SMU
SINGAPORE MANAGEMENT UNIVERSITY

Last lec to be tested

Session 7
Short Rate Models and Term Structure

Tee Chyng Wen

QF605 Fixed Income Securities

Term Structure Models

The **Market Models** and static replication method can handle the pricing of derivatives with **European payoffs**, such as caps, floors and European swaptions.

However, they are not able to handle derivatives with **path-dependent payoffs**, e.g. Bermudan or American option.

To value path-dependent products, we need a model of how the whole **term structure** (not just a single forward rate or bond) evolves. → keep track of all scenarios

One set of models specifies dynamics for the short rate under the risk-neutral measure. This then determines prices of zero coupon bonds, and hence, the entire term structure:

$$\underbrace{\mathbb{E}_t^* \left[e^{-\int_t^T r_u \, du} \right]}_{\text{money market acc}} = D(t,T) = e^{-\underbrace{R(t,T)}_{\text{zero rate (discounting rate)}}(T-t)}$$

Term Structure Models

A typical **short rate model** will take the following form:

$$dr_t = \mu_t \, dt + \sigma_t \, dW_t^*$$ → generic short rate model.

We begin by considering how two different features of the short rate model affect the spot curve that you obtain from the model:

1. the drift in the short rate (under $\mathbb{Q}^*$)
2. the volatility of the short rate (under $\mathbb{Q}^*$)

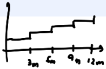
→ how they impact short rate
↓
so how that impacts term structure.

<u>1. Drift:</u> Suppose a (simplistic) short rate model specifies

$$dr_t = \mu \, dt,$$

where μ is a constant. The short rate grows linearly over time, and is deterministic. We could also write this as

$$\mu = \frac{dr_t}{dt}.$$



Drift in Short Rate Models

1st consider this

Example We consider a discrete approximation of positive μ. Suppose the initial $3m$ rate (with continuous compounding) is 5%. The next $3m$ rates will be 5.1%, 5.2%, 5.3%, $\cdots$ and so on.

so add

$$D(0,3m) = e^{-(0.05 \times 0.25)}$$

$$D(0,6m) = e^{-(0.05+0.051)\cdot 0.25} \quad \Rightarrow \quad R(0,6m) = 0.0505$$

$$D(0,9m) = e^{-(0.05+0.051+0.052)\cdot 0.25} \quad \Rightarrow \quad R(0,9m) = 0.051$$

$$D(0,12m) = e^{-(0.05+0.051+0.052+0.053)\cdot 0.25} \quad \Rightarrow \quad R(0,12m) = 0.0515.$$

if rates upward then term structure is upward.

Based on the calculation, we conclude that the term structure is upward sloping.

If μ is negative, then the term structure will be downward sloping.

Drift in Short Rate Models

Mathematically, we proceed as follows:

- First, we integrate the short rate SDE from 0 to t to obtain an expression for the short rate process: $\int_0^T dr_t = \int_0^T \mu \, dt$

$$r_t = r_0 + \mu t.$$ → short rate but not tradeable

- Next, we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T-t) + \frac{1}{2}\mu(T^2 - t^2) = r_t(T-t) + \frac{1}{2}\mu(T-t)^2$$

↳ for aesthetics

- We can now reconstruct the discount factor as

mentioned earlier this zerorate is the discount factor

$$e^{-R(t,T)(T-t)} = D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) - \frac{1}{2}\mu(T-t)^2}.$$

- Therefore, the spot curve in this stylized (simplified) model is given by

only drift now!

initial rate

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = \frac{1}{2}\mu(T-t) + r_t.$$

↳ more we add +ve μ → the $R(t,T)$ goes up

rate increase amt

Clearly, if $\mu > 0$, the spot curve is upward sloping, and if $\mu < 0$, the spot curve is downward sloping.

