

# NATURAL TRANSFORMATIONS, DUALITY, & EQUIVALENCES

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## 1. WHAT IS A NATURAL TRANSFORMATION?

Let's introduce the final essential categorical concept, the natural transformation. This is an extremely important concept, I believe Mac Lane has said that he defined the notion of category so that he could make precise a functor, and he defined a functor to make precise the notion of a natural transformation.

**Definition 1.1.** *Given two functors  $F$  and  $G$  from  $C$  to  $D$ , a **natural transformation**  $\eta$  from  $F$  to  $G$  is for each object  $x$  of  $C$ , an arrow  $\eta_x$  from  $Fx$  to  $Gx$  such that the following diagram commutes for all  $x, y, f$ :*

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow \eta_x & & \downarrow \eta_y \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

*We write  $\eta : F \Rightarrow G$  to denote a natural transformation.*

Natural transformation is a wonderful way of formalizing an intuitive sense of natural. For example, if  $V$  is a vector space over a field  $F$ , there is a dual vector space  $V^*$  which is the vector space of linear maps from  $V$  to  $F$ . Perhaps you know that if  $V$  is finite dimensional, it is isomorphic to its dual. However these aren't canonically isomorphic: in order to make an isomorphism, you have to **choose** a basis and then identify them. Natural transformation makes precise when this is canonical. For example, if  $\text{Vect}_F$  is the category of  $F$ -vector spaces, then  $(-)^*$ , the dual, is a contravariant functor from  $\text{Vect}_F$  to itself. On arrows,  $(-)^*$  does the same thing as the Hom functor  $C(-, F)$ . We can compose  $(-)^*$  with itself to get the covariant double dual functor  $(-)^{**}$ . If  $f$  is a map from  $V$  to  $W$ , then the double dual makes a map from  $V^{**}$  to  $W^{**}$  as follows: given a map  $g$  that takes maps  $h$  from  $V$  to  $F$  to  $F$ , we get the map  $f^{**}(g)$  that takes maps  $k : V \rightarrow F$  to  $g(k \circ f)$ . We can define a natural transformation  $\eta$  from  $1_{\text{Vect}_F}$  to the double dual  $(-)^{**}$ : given  $v \in V$ , we send it to the element of  $V^{**}$  that takes an element of  $V^*$ , and evaluates it at  $v$ . This is an isomorphism if the vector space is finite dimension, and note that

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it is canonical: there was no need to make any choices. Then, we should expect this collection of maps  $\eta_V, V \in \text{Vect}_F$  to be a natural transformation. And indeed it is, as one can check by following an element around the diagram that we want to commute:

$$\begin{array}{ccc} V & \xrightarrow{1_{\text{Vect}_F} f = f} & W \\ \downarrow \eta_V & & \downarrow \eta_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

Lets follow around an element  $v \in V$ :

$$\begin{array}{ccc} v & \xrightarrow{f} & f(v) \\ \downarrow \eta_V & & \downarrow \eta_W \\ h : h(g) = g(v) & \xrightarrow{f^{**}} & k : k(g) = h(g \circ f), k : k(g) = g(f(v)) \end{array}$$

Another example is the abelianization. Given a group  $G$ , we can define a subgroup called the commutator subgroup  $[G, G] = \{aba^{-1}b^{-1} | a, b \in G\}$ . The abelianization of  $G$  is the group  $G/[G, G]$ . This is a functor as if  $f : G \rightarrow H$  is a homomorphism, we can compose with the projection  $H \rightarrow H/[H, H]$  to get a map  $G \rightarrow H/[H, H]$ .  $[G, G]$  is in the kernel of this map, so we get then a map  $G/[G, G] \rightarrow H/[H, H]$ . This is the map that the abelianization sends  $f$  to. Now the projection  $\pi_G : G \rightarrow G/[G, G]$  is a natural transformation as the diagram below commutes (by definition):

