QF605 Additional Examples
Session 2: Interest Rate and Swap Market

1 Questions

1. The spot LIBOR rates are as follow:

Tenor	Rate
$1m$	1.15%
$2m$	1.20%
$3m$	1.25%
$6m$	1.40%
$9m$	1.55%
$12m$	1.75%

   Calculate:

   (a) The spot $3m$ discount factor $D(0, 3m)$.

   (b) The forward discount factor $D(3m, 6m)$.

   (c) The forward LIBOR rate $F(2m, 9m)$.

   (d) What rate would you show for a 2×12 FRA?

   note in formula sheet that these are fancy ways of Libor rate

   (e) If your view is that one-month later the spot $1m$ rate would still remain at 1.15%, how should you trade?

2. Consider the following continuously compounded zero rates:

Maturity	Zero Rate
$1y$	4%
$2y$	4.5%
$3y$	4.75%

   (a) Calculate the continuously compounded forward rates $F(0y, 1y)$, $F(1y, 2y)$ and $F(2y, 3y)$.

   (b) Show that the continuously compounded zero rate can be expressed as an arithmetic average of the corresponding forward rates.

3. Today's spot exchange rate for USD/SGD is $FX_0 = 1.42$. Furthermore, we observe the following zero-coupon bonds in the market:

$$D_{\text{SGD}}(0, T) = 0.98, \qquad D_{\text{USD}}(0, T) = 0.964.$$

   (a) What should be the forward value of the exchange rate at time T?

   (b) If we see that $FX_T = FX_0 = 1.42$, state an arbitrage.

4. We have the following continuously compounded zero rates under 30/360 day count.

Maturity	Rate
$3m$	1.10%
$6m$	1.40%
$12m$	1.75%
$18m$	1.90%
$24m$	2.00%

(a) A $2y$ fixed leg pays 1.75% semi-annually. What is the PV of this fixed leg?

(b) A $2y$ floating leg pays the $6m$ LIBOR rate semi-annually. What is the PV of this leg?

(c) What is the par swap rate for a $2y$ interest rate swap with semi-annual payment?

(d) Suppose you long a receiver swap at the par swap rate calculated in the previous question. Three months later, you observe the following zero rates in the market:

Maturity	Rate
$3m$	1.20%
$6m$	1.50%
$12m$	1.85%
$18m$	1.95%
$24m$	2.05%

What is the value of the receiver swap you have in your portfolio?

Note: Use linear interpolation on the zero rates if the maturity required is not provided.

5. We observe the following instruments in the swap market. All three interest rate swaps have semi-annual payment.

Instrument	Quote
$6m$ LIBOR	2%
$1y$ IRS	2.25%
$2y$ IRS	2.40%
$3y$ IRS	2.50%

(a) Determine the par swap rate for a $1.5y$ tenor interest rate swap with semi-annual payment.

(b) A *forward starting* swap with a $2y$ tenor starting at $t = 1y$ has the following cashflows:

Time (y)	Pay	Rec
1.5	Par Swap Rate	$6m$ LIBOR
2.0	Par Swap Rate	$6m$ LIBOR
2.5	Par Swap Rate	$6m$ LIBOR
3.0	Par Swap Rate	$6m$ LIBOR

Calculate the par swap rate for this forward starting swap.

Note: Use 30/360 day count convention. Apply linear interpolation on the discount factors if necessary.

6. In order to reduce the sensitivity of a bond portfolio with respect to small changes in the yield curve, it is desirable to have a portfolio with small dollar duration and dollar convexity. The process of obtaining a portfolio with zero dollar duration and dollar convexity is called portfolio immunization, and can be done by taking positions in other bonds available in the market. Let V be the value of a portfolio with dollar duration $D_{\$}(V)$ and dollar convexity $C_{\$}(V)$. Take positions of sizes B_1 and B_2, respectively, in two bonds with duration and convexity D_i and C_i for $i = 1, 2$.

   The value of the new (hedged) portfolio is

$$\Pi = V + B_1 + B_2.$$

   Note that $D_{\$}(B_1) = B_1 D_1$, $D_{\$}(B_2) = B_2 D_2$, $C_{\$}(B_1) = B_1 C_1$, $C_{\$}(B_2) = B_2 C_2$.

