ANALYTIC NUMBER THEORY

ISHAN LEVY

1. Asymptotics

Lemma 1.1. *If a_i is an indexed sequence of numbers ≥ 0, and $f(n) = \sum_{i\leq n} a_n$, $F(n) = \sum_{i\leq n} \log(i)a_i$. If $F(n) > 0$ for large n, then $\limsup_n \frac{f(n)\log(n)}{F(n)} \leq 1$. If $f(\lceil n^{1-\epsilon}\rceil) = o(f(n))$ as $n \to \infty$ for small $\epsilon > 0$, then $\log(n)f(n) \sim F(n)$.*

Proof. $F(n) = \sum_{i\leq n} log(i)a_i \leq \sum_{i\leq n} log(n)a_n$, showing the first statement. For the second, Observe that for any $\epsilon > 0$, we have $F(n) \geq \sum_{\lceil n^{1-\epsilon}\rceil}^{n} \log(n^{1-\epsilon})a_n = (1-\epsilon)\log(n)(f(n) - f(\lceil n^{1-\epsilon}\rceil))$, so $\frac{F(n)}{f(n)log(n)} \geq (1-\epsilon)(1-\frac{f(n^{1-\epsilon})}{f(n)})$, and letting $n\to\infty, \epsilon\to 0$ gives the result. $\square$

2. Dirichlet series and properties

A very powerful tool in mathematics for studying sequences of numbers is to study their generating functions. If a_n is a sequence, one way to turn a_n into a generating function is to consider $F(z) = \sum_0^\infty a_n z^n$. Properties of the sequence then relate to properties of the function, and vice versa. For example, we might expect the behavior of $F(z)$ as $z \to 1^-$ to be related to the asymptotics of the sequence. Also if F is analytic near 1, then the coefficients a_n can be obtained via Cauchy's integral formula. These generating functions are good at capturing additive properties of sequences, as $x^n x^m = x^{n+m}$.

Another type of generating function that can be made from sequences is an L-function. These capture more of the multiplicative structure of the sequence. These look like something of the form $L(s) = \sum_0^\infty \frac{a_n}{n^s}$, called a **Dirichlet series**. Both of these constructions are specializations of the more general construction ca where given two sequences λ_n, a_n, we can consider $\sum a_n e^{-\lambda_n s}$. The basic example is $\zeta(s) = \sum \frac{1}{n^s}$, the **Riemann zeta function**. Note the series converges absolutely and uniformly on compact sets for $\mathrm{Re}(s) > 1$.

As any good generating function should, $\zeta(s)$ tells us a great deal about the distribution of the primes. Below is a simple of example of how it can be used.

Theorem 2.1. *The series $\frac{1}{p}$ diverges, where p ranges over the positive primes in $\mathbb{Z}$.*

Proof. $\sum_1^\infty \frac{1}{n^s} = \prod_p (\frac{1}{1-\frac{1}{p^s}})$, which diverges as $s \to 1^+$. Taking the log of the right hand side we get $\sum_p -\log(1-\frac{1}{p^s}) = \sum_p \frac{1}{p} + \sum_{m\geq 2,p} \frac{1}{mp^{ms}}$. The second sum is $\leq \sum_{m\geq 2,p} \frac{1}{p^{ms}} = \sum_p \frac{1}{p^{2s}-p^s} \leq 2\sum_p \frac{1}{p^{2s}}$, which is finite as $s \to 1$. Thus $\sum_p \frac{1}{p}$ diverges. $\square$

First we will prove some lemmas about Dirichlet series. Note that a necessary condition for convergence at some point is that $|a_n|$ should be bounded by some polynomial. Here is a partial converse:

Lemma 2.2. *If* $|\sum_1^n a_n| = O(n^r)$, $r > 0$, *then the Dirichlet series for* a_n *converges uniformly for* $\mathrm{Re}(s) \geq r + \epsilon$ *for any* $\epsilon > 0$ *to a holomorphic function. If* $|a_n| = O(n^r)$, *then it converges absolutely and uniformly for* $\mathrm{Re}(s) > r + 1 + \epsilon$.

