PROJECTIVE MODULES OVER LOCAL RINGS

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1. Projective modules over local rings are free

First we start with an abstract construction. We should think of the category as some abelian category.

Definition 1.1. *A **colimit dévissage** of an object M in a cocomplete category C with a zero object 0 is a collection of subobjects of M_α indexed by the ordinals $\leq \alpha$ such that:*

(1) $M_0 = 0, M = M_\alpha$
(2) There exists for each $i < \alpha$ a N_i such that $M_{i+1} = N_i \coprod M_i$
(3) For each limit ordinal β, M_β is the coproduct of M_i for $i < \beta$.

Proposition 1.2. *If M_α is a colimit dévissage for M, then $M = \coprod_{i+1} M_{i+1}/M_i$.*

Proof. We will show inductively that for each $j \leq \alpha$, $M_j = \coprod_{i+1 \leq j} M_{i+1}/M_i$. For $i = 0$ this holds by (1). If it holds for i, it holds for $i+1$ by (2), and if it holds for everything less than a limit ordinal, it holds for that limit ordinal by (3). □

Definition 1.3. *A **Kaplansky dévissage** is a colimit dévissage in a category of modules such that each M_{i+1}/M_i is countably generated.*

Proposition 1.4. *If M is a sum of countably generated R-modules, and $M = P \oplus Q$, then P and Q are also sums of countably generated R modules.*

Proof. Let $M = \bigoplus_I N_j$ where N_j are countably generated. We will produce a Kaplansky dévissage M_α such that $M_i = P_i \oplus Q_i$ where $P_i = P \cap M_i, Q_i = Q \cap M_i$, and each M_i is a sum of the N_j. Then P_i, Q_i are Kaplansky dévissages for P, Q. For the construction, we only need to define M_{i+1} given $M_i \neq M$. Let N_j be the smallest j with N_j not contained in M_i, and suppose N_j is generated by $x_{11}, x_{12}, \ldots x_{1n}, \ldots$. Decompose $x_{11} = p_{11} + q_{11}$. Then p_{11}, q_{11} have nonzero component on finitely many N_k, which are generated by x_{2n}. Now do the same for x_{12} to get x_{3n}, and then the same for x_{21} to get x_{4n}. Proceeding this way going diagonally across the matrix x_{nm}, we can have each x_{nm} split as $p_{nm} + q_{nm}$ where p_{nm}, q_{nm} lie in N_j whose generators are in the x_{nm}. We add x_{nm} to M_i to get M_{i+1}. (3) of a colimit is satisfied as each M_i is a sum of the M_j. □

Corollary 1.5. *Every projective module is a sum of countably generated projective modules.*

Theorem 1.6. *Any projective module over a local ring is free.*

Proof. By above, we can assume that the projective module M is countably generated. Now we only to show that for any element x in M, there is a free direct summand of M containing x. For then we can do this to a set of generators of M and see that the module is free. To do this, Write $M \oplus N = F$ for F a free module, and choose a basis a_i for F such that $x = \sum_i r_i a_i, r_i \in R$ is a minimal representation of x.

$r_i \notin (r_j, j \neq i)$, or else we could remove a_i from our basis by adding an appropriate multiple of a_i to each of the $a_j, j \neq i$. Let m_i be the M component of a_i. It suffices to show that replacing the a_i with the m_i still yields a basis. To see this, if $m_i = \sum_j b_{ij} a_j$, observe that $\sum_i r_i a_i = \sum_i r_i m_i = \sum_{i,j} r_i b_{ij} a_j$, so $r_i = \sum_j r_j b_{ij}$, and b_{ii} must be a unit, and b_{ij} must not be a unit, so the determinant of (b_{ij}) is a unit. $\square$