

# QF605 Additional Examples

## Session 8: Short-Rate Models, Ho-Lee & Hull-White

### 1 Questions

1. Consider a stylized interest rate model

$$dr_t = \mu dt + \sigma dW_t^*,$$

where  $W_t^*$  is a standard Brownian motion under the risk-neutral measure  $\mathbb{Q}^*$ .

- (a) Determine the distribution, mean, and variance of the integral

$$\int_t^T r_u du.$$

- (b) Identify the expressions  $A(t, T)$  and  $B(t, T)$  in the following expectation:

$$D(t, T) = \mathbb{E}^* \left[ e^{-\int_t^T r_u du} \right] = e^{A(t, T) - r_t B(t, T)}.$$

- (c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

Determine the mean and variance of the integral

$$\int_0^T r_u du,$$

and use this to evaluate the expectation

$$D(0, T) = \mathbb{E}^* \left[ e^{-\int_0^T r_u du} \right].$$

3. Consider the Ho-Lee interest rate model

$$dr_t = \theta(t) dt + \sigma dW_t^*,$$

where  $W_t^*$  is a standard Brownian motion under the risk-neutral measure  $\mathbb{Q}^*$ . Show that

$$\theta(t) = -\frac{\partial^2}{\partial t^2} \log D(0, t) + \sigma^2 t.$$

4. Suppose we use a discrete ( $\Delta t = 1y$ ) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 1%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.95
$D(0, 2y)$	0.88

Determine the no-arbitrage values for  $\theta_0$ .

5. Suppose we use a discrete ( $\Delta t = 1y$ ) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 0.5%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.95123
$D(0, 2y)$	0.86936
$D(0, 3y)$	0.78663

Draw the Ho-Lee binomial tree and determine the no-arbitrage values for  $\theta_0$  and  $\theta_1$ .

## 2 Suggested Solutions

1. (a) First we integrate the stochastic differential equation from  $t$  to  $s$  to obtain:

$$r_s = r_t + \mu(s - t) + \int_t^s \sigma dW_u^*.$$

Next we integrate  $r_s$  from  $t$  to  $T$  to obtain

$$\begin{aligned} \int_t^T r_s ds &= r_t(T - t) + \mu \int_t^T (s - t) ds + \int_t^T \int_t^s \sigma dW_u^* ds \\ &= r_t(T - t) + \mu \left[ \frac{s^2}{2} - ts \right]_t^T + \int_t^T \int_u^T \sigma ds dW_u^* \\ &= r_t(T - t) + \frac{\mu}{2}(T - t)^2 + \int_t^T \sigma(T - u) dW_u^* \end{aligned}$$

Hence the mean is given by

$$\mathbb{E}^* \left[ \int_t^T r_s ds \right] = r_t(T - t) + \frac{\mu}{2}(T - t)^2, \quad \triangleleft$$

and the variance is given by

$$\begin{aligned} V \left[ \int_t^T r_s ds \right] &= \int_t^T \sigma^2(T - u)^2 du \\ &= \frac{\sigma^2(T - t)^3}{3} \quad \triangleleft \end{aligned}$$

- (b) Having identified the mean and variance of the short rate integral, we have

$$\begin{aligned} D(t, T) &= \mathbb{E}^* \left[ e^{-\int_t^T r_u du} \right] \\ &= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2} \frac{\sigma^2(T-t)^3}{3}} \end{aligned}$$

Comparing this against

$$D(t, T) = e^{A(t, T) - r_t B(t, T)},$$

we note that

$$\begin{aligned} A(t, T) &= -\frac{\mu}{2}(T - t)^2 + \frac{\sigma^2(T - t)^3}{6} \quad \triangleleft \\ B(t, T) &= (T - t) \quad \triangleleft \end{aligned}$$

- (c) For affine interest rate model, the zero coupon bond prices can be written as

$$D(t, T) = e^{A(t, T) - r_t B(t, T)},$$

for some deterministic functions of  $A(t, T)$  and  $B(t, T)$  of  $t$  and  $T$  only. This implies that

$$R(t, T) = \frac{1}{T - t} \left( -A(t, T) + r_t B(t, T) \right),$$

i.e. the zero (spot) rates are affine functions of the short rate.  $\triangleleft$