   Choose B_1 and B_2 such that the dollar duration and dollar convexity of the portfolio Π are equal to 0, i.e. such that

$$\begin{cases} B_1 D_1 + B_2 D_2 = -D_{\$}(V) \\ B_1 C_1 + B_2 C_2 = -C_{\$}(V) \end{cases}$$

   Suppose we have just invested \$1 million in a bond with duration 3.2 and convexity 16, and \$2.5 million in a bond with duration 4 and convexity 24.

   (a) What are the dollar duration and the dollar convexity of your portfolio?

   (b) If the yield curves go up by 10 basis points, estimate the new value of the portfolio.

   (c) You can buy or sell two other bonds, one with duration 1.6 and convexity 12, and another one with duration 3.2 and convexity 20. What positions would you take in these bonds to immunize your portfolio, i.e. to obtain a portfolio with zero dollar duration and dollar convexity?

2 Suggested Solutions

1. (a) The spot $3m$ discount factor is

$$\frac{1}{1+0.25\times 0.0125}\approx 0.996885. \quad \triangleleft$$

(b) The forward discount factor $D(3m,6m)$ is

formula sheet

$$D(3m,6m)=\frac{D(0,6m)}{D(0,3m)}=\frac{\frac{1}{1+0.5\times 0.014}}{\frac{1}{1+0.25\times 0.0125}}\approx 0.996152. \quad \triangleleft$$

(c) The forward LIBOR rate $F(2m,9m)$ is

$$(1+\Delta_{2m}L_{2m})(1+\Delta_{7m}F(2m,9m))=1+\Delta_{9m}L_{9m}$$

$$F(2m,9m)=\frac{1}{\frac{210}{360}}\left[\frac{1+\frac{270}{360}\times 0.0155}{1+\frac{60}{360}0.012}-1\right]\approx 1.6467\%. \quad \triangleleft$$

(d) We should show $F(2m,12m)$ to prevent arbitrage

$$(1+\Delta_{2m}L_{2m})(1+\Delta_{10m}F(2m,12m))=1+\Delta_{12m}L_{12m}$$

$$F(2m,12m)=\frac{1}{\frac{300}{360}}\left[\frac{1+\frac{360}{360}\times 0.0175}{1+\frac{60}{360}0.012}-1\right]\approx 1.85629\%. \quad \triangleleft$$

(e) If we think that the $1m$ spot rate will remain unchanged a month later, we should short the 1×2 FRA, since $F(1m,2m)>1.15\%$, we can borrow at 1.15% to deposit (lend) at $F(1m,2m)$ if we were right. $\triangleleft$ formula sheet

2. Note that the zero rates provided are *continuously compounded.*

(a) The first will be equal to the observed zero rate:

$$e^{F(0y,1y)\times 1}=e^{4\%\times 1}$$
$$\Rightarrow \quad F(0y,1y)=4\% \quad \triangleleft$$

By no-arbitrage, compounding in the forward rate should be identical to compounding in the zero rate for 2 years:

$$e^{F(0y,1y)\cdot 1}\times e^{F(1y,2y)\cdot 1}=e^{4.5\%\times 2}$$
$$\Rightarrow \quad F(1y,2y)=5\% \quad \triangleleft$$

And similarly, formula

$$e^{F(0y,1y)\cdot 1}\times e^{F(1y,2y)\cdot 1}\times e^{F(2y,3y)\cdot 1}=e^{4.75\%\times 3}$$
$$\Rightarrow \quad F(2y,3y)=5.25\% \quad \triangleleft$$

(b) Let r_n denote the continuously compounded zero rate for n years, and let F denote the continuously compounded forward rate. We have

$$e^{F(0,1)}\times e^{F(1,2)}\times\cdots\times e^{F(n-1,n)}=e^{r_n\times n}$$
$$\Rightarrow \quad r_n=\frac{F(0,1)+F(1,2)+\cdots+F(n-1,n)}{n} \quad \triangleleft$$

