

PAPER • OPEN ACCESS

Quantum Algorithms for Solving Linear Regression Equation

To cite this article: Kai Li *et al* 2021 *J. Phys.: Conf. Ser.* **1738** 012063

View the [article online](#) for updates and enhancements.

You may also like

- [A quantum algorithm for Toeplitz matrix-vector multiplication](#)
Shang Gao, , Yu-Guang Yang et al.
- [Near 100% spectral-purity photons from reconfigurable micro-rings](#)
Pingyu Zhu, , Yingwen Liu et al.
- [Experimental and Theoretical Research on the Vibrational Gaps in Two-Dimensional Three-Component Composite Thin Plates](#)
Yu Dian-Long, Liu Yao-Zong, Qiu Jing et al.

Quantum Algorithms for Solving Linear Regression Equation

Kai Li¹, Hongyi Dai², Feng Jing¹, Mei Gao¹, Bin Xue¹, Pu Wang¹ and Ming Zhang^{3*}

¹College of Information Communication, National University of Defense Technology, Xi'an, Shanxi, 710106, Peoples Republic of China

²Department of Physics, College of Liberal Arts and Sciences, National University of Defense Technology, Changsha, Hunan, 410073, Peoples Republic of China

³Department of Cognitive Science, College of Intelligence Science and Technology, National University of Defense Technology, Changsha, Hunan, 410073, Peoples Republic of China

*Corresponding author's e-mail: Luantuk@foxmail.com

Abstract. Linear regression is one of the most important and common analytical methods in mathematical statistics. The letter studies a general model of linear regression problem based on least squares method, and investigates the impact of quantum algorithms on the time complexity of solving linear regression problem when quantum algorithms can be implemented on quantum computers. With the help of the general model of linear regression problem in terms of least squares method, we propose a novel simplified quantum scheme for solving linear regression equation based on the sparsity-independent quantum singular value estimation algorithm. For a linear regression equation with dimension n , our scheme can reduce the time complexity from $O(Nn^2)$ to $O(\sqrt{N} \log N)$ when both the condition number κ of related matrices and the reciprocal of precision ϵ are small in size $O(\text{poly} \log N)$, where m is the dimension of the input u and N is the number of samples.

1. Introduction

The linear regression analysis[1] is one of the most common analytical methods for dealing with classical problems. However, solving linear regression problem will cost much computing resource in high-dimensional classical systems. Many studies have demonstrated that quantum algorithms has advantages for solving some problems with heavy calculation burden in comparison to traditional classical algorithms[2]. It seems interesting for us to explore how quantum algorithms will reduce the time complexity of solving linear regression equation when quantum algorithms can be implemented on quantum computers.

Quantum computation[3] can be implemented on quantum computers and perform high-efficiency calculations based on coherent superposition and parallel processing[4]. Two most famous quantum algorithms, the polynomial-time quantum algorithm for factoring[5-6] proposed by Shor in 1994 and the quantum algorithm for searching a database in time square-root its size[7] proposed by Grover in 1996, show the power of quantum computation. In 2009, A. W. Harrow et al. proposed the quantum algorithm for solving linear equations (HHL)[8], which is a remarkable development of quantum algorithms. And L. Wossnig et al. proposed a quantum singular value estimation algorithm(QSVE) to solve linear equations and matrix multiplication[9-10] in 2018, which is of significance as the basis of



matrixes computing. For matrix multiplication calculation, C. P. Shao proposed a new algorithm by swap test and analyzed other quantum algorithms[11] in 2018.

With the rapid progress of quantum computing, more and more new potential applications of quantum algorithms are already being explored. However, the application of quantum algorithms in actual classical control problems has not been fully studied so far. As one of important potential applications of quantum computation, the quantum algorithm of solving linear regression equation has not been deeply investigated so far.

The most common method of solving linear regression equation is least squares method[12-15]. For linear regression problem, the computational cost will go up dramatically with the increase of the dimension of the linear system. Because linear regression equation can be reduced to several matrix operations and there exist some corresponding quantum algorithms for matrix operations, it seems reasonable to develop an entire quantum scheme to improve the computational efficiency of solving linear regression equation. For matrix operations problems, quantum algorithms are capable to significantly reduce the cost of computing resources compared with classical algorithms. Therefore, we can use quantum computation[3, 16-17] as a computational tool to improve the computational efficiency of linear regression equation. Fortunately, we have obtained a novel simplified quantum scheme of solving linear regression equation by recognizing that a general model of linear regression equation in classical systems can be given in terms of least squares method. From the perspective of time complexity[18-20], we have demonstrated that quantum algorithms is superior to classical algorithms.

