Session 5
Constant Maturity Swap Payoffs

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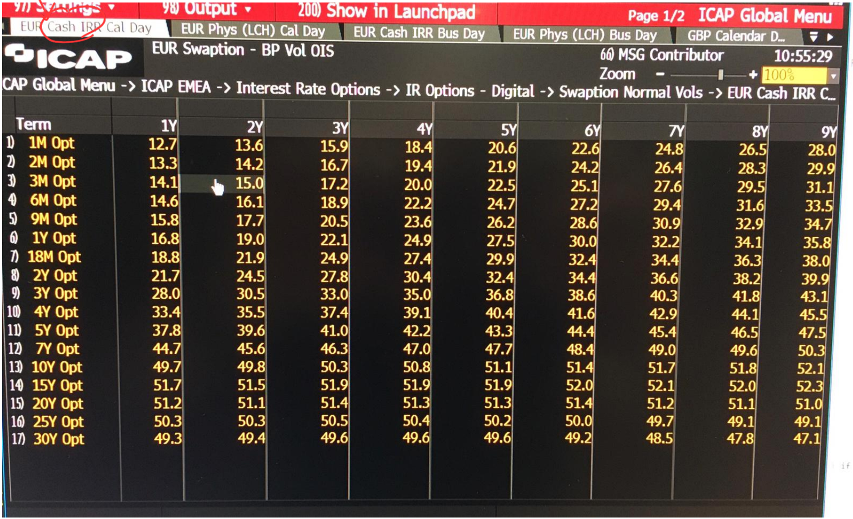
QF605 Fixed Income Securities

Swap-Settled Swaptions

Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1) 1M	12.70	13.60	15.90	18.40	20.60	22.60	24.80	26.50	28.00
2) 2M	13.30	14.20	16.70	19.40	21.90	24.20	26.40	28.30	29.90
3) 3M	14.10	15.00	17.20	20.00	22.50	25.10	27.60	29.50	31.10
4) 6M	14.60	16.10	18.90	22.20	24.70	27.20	29.40	31.60	33.50
5) 9M	15.80	17.70	20.50	23.60	26.20	28.60	30.90	32.80	34.70
6) 1Y	16.80	19.00	22.10	24.90	27.50	30.00	32.20	34.10	35.80
7) 18Y	18.80	21.90	24.90	27.40	29.90	32.40	34.40	36.30	38.00
8) 2Y	21.70	24.50	27.80	30.40	32.40	34.40	36.60	38.20	39.90
9) 3Y	28.00	30.50	33.10	35.00	36.80	38.60	40.30	41.80	43.10
10) 4Y	33.40	35.50	37.50	39.10	40.40	41.60	43.00	44.20	45.50
11) 5Y	37.80	39.70	41.10	42.20	43.30	44.40	45.40	46.40	47.40
12) 6Y	41.90	43.00	43.90	45.20	45.70	46.70	47.50	48.40	49.10
13) 7Y	44.70	45.60	46.30	47.10	47.70	48.40	49.00	49.60	50.20
14) 10Y	49.70	49.80	50.30	50.80	51.10	51.40	51.70	51.80	52.10
15) 12Y	51.30	50.90	51.00	51.40	51.60	52.00	52.30	52.50	52.50
16) 15Y	51.70	51.50	51.90	52.00	52.00	52.10	52.20	52.10	52.40
17) 20Y	51.20	51.10	51.40	51.30	51.30	51.50	51.30	51.20	51.20
18) 25Y	50.30	50.30	50.50	50.40	50.30	50.10	49.90	49.30	49.30
19) 30Y	49.30	49.40	49.60	49.60	49.70	49.30	48.60	48.00	47.40

