Additional Example 6

1 $\quad dX_t \doteq (r^D - r^F) X_t\, dt + \sigma X_t\, dW_t^D$

a. Ito's lemma. this looks like $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$

$$\therefore S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$$

$$\rightarrow X_T = X_0 e^{(r^D - r^F - \frac{1}{2}\sigma^2)T + \sigma W_T^D}$$

$$\therefore \mathbb{E}^D[X_T] = \mathbb{E}\left[X_0 e^{(r^D - r^F - \frac{1}{2}\sigma^2)T + \sigma W_T^D}\right]$$

$$= X_0 e^{(r^D - r^F)T - \cancel{\frac{1}{2}\sigma^2 T} + \cancel{\frac{1}{2}\sigma^2 T}}$$

$$= X_0 e^{(r^D - r^F)T}$$

b. $dX_t = (r^D - r^F) X_t\, dt + \sigma X_t\, dW_t^D$ $\qquad$ Let $Y_t = \dfrac{1}{X_t} = f(x)$

$$f(x) = \frac{1}{X_t} \qquad f'(x) = -\frac{1}{X_t^2} \qquad f''(x) = \frac{2}{X_t^3}$$

formula

$$dY_t = f'(x)\, dX_t + \frac{1}{2} f''(x) (dX_t)^2$$

$$= -\frac{1}{X_t^2}\left((r^D - r^F) X_t\, dt + \sigma X_t\, dW_t^D\right) + \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{X_t^3}\, X_t} \times \sigma^2 \cancel{X_t^2}\, dt$$

$$= -\frac{1}{X_t}\left[(r^D - r^F) dt + \sigma\, dW_t^D\right] + \frac{\sigma^2 dt}{X_t}$$

$$= \frac{r^F - r^D}{X_t} dt - \frac{\sigma\, dW_t^D}{X_t} + \frac{\sigma^2 dt}{X_t}$$

$$d\frac{1}{X_t} = \frac{1}{X_t}(r^F - r^D + \sigma^2) dt - \sigma \frac{1}{X_t}\, dW_t^D$$

formula

I want to remove this $\sigma^2 dt$
so need to cancel it out.

$$\therefore \; dW_t^D = dW_t^F + \sigma dt \leftarrow \text{Girsanov's formula}$$

Can I do it this way?

$$\therefore \; d\frac{1}{X_t} = \frac{1}{X_t}(r^F - r^D + \sigma^2)dt - \sigma\frac{1}{X_t}[dW_t^F + \sigma dt]$$

$$= \frac{1}{X_t}(r^F - r^D)dt + \cancel{\frac{1}{X_t}\sigma^2 dt} - \cancel{\frac{\sigma^2 dt}{X_t}} - \sigma\frac{1}{X_t}dW_t^F$$

$$d\frac{1}{X_t} = \frac{1}{X_t}(r^F - r^D)dt - \sigma\frac{1}{X_t}dW_t^F$$

formula

Let's take $dX_t = \mu X_t dt + \sigma X_t dW_t$

Let $Y_t = X_t B_t^F = f(X_t, B_t^F)$ $\qquad dB_t^F = r^F B_t^F dt$

$f(x,b) = xb$; $f_x = b$, $f_{xx} = 0$, $f_b = X$

$$dY_t = f_b(X_t, B_t^F)dB_t^F + f_x(X_t, B_t^F)dX_t + \frac{1}{2}f_{xx}(X_t, B_t^F)dX_t^2$$

$$= X_t r^F B_t^F dt + B_t^F(\mu X_t dt + \sigma X_t dW_t) + 0$$

$$= Y_t(r^F dt) + Y_t(\mu dt + \sigma dW_t)$$

$$= (r^F + \mu)Y_t dt + \sigma Y_t dW_t$$

Let $Z_t = \frac{Y_t}{B_t^D} = g(Y_t, B_t^D)$ $\quad dB_t^D = r^D B_t^D dt$

$$g(y,b) = \frac{y}{b} \; ; \; g_b = \frac{-y}{b^2} \; , \; g_y = \frac{1}{b} \; , \; g_{yy} = 0$$

$$dZ_t = g_b(Y_t, B_t^D)\, dB_t^D + g_Y(Y_t, B_t^D)\, dY_t + \frac{1}{2} g_{YY}(Y_t, B_t^D)(dY_t)^2$$