The rest of our letter are organized as follows. The Sect.2 analyzes a general model of linear regression problem based on least squares method. In the Sect.3, we make a time complexity comparison between classical algorithms and quantum algorithms. The Sect.4 summarizes the work of this letter.

2. Quantum acceleration algorithms of linear regression equation

2.1. Linear regression equation

Consider a general model of linear regression problem as follows. Features are measured for Nn variable h_{ij} ($i = 1, 2, \dots, N$ and $j = 1, 2, \dots, n$) and for N variable w_i ($i = 1, 2, \dots, N$) with the goal to establish a linear relationship between them. This can be represented mathematically as

$$w_i = y_1 h_{i1} + y_2 h_{i2} + y_3 h_{i3} + \dots + y_n h_{in} + e_i \quad (1)$$

Then we have

$$W = HY + e \quad (2)$$

where e is the error or residual, W is an N dimensional vectors and H is an $N \times n$ matrix. The goal is to estimate the parameter vector Y from a set of inputs h_{ij} and states w_i .

Suppose there are N observed samples where $N \geq n$, and

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, H = \begin{bmatrix} h_{11}, h_{12}, \dots, h_{1n} \\ \vdots \\ h_{N1}, h_{N2}, \dots, h_{Nn} \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad (3)$$

The approximate solution \hat{Y} of the equation (2) can be obtained by least squares methods:

$$\hat{Y} = (H^T H)^{-1} H^T W \quad (4)$$

So far, the linear regression problem is reduced to solving the equation (4) by classical or quantum algorithms.

Given the matrix H with singular value decomposition $H = \sum_i \sigma_i \mu_i \nu_i^T$, where σ_i is the singular value of H , $\mu_i \in R^N$ and $\nu_i \in R^{m+n}$ are the left and right singular vectors, and $\mu_i^T \mu_i = \nu_i^T \nu_i = 1, \mu_i^T \mu_j = \nu_i^T \nu_j = 0 (i \neq j)$. So we get:

$$H^T H = \sum_i \sigma_i \nu_i \mu_i^T \sum_i \sigma_i \mu_i \nu_i^T = \sum_i \sigma_i^2 \nu_i \nu_i^T \quad (5)$$

$H^T H$ is a nonsingular matrix with eigenvalues σ_i^2 and eigenvectors ν_i . Therefore, the eigenvalues of matrix $(H^T H)^{-1}$ are $1/\sigma_i^2$, and the eigenvectors are ν_i , that is

$$(H^T H)^{-1} = \sum_i \frac{1}{\sigma_i^2} \nu_i \nu_i^T \quad (6)$$

According to Eq. (6), we can easily get

$$\begin{aligned} (H^T H)^{-1} H^T &= \sum_i \frac{1}{\sigma_i^2} \nu_i \nu_i^T \sum_i \sigma_i \nu_i \mu_i^T \\ &= \sum_i \frac{1}{\sigma_i} \nu_i \mu_i^T \end{aligned} \quad (7)$$

The singular value σ'_i of matrix $(H^T H)^{-1} H^T$ can be obtained from the singular values σ_i of the matrix H ,

$$\sigma'_i = \frac{1}{\sigma_i} \quad (8)$$

2.2. Quantum scheme

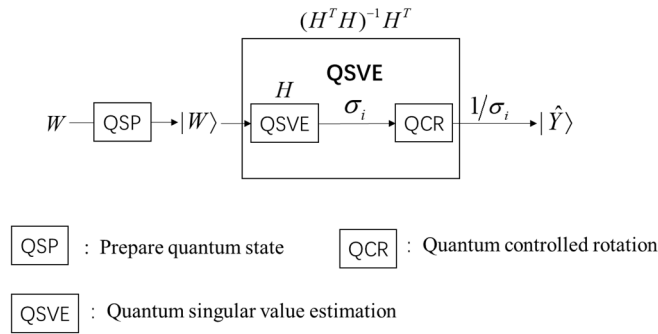


Figure 1. The flow chart of our quantum scheme.

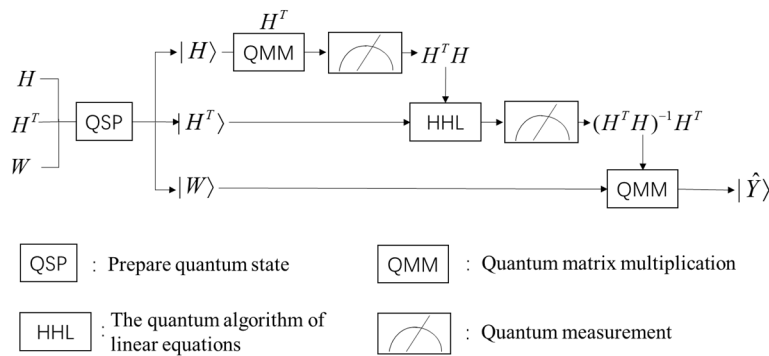


Figure 2. The flow chart of previous quantum scheme.

