SINGAPORE MANAGEMENT UNIVERSITY

Session 1: Bond Market and Bond Risk Management

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QF605 Fixed Income Securities

Remember that time is money.

— Benjamin Franklin (1706–1790)

Gentlemen prefer bonds.

— Andrew Mellon (1855-1937)

Money market
↳ deposit of < 1 year

Compounding

The convention used in the market is to treat all quoted interests as annualized, i.e. the total amount of interest paid for a whole year.

Simple/Linear Compounding Investing \$1 at the rate R for a period Δ yields

$$\text{Return} = 1 + \Delta \cdot R$$

always annualised rates (unless explicitly mentioned).

measures fraction of a year.

Discrete Compounding Interest is compounded at discrete frequency:

↳ like fixed deposit.

$$\text{Return} = \left(1 + \frac{R}{m}\right)^{m \times n}$$

money withdrawn

where m is the number of payments per year and n is the total number of years.

Continuous Compounding Interest is continuously compounded:

$$\text{Return} = e^{R \cdot T}$$

discounting : e^{-rT}

where T is the number of years. Continuous compounding is often easier to deal with mathematically.

↳ nothing in the market is continuously compounded.

Compounding

We can also turn this question around and ask how much money we need today to compound to \$1 at the maturity date.

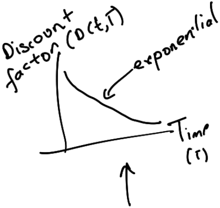
This brings in the concept of **discount factor**: $D(t,T)$, denoting the discount factor that discounts from T to back t, in other words, how much money we need at t to compound to \$1 at T.

Depending on the compounding convention, we have

$$D(0,\Delta) = \frac{1}{1+\Delta\cdot R}$$ Simple

$$D(0,n) = \frac{1}{\left(1+\frac{R}{m}\right)^{m\times n}}$$ Discrete

$$D(0,T) = e^{-R\cdot T}$$ Continuous



should be monotonic downward sloping.

- ⇒ Interest rate is generally positive $(R>0)$, so discount factors are generally less than 1.
- ⇒ **Liquidity Preference**: people generally prefer to receiver money earlier rather than later, hence $D(0,T)$ should be decreasing in T.

Compounding Frequency

Compound interest is the interest on interest, hence the frequency at which the interest is being compounded is important.

$N = 1$

Suppose we invest for one year at an interest rate of 8%. Different compounding frequency leads to different investment outcome.

Compound. freq.	$\frac{r}{m}$	$m \times N$	FV of \$100
Annual ← $m=1$	$8\%/1$	1×1	$100 \times (1 + 0.08)^1 = 108$
Semi-annual ← $m=2$	$8\%/2$	2×1	$100 \times (1 + 0.04)^2 = 108.16$
Quarterly ← $m=4$	$8\%/4$	4×1	$100 \times (1 + 0.02)^4 = 108.24$
Monthly ← $m=12$	$8\%/12$	12×1	$100 \times (1 + 0.00667)^{12} = 108.3$
Continuous ← e	8%	1	$100 \times e^{0.08 \times 1} = 108.3287$

R N/T

Money makes money. And the money that money makes, makes money.

— Benjamin Franklin (1706–1790)

Compound interest is the eighth wonder of the world.

— Attributed to Albert Einstein (1879–1955)

Effective Annual Rate

There is a distinction between stated annual interest rate and **effective annual rate (EAR)** — the interest actually earned over a year.

example on slide 5

$$r_{EAR} = \left(1 + \frac{r_S}{m}\right)^m - 1.$$

formula sheet

In our example, \$1 investment that earns 8.16% compounded annually gives the same FV as a \$1 investment that earns 8% compounded semiannually.

For an 8% stated annual interest rate with semi-annual compounding, the EAR is 8.16%

Special Case: **Bond Equivalent Yield** is conventionally used in the U.S. fixed-income market, and restates the yield in terms of semi-annual basis.

6 months

$$r_{BEY} = \left[\left(1 + \frac{r_S}{m}\right)^{\frac{m}{2}} - 1\right] \times 2$$

formula sheet

Example The effective annual yield on a fixed-income instrument is 9%. What is the yield on a bond equivalent basis? (ans. 8.81%)

$$1 + r_{EAR} = \left(\left(1 + \frac{rs}{m}\right)^m = \left(1 + \frac{r_{BEY}}{2}\right)^2\right) \text{sqrt}$$

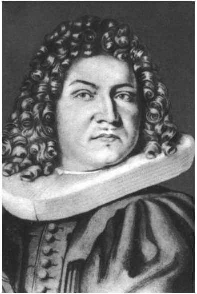
$$\therefore \left(1 + \frac{rs}{m}\right)^{m/2} = \left(1 + \frac{r_{BEY}}{2}\right)$$

$$\therefore \left(1 + \frac{rs}{m}\right)^{m/2} - 1 = \frac{r_{BEY}}{2}$$

$$\therefore r_{BEY} = \left[\left(1 + \frac{rs}{m}\right)^{m/2} - 1\right] \times 2$$ ← equation on slide 7.

Side Note: Exponential Constant

An account starts with \$1 and pays 100% per year. If the interest is credited once, at the end of the year, the value of the account at year end will be \$2. What happens if the interest is computed and credited more frequently during the year?



