Ho-Lee Model

The mean of this stochastic integral is given by

$$\mathbb{E}\left[\int_0^T r_u\, du\right] = r_0 T + \int_0^T \theta(s)(T-s)\, ds,$$

and the variance is given by

$$V\left[\int_0^T r_u\, du\right] = \int_0^T \sigma^2 (T-s)^2\, ds = \frac{1}{3}\sigma^2 T^3,$$

where we have used **Itô Isometry**.

Therefore, the **zero-coupon discount bond can be reconstructed** as

$-R(0,T)(T-0)$

$$e \quad \equiv D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u\, du}\right] = \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right].$$

Since we can express $D(0,T)$ in the form of $e^{A(0,T) - r_0 B(0,T)}$, we see that <u>Ho-Lee is an affine model</u>.

Ho-Lee Model

Fitting the initial term structure

From here we can work out that

$$
\begin{aligned}
\log D(0,T) &= -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3 \\
\frac{\partial}{\partial T}\log D(0,T) &= -r_0 - \int_0^T \theta(s)\,ds + \frac{1}{2}\sigma^2 T^2 \\
\frac{\partial^2}{\partial T^2}\log D(0,T) &= -\theta(T) + \sigma^2 T \\
\Rightarrow \quad \theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.
\end{aligned}
$$

This allows Ho-Lee model to fit the initial term structure $D(0,T)$ observed in the market.

$R(0,T)$

$$\log D(0,T) = -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3$$

$$\frac{\partial}{\partial T}\log D(0,T) = -r_0 - \left[\theta(T)\cancelto{0}{(T-T)} \cdot \cancelto{1}{\frac{dT}{dT}} - \theta(0)(T-0)\cdot \cancelto{0}{\frac{d0}{dT}} + \int_0^T \theta(s)\cdot 1\,ds \right] + \frac{1}{2}\sigma^2 T^2$$

$$\frac{\partial^2}{\partial T^2}\log D(0,T) = -0 - \left[\theta(T)\cdot \cancelto{1}{\frac{dT}{dT}} - \theta(0)\cdot \cancelto{0}{\frac{d0}{dT}} + \int_0^T \cancelto{0}{\frac{\partial}{\partial T}\theta(s)}\,ds \right] + \sigma^2 T$$

Ho-Lee Model

$dr_t = \theta(t)\,dt + \sigma\, dW_t^*$

We have shown that Ho-Lee model allows us to reconstruct the discount factor

$$e^{-R(t,T)(T-t)} \equiv D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6},$$
$$B(t,T) = T - t.$$

$dD(t,T) = ?$

What does Ho-Lee model tell us about the evolution of discount factors over time?

⇒ Note that the reconstructed discount factor is given as a function of time and short rate, i.e. $D(t,T) = f(t, r_t)$.

This means that we can use **Itô's formula** to derive the stochastic differential equation describing the evolution of the discount factors over time.

Ho-Lee Model

First, we work out the partial derivatives

$$f(t,x) = e^{A(t,T)-xB(t,T)}$$
$$f_t(t,x) = e^{A(t,T)-xB(t,T)}\left[\frac{\partial A(t,T)}{\partial t} - x\cdot\frac{\partial B(t,T)}{\partial t}\right]$$
$$f_x(t,x) = e^{A(t,T)-xB(t,T)}\Big[-B(t,T)\Big]$$
$$f_{xx}(t,x) = e^{A(t,T)-xB(t,T)}\Big[B(t,T)^2\Big],$$

where an application of **Leibniz's rule** yields

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6} \qquad ;\ B(t,T) = T-t$$
$$\frac{\partial A(t,T)}{\partial t} = \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}.$$

On the other hand, the time derivative for $B(t,T)$ is simply

$$\frac{\partial B(t,T)}{\partial t} = -1.$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2 (T-t)^3}{6}$$

$$\frac{\partial A(t,T)}{\partial t} = -\Bigg[\theta(T)\cancelto{0}{(T-T)}\cdot\cancelto{1}{\frac{dT}{dt}} - \theta(t)(T-t)\cdot\cancelto{1}{\frac{dt}{\partial t}}$$

$$+ \cancelto{0}{\int_t^T \frac{\partial}{\partial t}\theta(s)(T-s)\,ds}\Bigg] - \frac{\sigma^2}{2}(T-t)^2$$

$$dD(t,T) = D(t,T)\left[\cancel{\theta(t)(T-t)} - \cancel{\frac{\sigma^2(T-t)^2}{2}} + r_t\right]dt$$
$$-D(t,T)\,(T-t)\left(\cancel{\theta(t)dt} + \sigma\, dW_t^*\right)$$
$$+\frac{1}{2}D(t,T)\,\cancel{(T-t)^2\sigma^2 dt}$$

Ho-Lee Model

Applying **Itô's formula**, we obtain the following stochastic differential equation:

$$\begin{aligned}
dD(t,T) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2}f_{xx}(t,r_t)(dr_t)^2 \\
&= D(t,T)\left[\frac{\partial A(t,T)}{\partial t} - r_t\cdot\frac{\partial B(t,T)}{\partial t}\right]dt \\
&\qquad - D(t,T)(T-t)\Big(\theta(t)dt + \sigma dW_t^*\Big) \\
&\qquad + \frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt \qquad = \\
&= r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*.
\end{aligned}$$

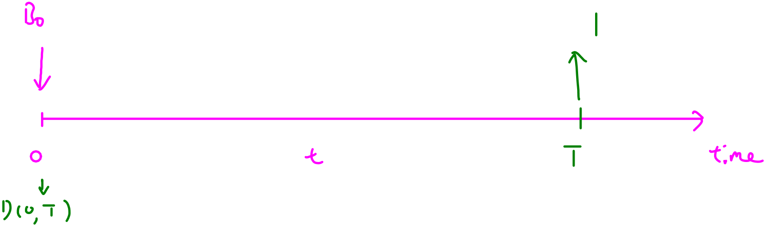
$$dr_t = \theta(t)\,dt + \sigma\, dW_t^*$$

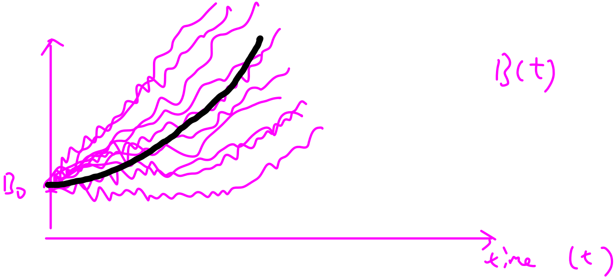
$$(dr_t)^2 = \sigma^2\, dt$$

Ho-Lee model : $dr_t = \theta(t)\,dt + \sigma\,dW_t^*$

Money-market Acc : $dB_t = r_t B_t\,dt$

Discount factor (zero coupon bond) : $dD(t,T) = r_t D(t,T)\,dt - \sigma(T-t)D(t,T)\,dW_t^*$





$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u \, du} \right]$$