



# Session 6

## Change of Numeraire Theorem & Convexity Correction

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## FX Rate Process

- While the previous example of using a stock process as a numeraire appears rather stylised, it is no longer so when we look at the foreign exchange (FX) market.
- FX products can be calculated based on either the **domestic** or the **foreign** currency denomination.
- In an arbitrage-free market, both values must be the same.
- Consider the following international economy: in the domestic market  $D$  there is a money-market account  $B_t^D$ , earning a risk-free interest rate of  $r^D$ ; in the foreign market  $F$  there is the corresponding money-market account  $B_t^F$  and interest rate  $r^F$ .

Let the FX rate follows a **geometric Brownian motion**, then the three price processes can be written as

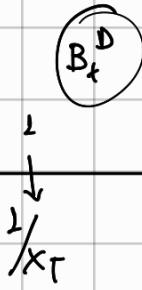
$$dB_t^F = r^F B_t^F dt$$

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

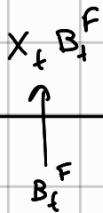
$$dB_t^D = r^D B_t^D dt.$$

Domestic (D)

(D)

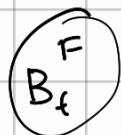


$$dB_t^D = r^D B_t^D dt$$



$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

Foreign  
(F)



$$dB_t^F = r^F B_t^F dt$$

$$\text{Let } Y_t = X_t B_t^F = f(X_t, B_t^F) \quad f(x, b) = xb \quad ; \quad f_x = b, \quad f_{bx} = 0, \quad f_b = x$$

$$dY_t = f_b(X_t, B_t^F) dB_t^F + f_x(X_t, B_t^F) dX_t + \frac{1}{2} f_{xx}(X_t, B_t^F) (dX_t)^2$$

$$= X_t r^F B_t^F dt + B_t^F (\mu X_t dt + \sigma X_t dW_t) + 0$$

$$= (r^F + \mu) X_t B_t^F dt + \sigma B_t^F X_t dW_t$$

$$dY_t = (r^F + \mu) Y_t dt + \sigma Y_t dW_t$$

$$\text{Let } Z_t = \frac{Y_t}{B_t^D} = g(Y_t, B_t^D) \quad g(y, b) = \frac{y}{b}, \quad g_b = -\frac{y}{b^2}, \quad g_y = \frac{1}{b}, \quad g_{yy} = 0$$

$$\begin{aligned} dZ_t &= g_b(Y_t, B_t^D) dB_t^D + g_y(Y_t, B_t^D) dY_t + \frac{1}{2} g_{yy}(Y_t, B_t^D) (dY_t)^2 \\ &= -\frac{Y_t}{(B_t^D)^2} \cdot r^D B_t^D dt + \frac{1}{B_t^D} ((r^F + \mu) Y_t dt + \sigma Y_t dW_t) + 0 \end{aligned}$$

$$= -r^D Z_t dt + (r^F + \mu) Z_t dt + \sigma Z_t dW_t$$

$$dZ_t = (r^F - r^D + \mu) Z_t dt + \sigma Z_t dW_t \quad \underline{\mathbb{P}} \xrightarrow{\text{empirical}} \text{measure (cuz } r^F - r^D \neq 0\text{)}$$

$$dZ_t = \sigma Z_t dW_t^D \quad \boxed{dW_t^D = dW_t + \frac{r^F - r^D + \mu}{\sigma} dt}$$

$$\underline{\mathbb{P}}^D \xrightarrow{\text{risk neutral measure}} \text{domestic PDV.}$$

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad \leftarrow \mathbb{P}$$

$$= \mu X_t dt + \sigma X_t (dW_t^D - \frac{r^F - r^D + \mu}{\sigma} dt) \leftarrow \underline{\mathbb{P}}$$

$$= \mu X_t dt + \sigma X_t dW_t^D + (-r^F + r^D) X_t dt - \mu X_t dt$$

$$= (r^D - r^F) X_t dt + \sigma X_t dW_t^D$$

*risky*  
maths show two views  
*riskless*

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t \\ \therefore X_T &= X_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \end{aligned}$$

$$\mathbb{E}[X_T] = X_0 e^{\mu T}$$

*risky*  
(not enforceable)

$$\begin{aligned} dX_t &= (r^D - r^F) X_t dt + \sigma X_t dW_t^D \\ X_T &= X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T + \sigma W_T^D} \end{aligned}$$

correct only from  
risk neutral  
perspective

continuous  
ver of this:

$\mathbb{E}^D[X_T] = X_0 e^{(r^D - r^F)T}$

enforceable

continuous  
ver of this:

$B_t^F X_t = \frac{(1+r^F)^t}{(1+r^D)^t} B_t^D X_t$

discrete ver

this uses economic  
logic (continuous  
ver uses pure  
maths).

