M 361K Homework 4

Ishan Shah

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6.3

13. Try to use L'Hospital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \to (\pi/2)^-$. Then evaluate directly by changing to sines and cosines.

Proof.

$$\begin{aligned}
\lim_{x \to (\pi/2)^-} \frac{\tan x}{\sec x} &= \lim_{x \to (\pi/2)^-} \frac{\sec^2 x}{\sec x \tan x} \\
&= \lim_{x \to (\pi/2)^-} \frac{\sec x}{\tan x} \\
&= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} \\
&= \lim_{x \to (\pi/2)^-} \frac{\tan x}{\sec x}
\end{aligned}$$

After iteratively using L'Hospital's Rule, we find a cycle which means that we cannot use L'Hospital's Rule to find the limit. Instead, we can us sines and cosines.

$$\begin{aligned}
\lim_{x \to (\pi/2)^-} \frac{\tan x}{\sec x} &= \lim_{x \to (\pi/2)^-} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \\
&= \lim_{x \to (\pi/2)^-} \sin x \\
&= 1
\end{aligned}$$

Thus, $\lim_{x \to (\pi/2)^-} \frac{\tan x}{\sec x} = 1$. □

14. Show that if $c > 0$, then $\lim_{x \to c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$.

Proof.

$$\begin{aligned}
\lim_{x \to c} \frac{x^c - c^x}{x^x - c^c} &= \lim_{x \to c} \frac{cx^{c-1} - c^x \ln c}{x^x(\ln x + 1) - 0} \\
&= \frac{cc^{c-1} - c^c \ln c}{c^c(\ln c + 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c^c(1-\ln c)}{c^c(1+\ln c)} \\
&= \frac{1-\ln c}{1+\ln c}
\end{aligned}
$$

Thus, $\lim_{x\to c} \frac{x^c - c^x}{x^x - c^c} = \frac{1-\ln c}{1+\ln c}$. □

6.4

11. If $x \in [0,1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

Use this to approximate $\ln 1.5$ with an error less than 0.01. Less than 0.001.

Proof. We can start by finding the first few terms of the Taylor series of $\ln(1+x)$. We need the derivatives of $\ln(1+x)$ up to n.

$$
\begin{aligned}
f(x) &= \ln(1+x) \implies f(0) = 0 \\
f'(x) &= \frac{1}{1+x} \implies f'(0) = 1 \\
f''(x) &= -\frac{1}{(1+x)^2} \implies f''(0) = -1 \\
&\vdots \\
f^{(n)}(x) &= (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \implies f^{(n)}(0) = (-1)^{n-1}(n-1)!
\end{aligned}
$$

Now, we can find the Taylor series of $\ln(1+x)$.

$$
\begin{aligned}
\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| &= \left| x^{n+1}\frac{f^{n+1}(\varepsilon)}{(n+1)!} \right| \\
&= \left| (-1)^n x^{n+1} \frac{n!}{(n+1)!(1+\varepsilon)^{n+1}} \right| \\
&= \left| \frac{x^{n+1}}{(n+1)(1+\varepsilon)^{n+1}} \right| \\
&< \frac{x^{n+1}}{n+1}
\end{aligned}
$$

Now, we can use this to approximate $\ln 1.5$. We can let $x = 0.5$ and bound our error with

$$\frac{x^{n+1}}{n+1}$$

- Error < 0.01: $n = 4$

$$\frac{0.5^5}{5} = 0.006 < 0.01$$
$$\ln 1.5 = 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} \approx 0.401$$

- Error < 0.001: $n = 7$

$$\frac{0.5^8}{8} = 0.0005 < 0.001$$
$$\ln 1.5 = 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} + \frac{0.5^5}{5} - \frac{0.5^6}{6} + \frac{0.5^7}{7} \approx 0.405$$

□

13. Calculate e correct to 7 decimal places.

Proof. We can use the Taylor series of e^x to approximate e.