$$= \frac{-Y_t}{\cancel{B_t^{D2}} B_t^D} r^D \cancel{B_t^D} dt + \frac{1}{B_t^D}\left((r^F+\mu) Y_t dt + \sigma Y_t dW_t\right) + 0$$

$$= -Z_t r^D dt + Z_t (r^F + \mu) dt + Z_t \sigma dW_t$$

$$dZ_t = (r^F - r^D + \mu) dt + \sigma Z_t dW_t$$

$$dZ_t = \sigma Z_t \left(dW_t + \frac{r^F - r^D + \mu}{\sigma} dt \right)$$

$$dZ_t = \sigma Z_t dW_t^F$$

$$dX_t = \mu X_t dt + \sigma X_t \left(dW_t^F - \frac{r^F - r^D + \mu}{\sigma} dt \right)$$

$$= \cancel{\mu X_t dt} + \sigma^2 X_t dW_t^F - (r^F - r^D) X_t dt - \cancel{\mu X_t dt}$$

$$dX_t = (r^D - r^F) dt + \sigma^2 X_t dW_t^F$$

$$S_t = \frac{1}{X_t}$$

c. $d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^f$

$$\frac{1}{X_T} = \frac{1}{X_0}e^{(r^f - r^D - \frac{1}{2}\sigma^2)T - \sigma W_T^f}$$

$$\mathbb{E}^F\left[\frac{1}{X_T}\right] = \mathbb{E}^f\left[\frac{1}{X_0}e^{(r^f - r^D - \frac{1}{2}\sigma^2)T - \sigma W_T^f}\right]$$

$$= \frac{1}{X_0}e^{(r^F - r^D \cancel{-\frac{1}{2}\sigma^2})T \cancel{+\frac{\sigma^2}{2}T}}$$

$$= \frac{1}{X_0}e^{(r^F - r^D)T}$$

For foreign investors r^F is domestic rates and r^D is foreign.

2a. $dL_i(t) = \sigma_i L_i(t)\,dW^{i+1}(t)$

$L_i(t) = L_i(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}(T)}$

$\mathbb{E}^i[L_i(T_i)]$ is directly not possible as the brownian motion is not a martingale in the i measure.

$$\therefore \mathbb{E}^i[L_i(T_i)] = \mathbb{E}^{i+1}\left[L_i(T_i)\cdot\frac{dQ^i}{dQ^{i+1}}\right]$$

$$= \mathbb{E}^{i+1}\left[L_i(T_i)\cdot\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}\right]$$

$$L_i(T) = \frac{1}{\Delta_i}\cdot\frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

$$1 + \Delta_i L_i(T) = \frac{D_i(t)}{D_{i+1}(t)}$$

$$= \mathbb{E}^{i+1}\left[L_i(T_i)\cdot\frac{D_i(T)/D_{i+1}(T)}{D_i(0)/D_{i+1}(0)}\right]$$

$$= \mathbb{E}^{i+1}\left[L_i(T_i) \times \frac{1+\Delta_i L_i(T)}{1+\Delta_i L_i(0)}\right]$$

$$= \mathbb{E}^{i+1}\left[\frac{L_i(T_i) + \Delta_i L_i(T_i)^2}{1+\Delta_i L_i(0)}\right]$$

$$= \frac{1}{1+\Delta_i L_i(0)} \times \left(\mathbb{E}^{i+1}\left[L_i(T_i)\right] + \Delta_i \mathbb{E}^{i+1}\left[L_i(T_i)^2\right]\right)$$

$$= \frac{1}{1+\Delta_i L_i(0)} \times \left(L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_1}\right)$$

$$= \frac{L_i(0)\left(1+\Delta_i L_i(0) e^{\sigma_i^2 T_i}\right)}{1+\Delta_i L_i(0)}$$

formula shee

b. $V_0 = D_i(0)\,\mathbb{E}^i\left[(L_i(T)-K)^+\right]$

$$= D_i(0)\,\mathbb{E}^{i+1}\left[\frac{dQ^i}{dQ^{i+1}} \times (L_i(T_i)-K)^+\right]$$

$$= D_i(0)\,\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)} \times (L_i(T_i)-K)^+\right]$$

$$= D_i(0)\,\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_{i+1}(T)}{D_i(0)/D_{i+1}(0)} \times (L_i(T_i)-K)^+\right] \qquad \frac{D_i(T)}{D_{i+1}(T)} \times \frac{D_{i+1}(0)}{D_i(0)}$$

$$= \cancel{D_i(0)} \times \frac{D_{i+1}(0)}{\cancel{D_i(0)}}\,\mathbb{E}^{i+1}\left[(1+\Delta_i L_i(T))(L_i(T_i)-K)^+\right]$$