3. (a) Let r^F and r^D denote the continuously compounded interest rate in the foreign and domestic economy, respective. Based on the definition of the exchange rate, we can arrive at

$$FX_T = FX_0 e^{(r^D - r^F)T}$$

formula sheet

using no-arbitrage argument. We can then write

sample paper q2

$$FX_T = FX_0 \cdot \frac{e^{-r^F T}}{e^{-r^D T}} = FX_0 \cdot \frac{D_{\text{USD}}(0,T)}{D_{\text{SGD}}(0,T)} = 1.3968 \quad \triangleleft$$

formula sheet

(b) If we see that $FX_T = FX_0 = 1.42$, enter into the FX forward to lock-in to this forward exchange rate. We long 1 unit of USD bond by shorting some SGD bond to generate a cash amount of \$1.36888 in SGD ($\because$ $0.964 \times 1.42 = 1.36888$). When the USD bond matures, we convert the \$1 USD back to SGD to get \$1.42 SGD. The short SGD bond position now becomes $-\$1.36888 \times \frac{1}{0.98} = \$1.3968$. The difference is the arbitrage. $\triangleleft$

As long as your answer make use of the over-estimated FX forward (1.42 instead of 1.3968), and use long/short bond position to lock in to risk-free return, it is considered as correct.

4. (a) Given the zero rates, the discount factors are

$$D(0,6m) = e^{-1.40\% \times 0.5} = 0.993$$
$$D(0,12m) = e^{-1.75\% \times 1} = 0.9827$$
$$D(0,18m) = e^{-1.90\% \times 1.5} = 0.9719$$
$$D(0,24m) = e^{-2.00\% \times 2} = 0.9608$$

The PV of the fixed leg paying 1.75% semi-annually can be calculated as

r fix

$$PV_{\text{fix}} = 0.5 \times \Big[D(0,6m) + D(0,12m) + D(0,18m) + D(0,24m)\Big] \times 1.75\%$$
$$= 0.0342 \quad \triangleleft$$

formula sheet

(b) The PV of the floating leg is

$$PV_{\text{flt}} = 1 - D(0,24m) = 0.03921 \quad \triangleleft$$

(c) The par swap rate can be calculated as

$$S = \frac{1 - D(0,24m)}{\sum_{i=1}^{4} 0.5 \times D(0,T_i)} = 2\% \quad \triangleleft$$

(d) After three months, the swap's cashflows are $3m$, $9m$, $15m$ and $21m$ away. First we linearly interpolate for the necessary zero rates:

$$R_{9m} = \frac{1.50\% + 1.85\%}{2} = 1.675\%$$

interpolation formula sheet

$$R_{15m} = \frac{1.85\% + 1.95\%}{2} = 1.9\%$$
$$R_{21m} = \frac{1.95\% + 2.05\%}{2} = 2.0\%$$

Based on these zero rates, we can work out the forward LIBOR rates:

$$e^{1.20\% \times 0.25} \cdot (1 + 0.5L(3m,9m)) = e^{1.675\% \times 0.75} \quad \Rightarrow \quad L(3m,9m) = 1.92\%$$
$$e^{1.675\% \times 0.75} \cdot (1 + 0.5L(9m,15m)) = e^{1.9\% \times 1.25} \quad \Rightarrow \quad L(9m,15m) = 2.25\%$$
$$e^{1.9\% \times 1.25} \cdot (1 + 0.5L(15m,21m)) = e^{2.0\% \times 1.75} \quad \Rightarrow \quad L(15m,21m) = 2.263\%$$

The PV of the floating leg is

$$
\begin{aligned}
PV_{\text{flt}} &= 0.5 \times \Big[D(0,3m)\cdot 1.4\% + D(0,9m)\cdot 1.92\% \\
&\qquad + D(0,15m)\cdot 2.25\% + D(0,21m)\cdot 2.263\%\Big] \\
&= 0.0383
\end{aligned}
$$

The PV of the fixed leg is

$$
\begin{aligned}
PV_{\text{fix}} &= 0.5 \times \Big[D(0,3m) + D(0,9m) + D(0,15m) + D(0,21m)\Big] \times 2\% \\
&= 0.0393
\end{aligned}
$$

So the value of the receiver swap is

formula shee

$$V_{\text{rec}} = PV_{\text{fix}} - PV_{\text{flt}} \approx 0.001 \quad \text{per \$1 notional.} \quad \triangleleft$$

