Session 4
LIBOR and Swap Market Models

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QF605 Fixed Income Securities

About Market Models

These models postulate a geometric Brownian motion for the market rates under consideration, such that the Black (1976) formula is recovered for the price of an European option on the market rate.

The Black formula is the market standard for calculating prices of European-style interest rate options.

— Antoon Pelsser

Martingales

Under the **risk-neutral valuation framework**, let V_t denote the value of a security at time t, we write

$$V_0 = e^{-rT}\mathbb{E}^*[V_T].$$

The expectation is taken under the risk-neutral measure associated with the risk-free bond numeraire.

This is valid because the asset ratio is a **martingale**

$$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right].$$

Under the risk-neutral measure, the best estimate based on the information at time t of the value of the discounted asset price at time T is the discounted asset price at time t:

$$\begin{aligned} M_t &= \mathbb{E}_t^*[M_T], \quad T > t \\ \therefore\ M_0 &= \mathbb{E}^*[M_T] \end{aligned}$$

Zero-Coupon Bond as Numeraire

Suppose the interest rate r is not a constant but a function of time (i.e. r_t), then under martingale pricing, we can value a financial contract V_t under the risk-neutral measure associated to the risk-free money market account numeraire as

$$V_t = \mathbb{E}^* \left[e^{-\int_t^T r_u du} V_T \right].$$

The expectation is evaluated under the probability measure $\mathbb{Q}^*$, which is associated to the **money market account numeraire** B_t.

Instead of using the value of the money market account B_t as a numeraire, the prices of **discount bonds** $D(t,T)$ **can also be used as a numeraire**.

A very convenient choice is to use the discount bond with maturity T as numeraire (co-inciding with the payoff time of the contract). A zero-coupon discount bond is given by

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} \right].$$

Zero-Coupon Bond as Numeraire

If we denote the probability measure associated to the numeraire $D(t,T)$ by $\mathbb{Q}^T$, we can apply the **change of numeraire theorem** to obtain

$$\frac{V_t}{D(t,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right].$$

However, at time T the price of the discount bond $D(T,T)=1$, and so

$$V_t = D(t,T)\mathbb{E}^T\left[V_T\right].$$

In words, by changing the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^T$, we have managed to express the expectation of the discounted payoff as a discounted expectation of the payoff.

$\Rightarrow$ We have therefore eliminated the problem of the correlation between the discounting term and the payoff term.

LIBOR Market Model

In the LIBOR market, we can choose to lend (deposit) capital and earn the LIBOR rate, which is the rate for unsecured borrowing and lending between banks.

If you lend into the LIBOR market for a period of length Δ, you earn $1+\Delta\cdot L$ one period later, where L denote the LIBOR rate you invested in.

Let $D(t,T)$ denote the value at time t of a discount bond which pays 1 at maturity T, the LIBOR rate and discount factor is related by

$$1=(1+\Delta\cdot L)\cdot D(0,\Delta).$$

Suppose we are at time t, and we commit into a forward LIBOR rate for the period $[T_i,T_{i+1}]$. We have the following relation

$$\begin{aligned} D(t,T_i) &= (1+\Delta_i L_t(T_i,T_{i+1}))D(t,T_{i+1}) \\ \Rightarrow \quad L_t(T_i,T_{i+1}) &= \frac{1}{\Delta_i}\frac{D(t,T_i)-D(t,T_{i+1})}{D(t,T_{i+1})}. \end{aligned}$$

LIBOR Market Model

$dF_t = \sigma F_t dW_t$

In most markets, only one specific LIBOR tenor is liquidly traded. In Singapore, this will be the 6m SIBOR or SOR rate.

In other words, for all practical purposes, $[T_i, T_{i+1}]$ are not arbitrary. Let us denote $L_i(t) = L_t(T_i, T_{i+1})$ and $D_i(t) = D(t, T_i)$. Now consider the process

formula —

$$\Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}.$$

This is a **ratio of marketed assets**. If we take the discount bond $D_{i+1}(t)$ as numeraire, then under the martingale measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$, the process $\Delta_i L_i(t)$ must be a martingale.

