# M 361K Homework 3

Ishan Shah

October 23, 2022

## 5.1

**3.** Let $a < b < c$. Suppose that $f$ is continuous on $[a, b]$, that $g$ is continuous on $[b, c]$, and that $f(b) = g(b)$. Define $h$ on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in [b, c]$. Prove that $h$ is continuous on $[a, c]$.

*Proof.* We have that $h$ is continuous from $[a, b) \cup (b, c]$ from the definition of $h$ and the continuity of $f$ and $g$. We now need to show that $h$ is continuous at $b$. Now, $\lim_{x\to b^-} = \lim_{x\to b^-} = f(b)$ and $\lim_{x\to b^+} = \lim_{x\to b^+} = g(b)$.

Because $f(b) = g(b)$, $\lim_{x\to b^-} h(x) = \lim_{x\to b^+} h(x) = h(b)$. Thus, $\lim_{x\to b} h(x) = h(b)$. Therefore, since $h$ is continuous from $[a, b) \cup [b] \cup (b, c]$, $h$ is continuous on $[a, c]$. □

**7.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $c$ and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of $c$ such that if $x \in V_\delta(c)$, then $f(x) > 0$.

*Proof.* Let $\epsilon = \frac{f(c)}{2} > 0$. Then, let there be some $\delta > 0$ such that there exists some $V_\delta(c) = (c - \delta, c + \delta)$. Then, let $x \in V_\delta(c) = (c - \delta, c + \delta)$. Then,

$$\begin{aligned} x \in (c - \delta, c + \delta) &\implies |f(x) - f(c)| < \epsilon \\ &\implies -\epsilon < f(x) - f(c) < \epsilon \\ &\implies f(c) - \frac{f(c)}{2} < f(x) \\ &\implies f(x) > 0 \end{aligned}$$

Thus, $f(x) > 0$ for all $x \in V_\delta(c)$. □

**12.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number $r$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

*Proof.* Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find some sequence of rational numbers $(x_n)$ that converges to some $x$ in $\mathbb{R}$. We know that $f$ is continuous at $x$, so we can say that $(f(x_n))$ converges to $f(x)$ and $f(x_n) = 0 \ \forall \ n \in \mathbb{N}$ because $f(r) = 0 \ \forall \ r \in \mathbb{Q}$. Thus, $f(x) = \lim f(x_n) = \lim(0) = 0$. □

## 5.2

**2.** Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function $f^n$ defined by $f^n(x) = (f(x))^n$, for $x \in A$, is continuous on $A$.

*Proof.* We can do a proof by induction where the base case is $n = 1$. Then, $f^1(x) = f(x)$ which is trivially continuous on $A$ by our original assumption.

Our inductive case is that $f^{n+1} = f^n f$ which is true by Theorem 5.2.1 which states that the product of two continuous functions is continuous. Thus, $f^{n+1}$ is continuous on $A$. □

**3.** Give an example of functions $f$ and $g$ that are both discontinuous at a point $c$ in $\mathbb{R}$ such that the sum $f + g$ is continuous at $c$ and the product $f \cdot g$ is continuous at $c$.

*Proof.* Let $f(x)$ and $g(x)$ be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x \neq c \end{cases}$$

Then, $f + g$ is continuous at $c$ because $f(c) + g(c) = 1$ and $f \cdot g$ is continuous at $c$ because $f(c) \cdot g(c) = 0$ for all values of $x$. □

**7.** Give an example of a function $f : [0, 1] \to \mathbb{R}$ that is discontinuous at every point of $[0, 1]$ but such that $|f|$ is continuous on $[0, 1]$.

*Proof.* The Density Theorem tells us that for every pair of rational numbers, there exists an irrational number between them, and vice versa. Thus, let $f$ be defined as follows:

$$f(x) = \begin{cases} -1 & \text{if } x = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \\ 1 & \text{if } x \neq \frac{p}{q} \text{ for all } p, q \in \mathbb{N} \end{cases}$$

Then, $|f|$ is 1 everywhere. Thus, $f$ is discontinuous at every point of $[0, 1]$ but $|f|$ is continuous on $[0, 1]$. □

**8.** Let $f, g$ be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers $r$. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

*Proof.* The Density Theorem states that there is a irrational number between every pair of rational numbers. Because $f(r) = g(r)$ for the rationals, $f(s) = g(s)$ for all irrational numbers $s$ in order to maintain continuity for $f$ and $g$ using the Density Theorem. Thus, $f(x) = g(x)$ for all $x \in \mathbb{R}$ is true. □

# 6.1

**3.** Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at $c$. Prove the following:

(a) If $\alpha \in \mathbb{R}$, then the function $\alpha f$ is differentiable at $c$.

*Proof.* We can use the definition of the derivative to show this.

$$\begin{aligned}(\alpha f)'(c) &= \lim_{x \to c} \frac{\alpha f(x) - \alpha f(c)}{x - c} \\ &= \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \alpha f'(c)\end{aligned}$$

□

(b) The function $f + g$ is differentiable at $c$ and $(f + g)'(c) = f'(c) + g'(c)$.

*Proof.*

$$\begin{aligned}(f+g)'(c) &= \lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c)\end{aligned}$$

□

**4.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^2$ for $x$ rational, $f(x) := 0$ for $x$ irrational. Show that $f$ is differentiable at $x = 0$, and find $f'(0)$.

*Proof.* Let $g(x) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$ so that $g(x) := x$ for $x$ rational and $g(x) := 0$ for $x$ irrational. To show that $f$ is differentiable at 0 we have to show that $\lim_{x \to 0} g(x) = f'(0)$ exists.

We know that $-|x| \leq g(x) \leq |x|$ for all $x \in \mathbb{R}$ and that $\lim_{x \to 0} -|x| = -0$ and $\lim_{x \to 0} |x| = 0$. Thus, $\lim_{x \to 0} g(x) = 0$ by Theorem 3.2.7 (Squeeze Theorem) so $f$ is differentiable at $x = 0$ and $f'(0) = 0$. □

**7.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c$ and that $f(c) = 0$. Show that $g(x) := |f(x)|$ is differentiable at $c$ if and only if $f'(c) = 0$.

*Proof.* Suppose that $g(x)$ is differentiable at $c$ and we want to show that $f'(c) = 0$.

$$\begin{aligned} g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{|f(x)| - |f(c)|}{x - c} \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ by the definition of } g \end{aligned}$$

Suppose that $f'(c) = 0$ and we want to show that $g(x)$ is differentiable at $c$. $\square$

**9.** Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative $f'$ is an odd function (that is, $f'(-x) = -f'(x)$ for all $x \in \mathbb{R}$). Also prove that if $g : \mathbb{R} \to \mathbb{R}$ is a differentiable odd function, then $g'$ is an even function.

# 6.2

**5.** Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$. Prove that $a^{1/n} - b^{1/n} < (a - b)^{1/n}$. *Hint: Show that $f(x) := x^{1/n} - (x - 1)^{1/n}$ is decreasing for $x \geq 1$, and evaluate $f$ at 1 and $a/b$.*

**6.** Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y$ in $\mathbb{R}$.

**8.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $(a, b)$. Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals $A$. *Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.*

**10.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $g(0) := 0$. Show that $g'(0) = 1$, but in every neighborhood of 0 the derivative $g'(x)$ takes on both positive and negative values. Thus $g$ is not monotonic in any neighborhood of 0.

**13.** Let $I$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on $I$. Show that if $f'$ is positive on $I$, then $f$ is strictly increasing on $I$.