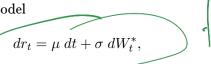
QF605 Additional Examples Session 7: Short-Rate Models and Term Structure

1 Questions

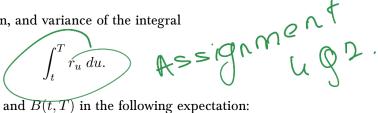
1. Consider a stylized interest rate model



formula

where W_t^* is a standard Brownian motion under the risk-neutral measure \mathbb{Q}^* .

(a) Determine the distribution, mean, and variance of the integral



(b) Identify the expressions A(t,T) and B(t,T) in the following expectation:

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right] = e^{A(t,T) - r_t B(t,T)}.$$

- (c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?
- 2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

Determine the mean and variance of the integral

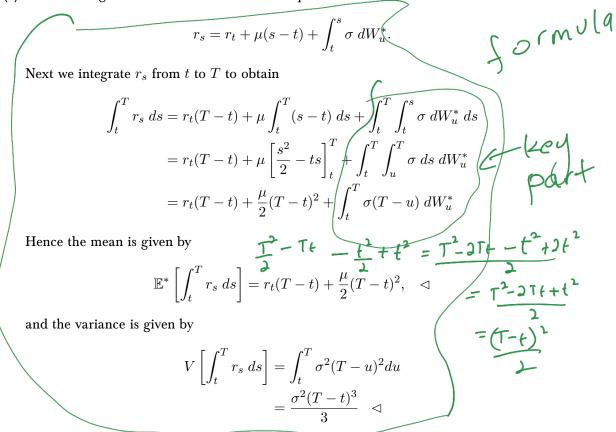
$$\int_0^T r_u du,$$

and use this to evaluate the expectation

$$D(0,T) = \mathbb{E}^* \left[e^{-\int_0^T r_u du} \right].$$

2 Suggested Solutions

1. (a) First we integrate the stochastic differential equation from t to s to obtain:



(b) Having identified the mean and variance of the short rate integral, we have

$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right]$$
$$= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2} \frac{\sigma^2(T-t)^3}{3}}$$

Comparing this against

$$D(t,T) = e^{A(t,T) - r_t B(t,T)}$$

we note that

$$A(t,T) = -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2(T-t)^3}{6}$$
 \triangleleft $B(t,T) = (T-t)$ \triangleleft

(c) For affine interest rate model, the zero coupon bond prices can be written as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)}$$

for some deterministic functions of A(t,T) and B(t,T) of t and T only. This implies that

$$R(t,T) = \frac{1}{T-t} \Big(-A(t,T) + r_t B(t,T) \Big),$$

i.e. the zero (spot) rates are affine functions of the short rate. \triangleleft

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

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We can solve this stochastic differential equation by applying Itô's formula to the function $f(r_t,t)=e^{\kappa t}r_t$, and the solution is given by

$$r_t = r_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)} dW_u^*$$

Integrating both sides from 0 to T, we have

$$\underbrace{ \int_0^T r_t \ dt = \int_0^T r_0 e^{-\kappa t} \ dt + \int_0^T \theta \left(1 - e^{-\kappa t}\right) \ dt + \underbrace{ \int_0^T \int_0^t \sigma e^{\kappa (u - t)} \ dW_u^* \ dt }_{\text{double integral}}. }$$

On the right hand side, the first and second integrals can be carried out directly. The double integral can be simplified by exchanging the order of integration (Fubini's Theorem):

Inner Integral $u: 0 \le u \le T$ Outer Integral $t: 0 \le t \le T$ Outer Integral $u: 0 \le u \le T$

So we have

$$\int_0^T \int_0^t \sigma e^{\kappa(u-t)} \ dW_u^* \ dt = \left(\int_0^T \int_u^T \sigma e^{\kappa(u-t)} \ dt \ dW_u^* \right)$$

$$= \int_0^T \left[-\frac{\sigma}{\kappa} e^{\kappa(u-t)} \right]_u^T \ dW_u^*$$

$$= \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)} \right) \ dW_u^*$$
e overall integral as:

So we can write the overall integral as:

$$\int_{0}^{T} r_{t} dt = \int_{0}^{T} r_{0} e^{-\kappa t} dt + \int_{0}^{T} \theta \left(1 - e^{-\kappa t}\right) dt + \frac{\sigma}{\kappa} \int_{0}^{T} \left(1 - e^{\kappa(u - T)}\right) dW_{u}^{*}$$

Taking expectation on both sides gives us the mean of this integral

$$t = \int_{0}^{\infty} r_{0}e^{-\kappa t} dt + \int_{0}^{\infty} \theta \left(1 - e^{-\kappa t}\right) dt + \frac{\theta}{\kappa} \int_{0}^{\infty} \left(1 - e^{\kappa(u - T)}\right) dW_{u}^{*}$$

$$\text{dion on both sides gives us the mean of this integral}$$

$$\mathbb{E}^{*} \left[\int_{0}^{T} r_{t} dt \right] = \int_{0}^{T} r_{0}e^{-\kappa t} dt + \int_{0}^{T} \theta \left(1 - e^{-\kappa t}\right) dt$$

$$= \frac{r_{0}}{\kappa} \left(1 - e^{-\kappa T}\right) + \theta T - \frac{\theta}{\kappa} \left(1 - e^{-\kappa T}\right).$$

Taking the variance, we obtain

$$V\left[\int_{0}^{T} r_{t} dt\right] = V\left[\int_{0}^{T} r_{0}e^{-\kappa t} dt + \int_{0}^{T} \theta\left(1 - e^{-\kappa t}\right) dt + \frac{\sigma}{\kappa} \int_{0}^{T} \left(1 - e^{\kappa(u - T)}\right) dW_{u}^{*}\right]$$

$$= V\left[\frac{\sigma}{\kappa} \int_{0}^{T} \left(1 - e^{\kappa(u - T)}\right) dW_{u}^{*}\right]$$

$$= \frac{\sigma^{2}}{\kappa^{2}} \int_{0}^{T} \left(1 - e^{\kappa(u - T)}\right)^{2} du \qquad \therefore \text{ Itô's Isometry}$$

$$= \frac{\sigma^{2}}{\kappa^{2}} \int_{0}^{T} \left(1 - 2e^{\kappa(u - T)} + e^{2\kappa(u - T)}\right) du$$

$$= \frac{\sigma^{2}}{\kappa^{2}} \left[T - \frac{2}{\kappa} \left(1 - e^{-\kappa T}\right) + \frac{1}{2\kappa} \left(1 - e^{-2\kappa T}\right)\right]$$

Finally, we can express the discount factor as

$$D(0,T) = \mathbb{E}^* \left[e^{-\int_0^T r_t \, dt} \right]$$

$$= \exp \left(\underbrace{-\frac{r_0}{\kappa} \left(1 - e^{-\kappa T} \right) - \theta T + \frac{\theta}{\kappa} \left(1 - e^{-\kappa T} \right)}_{\text{mean}} + \underbrace{\frac{1}{2} \cdot \underbrace{\frac{\sigma^2}{\kappa^2} \left[T - \frac{2}{\kappa} \left(1 - e^{-\kappa T} \right) + \frac{1}{2\kappa} \left(1 - e^{-2\kappa T} \right) \right]}_{\text{variance}} \right)$$