Since Δ_i is a constant, the process $L_i(t)$ must be a martingale under $\mathbb{Q}^{i+1}$. This gives rise to the **LIBOR Market Model (LMM)**

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t) \quad \Rightarrow \quad L_i(t) = L_i(0) \exp\left[-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}(t)\right],$$

$F_T = F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}$

where W^{i+1} is a Brownian motion under $\mathbb{Q}^{i+1}$.

$$dr_t = K(\theta - r_t)\,dt + \sigma\, dW_t^*, \quad \mathbb{Q}^*$$

$$D(t,T) = e^{-\int_t^T r_u\,du}$$

$$L_i(t) = \frac{1}{\Delta_i}\,\frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

$$\mathbb{E}^*[L_i(t)] = \mathbb{E}^*\left[\frac{1}{\Delta_i}\,\frac{e^{-\int_t^{T_i} r_u\,du} - e^{-\int_t^{T_{i+1}} r_u\,du}}{e^{-\int_t^{T_{i+1}} r_u\,du}}\right] \leftarrow \text{must be a martingale}$$

forward contract is a martingale only under $\mathbb{Q}^{i+1}$ measure $\left(\mathbb{E}^{i+1}[L_i(T)] = L_i(0)\right)$

Pricing a Caplet

The payoff of a **caplet** C_i at time T_{i+1} is given by

$$C_i(T_{i+1}) = \Delta_i(L_i(T_i) - K)^+.$$

Choosing D_{i+1} as a numeraire and working under the associated martingale measure $\mathbb{Q}^{i+1}$, we know that

$$\frac{C_i(0)}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{C_i(T_{i+1})}{D_{i+1}(T_{i+1})}\right]$$

$$\Rightarrow \quad C_i(0) = D_{i+1}(0)\Delta_i\mathbb{E}^{i+1}[(L_i(T_i) - K)^+].$$

The remaining steps required to derive a formula for a caplet price is identical to how we would handle a vanilla European option.

Pricing a Caplet

The LIBOR rate follows the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t),$$

where $W^{i+1}(t)$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$. The solution is given by

$F_T = F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}$

$$L_i(T) = L_i(0) e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W^{i+1}(T)}.$$

Evaluating the expectation, we obtain

$$\begin{aligned} C_i(0) &= D_{i+1}(0)\Delta_i \mathbb{E}^{i+1}[(L_i(T_i) - K)^+] \\ &= D_{i+1}(0)\Delta_i [L_i(0)\Phi(d_1) - K\Phi(d_2)], \end{aligned}$$

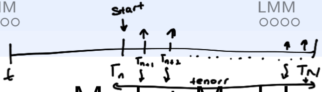
Libor is a whole curve so we have an additional index i →

Anything we know how to hedge is a martingale.

where

$$d_1 = \frac{\log \frac{L_i(0)}{K} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}, \quad d_2 = d_1 - \sigma_i \sqrt{T}.$$

$C = e^{-rT}[F\Phi(d_1) - K\Phi(d_2)]$



Make swap rate stochastic

Swap Market Model

forward starting swap.

Let us denote the **par swap rate** for the $[T_n, T_N]$ swap as $S_{n,N}$:

$$s = \frac{1 - D(0,T)}{\sum_i \Delta_{i-1} D(0,T_i)}$$

$$S_{n,N}(t) = \frac{D_n(t) - D_N(t)}{\sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)}.$$

N ← swap expires

formula

The term in the denominator is also called the **present value of a basis point** (PVBP)

$$P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t).$$

formula

Note that a one-period swap rate $S_{i,i+1}$ is equal to the LIBOR rate. We can now write the value of a payer and receiver swap as

formula

$$\left.\begin{aligned} \text{Payer Swap} &= P_{n+1,N}(t)(S_{n,N}(t) - K) \\ \text{Receiver Swap} &= P_{n+1,N}(t)(K - S_{n,N}(t)) \end{aligned}\right\} \text{what to make stochastic}$$

Receive float, Pay fix

$$\text{Payer Swap} = PV_{flt} - PV_{fix}$$

$$= \left[D_n(t) - D_N(t)\right] - \left[K \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)\right]$$

$$= \left[D_n(t) - D_N(t)\right] - \left[K P_{n+1,N}(t)\right]$$

$$= P_{n+1,N}(t) \left[\frac{D_n(t) - D_N(t)}{P_{n+1,N}(t)} - K\right]$$

option on swaprate using BS this formula tells what to make stochastic.

Pricing a Swaption

$$\frac{V(u)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V(T)}{P_{n+1,N}(T)}\right]$$

The PVBP is a portfolio of traded assets and has strictly positive value. It can therefore be used as a numeraire.

