M 361K: Real Analysis

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1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1) $a + b = b + a$.
- (A2) $(a + b) + c = a + (b + c)$.
- (A3) $\exists$ an element $o \in \mathbb{R}$ such that $a + o = o + a = a$.
- (A4) For each element $a \in \mathbb{R}$, $\exists$ an element $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.
- (M1) $ab = ba$.
- (M2) $(ab)c = a(bc)$.
- (M3) $\exists$ an element $1 \in \mathbb{R}$ such that $a * 1 = 1 * a = a$.
- (M4) For each element $a \in \mathbb{R} \setminus 0$, $\exists$ an element $\frac{1}{a} \in \mathbb{R}$ such that $a * \frac{1}{a} = \frac{1}{a} * a = 1$.
- (D) $a * (b + c) = a * b + a * c$.

Remark (Equality property of $\mathbb{R}$). If $a = b$ and $c = d$, then $a + c = b + d$ and $a * c = b * d$.

$\forall x, y, z \in \mathbb{R}$:

Theorem 1.1. If $x + z = y + z$ then $x = y$.

Proof.

$$\begin{aligned} x + z &= y + z \quad (A4) \\ (x + z) + (-z) &= (y + z) + (-z) \quad (A2) \\ x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\ x + 0 &= y + 0 \quad (A3) \\ x &= y \end{aligned}$$

□

Theorem 1.2. For any $x \in \mathbb{R}$, $x * 0 = 0$.

Proof.

$$\begin{aligned} x * 0 &= x * (0 + 0) \\ x * 0 &= x * 0 + x * 0 \\ x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\ 0 &= x * 0 + (x * 0 + (-x * 0)) \\ &= x * 0 + 0 \\ &= x * 0 \end{aligned}$$

□

Theorem 1.3. $-1 * x = -x$ i.e. $x + (-1) * x = 0$.

Proof.

$$\begin{aligned} x + (-1) * x &= x + x * (-1) \\ &= x * 1 + x * (-1) \\ &= x * (1 + (-1)) \\ &= x * 0 \\ &= 0 \end{aligned}$$

$\square$

Theorem 1.4 (Zero-product property)**.** $\forall x, y \in \mathbb{R}$, $x * y = 0 \iff x = 0 \lor y = 0$.

Proof. Let $x, y \in \mathbb{R}$, if $x = 0$ or $y = 0$, then $x * y = 0$. Suppose $x \neq 0$, then we must show $y = 0$. Since $x \neq 0$, $\frac{1}{x}$ exists. Thus, if:

$$\begin{aligned} xy &= 0 \\ \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\ (\frac{1}{x} * (xy)) * y &= 0 \\ 1 * y &= 0 \\ y &= 0 \end{aligned}$$

$\square$

1.2 Order Axioms

$\forall x, y \in \mathbb{R}$:

- (O1) One of $x < y$, $x > y$ or $x = y$ is true.
- (O2) If $x < y$ and $y < z$, then $x < z$.
- (O3) If $x < y$ then $x + z < y + z$.
- (O4) If $x < y$ and $z > 0$ then $xz < yz$.

Theorem 1.5. If $x < y$ then $-y < -x$.

Proof.

$$\begin{gathered} x < y \\ x + (-x + -y) < y + (-x + -y) \\ (x + -x) + -y < (y + -y) + -x \\ 0 + -y < 0 + -x \\ -y < -x \end{gathered}$$

$\square$

Theorem 1.6. If $x < y$ and $z > 0$ then $xz > yz$.

Proof. If $x < y$ and $z > 0$ then $-z < 0$. Thus, $x(-z) < y(-z)$. But,

$$\begin{aligned} x(-z) &= x(-1 * z) \\ &= (x * -1) * z \\ &= (-1 * x) * z \\ &= -1(x * z) \\ &= -x * z \end{aligned}$$

Similarly, $y(-z) = -y * z$. Thus, $-x * z < -y * z$, so $xz > yz$. $\square$

Remark (Completeness of $\mathbb{R}$). $\mathbb{R}$ is an ordered field. $\mathbb{R}$ is complete, while $\mathbb{Q}$ is not complete.

2 August 30

Theorem 2.1. $\sqrt{2}$ is irrational.

Proof. Suppose not. Suppose that $\sqrt{2}$ is rational. Then $\exists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}, n \neq 0$ and m and n share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus, m^2 is even and m is even. Then, $m = 2k$ for some $k \in \mathbb{Z}$. But, by substituting $m = 2k$ into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus, n^2 is even, so n is even. So, n is a perfect square, which is a contradiction. Thus, $\sqrt{2}$ is irrational. □

2.1 Upper and Lower Bounds

Theorem: Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s \forall s \in S$, m is called an **upper bound** for S. If $m \leq s \forall s \in S$, m is called a **lower bound** for S. **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for T.

Definition 2.1 (Supremum)**.** The least upper bound of a set is called the supremum of the set.

Definition 2.2 (Infimum)**.** The greatest lower bound of a set is called the infimum of the set.

2.2 Completeness Axiom

Definition 2.3 (Completeness axiom)**.** Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem 2.2. The set of natural numbers $\mathbb{N}$ is unbounded above.

Proof. Suppose not. Suppose that $\mathbb{N}$ is bounded above. If $\mathbb{N}$ were bounded above, it must have a supremum m. Since $\sup \mathbb{N} = m$, $m - 1$ is not an upper bound. Thus, $\exists n_0 \in \mathbb{N}$ such that $n_0 > m - 1$. But then, $n_0 + 1 > m$. This is a contradiction since $n_0 + 1 \in \mathbb{N}$. Thus, $\mathbb{N}$ is unbounded above. □

Theorem 2.3. If A and B are nonempty subsets of $\mathbb{R}$, let $C = \{x + y \mid x \in A, y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C = \sup A + \sup B$.

Proof. Let $\sup A = a$ and $\sup B = b$. Then if $z \in C, z = x + y$ for some $x \in A, y \in B$. Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom, $\exists$ a least upper bound of $C, c = \sup C$. It must be that $c \leq a+b$, so we must show $c \geq a + b$. Let $\varepsilon > 0$. Since $a = \sup A$, $a - \varepsilon$ is not an upper bound for A. $\exists x \in A$ such that $a - \varepsilon < x$. Likewise, $\exists y \in B$ such that $b - \varepsilon < y$. Then,

$$(a - \varepsilon) + (b - \varepsilon) = a + b - 2 * \varepsilon < x + y \leq c$$

Thus, $a + b < c + 2 * \varepsilon \forall \varepsilon > 0$. So, $a + b \leq c \therefore c = a + b$. □

3 September 6

3.1 Cardinality

Definition 3.1 (Cardinality)**.** The cardinality of a set A is the number of elements in A. We denote this as $|A|$. We say that two sets A and B have the same cardinality if and only if $\exists$ a bijection $f : A \to B$, or $|A| = |B|$.

