 3$ ,  $x_0 = 1$  and  $x_1 = 2$ . Hence,

$$\begin{aligned} M_1(f) &= 5 \\ m_1(f) &= 4 \end{aligned}$$

Let  $\Delta x_i = x_i - x_{i-1}$ . We then define  $U(f, \mathcal{P}) = \sum_{i=1}^n M_i \Delta x_i$  (the upper sum of  $f$  with respect to  $\mathcal{P}$ ) and  $L(f, \mathcal{P}) = \sum_{i=1}^n m_i \Delta x_i$  (the lower sum of  $f$  with respect to  $\mathcal{P}$ ). Now, define

$$\begin{aligned} U(f) &= \inf \{U(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\} \\ L(f) &= \sup \{L(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\} \end{aligned}$$

**Definition 15.4** (Riemann integrable). We say that  $f$  is Riemann integrable if and only if  $U(f) = L(f)$ . In this case, we write

$$\int_a^b f(x) dx = U(f) = L(f)$$

To show a function  $f$  is Riemann integrable on  $[a, b]$  given  $\varepsilon > 0$ , we only need to find one partition such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

## 16 November 3

### 16.1 Partitions

**Theorem 16.1.** Let  $f$  be bounded on  $[a, b]$  if  $\mathcal{P}$  and  $\mathcal{Q}$  are partitions of  $[a, b]$  such that  $\mathcal{Q}$  is a refinement of  $\mathcal{P}$ . Then,

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P})$$

*Proof.* We know that  $\mathcal{Q}$  will contain more points than  $\mathcal{P}$ .  $\mathcal{P}$  is described by  $m_k \cdot (x_k - x_{k-1})$  while  $\mathcal{Q}$  is described by  $m_x \cdot (x^* - x_{k-1}) + m_x \cdot (x_k - x^*)$ .  $\square$

**Theorem 16.2.** Let  $\mathcal{P}$  and  $\mathcal{Q}$  be partitions of  $[a, b]$ . Then

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q})$$

*Proof.* Let  $\mathcal{P}$  and  $\mathcal{Q}$  be partitions of  $f$ . Then  $\mathcal{P} \cup \mathcal{Q}$  is a refinement of  $\mathcal{P}$  and  $\mathcal{Q}$ . Thus,

$$L(f, \mathcal{P}) \leq (f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{P} \cup \mathcal{Q}) \leq U(f, \mathcal{Q})$$

$\square$

**Theorem 16.3.** Let  $f$  be bounded on  $[a, b]$ . Then,  $L(f) \leq U(f)$ .

*Proof.* Let  $\mathcal{P}$  and  $\mathcal{Q}$  be partitions of  $[a, b]$ . Then by the previous theorem,  $U(f, \mathcal{Q})$  is an upper bound for

$$S = \{L(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

So,  $U(f, \mathcal{Q})$  is at least as large as  $\sup S = L(f)$ . That is,  $L(f) \leq U(f, \mathcal{Q})$  for each partition  $\mathcal{Q}$ . Then,

$$L(f) \leq \inf\{U(f, \mathcal{Q}) : \mathcal{Q} \text{ is a partition of } [a, b]\} = U(f)$$

Therefore,  $L(f) \leq U(f)$ .  $\square$

**Example 16.1.**  $f(x) = x^2$  on  $[0, 1]$  with partition  $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ .

$$\begin{aligned} M_i &= \sup \left\{ f(x) : x \in \left[ \frac{i-1}{n}, \frac{i}{n} \right] \right\} = \left( \frac{i^2}{n^2} \right) \\ M_i &= \inf \left\{ f(x) : x \in \left[ \frac{i-1}{n}, \frac{i}{n} \right] \right\} = \left( \frac{i-1}{n} \right)^2 \\ U(f, \mathcal{P}_n) &= \sum_{i=1}^n M_i \cdot \Delta x_i = \sum_{i=1}^n \left( \frac{i}{n} \right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2 = \left[ \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right] \\ L(f, \mathcal{P}_n) &= \sum_{i=1}^n m_i \cdot \Delta x_i = \sum_{i=1}^n \left( \frac{i-1}{n} \right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n (i-1)^2 = \left[ \frac{1}{n^3} \cdot \frac{n(n-1)(2n-1)}{6} \right] \end{aligned}$$