Proof. Let $f_n = \sum_1^n a_n$. We will show that the partial sums are uniformly Cauchy by using summation by parts. $\sum_k^m \frac{a_n}{n^s} = \sum_k^m \frac{f_n - f_{n-1}}{n^s} = \frac{f_m}{m^s} - \frac{f_{k-1}}{(k-1)^s} + \sum_k^{m-1} f_n(\frac{1}{n^s} - \frac{1}{(n+1)^s})$. The first two terms go to 0 uniformly as $k \to \infty$, and the difference in the sum is equal to $\int_n^{n+1}(-s)x^{-(s+1)}$, which in absolute value is at most $|s|n^{-s+1}$. But then since $f_n = O(n^r)$, the last term in absolute value is at most $\sum_k^{m-1} \frac{cn^r}{n^{-(r+\epsilon+1)}}$, which is uniformly bounded in m and goes to 0 as $k \to \infty$ since $\epsilon + 1 > 1$. The last statement is clear. □

Lemma 2.3. *If* F, G *are Dirichlet series for* a_n, b_n *that converge, and* F, G *agree where they converge, then* $a_n = b_n$.

Proof. Let a_i be the first nonzero term of a_n. Then $F \sim \frac{a_i}{i^s}$ as $s \to \infty$, so this behavior determines i and a_i. By subtracting this term off, we can recover the rest of the sequence. We have used properties of F that agree with G, so $a_n = b_n$. □

Note by analyticity, if they agree on a small set, they agree.

Lemma 2.4. *Suppose that* F, G *are convergent Dirichlet series for* a_n, b_n. *Then* FG *is the Dirichlet series for* $a_n \star b_n$, *where* $\star$ *denotes the Dirichlet convolution.*

Proof. For sufficiently large s, F, G converge absolutely and uniformly. Then when we take their product, we can change the order of summation to get the result. □

3. Zeta functions & The class number formula

Given a number field K, let $j_K(n)$ be the number of ideals of norm n in $\mathcal{O}_K$. The **Dedekind zeta function** $\zeta_K(s)$ is the Dirichlet series for $j_K(n)$

Lemma 3.1. $\zeta_K(s)$ *converges for* $\mathrm{Re}(s) > 1$.

Proof. From the theorem on the distribution of ideals, it follows that $\sum_{i \leq n} j_K(i) = O(n)$, so that by Lemma 2.2 this is true. □

We will want to extend $\zeta_K(s)$ to $\mathrm{Re}(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$, so that we can better study its behavior near 1. First $\zeta(s)$ will be extended. Note that $\zeta(s)(1 - 2^{1-s})$ is the Dirichlet series for the sequence $(-1)^{n+1}$, which converges for $\mathrm{Re}(s) > 0$. This lets us extend $\zeta(s)$ to $\mathrm{Re}(s) > 0$, but it might have some singularities where $1 - 2^{1-s}$ is 0. To reduce the number of possible poles, $\zeta(s)(1 - 3^{1-s})$ also converges for $\mathrm{Re}(s) > 0$ for the same reason, but the only common zeros of $1 - 2^{1-s}$ and $1 - 3^{1-s}$ are $s = 1$. Thus $\zeta(s)$ can only have a simple pole at $s = 1$, and indeed it does since $1 - 2^{1-s}$ has a simple zero. Now for some κ, $\zeta_K(s) - \kappa\zeta(s)$ is the Dirichlet series for some sequence $f(n)$ with $\sum_1^n f(i) = O(n^{1-\frac{1}{[K:\mathbb{Q}]}})$ so we can use this to extend $\zeta_K(s)$. Note that as a consequence, $\zeta_K(s)$ also has a simple pole at 1.

Theorem 3.2. $\zeta_K(s)$ *is an analytic function in the region* $\mathrm{Re}(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$, *and has a simple pole with residue* $\frac{h_K 2^{r+s}\pi^s \operatorname{reg}(K)}{\omega_K\sqrt{|\operatorname{disc}(K)|}}$.

