Session 8
Ho-Lee & Hull-White Models

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QF605 Fixed Income Securities

Note: Integrating W_t wrt t

Consider the following integral:

$$
\begin{aligned}
\int_0^T W_t \, dt &= \int_0^T \int_0^t dW_u \, dt \\
&= \int_0^T \int_u^T dt \, dW_u \\
&= \int_0^T (T-u) \, dW_u \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} (T-t_i)(W_{t_{i+1}} - W_{t_i})
\end{aligned}
$$

As a deterministic function $(T - t_i)$ weighted sum of independent Brownian increment, this integral must be normally distributed.

$\Rightarrow$ It remains to determine the mean and variance of the integral.

Note: Integrating W_t wrt t

The mean is given by

$$\mathbb{E}\left[\int_0^T (T-u)\, dW_u\right] = 0$$

since all stochastic integral has zero-mean, and

$$\begin{aligned} V\left[\int_0^T (T-u)\, dW_u\right] &= \mathbb{E}\left[\int_0^T (T-u)^2\, du\right] \\ &= \int_0^T \left(T^2 - 2uT + u^2\right) du \\ &= \left[T^2 u - u^2 T + \frac{u^3}{3}\right]_0^T = \frac{T^3}{3} \end{aligned}$$

where we have used Itô's Isometry.

"Affine Term Structure Models"

Equilibrium Affine Models

stable in long run (will mean revert)

Definition It can be shown that in any **equilibrium short rate model** (e.g. Vasicek, CIR), the zero coupon bond prices can be reconstructed as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)}$$

for some deterministic functions $A(t,T)$ and $B(t,T)$ of t and T only.

This implies that the **spot curve or zero rate curve** can be written as

$$R(t,T) = \frac{1}{T-t}\Big(- A(t,T) + r_t B(t,T) \Big)$$

for this class of model.

$$= -\frac{(A(t,T))}{T-t} + r_t \left(\frac{B(t,T)}{T-t}\right)$$

$$= \alpha + \beta \cdot r_t$$

In this class of model, spot rates are affine functions of the short rate, and so this class is referred to as the class of **affine term structure models**.

Affine function is composed of a linear function plus a constant (translation).

Affine: $f(x) = a + bx$

Linear: $f(x) = bx$ (cannot have constant)

No-Arbitrage Affine Models

However, equilibrium models only have a few model parameters—there is no guarantee that we will be able to fit to the observed term structure.

Although it is possible to perform a least square optimization to match the observed discount factors as closely as possible, to prevent arbitrage, we must be able to fit exactly to liquid discount instruments.

Ho-Lee, and subsequently **Hull-White**, proposed to address this problem by letting the model parameters be deterministic function of time – this way, we can match any observed spot curve $R(t, T)$.

Standard terminology for these models is that these are **no-arbitrage short rate models**:

adjust constant θ to take in more params

$$\text{Ho-Lee:} \qquad dr_t = \theta(t)dt + \sigma dW_t^*.$$

$$\text{Hull-White:} \qquad dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t^*.$$

← Extended Vasicek model

The simplest no-arbitrage model is the Ho-Lee model: where we choose the deterministic function $\theta(t)$ to match the observed spot curve.

match θ(t) to all discount factors in the market.

"parsimonious"

Equilibrium models

feature: few parameters cuz describe long run ignore short run

cannot ignore if short term trading

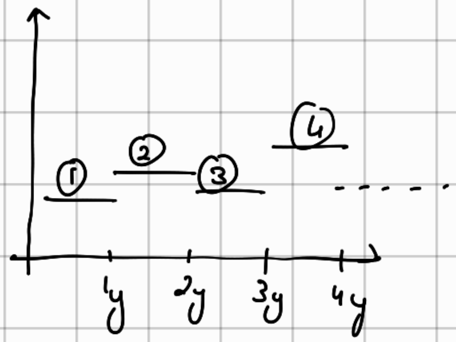
$dr_t = \mu dt + \sigma dW_t^*$ $\rightarrow$ μ, σ

