Sample Paper

1c. $D_0(1y, 2y) = \frac{D_0(0,2y)}{D_0(0,1y)} = \frac{\prod_{i=1}^{720} \frac{1}{(1+0.008\times\frac{1}{360})}}{\prod^{360} \frac{1}{1+0.008\times\frac{1}{360}}} = 0.9920$

$\bar{D}(1y,2y) = \frac{\bar{D}(0,2y)}{\bar{D}(0,1y)} = \frac{\frac{1}{(1+2\times 0.025)}}{\frac{1}{(1+0.020)}} = 0.9733$

$L(1y,2y) = \frac{\bar{D}(0,1y) - \bar{D}(0,2y)}{\Delta\,\bar{D}(0,2y)}$

formula sheet

b.

2a i $F_{x_T} = F_{x_0}\left(\frac{D_f(0,T)}{D_d(0,T)}\right)$ formula

ii $1.45 = \frac{1.5 \times 0.95}{D_d(0,T)}$

$D_d(0,T) = 0.9828$

b. $D(0,1) \times 104.5 = 101.5$

$D(0,1) = 0.971292$

$D(0,1) \times 4.5 + D(0,2y)\,105 = 102$ formula - Lecture 1

$D(0,2y) = \frac{102 - D(0,1)4.5}{105}$

$D(0,2y) = 0.929802$

$D(0,1) \times 4.5 + D(0,2y)\,105.5 = 102.46$

3. $\int_0^t dr_u = \int_0^t \mu\, du + \int_0^t \sigma\, dW_u^*$

$\therefore r_t - r_0 = \mu t + \int_0^t \sigma\, dW_u^*$

$\therefore r_t = r_0 + \mu t + \sigma W_t$

$\int_t^T r_u\, du = \int_t^T r_0\, du + \int_t^T \mu u\, du + \int_t^T \sigma W_u^*\, du$

$\int_t^T r_u\, du = r_0(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(W_u^* - W_t^*)\, du$

formula -

$$\mathbb{E}\left[\int_t^T r_u\,du\right] = r_0(T-t) + \frac{\mu}{2}(T-t)^2$$

$$V\left[\int_t^T r_u\,du\right] = \sigma^2 \frac{(T-t)^3}{3}$$

formula

$$\mathbb{E}^*\left[e^{-\int_t^T r_u\,du}\right] = e^{-r_0(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3}$$

b. $\therefore \; R(t,T) = r_0 + \frac{\mu}{2}(T-t) - \frac{\sigma^2}{6}(T-t)^2$

This is an affine short rate model as it is a linear func with a constant term.

4a. $\log\left(\frac{\alpha S_T}{\beta}\right)$

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T^*}$$

$$\therefore \log\left(\frac{\alpha S_T}{\beta}\right) = \log\left(\frac{\alpha}{\beta}\right) + \log(S_T)$$

$$= \log\left(\frac{\alpha}{\beta}\right) + \log\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T}\right) \qquad \sigma W_T \sim N(0, \sigma^2 T)$$

$$= \log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + (r - \tfrac{1}{2}\sigma^2)T + \sigma W_T$$

$$V_0 = e^{-rT}\,\mathbb{E}\left[\log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + (r-\tfrac{1}{2}\sigma^2)T + \underline{\sigma W_T}\right)$$

$$\therefore V_0 = e^{-rT}\left(\log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + (r - \tfrac{1}{2}\sigma^2)T\right)$$

6. $V_0 = e^{-rT}\int_0^\infty g(k) f(s)\, ds$ $\quad h''(k) = \frac{-1}{K^2}$ $\quad F = S_0 e^{rT}$

$$V_0 = e^{-rT} g(F) + \int_0^F g''(K) P(K)\, dK + \int_F^\infty g''(K) C(K)\, dK$$

← entire Replication in formula sheet

$$= e^{-rT}\log\left(\frac{\alpha F}{\beta}\right) - \int_0^F \frac{1}{K^2} P(K)\, dK - \int_F^\infty \frac{1}{K^2} C(K)\, dK$$

$$= e^{-rT}\left(\log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + rT\right) - \int_0^F \frac{P(K)}{K^2}\, dK - \int_F^\infty \frac{C(K)\, dK}{K^2}$$

5. if $d\frac{1}{Y_t} = \mu \frac{1}{X_t} dt + \sigma \frac{1}{X_t} dW_t$ $\quad$ Let $dX_t = \mu X_t dt + \sigma X_t dW_t$