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

We can solve this stochastic differential equation by applying Itô's formula to the function  $f(r_t, t) = e^{\kappa t} r_t$ , and the solution is given by

$$r_t = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u^*$$

Integrating both sides from 0 to  $T$ , we have

$$\int_0^T r_t dt = \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \underbrace{\int_0^T \int_0^t \sigma e^{\kappa(u-t)} dW_u^* dt}_{\text{double integral}}.$$

On the right hand side, the first and second integrals can be carried out directly. The double integral can be simplified by exchanging the order of integration (Fubini's Theorem):

$$\begin{array}{ll} \text{Inner Integral } u : & 0 \leq u \leq T \\ \text{Outer Integral } t : & 0 \leq t \leq T \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{Inner Integral } t : & u \leq t \leq T \\ \text{Outer Integral } u : & 0 \leq u \leq T \end{array}$$

So we have

$$\begin{aligned} \int_0^T \int_0^t \sigma e^{\kappa(u-t)} dW_u^* dt &= \int_0^T \int_u^T \sigma e^{\kappa(u-t)} dt dW_u^* \\ &= \int_0^T \left[ -\frac{\sigma}{\kappa} e^{\kappa(u-t)} \right]_u^T dW_u^* \\ &= \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \end{aligned}$$

So we can write the overall integral as:

$$\int_0^T r_t dt = \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^*$$

Taking expectation on both sides gives us the mean of this integral

$$\begin{aligned} \mathbb{E}^* \left[ \int_0^T r_t dt \right] &= \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt \\ &= \frac{r_0}{\kappa} (1 - e^{-\kappa T}) + \theta T - \frac{\theta}{\kappa} (1 - e^{-\kappa T}). \end{aligned}$$

Taking the variance, we obtain

$$\begin{aligned} V \left[ \int_0^T r_t dt \right] &= V \left[ \int_0^T r_0 e^{-\kappa t} dt + \int_0^T \theta (1 - e^{-\kappa t}) dt + \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \right] \\ &= V \left[ \frac{\sigma}{\kappa} \int_0^T (1 - e^{\kappa(u-T)}) dW_u^* \right] \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T (1 - e^{\kappa(u-T)})^2 du \quad \because \text{Itô's Isometry} \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T (1 - 2e^{\kappa(u-T)} + e^{2\kappa(u-T)}) du \\ &= \frac{\sigma^2}{\kappa^2} \left[ T - \frac{2}{\kappa} (1 - e^{-\kappa T}) + \frac{1}{2\kappa} (1 - e^{-2\kappa T}) \right] \end{aligned}$$

Finally, we can express the discount factor as

$$D(0, T) = \mathbb{E}^* \left[ e^{-\int_0^T r_t dt} \right]$$

$$= \exp \left( \underbrace{-\frac{r_0}{\kappa} (1 - e^{-\kappa T}) - \theta T + \frac{\theta}{\kappa} (1 - e^{-\kappa T})}_{\text{mean}} + \frac{1}{2} \cdot \underbrace{\frac{\sigma^2}{\kappa^2} \left[ T - \frac{2}{\kappa} (1 - e^{-\kappa T}) + \frac{1}{2\kappa} (1 - e^{-2\kappa T}) \right]}_{\text{variance}} \right)$$

3. Ho-Lee interest rate model is given by

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where  $W_t^*$  is a Brownian motion under the measure  $\mathbb{Q}^*$ . To fit the initial term structure, we require that

$$\theta(t) = -\frac{\partial^2}{\partial t^2} \log D(0, t) + \sigma^2 t.$$

To prove this, first write out the interest rate process in integral format

$$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*.$$

Next

$$\begin{aligned} \int_0^t r_u du &= \int_0^t r_0 du + \int_0^t \int_0^u \theta(s) ds du + \int_0^t \int_0^u \sigma dW_s^* du \\ &= r_0 t + \int_0^t \theta(s)(t-s)ds + \int_0^t \sigma(t-s)dW_s^*. \end{aligned}$$

The mean of this stochastic integral is given by

$$\mathbb{E} \left[ \int_0^t r_u du \right] = r_0 t + \int_0^t \theta(s)(t-s)ds,$$

and the variance is given by

$$V \left[ \int_0^t r_u du \right] = \int_0^t \sigma^2 (t-s)^2 ds = \frac{1}{3} \sigma^2 t^3,$$