Jacob Bernoulli (1655–1705)

As the payment frequency increases, the interest paid out needs to be divided by the number of payment, but the interest earned is multiplied by the same amount. Taking the limit yields:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Day Count Conventions

Calculating Δ is important because it tells us how much interest we have accrued over a given period.

Three very common ways of calculating this are ← but each market has their own convention

- **Act/Act**: Read "actual/actual", simply count the number of calendar days. ↳ $\Delta = \frac{\text{No. of days in period}}{\text{No. of days in that year}}$ } calendar days
- **30/360**: Read "thirty three-sixty", assumes that there are 30 days in a month and 360 days in a year.
- US market → **Act/360**: Read "actual three-sixty", each months has the right number of days, but there are only 360 days in a year.
- Act/365 → Asian markets

Business Day Convention If payment date falls on a non-business day, then we need to move the date using business day convention:

- **Actual**: on the actual day, regardless. ← not popular cuz money just sits there
- **Following(/Previous)**: rolled onto the following business day. → next business day is a new month (so becomes 6 months)
- **Modified Following(/Previous)**: rolled only the following business day, except if it is the next calendar month, in this case roll backwards. ← most popular choice

Eg 5month deposit expires on month end but it is a weekend. So it rolls back.

Day Count Conventions

Example A simple interest rate contract whereby you are paid $\Delta \times 4.5\%$ (Δ is the day count fraction) gives you the optionality to choose one of the following day count convention:

- 30/360
- Act/360
- Act/365
- Act/Act

The 4.5% rate accrues from 1-Jan-2025 to 30-June-2025. Which would you choose and why?

Day Count Conventions

Example Everything else being equal, should a $1m$ expiry call option on a non-dividend paying stock be more valuable on

(A) 15 Jan ⇐ coz more days. ⇐ more expensive option

(B) 15 Feb

cheaper coz less days

Basic Instruments in the Bond Market

Only the final value.

- **Zero-coupon Bonds**: These are pure discount instruments – making a single payment on the maturity date T.
  - ⇒ The final payment is commonly referred to as **principal, face value, or notional**.
  - ⇒ They tend to have very short maturities (e.g. treasury bills or commercial papers). issued by govt
    ↳ 15mon – 2 year
    from companies → cheaper cuz credit risk.
- **Coupon Bonds** In addition to the final notional, these also pay a periodic fixed coupon over the life of the bond.
  - ⇒ Tends to have longer maturity.
  - ⇒ US convention: zero-coupon bonds are called bills, bonds with maturity 2-10 years are called notes, bonds with maturities longer than 10 year are called bonds.

less common

- **Floating Rate Bonds** Coupon payments are not fixed but depend on some benchmark interest rate (floating).
  → prevailing interest rate at that time.
  - ⇒ E.g. quarterly compounded $3m$ benchmark rate plus 10 basis points.

Zero-coupon Bonds

The importance of the zero-coupon bonds is that we can write any deterministic set of cashflows as a linear multiple of such bonds.

For instance, suppose we agreed to lend a company a principal N at $T = 1$ and the company is to pay a fixed six-monthly annualized rate of 10% for 5 years, and then at $T = 6$ return the principal.

We can write this transaction in terms of zero-coupon bonds:

$$PV = -N \cdot D(0,1) + \sum_{i=1}^{10} 0.05 \cdot N \cdot D(0, 1 + 0.5i) + N \cdot D(0,6)$$

The convenient property is that—upon substituting the values of the zero-coupon bonds—we have a **present value** for the entire transaction.

If this number is 0, the loan is at fair value. This technique for valuing trades is very standard and is called **present-valuing (or PV)**.

Zero Rates

Zero rate (or spot rate) is the yield-to-maturity of a zero-coupon bond. From the price $D(t,T)$, we can calculate the continuously compounded **spot rate** $R(t,T)$ that is set at t and pays at T.

time 0 Time final

↳ zero-rate (2 time stamps)

↳ for final and start value calc (compound and discount)

Continuously compounded zero rate

$$D(t,T) = e^{-R(t,T)(T-t)} \quad \Rightarrow \quad \overbrace{R(t,T) = -\frac{\log D(t,T)}{T-t}}^{\text{Zero rate expressed as discount factor.}}.$$

By no-arbitrage, the zero rate should be the interest rate earned on any investment at time t and pays at T $\Rightarrow$ useful for discounting.

Discretely compounded zero rate

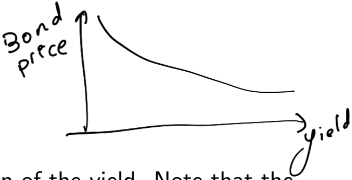
$$D(t,T) = \frac{1}{\left(1 + \frac{R(t,T)}{m}\right)^{(T-t)\times m}} \quad \Rightarrow \quad R(t,T) = m\left[D(t,T)^{-\frac{1}{m(T-t)}} - 1\right]$$

- $\Rightarrow$ If $R(t,T)$ is specified for all values of T, then we have a zero rate curve.
- $\Rightarrow$ $R(t,T)$ is the interest rate implicit in a discount factor/zero-coupon bond $D(t,T)$.

Yield-to-Maturity

For coupon bonds, **yield-to-maturity (YtM)** is a more convenient interest rate measure. The YtM is defined as the interest rate such that if all the different cash flows are discounted at this rate (y), then the resulting net present value is equal to the current price of the bond. Assuming continuous compounding:

$$B = \sum_{i=1}^{N} c_i e^{-y \cdot T_i}$$



Here c_i is the cash flow paid at time T_i.