Proof. Because of how we have shown that $\zeta_K(s)$ is analytic near 1, we only need to compute the residue of $\zeta(s)$ at 0. Note that the Dirichlet series for $(-1)^{s+1}$ at 1 is $\sum_i \frac{(-1)^{i+1}}{i} = \int_0^1 \sum_i (-1)^{i+1} x^{i+1} = \int_0^1 \frac{1}{1+x} = \log(2)$. On the other hand, the $s-1$ term in the series expansion of $(1-2^{1-s})$ is $\frac{1}{\log(2)}$. $\square$

Now we can write $\zeta_K(s)$ another way. But first a technical lemma.

Lemma 3.3. *Suppose that $a_i \in \mathbb{C}$ satisfy $|a_i| < 1$ and $\sum_i |a_i| \leq \infty$. Then $\prod_1^\infty (1-a_i)^{-1} = \sum_{S \subset \mathbb{N} \text{ finite}} \prod_S a_i$, where the sum is absolutely convergent.*

Proof. Note that $e^{-\sum_i |a_i|} = \prod_i e^{-|a_i|}$ converges and by Taylor's theorem is at least $\prod_i (1-|a_i|)$. Thus the inverse, which is the left hand side, absolutely converges, and the partial products show that it converges to the right hand side. $\square$

Proposition 3.4. *For* $\operatorname{Re}(s) > 1$, $\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1-N(\mathfrak{p})^{-s}}$.

Proof. These follow from Lemma 3.3, the fact that $\mathcal{O}_K$ is a Dedekind domain, and the fact that $\zeta_K(s)$ converges in that region. $\square$

For any subset of primes S, we can also consider $\zeta_{K,S}(s) = \prod_{\mathfrak{p} \subset S} \frac{1}{1-N(\mathfrak{p})^{-s}}$. Note that it converges for $\operatorname{Re}(s) > 1$ as well.

Lemma 3.5. *Let S be a set of primes in $\mathcal{O}_K$. Then $|\sum_{\mathfrak{p} \in S} \frac{1}{N(\mathfrak{p})^s} - \log(\zeta_{K,S}(s))| = O(1)$ for* $\operatorname{Re}(s) > 1$.

Proof. $\log(\zeta_K(s)) = \sum_{\mathfrak{p} \in S} -\log(1-N(\mathfrak{p})^s) = \sum_{\mathfrak{p} \in S} \frac{1}{N(\mathfrak{p}^s)} + \sum_{\mathfrak{p} \in S, m \geq 2} \frac{1}{m \mathfrak{p}^{ms}}$.The second term is at most $\sum_{\mathfrak{p}, m \geq 2} \frac{1}{N(\mathfrak{p})^{ms}} = \sum_{\mathfrak{p} \in S} \frac{1}{N(\mathfrak{p})^{2s} - N(\mathfrak{p})^s} \leq 2 \sum_{\mathfrak{p} \in S} N(\mathfrak{p})^{-2} < \infty$. $\square$

The **Dirichlet density** of a set of primes S is the limit as $s \to 1$, if it exists, of $\frac{\sum_{\mathfrak{p} \subset S} \frac{1}{N(\mathfrak{p})^s}}{\sum_{\mathfrak{p}} \frac{1}{N(\mathfrak{p})^s}}$. We can similarly define the upper and lower Dirichlet densities using lim inf and lim sup. It clearly satisfies natural properties one might expect. We say that S has **polar density** $\frac{n}{m}$ if $\zeta_{K,S}(s)^m$ has a pole of order n. Polar density also satisfies natural properties. The following is easy to see because of Lemma 3.5.

Lemma 3.6. *If the polar density exists, so does the Dirichlet density, and the two are equal.*

Lemma 3.7. *Let A be a set of primes with residual degree > 1 over $\mathbb{Z}$. Then $\zeta_{K,A}(s)$ is holomorphic for* $\operatorname{Re}(s) > \frac{1}{2}$ *and A has polar density* 0.

