Assignment 4

1. $h(K) = \dfrac{g(K)}{IRR(K)}$

$$h'(K) = \frac{IRR(K)\,g'(K) - g(K)\,IRR'(K)}{IRR(K)^2}$$

in this payoff $g''(K)$ is 0

$$h''(K) = \frac{IRR(K)\,g''(K) - IRR''(K)g(K) - 2\cdot IRR'(K)g'(K)}{IRR(K)^2} + \frac{2\cdot IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$= \frac{-IRR''(K)g(K) - 2IRR'(K)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2 g(K)}{IRR(K)^3}$$

$$\int_0^\infty h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK$$

$$= \cancel{\int_0^{K_1} h(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK} \rightarrow 0 + \int_{K_1}^{K_2} h_1(K)\frac{\partial^2 V^{pay}}{\partial K^2}dK + \int_{K_2}^{\infty} h_2(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK$$

$$= \int_{K_1}^{K_2} h_1(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK + \int_{K_2}^{\infty} h_2(K)\frac{\partial^2 V^{pay}(K)}{\partial K^2}dK$$

$$= \left[h_1(K)\frac{\partial V^{pay}(K)}{\partial K}\right]_{K_1}^{K_2} - \int_{K_1}^{K_2} h_1'(K)\frac{\partial V^{pay}(K)}{\partial K}dK + \left[h_2(K)\frac{\partial V^{pay}(K)}{\partial K}\right]_{K_2}^{\infty}$$

$$- \int_{K_2}^{\infty} h_2'(K)\frac{\partial V^{pay}(K)}{\partial K}(dK)$$

$$= h_1(K_2)\frac{\partial V^{pay}(K_2)}{\partial K} - \cancel{h_1(K_1)\frac{\partial V^{pay}(K_1)}{\partial K}} \rightarrow 0 - \int_{K_1}^{K_2} h_1'(K)\frac{\partial V^{pay}(K)}{\partial K}dK$$

The payoffs are diff (cannot cancel).

$$+ \cancel{h(\infty)\frac{\partial V^{pay}(\infty)}{\partial K}} \rightarrow 0 - h_2(K_2)\frac{\partial V^{pay}(K_2)}{\partial K} - \int_{K_2}^{\infty} h_2'(K)\frac{\partial V^{pay}(K)}{\partial K}dK$$

$$= h_1(K_2)\frac{\partial V^{pay}(K_2)}{\partial K} - \left[h_1'(K)V^{pay}(K)\right]_{K_1}^{K_2} + \int_{K_1}^{K_2} h_1''(K)V^{pay}(K)dK$$

$$- h_2(K_2)\frac{\partial V^{pay}(K_2)}{\partial K} - \left[h_2'(K)V^{pay}(K)\right]_{K_2}^{\infty} + \int_{K_2}^{\infty} h_2''(K)V^{pay}(K)dK$$

$$= h_1(k_2)\frac{\partial V^{pay}(k_2)}{\partial k} - \left(h_1'(k_2)V^{pay}(k_2) - h_1'(k_1)V^{pay}(k_1)\right]$$

$$+ \int_{k_1}^{k_2} h_1''(k)V^{par}(k)dk - h_2(k_2)\frac{\partial V^{pay}(k_2)}{\partial k}(k_2)$$

$$- \left[\cancel{h_1'(\infty)V^{pay}(\infty)} - h_2'(k_2)V^{pay}(k_1)\right] + \int_{k_2}^{\infty} h_2''(k)V^{pay}(k)dk$$