# FX Rate Process

From a domestic market point of view, there are two **marketed assets**: the domestic money-market account  $B_t^D$  and the value of the foreign money-market account in domestic terms, given by  $B_t^F X_t$ . Applying Itô's formula to the function  $Y_t = f(B_t^F, X_t) = B_t^F X_t$ , we obtain

$$dY_t = (r^F + \mu) Y_t dt + \sigma Y_t dW_t.$$

In the domestic market, we can use the domestic money-market account as a numeraire. Defining the process  $Z_t = g(B_t^D, Y_t) = \frac{Y_t}{B_t^D} = \frac{B_t^F X_t}{B_t^D}$ , we again apply Itô's formula to obtain

$$dZ_t = (r^F - r^D + \mu) Z_t dt + \sigma Z_t dW_t.$$

Applying Girsanov's theorem with  $\kappa = \frac{r^F - r^D + \mu}{\sigma}$  to obtain the equivalent martingale measure  $\mathbb{Q}^D$ , under which the process  $Z_t = \frac{B_t^F X_t}{B_t^D}$  is a martingale.

We see that under this  $\mathbb{Q}^D$  measure, the exchange rate process follows

$$dX_t = (r^D - r^F) X_t dt + \sigma X_t dW_t^D.$$

*Domestic perspective*

# FX Rate Process

Taking the perspective of the **foreign market**, we have two marketed assets:  $B_t^F$  and  $\frac{B_t^D}{X_t}$ . Using  $B_t^F$  as a numeraire, we can derive the process  $\frac{B_t^D}{X_t B_t^F}$  as follow

$$d \left( \frac{B_t^D}{X_t B_t^F} \right) = (r^D - r^F - \mu + \sigma^2) \left( \frac{B_t^D}{X_t B_t^F} \right) dt - \sigma \left( \frac{B_t^D}{X_t B_t^F} \right) dW_t.$$

Applying Girsanov's theorem with  $\kappa = -\frac{r^D - r^F - \mu}{\sigma} - \sigma$ , we obtain the unique equivalent martingale measure  $\mathbb{Q}^F$ . Under this measure, the exchange rate  $\frac{1}{X_t}$  follows the process

$$d \frac{1}{X_t} = (r^F - r^D) \frac{1}{X_t} dt - \sigma \frac{1}{X_t} dW_t^F.$$

foreign perspective

In the foreign economy, the trader will see exactly the same price for an FX option.

# Single-currency Change of Numeraire Theorem

Products requiring **convexity correction** are characterised by the fact that the underlying assets are paid at the wrong time or in the wrong denomination.

In this case, the forwards need to be adjusted to reflect the fact that they are paid incorrectly. This adjustment is known in the market as convexity correction.

We are already familiar with the **single-currency change of numeraire theorem**, which states that in an arbitrage-free economy, we have

$$\mathbb{Q}^N: \frac{V_0}{N_0} = \mathbb{E}^N \left[ \frac{V_T}{M_T} \right]$$

$$\mathbb{E}^N[H_T] = \mathbb{E}^M \left[ H_T \frac{N_T/N_0}{M_T/M_0} \right],$$

i- Space

$$\mathbb{Q}^M: \frac{V_0}{M_0} = \mathbb{E}^M \left[ \frac{V_T}{M_T} \right]$$

where

change from one to another  
→ Radon-Nikodym

risk-neutral measure  
where asset M as choice of numeraire

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{N_T/N_0}{M_T/M_0}$$

ratio of numeraire  
adjusts for drift  
(drift rate of numeraire)

Many of the products we are interested in will be multi-currency products. We can extend the single-currency theorem to include multi-currency economies.

$$Q^*: \frac{c_0}{B_0} = E^* \left[ \frac{c_T}{B_T} \right]$$

unique

$$Q^T: \frac{c_0}{D(0,T)} = E^T \left[ \frac{c_T}{D(T,T)} \right]$$

$$D(0,T) E^T [c_T] = B_0 E^* \left[ \frac{c_T}{B_T} \right] \quad D(T,T)$$

known today so far  
be in or out of the E

$$E^T [c_T] = E^* \left[ c_T \cdot \frac{1}{\frac{D(0,T)}{B_T/B_0}} \right]$$

$$= E^* \left[ c_T \cdot \frac{dQ^T}{dQ^*} \right]$$

$$Q^*: \frac{V_0^{\text{pay}}}{B_0} = E^* \left[ \frac{V_T^{\text{pay}}}{B_T} \right]$$

unique

$$V_0^{\text{pay}} = B_0 E^* \left[ \frac{P_{n+1,N}(T) (S_T - K)^+}{B_T} \right]$$

$$Q^{n+1,N}: V_0^{\text{pay}} = P_{n+1,N}(0) E^{n+1,N} \left[ \frac{P_{n+1,N}(T) (S_T - K)^+}{B_T} \right]$$

$$P_{n+1,N}(0) E^{n+1,N} \left[ (S_T - K)^+ \right] = B_0 E^* \left[ \frac{P_{n+1,N}(T) (S_T - K)^+}{B_T} \right]$$

$$E^{n+1,N} \left[ (S_T - K)^+ \right] = E^* \left[ (S_T - K)^+ \cdot \frac{P_{n+1,N}(T) / P_{n+1,N}(0)}{B_T / B_0} \right] = E^* \left[ (S_T - K)^+ \cdot \frac{dQ^{n+1,N}}{dQ^*} \right]$$

Rodon-Nikodym

What he means by 'unique'

1. Know how to hedge
2. Hedge  $\rightarrow$  eliminate risk  $\rightarrow$  risk-neutral measure ( $\mathbb{P}$ )
3. No-arbitrage: price = cost of hedge  
(unique)
4. martingale: choose numeraire  $B_t$   
so we don't need to know  $t$ ?  
$$\frac{V_0}{B_0} = \mathbb{E}^t \left[ \frac{V_T}{B_T} \right]$$

# Multi-currency Change of Numeraire Theorem

We can put convexity correction on a firm mathematical basis by showing that it can be interpreted as the side-effect of a change of numeraire.

