M 361K: Real Analysis

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1.1 Algebraic Axioms

$\forall a, b, c \in \mathbb{R}$

- (A1) $a + b = b + a$.
- (A2) $(a + b) + c = a + (b + c)$.
- (A3) $\exists$ an element $o \in \mathbb{R}$ such that $a + o = o + a = a$.
- (A4) For each element $a \in \mathbb{R}$, $\exists$ an element $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.
- (M1) $ab = ba$.
- (M2) $(ab)c = a(bc)$.
- (M3) $\exists$ an element $1 \in \mathbb{R}$ such that $a * 1 = 1 * a = a$.
- (M4) For each element $a \in \mathbb{R} \setminus 0$, $\exists$ an element $\frac{1}{a} \in \mathbb{R}$ such that $a * \frac{1}{a} = \frac{1}{a} * a = 1$.
- (D) $a * (b + c) = a * b + a * c$.

$\implies$ If $a = b$ and $c = d$, then $a + c = b + d$ and $a * c = b * d$ (assumed properties of equality).

Proofs $\forall x, y, z \in \mathbb{R}$

- If $x + z = y + z$ then $x = y$.

$$\begin{aligned} x + z &= y + z \quad (A4) \\ (x+z) + (-z) &= (y+z) + (-z) \quad (A2) \\ x + (z + (-z)) &= y + (z + (-z)) \quad (A4) \\ x + 0 &= y + 0 \quad (A3) \\ x &= y \end{aligned}$$

- For any $x \in \mathbb{R}$, $x * 0 = 0$

$$\begin{aligned} x * 0 &= x * (0 + 0) \\ x * 0 &= x * 0 + x * 0 \\ x * 0 + (-x * 0) &= (x * 0 + x * 0) + (-x * 0) \\ 0 &= x * 0 + (x * 0 + (-x * 0)) \\ &= x * 0 + 0 \\ &= x * 0 \end{aligned}$$

- $-1 * x = -x$ i.e. $x + (-1) * x = 0$

$$\begin{aligned} x + (-1) * x &= x + x * (-1) \\ &= x * 1 + x * (-1) \\ &= x * (1 + (-1)) \\ &= x * 0 \\ &= 0 \end{aligned}$$

- **Theorem:** $\forall x, y \in \mathbb{R}$, $x * y = 0 \iff x = 0 \lor y = 0$ (zero-product property).
  Proof: Let $x, y \in \mathbb{R}$, if $x = 0$ or $y = 0$, then $x * y = 0$. Suppose $x \neq 0$, then we must show $y = 0$. Since $x \neq 0$, $\frac{1}{x}$ exists. Thus, if:

$$\begin{aligned} xy &= 0 \\ \frac{1}{x} * (xy) &= \frac{1}{x} * 0 \\ (\frac{1}{x} * (xy)) * y &= 0 \\ 1 * y &= 0 \\ y &= 0 \blacksquare \end{aligned}$$

1.2 Order Axioms

$\forall x, y \in \mathbb{R}$

- (O1) One of $x < y$, $x > y$ or $x = y$ is true.

- (O2) If $x < y$ and $y < z$, then $x < z$.
- (O3) If $x < y$ then $x + z < y + z$.
- (O4) If $x < y$ and $z > 0$ then $xz < yz$.

Proofs

- If $x < y$ then $-y < -x$

$$\begin{gathered} x < y \\ x + (-x + -y) < y + (-x + -y) \\ (x + -x) + -y < (y + -y) + -x \\ 0 + -y < 0 + -x \\ -y < -x \end{gathered}$$

- **Theorem:** If $x < y$ and $z > 0$ then $xz > yz$.
  Proof: If $x < y$ and $z > 0$ then $-z > 0$. Thus, $x(-z) < y(-z)$. But,

$$\begin{aligned} x(-z) &= x(-1 * z) \\ &= (x * -1) * z \\ &= (-1 * x) * z \\ &= -1(x * z) \\ &= -x * z \end{aligned}$$

  Similarly, $y(-z) = -y * z$. Thus, $-x * z < -y * z$, so $xz > yz$. ■

$\implies \mathbb{R}$ is an ordered field. Reals are complete, rationals are not.

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Theorem: $\sqrt{2}$ is irrational.
Proof: Suppose not. Suppose that $\sqrt{2}$ is rational. Then $\exists m, n \in \mathbb{Z}$ such that $\sqrt{2} = \frac{m}{n}, n \neq 0$ and m and n share no common factors. Then,

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ 2n^2 &= m^2 \end{aligned}$$

Thus, m^2 is even and m is even. Then, $m = 2k$ for some $k \in \mathbb{Z}$. But, by substituting $m = 2k$ into the above equation, we get

$$\begin{aligned} 2n^2 &= (2k)^2 \\ 2n^2 &= 4k^2 \\ n^2 &= 2k^2 \end{aligned}$$

Thus, n^2 is even, so n is even. So, n is a perfect square, which is a contradiction. Thus, $\sqrt{2}$ is irrational. ■

2.1 Upper and Lower Bounds

Theorem: Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s \forall s \in S$, m is called an **upper bound** for S. If $m \leq s \forall s \in S$, m is called a **lower bound** for S. **Minimums** and **maximums** must exist in the set to be valid.

$$T = \{q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2}\}$$

- Lower bound: -420, -1
- Upper bound: 100, 5, 2
- Minimum: 0
- Maximum: No max

Because rationals are not complete, there is no upper bound for T.

Supremum: The least upper bound of a set is called the supremum of the set.
Infimum: The greatest lower bound is called the infimum of the set.

2.2 Completeness Axiom

Completeness Axiom: Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. That is, $\sup S$ exists and is a real number.

Theorem: The set of natural numbers $\mathbb{N}$ is unbounded above.
Proof: Suppose not. Suppose that $\mathbb{N}$ is bounded above. If $\mathbb{N}$ were bounded above, it must have a supremum m. Since $\sup \mathbb{N} = m$, $m - 1$ is not an upper bound. Thus, $\exists n_0 \in \mathbb{N}$ such that $n_0 > m - 1$. But then, $n_0 + 1 > m$. This is a contradiction since $n_0 + 1 \in N$. Thus, N is unbounded above. $\blacksquare$

Theorem: If A and B are nonempty subsets of $\mathbb{R}$, let $C = \{x + y \mid x \in A, y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C = \sup A + \sup B$.
Proof: Let $\sup A = a$ and $\sup B = b$. Then if $z \in C, z = x + y$ for some $x \in A, y \in B$. Then,

$$z = x + y \leq a + b = \sup A + \sup B$$

By the completeness axiom, $\exists$ a least upper bound of $C, c = \sup C$. It must be that $c \leq a+b$, so we must show $c \geq a + b$. Let $\epsilon > 0$. Since $a = \sup A$, $a - \epsilon$ is not an upper bound for A. $\exists x \in A$ such that $a - \epsilon < x$. Likewise, $\exists y \in B$ such that $b - \epsilon < y$. Then,

$$(a - \epsilon) + (b - \epsilon) = a + b - 2 * \epsilon < x + y \leq c$$

Thus, $a + b < c + 2 * \epsilon \forall \epsilon > 0$. So, $a + b \leq c \therefore c = a + b$.