If we use $P_{n+1,N}(t)$ as a numeraire, then under the measure $\mathbb{Q}^{n+1,N}$ associated to the numeraire $P_{n+1,N}(t)$, all $P_{n+1,N}$ rebased values must be martingales in an arbitrage-free world.

In particular, the par swap rate $S_{n,N}$ must be a martingale under $\mathbb{Q}^{n+1,N}$. The swap market model makes the assumption that $S_{n,N}$ is a lognormal martingale under $\mathbb{Q}^{n+1,N}$. We write down the process

formula

$dL_i(t) = \sigma_i \times L_i(t) \times dW^{i+1}(t)$

$dF_t : \sigma F_t dW_t$

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}(t),$$

where $W^{n+1,N}(t)$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. ← use PVBP as numeraire

swap rate → martingale

risk neutral

A **swaption** (short for swap option) gives the right to enter at time T_n into a swap with fixed rate K. A **receiver swaption** gives the right to enter into a receiver swap, and a **payer swaption** gives the right to enter into a payer swap.

Pricing a Swaption

expiry × tenor
2 × 10

10 × 10

Swaptions are often denoted as $T_n \times (T_N - T_n)$, where T_n is the option expiry date (and also the start of the underlying swap), and $T_N - T_n$ is the tenor of the underlying swap.

The payoff of a payer swaption is given by

$$V_{n,N}^{\text{payer}}(T_n) = \underbrace{[P_{n+1,N}(T)(S_{n,N}(T) - K)]^+}_{\text{bond price never negative}} = P_{n+1,N}(T)\left[S_{n,N}(T) - K\right]^+$$

bond price never negative → so $P_{n+1,N}(T)$ never -ve so no need max operator

Using $P_{n+1,N}$ as a numeraire, we can value the payer swaption under the measure $\mathbb{Q}^{n+1,N}$ → martingale and get black formula (simplify payoff expected)

$$\frac{V_{n,N}^{\text{payer}}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_{n,N}^{\text{payer}}(T_n)}{P_{n+1,N}(T_n)}\right] = E^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\left[S_{n,N}(T) - K\right]^+}{\cancel{P_{n+1,N}(T)}}\right]$$

$$\Rightarrow \quad V_{n,N}^{\text{payer}}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+].$$

The remaining steps required to derive a formula for a swaption is identical to how we would handle a vanilla European option.

Pricing a Swaption

The swap rate follows the stochastic differential equation

$dF = \sigma \times F \times dW_t$

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}(t),$$

where $W^{n+1,N}(t)$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. The solution is given by

$F_T = F_0 \times e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}$

$$S_{n,N}(T) = S_{n,N}(0) e^{-\frac{1}{2}\sigma^2_{n,N}T + \sigma_{n,N}W^{n+1,N}(T)}.$$

formula

Evaluating the expectation, we obtain

$$\begin{aligned} V^{payer}_{n,N}(0) &= P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+] \\ &= P_{n+1,N}(0)[S_{n,N}(0)\Phi(d_1) - K\Phi(d_2)], \end{aligned}$$

where

$$d_1 = \frac{\log \frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma^2_{n,N}T}{\sigma_{n,N}\sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N}\sqrt{T}. \quad \triangleleft$$