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$ $\rightarrow$ k, θ, σ

Non-equilibrium models:

$\theta(t)$: piecewise constant

gives more freedom to arrange and match discount factor (prevent arbitrage)



1 $\rightarrow$ 1y

1 + 2 $\rightarrow$ 2y

1 + 2 + 3 $\rightarrow$ 3y

1 + 2 + 3 + 4 $\rightarrow$ 4y

Ho-Lee Model

In the Ho-Lee interest rate model, the short rate follows:

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a Brownian motion under the measure $\mathbb{Q}^*$. To **fit the initial term structure**, we require that

$$\theta(T) = -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.$$

To show this, first write out the interest rate process by integrating both sides:

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*.$$

Next, integrate again to obtain an expression for the integrated rate:

$$\begin{aligned}\int_0^T r_u\,du &= \int_0^T r_0\,du + \int_0^T\int_0^u \theta(s)\,ds\,du + \int_0^T\int_0^u \sigma\,dW_s^*\,du\\ &= r_0T + \int_0^T \theta(s)(T-s)\,ds + \int_0^T \sigma(T-s)\,dW_s^*.\end{aligned}$$

Model: $dr_t = \theta(t)dt + \sigma dW_t^*$

① $$\int_0^t dr_s = \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

$$r_t - r_0 = \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

$$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$$

(Fubini's theorem)

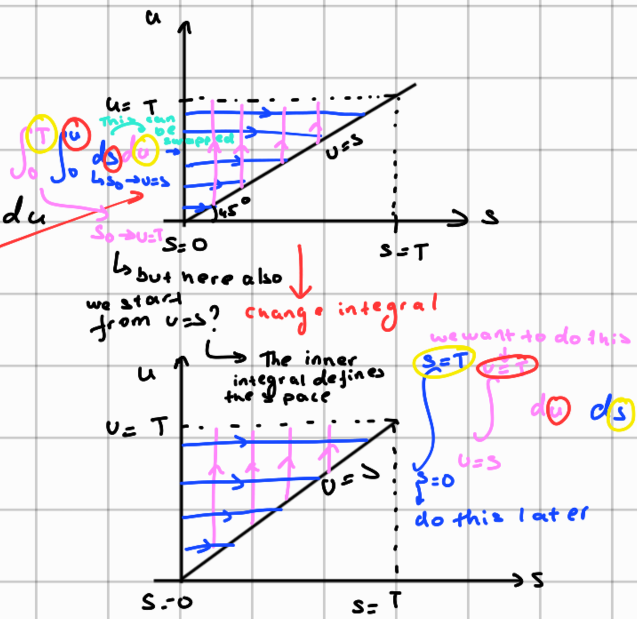
② $$\int_0^T r_u du = \int_0^T r_0 du + \int_0^T \int_0^t \theta(s)ds\,du + \int_0^T \int_0^t \sigma dW_s^* du$$

change integration order.

$$= r_0 T + \int_0^T \int_0^t \theta(s)ds\,du + \int_0^T \int_0^t \sigma dW_s^* du$$

$$= r_0 T + \int_0^T \int_s^T \theta(s)du\,ds + \int_0^T \int_s^T \sigma du\,dW_s^*$$

$$= r_0 T + \int_0^T \theta(s)(T-s)ds + \int_0^T \sigma(T-s)dW_s^*$$



Ho-Lee Model

The mean of this stochastic integral is given by

$$\mathbb{E}\left[\int_0^T r_u \, du\right] = r_0 T + \int_0^T \theta(s)(T-s)\, ds,$$

and the variance is given by

if starting from t instead of 0 start from t.

$$V\left[\int_0^T r_u \, du\right] = \int_0^T \sigma^2 (T-s)^2 \, ds = \frac{1}{3}\sigma^2 T^3,$$

where we have used **Itô Isometry**.

Therefore, the **zero-coupon discount bond can be reconstructed** as

$$e^{-R(0,T)(T-u)} = D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u \, du}\right] = \exp\left[-r_0 T - \underbrace{\int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3}_{\text{func of time .. so deterministic}}\right].$$

Since we can express $D(0,T)$ in the form of $e^{A(0,T)-r_0 B(0,T)}$, we see that Ho-Lee is an affine model.

