M 361K Homework 1

Ishan Shah

September 15, 2022

Section 2.1

Let $a, b \in \mathbb{R}$. Prove the following theorems:

1a. If $a + b = 0$, then $b = -a$.

Proof.

$$\begin{aligned} a + b &= 0 \\ a + (-a) + b &= 0 + (-a) \ (\mathbf{A4}) \\ 0 + b &= 0 + (-a) \\ b &= -a \end{aligned}$$

□

2b. $(-a) * (-b) = a * b$.

Proof.

$$\begin{aligned} (-a) * (-b) &= a * b \\ (-1 * a) * (-1 * b) &= 1 * a * 1 * b \\ (-1 * -1) * (a * b) &= (1 * 1) * (a * b) \\ 1 * (a * b) &= 1 * (a * b) \\ &= a * b \end{aligned}$$

□

2c. $1/(-a) = -(1/a)$.

Proof. We can use the proof from **2b** to simplify the negative signs.

$$\begin{aligned} 1/(-a) &= -(1/a) \\ 1 &= (-a) * (-1/a) \\ 1 &= a * (1/a) \end{aligned}$$

$$
\begin{aligned}
1 &= 1 \\
1/(-a) &= -(1/a)
\end{aligned}
$$

□

5. If $a \neq 0$ and $b \neq 0$, $1/(ab) = (1/a)(1/b)$.

Proof. We need to show that $(1/a)(1/b) * (ab) = 1$ and $(ab) * (1/a)(1/b) = 1$.

$$
\begin{aligned}
(1/a)(1/b) * (ab) &= (1/a)(1/b) * (ab) \\
&= (1/a) * (1/b) * (a * b) \\
&= (1/a * a) * (1/b * b) \ (\mathbf{M4}) \\
&= 1 * 1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
(ab) * (1/a)(1/b) &= (ab) * (1/a)(1/b) \\
&= (a * b) * (1/a) * (1/b) \\
&= (a * 1/a) * (b * 1/b) \ (\mathbf{M4}) \\
&= 1 * 1 \\
&= 1
\end{aligned}
$$

Using the existence of reciprocals property,

$$1/(ab) = (1/a)(1/b)$$

□

18. If for every $\epsilon > 0$ we have $a \leq b + \epsilon$, then $a \leq b$.

Proof. Suppose not. Suppose that $b < a$. Then, $0 < a - b$. Let $\epsilon = \frac{a-b}{2}$. Then,

$$
\begin{aligned}
a &\leq b + \frac{a-b}{2} \\
a &\leq \frac{a+b}{2} \\
2a &\leq a + b \\
a &\leq b
\end{aligned}
$$

We have that $a \leq b$ and $b < a$, which is a contradiction. Therefore, $a \leq b$. □

Section 2.3

5. Find the infimum and supremum, if they exist, of each of the following sets:

(a) $A := \{x \in \mathbb{R} : 2x + 5 > 0\}$

$$\begin{aligned} 2x + 5 &> 0 \\ 2x &> -5 \\ x &> -\frac{5}{2} \end{aligned}$$

- Infimum: $-\frac{5}{2}$
- Supremum: DNE

(b) $B := \{x \in \mathbb{R} : x + 2 \geq x^2\}$

$$\begin{aligned} x + 2 &\geq x^2 \\ x^2 - x - 2 &\leq 0 \\ (x+1)(x-2) &\leq 0 \end{aligned}$$

We have that $-1 \leq x \leq 2$.

- Infimum: -1
- Supremum: 2

(c) $C := \{x \in \mathbb{R} : x < 1/x\}$

$$\begin{aligned} x &< \frac{1}{x} \\ x - \frac{1}{x} &< 0 \\ \frac{x^2 - 1}{x} &< 0 \\ \frac{(x+1)(x-1)}{x} &< 0 \end{aligned}$$

This is upper bounded by 1.

- Infimum: DNE
- Supremum: 1

(d) $D := \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\}$

$$\begin{aligned} x^2 - 2x - 5 &< 0 \\ (x - (1 + \sqrt{6}))(x - (1 - \sqrt{6})) &< 0 \end{aligned}$$

We have that $1 - \sqrt{6} < x < 1 + \sqrt{6}$.

