

# NON-EMBEDDING RESULTS VIA SPANIER WHITEHEAD DUALITY

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What is the minimal dimension such that  $\mathbb{CP}^2$  or  $\mathbb{HP}^2$  can be embedded in Euclidean space? It turns out one gets the optimal answer by answering the corresponding stable problem, namely what is the minimal dimension such that a space with the stable homotopy type of  $\mathbb{CP}^2$  or  $\mathbb{HP}^2$  can be embedded in Euclidean space?

We can answer a generalization of this question as follows (due to Hilton and Spanier):

**Theorem 0.1.** *Let  $f$  be a map  $S^{m-1} \rightarrow S^n$ , and let  $C_f$  denote the cofibre. Suppose that  $f$  cannot be stably desuspended. Then the minimum dimension embedding of  $C_f$  is  $m + n + 1$ .*

In particular,  $\mathbb{CP}^2$  and  $\mathbb{HP}^2$  are cofibres of Hopf maps, which cannot stably be desuspended (a stable desuspension is another map of spheres of lower dimension agreeing stably with  $f$ ). Note that  $C_f$  can be embedded in  $S^{m+n+1}$  because the mapping cylinder of  $f$  embeds into the join of  $S^{m-1}$  and  $S^n$ , which is  $S^{m+n}$ , and the cylinder end can be coned off in  $S^{m+n+1}$ . So it suffices to show one cannot embed into anything smaller.

First we consider the simplest cases. If  $m < n + 1$ , then  $f$  is trivial, so, we must have  $n = 0$ , in which case the assertion is obvious. If  $m = n + 1$ , then  $n = 1$ . Then, one can use Alexander duality to observe that were there an embedding into  $S^3$ , then the complement would have zero dimensional homology that is not free.

Thus we can assume that  $n > 1, m > n + 1$ .

**Lemma 0.2.** *If  $n > 1, m > n + 1$ ,  $f$  can be stably desuspended iff  $C_f$  can be stably desuspended.*

*Proof.* Clearly the stable homotopy type of  $C_f$  depends only on that of  $f$ , proving one direction. On the other hand, if  $C_f$  can be stably desuspended to a space  $X$ , a homology decomposition of  $X$  will be the cofibre of a map  $g$  between spheres. After suspending enough, these will be of the same dimension, and since the map between the middle skeleton has to extend to a homotopy equivalence between the two spaces, the attaching maps differ by a unit (i.e an integer multiple), so  $f$  can be stably desuspended.  $\square$

The essential input of working stably is the following observation: the Spanier-Whitehead dual of the cofibre of  $C_f$  (denoted  $DC_f$ ) is  $\Sigma^{-m-n}C_{\pm f}$ , where the sign (unimportant) I think is  $(-1)^{mn}$ . To see this, the dual of a map  $f$  between spheres is  $\pm f$ , suspended to have the right degrees. We have a cofibre sequence,  $f : S^{m-1} \rightarrow S^n \rightarrow C_f$ , which taking duals gives a cofibre sequence  $DC_f \rightarrow S^{-n} \rightarrow S^{1-m}$ . Rearranging this shows that  $DC_f = \Sigma^{-m-n}C_{\pm f}$ . Now the complement of  $C_f$  inside  $S^{m+n}$  would be  $\Sigma^{m+n-1}DC_f = \Sigma^{-1}C_f$ ! This completes the proof via the lemma.

The same argument gives the slightly stronger version:

**Theorem 0.3.** *Let  $f$  be a map  $S^{m-1} \rightarrow S^n$ , and let  $C_f$  denote the cofibre. Let  $k$  be the maximum number of times that  $f$  can be stably desuspended. Then the minimum dimension embedding of  $C_f$  is  $m + n + 1 - k$ .*