They are not independent so cannot be split like that

why do I go wrong here

$$= D_{i+1}(0)\,\mathbb{E}^{i+1}\left[1+\Delta_i L_i(T_i)\right]\mathbb{E}\left[(L_i(T_i)-K)^+\right]$$

forward price

$$= D_{i+1}(0)\left(1+\Delta_i L_i(0)\right)\cdot\left(L_i(0)\Phi(d_1) - K\Phi(d_2)\right)$$

$$= D_{i+1}(0)L_i(0)\Phi(d_1) - D_{i+1}(0)K\Phi(d_2) + \Delta_i D_{i+1}(0)L_i(0)^2\Phi(d_1) - \Delta_i D_{i+1}(0)L_i(0)K\Phi(d_2)$$

$$= D_{i+1}(0)\left[L_i(0)\Phi(d_1) - K\Phi(d_2) + \Delta_i D_{i+1}(0)\left(L_i(0)^2\Phi(d_1) - L_i(0)K\Phi(d_2)\right)\right]$$

$$d_1 = \frac{\log\left(\frac{L_i(0)}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\log\left(\frac{L_i(0)}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

(b) The LIBOR-in-arrear caplet contract can be valued as

Redo this

$$
\begin{aligned}
V_0 &= D_i(0)\mathbb{E}^i[(L_i(T)-K)^+] \\
&= D_i(0)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}(L_i(T)-K)^+\right] \\
&= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}(L_i(T)-K)^+\right] \\
&= D_{i+1}(0)\mathbb{E}^{i+1}\left[(1+\Delta_i L_i(T))\cdot(L_i(T)-K)^+\right] \\
&= D_{i+1}(0)\left\{\mathbb{E}^{i+1}[(L_i(T)-K)^+] + \Delta_i\mathbb{E}^{i+1}[L_i(T)(L_i(T)-K)^+]\right\} \\
&= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right] \\
&\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2e^{\sigma_i^2T}\Phi\left(-x^*+2\sigma_i\sqrt{T}\right) - L_i(0)K\Phi\left(-x^*+\sigma_i\sqrt{T}\right)\right] \\
&= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right] \\
&\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2e^{\sigma_i^2T}\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{3\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right. \\
&\quad \left. - L_i(0)K\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right] \quad \triangleleft
\end{aligned}
$$

shortcut to solve this

3a. $\Delta_i L_i(T_{i+1})$

$$L_i(T) = L_i(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}(T)}$$

$$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{\Delta_i L_i(T)}{\cancel{D_{i+1}(T)}\, 1}\right]$$

$$V_0 = D_{i+1}(0)\,\Delta_i\,\mathbb{E}^{i+1}[L_i(T)]$$

$$= D_{i+1}(0)\,\Delta_i L_i(0)$$

if got space

b. $$\frac{V_0}{D(0,T_i)} = \mathbb{E}^i\left[\frac{\Delta_i L_i(T)}{\cancel{D(T_i,T_i)}\, 1}\right]$$

$$V_0 = D(0,T_i)\,\Delta_i\,\mathbb{E}^i\left[L_i(T)\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}\right]$$

calculated earlier

$$= D(0,T_i)\,\Delta_i\left(\frac{L_i(0)\left(1+\Delta_i L_i(0)e^{\sigma_i^2 T_i}\right)}{1+\Delta_i L_i(0)}\right)$$