The price of the bond can be regarded as a function of the yield. Note that the price of the bond $B(y)$ is a decreasing function of the yield y:

$$\frac{\partial B}{\partial y} = B'(y) = -\sum_{i=1}^{N} T_i c_i e^{-yT_i} < 0.$$

Therefore, if the yield of the bond goes up, then the price of the bond goes down, and vice versa.

Yield-to-Maturity

The **yield-to-maturity** or **internal rate of return** or **redemption yield** is the discount rate that makes the market price of a bond equal to the discounted value of its future cashflows.

Although continuous compounding is elegant, in practice we use discrete compounding to compute bond prices.

Compare the formula with zero rates and formula with the yield-to-maturity:

preferred method.

formula

this is risk

$$B = \sum_{i=1}^{N} \frac{c_i}{\left(1+\frac{y}{m}\right)^{m\times T_i}} = \sum_{i=1}^{N} \frac{c_i}{\left(1+\frac{R_i}{m}\right)^{m\times T_i}}$$

avg of zero rates

each individual zero rate

Notice that the yield is a blend or a kind of average of the different zero rates associated with the cashflows.

In other words, the yield must be between the highest and lowest zero rates.

Yield-to-Maturity

Consider a $1.5y$ semi-annual coupon bond with a coupon of 8.5%. Compare the two formulas for the bond:

$$1.043066 = \frac{0.0425}{\left(1+\frac{0.0554}{2}\right)^1} + \frac{0.0425}{\left(1+\frac{0.0545}{2}\right)^2} + \frac{1.0425}{\left(1+\frac{0.0547}{2}\right)^3}$$

$$1.043066 = \frac{0.0425}{\left(1+\frac{0.054704}{2}\right)^1} + \frac{0.0425}{\left(1+\frac{0.054704}{2}\right)^2} + \frac{1.0425}{\left(1+\frac{0.054704}{2}\right)^3}$$

$\Rightarrow$ Using the zero rates 5.54%, 5.45%, and 5.47%, the bond price is 1.043066 per dollar par value.

$\Rightarrow$ The implied yield of 5.4704% is a kind of average of the discount rates 5.54%, 5.45%, and 5.47%.

$\Rightarrow$ The YtM is useful because it provides a way of converting a price into something resembling an interest rate. This sometimes makes it easier to compare how cheap or expensive a bond is.

Yield-to-Maturity

Par Yield is the coupon rate that makes the value of the bond equal to its **face value**. In other words, par yield is the value of the coupon rate such that

$$B = \text{Face Value}$$

coupon = yield initially (nice whole number)

then the bond price changes on issuers credit risk.

Bonds are typically issued at par.

Under discrete compounding, a bond will trade at par whenever its yield-to-maturity is equal to the coupon rate.

Example Consider the following annual coupon bond:

$$B = \sum_{i=1}^{N} \frac{c_i}{(1+y)^i}.$$

Show that when the yield-to-maturity y is equal to its coupon rate ($c = 100y$), then $B = 100$.

$$B = \sum_{i=1}^{N} \frac{C_i}{(1+y)^i}$$

Last payment is final coupon + notional value

$$= \frac{C}{1+y} + \frac{C}{(1+y)^2} + \frac{C}{(1+y)^3} + \cdots + \underbrace{\frac{C}{(1+y)^N} + \frac{100}{(1+y)^N}}$$

geometric series.

$$= \frac{C}{1+y}\left[\frac{1 - \frac{1}{(1+y)^N}}{1 - \frac{1}{1+y}}\right] + \frac{100}{(1+y)^N}$$

formula

geometric series sum $= \frac{1-r^n}{1-r}$

$$r = \frac{1}{1+y} \; ; \; n = N$$

$$= \frac{C}{\cancel{1+y}}\left(\frac{1 - \frac{1}{(1+y)^N}}{\frac{\cancel{1}+y-\cancel{1}}{\cancel{1+y}}}\right) + \frac{100}{(1+y)^N}$$

$$= \frac{C}{y}\left(1 - \frac{1}{(1+y)^N}\right) + \frac{100}{(1+y)^N}$$

notion

100 x yield

$C = 100y$

$$= \frac{100\cancel{y}}{\cancel{y}}\left(1 - \frac{1}{(1+y)^N}\right) + \frac{100}{(1+y)^N}$$

$$= 100 - \cancel{\frac{100}{(1+y)^N}} + \cancel{\frac{100}{(1+y)^N}}$$

$$= \underline{\underline{100}}$$ ← Bond value

solve using Bond eq in slide 18.

Bootstrapping a Bond Curve

Example Suppose we have 3 coupon bonds (annual coupon):

Bond	Maturity	Coupon	Price
A	$1y$	5	101
B	$2y$	6.5	102
C	$3y$	7	102.5

What can we say about $D(0,1)$, $D(0,2)$, and $D(0,3)$ and the continuously compounded zero rates $R(0,1)$, $R(0,2)$, $R(0,3)$?

