QF605 Fixed-Income Securities
Solutions to Assignment 2

1. Given the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(0) dW^{n+1,N},$$

we integrate both sides from 0 to T to get

$$S_{n,N}(T) = S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0) W_T^{n+1,N}$$

Under the martingale valuation framework, we have

$$\begin{aligned}
\frac{V^{pay}_{n,N}(0)}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N}\left[\frac{V^{pay}_{n,N}(T)}{P_{n+1,N}(T)}\right] \\
V^{pay}_{n,N}(0) &= P_{n+1,N}(0)\mathbb{E}^{n+1,N}\left[(S_{n,N}(T)-K)^+\right] \\
&= P_{n+1,N}(0)\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_{n,N}(0)\left(1+\sigma_{n,N}\sqrt{T}x\right)-K\right)^+ \; e^{-\frac{x^2}{2}}\; dx \\
&= P_{n+1,N}(0)\cdot\frac{1}{\sqrt{2\pi}}\left(\int_{x^*}^{\infty}(S_{n,N}(0)-K)\; e^{-\frac{x^2}{2}}\; dx + \int_{x^*}^{\infty} S_{n,N}(0)\sigma_{n,N}\sqrt{T}x\; e^{-\frac{x^2}{2}}\; dx\right) \\
&= P_{n+1,N}(0)\left[(S_{n,N}(0)-K)\Phi(-x^*) + S_{n,N}(0)\sigma_{n,N}\sqrt{T}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x^*)^2}{2}}\right] \\
&= P_{n+1,N}(0)\left[(S_{n,N}(0)-K)\Phi(-x^*) + S_{n,N}(0)\sigma_{n,N}\sqrt{T}\phi(x^*)\right] \quad \triangleleft
\end{aligned}$$

2. (a) Solving the displaced-diffusion stochastic differential equation, we obtain

$$L_i(T_{i+1}) = \frac{L_i(0)}{\beta}e^{-\frac{1}{2}\beta^2\sigma^2 T_{i+1}+\beta\sigma W_{T_{i+1}}} - \frac{1-\beta}{\beta}L_i(0).$$

Taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}[L_i(T_{i+1}] &= \mathbb{E}^{i+1}\left[\frac{L_i(0)}{\beta}e^{-\frac{1}{2}\beta^2\sigma^2 T_{i+1}+\beta\sigma W_{T_{i+1}}} - \frac{1-\beta}{\beta}L_i(0)\right] \\
&= \frac{L_i(0)}{\beta} - \frac{1-\beta}{\beta}L_i(0) \\
&= L_i(0) \quad \triangleleft
\end{aligned}$$

(b) Let $T = T_{i+1}$, the payoff is positive when

$$\begin{aligned}
L_i(T) &> K \\
\frac{L_i(0)}{\beta}e^{-\frac{1}{2}\beta^2\sigma_i^2 T+\beta\sigma_i\sqrt{T}x} - \frac{1-\beta}{\beta}L_i(0) &> K
\end{aligned}$$

Now let

$$K' = K + \frac{1-\beta}{\beta} L_i(0), \qquad L_i(0)' = \frac{L_i(0)}{\beta}, \qquad \sigma_i' = \sigma_i \beta,$$

the inequality is given by

$$x > \frac{\log \frac{K'}{L_i(0)'} + \frac{(\sigma_i')^2 T}{2}}{\sigma_i' \sqrt{T}} = x^*$$

Now evaluating the expectation, we have

$$\begin{aligned}
\mathbb{E}[(L_i(T) - K)^+] &= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(\frac{L_i(0)}{\beta} e^{-\frac{\beta^2 \sigma_i^2 T}{2} + \beta \sigma_i \sqrt{T} x} - \frac{1-\beta}{\beta} L_i(0) - K \right) e^{-\frac{x}{2}} \, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(L_i(0)' e^{-\frac{(\sigma_i')^2 T}{2} + \sigma_i' \sqrt{T} x} - K' \right) e^{-\frac{x}{2}} \, dx \\
&= L_i(0)' \Phi \left(\frac{\log \frac{L_i(0)'}{K'} + \frac{(\sigma_i')^2 T}{2}}{\sigma_i' \sqrt{T}} \right) - K' \Phi \left(\frac{\log \frac{L_i(0)'}{K'} - \frac{(\sigma_i')^2 T}{2}}{\sigma_i' \sqrt{T}} \right) \quad \triangleleft
\end{aligned}$$

3. The receiver swaption payoff is given by

$$P_{n+1,N}(T)(K - S_{n,N}(T))^+$$

where $T = T_n$. If we write

$$\begin{aligned}
\frac{V_0^{rec}}{B_0} &= \mathbb{E}^* \left[\frac{V_T^{rec}}{B_T} \right] \\
V_0^{rec} &= \mathbb{E}^* \left[\frac{B_0}{B_T} P_{n+1,N}(T)(K - S_{n,N}(T))^+ \right]
\end{aligned}$$

and we cannot proceed any further. We need to change the measure to $\mathbb{Q}^{n+1,N}$:

$$\begin{aligned}
V_0^{rec} &= \mathbb{E}^{n+1,N} \left[\frac{B_0}{B_T} P_{n+1,N}(T)(K - S_{n,N}(T))^+ \cdot \frac{d\mathbb{Q}^*}{d\mathbb{Q}^{n+1,N}} \right] \\
&= \mathbb{E}^{n+1,N} \left[\frac{B_0}{B_T} P_{n+1,N}(T)(K - S_{n,N}(T))^+ \cdot \frac{B_T / B_0}{P_{n+1,N}(T) / P_{n+1,N}(0)} \right] \\
&= P_{n+1,N}(0) \mathbb{E}^{n+1,N} \left[(K - S_{n,N}(T))^+ \right] \quad \triangleleft
\end{aligned}$$