Additional Example 8

3. $dr_t = \theta(t)\,dt + \sigma\,dW_t^*$

$$\int_0^t dr_s = \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$\int_0^T r_u\,du = \int_0^T r_0\,du + \int_0^T \int_0^t \theta(s)\,ds\,du + \int_0^T \int_0^t \sigma\,dW_s^*\,du$$

$$\int_0^T r_u\,du = r_0 T + \int_0^T \int_s^t \theta(s)\,du\,ds + \int_0^T \int_s^t \sigma\,du\,dW_s^*$$

$$\int_0^T r_u\,du = r_0 T + \int_0^T \theta(s)(t-s)\,ds + \int_0^T \sigma(t-s)\,dW_s^*$$

$$\mathbb{E}\left[\int_0^T r_u\,du\right] = r_0 T + \int_0^T \theta(s)(T-s)\,ds$$

$$V\left[\int_0^T r_u\,du\right] = \frac{\sigma^2 T^3}{3}$$

$$D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u\,du}\right] = \exp\left(-r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{\sigma^2 T^3}{6}\right)$$

$$\log D(0,T) = -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{\sigma^2 T^3}{6}$$

$$\frac{\partial \log D(0,T)}{\partial T} = -r_0 - \int_0^T \theta(s)\,ds + \frac{\sigma^2 T^2}{2}$$

$$= -r_0 - \frac{\theta(T)^2}{2} + \frac{\sigma^2 T^2}{2}$$

$$\frac{\partial^2 \log D(0,T)}{\partial T^2} = -\theta(T) + \sigma^2 T$$

$$\therefore \theta(T) = \sigma^2 T - \frac{\partial^2}{\partial T^2} \log D(0, T)$$

$$= \frac{-\partial^2}{\partial T^2} \log D(0, T) + \sigma^2 T$$