Remark. This bijection holds true because cardinality is reflexive (via the identity function), symmetric (via the inverse function), and transitive (via composition).

Remark. The following examples demonstrate how to prove whether two sets have the same cardinality.

- $|\text{even integers}| = |\text{odd integers}|$: $f(2n) = 2n + 1$.
- $|\mathbb{Z}| = |\mathbb{Z}^+|$: $f(0) = 1$, $f(1) = 2$, $f(-1) = 3$, $f(2) = 4$, ...
- $|\mathbb{Q}^+| = |\mathbb{Z}^+|$: We can create a diagonal mapping by taking $\frac{n}{m}$ for counting numbers on the rows and columns.
- $|\mathbb{Q}| = |\mathbb{Z}^+|$: $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$, so we can repeat the diagonal mapping for $\mathbb{Q}^-$. This is because any subset of a countable set is countable.
- $|\mathbb{Q}| \neq |\mathbb{R}|$: For the real numbers, Cantor's Diagonal Argument proves the sets have different cardinality since no possible surjection exists.

In essence, if we show that there exists some one-to-one mapping between the two sets we can claim that $|A| = |B|$.

3.2 Countability

Definition 3.2 (Countable)**.** If a set is finite or has the same cardinality as $\mathbb{N}$ (i.e. $\mathbb{Z}^+$), we say that the set is countable.

Theorem 3.1. Any subset of a countable set is countable.

Theorem 3.2. Any set that contains an uncountable set is uncountable.

Theorem 3.3. If $[a_n, b_n] \forall n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, $\exists \delta \in \mathbb{R}$ such that $\delta \in I_n \forall n \in \mathbb{N}$.

Proof. $I_n \subseteq I_1 \forall n \in \mathbb{N}$. Thus, $a_n \subseteq b_1 \forall n \in \mathbb{N}$. So, b_n is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Let δ be the supremum of $\{a_n \mid n \in \mathbb{N}\}$. Thus, $a_n \leq \delta \forall n \in \mathbb{N}$.

We have now shown that $a_n \leq \delta \forall n \in \mathbb{N}$, and we need to show that $\delta \leq b_n \forall n \in \mathbb{N}$. This is left as an exercise for the reader. □

Remark. A nested sequence means that successive subsets contain the previous subset. For example, $[0, 1] \subseteq [0, 2] \subseteq [0, 3] \subseteq \dots$ is a nested sequence.

Theorem 3.4. $[0, 1]$ is uncountable.

Proof. Assume $[0, 1]$ is countable. That is, $[0, 1] = I = \{x_1, x_2, x_3, \ldots\}$. Select a closed interval $I_1 \subseteq I$ such that $x_1 \notin I_1$. Next, select a closed interval $I_2 \subseteq I_1$ such that $x_2 \notin I_2$, and so on. Then, we have

$$I_n \subseteq \ldots \subseteq I_2 \subseteq I_1 \subseteq I$$

and $x_n \notin I_n \forall n \in \mathbb{N}$. By **Theorem 3.3**, $\exists \delta \in I$ such that $\delta \in I_n \forall n \in \mathbb{N}$. This implies that $\delta \neq x_n \forall n \in \mathbb{N}$. Thus, $\delta \notin I$, which is a contradiction. Therefore, $[0, 1]$ is uncountable. $\square$

4 September 8

4.1 Limits of Sequences

Definition 4.1 (Limit of a sequence)**.** A sequence a_n is said to converge to a real number s, if for any $\varepsilon > 0$, $\exists$ a real number k such that for all $n \geq k$, the terms a_n satisfy $|a_n - s| < \varepsilon$.

Theorem 4.1. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Proof. We need to find some N such that $n > N \forall \varepsilon > 0$.

$$\begin{aligned} |\frac{1}{\sqrt{n}} - 0| &< \varepsilon \\ \frac{1}{\sqrt{n}} &< \varepsilon \\ \frac{1}{n} &< \varepsilon^2 \\ n &> \frac{1}{\varepsilon^2} \end{aligned}$$

Let $\varepsilon > 0$ and $N = \frac{1}{\varepsilon^2}$. Then, if $n > N$, we have that

$$\begin{aligned} |\frac{1}{\sqrt{n}} - 0| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{\frac{1}{\varepsilon^2}}} \\ &= \varepsilon \end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. □

Theorem 4.2. $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$.

Proof. Let $\varepsilon > 0$ and $N = \frac{1}{\varepsilon}$. Then, we have

$$\begin{aligned} |1 + \frac{1}{2^n} - 1| &< \varepsilon \\ |\frac{1}{2^n}| = \frac{1}{2^n} < \frac{1}{n} &< \frac{1}{\frac{1}{\varepsilon}} < \varepsilon \\ n &> \frac{1}{\varepsilon} \end{aligned}$$

Thus, $\lim_{n\to\infty} 1 + \frac{1}{2^n} = 1$. □

Theorem 4.3. Every convergent sequence is bounded.

Proof. Let S_n be a convergent sequence with a limit s and $\varepsilon = 1$. Then, there exists some N such that $|S_n - s| < 1$ when $n > N$. That is, $|S_n| < |s| + 1$.

Let $M = \max\{S_1, S_2, \ldots, S_n, |s| + 1\}$. Then, $|S_n| \leq M$, so S_n is bounded. □

Theorem 4.4. If a sequence converges, its limit is unique.

Proof. Suppose a sequence S_n converges to s and t. Let $\varepsilon > 0$. Then, $\exists N_1$ such that $|S_n - s| < \frac{\varepsilon}{2}$. For $n > N_1$, $\exists N_2$ such that $|S_n - t| < \frac{\varepsilon}{2}$. For $n > N_2$, let $N = m + \{N_1, N_2\}$. Then, for $n > N$, we have

$$\begin{aligned}
|s-t| &= |s + S_n - S_n - t| \\
&= |s - S_n + S_n - t| \\
&\leq |s - S_n| + |S_n - t| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
|s-t| &= \varepsilon
\end{aligned}$$

Thus, the limit is unique. □

5 September 13

5.1 Monotone Sequences

Definition 5.1 (Monotone sequence)**.** A sequence S_n of real numbers is said to be increasing $\iff S_n \leq S_{n+1} \; \forall \, n \in \mathbb{N}$ and decreasing $\iff S_n \geq S_{n+1} \; \forall \, n \in \mathbb{N}$.

Remark. The Fibonacci sequence is an example of an increasing sequence.

Definition 5.2 (Monotone convergence theorem)**.** A monotone sequence is convergent if and only if it is bounded.

Theorem 5.1. An increasing bounded sequence is convergent.