The comparison between our scheme and previous quantum algorithms is shown in Figure 1 and Figure 2. Our scheme can obtain singular value σ'_i of matrix $(H^T H)^{-1} H^T$ from σ_i based on QSVE in matrix H . Therefore, we do not have to build quantum gate circuits repeatedly compared to previous quantum algorithms, which can effectively reduce the time complexity of solving linear regression equation.

3. Algorithms complexity analysis

Then we analyzed the time complexity of the classical algorithms and quantum algorithms.

3.1. Time complexity analysis of classical algorithms

The time complexity of classical algorithms is estimated in three steps:

- Matrix multiplication $E = H^T H$. H is an $N \times n$ matrix. Therefore, the time complexity of calculating $E = H^T H$ is $O(Nn^2)$.
- Matrix inverse $F = E^{-1} H^T$. The time complexity of E^{-1} is $O(n^3)$ and the time complexity of matrix multiplication $E^{-1} H^T$ is $O(Nn^2)$.
- The matrix multiplication of matrices F and W . The time complexity of $Y = FW$ is $O(Nn)$.

3.2. Time complexity analysis of quantum algorithms

The time complexity of our scheme is analyzed in the following two parts: preparing quantum states, the quantum algorithms of matrix multiplication based on the quantum singular value estimation algorithm.

- The time complexity of preparing quantum states. Quantum states with dimension Nn can be prepared in time $O(\log(Nn)/\epsilon^2)$ when quantum states can be efficiently prepared, where ϵ is the precision.
- The time complexity of matrix multiplication. The time complexity of QAMM based on QSVE with dimension n is $O(\kappa^3 \sqrt{n}/\epsilon)$, κ is the condition number, and ϵ is the precision. Besides, the time complexity of QAMM by swap test is $O(\kappa \sqrt{n}/\epsilon)$. Thus, the time complexity of our quantum algorithm is $O(\kappa_3^3 \sqrt{N}/\epsilon)$, where κ_1 , κ_2 and κ_3 are the condition number of the corresponding matrices H , H^T and W .

Remark: the precision ϵ means that if s is the exact result and s' is the result getting from quantum algorithms, then $|s - s'| \leq \epsilon$.

To be more intuitive, we made a time complexity comparison between the classical algorithms and quantum algorithms. From Table 1 and Table 2, we know that the holistic complexity of classical algorithms is $O(Nn^2)$, while our scheme just needs $O(\sqrt{N} \log(N))$ when $\kappa, 1/\epsilon = O(\log(N))$ and quantum states can be efficiently prepared.

Table 1. Time complexity comparison of each step between classical and quantum algorithms for linear regression equation.

algorithm	time complexity	
preparing quantum states:	classical algorithms	quantum algorithms
i) $H \rightarrow H\rangle$	/	$O(\log(Nn)/\epsilon^2)$
ii) $W \rightarrow W\rangle$	/	$O(\log(N)/\epsilon^2)$
iii) $H^T \rightarrow H^T\rangle$	/	$O(\log(Nn)/\epsilon^2)$
Least squares methods	classical	quantum
i) $E = H^T H$	$O(Nn^2)$	$O(\kappa_1 \sqrt{N}/\epsilon)$
ii) $F = (E)^{-1} H^T$	$O(Nn^2)$	$O(\kappa_2^2 \log(n)/\epsilon)$
iii) $Y = FW$	$O(Nn)$	$O(\kappa_3 \sqrt{N}/\epsilon)$
Sum ($\kappa, 1/\epsilon = O(\log N)$)	$O(Nn^2)$	$O(\sqrt{N} \log N)$

Table 2. Time complexity of our scheme for linear regression equation

algorithm	time complexity
preparing quantum states $W \rightarrow W\rangle$	$O(\log(N)/\epsilon^2)$
Least squares method	$O(\kappa_3^3 \sqrt{N}/\epsilon)$
Sum	$O(\sqrt{N} \log N)$

4. Conclusion

For general classical linear regression problems, the letter proposed an entire quantum scheme based on least squares method. For an n th-order classical system, when the condition number κ and the reciprocal of precision ϵ are small in size $O(\text{poly } \log(N))$, our scheme can reduce the run time to $O(\sqrt{N} \log(N))$, which achieves significant acceleration in comparison to the run time $O(Nn^2)$ in traditional classical algorithms. Based on singular value estimation algorithm, our scheme is not constrained by the sparse condition.

