QF605 Additional Examples
Session 1: Bond Market and Bond Risk Management

1 Questions

1. A bank quotes an interest rate of 5% with quarterly compounding for a year. Calculate
   (a) the effective annual rate.
   (b) the bond equivalent rate.
   (c) the equivalent continuous compounding rate.

2. A 3y coupon bond pays an annual coupon of 5. The discrete zero curve is as follows:

Maturity	Zero Rate
$1y$	5%
$2y$	5%
$3y$	5%

   Determine:
   (a) The price of the bond.
   (b) The par yield of the bond.

3. Suppose we use continuous compounded yield to express the price of a coupon bond:

$$B = \sum_{i=1}^{N} c_i e^{-y \cdot T_i},$$

   where the coupon c_i is a fixed value c for all $T_1, T_2, \cdots, T_{N-1}$, and the last payment is $c_N = c + 100$. Derive an expression for the continuously compounded par yield of this bond.

4. A 5-year bond with a yield-to-maturity of 11%, continuously compounded, pays an 8% annual coupon.
   (a) Calculate the bond price.
   (b) Calculate the modified duration of this bond.
   (c) Use modified duration to calculate the change in bond price under a 0.2% decrease in its yield.
   (d) Calculate the convexity of this bond.
   (e) Recalculate the bond price with a 10.8% yield, and check this result with the price change estimation based on modified duration and convexity.

5. We hold on to the following bond position in our portfolio:

Bond	Position	Mod. Duration	Convexity
A	1.5 million	3.4	20
B	2.0 million	2.8	18

We observe 2 other bonds in the market

Bond	Mod. Duration	Convexity
C	2.9	18
D	1.4	10

What positions should we take on these 2 bonds to obtain a bond portfolio with 0 dollar duration and dollar convexity?

2 Suggested Solutions

1. (a) The effective annual rate is given by

$$r_{\text{EAR}} = \left(1 + \frac{r_S}{m}\right)^m - 1$$
$$= \left(1 + \frac{5\%}{4}\right)^4 - 1 \approx 5.095\% \quad \triangleleft$$

(b) The bond equivalent rate is given by

$$r_{\text{BEY}} = \left[\left(1 + \frac{r_S}{m}\right)^{\frac{m}{2}} - 1\right] \times 2$$
$$= \left[\left(1 + \frac{5\%}{4}\right)^{\frac{4}{2}} - 1\right] \times 2 \approx 5.031\% \quad \triangleleft$$

(c) Let r_c denote the continuously compounded rate, we have

$$e^{r_c} = \left(1 + \frac{r_S}{m}\right)^m$$
$$\Rightarrow \quad r_c = m \times \log\left(1 + \frac{r_S}{m}\right)$$
$$\approx 4.969\% \quad \triangleleft$$

2. Note that the zero rates provided are *discretely compounded.*

(a) The bond price can be calculated as

$$B = \frac{5}{(1+5\%)^1} + \frac{5}{(1+5\%)^2} + \frac{105}{(1+5\%)^3}$$
$$= 100. \quad \triangleleft$$

(b) A bond trades at par when its price is equal to the face value. For this bond, the par yield is 5%. $\triangleleft$ a formula

3. Let $T_{i+1} - T_i = \Delta T$, i.e. the difference in time between each cashflow. We have

$$B = c \cdot e^{-y\Delta T} + c \cdot e^{-2y\Delta T} + c \cdot e^{-3y\Delta T} + \cdots + c \cdot e^{-Ny\Delta T} + 100 \cdot e^{-Ny\Delta T}$$
$$= c \cdot e^{-y\Delta T}\left[1 + e^{-y\Delta T} + \cdots + e^{-(N-1)y\Delta T}\right] + 100 \cdot e^{-Ny\Delta T}$$
$$= c \cdot e^{-y\Delta T} \times \frac{1 - e^{-Ny\Delta T}}{1 - e^{-y\Delta T}} + 100 \cdot e^{-Ny\Delta T}$$
$$= c \times \frac{1 - e^{-Ny\Delta T}}{e^{y\Delta T} - 1} + 100 \cdot e^{-Ny\Delta T}$$

formula sheet

Now we set B to 100 to obtain

$$c \times \frac{1 - e^{-Ny\Delta T}}{e^{y\Delta T} - 1} = 100 \times \left(1 - e^{-Ny\Delta T}\right)$$
$$\Rightarrow \quad c = 100 \cdot \left(e^{y\Delta T} - 1\right).$$