IRR-Settled Swaptions

cash settled (IRR: Internal Rate of return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ICAP EUR Swaption - BP Vol OIS 60) MSG Contributor 10:55:29
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Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1) 1M Opt	12.7	13.6	15.9	18.4	20.6	22.6	24.8	26.5	28.0
2) 2M Opt	13.3	14.2	16.7	19.4	21.9	24.2	26.4	28.3	29.9
3) 3M Opt	14.1	15.0	17.2	20.0	22.5	25.1	27.6	29.5	31.1
4) 6M Opt	14.6	16.1	18.9	22.2	24.7	27.2	29.4	31.6	33.5
5) 9M Opt	15.8	17.7	20.5	23.6	26.2	28.6	30.9	32.9	34.7
6) 1Y Opt	16.8	19.0	22.1	24.9	27.5	30.0	32.2	34.1	35.8
7) 18M Opt	18.8	21.9	24.9	27.4	29.9	32.4	34.4	36.3	38.0
8) 2Y Opt	21.7	24.5	27.8	30.4	32.4	34.4	36.6	38.2	39.9
9) 3Y Opt	28.0	30.5	33.0	35.0	36.8	38.6	40.3	41.8	43.1
10) 4Y Opt	33.4	35.5	37.4	39.1	40.4	41.6	42.9	44.1	45.5
11) 5Y Opt	37.8	39.6	41.0	42.2	43.3	44.4	45.4	46.5	47.5
12) 7Y Opt	44.7	45.6	46.3	47.0	47.7	48.4	49.0	49.6	50.3
13) 10Y Opt	49.7	49.8	50.3	50.8	51.1	51.4	51.7	51.8	52.1
14) 15Y Opt	51.7	51.5	51.9	51.9	51.9	52.0	52.1	52.0	52.3
15) 20Y Opt	51.2	51.1	51.4	51.3	51.3	51.4	51.2	51.1	51.0
16) 25Y Opt	50.3	50.3	50.5	50.4	50.2	50.0	49.7	49.1	49.1
17) 30Y Opt	49.3	49.4	49.6	49.6	49.6	49.2	48.5	47.8	47.1

Swap-Settled Swaptions

The swaptions we have covered so far in our Market Model discussion are **swap-settled swaptions** — when you exercise, you enter into a swap contract with your counterparty.

The payoff of the swaptions are

So more convulated as if also depends on PV01

$$\text{Payer Swaption} = \Big[P_{n+1,N}(T)(S_{n,N}(T) - K)\Big]^+$$

PV01

$$\text{Receiver Swaption} = \Big[P_{n+1,N}(T)(K - S_{n,N}(T))\Big]^+$$

formula

where

$$P_{n+1,N}(T) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(T),$$

problem because banks based in diff countries dispute about the discount factor.

Upon exercising, we get

$$\text{Payer Swaption} = V^{flt}(T) - V^{fix}(T)$$

$$\text{Receiver Swaption} = V^{fix}(T) - V^{flt}(T)$$

IRR-Settled Swaptions

better option as it is only based on the swap rate and since we want to know swap rate distribution → better to use an instrument that is only dependent on our variable of interest

An **Internal-Rate-of-Return (IRR)-settled swaption** has the following payoff:

four

$$\left.\begin{cases}\text{Payer Swaption} = \Big[\text{IRR}(S_{n,N}(T))(S_{n,N}(T) - K)\Big]^+ \\ \text{Receiver Swaption} = \Big[\text{IRR}(S_{n,N}(T))(K - S_{n,N}(T))\Big]^+\end{cases}\right\}$$

model-free

where

$$\text{IRR}(S) = \sum_{i=1}^{(T_N - T_n)\times m} \frac{\frac{1}{m}}{\left(1 + \frac{S}{m}\right)^i}$$

day count fraction

swap rate ↳ market observable

and $\frac{1}{m} = \Delta$ is the day count fraction corresponding to the payment frequency (m) of the swap.

solves the dispute mention in previous slide

IRR-settled swaptions are settled in cash based on the value of the payoff observed on the maturity date.

Swap-settled swaptions are common in the USD market, while IRR-settled swaptions are common in the European (EUR & GBP) markets.