$\xrightarrow{} 0$; $\leftarrow k_1 - k_2$

no integration so this is less than k_1

$$V_0 = h_1(k_2)\frac{\partial V^{pay}(k_2)}{\partial k} - h_1'(k_2)V^{pay}(k_2) + h_1'(k_1)V^{pay}(k_1)$$

$$+ \int_{k_1}^{k_2} h_1''(k)V^{pay}(k)dk - h_2(k_2)\frac{\partial V^{pay}(k_2)}{\partial k} + h_2'(k_2)V^{pay}(k_2)$$

$$+ \int_{k_2}^{\infty} h_2''(k)V^{par}(k)dk$$

$$h_1(k_2) = \frac{g(k_2)}{IRR(k_2)} = \frac{k_2 - k_1}{IRR(k_2)}$$

$$h_2(k_2) = \frac{g(k_2)}{IRR(k_2)} = \frac{k_2 - k_1}{IRR(k_2)}$$

$g'(k_2) = 1$ ← with respect to h_1

$$h_1'(k_2) = \frac{IRR(k_2)g'(k_2) - g(k_2)IRR'(k_2)}{IRR(k_2)^2}$$

$$= \frac{IRR(k_2) - (k_2 - k_1)IRR'(k_2)}{IRR(k_2)^2}$$

$$h_1'(k_1) = \frac{IRR(k_1) - \cancel{(k_1 - k_1)}\,IRR'(k_1)}{IRR(k_1)^2} \quad (\to 0)$$

$$= \frac{1}{IRR(k_1)}$$

$g'(k) = 0$

$$h_2'(k_2) = \frac{-(k_2 - k_1)IRR'(k_2)}{IRR(k_2)^2}$$

$$\therefore\ V_0 = \cancel{\left(\frac{k_2-k_1}{IRR(k_2)}\right)} \frac{\partial V^{pay}(k_2)}{\partial k} - \cancel{\frac{IRR(k_2)-(k_2-k_1)IRR'(k_2)}{IRR(k_2)^2}}\, V^{pay}(k_2) + \frac{1}{IRR(k_1)} V^{pay}(k_1)$$

$$+ \int_{k_1}^{k_2} h_1''(k) V^{pay}(k)\,dk - \cancel{\frac{k_2-k_1}{IRR(k_2)}} \frac{\partial V^{pay}(k_2)}{\partial k} + \left(\cancel{\frac{-(k_2-k_1)IRR'(k_2)}{IRR(k_2)^2}}\right) V^{pay}(k_2)$$

$$+ \int_{k_2}^{\infty} h_2''(k) V^{pay}(k)\,dk$$

$$= -\frac{1}{IRR(k_2)} V^{pay}(k_2) + \frac{1}{IRR(k_1)} V^{pay}(k_1) + \int_{k_1}^{k_2} h_1''(k) V^{pay}(k)\,dk$$

$$+ \int_{k_2}^{\infty} h''(k)\, V^{pay}(k)\,dk$$

$$V_0 = \frac{1}{IRR(k_1)} V^{pay}(k_1) - \frac{1}{IRR(k_2)} V^{pay}(k_2)$$

$$+ \int_{k_1}^{k_2} \frac{-IRR''(k)(k-k_1) - 2IRR'(k)}{IRR(k)^2} + \frac{2IRR'(k)^2(k-k_1)}{IRR(k)^3}\, V^{pay}(k)\,dk$$

$$+ \int_{k_2}^{\infty} \frac{-IRR''(k)\,g(k_2-k_1)}{IRR(k)^2} + \frac{2IRR'(k)^2(k_2-k_1)}{IRR(k)^3}\, V^{pay}(k)\,dk$$

2. Model : $dr_t = \theta(t)dt + \sigma dW_t^*$

Step 1:

$$\int_0^t dr_s = \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

$$r_t - r_0 = \int_0^t \theta(s)\,ds + \int_0^t \sigma dW_s^*$$

$$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

Step 2:

$$\int_0^T r_u du = \int_0^T r_0 du + \int_0^T\int_0^t \theta(s)ds\,du + \int_0^T\int_0^T \sigma dW_s^* du$$

$$= r_0 T + \int_0^T\int_0^t \theta(s)ds\,du + \int_0^T\int_0^t \sigma dW_s^* du$$

$$= r_0 T + \int_0^T\int_s^T \theta(s)du\,ds + \int_0^T\int_s^T \sigma du\,dW_s^*$$

$$= r_0 T + \int_0^T \theta(s)(T-s)ds + \int_0^T \sigma(T-s)\,dW_s^*$$

$$\mathbb{E}\left[\int_0^T r_u du\right] = r_0 T + \int_0^T \theta(s)(T-s)ds$$

$$V\left[\int_0^T r_u du\right] = \int_0^T \sigma^2(T-s)^2 ds = \frac{1}{3}\sigma^2 T^3$$