This means that we will show how convexity correction can be understood mathematically as the expected value of an underlying asset under a different probability measure than its own martingale measure.

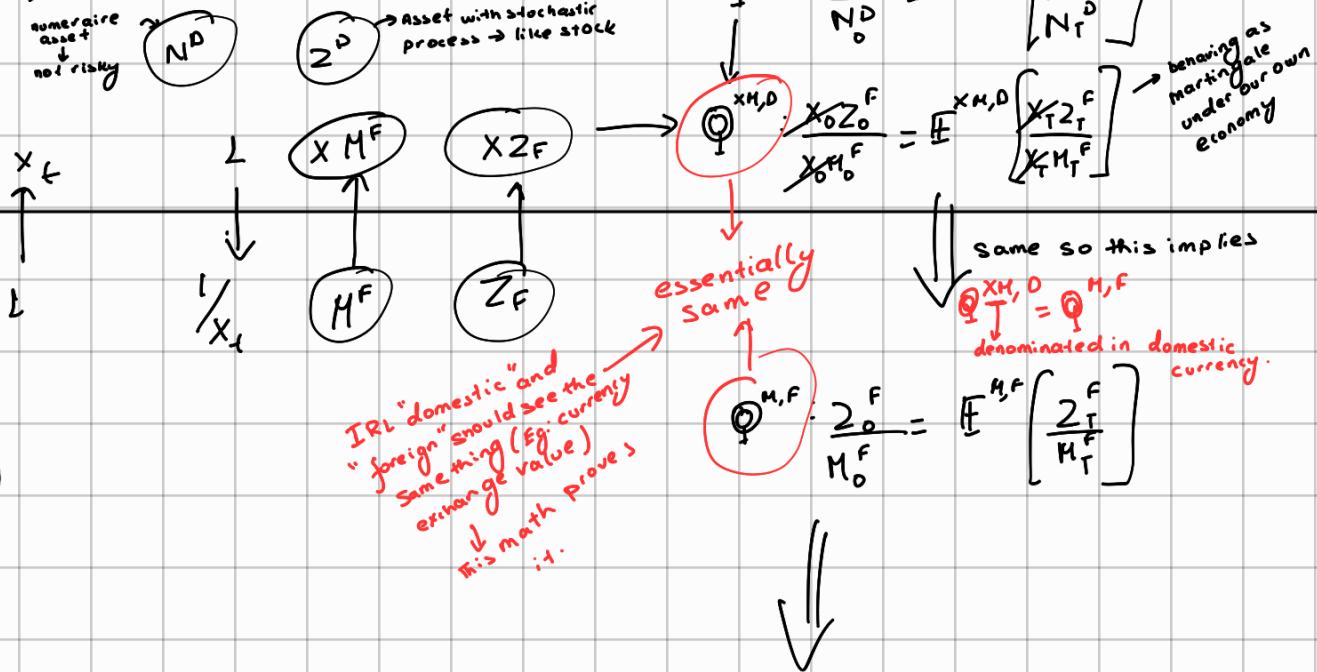
Suppose we have a domestic economy  $D$  and a foreign economy  $F$ , together with the exchange rate  $X_t$  that expresses the value at time  $t$  of one unit of foreign currency in terms of domestic currency.

This implies that for a numeraire  $N^D$  in the domestic economy there exists a unique martingale measure  $\mathbb{Q}^{D \rightarrow \text{Currency}}$  such that all  $N^D$  rebased traded assets in the domestic economy become martingales.

Note that we have 2 types of traded assets in the domestic economy:

- the domestic assets  $Z^D$
- the domestic value of the foreign assets which are given by  $X \cdot Z^F$

Domestic (D)



Foreign (F)

$$\mathbb{Q}^{N_x,F} \cdot \frac{Z_0^F}{N_0^F} = E^{N_x,F} \left[ \frac{Z_T^F}{N_T^F} \right]$$

Below this, another equation shows the relationship between  $\mathbb{Q}^{N_x,F}$  and  $\mathbb{Q}^{N,D}$ :

$$\mathbb{Q}^{N_x,F} = \mathbb{Q}^{N,D}$$

## Multi-currency Change of Numeraire Theorem

Let us now consider 2 numeraires, one in the domestic economy  $N^D$  and one in the foreign economy  $M^F$ .

As the exchange rate is strictly positive, the domestic value of the foreign numeraire in domestic denomination  $X \cdot M^F$  is a valid domestic numeraire.

Hence, there exists a unique martingale measure  $\mathbb{Q}^{X^M, D}$  such that all  $X^M$  rebased traded assets in the domestic economy become martingales.