Swaption Vols – ATM Vols

91) Asset 92) Actions 93) Settings Interest Rate Volatility Cube

USD | USD BVOL Cube (Default) | Bid | Date 10/29/15

1) Analyze Cube 2) Market Data

Swap Curve (23) US Dollar Index Tenor 3M Show Vol Black

View Strike ATM ← only ATM Discounting IBOR Show Strikes

Table Charts

tenor

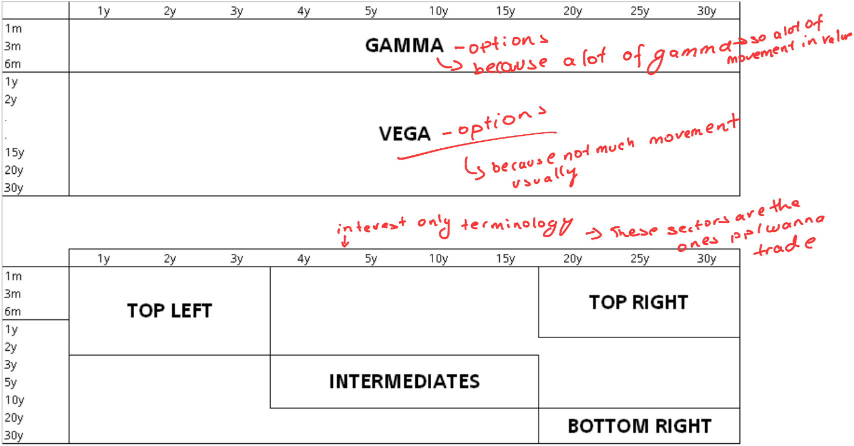
for 1y tenor option in 1m need to determine if enter option

expiring

implied vol

Expiry	1Yr	2Yr	3Yr	4Yr	5Yr	6Yr	7Yr	8Yr	9Yr	10Yr	12Yr	15Yr
1Mo	66.10	62.21	55.17	49.96	46.57	42.67	40.01	37.95	36.32	34.93	33.42	31.65
3Mo	70.23	63.36	57.30	52.53	48.96	44.96	42.20	40.23	38.56	37.12	35.62	33.86
6Mo	64.49	59.82	55.44	50.78	47.56	44.22	41.86	40.16	38.75	37.52	36.11	34.49
9Mo	61.24	56.74	52.60	48.62	45.77	43.00	40.94	39.50	38.25	37.09	35.75	34.21
1Yr	58.49	54.16	50.10	46.94	44.14	41.76	39.93	38.66	37.61	36.68	35.47	34.08
2Yr	52.60	48.17	44.87	42.30	40.20	38.71	37.53	36.65	35.84	35.19	34.21	32.99
3Yr	47.94	44.17	41.45	39.51	37.94	36.84	35.88	35.12	34.48	33.94	33.11	32.00
4Yr	43.43	40.52	38.55	37.09	35.81	34.97	34.30	33.72	33.27	32.92	32.17	31.12
5Yr	39.96	37.89	36.69	35.71	34.78	34.01	33.41	32.99	32.61	32.31	31.59	30.56
6Yr	37.44	36.00	35.03	34.21	33.40	32.80	32.32	31.97	31.66	31.40	30.73	29.77
7Yr	35.22	34.29	33.51	32.84	32.18	31.75	31.38	31.06	30.80	30.57	29.95	29.04
8Yr	33.81	32.87	32.23	31.74	31.25	30.91	30.60	30.33	30.09	29.90	29.32	28.48
9Yr	32.43	31.53	31.13	30.79	30.45	30.18	29.89	29.67	29.46	29.29	28.77	27.95
10Yr	31.21	30.41	30.19	29.98	29.75	29.52	29.26	29.07	28.88	28.72	28.24	27.45
12Yr	30.02	29.26	28.90	28.89	28.88	28.62	28.37	28.20	28.16	28.12	27.62	26.86
15Yr	28.25	27.56	27.43	27.33	27.36	27.26	27.19	27.20	27.23	27.23	26.76	25.99

