Assignment 2

1. Question: $dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(0)\, dW^{n+1,N}$

to

$$V_n^{pay}(0) = P_{n+1,N}\,\mathbb{E}^{n+1,N}\left[(S_{n,N}(T) - K)^+\right]$$

Answer:

$$\int_0^T dS_{n,N}(t) = \int_0^T \sigma_{n,N} S_{n,N}(0)\, dW^{n+1,N}(t)$$

$$S_{n,N}(T) - S_{n,N}(0) = S_{n,N}(0)\,\sigma_{n,N} \times (W^{n+1,N}(T) - 0)$$

$$S_{n,N}(T) = S_{n,N}(0) + S_{n,N}(0)\,\sigma_{n,N}\, W^{n+1,N}(T)$$

$$\frac{V_0}{N_0} = \mathbb{E}\left[\frac{V_T}{N_T}\right]$$

$$\therefore \frac{V^{payer}_{n,N}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V^{payer}_{n,N}(T_n)}{P_{n+1,N}(T_n)}\right]$$

$$\therefore V^{payer}_{n,N}(0) = P_{n+1,N}(0) \times \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\,(S_{n,N}(T) - K)^+}{\cancel{P_{n+1,N}(T)}}\right]$$

$$V^{payer}_{n,N}(0) = P_{n+1,N}(0)\,\mathbb{E}^{n+1,N}\left[(S_{n,N}(T) - K)^+\right]$$

$$V^{payer}_{n,N}(0) = P_{n+1,N}(0)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left(S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}x - K\right)^+ e^{-x^2/2}\,dx$$

$S(0) + \sigma_n S(0)\sqrt{T}x - K > 0$

$\sigma_n S(0)\sqrt{T}x > K - S(0)$

$S(0)\sqrt{T}x > K - S(0)$

$x > \frac{K - S(0)}{\sigma_n S(0)\sqrt{T}} = x^*$

$$= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}x - K\right)e^{-x^2/2}\,dx$$

$$= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}(S_{n,N}(0) - K)e^{-x^2/2}\,dx + \int_{x^*}^{\infty}\sigma_{n,N} S_{n,N}(0)\sqrt{T}x\,e^{-x^2/2}\,dx$$

$$= P_{n+1,N}(0)\left[(S_{n,N}(0)-K)\,\Phi(-x^*) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\int_{x^*}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-x^2/2}\,dx \right.$$

← I missed the phi here

$$= P_{n+1,N}(0)\left[(S_{n,N}(0)-K)\,\Phi(-x^*) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\,\phi(x^*) \right]$$

$$x^* = \frac{K - S_{n,N}(0)}{\sigma_{n,N} S_{n,N}(0)\sqrt{T}}$$

2 $$dL_i(t) = \sigma_i\left[\beta L_i(t) + (1-\beta)L_i(0)\right]dW_t^{i+1}$$

General displaced difusion formula:

$$dF_t = \sigma\left[\beta F_t + (1-\beta)F_0\right]dW_t^*$$

$$\therefore F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T^*} - \frac{1-\beta}{\beta}F_0$$

in this case F_t is $L_i(t)$, F_0 is $L_i(0)$, σ is σ_i and W_t^* is W_t^{i+1}

$$\therefore L_i(T) = \frac{L_i(0)}{\beta} e^{-\frac{\beta^2\sigma_i^2 T}{2} + \beta\sigma_i W_T^{i+1}} - \frac{1-\beta}{\beta}L_i(0)$$