↳ zero curve based on these list of bond,

↳ so how much I will get after 1+2+3 = 6 years

ans.: $D(0,1) = 0.9619$, $D(0,2) = 0.899$, $D(0,3) = 0.8362$

Bond A

$$101 = \underbrace{D(0,1) \times \overset{\text{coupon}}{5} + D(0,1) \times \overset{\text{the face value.}}{100}}$$

$$D(0,1) = \frac{101}{105}$$

get Discounted

then $D(0,1) = e^{-R(t,T)(T-t)}$

Bond B

$$102 = D(0,1) \times 6.5 + \underbrace{D(0,2) \times \overset{\overset{\text{final notional val.} \to 100 + \overset{\text{last coupon}}{6.5}}{\downarrow}}{106.5}}_{\text{final value/payment}}$$

Bond C

$$102.5 = D(0,1) \times 7 + D(0,2) \times 7 + \underbrace{D(0,3) \times \overset{100+7}{107}}_{\text{final payment}}$$

Bootstrapping a Bond Curve

Example Following the previous question, suppose we have another coupon bond D that pays a coupon of 3 annually, with 2 years left to maturity.

1. What should be the no-arbitrage price for this bond?
2. Suppose this bond trades at \$94. Form an arbitrage.

formula

Profit $= 95.483 - 94 = 1.483$

ans.: $B_D = 95.483$. $w_A = 0.0313$, $w_B = -0.967$.

short that bond.

1. $D(0,1) \times 3 + D(0,2) \times 103 = B_D$ ← cuz 2 years left to maturity.

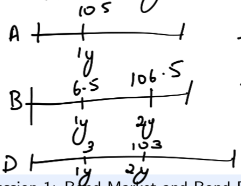
2. 1. Buy bond D at the cheaper price
   2. Offset using Bond A and B (C is useless as it expires in 3 years and D in 2).

$\text{portfolio} = B_D + w_A B_A + w_B B_B$ ← with incorrect price portfolio value is the profit

if correct it is 0.

$t=1$: $0 = 3 + w_A 105 + w_B(6.5)$

$t=2$: $0 = 103 + w_B(106.5)$



Bootstrapping a Bond Curve

Example Consider the following bonds from the same issuer:

Maturity (years)	Coupon	Price
0.25	0	97.5
0.50	0	94.9
1.00	0	90.0
1.50	8 (semi-annual)	96.0
2.00	12 (semi-annual)	101.6

annual
$\rightarrow$ final

Bootstrap the continuously compounded zero rate curve.

ans.: $D(0,1.5) = 0.85196,\ D(0,2) = 0.8056.$

$96 = D(0,0.5)\times 4 + D(0,1)\times 4 + D(0,1.5)\times 104$

$8/2 \leftarrow$ semi annual.

$101.6 = D(0,0.5)\times 6 + D(0,1)\times 6 + D(0,1.5)\times 6 + D(0,2)\times 106$

Case Study: On-the-run vs. Off-the-run

Brand new bond (latest piece)

On-the-run treasuries are the most recently issued US Treasury bonds of a particular maturity. Because on-the-run issues are the most liquid, they typically trade at a slight premium and therefore yield a little less than their **off-the-run** counterparts.

Long-Term Capital Management (LTCM) successfully exploited this price differential through an arbitrage strategy that involves selling on-the-run treasuries and buying off-the-run treasuries between 1994–1998. → short

→ long.

The idea is that this disparity in pricing should not persist. At the same time, there was an economic justification underpinning the disparity.

Convergence Arbitrage ← so the prices will become similar. ← so on-the-run becomes cheaper and off the run remains the same.

- Trade made between two different securities that tend to become more similar over time. Because they differ from one another, there is daily mark-to-market risk.
- But over the longer term the prices will tend to move closer and closer to each other because it will become increasingly obvious that the two securities are similar.

Clean Price *vs.* Dirty Price

quote clean prices because it doesn't change and easier to model.

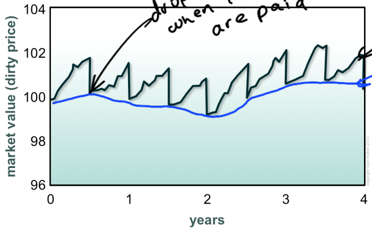
shown on broker screen.

The price of bonds quoted in the market are **clean prices**. That is, they are quoted without any accrued interest. The accrued interest is the amount of interest that has built up since the last coupon payment.

Trade on this (so the transaction amt)

The actual payment is called the **dirty price** and is the sum of the quoted clean price and the accrued interest. (AI)

The cash price of a coupon bond falls by the amount of the coupon just after the coupon has been paid, this means that the cash price have a "saw" pattern.



dirty price

clean price (drop points are when coupons are paid)

$$\text{Clean Price} = \text{Dirty Price} - \text{AI}$$

coupon.

$$\text{AI} = c \times \frac{\text{days since last coupon date}}{\text{days between coupon dates}}$$

floating: $\frac{C(r+0.01)}{1+r+0.01}$ ← eliminates sensitivity. fixed coupon: $\frac{C}{1+r}$ ← doesn't change; if r↑ then fixed coupon↓ and vice versa

Floating Rate Notes (FRN)

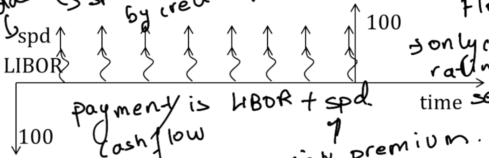
- **FRN** is a bond that pays a coupon linked to a variable interest rate index – typically LIBOR (or Euribor for EUR).
- FRN eliminates most of the interest rate sensitivity, making it almost a pure credit play – the price action of a FRN is driven mostly by the changes in the market-perceived credit quality of the note issuer.