Let $Y_t = \frac{B_t^D}{X_t}$

$$\frac{\partial Y}{\partial B_t^D} = \frac{1}{X_t} \qquad \frac{\partial Y}{\partial X_t} = \frac{-B_t^D}{X_t^2} \qquad \frac{\partial^2 Y}{\partial X_t^2} = \frac{2B_t^D}{X_t^3}$$

$$dY_t = \frac{\partial Y}{\partial B} dB_t + \frac{\partial Y}{\partial X} dX_t + \frac{1}{2}\frac{\partial^2 Y}{\partial X_t^2}(dX_t)^2$$

$$= r^D Y_t dt - Y_t(\mu dt + \sigma dW_t) + Y_t \sigma^2 dt$$

$$dY_t = (r^D - \mu + \sigma^2) dt + \sigma dW_t$$

Let $Z_t = \frac{Y_t}{B_t^F}$

$$\therefore \frac{\partial Z}{\partial Y} = \frac{1}{B_t^F} \qquad \frac{\partial Z}{\partial B} = \frac{-Y_t}{{B_t^F}^2}$$

$$\therefore dZ_t = \frac{\partial Z}{\partial Y} dY_t + \frac{\partial Z}{\partial B} dB_t$$

$$= Z_t((r^D - \mu + \sigma^2) dt + \sigma dW_t) - Z_t r^F dt$$

$$= Z_t\left((r^D - \mu + \sigma^2 - r^F) dt + \sigma dW_t\right)$$

$$d\frac{1}{X_t} = \mu\frac{1}{X_t}dt + \sigma\frac{1}{X_t}dW_t$$

$\mu = 0$ for martingale

$\therefore \mu = r^f - r^D - \sigma^2$

$\therefore dW_t = dW_t^F - \sigma dt$

$$\therefore d\frac{1}{X_t} = (r^f - r^D - \sigma^2)dt + \sigma(dW_t^F - \sigma d_t)$$

$$d\frac{1}{X_t} = (r^F - r^D)dt + \sigma dW_t^F$$

6a. $L_i(t) = L_i(0)e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^{i+1}}$ × $dL_i(t) = \sigma_i L_i(t)dW_t^{i+1}$ Market model

$L_t(T_i, T_{i+1}) = \frac{1}{\Delta_i}\frac{D(t, T_i) - D(t, T_{i+1})}{D(t, T_{i+1})}$ formula sheet

b. Discount factor $D_{i+1}(t, T_{i+1})$ ✓

Here K = 5%.

c. $V_0 = D_{i+1}(0)\,\mathbb{E}^{i+1}[100(\mathbb{1}_{L_i(T)>5\%})$

$$= D_{i+1}(0)\,100\left(\int_{x^*}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\right)$$ ← formula

$$= D_{i+1}(0)\,100(\Phi(-x^*))$$

$$= D_{i+1}(0)\,100\Phi\left(\frac{\ln\left(\frac{L_i(0)}{5\%}\right) - \frac{1}{2}\sigma_i^2 T}{\sigma\sqrt{T}}\right)$$

7. $\mathbb{E}[dW_t^{i+1} dW_t^F] = \rho dt$

$$\mathbb{E}^{i+1,D}[L_i^F(T)] = \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{dQ^{i+1,D}}{dQ^{i+1,F}}\right]$$

$$= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{D_{i+1}^D(T_{i+1})}{D_{i+1}^D(0)}}{\frac{X_T D_{i+1}^F(T_{i+1})}{X_0 D_{i+1}^F(T_{i+1})}}\right]$$

Formula Sheet

$$= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}}\right]$$

$$= \mathbb{E}^{i+1,F}\left[L_i^F(T) \frac{\cancel{\frac{1}{F_0}} e^{-\frac{\sigma_x^2 T}{2} + \sigma W_T^F}}{\cancel{1/F_0}}\right]$$

$$= \mathbb{E}^{i+1,F}\left[L_i^F(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma W_T^{i+1}} e^{-\frac{\sigma_x^2 T}{2} + \sigma W_T^F}\right]$$

$$= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} \mathbb{E}^{i+1,F}\left[e^{\sigma_i Z_T^{(1)} + \sigma_x(\rho Z_T^{(1)} + \sqrt{1-\rho^2} Z_T^{(2)})}\right]$$

$$= L_i^F(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_x^2 T}{2}} \mathbb{E}^{i+1,F}\left[e^{(\sigma_i + \sigma_x \rho) Z_T^{(1)} + \sqrt{1-\rho^2} Z_T^{(2)}}\right]$$