IRR-Settled Swaptions

The Market Model used to value IRR-settled swaptions is:

$$V_{n,N}(0) \approx D(0,T) \cdot \mathrm{IRR}(S_{n,N}(0)) \cdot \mathrm{Black}(S_{n,N}(0), K, \sigma_{n,N}, T)$$

Historical Note:

- In the USD market, participants agree on the value of the PV01 $P_{n+1,N}$, i.e. there is no dispute on the discount factors.
- In the earlier days, market participants disagree on the PV01 value in the Euro and Sterling market.
- To avoid ambiguity, market participants agree to use the IRR formula to discount cashflows in the EUR and GBP market.
- The rational was that since $D(0,T) = \frac{1}{(1+r)^T}$, a good approximation would be to use the observed swap rate $S_{n,N}(T)$ for discounting.

Swap-settled:

$\mathbb{Q}^{n+1,N}$

$$\frac{V^{pay}_{swap}(t)}{P_{n+1,N}(t)} = \mathbb{E}^{n+1,N}\left[\frac{V^{pay}_{swap}(T)}{P_{n+1,N}(T)}\right]$$

$$= \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\,(S_{n,N}(T)-K)^+}{\cancel{P_{n+1,N}(T)}}\right]$$

formula

IRR-settled:

$\mathbb{Q}^{n+1,N}$

$$\frac{V^{Pay}_{IRR}(t)}{P_{n+1,N}(t)} = \mathbb{E}^{n+1,N}\left[\frac{V^{Pay}_{IRR}(T)}{P_{n+1,N}(T)}\right]$$

$$= \mathbb{E}^{n+1,N}\left[\frac{IRR(S_{n,N}(T))\,(S_{n,N}(T)-K)^+}{P_{n+1,N}(T)}\right]$$

unsolvable (DANGER!)

↓

no easy way arround it

↓

slightly inaccurate assumption to get black formula.

$\mathbb{Q}^T$:

$$\frac{V^{Pay}_{IRR}(t)}{D(t,T)} = \mathbb{E}^T\left[\frac{V^{Pay}_{IRR}(T)}{\cancel{D(T,T)}\rightarrow 1}\right]$$

at end or maturity

right measure

should have bee $dS = \sigma S dW^{n+1,N}$

$$V^{pay}_{IRR}(0) = D(t,T)\,\mathbb{E}^T\left[IRR(S_{n,N}(T))\cdot(S_{n,N}(T)-K)^+\right]$$

Again stuck

↳ so simplify

wrong measure

$$\simeq D(t,T)\,IRR(S_{n,N}(0))\,\mathbb{E}^{T}\left[S_{n,N}(T)-K\right]^+$$

use today's swap rate to do discounting

here red variables in the equation are the wrong assumptions that the market undertapes to get it to black model.

$$\simeq D(t,T)\,IRR(S_{n,N}(0))\,\mathbb{E}^{n+1,N}\left[S_{n,N}(T)-K\right]$$

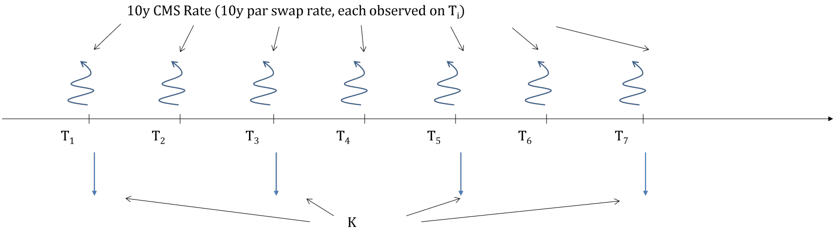
formula

Constant Maturity Swap

A **constant maturity swap** (CMS) pays a swap rate rather than a LIBOR rate on its floating leg.

⇒ Can be either quoted **in arrears** or **in advance**.

⇒ The payment can be **capped** or **floored**.

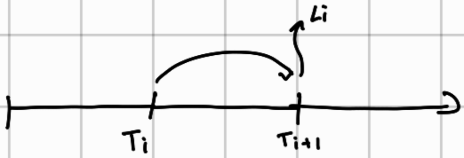


CMS is an instrument having cashflows "paid at the wrong timing":

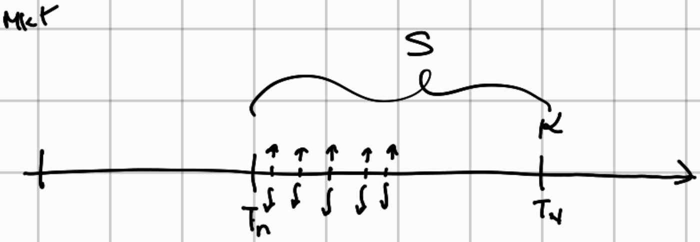
⇒ A $10y$ CMS rate to be paid one year later is not exactly equal to the forward swap rate $S_{1y,10y}$.