Term structure is upward

Model: $dr_t = \mu dt$

① $\int_0^t dr_u = \int_0^t \mu \, du$

$r_t - r_0 = \mu t$

$r_t = r_0 + \mu t$

② $\int_t^T r_u du = \int_t^T r_0 du + \int_t^T \mu u \, du$

$= r_0 (T-t) + \mu \left[\frac{u^2}{2}\right]_t^T$

$= r_0 (T-t) + \mu^2 \frac{T^2 - t^2}{2}$

$= r_0 (T-t) + \mu \cdot \frac{T^2 - t^2 - 2Tt + 2Tt + t^2 - t^2}{2}$ grouped based on colours

$= r_0 (T-t) + \mu \frac{2Tt - 2t^2}{2} + \frac{(T-t)^2}{2}$

$= r_0 (T-t) + \mu t (T-t) + \mu \frac{(T-t)^2}{2}$

$= (r_0 + \mu t)(T-t) + \mu \frac{(T-t)^2}{2}$

earlier calced

$= r_t (T-t) + \mu \frac{(T-t)^2}{2}$

Volatility in Short Rate Models

2. Volatility: Suppose a (simplistic) short rate model specifies

$$\rightarrow dr_t = \sigma dW_t^* \quad ; \quad \Delta r_t = \sigma \Delta W_t^*$$

where σ is a constant, and W_t^* is a Brownian motion under $\mathbb{Q}^*$.

The short rate follows a random walk without drift under $\mathbb{Q}$, where σ affects the variance of the "error term".

In discrete term, we have

$$r_{t+\Delta t} \approx r_t + \Delta r_t = r_t + \sigma \Delta W_t^*$$

↳ discretize it to see how it might affect the term structure

Volatility in Short Rate Models

Example We consider a discrete approximation of this short rate model with small σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are describe by a tree, where each period the short rate can move up or down by 0.5%. The risk-neutral probability of an up/down move is always 0.5.

A 2-period tree looks as follows:

$r{=}5.5\%$

$D(1y{,}2y){=}0.94649 = e^{-0.055 \cdot 1}$

$r{=}5\%$

$D(0{,}2y){=}0.90485$

$r{=}4.5\%$

$D(1y{,}2y){=}0.956 = e^{-0.045 \cdot 1}$

forward discount factors.

↳ discounting in future for a further future val

↳ lower rates higher discount factors.

$$D(0,2y) = \mathbb{E}^*\left[D(0,1y)\, D(1y,2y)\right]$$

$$= D(0,1y) \cdot \mathbb{E}^*\left[D(1y,2y)\right]$$

$e^{-0.05 \cdot 1}$

$$= D(0,1y) \cdot \left[\frac{1}{2} \cdot .94649 + \frac{1}{2}\, 0.956\right.$$

Volatility in Short Rate Models

A 3-period tree looks as follows:

$r{=}6\%$
$D(2y{,}3y){=}0.94176$

$D(1y,2y)\cdot E[D(2y,3y)]$
↓
$r{=}5.5\%$
$D(1y{,}3y){=}0.89585$
$\hookrightarrow e^{-0.055\cdot 1} \times \left[\frac{1}{2}\times 0.94146 + \frac{1}{2}\times 0.95123\right]$

$r{=}5\%$
$D(0{,}3y){=}0.86076$

$r{=}5\%$
$D(2y{,}3y){=}0.95123$

$r{=}4.5\%$
$D(1y{,}3y){=}0.91394$

$r{=}4\%$
$D(2y{,}3y){=}0.96079$

From the zero coupon bond prices, we work out the spot rates:

$$R(0,1y) = 5\%,\ D(0,2y) = 0.90485 \quad\Rightarrow\quad R(0,2y) = 4.9994\%$$
$$D(0,3y) = 0.86076 \quad\Rightarrow\quad R(0,3y) = 4.9979\%$$

term structure upward sloping here

Volatility in Short Rate Models

Example We now consider a discrete approximation of the short rate model with large σ. Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are described by a tree, where each period the short rate can move up or down by 2%, and the risk-neutral probability of an up/down move is always $\frac{1}{2}$.