Ho-Lee Model

Fitting the initial term structure

From here we can work out that

$$
\begin{aligned}
\log D(0,T) &= -r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3 \\
\frac{\partial}{\partial T}\log D(0,T) &= -r_0 - \int_0^T \theta(s)\, ds + \frac{1}{2}\sigma^2 T^2 \\
\frac{\partial^2}{\partial T^2}\log D(0,T) &= -\theta(T) + \sigma^2 T \\
\Rightarrow \quad \theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.
\end{aligned}
$$

This allows Ho-Lee model to fit the initial term structure $D(0,T)$ observed in the market.

Ho-Lee Model

$dr_t = \theta(t)dt + \sigma dW_t^*$

We have shown that Ho-Lee model allows us to reconstruct the discount factor

$$e^{-R(t,T)(T-t)} \equiv D(t,T) = e^{\overbrace{A(t,T)}^{\text{time}} - \overbrace{r_t B(t,T)}^{\text{short rate}\to\text{stochastic}}},$$

(discount factor is a bivariate func of time & short rate (stochastic process))

where

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6},$$
$$B(t,T) = T - t.$$

What does Ho-Lee model tell us about the evolution of discount factors over time?

⇒ Note that the reconstructed discount factor is given as a function of time and short rate, i.e. $D(t,T) = f(t, r_t)$.

This means that we can use **Itô's formula** to derive the stochastic differential equation describing the evolution of the discount factors over time.

Ho-Lee Model

First, we work out the partial derivatives

formula

$$
\begin{aligned}
f(t,x) &= e^{A(t,T)-xB(t,T)}\\
f_t(t,x) &= e^{A(t,T)-xB(t,T)}\left[\frac{\partial A(t,T)}{\partial t} - x\cdot\frac{\partial B(t,T)}{\partial t}\right]\\
f_x(t,x) &= e^{A(t,T)-xB(t,T)}\left[-B(t,T)\right]\\
f_{xx}(t,x) &= \overbrace{e^{A(t,T)-xB(t,T)}}^{D(t,T)}\left[B(t,T)^2\right],
\end{aligned}
$$

where an application of **Leibniz's rule** yields

diff on integration sign [Leibniz's rule].

$$
A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6} \qquad ;\ B(t,T) = T - t
$$

$$
\frac{\partial A(t,T)}{\partial t} = \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}.
$$

On the other hand, the time derivative for $B(t,T)$ is simply

formula

$$
\frac{\partial B(t,T)}{\partial t} = -1.
$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6}$$

formula

diff with respect to t.

$$\frac{\partial A(t,T)}{\partial t} = -\left[\theta(T)\cancel{(T-T)}^{0}\cdot\cancel{\frac{dT}{dt}}^{1} - \theta(t)(T-t)\cdot\cancel{\frac{dt}{dt}}^{1} + \int_t^T \cancel{\frac{\partial}{\partial t}\theta(s)(T-s)}^{0}\,ds\right] - \frac{\sigma^2}{2}(T-t)^2$$

$$= \theta(t)(T-t) - \frac{\sigma^2}{2}(T-t)^2$$

Ho-Lee Model

$$d(D(t,T)) = D(t,T)\left[\theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2} + r_t\right]dt$$
$$- D(t,T)(T-t)\left(\theta(t)dt + \sigma dW_t^*\right)$$
$$+ \tfrac{1}{2} D(t,T)(T-t)^2\sigma^2 dt \;=\; r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*$$

Applying **Itô's formula**, we obtain the following stochastic differential equation:

formula

$f_t(t,x)$ $\quad f_x(t,y)$ $\quad$ Ho Lee model $\quad f_{xx}(t,x)$

$$\begin{aligned}
dD(t,T) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2}f_{xx}(t,r_t)(dr_t)^2 \\
&= D(t,T)\left[\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t}\right]dt \\
&\quad - D(t,T)(T-t)\Big(\theta(t)dt + \sigma dW_t^*\Big) \\
&\quad + \frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt \\
&= r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*.
\end{aligned}$$

$\frac{\partial A(t,T)}{\partial t} \to \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}$, $\quad \frac{\partial B(t,T)}{\partial t} \to -1$

$dr_t = \theta(t)dt + \sigma dW_t$

$(dr_t)^2 = \sigma^2 dt$

Ho-Lee model: $dr_t = \theta(t)dt + \sigma dW_t^*$

r is stochastic not deterministic

Regardless of the absence of dW_t this is a stochastic process cuz of r_t.