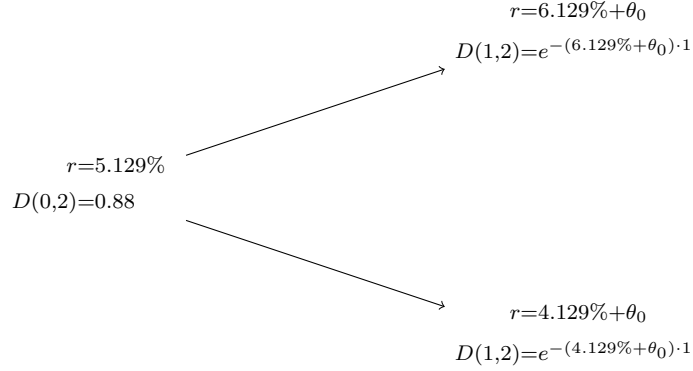
where we have used Itô Isometry. Therefore, the zero-coupon discount bond is given by

$$D(0, t) = \mathbb{E}[e^{-\int_0^t r_u du}] = \exp \left[ -r_0 t - \int_0^t \theta(s)(t-s)ds + \frac{1}{6} \sigma^2 t^3 \right].$$

From here we can work out that

$$\begin{aligned} \log D(0, t) &= -r_0 t - \int_0^t \theta(s)(t-s)ds + \frac{1}{6} \sigma^2 t^3 \\ \frac{\partial}{\partial t} \log D(0, t) &= -r_0 - \int_0^t \theta(s)ds + \frac{1}{2} \sigma^2 t^2 \\ \frac{\partial^2}{\partial t^2} \log D(0, t) &= -\theta(t) + \sigma^2 t \\ \Rightarrow \theta(t) &= -\frac{\partial^2}{\partial t^2} \log D(0, t) + \sigma^2 t \quad \triangleleft \end{aligned}$$

4. Given the first discount factor  $D(0, 1y) = 0.95$ , we can work out the discretized short rate starts at  $r = -\log(0.95) = 5.129\%$ . The binomial tree process is given by

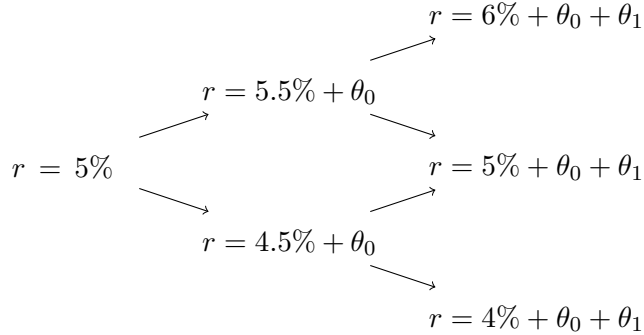


$$\begin{aligned}
 D(0, 2y) &= D(0, 1y) \times D(1y, 2y) \\
 0.88 &= 0.95 \times 0.5 \times \left( e^{-(6.129\%+\theta_0)} + e^{-(4.129\%+\theta_0)} \right) \\
 \Rightarrow \theta_0 &= 0.0253 \quad \triangleleft
 \end{aligned}$$

5. First work out the initial short rate:

$$D(0, 1y) = 0.95123 = e^{-R(0,1) \times 1} \quad \Rightarrow \quad R(0, 1) = r_0 = 5\%$$

The 3-period binomial tree for the short-rate under Ho-Lee model is as follows:



Using  $D(0, 2y)$ , we write

$$\begin{aligned}
 D(0, 2y) &= D(0, 1y) \times \mathbb{E}^*[D(1y, 2y)] \\
 &= D(0, 1y) \times \left[ \frac{1}{2} \times e^{-(\theta_0+0.055)} + \frac{1}{2} \times e^{-(\theta_0+0.045)} \right] \\
 \Rightarrow \theta_0 &= 0.04 \quad \triangleleft
 \end{aligned}$$

Next, we note that

$$\begin{cases} D_u(1, 3) = e^{-(\theta_0+0.055)} \times \left[ \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.06)} + \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.05)} \right] \\ D_d(1, 3) = e^{-(\theta_0+0.045)} \times \left[ \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.05)} + \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.04)} \right] \end{cases}$$

And hence

$$D(0, 3y) = D(0, 1y) \mathbb{E}^*[D(1, 3)] = D(0, 1y) \times \left[ \frac{1}{2} \times D_u(1, 3) + \frac{1}{2} \times D_d(1, 3) \right]$$
$$\Rightarrow \theta_1 = 0.01 \quad \triangleleft$$