PROJECTIVE MODULES OVER LOCAL RINGS

ISHAN LEVY

1. Projective modules over local rings are free

First we start with an abstract construction. We should think of the category as some abelian category.

Definition 1.1. *A* ***colimit dévissage*** *of an object M in a cocomplete category C with a zero object 0 is a collection of subobjects of M_α indexed by the ordinals $\leq \alpha$ such that:*

1. $M_0 = 0, M = M_\alpha$
2. *There exists for each $i < \alpha$ a N_i such that $M_{i+1} = N_i \coprod M_i$*
3. *For each limit ordinal β, M_β is the coproduct of M_i for $i < \beta$.*

Proposition 1.2. *If M_α is a colimit dévissage for M, then $M = \coprod_{i+1} M_{i+1}/M_i$.*

Proof. We will show inductively that for each $j \leq \alpha$, $M_j = \coprod_{i+1 \leq j} M_{i+1}/M_i$. For $i = 0$ this holds by (1). If it holds for i, it holds for $i+1$ by (2), and if it holds for everything less than a limit ordinal, it holds for that limit ordinal by (3). □

Definition 1.3. *A* ***Kaplansky dévissage*** *is a colimit dévissage in a category of modules such that each M_{i+1}/M_i is countably generated.*

Proposition 1.4. *If M is a sum of countably generated R-modules, and $M = P \oplus Q$, then P and Q are also sums of countably generated R modules.*

Proof. Let $M = \bigoplus_I N_j$ where N_j are countably generated. We will produce a Kaplansky dévissage M_α such that $M_i = P_i \oplus Q_i$ where $P_i = P \cap M_i, Q_i = Q \cap M_i$, and each M_i is a sum of the N_j. Then P_i, Q_i are Kaplansky dévissages for P, Q. For the construction, we only need to define M_{i+1} given $M_i \neq M$. Let N_j be the smallest j with N_j not contained in M_i, and suppose N_j is generated by $x_{11}, x_{12}, \dots x_{1n}, \dots$. Decompose $x_{11} = p_{11} + q_{11}$. Then p_{11}, q_{11} have nonzero component on finitely many N_k, which are generated by x_{2n}. Now do the same for x_{12} to get x_{3n}, and then the same for x_{21} to get x_{4n}. Proceeding this way going diagonally across the matrix x_{nm}, we can have each x_{nm} split as $p_{nm} + q_{nm}$ where p_{nm}, q_{nm} lie in N_j whose generators are in the x_{nm}. We add x_{nm} to M_i to get M_{i+1}. (3) of a colimit is satisfied as each M_i is a sum of the M_j. □

Corollary 1.5. *Every projective module is a sum of countably generated projective modules.*

Theorem 1.6. *Any projective module over a local ring is free.*

Proof. By above, we can assume that the projective module M is countably generated. Now we only to show that for any element x in M, there is a free direct summand of M containing x. For then we can do this to a set of generators of M and see that the module is free. To do this, Write $M \oplus N = F$ for F a free module, and choose a basis a_i for F such that $x = \sum_i r_i a_i, r_i \in R$ is a minimal representation of x.

$r_i \notin (r_j, j \neq i)$, or else we could remove a_i from our basis by adding an appropriate multiple of a_i to each of the $a_j, j \neq i$. Let m_i be the M component of a_i. It suffices to show that replacing the a_i with the m_i still yields a basis. To see this, if $m_i = \sum_j b_{ij} a_j$, observe that $\sum_i r_i a_i = \sum_i r_i m_i = \sum_{i,j} r_i b_{ij} a_j$, so $r_i = \sum_j r_j b_{ij}$, and b_{ii} must be a unit, and b_{ij} must not be a unit, so the determinant of (b_{ij}) is a unit. $\square$