Proof. Suppose S_n is a bounded increasing sequence. Let S be the set $\{S_n \mid n \in \mathbb{N}\}$. By the completeness axiom, $\sup S$ exists. Let $s = \sup S$. We claim $\lim_{n\to\infty} S_n = s$. Given $\varepsilon > 0, s - \varepsilon$ is not an upper bound for S.
Thus, $\exists \, N \in \mathbb{N}$ such that $S_N > s - \varepsilon$. Furthermore, since S_n is increasing and s is an upper bound for S, we have $s - \varepsilon < S_N \leq S_n \leq s \; \forall \, n \geq N$. □

Remark. This is an elementary proof because it only uses axioms to make the conclusion.

Ex. $S_{n+1} = \sqrt{1 + S_n}, S_1 = 1$.

Theorem 5.2. If S_n is an unbounded increasing sequence, then $\lim_{n\to\infty} S_n = \infty$.

Proof. Let S_n be an increasing unbounded sequence. Then, $\{S_n \mid n \in \mathbb{N}\}$ is not bounded above, but S is bounded below by S_1. Thus, given $M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that $S_N > M$. But since S_n is increasing, $S_n > M \; \forall \, n > N$. Thus, $\lim_{n\to\infty} S_n = \infty$. □

6 September 15

6.1 Cauchy Sequences

Definition 6.1 (Cauchy sequence)**.** A sequence of real numbers S_n is called a Cauchy sequence if and only if for each $\varepsilon > 0$, $\exists N$ such that $m, n > N \implies |S_m - S_n| < \varepsilon$.

Remark. This means the elements of the sequence get closer to each other as N increases.

Theorem 6.1. Every convergent sequence is Cauchy.

Proof. Let S_n be a convergent sequence. Then $\exists N$ such that $n > N \implies |S_n - s| < \frac{\varepsilon}{2}$ for some $s \in \mathbb{R}$. Then, for $n, m > N$, we have

$$\begin{aligned} |S_n - S_m| &= |S_n - s + s - S_m| \\ &\le |S_n - s| + |s - S_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, S_n is Cauchy. $\square$

Theorem 6.2. A sequence of real numbers is Cauchy if and only if it is convergent.

Remark. We cannot prove this yet.

7 September 20

7.1 Empty Set

Theorem 7.1. The empty set is a subset of any set.

Proof. Suppose not. That is, suppose $\exists A$ such that $\emptyset \not\subset A$. Thus, $\exists x \in \emptyset$ such that $x \notin A$. This is a contradiction because the empty set has no elements. Therefore, $\emptyset \subset A$. $\square$

Theorem 7.2. There is only one set with no elements.

Proof. Suppose not. That is, suppose $\exists$ two empty sets E_1, E_2. Then $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$. Thus, $E_1 = E_2$. This is a contradiction because E_1 and E_2 are two different sets. Therefore, there is only one empty set. $\square$

Remark (Closedness of $\emptyset$)*.* The empty set is open and closed (vacuously true).

7.2 Topology

Let $S \subseteq \mathbb{R}$ for the following definitions.

Definition 7.1 (Neighborhood)**.** A neighborhood of x in S can be thought of an varepsilon-sized ball around x, i.e. $N(x, \varepsilon) = \{y \in R \mid 0 \leq |x - y| < \varepsilon\}$.

Definition 7.2 (Deleted neighborhood)**.** A deleted neighborhood is the same as a neighborhood except that x is not included, i.e. $N^\star(x, \varepsilon) = \{y \in R \mid 0 < |x - y| < \varepsilon\}$.

Definition 7.3 (Accumulation point)**.** $x \in \mathbb{R}$ is an accumulation point of S if and only if every deleted neighborhood of x contains a point of S.

Remark. $(0, \infty)$ has accumulation points $[0, \infty)$. $(0, 1)$ does not contain all of its accumulation points since 0 and 1 are both accumulation points of the set.

Theorem 7.3. $S \in \mathbb{R}$ is closed if and only if S contains all of its accumulation points.

Proof. Suppose S is closed. Let x be an accumulation point of S. If $x \notin S$, then $x \in S^\complement$. Thus, $\exists$ a neighborhood N of x such that $N \subseteq S^\complement$. But $N \cap S = \emptyset$, which contradicts x being an accumulation point of S.

Conversely, suppose S contains all of its accumulation points. Let $x \in S^\complement$, then x is not an accumulation point of S. Thus, $\exists N^\star(x, \varepsilon)$ that misses S. Since $x \notin S$, $N(x, \varepsilon)$ misses S. Therefore, $S^\complement$ is open, which means S is closed. $\square$

Theorem 7.4. If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a max.

Proof. Let $s = \sup S$. Then, s is an accumulation point of S. Since S is closed, $s \in S$. Thus, s is a max of S. $\square$

Definition 7.4 (Interior point)**.** $x \in S$ is an interior point of S if and only if $\exists N(x, t)$ such that $N(x, t) \subset S$.

Definition 7.5 (Boundary point)**.** $x \in S$ is a boundary point of S if and only if every neighborhood N of x has $N \cap S \neq \emptyset$ and $N \cap S^\complement \neq \emptyset$.

7.3 Closure

Definition 7.6 (Open set)**.** *S is an open set if and only if every point in S is an interior point of S. $\forall x \in S, \exists$ a neighborhood $N(x, \varepsilon)$ for some $\varepsilon > 0$ such that $N(x, \varepsilon) \subseteq S$.*

Definition 7.7 (Closed set)**.** *S is a closed set if and only S contains at least one of its boundary points. Additionally, $S^{\complement}$ must be an open set.*

Remark (Closure of $\mathbb{R}$). $\mathbb{R}$ is open because all of its points are interior points. $\mathbb{R}$ is also closed because $\mathbb{R}$ has no boundary points, therefore implying that it contains at least one of its boundary points (vacuously true).

Theorem 7.5. The union of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then $\exists$ a neighborhood N_1 of x such that $N_1 \subseteq A$. But then, $N_1 \subseteq A \cup B$. If $x \in B$, then $\exists$ a neighborhood N_2 of x such that $N_2 \subseteq B$. But then, $N_2 \subseteq A \cup B$.

Thus, in either case, $\exists$ a neighborhood N of x such that $N \subseteq A \cup B$. Therefore, $A \cup B$ is open. □

Theorem 7.6. An arbitrary union of open sets is open.

Proof. Let $A_1, A_2, \dots, A_n$ be open sets. Let $x \in \bigcup_{i=1}^{n} A_i$. Then $x \in A_i$ for some i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcup_{i=1}^{n} A_i$. Therefore, $\bigcup_{i=1}^{n} N_i \subseteq \bigcup_{i=1}^{n} A_i$.

Thus, $\bigcup_{i=1}^{n} N_i$ is a neighborhood of x such that $\bigcup_{i=1}^{n} N_i \subseteq \bigcup_{i=1}^{n} A_i$. Therefore, $\bigcup_{i=1}^{n} A_i$ is open. □

Theorem 7.7. The intersection of two open sets is open.