Money-market Acc: $dB_t = r_t B_t \, dt$ ← no adjustment or stability

both have to be same interest otherwise arbitrage

bond more instantaneous reaction to rate change.

missing in the model mkt acc

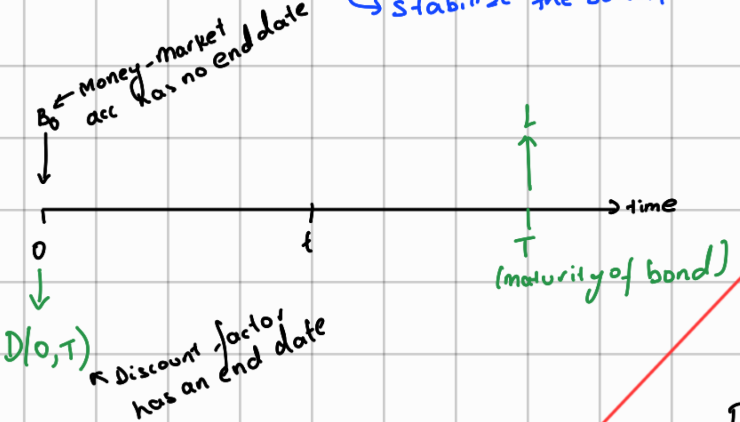
Discount factor [zero coupon bond]: $dD(t,T) = r_t D(t,T)dt - \sigma(T-t)D(t,T)dW_t^*$

As time goes by $t \to T$ so $T-t \to 0$ so brownian increment factor goes down (switches off) to 0 and no more uncertainty and it discount factor goes to 1.

drive discount factor, rates ↑ discount factor should go ↓ but if rates ↗ next time compounded more, so the $-dW_t^*$ term adjusts a bit.

↳ stabilise the bond price

So adds to the stability

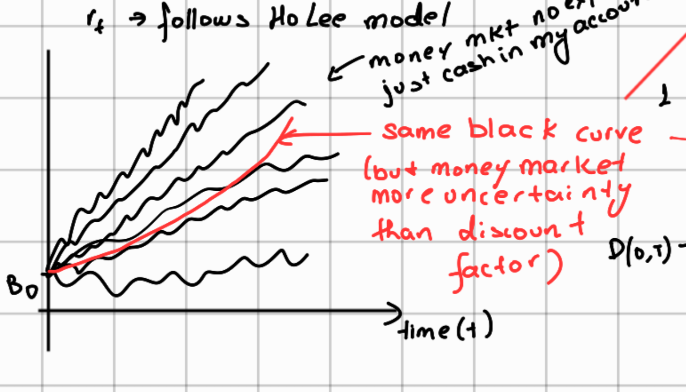


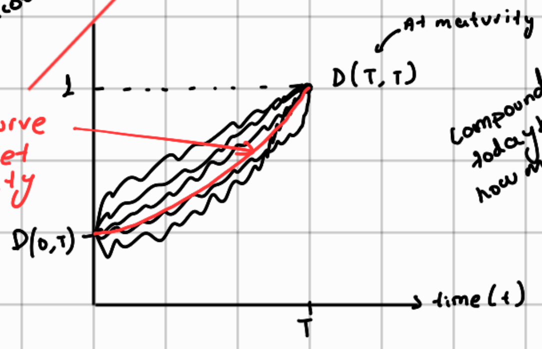
$D(t,T) = \mathbb{E}^*[e^{-\int_t^T r_u du}]$

