Additional Example 7

1. $dr_t = \mu dt + \sigma dW_t^*$

a. Step 1:

$$\int_t^s dr_v = \int_t^s \mu du + \int_t^s \sigma dW_u^*$$

$$r_s - r_t = \int_t^s \mu du + \int_t^s \sigma dW_u^*$$

$$r_s = r_t + \int_t^s \mu du + \int_t^s \sigma dW_u^*$$

$$r_s = r_t + \mu(s-t) + \int_t^s \sigma dW_u^*$$

$$\int_t^T r_s ds = \int_t^T r_t du + \int_t^T \mu(s-t) ds + \int_t^T \int_t^s \sigma dW_u^* ds$$

$$\int_t^T r_s ds = r_t(T-t) + \mu\left[\frac{(s-t)^2}{2}\right]_t^T + \int_t^T \int_u^T \sigma ds dW_u^*$$

$$\int_t^T r_s ds = r_t(T-t) + \mu\left[\frac{(T-t)^2}{2} - \cancel{\frac{(t-t)^2}{2}}^{0}\right] + \int_t^T \sigma(T-u) dW_u^*$$

$$\int_t^T r_s ds = r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(T-u) dW_u^*$$

$$\mathbb{E}^*\left[\int_t^T r_s ds\right] = r_t(T-t) + \frac{\mu}{2}(T-t)^2$$

$$V\left[\int_t^T r_s ds\right] = V\left[\int_t^T \sigma(T-u) dW_u^*\right] = \sigma^2\frac{(T-t)^3}{3}$$

$$\int_t^T r_s ds \sim N\left(r_t(T-t) + \frac{\mu}{2}(T-t)^2, \frac{\sigma^2}{3}(T-t)^3\right)$$

$$e^{\frac{1}{2}\sigma^2} \qquad \sigma^2 = Var = \frac{\sigma^2 (T-t)^3}{3}$$

b. $D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u \, du}\right]$

$$= \mathbb{E}^*\left[e^{-\left(r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma (T-u)\, dw_u^*\right)}\right]$$

$$= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2}\sigma^2 \frac{(T-t)^3}{3}}$$

$$= e^{\frac{1}{2}\sigma^2 \frac{(T-t)^3}{3} - \frac{\mu}{2}(T-t)^2 - r_t(T-t)}$$

$$A(t,T) = \frac{1}{2}\sigma^2 \frac{(T-t)^3}{3} - \frac{\mu}{2}(T-t)^2 = -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2 (T-t)^3}{6}$$

$$B(t,T) = (T-t)$$

c. Affine models are linear functions with a constant. The above is an affine model with $r B(t,T)$ as the linear func and $A(t,T)$ as the constant.

2. $dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

$$r_t = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(u-t)} \, dW_u^*$$

$$\int_0^T r_t \, dt = \int_0^T r_0 e^{-kt} \, dt + \int_0^T \theta(1 - e^{-kt}) \, dt + \int_0^T \int_0^t \sigma e^{k(u-t)} \, dW_u^* \, dt$$

need to understand

fubini's switch

$$\int_0^T r_t \, dt = \left[\frac{-r_0}{k} e^{-kt}\right]_0^T + \left[\theta\left(t + \frac{e^{-kt}}{k}\right)\right]_0^T + \int_0^T \int_u^T \sigma e^{k(u-t)} \, dt \, dW_u^*$$

$$\int_0^T r_t \, dt = \left(\frac{-r_0}{k} e^{-kT} - \left(\frac{-r_0}{k} e^{-0}\right)\right) + \theta\left(T + \frac{e^{-kT}}{k}\right) - \theta\left(0 + \frac{e^{0}}{k}\right)$$

$$+\sigma\int_0^T\left[\frac{-e^{\kappa(u-t)}}{\kappa}\right]_u^T dW_u^*$$

$$e^{\kappa u-\kappa t} = \frac{e^{\kappa u}}{\kappa}\times e^{-\kappa t}\kappa = -\frac{e^{\kappa(u-t)}}{\kappa}$$

$$\int_0^T r_t\,dt = \frac{-r_0e^{-\kappa T}}{\kappa}+\frac{r_0}{\kappa}+\theta T+\frac{\theta e^{-\kappa T}}{\kappa}-0-\frac{\theta}{\kappa}+\sigma\int_0^T\frac{-e^{\kappa(u-T)}}{\kappa}+\frac{e^{\kappa(u-u)}}{\kappa}dW_u^*$$

$$=\frac{r_0}{\kappa}\left(1-e^{-\kappa T}\right)+\theta T+\frac{\theta}{\kappa}\left(e^{-\kappa T}-1\right)+\frac{\sigma}{\kappa}\int_0^T 1-e^{\kappa(u-T)}dW_u^*$$

$$\mathbb{E}\left[\int_0^T r_t\,dt\right]=\frac{r_0}{\kappa}\left(1-e^{-\kappa T}\right)+\theta T+\frac{\theta}{\kappa}\left(e^{-\kappa T}-1\right)$$

$$V\left[\int_0^T r_t\,dt\right]=V\left[\int_0^T\frac{\sigma}{\kappa}\left(1-e^{\kappa(u-T)}\right)dW_u^*\right]$$

$$=\int_0^T\frac{\sigma^2}{\kappa^2}\left(1-e^{\kappa(u-T)}\right)^2du$$

$$=\frac{\sigma^2}{\kappa^2}\int_0^T 1-2e^{\kappa(u-T)}+e^{2\kappa(u-T)}du$$

$$=\frac{\sigma^2}{\kappa^2}\times\left[T-\left(+\frac{2}{\kappa}e^{\kappa(u-T)}\right]_0^T+\left(+\frac{e^{2\kappa(u-T)}}{2\kappa}\right]_0^T\right]$$

$$=\frac{\sigma^2}{\kappa^2}\left[T-\left(+\frac{2}{\kappa}\left(e^{\cancel{\kappa T-\kappa T}}-e^{\kappa(0-T)}\right)\right]\right.$$
$$\left.+\left(\frac{+1}{2\kappa}\left(e^{2\kappa(T-T)}-e^{2\kappa(0-T)}\right)\right)\right]$$

$$= \frac{\sigma^2}{k^2}\left(T - \frac{2e}{k}^{ku}\left(1 - e^{-kT}\right) + \frac{1}{2k}\left(1 - e^{-2kT}\right)\right)$$

$$D(0,T) = \mathbb{E}^*\left[e^{-\int_0^T r_u du}\right]$$

$$= e^{-\left(\frac{r_0}{k}\left(1 - e^{-kT}\right) + \theta T + \frac{\theta}{k}\left(e^{-kT} - 1\right)\right) + \frac{1}{2}\frac{\sigma^2}{k^2}\left(T - \frac{2e}{k}^{ku}\left(1 - e^{-kT}\right) + \frac{1}{2k}\left(1 - e^{-2kT}\right)\right)}$$