Proof. Let A and B be open sets. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus, $\exists$ neighborhoods $N_1(x, \varepsilon_1)$ and $N_2(x, \varepsilon_2)$. Let $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$. Then $N_1(x, \varepsilon) \subseteq A$ and $N_2(x, \varepsilon) \subseteq B$.

Thus, $N(x, \varepsilon) \subseteq A \cap B$. Therefore, $A \cap B$ is open. □

Theorem 7.8. A finite intersection of open sets is open.

Proof. Let $A_1, A_2, \dots, A_n$ be open sets. Let $x \in \bigcap_{i=1}^{n} A_i$. Then $x \in A_i$ for all i. Let N_i be a neighborhood of x such that $N_i \subseteq A_i$. Then $N_i \subseteq A_i \subseteq \bigcap_{i=1}^{n} A_i$. Therefore, $\bigcap_{i=1}^{n} N_i \subseteq \bigcap_{i=1}^{n} A_i$.

Thus, $\bigcap_{i=1}^{n} N_i$ is a neighborhood of x such that $\bigcap_{i=1}^{n} N_i \subseteq \bigcap_{i=1}^{n} A_i$. Therefore, $\bigcap_{i=1}^{n} A_i$ is open. □

Theorem 7.9. An arbitrary intersection of open sets is open.

Remark (Counterexample). $\bigcap_{i=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \emptyset$.

8 September 22

8.1 Set Covers

Definition 8.1 (Open cover)**.** An open cover F of some subset $S \in \mathbb{R}$ is a collection of open sets whose union contains S.

Remark. If $E \subseteq F$ and E also covers S, we call E a **subcover**.

Definition 8.2 (Compact)**.** A set S is said to be compact is and only if whenever S is contained in the union of a family F of open sets, then it is contained in a finite number of the sets in F (every open cover has a finite subcover).

Remark. It is hard to show that a set is compact since we have to consider *every* open cover.

Theorem 8.1 (Heine-Borel)**.** A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.

Proof. Let S be a compact set. Observe the open cover $(-n, n)\forall n \in \mathbb{N}$. Since S is compact, $\exists$ a finite subcover $(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)$. $\exists$ one of these sets such that $\bigcup_{i=1}^{k}(-n_i, n_i) = (-n_m, n_m)$ for some $m = 1, 2, \ldots k$. Thus, $S \subseteq (-n_m, n_m)$, so S is bounded.

Let S be a compact set. Suppose S is not closed. Let p be a boundary point of S, and Let $U_n = \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}]\forall n \in \mathbb{N}$. $S \subseteq \bigcup U_n = \mathbb{R}\ p$. $\exists$ a finite subcover $n_1, n_2, \ldots, n_k$ such that $S \subseteq \bigcup_{i=1}^{k} U_{n_i}$. $\exists k$ such that $S \subseteq U_{n_k}$. But, this is a contradiction with p being a boundary point. Therefore, S is closed.

The proof in the other direction is similar, yet non-trivial. $\square$

Theorem 8.2 (Bolzano-Weierstrass)**.** If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then $\exists$ at least one accumulation point of S.

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Suppose S has no accumulation points, then S is closed. By Heine-Borel, S must be compact. Define neighborhoods N_x such that $N_x(x) \cap S = x\forall x \in S$. Clearly, $S \subseteq \bigcup_x N_x$. But, the collection of all N_x must contain a finite subcover. That is,

$$S \subseteq N_{x_1} \cup N_{x_2} \cup \ldots \cup N_{x_k}$$

for some $k \in \mathbb{N}$. This contradicts that S is infinite. Therefore, S has an accumulation point. $\square$

8.2 Cauchy Convergence

Theorem 8.3. Every Cauchy sequence is convergent.

Proof. S_n is Cauchy, so $S = \{S_n \mid n \in \mathbb{N}\}$. By Bolzano-Weierstrass, $\exists$ an accumulation point s of S. We claim that $S_n \to s$. Given $\varepsilon > 0$, $\exists\ N$ such that $m, n > N$. Then $|S_m - S_n| < \frac{\varepsilon}{2}$. $(S - \frac{\varepsilon}{2}, S + \frac{\varepsilon}{2})$ contains an infinite number of points.

$\exists m > N$ such that $S_m \in N(s, \frac{\varepsilon}{2})$. But then, $|S_n - s| = |S_n - S_m + S_m - s| \leq |S_n - S_m| + |S_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $S_n \to s$. $\square$

Theorem 8.4. Let x_n be a sequence of non-negative real numbers. $\sum x_n$ converges if S_k, the sequence of partial sums is bounded.

Proof. $\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k$. S_k is increasing and bounded, it is convergent by the monotone convergence theorem. $\square$

9 September 27

9.1 Limits of Functions

Definition 9.1 (Limit of a function)**.** Let $f : D \to \mathbb{R}$ and let c be an accumulation point of the function. Then, $\lim_{x \to c} f(x) = L$ if and only if given $\varepsilon > 0, \exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Remark. Suppose we want to show that $\lim_{x \to 2} S_x + 1 = 11$. We are looking for some $\delta > 0$ such that $0 \leq |x - 2| < \delta$ and $|S_x + 1 - 11| < \varepsilon$. This is structured similarly to proofs of limits of sequences.

Additionally, the limit must go to an accumulation point of the function because we cannot find the limit of a value outside the function's domain.

Theorem 9.1. $\lim_{x \to 5} 10x + 2 = 52$.

Proof. We need to find some $\delta > 0$ such that whenever $0 < |x - 5| < \delta$, $|10x + 2 - 52| < \varepsilon$.

$$\begin{aligned} |10x - 50| &< \varepsilon \\ 10|x - 5| &< \varepsilon \\ |x - 5| &< \frac{\varepsilon}{10} \end{aligned}$$

Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{10}$. Then, whenever $0 < |x - 5| < \delta$, we have $|10x + 2 - 52| = |10x - 50| = 10|x - 5| < 10 * \frac{\varepsilon}{10} = \varepsilon$. ☐

Theorem 9.2. $\lim_{x \to 3} x^2 + 2x + 6 = 21$.

Proof. We need to find some $\delta > 0$ such that whenever $0 < |x - 3| < \delta$, $|(x^2 + 2x + 6) - 21| < \varepsilon$.

$$\begin{aligned} |x^2 + 2x + 6 - 21| &< \varepsilon \\ |x^2 + 2x - 15| &< \varepsilon \\ |x + 5||x - 3| &< \varepsilon \end{aligned}$$

If $\delta < 1 \implies |x + 5||x - 3| < 9|x - 3| < \varepsilon$. Thus $|x - 3| < \frac{\varepsilon}{9}$. We let $\delta = \min\{1, \frac{\varepsilon}{9}\}$.

Given $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{9}\}$. Then, whenever $0 < |x - 3| < \delta$, we have that $|x + 5| < 9$, thus, $|(x^2 + 2x + 6) - 21| = |x^2 + 2x - 15| = |x + 5||x - 3| < \min\{1, \frac{\varepsilon}{9}\} * \frac{\varepsilon}{9} = \varepsilon$. ☐

Remark. These proofs have two phases. First, we determine some δ as an upper bound. Then, we show how this choice of δ implies the limit is bounded by some ε.