fixed coupon is sensitive b/c of discounting using interest rates

traders speculate on credit rate improvements

- While the senior short-term floaters of AA-rated banks pay a coupon close to LIBOR and trade at a price close to par, in the credit markets, many floaters are issued by corporates with much lower credit ratings.
- Many AA-rated banks also issue FRN that are subordinate in the capital structure.

spread doesn't change ↳ spread decided by credit quality ∈ spd for secure < spd for risky



floating → LIBOR

payment is LIBOR + spd ← risk premium. cash flow

Floating → only credit rating sensitivity | fixed → interest rate sensitive → credit rating sensitivity

Floating Rate Notes (FRN)

I need to rewatch this

- Either way, investors require a higher yield to compensate them for the increased **credit risk**. At the same time, the coupons of the bonds must be discounted at a higher interest rate than LIBOR to account for this higher credit risk.
- Therefore, in order to issue the note at par, the coupon on the FRN must be set at a fixed spread over LIBOR, this is known as the **par floater spread**.
- Example: 2y FRN with semi-annual payment paying LIBOR + 50bps.
- FRN has a much lower interest rate sensitivity than fixed-rate bonds. If LIBOR moves up, the increment in coupons will be offset by the higher discount rate and vice versa.
- On coupon dates, whether the price of a FRN is above or below par is determined by its par floater spread. If it is above the fixed spread in the coupon, then it will trade below par, and vice versa.

Floating Rate Notes (FRN)

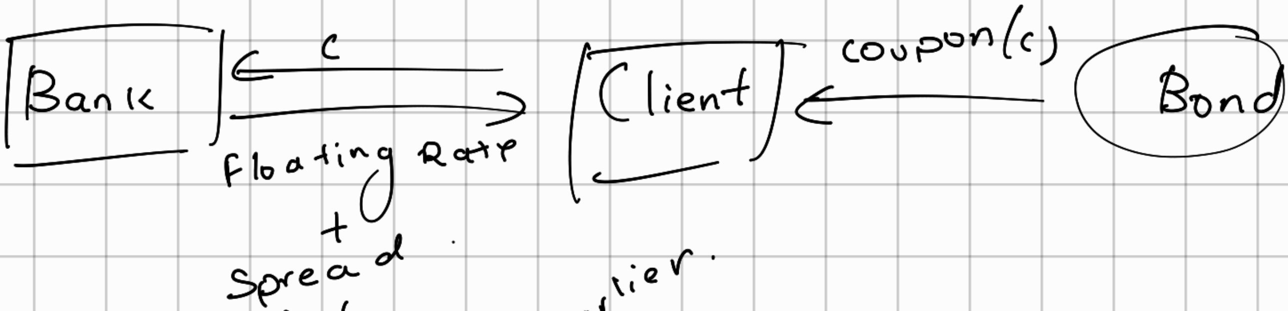
there is tiny interest rate exposure cuz. payment to be paid on t+1, the interest rate is fixed at time t

next coupon payment

period-10 coupon payment

- Between coupon dates, the LIBOR component of the impending coupon payment is fixed and its value is known today.
- However, it is discounted with LIBOR + spread, giving us interest rate exposure for this single cashflow – this is known as **reset risk**. It is usually small and drops to 0 as we approach the payment date.
- A large proportion of the FRN is issued by banks to satisfy their bank capital requirements.
- A large number of corporate and emerging market bonds are issued in FRN format.
- FRN is a way for credit investor to buy bond and take exposure to credit without taking exposure to interest rate movements.
- Most bonds are fixed rate, and so incorporate a significant interest rate sensitivity. We can turn them into pure credit exposure using asset swaps.

Asset Swap



fixed earlier.

if bond defaults client pays bank.

Floating Rate Notes (FRN)

By no-arbitrage, just after an interest payment has been made, the price of a floating rate bond must be equal to par if credit outlook remains constant.

Consider the following trading strategy, which requires an initial amount of cash equal to par:

- ⇒ Invest the cash at the floating rate, until the date of the next coupon payment on the floating rate bond.
- ⇒ When you receive the cash and interest, reinvest the cash amount at the new floating rate until the next coupon payment date.
- ⇒ At maturity of the floating rate bond, you receive the cash equal to par, plus the last interest payment.

Since the payoff of this strategy is equal to the payoff of buying the floating rate bond, the price of the bond must be equal to par.

Just before a coupon is paid, the price of the bond must be equal to par plus the value of the coupon.

Bond Duration

Duration and **convexity** are two of the most important parameters to estimate when investing in a bond, other than its yield.

⇒ Bond price is a decreasing and convex function of the bond yield.

The most frequently used bond duration measure in practice is **modified duration**.