- Infimum: $1 - \sqrt{6}$
- Supremum: $1 + \sqrt{6}$

7. If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.

Proof. Let S be any nonempty subset of R with some upper bound u. By the completeness axiom, there exists some least upper bound $\sup S$. Then, $\sup S \le u$ by the definition of supremum. Since S contains u, we have that $u \le \sup S$. Therefore, $\sup S = u$. $\square$

10. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$

Proof. Let $a = \sup A$, $b = \sup B$, and $c = \sup\{a, b\}$. Then, c is an upper bound of $A \cup B$. That is, $\forall x \in A, x \le a \le c$ and $\forall x \in B, x \le b \le c$. Let d be any upper bound of $A \cup B$. Then, $a \le d$ and $b \le d$. Therefore, $c \le d$. Therefore, c is the supremum of $A \cup B$ and $\sup(A \cup B) = \sup\{\sup A, \sup B\}$. $\square$

Section 2.5

2. If $S \subseteq \mathbb{R}$ is nonempty, show that S is bounded if and only if there exists a closed, bounded interval I such that $S \subseteq I$.

Proof. Suppose S is bounded. Then, S has an lower bound a and a upper bound b. That is, $\forall x \in S, a \le x \le b$, so $x \in [a, b]$. Therefore, $S \subseteq I$ where $I = [a, b]$.

Suppose there exist a closed, bounded interval $I = [a, b]$ such that $S \subseteq I$. Then, $\forall x \in S, x \in I$, so $a \le x \le b$. Therefore, S is bounded above and below. $\square$

Section 3.1

4. For any $b \in \mathbb{R}$, prove that $\lim(b/n) = 0$.

Proof. If $b = 0$, the limit is obviously 0. When $b \neq 0$, we have that for any $\epsilon > 0$, $\frac{\epsilon}{|b|} > 0$. We know $\exists$ some n_0 such that $\frac{1}{n_0} < \frac{\epsilon}{|b|}$. $\forall n \ge n_0, \frac{1}{n} < \frac{\epsilon}{|b|}$, so $|\frac{b}{n} - 0| < \epsilon, \forall n \ge n_0$. Therefore, $\lim(b/n) = 0$. $\square$

8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the converegence of $(|x_n|)$ need not imply the convergence of (x_n).

Proof. $|x_n - 0| = ||x_n| - 0|$. Thus, for $\epsilon > 0, |x_n - 0| < \epsilon$ if and only if $||x_n| - 0| < \epsilon$. This implies that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

An example of this is the sequence $x_n = \{1, -1, 1, -1, \dots\}$. This sequence is not convergent, but $|x_n| = \{1, 1, 1, 1, \dots\}$ is convergent. $\square$

13. Show that $\lim(1/3^n) = 0$.

Proof. Since $n \leq 3^n \iff \frac{1}{3^n} \leq \frac{1}{n}$, we have that

$$|\frac{1}{3^n} - 0| \leq \frac{1}{n}$$

Because we know that $\lim_n \frac{1}{n} = 0$, we have that $\lim_n \frac{1}{3^n} = 0$. $\square$

Section 3.2

2. Give an example of two divergent sequences X and Y such that:

(a) Their sum $X + Y$ converges.

(b) Their product XY converges.

Proof. Let $X = \{1, 0, 1, 0, 1, \ldots\}$ and $Y = \{0, 1, 0, 1, 0, \ldots\}$. Then, $X + Y = \{1, 1, 1, 1, 1, \ldots\}$ and $XY = \{0, 0, 0, 0, 0, \ldots\}$. Thus, both $X + Y$ and XY converge. $\square$

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why **Theorem 3.2.3** cannot be used.

Proof. Suppose that (b_n) is a bounded sequence and $\lim(a_n) = 0$. Now, let $|b_n| \leq M$ for some $M \geq 0$. Then, $|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|a_n|$. Since $\lim(a_n) = 0$, we have that $\lim(a_n b_n) = 0$. $\square$

Remark. **Theorem 3.2.3** cannot be used here because it only applies when both sequences converge. We know that a_n is convergent, but b_n is not necessarily convergent since not all bounded sequences converge.

22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\epsilon > 0$ there exists M such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

Proof. We have that $|x_n - y_n| < \epsilon = |(y_n - x_n) - 0| < \epsilon$. This means that $\lim(y_n - x_n) = 0$. We also know that $y_n = (y_n - x_n) + x_n$. Because x_n is convergent, this means that $|(y_n - x_n) + x_n| < \epsilon \implies |y_n| < \epsilon$ so (y_n) is convergent. $\square$