⇒ **Convexity correction** is required to obtain the right price.

Libor Mkt

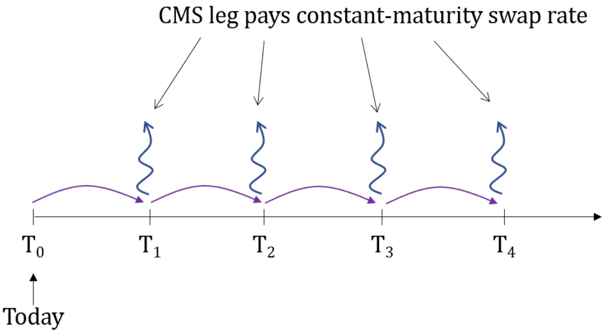


Swap Mkt



CMS Leg

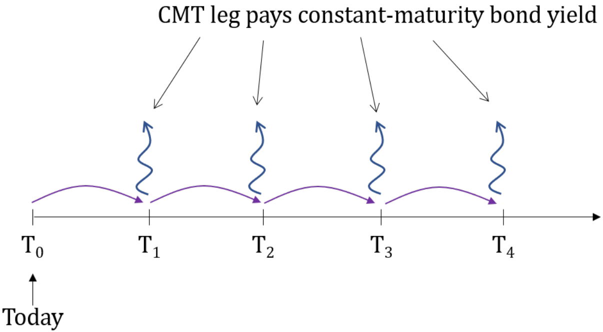
A CMS leg pays the constant-maturity swap rate periodically over time:



The CMS rate you receive at time T_{i+1} the par swap rate in the market at T_i.

CMT Leg

A closely related product is CMT, which pays the constant-maturity bond yield periodically over time:



The CMT bond yield you receive at time T_{i+1} the bond yield in the market at T_i.

CMS (or CMT) products give you an easy way to gain exposure to fixed-length longer-term interest rates.

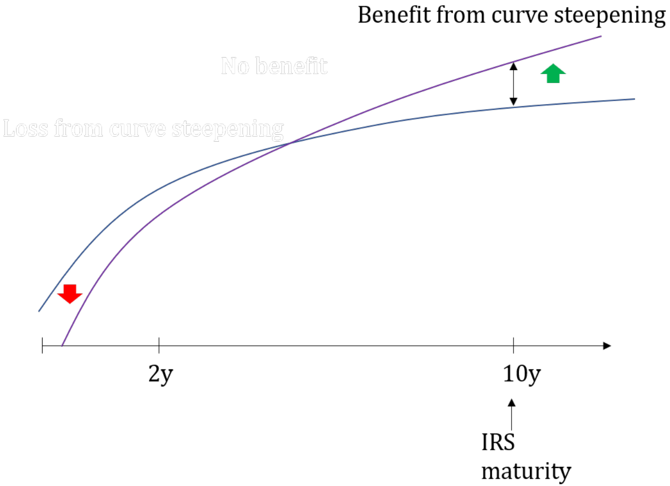
$\Rightarrow$ You can use it to express a view on a fixed point on the yield curve.

In contrast, if you use an IRS, then your exposure will progressively become shorter-term over time.