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & G/[G, G] \\ \downarrow & & \downarrow \\ H & \xrightarrow{\pi_H} & H/[H, H] \end{array}$$

As a third example, consider the category  $\omega$  which is the poset category of  $\mathbb{N}$  with the usual ordering. Consider a diagram consisting of a sequence of sets  $S_n$  with injective maps from  $S_n \rightarrow S_{n+1}$ . This can be thought of as a sequence of sets increasing in size (each containing the previous). Recall that diagrams are just functors, and in this case,  $\omega$  is the category for which this is a functor (we can call this functor  $F$ ). Let  $\widehat{\cup S_i}$  be the constant functor taking  $\omega$  to  $\cup S_i$ , and all the arrows to the identity. Then consider the natural transformation  $\eta : F \Rightarrow \widehat{\cup S_i}$  that sends each  $S_i$  with the subset it corresponds to in the union. I leave this as an easy exercise to check that this is a natural transformation (draw it!). This kind of natural transformation is called a **cocone** (this will be discussed in more depth when we do (co)limits).

Finally, consider the determinant of a (invertible) matrix,  $\det^n$ . I claim this is a natural transformation. Consider the two functors from  $\text{CRing}$  to  $\text{Grp}$ : one taking

$K$  to  $GL_n(K)$ , and the other taking it to  $K^*$  (check that these are functors). Then  $\det_K^n$  is for each element of  $\text{CRing}$  a map from  $GL_n(K)$  to  $K^*$  sending a linear transformation to its determinant. The diagram is the same as always:

$$\begin{array}{ccc} GL_n F & \xrightarrow{\det_F^n} & F^* \\ \downarrow GL_n f & & \downarrow f^* \\ GL_n K & \xrightarrow{\det_K^n} & K^* \end{array}$$

Given two categories  $C, D$ , we can form the **product category**,  $C \times D$  where the objects are pairs of objects, the arrows are pairs of arrows, and composition is defined as usual.

Now consider the contravariant powerset  $\text{Set}(-, 2)$  (2 is a set with two elements, we can view this functor as  $2^{(-)}$ ). As an exercise, try to find all the natural transformations from this functor to itself (this will come up again in a later lecture).

**Definition 1.2.** Suppose  $F, G, H$  are functors in  $\text{Cat}(C, D)$ . Then if  $\eta : F \Rightarrow G$  and  $\nu : G \Rightarrow H$  are natural transformations, then we can form the **vertical composite**,  $\nu \cdot \eta$ , a natural transformation from  $F$  to  $H$ , defined by  $\nu \cdot \eta_a = \nu_a \circ \eta_a$ .

We can check this is a natural transformation via the following diagram:

$$\begin{array}{ccccc} Fa & \xrightarrow{\eta_a} & Ga & \xrightarrow{\nu_a} & Ha \\ \downarrow Ff & & \downarrow Gf & & \downarrow Hf \\ Fb & \xrightarrow{\eta_b} & Gb & \xrightarrow{\nu_b} & Hb \end{array}$$

This turns  $\text{Cat}(C, D)$  into a category, which we call the functor category. We can write this as  $D^C$ . An isomorphism in  $\text{Cat}(C, D)$  is called a natural isomorphism. Alternatively, it is a natural transformation  $\eta$  where each  $\eta_a$  is an isomorphism.

I use the word vertical composite, because there is also a horizontal composite. It can be seen as follows:

Given the diagram below, we would like to create a natural transformation  $\nu\eta : F' \circ F \Rightarrow G' \circ G$  sometimes written  $\nu \circ \eta$ .

$$\begin{array}{ccccc} & F & & F' & \\ & \curvearrowright & & \curvearrowright & \\ C & & D & & E \\ & \Downarrow \eta & & \Downarrow \nu & \\ & \curvearrowleft & & \curvearrowleft & \\ & G & & G' & \end{array}$$