Then,  $\lim_{n \rightarrow \infty} U(f, \mathcal{P}_n) = \frac{1}{3}$  and  $\lim_{n \rightarrow \infty} L(f, \mathcal{P}_n) = \frac{1}{3}$ . Thus,  $U(f) \leq \frac{1}{3}$  and  $L(f) \geq \frac{1}{3}$ . Because  $L(f) \leq U(f)$ , we have that  $L(f) = U(f) = \frac{1}{3}$ .

Since  $L(f) = U(f)$ , this function is Riemann integrable. Therefore,

$$\int_0^1 x^2 = \int_0^1 x^2 dx = \frac{1}{3}$$

**Theorem 16.4.** Let  $f$  be a bounded function on  $[a, b]$ . Then,  $f$  is Riemann integrable if and only if given an  $\varepsilon > 0$ ,  $\exists$  a partition of  $[a, b]$  such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

*Proof.* If  $f$  is Riemann integrable, since  $\varepsilon > 0$ ,  $\exists$  a partition  $\mathcal{P}_1$  such that

$$L(f, \mathcal{P}_1) > L(f) - \frac{\varepsilon}{2}$$

Similarly,  $\exists \mathcal{P}_2$  such that

$$U(f, \mathcal{P}_2) < U(f) + \frac{\varepsilon}{2}$$

Let  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ . Then,

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &\leq U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) \\ &< \left( U(f) + \frac{\varepsilon}{2} \right) - \left( L(f) - \frac{\varepsilon}{2} \right) \\ &= U(f) - L(f) + \varepsilon \\ &= \varepsilon \end{aligned}$$

Therefore,  $f$  is Riemann integrable.

Conversely, given  $\varepsilon > 0$ , suppose  $\exists \mathcal{P}$  such that  $U(f, \mathcal{P}) < L(f, \mathcal{P}) + \varepsilon$ . Then,

$$U(f, \mathcal{P}) \leq U(f, \mathcal{P}) < L(f, \mathcal{P}) + \varepsilon \leq L(f) + \varepsilon$$

Therefore,  $U(f) \leq L(f)$ . But then  $L(f) = U(f)$ , so  $f$  is Riemann integrable.  $\square$

*Remark.* Generally, we just need to find some partition  $\mathcal{P}$  such that  $U(f, \mathcal{P})$  and  $L(f, \mathcal{P})$  are within  $\varepsilon$  of each other.

**Theorem 16.5.** Show that  $f(x) = x$  is Riemann integrable on  $[0, 1]$ .

*Proof.* We need to find a partition  $\mathcal{P}$  such that for every  $\varepsilon > 0$ ,

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

Define  $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ . Then,

$$\begin{aligned} U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) &= \sum_{i=1}^n M_i \Delta x_i - \sum_{i=1}^n m_i \Delta x_i \\ &= \sum_{i=1}^n \left( \frac{i}{n} \right) \cdot \frac{1}{n} - \sum_{i=1}^n \left( \frac{i-1}{n} \right) \cdot \frac{1}{n} \\ &= \frac{1}{n^2} \left( \sum_{i=1}^n i - \sum_{i=1}^n (i-1) \right) \\ &= \frac{1}{n^2} \left( \frac{n(n+1)}{2} - \frac{n(n-1)}{2} \right) \\ &= \frac{1}{n^2} \cdot n \\ &= \frac{1}{n} < \varepsilon \implies n > \frac{1}{\varepsilon} \end{aligned}$$

Thus, we should choose some  $n > \frac{1}{\varepsilon}$  so  $\mathcal{P} = \mathcal{P}_n$ . Therefore,  $f$  is Riemann integrable.  $\square$