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \end{aligned}$$

We only need the first 11 terms of the Taylor series to get e correct to 7 decimal places.

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} + \frac{1}{3628800} \\ &= 2.7182818 \end{aligned}$$

□

7.1

2. If $f(x) := x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 1, and the tags are selected as indicated. $\mathcal{P}_2 := (0, 2, 3, 4)$.

- $\dot{\mathcal{P}}_2$ with the tags at the left endpoints of the subintervals.

$$\begin{aligned} S(f, \dot{\mathcal{P}}) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= f(0)(x_1 - x_0) + f(2)(x_2 - x_1) + f(3)(x_3 - x_2) \\ &= 0^2(2 - 0) + 2^2(3 - 2) + 3^2(4 - 3) \\ &= 0 \cdot 2 + 4 \cdot 1 + 9 \cdot 1 = 13 \end{aligned}$$

- $\dot{\mathcal{P}}_2$ with the tags at the right endpoints of the subintervals.

$$\begin{aligned}
S(f, \dot{\mathcal{P}}) &= \sum_{i=1}^{n} f\left(t_i\right)\left(x_i - x_{i-1}\right) \\
&= f(2)\left(x_1 - x_0\right) + f(3)\left(x_2 - x_1\right) + f(4)\left(x_3 - x_2\right) \\
&= 2^2(2-0) + 3^2(3-2) + 4^2(4-3) \\
&= 4 \cdot 2 + 9 \cdot 1 + 16 \cdot 1 = 33
\end{aligned}$$

6b. Let $h(x) := 2$ if $0 \leq x < 1, h(1) := 3$ and $h(x) := 1$ if $1 < x \leq 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

Proof. If we define the integral as the area under the curve, we can estimate that the integral of h is $L = 3$. Now, let $\mathcal{P}_n = \{[0, 2/n], [2/n, 4/n], \ldots, [2/(n-1), n/2]\}$. Then, the Riemann integral is given by

$$\begin{aligned}
S(f, \mathcal{P}_n) &= \sum_{i=1}^{n} f(x_i) \Delta x_i \\
&= \sum_{i=1}^{n/2} 2 \cdot 2/n + \sum_{i=n/2+1}^{n} 1 \cdot 2/n \\
&= 2 \cdot 2/n \cdot n/2 + 1 \cdot 2/n \cdot n/2 \\
&= 2 + 1 \\
&= 3
\end{aligned}$$

Then, for any $\varepsilon > 0$, $|S(f, \mathcal{P}_n) - L| = |3 - 3| = 0 < \varepsilon$. Thus, h is Riemann integrable on $[0,2]$ and the integral of h is 3. $\square$

8. If $f \in \mathcal{R}[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$, show that $\left|\int_a^b f\right| \leq M(b-a)$.

Proof. We have that $-M \leq f(x) \leq M$ for all $x \in [a,b]$. Therefore, we can write

$$-M(b-a) = \int_a^b -M \leq \int_a^b f \leq \int_a^b M = M(b-a)$$

Thus, $\left|\int_a^b f\right| \leq M(b-a)$. $\square$

10. Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $\lim_n S(g; \dot{\mathcal{P}}_n)$ exists.

Proof. Because $g(x)$ is not bounded on $[0,1]$, $g \notin \mathcal{R}[0,1]$. Now, let

$$\dot{\mathcal{P}}_n = \{([\frac{i-1}{n}, \frac{i}{n}], \frac{i}{n})\}_{i=1}^{n} \forall n \in \mathbb{N}$$

Then, $||\dot{\mathcal{P}}_n|| = \frac{1}{n}$ so $\lim_{n\to\infty} ||\dot{\mathcal{P}}_n|| = \infty$. Now, $S(g, \dot{\mathcal{P}}_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = 0$ since $g(x) = 0$ for all rational $x \in [0,1]$. Thus, $\lim_n S(g, \dot{\mathcal{P}}_n) = 0$. $\square$