What is the relation between the probability measure  $\mathbb{Q}^{X^M, D}$  in the domestic economy and  $\mathbb{Q}^{M, F}$  in the foreign economy? All  $X \cdot M^F$  rebased traded assets in the domestic economy are martingales under  $\mathbb{Q}^{X^M, D}$ .

This would mean that the domestic value of the foreign traded assets

$$\frac{X \cdot Z^F}{X \cdot M^F} = \frac{Z^F}{M^F}$$

are martingales also. This implies that  $\mathbb{Q}^{X^M, D}$  and  $\mathbb{Q}^{M, F}$  are the same probability measure. So under the measure  $\mathbb{Q}^{M, F}$  all  $X \cdot M^F$  rebased traded assets are martingales in the domestic economy and all  $M^F$  rebased traded assets are martingales in the foreign economy.



# Multi-currency Change of Numeraire Theorem

From the **domestic economy perspective** we have  $N^D$  and  $X \cdot M^F$  as domestic numeraires. Hence, we can apply the single-currency change of numeraire theorem which yields

$$\text{because } \Phi^{N,F} = \Phi^{X,D}$$

$$\frac{d\mathbb{Q}^{N,D}}{d\mathbb{Q}^{M,F}} = \frac{\frac{N_T^D}{N_0^D}}{\frac{X_T M_T^F}{X_0 M_0^F}}$$

why adjust  
adjusted by rate

Convexity correction  
↳ payment in wrong currency and schedule.

**formula**

From the **foreign economy perspective**, we have  $\frac{1}{X} \cdot N^D$  and  $M^F$  as foreign numeraires. Hence, we can apply the single currency change of numeraire theorem to this case as well which yields

$$\frac{d\mathbb{Q}^{N,D}}{d\mathbb{Q}^{M,F}} = \frac{\frac{\frac{1}{X_T} N_T^D}{\frac{1}{X_0} N_0^D}}{\frac{M_T^F}{M_0^F}} \Rightarrow \Phi^{N,F} = \Phi^{N,D}$$

why more vol stock options more expensive?  
↳ higher vol → needs rebalancing more frequently

also the reason why Libor in arrears more expensive

**formula**

Note that the two expressions are identical with simple rearrangement. This is the **multi-currency change of numeraire theorem**. Note also that the single-currency change of numeraire theorem can be seen as a special case of the multi-currency theorem with  $X = 1$  and  $F = D$ .

# Lehman Brother's 100 million dollar formula

## Convexity correction

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{D_i(T_i)/D_i(0)}{D_{i+1}(T_i)/D_{i+1}(0)}$$

- Discovered the right formula
- Gained long exposures to LIBOR-in-Arrears
- "80 cents on the dollar"

↗ bought a lot on  
 ↗ cheap then published  
 ↗ not martingale  
 ↗ (but drifting)  
 ↗ wrong measure  
 $\mathbb{E}^i[L_i(T_i)] = \frac{L_i(0) + \Delta_i L_i(0)^2 e^{\sigma^2 T_i}}{1 + \Delta_i L_i(0)}$

depends on Libor vol.





$$V_0 = D(0, T_{i+1}) \cdot L_i(0) \rightarrow \text{fixed in advance paid in arrears}$$

Labor-in-arrears  
(LIA)