$\frac{L_i(0)}{\beta}$ can be expanded as $L_i(0) + \frac{(1-\beta)}{\beta}L_i(0)$

$$\therefore L_i(T) + \frac{(1-\beta)}{\beta}L_i(0) = \left(L_i(0) + \frac{1-\beta}{\beta}L_i(0)\right)\exp\left(-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i W_T^{i+1}\right)$$

$$\therefore L_i(T) = \left(L_i(0) + \frac{1-\beta}{\beta}L_i(0)\right)\left(\exp\left(-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i W_T^{i+1}\right)\right) - \frac{1-\beta}{\beta}L_i(0)$$

a. $\mathbb{E}^{i+1}[L_i(T_i)] = \left(L_i(0) + \frac{1-\beta}{\beta} L_i(0)\right) \exp\left(-\frac{1}{2}(\beta\sigma_i)^2 T\right) \mathbb{E}[\beta\sigma_i W_T^{i+1}]$

$$- \frac{1-\beta}{\beta} L_i(0)$$

$$= L_i(0) + \frac{1-\beta}{\beta} L_i(0)\, \cancel{e^{-\frac{1}{2}(\beta\sigma_i)^2 T}} \cdot \cancel{e^{\frac{1}{2}(\beta\sigma_i)^2 T}} - \frac{1-\beta}{\beta} L_i(0)$$

$$= L_i(0) + \cancel{\frac{1-\beta}{\beta} L_i(0)} - \cancel{\frac{1-\beta}{\beta} L_i(0)}$$

$$= L_i(0)$$

b. $\left(L_i(0) + \frac{1-\beta}{\beta}(L_i(0))\right) e^{-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i\sqrt{T}x} - \frac{1-\beta}{\beta} L_i(0) - K > 0$

$$\left(L_i(0) + \frac{1-\beta}{\beta} L_i(0)\right) e^{-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i\sqrt{T}x} > K + \frac{1-\beta}{\beta} L_i(0)$$

$$e^{-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i\sqrt{T}x} > \frac{K + \frac{1-\beta}{\beta} L_i(0)}{L_i(0) + \frac{1-\beta}{\beta} L_i(0)}$$

$$e^{-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i\sqrt{T}x} > \frac{K\beta + (1-\beta)L_i(0)}{\cancel{\beta}} \times \frac{\cancel{\beta}}{\cancel{L_i(0)\beta} + L_i(0) - \cancel{\beta L_i(0)}}$$

$$-\frac{1}{2}(\beta\sigma_i)^2 T + \beta\sigma_i\sqrt{T}x > \ln\left(\frac{K\beta + (1-\beta)L_i(0)}{L_i(0)}\right)$$

$$x > \frac{\ln\left(\frac{K\beta + (1-\beta)L_i(0)}{L_i(0)}\right) + \frac{1}{2}(\beta\sigma_i)^2 T}{\beta\sigma_i\sqrt{T}} = x^*$$

$$\mathbb{E}^{i+1}\left[\left(L_i(T_i)-K\right)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[L_i(0)+\left(\frac{1-\beta}{\beta}\right)L_i(0)e^{-\frac{1}{2}(\beta\sigma_i)^2T+\beta\sigma_i\sqrt{T}x}\right.$$

$$\left. -\frac{1-\beta}{\beta}L_i(0)-K\right]^+ e^{-\frac{x^2}{2}}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(L_i(0)+\left(\frac{1-\beta}{\beta}\right)\right)L_i(0)e^{-\frac{1}{2}\beta^2\sigma_1^2T+\beta\sigma_i\sqrt{T}x}e^{-\frac{x^2}{2}}dx$$

$$-\int_{x^*}^{\infty}\frac{1-\beta}{\beta}L_i(0)e^{-\frac{x^2}{2}}dx-\int_{x^*}^{\infty}Ke^{-x^2/2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\frac{L_i(0)}{\beta}e^{\frac{-\beta^2\sigma_i^2}{2}T+\beta\sigma_i\sqrt{T}x}e^{-\frac{x^2}{2}}dx$$

$$-\left(\frac{1-\beta}{\beta}L_i(0)+K\right)\int_{x^*}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

$$=\frac{L_i(0)}{\beta}e^{\frac{-\beta^2\sigma_i^2T}{2}}\int_{x^*}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\beta\sigma_i\sqrt{T}x}e^{-x^2/2}dx$$

$$-\left(\frac{1-\beta}{\beta}L_i(0)+K\right)\Phi(-x^*)$$

$$=\frac{L_i(0)}{\beta}e^{-\frac{\beta^2\sigma_i^2T}{2}}\int_{x^*}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x^2-2\beta\sigma_i\sqrt{T}x+(\beta\sigma_i\sqrt{T})^2-(\beta\sigma_i\sqrt{T})^2}{2}\right)}dx$$