↳ we expect $D(T,T) = 1$ so at present the integration of rates is what so it eventually goes to 1.

compound today by how much.

r_t → follows Ho Lee model





Ho-Lee Binomial Tree

$dr_t = \theta(t)\,dt + \sigma dW_t^*$

$\int_0^t dr_s = \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$

Integrating the Ho-Lee model from 0 to t, we obtain:

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \sigma W_t^*$$

either with equal prob. $\begin{cases} +1 \\ -1 \end{cases}$

Example Ho-Lee binomial tree $r_t = r_0 + \sum_{i=0}^{n-1} \theta_i \Delta t + \sum_{i=0}^{n-1} \sigma \sqrt{\Delta t}\, X_i$ ← discrete version of continuous model.

- Suppose we have an initial 1-year rate of $R = 5\%$ (with continuous compounding). $\Delta t = 1$, $r_0 = 5\%$.

  drift controlled by θ_i

- We assume that the probability of a rate increase/decrease is $\frac{1}{2}$.

  $\sigma\sqrt{\Delta t} = 1\%$

- At every node, we assume that the rate randomly increases by 1% or decreases by -1% (this is determined by the volatility of the short rate).
- Start at time $t = 0$. At time $s + 1$, we also add a deterministic amount $\sum_{u=0}^{u=s} \theta_u$ to the rate, in all nodes.

  ↳ choose θ to to match R

- We then choose the θ_u to ensure that we can match the observed spot rates $R(0,2)$, $R(0,3)$, $\cdots$.

Ho-Lee Binomial Tree

The 2-period binomial tree model looks as follows:

$r_0 + \sigma\sqrt{\Delta t}$

$r = 6\% + \theta_0$

$D(1,2) = e^{-(6\% + \theta_0)\cdot 1}$

$r = 5\%$

$D(0,2) = \ldots$

$r_0 - \sigma\sqrt{\Delta t}$

$r = 4\% + \theta_0$

$D(1,2) = e^{-(4\% + \theta_0)\cdot 1}$

We can choose θ_0 to match the observed spot rate $R(0,2)$.

Ho-Lee Binomial Tree

The 3-period binomial tree model looks as follows:

$$
\begin{tikzcd}
 & & \begin{array}{c} r{=}7\%{+}\theta_0{+}\theta_1 \\ D(2{,}3){=}e^{-(7\%+\theta_0+\theta_1)\cdot 1} \end{array} \\
 & \begin{array}{c} r{=}6\%{+}\theta_0 \\ D(1{,}3){=}\ldots \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} r{=}5\% \\ D(0{,}3){=}\ldots \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} r{=}5\%{+}\theta_0{+}\theta_1 \\ D(2{,}3){=}e^{-(5\%+\theta_0+\theta_1)\cdot 1} \end{array} \\
 & \begin{array}{c} r{=}4\%{+}\theta_0 \\ D(1{,}3){=}\ldots \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} r{=}3\%{+}\theta_0{+}\theta_1 \\ D(2{,}3){=}e^{-(3\%+\theta_0+\theta_1)\cdot 1} \end{array}
\end{tikzcd}
$$

We can then choose θ_1 to match the observed spot rate $R(0,3)$.

Slide 13

$r_0 \longrightarrow$ match $D(0,1)$

$\theta_0 \longrightarrow$ match $D(0,2)$

already fixed

$$D(0,2) = \mathbb{E}^*[\underset{\downarrow}{D(0,1)} \cdot D(1,2)]$$

$$= D(0,1)\,\mathbb{E}^*[D(1,2)]$$

$$D(0,2) = e^{-.05} \cdot \left[\frac{1}{2} \times e^{-0.06-\theta_0} + \frac{1}{2} e^{-0.04-\theta_0}\right]$$