We can do this by considering the following diagram:

$$\begin{array}{ccc} F'Fa & \xrightarrow{F'\eta_a} & F'Ga \\ \downarrow \nu_{Fa} & & \downarrow \nu_{Ga} \\ G'Fa & \xrightarrow{G'\eta_a} & G'Ga \end{array}$$

This commutes as  $\nu$  is natural for  $\eta_a$ . This suggests the following definition:

**Definition 1.3.** Suppose  $F, G, F', G', \eta, \nu$  are as above, we can form the **horizontal composite**  $\nu\eta : F' \circ F \rightarrow G' \circ G$  so that  $(\nu\eta)_a = \nu_{Ga} \circ F'\eta_a = G'\eta_a \circ \nu_{Fa}$ .

It remains to check this is a natural transformation, but this should be obvious if you draw the appropriate diagram (for a natural transformation). If  $F : C \rightarrow D$ ,  $G, H : D \rightarrow E$  are functors and  $\eta : G \Rightarrow H$  a natural transformation then the natural transformation  $\eta F$  denotes the horizontal composite  $\eta 1_F$ , and similarly if  $J : E \rightarrow X$  is a functor, then  $J\eta$  denotes  $1_J\eta$ .

Horizontal composites and vertical composites are related through the interchange law, which says  $(\tau\eta) \cdot (\tau'\eta') = (\tau \cdot \tau')(\eta \cdot \eta')$ . It can be described as the diagram below:

$$\begin{array}{ccc} C & \xrightarrow{\quad} & D \circ D \xrightarrow{\quad} E \\ \downarrow \eta' & & \downarrow \tau' \\ C & \xrightarrow{\quad} & D \circ D \xrightarrow{\quad} E \\ \downarrow \eta & & \downarrow \tau \end{array} = \begin{array}{ccc} C & \xrightarrow{\quad} & D \xrightarrow{\quad} E \\ \downarrow \eta' & & \downarrow \tau' \\ C & \xrightarrow{\quad} & D \xrightarrow{\quad} E \\ \downarrow \eta & & \downarrow \tau \end{array}$$

We can prove it using the diagram below. Let  $\eta' : F \Rightarrow G, \eta : G \Rightarrow H, \tau' : F' \Rightarrow G', \tau : G' \Rightarrow H'$  in the figure above.

$$\begin{array}{ccccccc} & & F'Ga & \xrightarrow{F'\eta_a} & F'Ha & \xrightarrow{\tau'_{Ha}} & G'Ha \\ & \nearrow F'\eta'_a & \parallel & & & & \searrow \tau_{Ha} \\ F'Fa & & & & & & \\ & \searrow F'\eta'_a & & & & & \\ & & F'Ga & \xrightarrow{\tau'_{Ga}} & G'Ga & \xrightarrow{G'\eta_a} & G'Ha \\ & & & & & & \nearrow \tau_{Ha} \\ & & & & & & H'Ha \end{array}$$

The path on the top is the natural transformation  $(\tau \cdot \tau')(\eta \cdot \eta')$ , and the path on the bottom is  $(\tau\eta) \cdot (\tau'\eta')$ . The middle rectangle commutes as  $\tau'$  is a natural transformation.

As a final note, there is an analogy between natural transformations and homotopies.

If  $X$  and  $Y$  are topological spaces, and  $f$  and  $g$  are maps (continuous, as always) from  $X$  to  $Y$ , a homotopy from  $f$  to  $g$  is a map from  $X \times [0, 1]$  to  $Y$  that at 0 restricts to  $f$  and at 1 restricts to  $g$ . The definition of a natural transformation can be presented analogously: Let  $2$  be the category with 2 objects, called 0 and 1 and

one non identity arrow from 0 to 1 (we can say the arrow category, as this is the category that represents the diagram consisting of a generic arrow).

If  $C$  and  $D$  are categories, and  $F$  and  $G$  are functors from  $C$  to  $D$ , a natural transformation is a functor from  $F$  to  $G$  is a functor from  $C \times 2$  to  $D$  that on 0 restricts to  $F$  and on 1 restricts to  $G$ .