$V_{pay} = PV_{flt} - PV_{fix}$

5. (a) We need to know the discount factors $D(0,6m)$, $D(0,1y)$, and $D(0,1.5y)$ to calculate the par swap rate for a $1.5y$ tenor swap. We have

$$D(0,6m) = \frac{1}{1 + 0.5 \times 2.0\%} = 0.99$$

Using the $1y$ IRS quote, we can write

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$$
\begin{aligned}
PV_{\text{fix}} &= 0.5 \cdot \Big[D(0,6m) + D(0,1y)\Big] \cdot 2.25\% \\
PV_{\text{flt}} &= D(0,6m)\cdot 0.5 \cdot \underset{L(0,6m)}{2.0\%} + D(0,1y)\cdot 0.5 \cdot L(6m,12m) \\
&= D(0,6m)\cdot 0.5 \cdot 2.0\% + D(0,1y)\cdot 0.5 \cdot \frac{1}{0.5}\,\frac{D(0,6m) - D(0,1y)}{D(0,1y)} \\
&= D(0,6m)\cdot 0.5 \cdot 2.0\% + D(0,6m) - D(0,1y) \\
&= D(0,6m)(1 + 0.5 \cdot 2.0\%) - D(0,1y) \\
&= 1 - D(0,1y)
\end{aligned}
$$

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Equating the PV of the fixed and floating leg, we can solve for $D(0,1y) = 0.9779$. Using the $2y$ IRS quote, we can write

$$
\begin{aligned}
PV_{\text{fix}} &= PV_{\text{flt}} \\
0.5 \cdot \Big[D(0,6m) + D(0,1y) + D(0,1.5y) + D(0,2y)\Big] \cdot 2.4\% &= 1 - D(0,2y)
\end{aligned}
$$

for finding 2 missing swap rates

We linearly interpolate for $D(0,1.5y)$ between $D(0,1y)$ and $D(0,2y)$, so that

$$D(0,1.5y) = \frac{D(0,1y) + D(0,2y)}{2} = 0.48895 + 0.5D(0,2y)$$

Substituting back, we obtain

$$D(0,2y) = 0.9536, \qquad D(0,1.5y) = 0.96575.$$

The par swap rate for a $1.5y$ tenor swap with semi-annual payment is

$$S = \frac{1 - D(0,1.5y)}{0.5 \cdot [D(0,6m) + D(0,1y) + D(0,1.5y)]} = 2.335\% \quad \triangleleft$$

normal par swap

(b) For the forward starting swap with a $2y$ tenor starting at $t = 1y$, we need the discount factors: $D(0,1y)$, $D(0,1.5y)$, $D(0,2y)$, $D(0,2.5y)$, and $D(0,3y)$. Again, we shall interpolate for the $D(0,2.5y)$ discount factor linearly as follows:

$$D(0,2.5y) = \frac{D(0,2y) + D(0,3y)}{2} = 0.4768 + 0.5D(0,3y).$$

Using the $3y$ IRS quote, we write

$$PV_{\text{fix}} = 0.5 \cdot \Big[D(0,6m) + D(0,1y) + D(0,1.5y) + D(0,2y) + D(0,2.5y) + D(0,3y)\Big] \cdot 2.5\%$$
$$PV_{\text{flt}} = 1 - D(0,3y)$$

Setting the PV equal, we obtain

formula for forward swap

$$D(0,3y) = 0.928, \qquad D(0,2.5y) = 0.941.$$

The par swap rate for this forward swap is

$$S = \frac{D(0,1y) - D(0,3y)}{0.5 \times [D(0,1.5y) + D(0,2y) + D(0,2.5y) + D(0,3y)]} = 2.63\% \quad \triangleleft$$

6. (a) We have

$$D_\$(V) = B_1D_1 + B_2D_2 = \$13,200,000$$
$$C_\$(V) = B_1C_1 + B_2C_2 = \$76,000,000$$

(b) We have

$$\Delta V \approx -D_\$(V)\Delta y + \frac{C_\$(V)}{2} \cdot (\Delta y)^2$$
$$\approx -\$13,162$$

Hence $V' \approx V + \Delta V = \$3,500,000 - \$13,162 = \$3,486,838$.

(c) Let B_3 and B_4 be the values of the positions taken in the two bonds. We have $\Pi = V + B_3 + B_4$, and

$$D_\$(\Pi) = \$13.2mil + D_3B_3 + D_4B_4$$
$$= \$13.2mil + 1.6B_3 + 3.2B_4 = 0$$
$$C_\$(\Pi) = \$76mil + 12B_3 + 20B_4. = 0$$

if got space

Solving, we obtain $B_3 = \$3.25mil$ and $B_4 = -\$5.75mil$.