Additional Example S

1. $dL_i(t) = \sigma_i L_i(t)\, dW_t^{i+1}$

$$L_i(T) = L_i(0)\exp\left(-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}\right)$$

$$V_0 = D_{i+1}(0) \cancel{\Delta}\, \underline{\mathbb{E}^{i+1}\left[L_i(T_i) + L_i^2(T_i)\right]}$$

$$\mathbb{E}^{i+1}[L_i] = L_i(0) e^{-\frac{\sigma^2 T}{2}} \mathbb{E}\left[e^{\sigma W_T}\right]$$
$$= L_i(0) \cancel{e^{-\frac{\sigma^2 T}{2}}} \cdot \cancel{e^{\frac{\sigma^2 T}{2}}}$$
$$= L_i(0)$$

$$\mathbb{E}^{i+1}[L_i^2 | T_i] = \mathbb{E}^{i+1}\left[L_i(0)^2 e^{-\sigma^2 T + 2\sigma W_T}\right]$$
$$= L_i(0)^2 e^{-\sigma^2 T} \mathbb{E}\left[e^{2\sigma W_T}\right]$$
$$= L_i(0)^2 e^{-\sigma^2 T} e^{2\sigma^2 T}$$
$$= L_i(0)^2 e^{\sigma^2 T}$$

why I don't quite get this.

no Δ

$$\therefore V_0 = D_{i+1}(0) \cancel{\Delta}\left(L_i(0) + L_i(0)^2 e^{\sigma^2 T}\right)$$

2. $dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t)\, dW^{n+1,N}$

$$S_{n,N}(T) = S_{n,N}(0)\exp\left(-\frac{\sigma_{n,N}^2 T}{2} + \sigma_{n,N} W_T^{n+1,N}\right)$$

$$\frac{V_{n,N}^{dig}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_{n,N}^{dig}(T)}{P_{n+1,N}(T)}\right]$$

$$V_{n,N}^{dig}(0) = P_{n+1,N}(0)\, \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)} \mathbb{1}_{S_{n,N}(T) > K}}{\cancel{P_{n+1,N}(T)}}\right]$$
$$= P_{n+1,N}(0)\, \mathbb{E}^{n+1,N}\left[\mathbb{1}_{S_{n,N}(T) > K}\right]$$
$$= P_{n+1,N}(0)\, \Phi(d_2)$$

$$d_2 = \frac{\ln\left(\frac{S_{n,N}(0)}{K}\right) - \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}$$

3. $$\begin{cases} V^{pay}(t) = D(t,T)\int_K^{\infty} IRR(S)(S-K)\,f(S)\,dS \\ V^{rec}(t) = D(t,T)\int_0^{K} IRR(S)(K-S)\,f(S)\,dS \end{cases}$$

$$V(K) = D(t,T)\,\mathbb{E}^T\left[IRR(S)(S-K)^+\right]$$
$$= D(t,T)\int_K^{\infty} IRR(S)(S-K)\,f(S)\,dS$$

$$\frac{\partial V}{\partial K} = -D(t,T)\int_K^{\infty} IRR(S)\,f(S)\,dS$$

$$\frac{\partial^2 V}{\partial K^2} = D(t,T)\,IRR(K)\,f(K)$$

$$f(K) = \frac{1}{D(t,T)\,IRR(K)}\,\frac{\partial^2 V(K)}{\partial K^2}$$

$$f(K)\begin{cases} \frac{1}{D(0,T)}\,\frac{1}{IRR(K)} \times \frac{\partial^2 V^{pay}}{\partial K^2} & \text{when } K > S_{n,N}(0) \\ \frac{1}{D(0,T)}\,\frac{1}{IRR(K)} \times \frac{\partial^2 V^{rec}(K)}{\partial K^2} & \text{when } K < S_{n,N}(0) \end{cases}$$