A 2-period tree looks as follows:

$$
\begin{array}{ccc}
 & & r=7\% \\
 & \nearrow & D(1y,2y)=0.93239 \\
r=5\% & & \\
D(0,2y)=0.90502 & & \\
 & \searrow & r=3\% \\
 & & D(1y,2y)=0.97045
\end{array}
$$

back calc

Volatility in Short Rate Models

A 3-period tree looks as follows:

r=9%
$D(2y,3y)$=0.91393

r=7%
$D(1y,3y)$=0.86953

r=5%
$D(0,3y)$=0.86157

r=5%
$D(2y,3y)$=0.95123

r=3%
$D(1y,3y)$=0.94195

r=1%
$D(2y,3y)$=0.99005

From the zero coupon bond prices, we work out the spot rates:

$$R(0,1y) = 5\%, \;\; D(0,2y) = 0.90502 \quad \Rightarrow \quad R(0,2y) = 4.99\%$$
$$D(0,3y) = 0.86157 \quad \Rightarrow \quad R(0,3y) = 4.9667\%$$

why

no drift
↓
more downward sloping term structure.

Volatility in Short Rate Models

Main Conclusions

1. Volatility of the short rate by itself produces a slightly downward sloping spot curve.

2. The higher the volatility, the more negative the slope of the spot curve.

→ answers the why in previous ques.

3. This is a consequence of Jensen's inequality and the fact that $f(x) = e^{-x}$ and $f(x) = \frac{1}{1+x}$ are convex in x.

↳ why option payoff have time value.
↳ interview ques.

Jensen's inequality states that

$$\mathbb{E}^*[e^{-r_t \cdot \Delta t}] \geq e^{-\mathbb{E}^*[r_t] \cdot \Delta t}$$

formula

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]) \text{ if } f \text{ is convex}$$
$$\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]) \text{ if } f \text{ is concave}$$

$\uparrow \quad \mathbb{E}^*\left[e^{-r_t \cdot \Delta t}\right] \geq e^{-\mathbb{E}^*[r_t] \cdot \Delta t}$

$\nearrow$

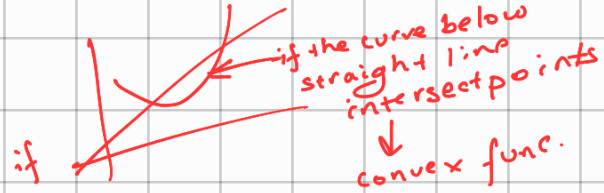
$\uparrow \quad D(t, t+\Delta t)$

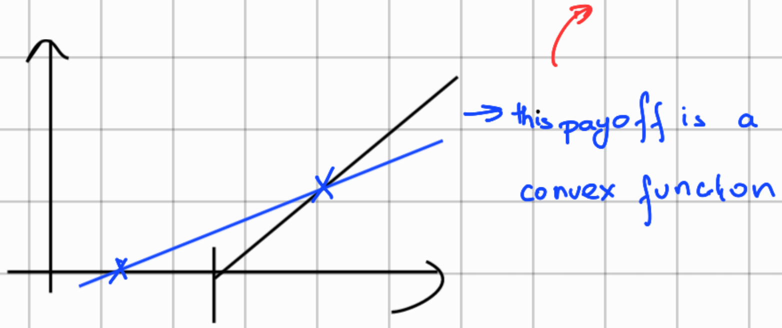
$\uparrow$

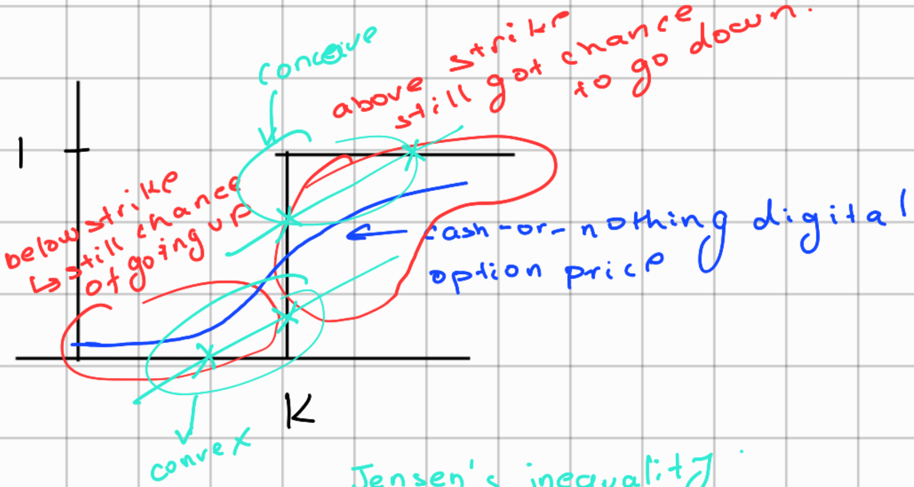
$e^{-\underbrace{R(t,t+\Delta t)} \cdot \Delta t}$