From here we can derive the par yield to be

$$y = \frac{1}{\Delta T} \log\left(\frac{c}{100} + 1\right). \quad \triangleleft$$

4. (a) The bond price is given by

$$B = 8 \times \left(e^{-0.11\times1} + e^{-0.11\times2} + e^{-0.11\times2} + e^{-0.11\times2} e^{-0.11\times5}\right) + 100 \times e^{-0.11\times5}$$
$$= 86.801 \quad \triangleleft$$

(b) Since the bond yield is continuously compounded, the modified duration is given by

$$D = \frac{1}{86.801} \times \left(1 \cdot 8 \cdot e^{-0.11\cdot1} + 2 \cdot 8 \cdot e^{-0.11\cdot2} + 3 \cdot 8 \cdot e^{-0.11\cdot3} + 4 \cdot 8 \cdot e^{-0.11\cdot4} + 5 \cdot 108 \cdot e^{-0.11\cdot5}\right)$$
$$= 4.256 \quad \triangleleft$$

modified.

(c) If the yield moves by $\Delta y = -0.2\%$, then

$$\frac{\Delta B}{B} \approx -D\Delta y = -4.256 \times (-0.2\%) = 0.85\% \quad \triangleleft$$
$$\therefore \quad \Delta B \approx -D\Delta y B = -4.256 \times (-0.2\%) \times 86.801 = 0.73885 \quad \triangleleft$$

(d) Since the bond yield is continuously compounded, the convexity is given by

$$C = \frac{1}{86.801} \times \left(1^2 \cdot 8 \cdot e^{-0.11\cdot1} + 2^2 \cdot 8 \cdot e^{-0.11\cdot2} + 3^2 \cdot 8 \cdot e^{-0.11\cdot3} + 4^2 \cdot 8 \cdot e^{-0.11\cdot4} + 5^2 \cdot 108 \cdot e^{-0.11\cdot5}\right)$$
$$= 19.871 \quad \triangleleft$$

(e) If the bond yield is 10.8%, the price should be

$$B = 8 \times \left(e^{-0.108\times1} + e^{-0.108\times2} + e^{-0.108\times3} + e^{-0.108\times4} + e^{-0.108\times5}\right) + 100 \times e^{-0.108\times5}$$
$$= 87.5434 \quad \triangleleft$$

Using both modified duration and convexity, the bond price will change by

$$\begin{aligned}\Delta B &\approx -D\Delta y B + \frac{1}{2}C(\Delta y)^2 B \\ &\approx -4.256 \times (-0.002) \times 86.801 + \frac{1}{2} \times 19.871 \times (-0.002)^2 \times 86.801 \\ &\approx 0.7423 \quad \triangleleft\end{aligned}$$

which compares very closely to the actual price change of

$$87.5434 - 86.801 = 0.7424$$

5. First we work out the dollar duration and dollar convexity of the holding on bonds A and B:

$$D_\$(V) = 1.5 \times 3.4 + 2 \times 2.8 = 10.7$$
$$C_\$(V) = 1.5 \times 20 + 2 \times 18 = 66$$

With bonds C and D, we solve the simultaneous equations:

$$\begin{cases} D_\$(V) + B_c \times 2.9 + B_d \times 1.4 = 0 \\ C_\$(V) + B_c \times 18 + B_d \times 10 = 0 \end{cases}$$

Solving, we obtain: $B_C = -3.8421$ and $B_D = 0.31579$. $\triangleleft$