$$-\left(\frac{1-\beta}{\beta}L_i(0)+K\right)\Phi(-x^*)$$

$$= \frac{L_i(0)}{\beta} e^{\cancel{-\frac{\beta^2\sigma_i^2 T}{2}} + \cancel{\frac{\beta^2\sigma_i^2 T}{2}}} \int_{x^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x^2 - 2\beta\sigma_i\sqrt{T} + (\beta\sigma_i\sqrt{T})^2)}{2}} dx$$

$$- \left(\frac{1-\beta}{\beta} L_i(0) + K\right) \Phi(-x^*)$$

$$= \frac{L_i(0)}{\beta} \int_{x^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x - \beta\sigma_i\sqrt{T})^2}{2}} dx - \left(\frac{1-\beta}{\beta} L_i(0) + K\right) \Phi(-x^*)$$

$$= \frac{L_i(0)}{\beta} \Phi(-x^* + \beta\sigma_i\sqrt{T}) - \left(\frac{1-\beta}{\beta} L_i(0) + K\right) \Phi(d_2)$$

$$\mathbb{E}^{i+1}[(L_i(T_i) - K)^+] = \frac{L_i(0)}{\beta} \Phi(d_1) - \left(\frac{1-\beta(L_i(0)}{\beta} + K\right) \Phi(d_2)$$

formula

$$d_1 = -x^* + \beta\sigma_i\sqrt{T} = \frac{\ln\left(\frac{L_i(0)}{K\beta + (1-\beta)L_i(0)}\right) + \frac{1}{2}(\beta\sigma_i)^2 T}{\beta\sigma_i\sqrt{T}}$$

$$d_2 = d_1 - \beta\sigma_i\sqrt{T} = -x^*$$

3. $\Phi^* \to B_t = B_0 e^{\int_0^t r_u du}$ → payoff mature at T struck at K
→ receiver swaption

$$\Phi^* : \frac{V_0^{rec}}{B_0} = \mathbb{E}^*\left[\frac{V_T^{rec}}{B_T}\right]$$

$$V_0 = B_0 \mathbb{E}^*\left[\frac{V_T^{rec}}{B_T}\right] \qquad V_T^{rec} = P_{n+1,N}(T) \cdot (K - S_{n,N}(T))^+$$

$$= \mathbb{E}^*\left[\frac{\cancel{B_0} \times P_{n+1,N}(T) \cdot (K - S_{n,N})^+}{\cancel{B_0} \cdot e^{\int_0^t r_u du}}\right]$$

→ At Φ^* r_u is stochastic so it is hard to evaluate further

$$= \mathbb{E}^*\left[\frac{P_{n+1,N}(T) \cdot (K - S_{n,N})^+}{e^{\int_0^t r_u du}}\right]$$

Now redo under $\mathbb{Q}^{n+1,N}$ where PVBP is $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$

$$\frac{V_0^{rec}}{B_0} = \mathbb{E}^{n+1,N}\left[\frac{V_T^{rec}}{B_T} \times \frac{d\mathbb{Q}^*}{d\mathbb{Q}^{n+1,N}}\right]$$

$$V_0^{rec} = \cancel{B_0}\,\mathbb{E}^{n+1,N}\left[\frac{V_T^{rec}}{\cancel{B_T}} \times \frac{\cancel{B_T}/\cancel{B_0}}{P_{n+1,N}(T)/P_{n+1,N}(0)}\right]$$

$$= \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)}\cdot(k-S_{n,N})^+}{} \times \frac{P_{n+1,N}(0)}{\cancel{P_{n+1,N}(T)}}\right]$$

$$= \mathbb{E}^{n+1,N}\left[(k-S_{n,N})^+ P_{n+1,N}(0)\right]$$

$$= P_{n+1,N}(0)\,\mathbb{E}^{n+1,N}\left[(k-S_{n,N})^+\right]$$