# 17 November 8

## 17.1 Tagged Partitions

**Definition 17.1** (Tagged partition).  $\dot{\mathcal{P}}$  is a tagged partition of the form  $\{([x_{i-1}, x_i], t_i)\}_{i=1}^n$  where  $t_i \in [x_{i-1}, x_i]$ . Let  $\mathcal{P}$  be a tagged partition of  $[a, b]$ . Then, define the Riemann sum of  $f$  with respect to  $\mathcal{P}$  on  $[a, b]$  as

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i) \cdot (x_i - x_{i-1})$$

*Remark.*  $\|\dot{\mathcal{P}}\| = \|\mathcal{P}\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$

**Definition 17.2** (Riemann integrable). A function  $f : [a, b] \rightarrow R$  is said to be Riemann integrable on  $[a, b]$  if  $\exists$  a number  $L$  such that  $\forall \varepsilon > 0, \exists \delta > 0$  such that if  $\dot{\mathcal{P}}$  is any partition with  $\|\dot{\mathcal{P}}\| < \delta$ , then

$$|S(f, \dot{\mathcal{P}}) - L| < \varepsilon$$

In this case we say that  $\int_a^b f = \int_a^b f(x)dx = L$ .

**Theorem 17.1.** Every constant function is Riemann integrable on  $[a, b]$ .

*Proof.* Given  $\varepsilon > 0$ , we need to find  $\delta$  such that  $\|\dot{\mathcal{P}}\| < \delta \implies |S(f, \dot{\mathcal{P}}) - k(b-a)| < \varepsilon$ . We have that

$$\begin{aligned} S(f, \dot{\mathcal{P}}) &= f(t_1) \cdot \Delta x_1 \\ &= k(b-a) \\ |S(f, \dot{\mathcal{P}}) - k(b-a)| &= |k(b-a) - k(b-a)| = 0 < \varepsilon \end{aligned}$$

So, we can choose some  $\delta$  to satisfy this condition. Thus, every constant function is Riemann integrable.  $\square$

**Lemma 17.1.** Let  $k \in R$  and  $\dot{\mathcal{P}}$  be a tagged partition, then

$$S(kf, \dot{\mathcal{P}}) = kS(f, \dot{\mathcal{P}})$$

**Theorem 17.2.** Let  $k \in R$  and  $f \in \mathcal{R}[a, b]$ , then

$$\int_a^b kf = k \int_a^b f$$

*Proof.* Given  $\varepsilon > 0$ , we need to find  $\delta$  such that  $\|\dot{\mathcal{P}}\| < \delta \implies |S(kf, \dot{\mathcal{P}}) - k \int_a^b f| < \varepsilon$ . Since  $f \in \mathcal{R}[a, b]$ ,  $\exists \delta$  such that  $\|\dot{\mathcal{P}}\| < \delta \implies |S(f, \dot{\mathcal{P}}) - \int_a^b f| < \frac{\varepsilon}{|k|}$ . Then,  $\forall \dot{\mathcal{P}}$  such that  $\|\dot{\mathcal{P}}\| < \delta$ , we have that

$$\left| S(kf, \dot{\mathcal{P}}) - k \int_a^b f \right| = \left| kS(f, \dot{\mathcal{P}}) - k \int_a^b f \right|$$

$$\begin{aligned}
&= \left| kS(f, \dot{\mathcal{P}}) - \int_a^b kf \right| \\
&< \frac{\varepsilon}{|k|} \cdot |k| = \varepsilon
\end{aligned}$$

□

*Remark.* Note that  $\mathcal{R}[a, b]$  is the set of all Riemann integrable functions on  $[a, b]$ .

**Theorem 17.3.** (On exam 2) If  $f, g \in \mathcal{R}[a, b]$ , then

$$f + g \in \mathcal{R}[a, b]$$

*Proof.* We can either find  $p$  such that  $U(f + g, \mathcal{P}) - L(f + g, \mathcal{P}) < \varepsilon$  to show that  $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$   
OR we can find  $|S(f + g, \dot{\mathcal{P}}) - \int_a^b f + \int_a^b g| < \varepsilon$  to show that  $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ . □

**Theorem 17.4.** (On exam 2)

$$\left| S(f + g, \dot{\mathcal{P}}) - \left( \int_a^b f + \int_a^b g \right) \right| < \varepsilon$$

## 18 November 10

### 18.1 Fundamental Theorem of Calculus

**Definition 18.1** (Fundamental theorem of calculus).  $f(x) : D \rightarrow \mathbb{R}$  is uniformly continuous if and only if given  $\varepsilon > 0$ ,  $\exists \delta$  such that  $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ .

*Remark.* We are closer to proving this!

**Theorem 18.1.** A continuous function on a closed interval  $[a, b]$  is uniformly continuous.

*Remark.* To be proved.

**Theorem 18.2.** Let  $f$  be continuous on  $[a, b]$ . Then,  $f$  is Riemann integrable on  $[a, b]$ .

*Proof.* Since  $f$  is continuous on  $[a, b]$ ,  $\exists \delta > 0$  such that when  $|x - y| < \delta$ ,  $|f(x) - f(y)| < \frac{\varepsilon}{b-a} \forall \varepsilon > 0$ . Let  $\mathcal{P}$  be a partition of  $[a, b]$  such that  $\Delta x_i < \delta \forall i$ . On each subinterval  $[x_i, x_{i+1}]$ ,  $f$  will obtain a maximum and minimum value at  $s_i$  and  $t_i$  respectively. Furthermore,  $|s_i - t_i| < \delta$ , so

$$0 \leq M_i - m_i = f(t_i) - f(s_i) < \frac{\varepsilon}{b-a} \forall i$$

Then,

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{i=1}^n (M_i - m_i) \Delta x_i < \sum_{i=1}^n \frac{\varepsilon}{b-a} \Delta x_i = \frac{\varepsilon}{b-a} (b-a) = \varepsilon$$

□

**Theorem 18.3.** If  $f \in R[a, c]$  and  $f \in R[c, b]$ , then  $f \in R[a, b]$  and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

*Proof.* Given  $\varepsilon > 0$ ,  $\exists$  a partition  $\mathcal{P}_1$  of  $[a, c]$  and  $\mathcal{P}_2$  of  $[c, b]$  such that  $U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) < \frac{\varepsilon}{2}$  and  $U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2) < \frac{\varepsilon}{2}$ . Then, define  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ . Then,  $\mathcal{P}$  is a partition of  $[a, b]$  and

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &= U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) - L(f, \mathcal{P}_2) \\ &= U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

So,  $f \in R[a, b]$ . Furthermore,

$$\begin{aligned} \int_a^b f &\leq U(f, \mathcal{P}) = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) \\ &< L(f, \mathcal{P}_1) + L(f, \mathcal{P}_2) + \varepsilon \\ &\leq \int_a^c f + \int_c^b f + \varepsilon \end{aligned}$$

Similarly,

$$\begin{aligned}
\int_a^b f &\geq L(f, \mathcal{P}) = L(f, \mathcal{P}_1) + L(f, \mathcal{P}_2) \\
&> U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) - \varepsilon \\
&\geq \int_a^c f + \int_c^b f - \varepsilon
\end{aligned}$$

Therefore,  $\int_a^b f = \int_a^c f + \int_c^b f$ . □

**Theorem 18.4.** If  $f$  is Riemann integrable on  $[a, b]$  and  $g$  is continuous on  $[c, d]$  when  $f([a, b]) \subseteq [c, d]$ , then  $g \circ f$  is Riemann integrable on  $[a, b]$ .

*Remark.* To be proved.

**Theorem 18.5.** Let  $f$  be Riemann integrable on a closed interval  $[a, b]$ . Then,  $|f|$  is Riemann integrable on  $[a, b]$  and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

*Proof.*  $|x|$  is continuous so we can apply the previous theorem. Then,

$$\begin{aligned}
-|f(x)| &\leq f(x) \leq |f(x)| \\
-\int_a^b |f| &\leq \int_a^b f \leq \int_a^b |f| \text{ because } \frac{M_i}{m_i}
\end{aligned}$$

Thus,  $\left| \int_a^b f \right| \leq \int_a^b |f|$ . □