4. $L_i^D(T_i) \to L_i^D(T_{i+1})$ Vol σ_i

$$dF_t = \sigma_x F_t \, dW_t^D$$

$$F_T = F_0 e^{-\frac{\sigma_x^2 T}{2} + \sigma_x W_T^D}$$

$$\frac{\frac{\frac{1}{X_T} N_T^D}{\frac{1}{X_0} N_0^D}}{\frac{M_T^F}{M_0^F}}$$

$$\mathrm{Corr}(W_t^D, W_t^{i+1}) = \rho$$

$$\mathbb{E}^{i+1,F}[L_i^D(T)] = ?$$

$$\frac{\frac{1}{X_T} N_T^D}{\frac{1}{X_0} N_0^D} \times \frac{M_0^F}{M_T^F} = \frac{1}{X_T}\frac{N_T^D}{M_T^F} \times \frac{M_0^F}{\frac{1}{X_0} N_0^D} = \frac{N_T^D}{X_T M_T^F} \times \frac{X_0 M^F}{N_0^D}$$

$$\mathbb{E}^{i+1,F}[L_i^F(T)] = \mathbb{E}^{i+1,D}\left[L_i^D(T) \cdot \frac{dQ^{i+1,F}}{dQ^{i+1,D}}\right]$$

$$= \mathbb{E}^{i+1,D}\left[L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1,D}} \times \frac{\frac{\frac{1}{X_T} D_{i+1}^D(T_{i+1})}{\frac{1}{X_0} D_{i+1}^D(0)}}{\frac{D_{i+1}^F(T_i+1)}{D_{i+1}^F(0)}}\right]$$

$$= \mathbb{E}^{i+1,D}\left[L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1,D}} \times \frac{D_{i+1}^D(T_{i+1})}{X_T D_{i+1}^F(T_{i+1})} \times \frac{X_0 D_{i+1}^F(0)}{D_{i+1}^D(0)}\right]$$

$$= \mathbb{E}^{i+1,D}\left[L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1,D}} \times \frac{1}{F_{T+1}} \times F_0\right]$$

$$= \mathbb{E}^{i+1,D}\left(L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1,D}} \frac{\times \cancel{F_0}}{\cancel{F_0} e^{-\frac{\sigma_x^2 T}{2} + \sigma_i W_T^D}}\right)$$

$$= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} - \frac{\sigma_x^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{\sigma_i W_T^{i+1,D} + \sigma_x W_T^D}\right]$$

$$= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} - \frac{\sigma_x^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{\sigma_i Z_T^{(1)} + \sigma_x(\rho Z_T^{(1)} + \sqrt{1-\rho^2} Z_T^{(2)})}\right]$$

$$= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} - \frac{\sigma_x^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{(\sigma_i + \sigma_x \rho) Z_T^{(1)}}\right] \mathbb{E}^{i+1,D}\left[e^{\sigma_x \sqrt{1-\rho^2} Z_T^{(2)}}\right]$$

$$= L_i^D(0) e^{\cancel{-\frac{\sigma_i^2 T}{2}} \cancel{-\frac{\sigma_x^2 T}{2}}} \cdot e^{\cancel{\frac{\sigma_i^2 T}{2}} + \frac{\cancel{2}\sigma_i \sigma_x \rho T}{\cancel{2}} + \cancel{\frac{\sigma_x^2 \rho^2 T}{2}}} \cdot e^{\cancel{\frac{\sigma_x^2}{2}} T - \cancel{\frac{\sigma_x^2 \rho^2}{2}} T}$$

$$E^{i+1,F}[L_i^{D}(T)] = L_i^{D}(0)e^{\sigma_i \sigma_x \rho T}$$

5. $dw_t^f \, dw_t^g = \rho_{fg} dt$

$dw_t^g \, dw_t^n = \rho_{gn} dt$

$dw_t^f \, dw_t^n = \rho_{fn} dt$

$$dw_t^f = \alpha_{11} dZ_t^{(1)}$$
$$dw_t^g = \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}$$
$$dw_t^n = \alpha_{13} dZ_t^{(1)} + \alpha_{23} dZ_t^{(2)} + \alpha_{33} dZ_t^{(3)}$$

$$V[dw_t^f] = V[\alpha_{11} dZ_t^{(1)}]$$
$$dt = \ = \alpha_{11}^2 dt$$

$\therefore \alpha_{11} = 1$

$Z_t^{(1)}$ and $Z_t^{(2)}$ are independent so Cov 0.