Proof. Split A up by residual degree f, and note that the terms corresponding zeta function is bounded by $\zeta_K(fs)^n$, so by Lemma 2.2 each is holomorphic on $\operatorname{Re}(s) > \frac{1}{f}$. $\square$

Corollary 3.8. *The primes in K that split completely in a Galois extension L have polar density $\frac{1}{[M:K]}$ where M is the Galois closure of L/K.*

Proof. First assume L is Galois. WLOG, we can restrict to primes that have residual degree 1 over $\mathbb{Q}$. But then this is clear, since if A are the primes of residual degree 1 in L and B are the primes of residual degree 1 splitting completely, then $\zeta_{L,A}(s)$ is the $[L:K]^{th}$ power of the $\zeta_{K,B}(s)$, so the result follows. Now we can remove the Galois assumption by noting that a prime splits completely in L iff it splits completely in the Galois closure. $\square$

Corollary 3.9. *Let H be a subgroup of $\mathbb{Z}/m\mathbb{Z}^\times$. Then the primes congruent to H have polar density $\frac{|H|}{m}$.*

Proof. Apply the previous result to the corresponding abelian extension of $\mathbb{Q}$. □

A **Dirichlet character** mod n (usually denoted χ) is a totally multiplicative function on the natural numbers factoring through $\mathbb{Z}/n\mathbb{Z}$, and supported on $\mathbb{Z}/n\mathbb{Z}^\times$. The character corresponding to the trivial homomorphism is called the trivial character, and denoted 1. We can define $L(s,\chi) = \sum_1^\infty \frac{\chi(n)}{n^s}$ to be the Dirichlet L-series.

Lemma 3.10. *$L(s,\chi)$ is holomorphic on $\mathrm{Re}(s) > 0$ when χ is nontrivial, and has a simple pole when χ is trivial.*

Proof. Note that $\sum_{\mathbb{Z}/n\mathbb{Z}} \chi(a) = 0$ for a nontrivial character mod n, so that $\sum_1^m \chi(a) = O(1)$, giving the first result. $L(s,1) = \zeta(s)\prod_{p|n}(1-p^{-s})$, giving the second result. □

More generally, given an abelian extension L of a number field K, we can consider characters $\chi \in \hat{G}$ on the Galois group G, and we can define a corresponding character on the set of ideals via the Artin map composed with the character. For simplicity, let $\chi(I)$ be shorthand for $\chi((\frac{L/K}{I}))$. Then $L_K(s,\chi) = \prod_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O}_K)^{ur}}(1-\frac{\chi(\mathfrak{p})}{\mathfrak{p}^s}) = \sum_{I\in I^{ur}} \frac{\chi(I)}{N(I)^s}$.

Proposition 3.11. *Let $f_\mathfrak{p}$ be the residual degree of every prime over $\mathfrak{p}$ in L, and let $r_\mathfrak{p}$ be the number of factors $\mathfrak{p}$ splits into. $\zeta_L(s) = \prod_{\mathfrak{p}\notin \mathrm{Spec}(\mathcal{O}_K)^{ur}}(1-N(\mathfrak{p})^{-f_\mathfrak{p} s})^{r_\mathfrak{p}} \prod_{\chi\in\hat{G}} L(s,\chi)$. $\zeta_K(s,1) = \zeta_{K,\mathrm{Spec}(\mathcal{O}_K)^{ur}}(s)$.*

Proof. The last identity is obvious, so we'll focus on the first. We can split the factors of $(1-N(\mathfrak{P})^{-s})^{-1}$ on the left hand side into Galois orbits. For each prime $\mathfrak{p}$, we will get $\prod_{\mathfrak{P}/\mathfrak{p}}(1-\frac{1}{N(\mathfrak{P})^s})^{-r_\mathfrak{p}}$, and for the unramified primes, this factors into $\prod_{\chi\in\hat{G}}(1-\frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s})^{-1}$ by basic facts about characters of abelian groups, and how Frobenius relates to splitting. □

Corollary 3.12. *If χ is a nontrivial Dirichlet character, then $L(1,\chi) \neq 0$.*