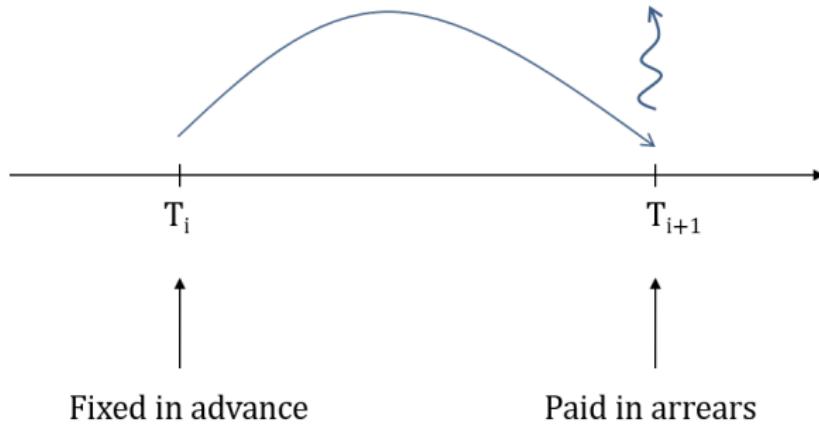
Exotics

fixed and paid in arrears  $\Rightarrow$  cannot replicate

[came out in 90's, V-curve very steep]  
Want to be paid immediately

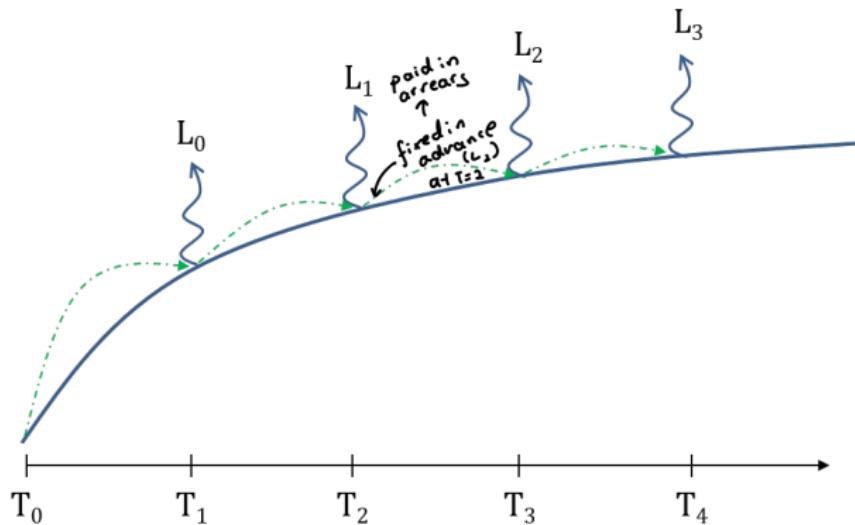
$$V_0 \approx D(0, T_i) \cdot L_i(0) \rightarrow \text{underprices}$$

In a standard LIBOR product, the interest rate is fixed in advance, paid in arrears.

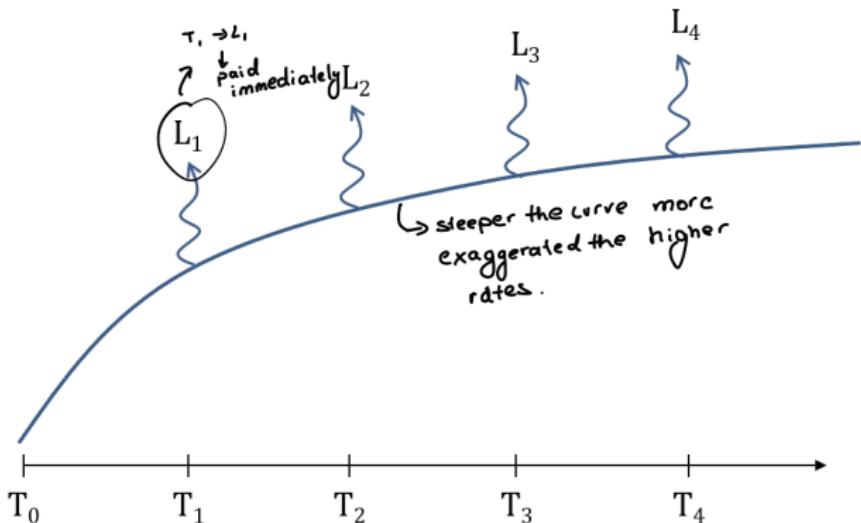
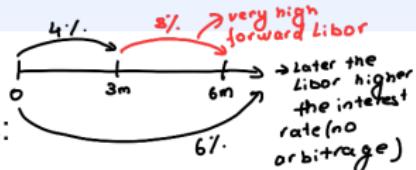


The rate you receive at time  $T_{i+1}$  is fixed at  $T_i$ .

Floating leg of a standard interest rate swaps:  $\rightarrow$  vanilla



## Floating leg of an in-arrears interest rate swaps:



## In-arrears Products: Historical Context

In the early 90s (1992-1993), the yield curve was extremely steep, prompting the development of LIBOR-in-arrears products.

- ⇒ In contrast to standard IR products, the floating leg of in-arrears products is fixed and paid at the same time.
- ⇒ This is meant to be the rate for the next period for standard products.

If the yield curve is upward sloping, then the expected coupons of the in-arrears floating leg will be higher.

- ⇒ Long payer swap with in-arrears floating leg to capitalize.

On the other hand, if you think that LIBOR will not rise as fast as the curve predicts, you can long a receiver in-arrears swap to capitalize on your view.

## In-arrears Products: Historical Context

Generally speaking, corporate clients or financial institutions hold receiver swaps.

↳ not market makers  
↳ more receiver swaps in portfolio  
↳ more received

↳ issue bonds  
(pay fixed → receive swaps)

If the yield curve is steep, the rate they receive on the fixed leg of the in-arrears product is higher – more attractive.

↳ in arrears LIBOR is worth more than standard vanilla LIBOR.

⇒ Their view is that the in-arrears rates are overestimated, the realized rates will be lower.

Similarly, if the view is that LIBOR will be falling, a receiver swap with an in-arrears floating leg is also more beneficial.

⇒ The rate will be lower at the end of each period than at the beginning.

## LIBOR-in-Arrears

In a standard vanilla LIBOR payment, the LIBOR interest rate  $L_i$  is observed at time  $T_i$  and paid at the end of the accrual period at time  $T_{i+1}$ , as  $\Delta_i L_i(T_i)$ , where  $\Delta_i$  denotes the daycount or accrual fraction.

The forward LIBOR rate  $L_i(t)$  is defined as

$$L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{\Delta_i D_{i+1}(t)},$$

where  $D_i$  denotes the discount factor maturing at time  $T_i$ .

Hence if we choose  $D_{i+1}$  as the numeraire then under the associated martingale measure  $\mathbb{Q}^{i+1}$ , the forward LIBOR rate  $L_i$  is a martingale.