4. Let $F = S_{n,N}(0)$ $\qquad h(K) = \frac{g(K)}{IRR(K)}$

$$V_0 = D(0,T)\,\mathbb{E}[g(K)]$$
$$= D(0,T)\int_0^{\infty} g(K)\,f(K)\,dK$$

$g(K) = h(K) \times IRR(K)$

$IRR(K) \times D(0,T) \times f(K) = \frac{\partial^2 V}{\partial K^2}$

$$= \int_0^F h(K) \frac{\partial^2 V^{rec}(K)}{\partial K^2} dK + \int_F^\infty h(K) \frac{\partial^2 V^{pay}(K)}{\partial K^2} dK$$

$$\int_F^\infty h(K) \frac{\partial^2 V^{pay}(K)}{\partial K^2} dK \qquad \int u \cdot v'$$

$$= \left[h(K) \frac{\partial V^{pay}(K)}{\partial K} \right]_F^\infty - \int_F^\infty h'(K) \frac{\partial V^{pay}(K)}{\partial K} dK$$

$$= \left[\cancel{h(\infty) \frac{\partial V^{pay}(\infty)}{\partial K}}^{\to 0} - h(F) \frac{\partial V^{pay}(F)}{\partial K} \right] - \left[h'(K) V^{pay}(K) \right]_F^\infty + \int_F^\infty h''(K) V^{pay}(K) dK$$

$$= -h(F) \frac{\partial V^{pay}(F)}{\partial K} - \left[\cancel{h'(\infty) V^{pay}(\infty)}^{\to 0} - h'(F) V^{pay}(F) \right] + \int_F^\infty h''(K) V^{pay}(K) dK$$

$$= -h(F) \frac{\partial V^{pay}(F)}{\partial K} + h'(F) V^{pay}(F) + \int_F^\infty h''(K) V^{pay}(K) dK$$

$$\int_0^F h(K) \frac{\partial^2 V^{rec}(K)}{\partial K^2} dK = h(F) \frac{\partial V^{rec}(F)}{\partial K} - h'(F) V^{rec}(F) + \int_0^F h''(K) V^{rec}(K) dK$$

imp it is written like this for put call

$$\therefore V_0 = -h(F) \left[\frac{\partial V^{pay}(F)}{\partial K} - \frac{\partial V^{rec}(F)}{\partial K} \right] + h'(F) \left(V^{pay}(F) - V^{rec}(F) \right) + \int_0^F h''(K) V^{pay}(K) dK + \int_F^\infty h''(K) V^{rec}(K) dK$$

$$V^{pay}(K) - V^{rec}(K) = D(0,T)\,\mathbb{E}\left[IRR(S)(S-K)^+\right] - D(0,T)\,\mathbb{E}\left[IRR(S)(K-S)^+\right]$$

$$= D(0,T)\, IRR(S)\,(S-K)$$

$K = F = S_{n,N}$ then $V^{pay}(F) - V^{rec}(F) = 0$

How to get this? $\rightarrow$ $\dfrac{\partial V^{pay}(K)}{\partial K} - \dfrac{\partial V^{rec}(K)}{\partial K} = -D(0,T)\, IRR(S)$

$$\therefore V_0 = -h(F)\left(-D(0,T)\, IRR(F)\right) + h'(F)\left[V^{pay}_{(F)} - V^{rec}_{(F)}\right] + \int_0^F h''(K)\, V^{rec}(K)\, dK + \int_F^\infty h''(K)\, V^{pay}(K)\, dK$$

$$= D(0,T)\, g(F) + h'(F)\left[V^{pay}(F) - V^{rec}(F)\right] + \int_0^F h''(K)\, V^{rec}(K)\, dK + \int_F^\infty h''(K)\, V^{pay}(K)\, dK$$

for CMS $g(F) = F$ so $V^{pay}(F) - V^{rec}(F) = 0$

$$V_0 = D(0,T)\, g(F) + \int_0^F h''(K)\, V^{rec}(K)\, dK + \int_F^\infty h''(K)\, V^{pay}(K)\, dK$$