Theorem 9.3. Let $f : D \to \mathbb{R}$ and c is an accumulation point of D. Then, $\lim_{x \to c} f(x) = L$ if and only if for every sequence $S_n \in D$ such that $S_n \to c$, $S_n \neq c \forall n$, then $f(S_n)$ converges to L.

Proof. $\lim_{x \to c} f(x) = L$ and $S_n \to L \implies f(S_n) \to L$. We need to find N such that $n > N$ and $|f(S_n) - L| < \varepsilon$. We know that $\exists \delta$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$ and $\exists N$ such that $n > N \implies |S_n - c| < \delta$. Thus, for $n > N$ we have $|f(S_n) - L| \in \varepsilon$.

Suppose L is not the limit of f as x approaches c. We must find (S_n) that converges to c, but $f(S_n)$ does not converge to L (contrapositive). $\exists \varepsilon > 0$ such that $\forall \delta > 0, 0 < |x - c| < \delta \implies |f(x) - L| \geq \varepsilon$. For each $n \in N, \exists S_n \in D$ such that $0 < |S_n - c| < \frac{1}{n}$ and $|f(S_n) - L| \geq \varepsilon$. Then, $S_n \to c$, but $f(S_n) \not\to L$. This is a contradiction. ☐

10 September 29

10.1 Sums of Limits

Theorem 10.1. Let $lim_{x \to c} f(x) = L$, $lim_{x \to c} g(x) = M$. Then, $lim_{x \to c}(f+g)(x) = L + M$.

Proof (Definition 9.1). Given $\varepsilon > 0$, let $\delta_1 > 0$ be such that $0 < |x-c| < \delta_1 \implies |f(x)-L| < \frac{\varepsilon}{2}$. Let $\delta_2 > 0$ be such that $0 < |x-c| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$.

Let $\delta = min\{\delta_1, \delta_2\}$. Then, for $0 < |x-c| < \delta$, we have

$$|f(x)+g(x)-(L+M)| = |(f(x)-L)+(g(x)-M)| \leq |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

□

Proof (Theorem 9.3). Let $lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$, and S_n be a sequence of real numbers such that $S_n \to c$. Then,

$$\lim_{n\to\infty}(f+g)(S_n) = \lim_{n\to\infty} f(S_n) + g(S_n) = \lim_{n\to\infty} f(S_n) + \lim_{n\to\infty} g(S_n) = L + M$$

Thus, $\lim_{x \to c}(f+g)(x) = L + M$. □

Remark. This is true for $-$, $\times$, and $\div$ as well.

Definition 10.1 (Sequential criterion for functional limits)**.** $\lim_{x \to c} f(x) = L$ if and only if whenever $S_n \to c$, $\lim_{n\to\infty} f(S_n) = L$.

Theorem 10.2. Let $k \in \mathbb{R}$. If $\lim_{x\to c} f(x) = L$, then $\lim_{x \to c} kf(x) = kL$.

Proof. Let $\lim_{x\to c} f(x) = L$, $k \in \mathbb{R}$, and S_n be a sequence of real numbers such that $S_n \to c$. Then,

$$\lim_{n\to\infty} kf(S_n) = k \lim_{n\to\infty} f(S_n) = kL$$

Thus, $\lim_{x\to c} kf(x) = kL$. □

10.2 Continuity of Functions

Definition 10.2 (Continuous function)**.** A function f is continuous at $x = c$ if and only if $\lim_{x\to c} f(x) = f(c)$. Let s be an accumulation point of the domain $f : D \to \mathbb{R}$. Then, f is continuous at s if and only if for each $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $0 < |x-s| < \delta$, $|f(x) - f(s)| < \varepsilon$.

Remark. Let $f(x) = x\sin(\frac{1}{x})$ where $x \neq 0$, $f(0) = 0$. If we want to show that this function is continuous, we need to find some $\delta > 0$ such that $|x| < \delta \implies |f(x) - f(0)| < \varepsilon$. Let $\delta = \varepsilon$, then when $|x| < \delta$, $|f(x) - f(0)| = |x\sin(\frac{1}{x}) - 0| = |x\sin(\frac{1}{x})| \leq |x| < \varepsilon$.

Theorem 10.3. If f and g are continuous at $x = c$, then $f+g$ is also continuous at $x = c$.

Proof. Let f and g be continuous at c and S_n be a sequence of real numbers such that $S_n \to c$. Then,

$$\lim_{n\to\infty}(f+g)(S_n) = \lim_{n\to\infty} f(S_n) + \lim_{n\to\infty} g(S_n) = f(c) + g(c)$$

Thus, $\lim_{x\to c}(f+g)(x) = (f+g)(c)$. □

Theorem 10.4. Let $f : D \to E$ be continuous at $x = c$ and let $g : E \to R$ be continuous at $x = f(c)$. Then, the composition $g \circ f$ is continuous at $x = c$.

Proof. This is left as an exercise for the reader. □

11 October 6

11.1 Derivatives

Definition 11.1 (Derivative)**.** Let f be a real-valued function defined on an open interval containing c. We say f is differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. We call this limit $f'(c)$.

Theorem 11.1. *If f is differentiable at c, then f is continuous at c.*

Proof. Let f be defined on some interval I containing c. Then if f is differentiable at c, if and only if for $x \neq c$,

$$f(x) = (x-c)\frac{f(x)-f(c)}{x-c} + f(c)$$

Then, $\lim_{x\to c} f(x) = \lim_{x\to c}(x-c)\frac{f(x)-f(c)}{x-c} + f(c) = \lim_{x\to c}(x-c)f'(c) + f(c) = f(c)$. Therefore, f is continuous at c. $\square$

Derivative Rules

- $\frac{d}{dx}kf = k\frac{df}{dx}$
- $\frac{d}{dx}f + g = \frac{df}{dx} + \frac{dg}{dx}$
- $\frac{d}{dx}f \cdot g = \frac{df}{dx}g + \frac{dg}{dx}f$
- $\frac{d}{dx}\frac{f}{g} = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}$

Theorem 11.2 (Product rule)**.**

$$(fg)' = f'g + fg'$$

Proof. Suppose f and g are differentiable at c. Then,

$$\begin{aligned}
\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c} &= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c} \\
&= \lim_{x\to c}\frac{f(x)(g(x)-g(c))}{x-c} + \frac{g(x)(f(x)-f(c))}{x-c}. \\
&= f(c)g'(c) + g(c)f'(c)
\end{aligned}$$