In a **LIBOR-in-Arrears** contract, the interest payment at time  $T_i$  is based on  $L_i(T_i)$ . Hence, the LIBOR contract is fixed only at the end of the interest rate period, i.e. fixed in arrears. We want to evaluate

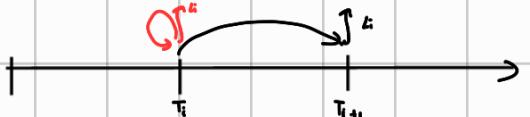
$$V^{\text{LIA}}(0) = D_i(0) \mathbb{E}^i [L_i(T_i)],$$
↳ formula

where  $L_i$  is not a martingale under the measure  $\mathbb{Q}^i$ .



$$\textcircled{Q}^{i+1}: dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t)$$

$$L_i(t) = L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}(T)}$$



$$\frac{V_0}{D_{i+1}(0)} = E^{i+1} \left[ \frac{V_T}{D_{i+1}(T)} \right] = E^{i+1}[L_i(T)]$$

$$\Rightarrow V_0 = D_{i+1}(0) L_i(0)$$

$$\frac{V^{LA}(0)}{D_i(0)} = E^i \left[ \frac{V^{LA}(T)}{D_i(T)} \right]$$

no +1 because all immediately

$$V^{LA}(0) = D_i(0) E^i[L_i(T)]$$

cannot do anything further  
because  $W_T$  is in  $i+1$  measure  
but expectation is in  $i$ . So not a  
martingale and the brownian is drifting.

$$\text{approximation} \approx D_i(0) E^{i+1}[L_i(T)]$$

$$\approx D_i(0) L_i(0)$$

↳ only diff from previous is discount by 1 period less

↳ get paid faster.

problem statement → Lehmann

$$E^i[L_i(T)] \neq E^{i+1}[L_i(T)]$$

$$= E^{i+1} \left[ L_i(T) \cdot \frac{dQ^i}{dQ^{i+1}} \right]$$

Radon-Nikodym derivative  
↳ handle the drift

$$= E^{i+1} \left[ L_i(T) \cdot \frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)} \right]$$

$$L_i(T) = \frac{1}{\Delta_i} \cdot \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

$$1 + \Delta_i L_i(T) = \frac{D_i(t)}{D_{i+1}(t)}$$

↑ or move ↓

$$= E^{i+1} \left[ L_i(T) \cdot \frac{D_i(T)/D_{i+1}(T)}{D_i(0)/D_{i+1}(0)} \right]$$

$$= E^{i+1} \left[ L_i(T) \cdot \frac{1 + \Delta_i L_i(T)}{1 + \Delta_i L_i(0)} \right]$$

$$= E^{i+1} \left[ \frac{L_i(T) + \Delta_i (L_i(T))^2}{1 + \Delta_i L_i(0)} \right] = \frac{1}{1 + \Delta_i L_i(0)} \cdot \left( E^{i+1}[L_i(T)] + \Delta_i E^{i+1}[(L_i(T))^2] \right)$$

$$= \frac{1}{1 + \Delta_i L_i(0)} \cdot \left( L_i(0) + \Delta_i (L_i(0))^2 e^{\sigma_i^2 T_i} \right)$$

$$= \frac{L_i(0) \left( 1 + \Delta_i L_i(0) e^{\sigma_i^2 T_i} \right)}{1 + \Delta_i L_i(0)}$$

# LIBOR-in-Arrears

To calculate the convexity correction, we proceed as follow. We write down the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{D_i(T_i)/D_i(0)}{D_{i+1}(T_i)/D_{i+1}(0)}. \quad \clubsuit$$

Using the definition of LIBOR rate, it simplifies to

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{1 + \Delta_i L_i(T_i)}{1 + \Delta_i L_i(0)}.$$

Using this expression, we can now evaluate the expectation as

$$\begin{aligned}\mathbb{E}^i[L_i(T_i)] &= \mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} L_i(T_i)\right] \\ &= \frac{1}{1 + \Delta_i L_i(0)} \mathbb{E}^{i+1}[L_i(T_i)(1 + \Delta L_i(T_i))] \\ &= \frac{\mathbb{E}^{i+1}[L_i(T_i)] + \Delta_i \mathbb{E}^{i+1}[L_i(T_i)^2]}{1 + \Delta_i L_i(0)}.\end{aligned}$$

# LIBOR-in-Arrears

Recall our **LIBOR Market Model** for the forward Libor rate:

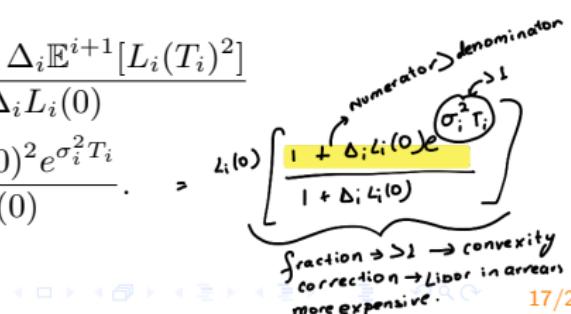
$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}.$$

The solution is given by

$$L_i(T) = L_i(0) \exp \left[ -\frac{1}{2} \sigma_i^2 T + \sigma_i W^{i+1} \right].$$

Substituting back to our expectation, we obtain

$$\begin{aligned} \mathbb{E}^i[L_i(T_i)] &= \frac{\mathbb{E}^{i+1}[L_i(T_i)] + \Delta_i \mathbb{E}^{i+1}[L_i(T_i)^2]}{1 + \Delta_i L_i(0)} \\ &= \frac{L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}}{1 + \Delta_i L_i(0)}. \end{aligned}$$



# Cholesky Decomposition of Brownian Motions

We have a bunch of brownian motions which are correlated.  
 ↓  
 decompose to brownian motion that are independent.