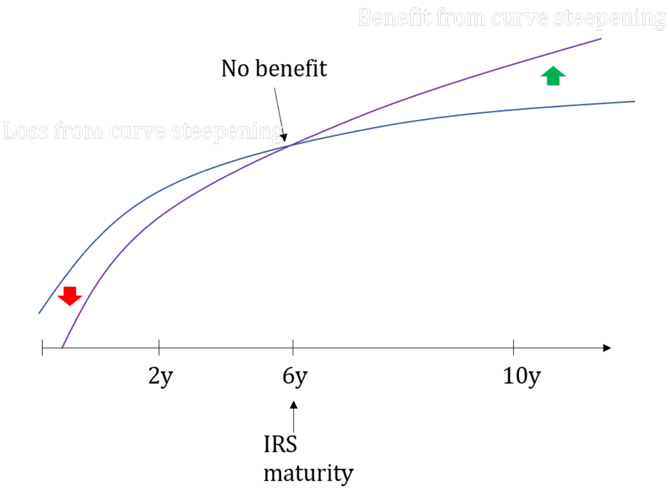
For example, suppose you think that the yield curve will steepen, so that $10y$ swap rate will increase, while $2y$ swap rate will decrease.

To this end, you long a $10y$ payer IRS. If the yield curve steepens, you benefit. If the yield curve flattens, you lose.

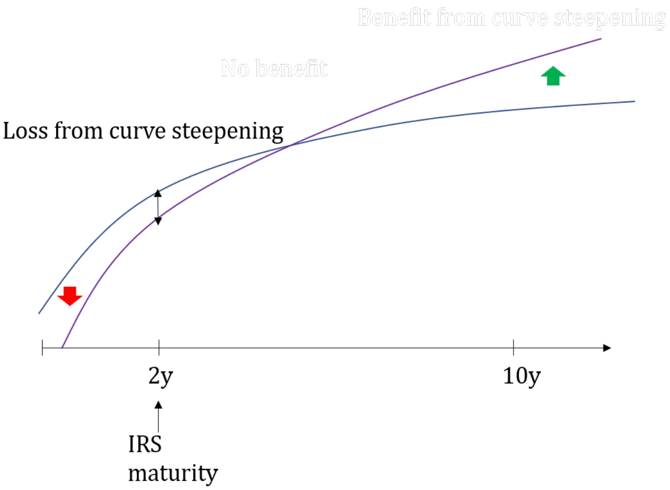
Trade day (initial)



4 years later



8 years later



Insurance companies or pension funds have long dated obligations — generally speaking, the exposure does not age time.

Year	Exposure
2y	-
5y	-
10y	-
15y	-
20y	-
30y	-
40y	-
50y	-

If they use IRS to hedge their exposure, the IRS sensitivity will progressively become shorter-term over time.

For hedge funds and other institutional clients, they use CMS products to speculate on the movement of the yield curve.

- Receive long-maturity CMS rate if they think yield will steepen
  $\Rightarrow$ Spread trade: Receive 10y pay 2y CMS

- Pay long-maturity CMS rate if they think yield will flatten
  $\Rightarrow$ Spread trade: Pay 10y receive 2y CMS

- CMS spread options

Risk-Neutral Density of the Forward Swap Rate

The value of a **CMS payoff** is a function of the distribution of the swap rate.

The standard practice in the market is to use the static-replication method to obtain a **model-independent convexity correction**.

Let us begin with an IRR-settled payer swaption:

Zero-coupon bond (general) → if deterministic then $= e^{-rT}$

$$V^{pay}(K) = D(t,T)\int_K^\infty \text{IRR}(s)\cdot(s-K)\, f(s)\, ds$$

↳ model free from stochastic session 8

Differentiating the formula twice yields

formula

$$\frac{\partial V^{pay}(K)}{\partial K} = -D(t,T)\int_K^\infty \text{IRR}(s)\, f(s)\, ds$$

$\frac{\partial c}{\partial K^2} = e^{-rT} f(K)$ vs

$$\boxed{\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(t,T)\,\text{IRR}(K)\, f(K)}$$

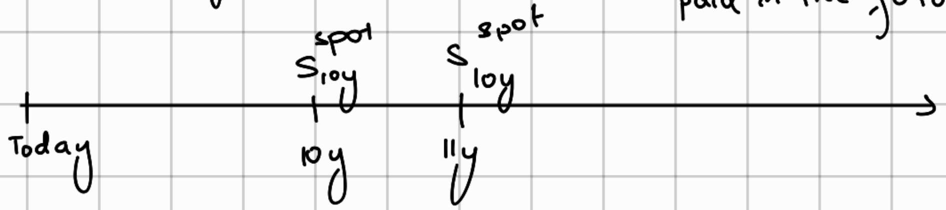
Carr-Madan static replication formula.