$\square$

Theorem 11.3 (Quotient rule)**.**

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

Proof. Let f and g be differentiable at c. Then,

$$\begin{aligned}
\lim_{x\to c}\frac{\frac{f}{g}(x)-\frac{f}{g}(c)}{x-c} &= \lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c}\\
&= \lim_{x\to c}\frac{\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)}}{x-c}\\
&= \lim_{x\to c}\frac{f(x)g(c)-g(c)f(c)+g(c)f(c)-f(c)g(x)}{(x-c)g(x)g(c)}\\
&= \lim_{x\to c}\frac{g(c)\frac{f(x)-f(c)}{(x-c)}+f(c)\frac{g(x)-g(c)}{(x-c)}}{g(c)g(x)}\\
&= \lim_{x\to c}\frac{g(c)f'(c)-f(c)g'(c)}{g^2(c)}Aa
\end{aligned}$$

□

Theorem 11.4 (Power rule)**.**

$$(x^n)' = nx^{n-1}f' \ \forall\ n\in\mathbb{N}$$

Proof by induction. $p(n) = (x^n)' = nx^{n-1}f'$.
$p(1)$: $f(x) = x$. $\lim_{x\to c}\frac{x-c}{x-c} = 1 = 1\cdot x^0$.
$p(k)\to p(k+1)$:

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k\cdot x\\
&= (\frac{d}{dx}x^k)\cdot x + x^k(\frac{d}{dx}x)\\
&= kx^{k-1}\cdot x + x^k\cdot 1\\
&= kx^k + x^k\\
&= (k+1)x^k
\end{aligned}$$

□

Theorem 11.5 (Chain rule)**.**

$$g(f(x))' = g'(f(x))\cdot f'(x)$$

Proof.

$$\begin{aligned}
\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\frac{f(x)-f(c)}{x-c}\\
&= g'(f(x))f'(x)
\end{aligned}$$

□

Remark. This will not hold if $f(x) = f(c)$. This is not the full proof.

12 October 13

12.1 Differentiability and Continuity

Theorem 12.1. Let f be defined on an interval I containing c. Then, f is differentiable at c if and only if $\exists$ a function φ on I such that φ is continuous at c and

$$f(x) - f(c) = \varphi(x)(x - c) \forall x \neq c$$

In this case, we have $\varphi(c) = f'(c)$.

Remark. Let $f(x) = x^3$. Then, $f(x) - f(c) = x^3 - c^3 = (x^2 + xc + c^2)(x - c)$. $\phi(c) = c^2 + c \cdot c + c^2 = 3c^2 = f'(c)$.

Proof. If $f'(c)$ exists, we can define φ as

$$\varphi(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

Then, φ is continuous. Since $\lim_{x \to c} \varphi(x) = f'(c) = \varphi(c)$. Thus, the function is differentiable. If $x = c$, the equation from the theorem holds as $0 = 0$.

Assume φ is continuous at c and satisfies the equation. Then, continuity of φ implies $\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \implies \varphi(c) = f'(c)$ since f is differentiable. ☐

Theorem 12.2 (Chain rule)**.**

$$g(f(c))' = g'(f(c)) \cdot f'(c)$$

Proof. Let $c \in I$. f is continuous at c. Define

$$\varphi(x) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$$

Thus, φ is continuous at c. Then,

$$\begin{aligned} \lim_{x \to c} \varphi(f(x)) &= \varphi(f(c)) = g'(f(c)) \\ g(y) - g(f(c)) &= \varphi(y)(y - f(c)) \\ g(f(x)) - g(f(c)) &= \varphi(f(x))(f(x) - f(c)) \\ \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} &= \lim_{x \to c} \frac{\varphi(f(x))(f(x) - f(c))}{x - c} \\ g'(f(c)) &= \lim_{x \to c} \varphi(f(x)) \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ g'(f(c)) &= g'(f(c)) \cdot f'(c) \end{aligned}$$

Thus, the chain rule holds. ☐

Theorem 12.3. If S is a nonempty compact subset of $\mathbb{R}$, S has a max and a min.

Proof. Let $m = \sup S$ exist by the completeness axiom. Given $t > 0$, $\exists x$ such that $m - t < x < m$. Then, m is an accumulation point of S. But S is closed by Heine-Borel. Thus, $m \in S$.

The same proof holds for the min. $\square$

Theorem 12.4. If f is continuous and D is compact, then $f(D)$ is compact. (Note: this will be on the final).

Proof. We know that the inverse of a continuous function is continuous (final exam proof) and that if an open set is continuous its inverse is also continuous (exam 2 proof).

Take an open cover $U = \{u_i\}$ of $f(D)$. Then, $f^{-1}(u_i)$ is an open cover for D. But, only a finite number are needed $(\{u_1, u_2, \ldots, u_n\})$. Then, $(\{f(u_1), f(u_2), \ldots, f(u_n)\})$ is a finite subcover of u_i for $f(D)$. $\square$

Theorem 12.5. Let D be compact and suppose $f : D \to \mathbb{R}$ is continuous, then f assumes a min and a max.

Proof. Since D is compact, $f(D)$ is compact. Thus, $f(D)$ has a min y_1 and a max y_2. Since $y_1, y_2 \in f(D), \exists x_1, x_2 \in D$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus, $f(x_1) \leq f(x) \leq f(x_2) \forall x \in D$. $\square$

Theorem 12.6. If f is differentiable on an (a, b) and f assumes a max or min for some $c \in (a, b)$, then $f'(c) = 0$.

Proof. Suppose f assumes its max is at c. That is to say $f(x) \leq f(c) \forall x \in (a, b)$. Let x_n be a sequence converging to c such that $a < x_n < c$. Then,

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges to $f'(c)$. But, each term is nonnegative. Therefore, the derivative is nonnegative $\implies f'(c) \geq 0$. Now, define y_n as a sequence converging to c such that $c < y_n < b$.

If we look at the sequence $\frac{f(y_n) - f(c)}{y_n - c}$, we see that it converges to $f'(c)$. But, each term is nonpositive. Therefore, the derivative is nonpositive, so $f'(c) \leq 0 \therefore 0 \leq f'(c) \leq 0$, so we must have that $f'(c) = 0$. $\square$

13 October 20

13.1 Mean Value Theorem

Theorem 13.1 (Rolle's theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b), and let $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous and $[a, b]$ is compact, $\exists x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2) \forall x \in [a, b]$. If x_1 and x_2 are the endpoints of the interval, then f is a compact function, thus $f'(c) = 0 \forall c \in (a, b)$. Otherwise, f contains a max at $x_2 \therefore f'(x_2) = 0$. Thus $\exists c \in (a, b)$ such that $f'(c) = 0$. $\square$

Theorem 13.2 (Mean value theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $g(x)$ be defined as $g(x) = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$. Let $h(x)$ be the distance from the graph of $f \circ g$. That is, $h = f - g$. Then, h is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $h(a) = h(b) = 0$.

