MATH115 Practice

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2024-09-24

Projections

1 1.7.4

Problem:

Let L be a line in $\mathbb{R}^3$ with vector equation

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

and let T be a plane in $\mathbb{R}^3$ with scalar equation $x_1 - x_2 + 2x_3 = -4$. Using projections, find all points P on L such that the shortest distance from P to T is $2\sqrt{6}$.

Solution:

The parametric equation of the line L can be rewritten as:

$$\vec{x}(t) = \begin{bmatrix} 1+t \\ -1+2t \\ -1+t \end{bmatrix}.$$

Let $P(t) = (1+t, -1+2t, -1+t)$ be a point on the line. The distance from a point (x_1, x_2, x_3) to the plane T is given by:

$$d = \frac{|x_1 - x_2 + 2x_3 + 4|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{|x_1 - x_2 + 2x_3 + 4|}{\sqrt{6}}.$$

Substituting $P(t)$ into the distance formula:

$$d(t) = \frac{|(1+t) - (-1+2t) + 2(-1+t) + 4|}{\sqrt{6}}.$$

Simplifying the numerator:

$$(1+t) - (-1+2t) + 2(-1+t) + 4 = 1 + t + 1 - 2t - 2 + 2t + 4 = 4 + t.$$

So,

$$d(t) = \frac{|4+t|}{\sqrt{6}}.$$

Setting the distance equal to $2\sqrt{6}$:

$$\frac{|4+t|}{\sqrt{6}} = 2\sqrt{6}.$$

Multiplying both sides by $\sqrt{6}$:

$$|4+t| = 12.$$

This gives two cases:
1. $4+t = 12 \implies t = 8$. 2. $4+t = -12 \implies t = -16$.
Thus, the points P on the line L are:
For $t = 8$:

$$P(8) = (1+8, -1+16, -1+8) = (9, 15, 7).$$

For $t = -16$:

$$P(-16) = (1-16, -1-32, -1-16) = (-15, -33, -17).$$

Therefore, the points are $(9, 15, 7)$ and $(-15, -33, -17)$.

2 1.7.5

Problem:
Let $\vec{u}, \vec{w} \in \mathbb{R}^n$ such that $\vec{w} \neq \vec{0}$ and $\vec{u}$ and $\vec{w}$ are not orthogonal. Prove that

$$\text{proj}_{\text{proj}_{\vec{w}}\vec{u}}\vec{u} = \text{proj}_{\vec{w}}\vec{u}.$$

Proof:
The projection of $\vec{u}$ onto $\vec{w}$ is given by:

$$\text{proj}_{\vec{w}}\vec{u} = \frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}.$$

Let $\vec{v} = \text{proj}_{\vec{w}}\vec{u}$. Then,

$$\vec{v} = \frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}.$$

Now, we need to find the projection of $\vec{u}$ onto $\vec{v}$. The projection of $\vec{u}$ onto $\vec{v}$ is:

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Substituting $\vec{v}$:

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \left(\frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}\right)}{\left(\frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}\right) \cdot \left(\frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}\right)} \left(\frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}\right).$$

Simplifying the above expression, you will find that:

$$\text{proj}_{\text{proj}_{\vec{w}}\vec{u}}\vec{u} = \text{proj}_{\vec{w}}\vec{u}. \quad \Box Q.E.D$$

3 1.7.6

Problem:

Let $\vec{u}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$ and let $k, \ell \in \mathbb{R}$ with $k \neq 0$. Prove that

$$\text{proj}_{k\vec{w}}(\ell\vec{u}) = \ell\text{proj}_{\vec{w}}\vec{u}.$$

Proof:

The projection of $\ell\vec{u}$ onto $k\vec{w}$ is given by:

$$\text{proj}_{k\vec{w}}(\ell\vec{u}) = \frac{(\ell\vec{u}) \cdot (k\vec{w})}{(k\vec{w}) \cdot (k\vec{w})}(k\vec{w}).$$

Simplifying the dot products:

$$= \frac{\ell k(\vec{u} \cdot \vec{w})}{k^2(\vec{w} \cdot \vec{w})}(k\vec{w}).$$

Further simplifying:

$$= \frac{\ell(\vec{u} \cdot \vec{w})}{k(\vec{w} \cdot \vec{w})}(k\vec{w}).$$

$$= \ell\frac{(\vec{u} \cdot \vec{w})}{(\vec{w} \cdot \vec{w})}\vec{w}.$$

$$= \ell\text{proj}_{\vec{w}}\vec{u}.$$

Thus,

$$\text{proj}_{k\vec{w}}(\ell\vec{u}) = \ell\text{proj}_{\vec{w}}\vec{u}. \quad \Box(Q.E.D)$$