Proof. This follows from the above proposition, the fact that $L(s,\chi)$ are holomorphic at 1 for nontrivial χ, and the fact that $L(s,1), \zeta_{\mathbb{Q}[\zeta_m]}(s)$ have a simple pole at 1. □

From the proposition, it follows that if K is a subfield of a cyclotomic field (i.e. any abelian extension) where primes dividing m ramify, then the residue at 1 of $\zeta_K(s)$ is also given by $\prod_{p\nmid m}(1-\frac{1}{p^{f_p}})^{r_p}(1-\frac{1}{p})^{-1}\prod_{1\neq\chi\in\hat{G}} L(1,\chi)$, giving a class number formula. To compute this, we will need to evaluate $L(1,\chi)$ for nontrivial characters.

Theorem 3.13. *Let χ be a nontrivial Dirichlet character mod m. Then $L(1,\chi) = -\frac{1}{m}\sum_1^{m-1}\tau_k(\chi)\log(1-\omega^{-k}))$ where $\tau_k(\chi)$ is the Gauss sum $\sum_{\mathbb{Z}/m\mathbb{Z}^\times}\chi(a)\omega^{ak}$, and ω a primitive m^{th} root of unity.*

Proof. $L(s,\chi) = \sum_a \frac{\chi(a)}{a^s} = \sum_{b\in\mathbb{Z}/m\mathbb{Z}^\times}\chi(b)\sum_{a\equiv b \pmod m}\frac{\chi(a)}{a^s} = \frac{1}{m}\sum_{b\in\mathbb{Z}/m\mathbb{Z}^\times}\chi(b)\sum_a \frac{\sum_0^{m-1}\omega^{(b-a)k}}{a^s} = \frac{1}{m}\sum_1^{m-1}\tau_k(\chi)\sum_a\frac{\omega^{-ak}}{a^s}$, and evaluating at $s=1$ gives the result. □

We can simplify this formula as follows: If χ is a character mod m and is not induced from one mod $n \neq m$, then it is **primitive**. If χ mod m is induced from χ' mod n, then we have the formula $L(s,\chi) = L(s,\chi')\prod_{p|m,p\nmid n}(1-\frac{\chi'(p)}{p^s})$, and every character is induced from a primitive one, so we only need to be able to compute $L(s,\chi)$ for primitive characters. Let

$\tau(\chi) = \tau_1(\chi)$. Then for any character mod m, if $(k,m) = 1$, it is easy to see $\tau_k(\chi) = \overline{\chi}(k)\tau(\chi)$. More generally, $\tau_k(\chi) = \chi(1 + i\frac{m}{(m,k)})\tau_k(\chi)$, and so if χ is primitive and $(k,m) > 1$, then we have $\tau_k(\chi) = 0$.

Thus $L(1,\chi) = -\frac{\tau(\chi)}{m}\sum_{\mathbb{Z}/m\mathbb{Z}^\times}\overline{\chi}(k)\log(1-\omega^{-k}))$.

4. Distribution of primes

Theorem 4.1 (Dirichlet's Theorem)**.** *The polar density of primes in each relatively prime congruence class mod m are $\frac{1}{\phi(m)}$.*

Proof. Let $(a,m) = 1$, $\mathrm{Re}(s) > 1$. $\sum_{p\equiv a \pmod m}\frac{1}{p^s} = \frac{1}{\phi(m)}\sum_p \frac{\sum_\chi \chi(a^{-1}p)}{p^s} = \frac{1}{\phi(m)}\sum_\chi \overline{\chi}(a)\sum_p \frac{\chi(p)}{p^s} = \frac{1}{\phi(m)}\sum_\chi \overline{\chi}(a)\log(L(s,\chi)) + O(1)$. Now letting s near 1 and applying Corollary 3.12, it follows that all the terms in the sum are $O(1)$ except for $\log(L(s,1))$, which is $\log(\zeta(s)) + O(1)$, so we get $= \frac{1}{\phi(m)}\log(\zeta(s)) + O(1)$. □

One should note that the proof above works for any cyclotomic extension of number fields without any change other than restricting to primes with residual degree 1 over $\mathbb{Q}$.