This can be rewritten as

$$f(K) = \frac{\partial^2 V^{pay}(K)}{\partial K^2} \times \frac{1}{D(t,T)\,\text{IRR}(K)}$$

CMS

we want to know the distribution of the swap of the 10year swap rate at 10 and 11 years in the future $\rightarrow$ get expected value of the swap rate paid in the future.



Static Replication Approach

We will also obtain the same result by differentiating the IRR-settled receiver swaption formula twice.

Suppose we want to value a contract paying $g(S_{n,N}(T))$ at time T, we let

$\frac{V_0}{D(0,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right]$

↳ on maturity $D(T,T)$ becomes 1.

$$h(K) = \frac{g(K)}{\text{IRR}(K)},$$

and write (let $F = S_{n,N}(0)$ denote the forward swap rate)

we know (previous slide: $f(K) = \frac{\partial^2 V}{\partial K^2} \cdot \frac{1}{D(0,T)\cdot \text{IRR}(K)}$

$$V_0 = D(0,T) \underbrace{\int_0^\infty g(K)\, f(K)\, dK}_{V_T}$$

$$= \int_0^F h(K)\, \frac{\partial^2 V^{rec}(K)}{\partial K^2}\, dK + \int_F^\infty h(K)\, \frac{\partial^2 V^{pay}(K)}{\partial K^2}\, dK$$

Integration-by-parts twice, we will get

$$V_0 = D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)]$$
$$+ \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

Static Replication of CMS Payoffs

Using **quotient rule**, the first and second order derivatives of $h(K)$ are given by:

$$\begin{aligned}
h(K) &= \frac{g(K)}{\mathrm{IRR}(K)} \\
h'(K) &= \frac{\mathrm{IRR}(K)g'(K) - g(K)\mathrm{IRR}'(K)}{\mathrm{IRR}(K)^2} \\
h''(K) &= \frac{\mathrm{IRR}(K)g''(K) - \mathrm{IRR}''(K)g(K) - 2\cdot\mathrm{IRR}'(K)g'(K)}{\mathrm{IRR}(K)^2} \\
&\quad + \frac{2\cdot\mathrm{IRR}'(K)^2 g(K)}{\mathrm{IRR}(K)^3}.
\end{aligned}$$

formula

CMS Rate

Example Show that a CMS rate payment for the swap rate $S_{n,N}(T)$ at time T can be valued as (where $F = S_{n,N}(0)$)

formula

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

using static replication, where

$$h''(K) = \frac{-\text{IRR}''(K)\cdot K - 2\cdot\text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2\cdot K}{\text{IRR}(K)^3}.$$

formula

$g(S_{n,N}) = S$

$$\frac{V_0^{CMS}}{D(0,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right] = D(0,T)\mathbb{E}^T\left[g(S_{n,N}(T))\right]$$ ← slide 17, 18 for sol.

$g(s)=s,\ g'(s)=1,\ g''(s)=0$

same here

CMS Caplet

Example Show that an at-the-money (ATM) CMS caplet struck at the forward swap rate $L = S_{n,N}(0) = F$ maturing at T can be valued as

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

using static replication, where

$$h'(K) = \frac{\text{IRR}(K) - \text{IRR}'(K) \cdot (K - L)}{\text{IRR}(K)^2}$$

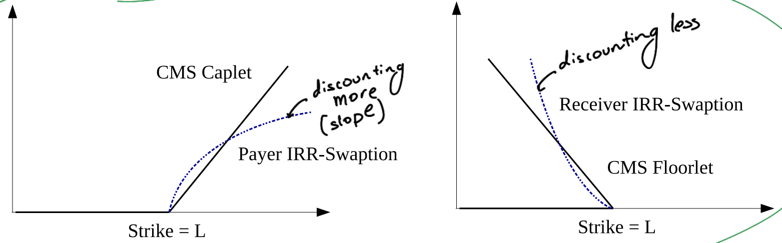
$$h''(K) = \frac{-\text{IRR}''(K)(K-L) - 2 \cdot \text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 \cdot (K-L)}{\text{IRR}(K)^3}$$

Use Michelle's pic

CMS Replication – Intuition

Note that the CMS caplet payoff and the IRR-settled payer swaption payoff are both functions of the same swap rate $S_{n,N}(T)$'s distribution.