By Rolle's Theorem, $\exists c \in (a, b)$ such that $h'(c) = 0$. Thus,

$$0 = h'(c) = f'(c) - g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Therefore, $f'(c) = \frac{f(b)-f(a)}{b-a}$. $\square$

Theorem 13.3. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then if $f'(x) = 0 \forall x \in (a, b)$, then f is constant on $[a, b]$.

Proof. Suppose f is not constant. Then, $\exists x_1, x_2$ such that $a \leq x_1 < x_2 \leq b$ and $f(x_1) \neq f(x_2)$. By the Mean Value Theorem, $\exists c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

But, this is a contradiction. Therefore, f is constant on $[a, b]$. $\square$

Theorem 13.4. Let f be differentiable on an interval I. If $f'(x) > 0 \forall x \in I$, then f is strictly increasing on I.

Proof. Suppose $f'(x) > 0 \forall x \in I$ and $x_1, x_2 \in I$ such that $x_1 < x_2$. Mean Value Theorem implies that $\exists c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Which is to say that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Thus, $f(x_2) - f(x_1)$ is positive since $f'(c)$ and $(x_2 - x_1)$ are both positive. Therefore, f is increasing. $\square$

13.2 Intermediate Value Theorem

Theorem 13.5 (Intermediate value theorem)**.** Let f be continuous on $[a, b]$ and suppose $f(a) < 0 < f(b)$. Then $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof. Let c be the largest value for which $f(x) \leq 0$. Let $S = \{x \in [a, b] \mid f(x) \leq 0\}$. Since $a \in S, S$, is nonempty. Thus, $\sup S = c$ exists.

We claim that $f(c) = 0$. Suppose $f(c) < 0$, then $\exists$ a neighborhood U of c such that $f(x) < 0 \forall x \in U \cap [a, b]$. Now, $c \neq b$ since $f(a) < 0 < f(b)$. Thus, U contains a point p such that $c < p < b$ where $f(p) < 0$. But, this is a contradiction since $p \in S$ and $p > c$. Therefore, $f(c) \not< 0$.

Similarly, suppose $f(c) > 0$. We can follow this proof in the other direction to show that $f(c) = 0$. ◻

Remark. This is the baby version of the intermediate value theorem. The full version will be asked on exam 2.

14 October 25

14.1 Cauchy Mean Value Theorem

Theorem 14.1 (Cauchy mean value theorem)**.** Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then, $\exists$ at least one $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Proof. Let $h(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x) \forall x \in [a, b]$.
Note that

$$\begin{aligned} h(a) &= (f(b) - f(a))g(a) - (g(b) - g(a))f(a) = 0 \\ &= f(b)g(a) - f(a)g(b) \end{aligned}$$

and

$$\begin{aligned} h(b) &= (f(b) - f(a))g(b) - (g(b) - g(a))f(b) = 0 \\ &= f(b)g(a) - f(a)g(b) \end{aligned}$$

Thus, h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b)$. Therefore, by Rolle's theorem, $\exists c \in (a, b)$ such that $h'(c) = 0$. That is to say,

$$h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) = 0$$

which implies the desired equality. $\square$

14.2 L'Hospital's Rule

Theorem 14.2 (L'Hospital's rule)**.** Let f and g be continuous on $[a, b]$ and differentiable on (a, b) and $f(c) = g(c) = 0$. Also suppose that $g'(c) \neq 0$ in some neighborhood of c.
If

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L$$

Proof. Let x_n be a sequence that converges to c. By the Cauchy mean value theorem $\exists$ a sequence c_n such that c_n is between x_n and c for each n and

$$(f(x_n) - f(c))g'(c_n) = (g(x_n) - g(c))f'(c_n)$$

Thus,

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)}$$

Furthermore, since $x_n \to c$ and $c_n \to c$, we have that if $\lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = L$, then $\lim_{h\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{x\to c} \frac{f(x)}{g(x)} = L$. $\square$

14.3 Taylor's Theorem

Theorem 14.3 (Taylor's theorem)**.** Let f and its first n derivatives be continuous on $[a,b]$ (implying that they are also differentiable). Let $x_0 \in [a,b]$. Then, for each $x \in [a,b]$ with $x \neq x_0$, $\exists$ a c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(c)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Proof. Let x_0 and x be given and let $J = [x_0, x]$ or $[x, x_0]$. We will define F on J as follows:

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2!}f''(t) - \cdots - \frac{(x-t)^n}{(n)!}f^{(n)}(t)$$

Note that

$$F'(t) = \frac{-(x-t)^n}{n!}f^{(n+1)}(t)$$

and define G by

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0)$$

Note that $G(x_0) = 0 = G(x)$. Then, by Rolle's Theorem, $\exists c$ between x and x_0 such that $G'(c) = 0$. That is,

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}}F(x_0)$$

Hence,

$$\begin{aligned} F(x_0) &= -\left(\frac{1}{n+1}\right)\left(\frac{(x-x_0)^{n+1}}{(x-c)^n}\right)F'(c) \\ &= \left(\frac{1}{n+1}\right)\left(\frac{(x-x_0)^{n+1}}{(x-c)^n}\right)\left(\frac{(x-c)^n}{n!}\right)f^{(n+1)}(c) \\ &= \left(\frac{(x-x_0)^{n+1}}{(n+1)!}\right)f^{(n+1)}(c) \end{aligned}$$

which implies the desired equality. $\square$

15 October 27

15.1 Applications of Taylor's Theorem

Definition 15.1 (Taylor polynomial)**.** We denote a Taylor polynomial $\mathcal{P}_n(x)$ as

$$\mathcal{P}_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f^{(n)(x_0)}}{n!}(x - x_0)^n$$

and a remainder term $R_n(x)$ with some $c \in \mathbb{R}$ where $x_0 <= c <= x$ as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Example 15.1. Estimate e^6 on $[-1, 1]$ using a Taylor polynomial. Let $f(x) = e^x$, $x_0 = 0$ and $n = 5$.

$$\begin{aligned} e^x &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \end{aligned}$$

You can place an upper bound on the remainder term on the interval $[-1, 1]$ ($c = 1$ maxes out $f'(c)$ and $x = 1$ maxes out x^6).

$$|R_5(x)| = \left|\frac{f^{(6)}(c)}{6!}x^6\right| = \frac{|f^6(c)|}{6!}\left|x^6\right| \leq \frac{e \cdot 1}{6!}$$

Example 15.2. Estimate $\cos(1)$ to within $1/1000$ using a Taylor polynomial.
Take $x_0 = 0$, on $[-1, 1]$. We need

$$|R_n(x)| = \left|\frac{f^{n+1}(c)}{(n+1)!}x^{n+1}\right| \leq \frac{1}{1000}$$

If you find the Taylor polynomial of cosine to the 6th degree,

$$\begin{gathered} \cos(0) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ \therefore \left|-\frac{x^6}{6!}\right| \leq \frac{1}{1000} \text{ on } [-1, 1] \end{gathered}$$

Hence, this is a good enough approximation that estimates $\cos(x)$ on $[-1, 1]$ within an error of $1/1000$.