Check that these two definitions of natural transformations are equivalent and note the similarity with homotopies. In a way, a natural transformation is categorification of homotopy.

Finally let's end with an interesting non-example. Let  $\text{FinSet}_g$  be the category of finite sets and bijections between them. Consider two functors to  $\text{Set}$ , the first,  $\text{Aut}$ , takes  $X$  to the set of bijections from  $X$  to itself, on maps, it takes  $f : X \rightarrow Y$  to the function that takes  $\phi : X \rightarrow X$  to  $f \circ \phi \circ f^{-1} : Y \rightarrow Y$ . The second,  $\text{Ord}$ , takes  $X$  to the set of total orders on  $X$ , and on maps takes  $f$  to the total order on  $Y$  induced by the bijection. These two functors send isomorphic objects to isomorphic sets, but are not naturally isomorphic: in fact, there isn't even a natural transformation between them! For, let's consider  $f$ , the nontrivial bijection from a set  $\{a, b\}$  to itself. If there was a natural transformation, we would have

$$\begin{array}{ccc} \{1, (a, b)\} & \xrightarrow{\text{Aut}(f)} & \{1, (a, b)\} \\ \downarrow \eta_b & & \downarrow \eta_c \\ \{a < b, b < a\} & \xrightarrow{Ff} & \{a < b, b < a\} \end{array}$$

$\text{Aut}(f)$  is the identity, but  $Ff$  is not, so this diagram cannot commute.

The fact that this bijection is not natural has an interesting interpretation in the context of a combinatorics problem. In particular, let's count the number of trees on a set of  $n$  elements, which we'll call  $T_n$ . Let  $|\cdot|$  denote cardinality of a set. Consider the product  $T_n \times n \times n$ , consisting of a tree on the set  $n$ , as well as a head and a tail (shown in Fig 2).

Note that since there is a unique path between any two points in a tree, we can draw an arrow from the tail to the head, yielding a total ordering on a subset of 1 to  $n$ , ie. a skeleton, as well as trees coming out of each point. Note that the skeleton and the trees coming out of each point completely determine  $T_n \times n \times n$ . Then as total orders are in bijections with permutations, we can consider the set of permutations with trees coming out of them, a typical example in the figure below:

These are in bijection with functions from the set of  $n$  elements to itself, as a function determines such a tree by writing where everything goes, which eventually (after applying the function enough times) determines the cycles and the trees coming out of them. Thus  $T_n \times n \times n$  is in bijection with the set of functions from  $\{1, \dots, n\}$  to itself, which is  $n^n$ . Thus  $|T_n| = n^{n-2}$  (This is known as Cayley's Theorem). Perhaps

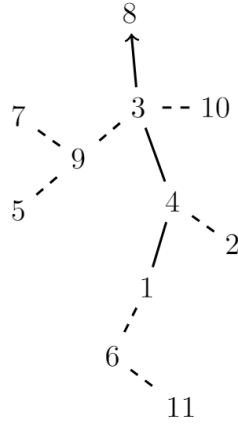


FIGURE 1. A tree, on 11 elements, with a skeleton, indicated by the bold lines, is determined by the total ordering on the skeleton and the trees coming out of each point on the skeleton.

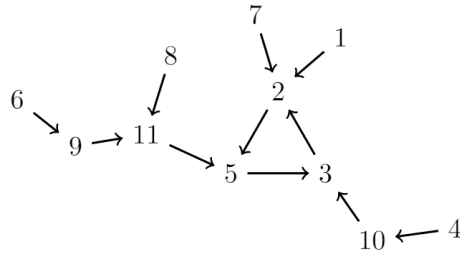


FIGURE 2. A permutation on 2, 3, and 5 with trees coming out of it.

the reason this proof does something nontrivial is because it used this bijection which was unnatural.