Beyond the strike rate of L, CMS caplet payoff is **linear** and payer swaption payoff is **concave** of the swap rate $S_{n,N}(T)$: formula



Since swaptions are vanilla derivatives and more liquid, we can **replicate the CMS caplet** payoff using a basket of IRR-settled payer swaptions with increasing strikes starting with the CMS caplet strike L.

At time T

CMS caplet $= (S - L)^+$

IRR payer $= IRR(S)\,(S - L)^+$

$$= \sum_i \frac{\frac{1}{m}}{\left(1 + \frac{S}{m}\right)^i} \cdot (S - L)^+$$

CMS Replication – Intuition

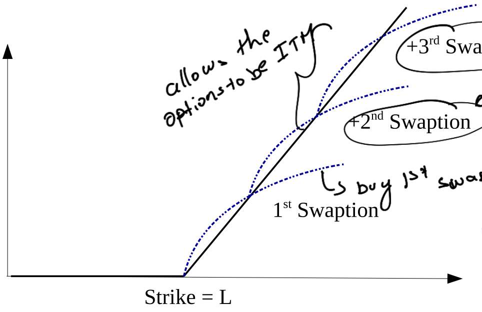
Using a series of IRR-settled payer swaptions, we can **statically replicate** the CMS caplet as follows:

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

$$\approx V^{pay}(L)h'(L) + \sum_{i=1}^{\infty} h''(L+i\cdot\Delta K)\, V^{pay}(L+\, i\cdot\Delta K)\, \Delta K$$