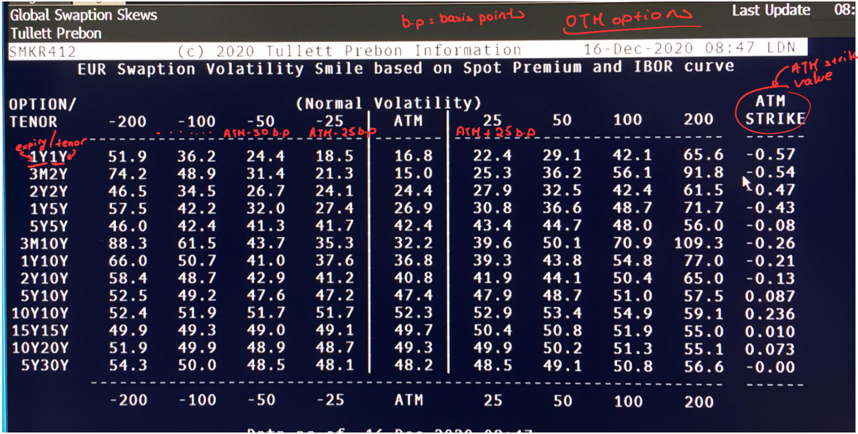
Swaption ATM Vols



Swaption ATM Vols

Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1) 1M	12.70	13.60	15.90	18.40	20.60	22.60	24.80	26.50	28.00
2) 2M	13.30	14.20	16.70	19.40	21.90	24.20	26.40	28.30	29.90
3) 3M	14.10	15.00	17.20	20.00	22.50	25.10	27.60	29.50	31.10
4) 6M	14.60	16.10	18.90	22.20	24.70	27.20	29.40	31.60	33.50
5) 9M	15.80	17.70	20.50	23.60	[illegible]	[illegible]	30.90	32.80	34.70
6) 1Y	16.80	19.00	22.10	24.90	27.50	30.00	32.20	34.10	35.80
7) 18Y	18.80	21.90	24.90	27.40	29.90	32.40	34.40	36.30	38.00
8) 2Y	21.70	24.50	27.80	30.40	32.40	34.40	36.60	38.20	39.90
9) 3Y	28.00	30.50	33.10	35.00	36.80	38.60	40.30	41.80	43.10
10) 4Y	33.40	35.50	37.50	39.10	40.40	41.60	43.00	44.20	45.50
11) 5Y	37.80	39.70	41.10	42.20	43.30	44.40	45.40	46.40	47.40
12) 6Y	41.90	43.00	43.90	45.20	45.70	46.70	47.50	48.40	49.10
13) 7Y	44.70	45.60	46.30	47.10	47.70	48.40	49.00	49.60	50.20
14) 10Y	49.70	49.80	50.30	50.80	51.10	51.40	51.70	51.80	52.10
15) 12Y	51.30	50.90	51.00	51.40	51.60	52.00	52.30	52.50	52.50
16) 15Y	51.70	51.50	51.90	52.00	52.00	52.10	52.20	52.10	52.40
17) 20Y	51.20	51.10	51.40	51.30	51.30	51.50	51.30	51.20	51.20
18) 25Y	50.30	50.30	50.50	50.40	50.30	50.10	49.90	49.30	49.30
19) 30Y	49.30	49.40	49.60	49.60	49.70	49.30	48.60	48.00	47.40

Swaption Vols – Smile/Skew



Global Swaption Skews
Tullett Prebon
Last Update 08:

SMKR412 (c) 2020 Tullett Prebon Information 16-Dec-2020 08:47 LDN

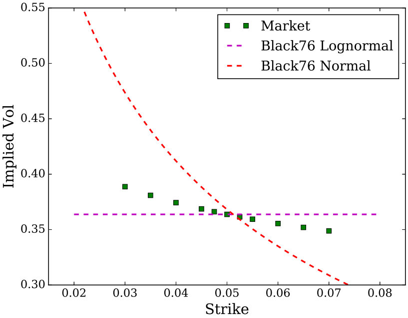
EUR Swaption Volatility Smile based on Spot Premium and IBOR curve

(Normal Volatility)

OPTION/ TENOR	-200	-100	-50	-25	ATM	25	50	100	200	ATM STRIKE
1Y1Y	51.9	36.2	24.4	18.5	16.8	22.4	29.1	42.1	65.6	-0.57
3M2Y	74.2	48.9	31.4	21.3	15.0	25.3	36.2	56.1	91.8	-0.54
2Y2Y	46.5	34.5	26.7	24.1	24.4	27.9	32.5	42.4	61.5	-0.47
1Y5Y	57.5	42.2	32.0	27.4	26.9	30.8	36.6	48.7	71.7	-0.43
5Y5Y	46.0	42.4	41.3	41.7	42.4	43.4	44.7	48.0	56.0	-0.08
3M10Y	88.3	61.5	43.7	35.3	32.2	39.6	50.1	70.9	109.3	-0.26
1Y10Y	66.0	50.7	41.0	37.6	36.8	39.3	43.8	54.8	77.0	-0.21
2Y10Y	58.4	48.7	42.9	41.2	40.8	41.9	44.1	50.4	65.0	-0.13
5Y10Y	52.5	49.2	47.6	47.2	47.4	47.9	48.7	51.0	57.5	0.087
10Y10Y	52.4	51.9	51.7	51.7	52.3	52.9	53.4	54.9	59.1	0.236
15Y15Y	49.9	49.3	49.0	49.1	49.7	50.4	50.8	51.9	55.0	0.010
10Y20Y	51.9	49.9	48.9	48.7	49.3	49.9	50.2	51.3	55.1	0.073
5Y30Y	54.3	50.0	48.5	48.1	48.2	48.5	49.1	50.8	56.6	-0.00
	-200	-100	-50	-25	ATM	25	50	100	200	

Swaption Vol Calibration

Suppose the implied volatility across strike for a given swaption maturity and tenor is given by the green markers in the following figure:



The at-the-money volatility is 0.36, and the forward swap rate is 0.05.