$$= L_i^F(0) \cancel{e^{-\frac{\sigma_i^2 T}{2}}} \cancel{e^{-\frac{\sigma_x^2 T}{2}}} \cancel{e^{\frac{\sigma_i^2 T}{2}}} e^{\frac{2\sigma_i \sigma_x \rho T}{2}} \cancel{e^{\frac{\sigma_x^2 \rho^2 T}{2}}} \cancel{e^{\frac{\sigma_x^2 T}{2}}} \cancel{e^{-\frac{\sigma_x^2 \rho^2 T}{2}}}$$

$$= L_i^F(0) e^{\sigma_i \sigma_x \rho T}$$

8. $L_i(T) = L_i(0) e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}$

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}\left[\frac{V_T}{\cancel{D_{i+1}(T)} 1}\right]$$

$$V_0 = D_{i+1}(0)\, \mathbb{E}[L_i(T)^2]$$

$$= D_{i+1}(0)\, \mathbb{E}[L_i^2(0) e^{-\sigma_i^2 T + 2\sigma_i W_T^{i+1}}]$$

$$= D_{i+1}(0) L_i^2(0) e^{-\sigma_i^2 T + 2\sigma_i^2 T}$$

$$= D_{i+1}(0) L_i^2(0) e^{\sigma_i^2 T}$$

b. $V_0 = D(0,T)\, L_i(0)^2 + \int_0^{L_i(0)} h''(K)\, V^f(K)\, dK + \int_{L_i(0)}^{\infty} h''(K)\, V^c(K)\, dK$

Lec5 → slide 16, 17, 18, 19 → formula

Let $h(F) = L_i^2(0)$ $\qquad h'(F) = 2L_i(0)$ $\qquad h''(F) = 2$

$F = L_i(0)$

$$V_0 = D(0,T)\, h(F) + \int_0^F h''(K)\, V^f(K)\, dK + \int_F^{\infty} h''(K)\, V^c(K)\, dK$$

$$= D(0,T)\, L_i(0)^2 + \int_0^{L_i(0)} 2\, V^f(K)\, dK + \int_{L_i(0)}^{\infty} 2\, V^c(K)\, dK$$

Stochastic Lec 8 → slide 17, 18, 19.

a. $\int_0^t dr_s = \int_0^t \theta(s)\, ds + \int_0^t \sigma\, dW_s^*$

$$r_t = r_0 + \int_0^t \theta(s)\, ds + \int_0^t \sigma\, dW_s^*$$

$$\int_t^T r_u\, du = \int_t^T r_0\, du + \int_t^T \int_0^t \theta(s)\, ds\, du + \int_t^T \int_0^t \sigma\, dW_s^*\, du$$

$$= r_0(T-t) + \int_t^T \int_s^T \theta(s)\, du\, ds + \int_t^T \int_s^T \sigma\, du\, dW_s^*$$

$$= r_0(T-t) + \int_t^T \theta(s)(T-s)\, ds + \int_t^T \sigma(T-s)\, dW_s^*$$

$$\mathbb{E}\left[\int_t^T r_u\, du\right] = r_0(T-t) + \int_t^T \theta(s)(T-s)\, ds$$

$$V\left[\int_t^T r_u\, du\right] = \int_t^T \sigma^2 (T-s)^2\, ds = \left[\frac{\sigma^2 (T-s)^3}{3}\right]_t^T$$

$$= \frac{\sigma^2 (T-t)^3}{3}$$

$$\int_t^T r_u\,du \sim N\left(r_0(T-t) + \int_t^T \theta(s)(T-s)\,ds,\ \frac{\sigma^2}{3}(T-t)^3\right)$$

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u\,du}\right] = e^{-r_0(T-t) - \int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2}{6}(T-t)^3}$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2}{6}(T-t)^3$$

if got space

$$B(t,T) = (T-t)$$

session 4 slide 15 ← formula

10a. θ → is short rate , r_t → mean rate

if $\theta > r_t$ then it drifts upwards towards mean

if $\theta < r_t$ then it drifts downwards towards mean. if $\theta = r_t$ no drift.

K is speed of mean reversion

formula

b. $r_t = r_0 e^{-Kt} + \theta(1-e^{-Kt}) + \sigma\int_0^t e^{K(u-T)}\,dW_u^*$

$$\lim_{t\to\infty} \mathbb{E}(r_t) = r_0\underbrace{e^{-Kt}}_{\to 0} + \theta(1-\underbrace{e^{-Kt}}_{\to 0})$$

$$= \theta$$

$$\lim_{t\to\infty} V(r_t) = \frac{\sigma^2}{2K}(1-\underbrace{e^{-2Kt}}_{\to 0})$$

$$= \frac{\sigma^2}{2K}$$

formula