We can improve the results of Theorem 3.8. First suppose that L/K has cyclic Galois group of order n.

Lemma 4.2. *The Dirichlet density of elements of order $d|n$ is $\frac{\phi(d)}{n}$.*

Proof. The density of elements of order dividing d is $\frac{d}{n}$ by Theorem 3.8. But then by Möbius inversion, we are done. □

Theorem 4.3 (Frobenius Density Theorem)**.** *Let L be a Galois extension of K with Galois group G, and let $\sigma \in G$ be an element of order n. Then the set of primes in K with Frobenius σ^k has Dirichlet density $c\frac{\phi(n)}{|G|}$, where c is the index of the normalizer of $\langle\sigma\rangle$ in G.*

Proof. We will ignore ramifying primes, and those in L^σ and K with inertial degree > 1 over $\mathbb{Q}$. Let L^σ be the fixed field of σ. By the previous lemma, the set A of primes in L^σ with Frobenius σ^k has polar density $\frac{\phi(d)}{n}$. Now let B be the set of prime in K with Frobenius σ^k for some prime over them. Each prime in B has $\frac{|G|}{n}$ primes above it in L, and the Galois group acts on these transitively, which acts on the decomposition group transitively by conjugation. Thus $\frac{|G|}{nc}$ primes above the prime in B must have a Frobenius that works. Each of these gives a different element of A that restricts to B, so the restriction map from A to B is $\frac{|G|}{nc}$ to 1. Looking at the level of zeta functions for A, B, since everything is inertial degree 1 over $\mathbb{Q}$, we immediately get that the polar density of B is $\frac{c\phi(n)}{|G|}$. □

The Chebotarev Density Theorem is a common generalization of both of the previous theorems. Here is a relatively simple approach to the Dirichlet density version of the theorem:

Theorem 4.4 (Chebotarev Density Theorem)**.** *Let L/K be a Galois extension of number fields, and let $[\sigma]$ be a conjugacy class in the Galois group. The Dirichlet density of primes in the class $[\sigma]$ is $\frac{[\sigma]}{|G|}$.*

Proof. First we'll reduce to the case of a cyclic extension using the same technique as in the previous. Given a prime p with Frobenius $[\sigma]$, note that it splits into $\frac{|G|}{o(\sigma)}$ primes in L, and exactly $\frac{1}{[\sigma]}$ of those have Frobenius actually σ. Combining this with the Dirichlet density for the cyclic case, along with ignoring primes ramifying or having nontrivial residual degree over $\mathbb{Q}$, we get the result.

Next, we will reduce to the case that L is a cyclotomic extension of K, which was proven in the remark after Dirichlet's theorem. If L is a cyclic extension, note that we only need to show that $\frac{1}{|G|}$ is a lower bound on the lower Dirichlet density as the same lower bound on the rest of the elements of G will give the desired upper bound. This will be shown as follows: pick a prime m linearly disjoint from K, and consider $L[\zeta_m]$, whose Galois group can be identified with $G \times \mathbb{Z}/m\mathbb{Z}^\times$. Then if $a \in \mathbb{Z}/m\mathbb{Z}^\times$ is an element with n dividing its order, then $\langle(\sigma, a)\rangle \cap G \times \{1\}$ is a trivial group, which by Galois theory means that $L[\zeta_m]/L[\zeta_m]^{(\sigma,a)}$ is a cyclotomic extension, so the density of primes for (σ, a) is what we want. In addition, the sum of the lower Dirichlet densities for the elements (σ, a) as a ranges in $\mathbb{Z}/m\mathbb{Z}^\times$ is at most the lower density of σ. If H_m is the number of elements elements of $\mathbb{Z}/m\mathbb{Z}^\times$ with n dividing its order, then the lower Dirichlet density is at least $\frac{H_m}{(m-1)|G|}$. Now we can choose by Dirichlet's theorem $m \equiv 1 \pmod{n^k}$ for large k so that $\frac{H_m}{m-1} \to 1$. □