M 361K Homework 2

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3.3

5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}\ \forall\ n \in \mathbb{N}$. Show that (y_n) converges and find the limit.

8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n\ \forall\ n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.

12. Establish the convergence and find the limits of the following sequences.

(a) $((1 + 1/n)^{n+1})$

(b) $((1 + 1/n)^2 n)$

3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.

4b. Show that the sequence $(\sin n\pi/4)$ is divergent.

10. Let (x_n) be a bounded subsequence and for each $n \in \mathbb{N}$, let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

3.5

2. Show directly from the definition that the following are Cauchy sequences.

(a) $(\frac{n+1}{n})$

(b) $(1 + \frac{1}{2!} + \cdots + \frac{1}{n!})$

7. Let (x_n) be a Cauchy sequence such that x_n is an inteeger for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.

8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.

4.1

2. Determine a condition on $|x - 4|$ that will assure that:

(a) $|\sqrt{x} - 2| < \frac{1}{2}$

(b) $|\sqrt{x} - 2| < 10^{-2}$

6. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose that there exists constants K and L such that $|f(x) - L| \leq K|x - c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

9a. Use either the ϵ-δ definition of a limit or the Sequential Criterion for limits to establish that $\lim_{x \to 2} \frac{1}{1-x} = -1$.

10a. Use the definition of the limit to show that $\lim_{x \to 2}(x^2 + 4x) = 12$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

4.2

1. Apply theorem 4.2.4 to determine the following limits:

(a) $\lim_{x \to 1}(x + 1)(2x + 3)$ where $(x \in \mathbb{R})$

(b) $\lim_{x \to 1} \frac{x^2+2}{x^2-2}$ where $(x > 0)$

(c) $\lim_{x \to 2}(\frac{1}{x+1} - \frac{1}{2x})$ where $(x > 0)$

(d) $\lim_{x \to 0} \frac{x+1}{x^2+2}$ where $(x \in \mathbb{R})$

4. Prove that $\lim_{x \to 0} \cos(1/x)$ does not exist but that $\lim_{x \to 0} x \cos(1/x) = 0$.

6. Use the definition of the limit to prove the first assertion in Theorem 4.2.4(a).

10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

11. Determine whether the following limits exist at $\mathbb{R}$.

(a) $\lim_{x \to 0} \sin(1/x^2)$ where $(x \neq 0)$.

(b) $\lim_{x \to 0} x \sin(1/x^2)$ where $(x \neq 0)$.

(c) $\lim_{x \to 0} \operatorname{sgn} \sin(1/x)$ where $(x \neq 0)$.

(d) $\lim_{x \to 0} \sqrt{x} \sin(1/x^2)$ where $(x > 0)$.