go as high as available

formula



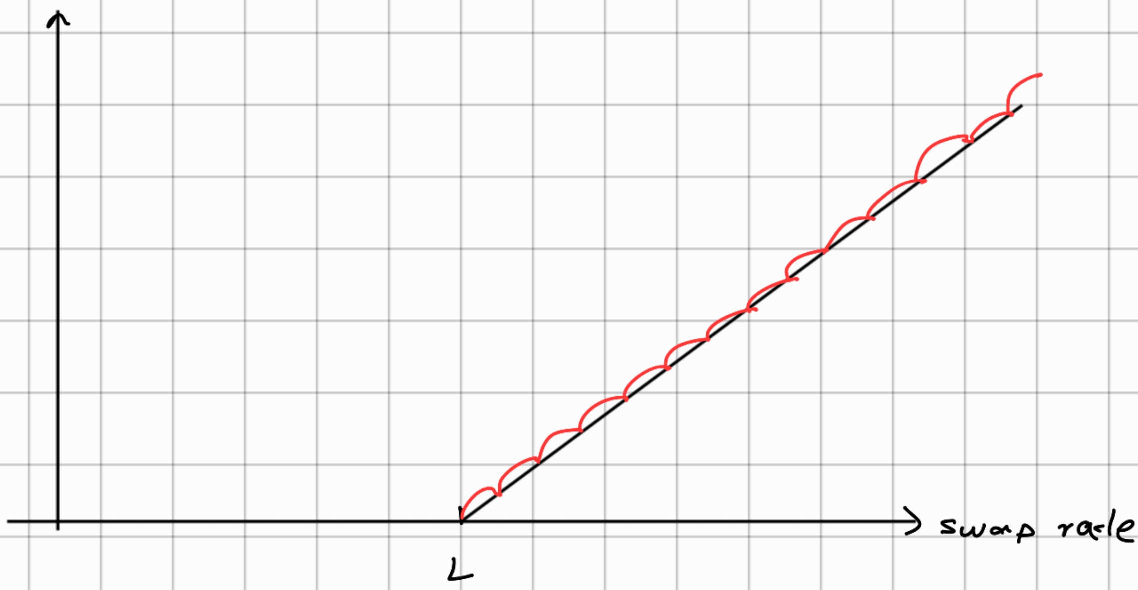
allows the options to be ITM

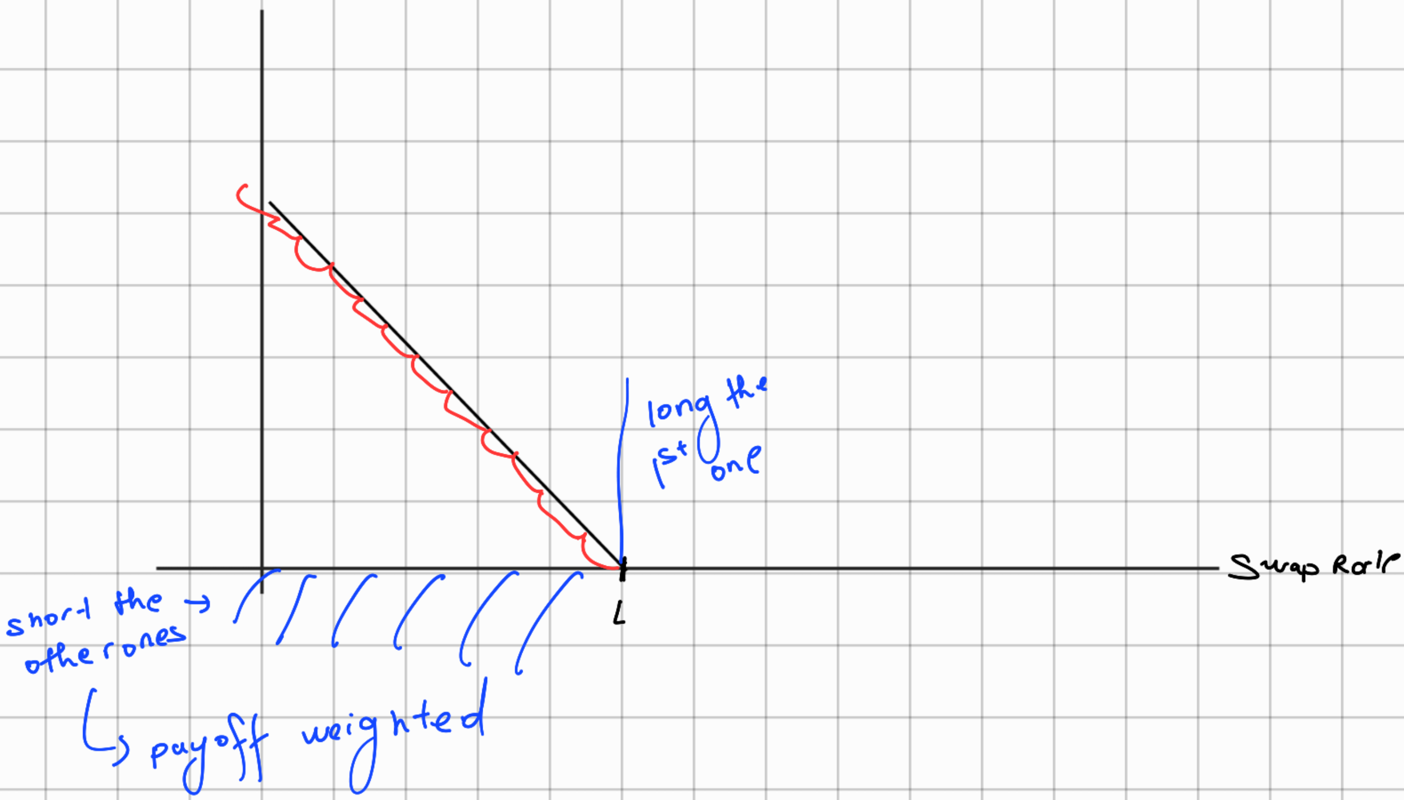
At higher strike than 1 and 2

At higher strike

buy 1st swaption

more the swaption the jump (blue line overlap with black)
↳ but blue always above (greater than) black.


swap rate
L


long the 1st one
Swap Rate
L
short the → other ones
↳ payoff weighted