Remark. You can estimate π with $\tan^{-1}(1)$ because it equals $\pi/4$.

Theorem 15.1. *e is irrational.*

Proof. We know that $e < 3$. Then, we have that

$$0 < e - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) < \frac{3}{(n+1)!} < \frac{3}{(n+1)!}$$

We can assume $e = \frac{a}{b}$ where $b \neq 0$ and $a, b \in \mathbb{Z}$. Then,

$$0 < \frac{a}{b} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}$$

Let M be the middle term. Then, take $n > \max\{b, 3\}$. Finally, we have

$$0 < M < a - n!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}\right) < \frac{3}{n+1} < \frac{3}{4}$$

This is a contradiction because there is no integer between 0 and $\frac{3}{4}$ even though M is an integer. □

15.2 Riemann Integrability

Definition 15.2 (Partition)**.** Let $[a, b]$ be an interval in $\mathbb{R}$. A partition $\mathcal{P}$ of $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Definition 15.3 (Upper and lower sums)**.** Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, and let

$$M_i(f) = \sup\{f(x) : [x_{i-1}, x_i]\}$$
$$m_i(f) = \inf\{f(x) : [x_{i-1}, x_i]\}$$

For example, $f(x) = x + 3, x_0 = 1$ and $x_1 = 2$. Hence,

$$M_1(f) = 5$$
$$m_1(f) = 4$$

Let $\Delta x_i = x_i - x_{i-1}$. We then define $U(f, p) = \sum_{i=1}^{n} M_i \Delta x_i$ (the upper sum of f with respect to $\mathcal{P}$) and $L(f, p) = \sum_{i=1}^{n} m_i \Delta x_i$ (the lower sum of f with respect to $\mathcal{P}$). Now, define

$$U(f) = \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$
$$L(f) = \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

Definition 15.4 (Riemann integrable)**.** We say that f is Riemann-integrable if and only if $U(f) = L(f)$. In this case, we write

$$\int_a^b f(x)dx = U(f) = L(f)$$

To show a function f is Riemann-integrable on $[a, b)$ given $\varepsilon > 0$, we only need to find one partition such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$$

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Theorem 15.2. Let f be bounded on $[a,b]$ if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a,b]$ such that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Then,

$$L(f,\mathcal{P}) \leq L(f,\mathcal{Q}) \leq U(f,\mathcal{Q}) \leq U(f,\mathcal{P})$$

Proof. We know that $\mathcal{Q}$ will contain more points than $\mathcal{P}$. $\mathcal{P}$ is described by $m_k \cdot (x_k - x_{k-1})$ while $\mathcal{Q}$ is described by $m_x \cdot (x^\star - x_{k-1}) + m_x \cdot (x_k - x^\star)$. $\square$

Theorem 15.3. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a,b]$. Then

$$L(f,\mathcal{P}) \leq U(f,\mathcal{Q})$$

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of f. Then $\mathcal{P} \cup \mathcal{Q}$ is a refinement of $\mathcal{P}$ and $\mathcal{Q}$. Thus,

$$L(f,\mathcal{P}) \leq (f,\mathcal{P} \cup \mathcal{Q}) \leq U(f,\mathcal{P} \cup \mathcal{Q}) \leq U(f,\mathcal{Q})$$

$\square$

Theorem 15.4. Let f be bounded on $[a,b]$. Then, $L(f) \leq U(f)$.

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a,b]$. Then by the previous theorem, $U(f,\mathcal{Q})$ is an upper bound for

$$S = \{L(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}$$

So, $U(f,\mathcal{Q})$ is at least as large as $\sup S = L(f)$. That is, $L(f) \leq U(f,\mathcal{Q})$ for each partition $\mathcal{Q}$. Then,

$$L(f) \leq \inf\{U(f,\mathcal{Q}) : \mathcal{Q} \text{ is a partition of } [a,b]\} = U(f)$$

Therefore, $L(f) \leq U(f)$. $\square$

Example 15.3. $f(x) = x^2$ on $[0,1]$ with partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$.

$$\begin{aligned}
M_i &= \sup\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = \left(\frac{i^2}{n^2}\right) \\
M_i &= \inf\left\{f(x) : x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = \left(\frac{i-1}{n}\right)^2 \\
U(f,\mathcal{P}_n) &= \sum_{i=1}^{n} M_i \cdot \Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \left[\frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right] \\
L(f,\mathcal{P}_n) &= \sum_{i=1}^{n} m_i \cdot \Delta x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} (i-1)^2 = \left[\frac{1}{n^3} \cdot \frac{n(n-1)(2n-1)}{6}\right]
\end{aligned}$$

Then, $\lim_{n\to\infty} U(f,\mathcal{P}_n) = \frac{1}{3}$ and $\lim_{n\to\infty} L(f,\mathcal{P}_n) = \frac{1}{3}$. Thus, $U(f) \leq \frac{1}{3}$ and $L(f) \geq \frac{1}{3}$. Because $L(f) \leq U(f)$, we have that $L(f) = U(f) = \frac{1}{3}$.

Since $L(f) = U(f)$, this function is Riemann-integrable. Therefore,

$$\int_0^1 x^2 = \int_0^1 x^2 dx = \frac{1}{3}$$

Theorem 15.5. Let f be a bounded function on $[a, b]$. Then, f is Riemann-integrable if and only if given an $\epsilon > 0$, $\exists$ a partition of $[a, b]$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

Proof. If f is Riemann-integrable, since $\epsilon > 0$, $\exists$ a partition $\mathcal{P}_1$ such that

$$L(f, \mathcal{P}_1) > L(f) - \frac{\epsilon}{2}$$

Similarly, $\exists \mathcal{P}_2$ such that

$$U(f, \mathcal{P}_2) < U(f) + \frac{\epsilon}{2}$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then,

$$\begin{aligned}
U(f, \mathcal{P}) - L(f, \mathcal{P}) &\leq U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) \\
&< \Big(U(f) + \frac{\epsilon}{2}\Big) - \Big((L(f) - \frac{\epsilon}{2})\Big) \\
&= U(f) - L(f) + \epsilon \\
&= \epsilon
\end{aligned}$$

Therefore, f is Riemann-integrable.

Conversely, given $\epsilon > 0$, suppose $\exists \mathcal{P}$ such that $U(f, \mathcal{P}) < L(f, \mathcal{P}) + \epsilon$. Then,

$$U(f, \mathcal{P}) \leq U(f, \mathcal{P}) < L(f, \mathcal{P}) + \epsilon \leq L(f) + \epsilon$$

Therefore, $U(f) \leq L(f)$. But then $L(f) = U(f)$, so f is Riemann-integrable. $\square$

Remark. Generally, we just need to find some partition $\mathcal{P}$ such that $U(f, \mathcal{P})$ and $L(f, \mathcal{P})$ are within ϵ of each other.