$\downarrow$

Discount factor increases, zero rate has to be lower.







Jensen's inequality.

Volatility in Short Rate Models

Mathematically, we proceed as follows:

- First, integrate the SDE from 0 to t to obtain the short rate process:

$$r_t = r_0 + \sigma W_t^*, \qquad \text{where } r_t \sim N(r_0, \sigma^2 t)$$

- Next we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T-t) + \sigma \int_t^T W_u^* \, du = r_t(T-t) + \sigma \int_t^T (W_u^* - W_t^*) \, du.$$

Recall that in the previous term, we have demonstrated that by applying Itô's formula to the function $X_t = f(t, W_t) = tW_t$, we can write

$$\int_0^T W_u \, du = \int_0^T (T-u) \, \underline{dW_u},$$

↳ mean = 0
so $\mathbb{E}[\cdot] = 0$

$$V = \int_0^T \left((T-u)\, dW_u\right)^2 = \int_0^T (T-u)^2 \, du = \frac{(T-0)^3}{3} = \frac{T^3}{3}$$

so that this integral is normally distributed, with mean and variance:

$$\mathbb{E}\left[\int_0^T W_u \, du\right] = 0, \qquad V\left[\int_0^T W_u \, du\right] = \frac{T^3}{3}.$$

Model: $dr_t = \sigma dW_t^*$

① $\int_0^t dr_u = \int_0^t \sigma dW_t^*$

$r_t = r_0 + \sigma W_t^*$ ← integrate once more.

② $\int_t^T r_u du = \int_t^T r_0 du + \int_t^T \sigma W_u^* du$

→ constant

not very elegant cause then need to keep track of historical data

$= r_0(T-t) + \sigma \int_t^T W_u^* du$

$= r_0(T-t) + \sigma W_t^* \cdot (T-t) - \sigma W_t^*(T-t) + \sigma \int_t^T W_u^* du$

$= (T-t)\overbrace{(r_0 + \sigma W_t^*)}^{=r_t} - \sigma W_t^*(T-t) + \sigma \int_t^T W_u^* du$

$= r_t(T-t) - \sigma \int_t^T W_t^* du + \sigma \int_t^T W_u^* du$

doesn't move as it doesn't use dummy variable u. So can inside or outside

→ Term 1: contribute to mean

$= r_t(T-t) + \sigma \int_t^T (W_u^* - W_t^*) du$

Term 2: contribute to variance

based on

$\int_0^T W_u du = \int_0^T (T-u) dW_u$

$\mathbb{E}\left[\int_0^T W_u du\right] = 0$

$V\left[\int_0^T (T-u) dW_u\right] = \int_0^T ((T-u) dW_u)^2 = \int_0^T (T-u)^2 du$

$= \frac{(T-0)^3}{3} = \frac{T^3}{3}$

$\therefore \int_t^T (W_u^* - W_t) du$

$\mathbb{E}\left[\int_t^T r_u du\right] = r_t(T-t)$ → only 1st term contributes to mean but not variance

so doesn't contribute to the mean

$\mathbb{E}\left[\int_t^T (W_u^* - W_t) du\right] = 0$

→ second term contributes to var

$V\left[\int_t^T r_u du\right] = V\left[\sigma \int_t^T (W_u^* - W_t) du\right] = \sigma^2 \frac{(T-t)^3}{3}$