Given two stochastic processes that are correlated with a correlation of  $\rho$ , it is often beneficial to be able to decompose them and express them as independent Brownian motions. Consider two correlated SDEs

$$\begin{aligned} dX_t &= \mu_X X_t dt + \sigma_X X_t dW_t^X \\ dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, \end{aligned}$$

where  $dW_t^X dW_t^Y = \rho dt$ . We can decompose the two correlated Brownian processes as follow:

$$\begin{aligned} dW_t^X &= \alpha_{11} dZ_t^{(1)} \\ dW_t^Y &= \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}, \end{aligned}$$

where  $dZ_t^{(1)} dZ_t^{(2)} = 0$ , and

$$\alpha_{11} = 1, \quad \alpha_{12} = \rho, \quad \alpha_{22} = \sqrt{1 - \alpha_{12}^2}.$$

After we know of Cholesky decomposed.

# Cholesky Decomposition of Brownian Motions

This is because of moment matching. First, the variance relationship is

I need to understand this again.

$$V[dW_t^X] = V[\alpha_{11}dZ_t^{(1)}] \rightarrow \text{cov mapped together}$$

$$dt = \alpha_{11}^2 dt \Rightarrow \alpha_{11} = 1.$$

Next, the covariance relationship is

$$\text{Cov}[dW_t^X, dW_t^Y] = \text{Cov}[\alpha_{11}dZ_t^{(1)}, \alpha_{12}dZ_t^{(1)} + \alpha_{22}dZ_t^{(2)}]$$

$$\rho dt = \text{Cov}[\alpha_{11}dZ_t^{(1)}, \alpha_{12}dZ_t^{(1)}] + \text{Cov}[\alpha_{11}dZ_t^{(1)}, \alpha_{22}dZ_t^{(2)}]$$

formula  $\Rightarrow \alpha_{11}\alpha_{12} dt \Rightarrow \alpha_{12} = \rho$  why 0?

connect

And finally,

$$V[dW_t^Y] = V[\alpha_{12}dZ_t^{(1)} + \alpha_{22}dZ_t^{(2)}]$$

$$dt = \alpha_{12}^2 dt + \alpha_{22}^2 dt \Rightarrow \alpha_{22} = \sqrt{1 - \rho^2}.$$

# Forward Exchange Rate Process

Earlier, we mentioned that from the domestic investor's perspective, the spot exchange rate follows

$$dX_t = (r^D - r^F)X_t dt + \sigma_X X_t dW_t^D, \rightarrow \text{domestic spot}$$

while from the foreign investor's perspective, the spot FX rate follows

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t} dt + \sigma_X \frac{1}{X_t} dW_t^F. \rightarrow \text{foreign spot}$$

*for foreigners*  
*our foreign is their domestic*

We have also covered that the forward exchange rate can be written as

$$\mathbb{E}^D[X_T] = \mathbb{E}^D \left[ X_t e^{\left( r^D - r^F - \frac{\sigma_X^2}{2} \right)(T-t) + \sigma_X (W_T^D - W_t^D)} \right]$$

$$= X_t e^{(r^D - r^F)(T-t)}$$

*we write as discount factors because written  
spot FX rate as r indicates interest rates are deterministic.  
But they are not as rates can change*

Let  $F_t = X_t e^{(r^D - r^F)(T-t)}$  denote the forward exchange rate process (maturity at  $T$ ), we can use Itô's formula to show that

$$dF_t = \sigma_X F_t dW_t^D.$$

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

$$\begin{aligned} F_t &= S_t e^{r(1-t)} && \rightarrow \text{forward contract} \\ &= f(t, S_t) && \hookrightarrow \text{drifting at risk free rate (stock price based)} \end{aligned}$$

$$dF_t = \sigma f_t dW_t^* \rightarrow \text{martingale.}$$

$\hookrightarrow$  if forward price

# Forward Exchange Rate Process

Let  $D^D(t, T)$  denote the LIBOR discount factor in the domestic economy, and  $D^F(t, T)$  denote the LIBOR discount factor in the foreign economy. Let us express the forward exchange rate  $F_t$  (maturing at  $T$ ) as:

$$F_t = X_t \cdot \frac{D^F(t, T)}{D^D(t, T)}$$

formula

By the same argument, we also have

$$\mathbb{E}^F \left[ \frac{1}{X_T} \right] = \frac{1}{X_t} e^{(r^F - r^D)(T-t)}$$

and the forward exchange rate (maturity at  $T$ ) from the foreign investor's perspective as

$$d\frac{1}{F_t} = \sigma_X \frac{1}{F_t} dW_t^F.$$

formula

Therefore, we express it as

$$\frac{1}{F_t} = \frac{1}{X_t} \cdot \frac{D^D(t, T)}{D^F(t, T)}$$

→ want to receive foreign LIBOR in domestic currency

## Pricing Quanto LIBOR

In a quanto LIBOR contract, a foreign LIBOR rate  $L_i^F$  is observed at  $T_i$  and is paid in domestic denomination at time  $T_{i+1}$ . Suppose the Brownian motions  $W_t^{i+1}$  and  $W_t^F$  have a correlation of  $\rho$ .

