Assignment 3

1a. We know:

$$1 = (1 + \Delta \cdot L) \cdot D(0, \Delta)$$

$$\therefore \; D(t, T_i) = (1 + \Delta_i L_i(T_i, T_{i+1})) D(t, T_{i+1})$$

Hence: $$L_t(T_i, T_{i+1}) = \frac{1}{\Delta_i} \frac{D(t, T_i) - D(t, T_{i+1})}{D(t, T_{i+1})}$$

for all practical purpose $[T_i, T_{i+1}]$ are not arbitrary.

So Let's assume $L_i(t) = L_t(T_i, T_{i+1})$ and $D_i(t) = D(t, T_i)$

$$\therefore \; \Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

since Δ_i is a constant, the process $L_i(t)$ must be a martingale under $\mathbb{P}^{i+1}$ measure with $D_{i+1}(t)$ as the numeraire.

Thus the LIBOR Market Model (LMM) is:

$$d L_i(t) = \sigma_i L_i(t) dW^{i+1}(t) \Rightarrow L_i(t) = L_i(0) \exp\left[-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}(t)\right]$$

b. A contract pays $\therefore \; V_T = \Delta_i \sqrt{L_i(T)}$ $\qquad T = T_{i+1}$

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{\cancelto{1}{D_{i+1}(T)}}\right]$$

$$V_0 = D_{i+1}(0) \, \mathbb{E}^{i+1}\left[\Delta_i \sqrt{L_i(T)}\right]$$

$$= D_{i+1}(0) \Delta_i \, \mathbb{E}^{i+1}\left[\sqrt{L_i(T)}\right]$$

$$= D_{i+1}(0) \Delta_i \mathbb{E}^{i+1}\left[L_i(0)^{1/2} \exp\left(-\frac{1}{4}\sigma_i^2 T + \frac{1}{2}\sigma_i W^{i+1}(T)\right)\right]$$

$$= D_{i+1}(0) \Delta_i \sqrt{L_i(0)} \, e^{-\frac{1}{4}\sigma_i^2 T} \, \mathbb{E}\left[e^{\frac{1}{2}\sigma_i W^{i+1}(T)}\right]$$

$$= D_{i+1}(0) \Delta_i \sqrt{L_i(0)} \, e^{-\frac{1}{4}\sigma_i^2 T} \, e^{\frac{1}{8}\sigma_i^2 T}$$

$$= D_{i+1}(0)\,\Delta_i\sqrt{L_i(0)}\,e^{-\frac{1}{8}\sigma_i^2 T}$$

$$= D(0,T_{i+1})\,\Delta_i\sqrt{L_i(0)}\,e^{-\frac{\sigma_i^2 T}{8}}$$

c. $\begin{cases} \$1 & \text{if } k_1 \le L_i(T) \le k_2 \\ 0 & \text{otherwise} \end{cases}$ $\qquad T = T_{i+1}$

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{\cancel{D_{i+1}(T)}\ 1}\right]$$

formula sheet

$$= D_{i+1}(0)\,\mathbb{E}^{i+1}\left[\mathbb{1}_{k_1 \le L_i(T) \le k_2}\right]$$

$$k_1 \le L_i(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i\sqrt{T}x} \le k_2$$

$$= D_{i+1}(0)\int_{-\infty}^{\infty} \mathbb{1}_{k_1 \le L_i(T) \le k_2}$$

$$\therefore \frac{\ln\left(\frac{k_1}{L_i(0)}\right) + \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}} \le x = x^L$$

$$\therefore \frac{\ln\left(\frac{k_2}{L_i(0)}\right) + \frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}} \ge x = x^H$$

$$= D_{i+1}(0)\int_{x^L}^{x^H} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$

$$= D_{i+1}(0)\left[\Phi(x^H) - \Phi(x^L)\right]$$

$$= D(0,T_{i+1})\left(\Phi(x^H) - \Phi(x^L)\right)$$

2a. The numeraire is $P_{n+1,N}(t)$ under $\mathbb{Q}^{n+1,N}$ measure. $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$ $\xrightarrow{}$ PVBP

b. $$\frac{V_0}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_T}{P_{n+1,N}(T)}\right]$$

$$V_0 = P_{n+1,N}(0)\,\mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\,S_{n,N}(T)\,\mathbb{1}_{S_{n,N}(T)>K}}{\cancel{P_{n+1,N}(T)}}\right]$$

$$S_{n,N}(0)e^{-\frac{\sigma_{n,N}^2 T}{2} + \sigma_{n,N}\sqrt{T}x} > K$$

$$x > \frac{\ln\left(\frac{K}{S_{n,N}(0)}\right) + \frac{\sigma_{n,N}^2 T}{2}}{\sigma_{n,N}\sqrt{T}} = x^*$$

$$= P_{n+1,N}(0)\,\mathbb{E}^{n+1,N}\left[S_{n,N}(T)\,\mathbb{1}_{S_{n,N}(T)>K}\right]$$

$$= P_{n+1,N}\ \mathbb{E}^{n+1,N}\left[S_{n,N}(0) e^{-\frac{\sigma_{n,N}^2}{2}T + \sigma_{n,N}\sqrt{T}x}\ \mathbb{1}_{S_{n,N}(T)>K} \right]$$

$$= P_{n+1,N} S_{n,N}(0) e^{-\frac{\sigma_{n,N}^2 T}{2}}\ \mathbb{E}^{n+1,N}\left[e^{\sigma_{n,N}\sqrt{T}x} \cdot \mathbb{1}_{S_{n,N}(T)>K} \right]$$

Skipped steps don't show completion of square → stochastic assignment 2

$$= P_{n+1,N} S_{n,N}(0) \cancel{e^{-\frac{\sigma_{n,N}^2 T}{2}}} \cdot \cancel{e^{\frac{\sigma_{n,N}^2 T}{2}}} \int_{x^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma_{n,N}\sqrt{T})^2}{2}}\, dx$$

Stochastic Assignment 2

$$= P_{n+1,N} S_{n,N}(0)\, \Phi(-x^* + \sigma\sqrt{T})$$

$$= P_{n+1,N} S_{n,N}(0)\, \Phi(d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_{n,N}(0)}{K}\right) + \frac{\sigma_{n,N}^2 T}{2}}{\sigma_{n,N}\sqrt{T}}$$

C. $S_{n,N}(t)$ is a martingale under $\mathbb{Q}^{n+1,N}$ measure. However, $S_{n,N}(T)$ pays at time T. So to value this we need to compute the expectation under $\mathbb{Q}$ measure. We would thus need to try discounting using the money market account under the $\mathbb{Q}$ measure. However, under this $W^{n+1,N}$ is no longer a standard brownian process and has a drift. Convexity correction allows us to compensate for it.