3. $D(0,1y) = 0.9656$

$$D(0,1y) = e^{-R(0,1y)\times 1}$$

$$\therefore \; -R(0,1y) = \ln(0.9656)$$

$$\therefore \; R(0,1y) = -\ln(0.9656)$$

$$= 0.0350056$$

$$r = 0.0350056 + .005 + \theta_0$$

$$D(1,2) = e^{-(0.0400056 + \theta_0)}$$

$$D(0,2y) \xrightarrow{0.5} D(1,2) \qquad D(0,2y) \xrightarrow{0.5} D(1,2)$$

$$r_0 = 0.0350056$$

$$D(1,2) = e^{-(0.0300056 + \theta_0)}$$

$$r = 0.0350056 - .005 + \theta_0$$

$$D(0,2y) = D(0,1y)\, \mathbb{E}\left[D(1,2)\right]$$

$$0.9224 = 0.9656 \cdot \left[\frac{1}{2} \times e^{-0.0400056-\theta_0} + \frac{1}{2} e^{-0.0300056-\theta_0}\right]$$

$$\theta_0 = -\ln\left[\frac{2 \times 0.9224}{0.9656 \times \left(e^{-0.0400056} + e^{-0.0300056}\right)}\right]$$

$$\theta_0 = 0.0107476$$

$r = 0.0450056 + \theta_0 + \theta_1$

$D(2,3) = e^{-(0.0557832 + \theta_1)}$

$r = 0.0400056 + 0.0107776$

$D(1,3) = \ldots$

$r = 0.0350056 + \theta_0 + \theta_1$

$D(2,3) = e^{-(0.0457832 + \theta_1)}$

$D(0,3)$

$r_0 = 0.0350056$

$D(1,3) = \ldots$

$r = 0.0300056 + 0.0107776$

$D(2,3) = e^{-(0.0357832 + \theta_1)}$

$r = 0.0250056 + \theta_0 + \theta_1$

$$D(0,3y) = \mathbb{E}^*\left[D(0,1) \cdot D(1,3)\right]$$

$$= D(0,1)\, \mathbb{E}^*\left[D(1,2)\ \mathbb{E}^*\left[D(2,3)\right]\right]$$

$$\frac{D(0,3y)}{D(0,1y)} = \left[\frac{1}{2} e^{-0.0507832} \cdot \left[\frac{1}{2} e^{-0.0557832 - \theta_1} + \frac{1}{2} e^{-0.0457832 - \theta_1}\right]\right.$$

$$\left. + \frac{1}{2} e^{-0.0407832} \cdot \left[\frac{1}{2} e^{-0.0457832 - \theta_1} + \frac{1}{2} e^{-0.0357832 - \theta_1}\right]\right]$$

$$\frac{D(0,3y)}{D(0,1y)} = \left[\frac{1}{4} e^{-0.1065664 - \theta_1} + \frac{1}{4} e^{-0.0965664 - \theta_1}\right.$$

$$\left. + \frac{1}{4} e^{-0.0865664 - \theta_1} + \frac{1}{4} e^{-0.0765664 - \theta_1}\right]$$

$$\theta_1 = -\ln\left[\frac{4\, D(0,3y)}{D(0,1y)\left(e^{-0.1065664} + e^{-0.0965664} + e^{-0.0865664} + e^{-0.0765664}\right)}\right]$$

$\theta_1 = -0.0103127$