To price this contract, we apply our multi-currency convexity correction formula

$$\begin{aligned}
 \mathbb{E}^{i+1,D} [L_i^F(T)] &= \mathbb{E}^{i+1,F} \left[ L_i^F(T) \cdot \frac{d\mathbb{Q}^{i+1,D}}{d\mathbb{Q}^{i+1,F}} \right] \\
 &\stackrel{\text{if } F \rightarrow \text{the expected = today's Libor.}}{=} \mathbb{E}^{i+1,F} \left[ L_i^F(T) \cdot \frac{\frac{D_{i+1}^D(T_{i+1})}{D_{i+1}^D(0)}}{\frac{X_T D_{i+1}^F(T_{i+1})}{X_0 D_{i+1}^F(0)}} \right] \xrightarrow{\frac{d\mathbb{Q}^{N,D}}{d\mathbb{Q}^{N,F}}} \text{slide 8} \\
 &\stackrel{\text{Receive foreign Libor in domestic currency at time } i+1}{=} \mathbb{E}^{i+1,F} \left[ L_i^F(T) \cdot \frac{\frac{D_{i+1}^D(T_{i+1})}{X_T D_{i+1}^F(T_{i+1})}}{\frac{D_{i+1}^D(0)}{X_0 D_{i+1}^F(0)}} \right] \xrightarrow{\frac{1}{F(t)}} \text{slide 21} \\
 &= \mathbb{E}^{i+1,F} \left[ L_i^F(T) \cdot \frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}} \right]
 \end{aligned}$$

Formula

$$\begin{aligned}
E^{i+1, D}[L_i^F(T)] &= E^{i+1, F} \left[ L_i^F(T) \cdot \frac{\frac{1}{F_T}}{\frac{1}{F_0}} \right] \\
&= E^{i+1, F} \left[ L_i^F(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i w_T^{i+1, F}} \right] \times \frac{\frac{1}{F_0} e^{-\frac{\sigma_x^2 T}{2} + \sigma_x w_T^F}}{\frac{1}{F_0}} \\
&= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} E^{i+1, F} \left[ e^{\sigma_i w_T^{i+1, F} + \sigma_x w_T^F} \right] \\
&\quad \text{can't separate and do expectation as they are not independent} \\
&= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} E^{i+1, F} \left[ e^{\sigma_i Z_T^{(1)} + \sigma_x (P Z_T^{(1)} + \sqrt{1-P^2} Z_T^{(2)})} \right] \\
&= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} E^{i+1, F} \left[ e^{(\sigma_i + \sigma_x P)^2 \frac{T}{2}} e^{\sigma_x^2 (1-P^2) \frac{T}{2}} \right] \\
&= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} \cdot e^{\frac{\sigma_i^2 T + 2\sigma_i \sigma_x P T + \sigma_x^2 P^2 T}{2}} e^{\frac{\sigma_x^2 T - \sigma_x^2 P^2 T}{2}}
\end{aligned}$$

↓  
↓  
↓  
↓

$$E^{i+1, D}[L_i^F(T)] = L_i^F(0) e^{\sigma_i \sigma_x P T}$$

$P > 0$  : When foreign  $\rightarrow L_i^F$  goes up  $\rightarrow \frac{1}{F_T} \uparrow \rightarrow$  foreign currency depreciates.

$L_i^F$  goes down  $\rightarrow \frac{1}{F_T} \downarrow \rightarrow$  foreign currency appreciates

so quanto Libor  $L_i^F(0) e^{\sigma_i \sigma_x P T} >$  vanilla Libor,  $L_i^F(0)$  (more expensive)  
 depends on how tedious to hedge.

# Pricing Quanto LIBOR

Substituting for  $L_i^F(T)$  and the forward exchange rate, the expectation to evaluate becomes

$$\mathbb{E}^{i+1,F} \left[ L_i^F(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}} \cdot \frac{1}{F_0} e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W^F} \right] \times F_0.$$

Next, we apply Cholesky decomposition:

$$W^{i+1} \rightarrow Z_1$$

$$W^F \rightarrow \rho Z_1 + \sqrt{1 - \rho^2} Z_2$$

where  $Z_1 \perp Z_2$ .

Finally, the convexity corrected foreign LIBOR rate paid in domestic denomination is given by

$$\tilde{L}_i^F(T) = L_i^F(0) e^{\rho \sigma_X \sigma_i T}.$$