Our scheme does not have to build quantum gate circuits repeatedly, which can effectively reduce the time complexity of linear regression. Due to the significance of linear regression in classical problems, our scheme will have a larger application scope, such as in mathematical statistics, artificial intelligence[21-22], etc. In summary, it is shown that the letter proposed an entire quantum algorithm for solving linear regression equation, and enrich the application area of quantum computation.

Acknowledgments

This work was supported in part by the National Key R&D Program of China under Grant 2018YFE0100500, in part by the National Natural Science Foundation of China under Grant 61871387 and 61871471.

References

- [1] Geladi, P., Kowalski, B. R. (1985) Partial least-squares regression: a tutorial. *Analytica Chimica Acta*, 185(1):1-17. Mehra, R. K., Lainiotis, D. G. (1976) *System identification*. Academic Press.
- [2] Ljung, L. (1999) *System identification: theory for the user*. Epfl, 16(1): 9-11.
- [3] Nielsen, M. A., Chuang, I. L. (2000) *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press.
- [4] Wiebe, N., Braun, D., Lloyd, S. (2012) Quantum Algorithm for Data Fitting. *Physical Review Letters*, 109(5): 050505.
- [5] Shor, P. W. (1994) Algorithms for quantum computation: discrete logarithms and factoring. *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, 1994: 124-134.
- [6] Shor, P. W. (1999) Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Review*, 41(2): 303-332.
- [7] Grover, L. K. (1996) A fast quantum mechanical algorithm for database search. *Proceedings of the 28th ACM Symposium on Theory of Computing ACM*, 1996: 212-219.
- [8] Harrow, A. W., Hassidim, A., Lloyd, S. (2009) Quantum algorithm for linear systems of equations. *Physical Review Letters*, 103(15): 150502.
- [9] Wossnig, L., Zhao, Z. K., Prakash, A. (2018) Quantum Linear System Algorithm for Dense Matrices. *Physical Review Letters*, 120: 050502.
- [10] Kerenidis, I., Prakash, A. (2017) Quantum Recommendation System. *Proceedings of 8th Innovations in Theoretical Computer Science Conference, Dagstuhl, Germany*, 49: 1.
- [11] Shao, C. P. (2018) Quantum algorithms to matrix multiplication. arXiv: 1803.01601[quant-ph].
- [12] Koch, G. G., Landis, J. R., Freeman, J. L., et al. (1977) A general methodology for the analysis of experiments with repeated measurement of categorical data. *Biometrics*, 33(1):133-158.

- [13] Marquardt, D. W. (1963) An Algorithm for Least-Squares Estimation of Nonlinear Parameters. *Journal of the Society for Industrial and Applied Mathematics*, 11(2): 431-441.
- [14] Suykens, J. A. K., Vandewalle, J. (1999) Least Squares Support Vector Machine Classifiers[J]. *Neural Processing Letters*, 9(3):293-300.
- [15] Chin, W. W., Marcoulides, G. (1998) The Partial Least Squares Approach to Structural Equation Modeling[J]. *Advances in Hospitality and Leisure*, 8(2).
- [16] Emily, G., Mark, H. (2018) *Quantum Computing: Progress and Prospects*. The National Academy press.
- [17] Coles, P. J., Eidenbenz, S. et al. (2018) *Quantum Algorithm Implementations for Beginners*, arXiv: 1804.03719.
- [18] Li, K., Yang, L., et al. (2018) Impact of quantum algorithms on time complexity of classical-data problems. *Chinese Control Conference (international)*.
- [19] Li, K., Dai, H. Y., and Zhang, M. (2020) Quantum algorithms of state estimators in classical control systems, *Science China Information Sciences*, 2020(9).
- [20] Yang, L., Li, K., Dai, H. Y., Zhang, M. (2019) A novel scheme of quantum state tomography based on quantum algorithms (in Chinese), *Acta Physica Sinica*.
- [21] Xue, B., Tong, N. N. (2019) DIOD: Fast and Efficient Weakly Semi-Supervised Deep Complex ISAR Object Detection. *IEEE Transactions on Cybernetics*, 49(11): 3991-4003.
- [22] Xue, B., Tong, N. N. (2020) Real-world ISAR Object Recognition using Deep Multimodal Relation Learning. *IEEE Transactions on Cybernetics*, 50(10): 4256–4267.