M 361K Homework 3

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5.1

3. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in [b, c]$. Prove that h is continuous on $[a, c]$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.

Proof. Let $\epsilon = \frac{f(c)}{2} > 0$. Then, let there be some $\delta > 0$ such that there exists some $V_\delta(c) = (c - \delta, c + \delta)$. Then, let $x \in V_\delta(c) = (c - \delta, c + \delta)$. Then,

$$\begin{aligned} x \in (c - \delta, c + \delta) &\implies |f(x) - f(c)| < \epsilon \\ &\implies -\epsilon < f(x) - f(c) < \epsilon \\ &\implies f(c) - \frac{f(c)}{2} < f(x) \\ &\implies f(x) > 0 \end{aligned}$$

Thus, $f(x) > 0$ for all $x \in V_\delta(c)$. □

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find some sequence of rational numbers (x_n) that converges to some x in $\mathbb{R}$. We know that f is continuous at x, so we can say that $(f(x_n))$ converges to $f(x)$ and $f(x_n) = 0 \ \forall \ n \in \mathbb{N}$ because $f(r) = 0 \ \forall \ r \in \mathbb{Q}$. Thus, $f(x) = \lim f(x_n) = \lim(0) = 0$. □

5.2

2. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on A.

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that the sum $f+g$ is continuous at c and the product $f \cdot g$ is continuous at c.

Proof. Let $f(x)$ and $g(x)$ be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x \neq c \end{cases}$$

Then, $f+g$ is continuous at c because $f(c)+g(c) = 1$ and $f \cdot g$ is continuous at c because $f(c) \cdot g(c) = 0$ for all values of x. □

7. Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$ but such that $|f|$ is continuous on $[0,1]$.

Proof. The Density Theorem tells us that for every pair of rational numbers, there exists an irrational number between them, and vice versa. Thus, let f be defined as follows:

$$f(x) = \begin{cases} -1 & \text{if } x = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \\ 1 & \text{if } x \neq \frac{p}{q} \text{ for all } p, q \in \mathbb{N} \end{cases}$$

Then, $|f|$ is 1 everywhere. Thus, f is discontinuous at every point of $[0,1]$ but $|f|$ is continuous on $[0,1]$. □

8. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

Proof. The Density Theorem states that there is a irrational number between every pair of rational numbers. Because $f(r) = g(r)$ for the rationals, $f(s) = g(s)$ for all irrational numbers s in order to maintain continuity for f and g using the Density Theorem. Thus, $f(x) = g(x)$ for all $x \in \mathbb{R}$ is true. □

6.1

3. Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Prove the following:

(a) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c.

(b) The function $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for x rational, $f(x) := 0$ for x irrational. Show that f is differentiable at $x = 0$, and find $f'(0)$

7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

9. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an odd function (that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$). Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then g' is an even function.

6.2

5. Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a - b)^{1/n}$. *Hint: Show that* $f(x) := x^{1/n} - (x - 1)^{1/n}$ *is decreasing for* $x \geq 1$, *and evaluate* f *at 1 and* a/b.

6. Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all x, y in $\mathbb{R}$.

8. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A. *Hint: Use the definition of* $f'(a)$ *and the Mean Value Theorem.*

10. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of 0 the derivative $g'(x)$ takes on both positive and negative values. Thus g is not monotonic in any neighborhood of 0.

13. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Show that if f' is positive on I, then f is strictly increasing on I.