$$\mathrm{Cov}[dw_t^f, dw_t^g] = \mathrm{Cov}[\alpha_{11} dZ_t^{(1)}, \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}]$$
$$\rho_{fg} dt = \mathrm{Cov}[\alpha_{11} dZ_t^{(1)}, \alpha_{12} dZ_t^{(1)}] + \underbrace{\cancel{\mathrm{Cov}[\alpha_{11} dZ_t^{(1)}, \alpha_{22} dZ_t^{(2)}]}}_{0}$$

$$\rho_{fg} \cancel{dt} = \alpha_{11}\alpha_{12} \cancel{dt}$$

$\alpha_{12} = \rho_{fg}$

$$V[dw^g] = V[\alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}]$$
$$dt = \alpha_{12}^2 dt + \alpha_{22}^2 dt$$
$$1 = \rho_{fg}^2 + \alpha_{22}^2 dt$$

$\therefore \alpha_{22} dt = \sqrt{1-\rho_{fg}^2}$

$$\operatorname{Cov}(dW_t^g, dW_t^h) = \operatorname{Cov}[\alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}, \alpha_{13} dZ_t^{(1)} + \alpha_{23} dZ_t^{(2)} + \alpha_{33} dZ_t^{(3)}]$$

$$\rho_{gh} dt = \operatorname{Cov}(\alpha_{12} dZ_t^{(1)}, \alpha_{13} dZ_t^{(1)}) + \underbrace{\operatorname{Cov}[\alpha_{12} dZ_t^{(1)}, \alpha_{23} dZ_t^{(2)}]}_{\to 0}$$

$$+ \underbrace{\operatorname{Cov}[\alpha_{12} dZ_t^{(1)}, \alpha_{33} dZ_t^{(3)}]}_{\to 0} + \underbrace{\operatorname{Cov}(\alpha_{22} dZ_t^{(2)}, \alpha_{13} dZ_t^{(1)})}_{\to 0}$$

$$+ \operatorname{Cov}[\alpha_{22} dZ_t^{(2)}, \alpha_{23} dZ_t^{(2)}] + \underbrace{\operatorname{Cov}(\alpha_{22} dZ_t^{(2)}, \alpha_{33} dZ_t^{(3)})}_{\to 0}$$

$$\rho_{gh} dt = \alpha_{12}\alpha_{13} dt + \alpha_{22}\alpha_{23} dt$$

$$\rho_{gh} = \rho_{fg}\alpha_{13} + \sqrt{1-\rho_{fg}^2}\,\alpha_{23}$$

$$\operatorname{Cov}(dW_t^f, dW_t^h) = \operatorname{Cov}(\alpha_{11} dZ_t^{(1)}, \alpha_{13} dZ_t^{(1)} + \alpha_{23} dZ_t^{(2)} + \alpha_{33} dZ_t^{(3)})$$

$$\rho_{fh} dt = \operatorname{Cov}(\alpha_{11} dZ_t^{(1)}, \alpha_{13} dZ_t^{(1)}) + \underbrace{\operatorname{Cov}(\alpha_{11} dZ_t^{(1)}, \alpha_{23} dZ_t^{(2)})}_{\to 0}$$
$$+ \underbrace{\operatorname{Cov}(\alpha_{11} dZ_t^{(1)}, \alpha_{33} dZ_t^{(3)})}_{0}$$

$$\rho_{fh} = 1\,\alpha_{13}$$

$$\alpha_{13} = \rho_{fh}$$

$$\therefore \rho_{gh} = \rho_{fg}\rho_{fh} + \sqrt{1-\rho_{fg}^2}\,\alpha_{23}$$

$$\therefore \alpha_{23} = \frac{\rho_{gh} - \rho_{fg}\rho_{fh}}{\sqrt{1-\rho_{fg}^2}} = \frac{\rho_{gh} - \alpha_{12}\alpha_{13}}{\alpha_{22}}$$

$$V(dW_t^h) = V[\alpha_{13} dZ_t^{(1)} + \alpha_{23} dZ_t^{(2)} + \alpha_{33} dZ_t^{(3)}]$$
$$dt = \alpha_{13}^2 dt + \alpha_{23}^2 dt + \alpha_{33}^2 dt$$

$$\alpha_{33} = \sqrt{1-\alpha_{13}^2-\alpha_{23}^2}$$