Volatility in Short Rate Models

- Applying this results to our integrated short rate process, we note that

$$\mathbb{E}\left[\int_t^T r_u \, du\right] = r_t(T-t)$$

$$V\left[\int_t^T r_u \, du\right] = V\left[\sigma \int_t^T (W_u^* - W_t^*) \, du\right] = \frac{\sigma^2 (T-t)^3}{3},$$

and hence

$$\int_t^T r_u \, du \sim N\left(r_t(T-t), \frac{\sigma^2}{3}(T-t)^3\right).$$

- We can now reconstruct the discount factor as

$$\mathbb{E}\left[e^{\theta x}\right] = e^{\mu\theta + \frac{\sigma^2\theta^2}{2}}$$

$$\theta = -1 \quad x = \int_t^T r_u \, du$$

$$\mu = \mathbb{E}\left[\int_t^T r_u \, du\right] = r_t(T-t)$$

$$\sigma^2 = V\left[\int_t^T r_u \, du\right] = \frac{\sigma^2 (T-t)^3}{3}$$

$$\theta^2 = 1$$

$$e^{-R(t,T)(T-t)} \equiv D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u \, du}\right].$$

$$= \mathbb{E}_t^*\left[e^{-\left(r_t(T-t) + \sigma\int_t^T (W_u^* - W_t)du\right)}\right]$$

$$= e^{-r_t(T-t) + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}}$$

$$= e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}$$

Volatility in Short Rate Models

- We know how to evaluate the expectation of a lognormal random variable. If $X \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

- Using this, we have

$$e^{-R(t,T)(T-t)} \equiv D(t,T) = \mathbb{E}_t^*\left[e^{-\int_t^T r_u du}\right] = e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}.$$

- Finally, we can express the zero rate $R(t,T)$ as follows:

only vol no drift

$$R(t,T) = -\frac{1}{T-t}\log D(t,T) = r_t - \frac{\sigma^2}{6}(T-t)^2.$$

- The further we look ahead (larger $T-t$), the larger the accumulated uncertainty, and hence the lower the corresponding spot rate. Also, the higher σ, the lower all spot rates.

General : $dr_t = \mu_t dt + \sigma_t dW_t^*$

easy : $dr_t + \mu dt + \sigma dW_t^*$

Vasicek Model

The Vasicek model for interest rate is a classic short rate model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*$$

Here, κ is the mean reversion coefficient, θ is the long run mean of the short rate, and σ is the volatility of the short rate. Vasicek model is mean reverting.

Applying Itô's formula to $f(r_t, t) = r_t e^{\kappa t}$, we can show that

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-T)} dW_u^*$$

We conclude that r_t is normally distributed, with a mean of

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right)$$

and a variance of

$$V[r_t] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

Vasicek Model

Once again, we can now write the integrated short rate process under Vasicek model as

$$\int_t^T r_u \, du \sim N\left(\mathbb{E}\left[\int_t^T r_u \, du\right], V\left[\int_t^T r_u \, du\right]\right).$$

This in turn allows us to reconstruct the discount factor as follows:

$$D(t,T) = \mathbb{E}\left[e^{-\int_t^T r_u \, du}\right].$$

Vasicek Model

Let $R(t,T)$ denote the zero rate covering the period $[t,T]$, so that

$$D(t,T) = e^{-R(t,T)(T-t)}.$$

After some algebra (see Session 7 Additional Examples Q2), we find that we can write

$$D(t,T) = e^{A(t,T)-B(t,T)r_t},$$

or (equivalently)

$$R(t,T) = \frac{1}{T-t}\Big[-A(t,T)+B(t,T)r_t\Big]$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right) \\ A(t,T) &= \frac{[B(t,T)-(T-t)](\kappa^2\theta-\frac{1}{2}\sigma^2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa} \end{aligned}$$

Cox-Ingersoll-Ross Model ← FY1 (no exam on this)

In any model in which the short rate is normally distributed (including the Vasicek model), there is always a non-zero probability that the short rate is negative.

An alternative model will be the Cox-Ingersoll-Ross (CIR) model:

introduced to prevent interest rate to become -ve.

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^*$$

↳ non central chi-square

lowest it goes is to 0. but then brownian disappears

However, r_t is non-centrally χ^2-distributed in the CIR model.