Extension to the Black Model

An immediate and straightforward extension is the Black Normal model:

$$dS_{n,N}(t) = \sigma_{n,N} dW^{n+1,N}(t).$$

This is an arithmetic Brownian motion.

If the implied volatility skew we observed in the market is between normal and lognormal, then we can make use of the displaced-diffusion (shifted lognormal) model:

$$dS_{n,N}(t) = \sigma_{n,N}[\beta S_{n,N}(t) + (1-\beta)S_{n,N}(0)]dW^{n+1,N}(t).$$

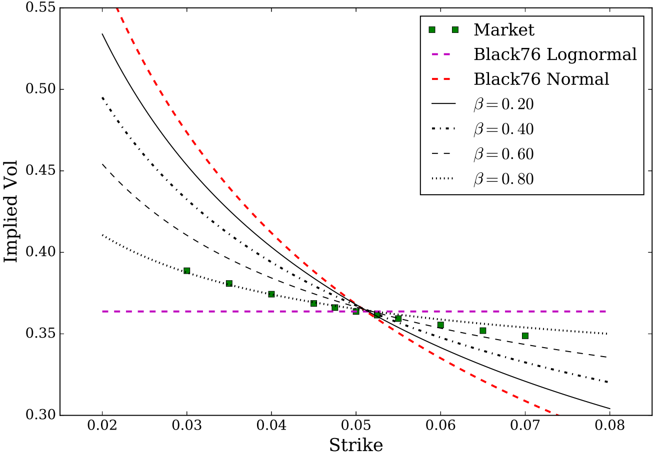
Recall that the solution is given by

$$S_{n,N}(T) = \frac{S_{n,N}(0)}{\beta} e^{\sigma_{n,N}\beta W^{n+1,N}(T) - \frac{\sigma^2_{n,N}\beta^2 T}{2}} - \frac{1-\beta}{\beta} S_{n,N}(0)$$

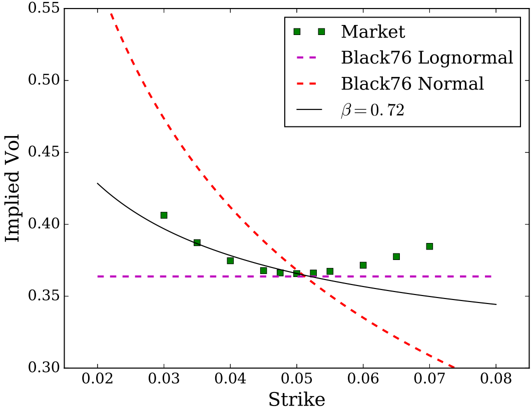
The swaption price under the displaced-diffusion model is

$$V_{n,N}(0) = P_{n+1,N}(0)\text{Black}\left(\frac{S_{n,N}(0)}{\beta},\ K + \frac{1-\beta}{\beta}S_{n,N}(0),\ \sigma\beta,\ T\right)$$

Swaption Vol Calibration – Displaced Diffusion

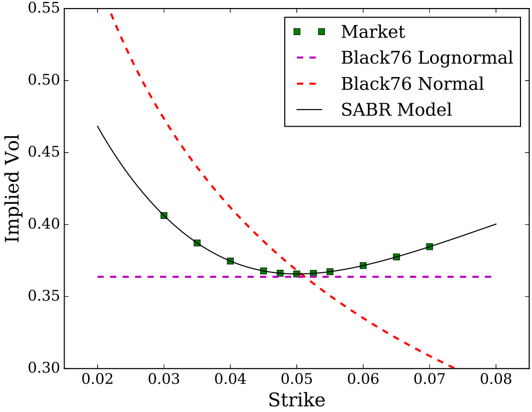


SABR Model



Displaced-diffusion model can only fit to implied volatility skew – there will be mismatch if the implied volatility surface also exhibit "smile" characteristic.

SABR Model



SABR model is able to fit both skew and smile in the implied volatility surface – this is the standard volatility model used in fixed-income market.