The **modified duration** of a bond is the rate of change of the price of the bond with respect to the yield of the bond, normalized by the price of the bond, and with opposite sign, i.e.

yield is negative so this -ve sign turns the number to +ve

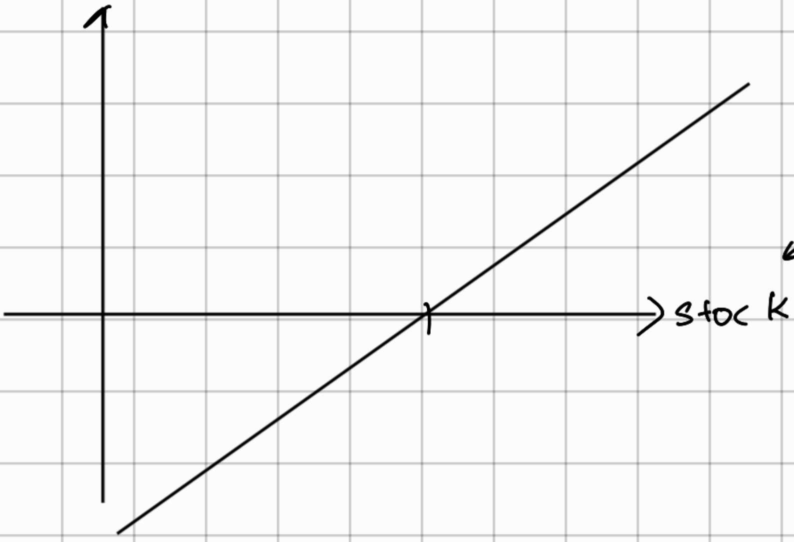
$$D = -\boxed{\frac{1}{B}}\boxed{\frac{\partial B}{\partial y}}.$$

gradient or sensitivity to yield changes

capture bond price change

The price B of the bond is considered a function of the yield y and with cash flows c_i at time t_i.

Modified duration measures the rate of return of the bond price for small changes in the bond yield



delta one product

↓

no higher order risk (only gradient, rest derivative orders are 0)

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right]$$

bond duration → delta

vega

$$C(s+\Delta s, \sigma+\Delta\sigma) = C(s,\sigma) + \frac{\partial C}{\partial s}\cdot\Delta s + \frac{\partial C}{\partial \sigma}\cdot\Delta\sigma + \frac{1}{2!}\left[\frac{\partial^2 C}{\partial s^2}\cdot(\Delta s)^2 + 2\frac{\partial^2 C}{\partial s \partial \sigma}\Delta s\Delta\sigma + \frac{\partial^2 C}{\partial \sigma^2}(\Delta\sigma)^2\right] + \cdots$$

call option greeks.

gamma ↑ Bond complexity

vanna

volga ↑ volatility's gamma. also called vomma

→ delta and vega should be more than sufficient to calculate portfolio sensitivity.

duration

$$B(y+\Delta y) = B(y) + \frac{\partial B}{\partial y}\cdot\Delta y + \underbrace{\frac{1}{2!}\frac{\partial^2 B}{\partial y^2}(\Delta y^2)}_{\text{convexity.}} + \cdots$$

↑ Bond.

Bond Duration

Under continuous compounding, we have

$$B = \sum_{i=1}^{n} c_i e^{-yt_i} \quad \Rightarrow \quad \frac{\partial B}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i},$$

1st derivative of B

and it follows that the modified duration D of the bond is

substitute in $-\frac{1}{B}\frac{\partial B}{\partial Y}$

formula → $$D = \frac{1}{B}\sum_{i=1}^{n} t_i c_i e^{-yt_i}.$$

continuous compounding is in reality bond are discreetely compounded. because impossible to reinvest continuously.

From here we see that this is just a "time weighted" average of the of the cashflows' NPV.

Example Suppose the bond yield is 4% and continuously compounded, what is the modified duration of a zero coupon bond paying maturing at the end of the 2^{nd} year. Ans.: 2

1 payment

Example A $2y$ coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually. Calculate the modified duration of this bond. Ans.: 1.9396

Examples

1. $D = \frac{1}{B} \cdot \sum_{i=1}^{n} t_i c_i e^{-y \cdot t_i}$

$B = \underline{100 \cdot e}^{-0.04\% \times 2}$ ← formula from earlier slides).

$\therefore \; D = \frac{1}{\cancel{B}} \times \cancel{(2 \times 100 \times e^{-0.04 \cdot 2}}$

Ans: duration of the bond is 2years.

→ Property : duration of a zero-coupon bond is the maturation period.

2. $D = \frac{1}{B} \times \Big[1 \times c \times e^{-r \times 1}$

$+ 2 \times c \times e^{-r \times 2}$

$\vdots$

$+ n \times (c + 100) \times e^{-r \times n} \Big]$

Bond price → B

c is coupon

n is year

100: Face value

$D = \frac{1}{102} + \left[\left(1 \times 6.5 \times e^{-5.2756 \times 1}\right) + \left(2 \times 106.5 \times e^{-5.2756 \times 2}\right) \right]$

Bond Duration

The **Macaulay Duration** of a bond is defined as the weighted averaged of the cash flow times, with weights equal to the value of the corresponding cash flow discounted with respect to the yield of the bond, i.e.

multiply t_i because payment schedule is important.

$$D_{\text{Mac}} = \frac{1}{B} \sum_{i=1}^{n} t_i \underbrace{c_i \text{Discount}(t_i, y)}_{\text{bond price}},$$

we got it almost right. but ours it a bit more clearer.

where Discount(t_i, y) is the discount factor corresponding to time t_i, computed with respect to the yield of the bond.

Based on the definition, we see that if the bond yield is continuously compounded, then the Macaulay duration is exactly equal to the modified duration.

If the bond yield is discretely compounded (as is the case in practice), the modified duration is:

formula (if space)

modified duration

Macaulay duration

$$D = \frac{D_{\text{Mac}}}{1 + \frac{y}{m}}.$$

a small difference between the two durations.

$$B(y) = \sum_{i=1}^{n} c_i \cdot \frac{1}{\left(1+\frac{y}{m}\right)^{m \times t_i}}$$

$$= \sum_{i=1}^{n} c_i \left(1+\frac{y}{m}\right)^{-m \times t_i}$$

$$\frac{\partial B}{\partial y} = -\cancel{m} \sum_{i=1}^{n} t_i c_i \left(1+\frac{y}{m}\right)^{-mt_i - 1} \times \frac{1}{\cancel{m}}$$

$$= -\sum_{i=1}^{n} t_i c_i \left(1+\frac{y}{m}\right)^{-mt_i - 1}$$

$$= -\sum_{i=1}^{n} \frac{t_i c_i}{\underbrace{\left(1+\frac{y}{m}\right)^{mt_i}}_{\text{discount factor}}} \cdot \frac{1}{\left(1+\frac{y}{m}\right)}$$

$$-\frac{1}{B} \cdot \frac{\partial B}{\partial y} = +\frac{1}{B} \sum_{i=1}^{n} \frac{t_i c_i}{\left(1+\frac{y}{m}\right)^{m \times t_i}} \cdot \frac{1}{1+\frac{y}{m}}$$

number of years

no. of payment terms in a year

cannot separate summation.

this how you will do when you compound discretely

Bond Duration

- Macaulay duration estimates the point in time when the future value of the bond would remain unchanged for small parallel changes in the zero rate curve.

- Macaulay duration is defined as the "time weighted" average of the cashflows NPV.

- For continuous compounding, the discount factors are $\text{Discount}(t_i, y) = e^{-yt_i}$, and it follows that $D_{Mac} = D$.

  $\Rightarrow$ In other words, the Macaulay duration and the modified duration of a bond have the same value if interest is compounded continuously.

  $\Rightarrow$ This is not the case if interest is compounded discretely, when the modified duration of a bond is smaller than its Macaulay duration.

Using Modified Duration

Let Δy be the change in the yield of the bond, and let

$$\Delta B = B(y + \Delta y) - B(y)$$

be the corresponding change in the price of the bond. The discretized version of the modified duration formula becomes

$$D \approx -\frac{1}{B} \cdot \frac{B(y+\Delta y) - B(y)}{\Delta y} = -\frac{\Delta B}{B \cdot \Delta y}$$

change in bond price

change in yield.

$$\Rightarrow \frac{\Delta B}{B} \approx -D\Delta y$$

Return. original bond price Return in bond price

if yield goes up Δy is +ve then $-D\Delta y$ is more -ve so lower returns.

In other words, the return $\frac{\Delta B}{B}$ of the bond can be approximated by the duration of the bond multiplied by the parallel shift in the yield curve, with opposite sign.

For very small changes in the yield of the bond, this approximation formula is accurate. For larger changes, convexity is used to better capture the effect of changes in the bond yield.

Bond Convexity

The **Convexity** C of a bond with price B and yield y is defined as

$$C = \frac{1}{B}\boxed{\frac{\partial^2 B}{\partial y^2}}.$$

← second derivative always +ve.

Under continuous compounding, we can see that

formula. →

$$C = \frac{1}{B}\sum_{i=1}^{n} t_i^2 c_i e^{-y t_i}.$$

B(y) is convex function of y.

Let D and C be the modified duration and the convexity of a bond with yield y and value $B = B(y)$. Then

Modified duration.

$$\frac{\Delta B}{B} \approx -D\Delta y + \frac{1}{2}C\cdot(\Delta y)^2,$$

formula (if space)

where $\Delta B = B(y+\Delta y) - B(y)$.

$$B(y+\Delta y) = B(y) + \frac{\partial B}{\partial y}\cdot \Delta y + \frac{1}{2!}\frac{\partial^2 B}{\partial y^2}(\Delta y^2) + \cdots$$

$$\frac{B(y+\Delta y) - B(y)}{B(y)} = \underbrace{\boxed{\frac{1}{B(y)}\frac{\partial B}{\partial y}}}_{-D}\Delta y + \frac{1}{2}\underbrace{\boxed{\frac{1}{B(y)}\frac{\partial^2 B}{\partial y^2}}}_{C}(\Delta y)^2 + \cdots$$

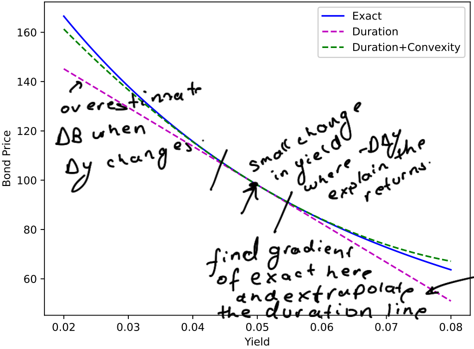
$$= -D(\Delta y) + \frac{1}{2}C(\Delta y)^2$$

Bond Convexity

We can use modified duration and convexity to estimate the changes in bond price due to parallel yield movements:

1^{st}-order: $$B(y+\Delta y) \approx B(y) - D\Delta y B(y)$$

2^{nd}-order: $$B(y+\Delta y) \approx B(y) - D\Delta y B(y) + \frac{1}{2}C \cdot (\Delta y)^2 B(y). \quad \triangleleft$$



overestimate ΔB when Δy changes.

small change in yield where $-D\Delta y$ explain the returns.

find gradient of exact here and extrapolate the duration line

1st order derivative ↳ straigth line and duration

Bond Convexity

Note that regardless of whether the yield goes up or down, a bond with higher convexity provides a better return when the bond yield moves.

Note also that an approximate value for the percentage return $\frac{\Delta B}{B}$ of a long bond position can be computed using this formula without requiring specific knowledge of the cash flows of the bond.

→ quick way to do quick intraday PnL.

A practical note about bond trading—for two bonds with the same duration, the bond with higher convexity provides a higher return for small changes in the yield curve.

- $\Rightarrow$ Regardless of whether the yield goes up or down, bond with higher convexity provides a higher return, and is the better investment, all other things being considered equal.

Bond Convexity

Example A $2y$ coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually.

$\hookrightarrow y$

1. Calculate the convexity of this bond. ans.: 3.8187
2. Suppose the yield goes up by 2 basis points, calculate the return $\frac{\Delta B}{B}$ of this bond. ans.: -0.037%
3. Compare the bond return to the approximation we obtain using modified duration and convexity. ans.: -0.038%

1. $C = \frac{1}{102} \times \left[\left(1^2 \cdot 6.5 \cdot e^{-y \cdot 1}\right) + \left(2^2 \cdot 106.5 \cdot e^{-y \cdot 2}\right)\right]$ $y = 5.2756\%$

2. $B(5.2756\%) = 102$ $\frac{? - 102}{102} = \_\_\_\_$

$B(5.2956\%) = ?$

you know the formula

$B = \sum_i^n c_i e^{-y t_i}$

3. $\frac{\Delta B}{B} \approx -D \cdot \Delta y + \frac{1}{2} \cdot C (\Delta y)^2$

from earlier example

Dollar Duration

specific to a particular bond because of $\frac{1}{B}$ } so cannot add the bonds value tgt.

Modified duration and convexity are not well suited for analyzing bond portfolios since they are non-additive.

because $\frac{1}{B}$ is percentage change.

⇒ The modified duration and the convexity of a portfolio made of positions in different bonds are not equal to the sum of the modified durations or of the convexities, respectively, of the bond positions.

Dollar duration and **dollar convexity** are additive and can be used to measure the sensitivity of bond portfolios with respect to parallel changes in the zero rate curves.

Dollar Duration of a bond is defined as

$$D_\$ = -\frac{\partial B}{\partial y},$$

} actual change no longer a return.

and measures the sensitivity of the bond price with respect to small changes of the bond yield. It is easy to see that $D_\$ = B \times D$.

← formula

Dollar Convexity

Dollar Convexity of a bond is defined as

$$C_{\$} = \frac{\partial^2 B}{\partial y^2},$$

and measures the sensitivity of the dollar duration of a bond with respect to small changes of the bond yield. It is easy to see that $C_{\$} = B \times C$. formula

Note that the change in dollar amount of a bond is related to dollar duration and dollar convexity as

$$\Delta B \approx -D_{\$} \Delta y + \frac{C_{\$}}{2} \cdot (\Delta y)^2$$

$$B \times \left(\frac{\Delta B}{B} = -D \cdot \Delta y + \frac{1}{2} \cdot C (\Delta y)^2 \right)$$

$$\Delta B = -D_{\$} \cdot \Delta y + \frac{1}{2} \cdot C_{\$} (\Delta y)^2$$

Duration *vs* Convexity

The duration of a bond is a measure of how long on average the holder of the bond has to wait before receiving cash payments.

- A zero-coupon bond that lasts n years has a duration of n years.
- However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n.
- The duration is therefore a weighted average of the times when payments are made.

The convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is least when the payments are concentrated around one particular point in time.

more payments $\uparrow$ convexity cuz multiply more ti

- By matching convexity as well as duration, a company can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.

DV01

$D_\$$ 0.01%

DV01 stands for **"dollar value of one basis point"**, and is often used instead of dollar duration when quoting the risk associated with a bond position or with a bond portfolio.

The DV01 of a bond measures the change in the value of the bond for a decrease of one basis point (i.e. 0.01% or 0.0001) in the yield of the bond:

$$DV01 = \frac{D_\$}{10,000}.$$

0.01% = 1 basis point $= \frac{.01}{100} = \frac{1}{10000}$

The DV01 of a bond is always positive, since a decrease in the bond yield results in an increase in the value of the bond:

$$\Delta B \approx DV01 \text{ for } \Delta y = -0.0001.$$

$D_\$ \approx \frac{-\Delta B}{\Delta y}$

$\Delta B = -D_\$ \Delta y$

$\Delta y = -0.0001 = -\frac{1}{10000}$

$\boxed{\Delta B = \frac{D_\$}{10,000} = DV01}$

DV01 *vs* Modified Duration

⇒ DV01 is useful in a hedging context – when you want to offset price movement between your portfolio with a hedging instrument.

⇒ Modified Duration is useful when comparing how sensitive different bonds are to yields.

DV01

Example A $2y$ coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually. Calculate the DV01 of this bond.

from previous explanation.

ans.: 0.0198

Formula

$$DV01 = \frac{D\$}{10{,}000} = \frac{D \times B}{10000}$$