formula

Solve for θ_0 to match $D(0,2)$

Slide 14

$r_0 \rightarrow D(0,1)$; $\theta_0 \rightarrow D(0,2)$

formula

$$D(0,3) = \mathbb{E}^*[D(0,1) \cdot D(1,3y)]$$

$$= D(0,1)\; \mathbb{E}^*[D(1,3)]$$

law of iterated expectation

$$= D(0,1)\,\mathbb{E}^*[D(1,2)\,\mathbb{E}^*[D(2,3)]]$$

$$= e^{-0.05} \times \left[\frac{1}{2} \cdot e^{-0.06-\theta_0} \cdot \left[\frac{1}{2} \cdot e^{-0.07-\theta_0-\theta_1} + \frac{1}{2} e^{-0.05-\theta_0-\theta_1}\right]\right.$$

$$\left. + \frac{1}{2} e^{-0.04-\theta_0} \cdot \left[\frac{1}{2} e^{-0.05-\theta_0-\theta_1} + \frac{1}{2} e^{-0.03-\theta_0-\theta_1}\right]\right]$$

Ho-Lee Binomial Tree

Example Consider the same Ho-Lee binomial tree given in the previous example. Suppose we observe the following in the interest rate market:

Instrument	Value
$D(0,1y)$	0.95123 → get r_0
$D(0,2y)$	0.90
$D(0,3y)$	0.86

Determine the no-arbitrage value of θ_0 and θ_1.

ans.: $\theta_0 = 0.00556$, $\theta_1 = -0.01$.

No-Arbitrage Models

Not examinable after this pt.

We should always use **no-arbitrage affine short rate models** because they allow us to fit the zero curve exactly.

The notion is that if we are to hedge our exposure using bonds and swaps, our model must at least be able to price them correctly.

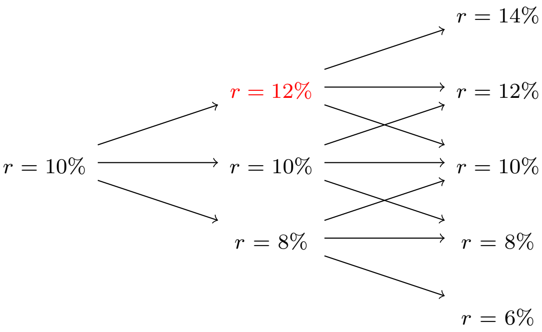
In practice, short rate models like Hull-White are more frequently implemented using **trinomial trees** (with 3 rather than 2 branches).

Ho-Lee doesn't mean revert. Hull-White mean reverts

The extra branch makes it easier to capture features like **mean reversion**. (Vasicek)

- ⇒ Time is discretized into steps of size Δt.
- ⇒ The underlying variable that evolves across the tree is the continuously compounded Δt-period short rate.
- ⇒ A key difference to binomial/trinomial tree models of the stock price is that discounting varies across branches.

Trinomial Tree



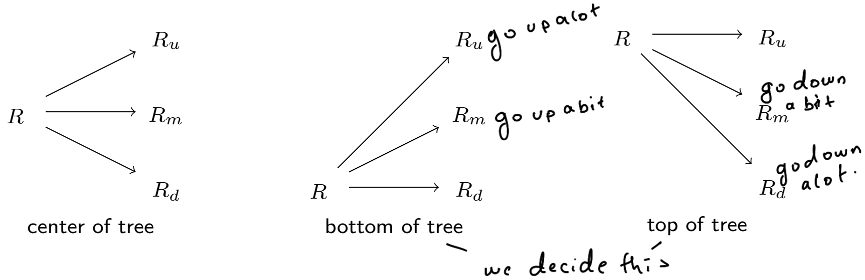
Suppose at the node indicated in red, the probabilities are $p_u = 0.25$, $p_m = 0.5$, and $p_d = 0.25$. Then at that node, a claim that pays off $100(r - 0.11)^+$ at the third date is worth $(\Delta t = 1)$

$$e^{-0.12\cdot\Delta t}\Big(0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0\Big) = 1.11$$

Hull-White Trinomial Tree

It is sometimes useful to use non-standard branching at the top and bottom of the tree.

Especially in models with mean reversion, this can improve the numerical stability of the procedure.



Hull-White Trinomial Tree

A trinomial tree with non-standard branching at the top and bottom: