WHAT IS ALGEBRAIC GEOMETRY?

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1. Motivation

Algebraic geometry is the study of the geometry that arises from studying polynomial functions, and spaces that are defined using polynomial equations. One reason to be interested in algebraic geometry is that it is closely related to complex geometry. For example, let's classify meromorphic functions on $\mathbb{CP}^1$, the Riemann sphere.

Proposition 1.1. *A meromorphic function on $\mathbb{CP}^1$ is given by a function in $\mathbb{C}(x)$.*

Proof. By compactness of $\mathbb{CP}^1$ and the fact that zeros are isolated, there are only finitely many zeros and poles of the rational function f. We can multiply f by a rational function so that the result is something that either only has poles or zeros, and by inverting it we can assume it only has zeros. The result is a holomorphic function on $\mathbb{CP}^1$, so lands in a compact subset of $\mathbb{C}$, hence is constant by Louisville's theorem or the open mapping theorem. $\square$

Indeed, like this example, many compact complex manifolds behave algebraically. For example, let's consider complex curves (Riemann surfaces) inside $\mathbb{CP}^2$. The points of $\mathbb{CP}^2$ are denoted $[x, y, z]$, where $x, y, z \in \mathbb{C}$ are not all 0, and $[x, y, z] = [x\lambda, y\lambda, z\lambda]$ One way to construct complex submanifolds of $\mathbb{CP}^2$ is to take a polynomial f that is homogeneous in the variables x, y, z, and consider the submanifold of $\mathbb{CP}^2$ that is the zero set of this homogeneous polynomial. For the general f, the resulting subset will indeed be everywhere smooth. For example this yields $\mathbb{CP}^1$ if f is a linear or quadratic form, and an elliptic curve if f is a cubic. In fact, any closed complex submanifold of $\mathbb{CP}^2$ is given by the zero set for some f.

Another main realm in which algebraic geometry is useful is in number theory. For example let's try to find the points on the circle $x^2+y^2 = 1$ inside $\mathbb{Q}^2$. In order to find all the solutions to this polynomial equation, we start by noticing $(1, 0)$ is a solution. We can use this point to make a projection map from V to $\mathbb{QP}^1$ as follows: for any point $(a, b) \in V$, we can take the line through $(1, 0)$ and (a, b), given by $b(x-1) = (1-a)y$. This line gives a point in $\mathbb{QP}^1$, namely $(b : 1 - a)$. Conversely given a point $(c : d) \in \mathbb{QP}^1$, we can consider the common solutions to $c(x-1) = dy$ and $x^2+y^2 = 1$, and recover our solution as the second solution to this pair of polynomial equations. For example if $d = 0$, $(1, 0)$ is a solution of multiplicity 2, meaning that $\mathbb{Q}[x, y]/(x^2 + y^2 - 1, x - 1) = \mathbb{Q}[x, y]/(x - 1, y^2)$ is a two dimensional $\mathbb{Q}$-vector space, so the solution corresponding to this is $(1, 0)$. If $d \neq 0$, then $x^2 + \frac{c^2}{d^2}(x - 1)^2 = 1$, and along with $x = 1$, we get $x = \frac{c^2-d^2}{c^2+d^2}$ as a solution, hence the pair $(\frac{c^2-d^2}{c^2+d^2}, \frac{-2cd}{c^2+d^2})$ is a solution. This parameterize all the solutions of $x^2 + y^2 = 1$ in terms of point in $\mathbb{QP}^1$. Note that the formula works for $(1 : 0)$ as well.

Exercise 1.1.1. *Classify the rational solutions to $y^2 = x^3 + x^2$.*

We have classified these solutions by making a parameterization of our solutions in terms of $\mathbb{QP}^1$, noting that this gives all the solutions because there is a projection map that is the inverse. Both the projection map and the parameterization map was very nice in the sense that it is given by rational functions.

2. Affine Geometry

These examples suggest that it might be good to systematically study solutions to polynomial equations as spaces: complex manifolds sometimes already behave like they are solution spaces to polynomial equations, and we have seen that we can use geometric maps between $\mathbb{QP}^1$ and the conic $x^2 + y^2 = 1$ to classify rational points. The first object that lets us do this is a variety. We will begin for simplicity by talking about with affine varieties over the field $\mathbb{C}$.

Definition 2.1. *An **affine variety** over $\mathbb{C}$ is a subset of $\mathbb{C}^n$ that is a solution to polynomial equations in n variables.*

By the Hilbert Basis theorem, any affine variety can be thought of as the solution of only finitely many polynomial equations.

Definition 2.2. *A **morphism** (or regular map) of affine varieties $g : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ is a function from U to V that is given by m polynomials in n variables.*

For an example of a morphism, we can project the hyperbola $xy = 1$ to the line $\mathbb{A}^1$ via the polynomial x. This hits every point once on $\mathbb{A}^1$ except for 0. Since any quadratic form over $\mathbb{C}$ is equivalent to xy,

In this way, we can turn affine varieties into a category. The most basic example is $\mathbb{A}^n$, which is just all of $\mathbb{C}^n$.

A variety comes with a topology inherited from $\mathbb{C}^n$, but it is not the usual topology. Namely, we say that any subset that is the common zeros of some polynomials is closed in the **Zariski topology**. A morphism of varieties is continuous with respect to this topology. Note that there is a natural basis for the open sets, namely, the **principle open sets** $D(f)$, the set on which f is nonzero for any polynomial.

Now given an affine variety V, we can form its **coordinate ring** $\mathbb{C}[V]$, by taking the ideal of polynomials I that vanish on V and setting $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/I$. A polynomial is the same as a map to $\mathbb{A}^1$, so this ring is just $\mathrm{Hom}(V, \mathbb{A}^1)$. Thus, when we have a morphism $f : U \to V$, we get a map $f^* : \mathbb{C}[V] \to \mathbb{C}[U]$ by $f^*(v) = v \circ f$, where we interpret v as a morphism.

Note that the $\mathbb{C}[V]$ are always finitely generated reduced $\mathbb{C}$ algebras (the inclusion map to $\mathbb{A}^n$ induces the desired surjection $\mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}[V]$), and the morphisms between them are $\mathbb{C}$ algebra morphisms. In fact, these two categories are equivalent, meaning studying the algebra of polynomial functions on V is just as good as studying V. Given a reduced finitely generated $\mathbb{C}$ algebra, we get an affine variety by choosing a surjection from $\mathbb{C}[x_1, \ldots, x_m]$, and the kernel is an ideal whose common zero set is a variety. Given a $\mathbb{C}$ algebra morphism $A \to B$, we can lift the map in the diagram:

$$\begin{tikzcd}
\mathbb{C}[x_1, \ldots, x_n] \arrow[r, dashed, "f'"] \arrow[d] \arrow[dr, "f"] & \mathbb{C}[y_1, \ldots, y_m] \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

The lift gives some polynomials defining a morphism from the corresponding varieties. The proof that this is an equivalence of categories amounts to the fact that the set of polynomials that vanish on any variety is the same as the radical of the ideal generated by any set of polynomials that define the variety. This is Hilbert's Nullstellensatz. It is this theorem that requires the field to be algebraically closed (though there are versions for non-algebraically closed fields). For example $x^2 + y^2 = -1$ has no real points, but its radical is not the trivial ideal. The way to fix this over a non algebraically closed field is to allow "points" in finite extensions of $\mathbb{Q}$, not just $\mathbb{Q}^n$. For example, in $\mathbb{A}^n$ over the field $\mathbb{R}$, the points are points in $\mathbb{C}^n$ up to complex conjugation. The points can be seen on the level of the rings.

Exercise 2.2.1. *Classify the maximal ideals of* $\mathbb{C}[x], \mathbb{R}[x]$*, and show that they correspond to the points in* $\mathbb{A}^1$.

We can talk about geometric properties of a variety. From the topology, we can make sense of an **irreducible variety**. Namely, this is a variety that is not the union of any two closed proper subsets of itself. Any variety breaks up into finitely many irreducible components. A line is irreducible, but a pair of lines isn't. When doing algebraic geometry people often restrict to irreducible varieties. The **dimension** of a variety is 1 less than the maximal chain of irreducible subvarieties of V. Varieties of dimension 1 are curves, 2 are surfaces, and 3 are 3-folds.

A irreducible variety often gives a complex manifold. Namely, we can treat the variety V as a subset of $\mathbb{C}^n$ with the complex topology. Then if the ideal of polynomials vanishing on a variety is given by $f_1, \ldots f_n$, and the Jacobian of partial derivatives of the f_i has rank $\operatorname{codim}(V)$ at each point of V, then by the inverse function theorem V is a closed complex submanifold of V. If the Jacobian has the right rank at a point, that point is called **smooth**. The resulting complex manifold will be dimension $\dim(V)$. Points that aren't smooth are called **singular**. For any variety, the singular points form a closed subset of smaller dimension. To see it is closed, note that the set of singular points is given by the vanishing of the determinants of minors of a matrix of polynomials. To see it is smaller dimension, the singular points on any irreducible component is proper since partial derivatives lower the degree of the polynomial, and the intersection of any variety with any other variety that doesn't contain it is also proper closed subset of it, and any points that aren't in this intersection are smooth iff they are smooth on the irreducible component.

3. Projective Geometry

Let's try to do geometry on $\mathbb{A}^2$. If we take two curves in $\mathbb{A}^2$, how many times do they intersect? This is an elementary problem when one of the curves is a line. After a linear change of variables, the line can be assumed to be $x = 0$. If the other curve is given by $f(x, y) = 0$, then the number of roots of $f(0, y)$ is the number of intersection points. Unfortunately, this depends on our line, even when the intersection points are counted with multiplicity. The way to fix this is to do projective geometry instead. This can be thought of as compactifying $\mathbb{A}^2$. Namely, we define $\mathbb{P}^n$ to be the set of points $[x_0 : \cdots : x_n]$ not all 0, where $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$. We say that a rational function on $\mathbb{P}^n$ is a quotient of two homogeneous polynomials in $x_0, \ldots x_n$ of the same degree. Note that this is well defined up to scalar multiplication since the degree of the top and bottom are the same.

We say that a rational function is **regular** at a point p if there is a way to write the homogeneous polynomial as a function that you can evaluate at that point. For example, the function x/y on $\mathbb{P}^1$ is regular on all points $[x : y]$ where $y \neq 0$. We can put a Zariski topology on $\mathbb{P}^n$. Namely, given an ideal of polynomials in $\mathbb{C}[x_0, \dots, x_n]$ generated by homogeneous elements, it makes sense to talk about the points of $\mathbb{P}^n$ in which they all vanish. We will define these to be the closed sets, and say that they are **projective varieties**. Note that a projective variety can be thought of as the closure of some affine variety in projective space. This is because if we consider the points in $\mathbb{P}^n$ where some x_i are not 0, we can make each point of the form $[x_1, \dots, 1, x_{i+1}, \dots x_n]$, which is exactly the points of $\mathbb{A}^n$. We call this the **affine chart** $\mathbb{A}^n_i$. We can intersect any projective variety with an affine chart to get an affine variety. For example, consider the subset of $\mathbb{P}^2$ given by $y^2z = x^3 + axz^2 + bz^3$ for some $a, b \in \mathbb{C}$, which is called an elliptic curve. In the affine chart where $z \neq 0$, we can find the equation for the elliptic curve by setting $z = 1$, where we get $y^2 = x^3 + ax + b$.

Exercise 3.0.1. *Show that an elliptic curve is smooth.*

Similarly as in the affine case we can define smoothness and dimension. These notions are compatible in the sense that the dimension of a projective variety is the maximal dimension of it on any affine chart, and a point is smooth if it is smooth in any (or each) affine chart. This condition can also be checked projectively again via the matrix of partial derivatives. For example, if we have a **projective hypersurface**, which is a variety given by one equation $F = 0$, then (in characteristic 0) the singular points are the subvariety simultaneous zero of $\frac{\partial F}{\partial x_i} = 0$. Analogously to the affine case, a smooth projective variety gives a compact complex manifold by replacing the Zariski topology with the complex topology.

Definition 3.1. *A rational map between projective varieties $A \subset P^n, B \subset P^m$ is a collection of $m+1$ rational functions $f_0, \dots f_m$ on $\mathbb{P}^n$ that are regular on a dense open set of A, and so that $[f_0 : \dots : f_m]$ sends points of A to points of B where it is regular. Two rational functions are the same if they are the same function on the intersection.*

A rational map that is everywhere regular is called a **regular map** or morphism. There are two different categories that we can make from these notions. For one, we consider projective varieties with **dominant** rational maps, namely rational maps where the image contains an open dense set (this is necessary in order to be able to compose rational maps). The notion of isomorphism in this category is called **birational equivalence**.

The other category we can make is the one with projective varieties and morphisms. Here we just call the notion of isomorphism isomorphism.

We already saw an example of a regular map. We can consider the projectivized conic $x^2 + y^2 = z^2$, and consider the projection map that we did at the beginning to $\mathbb{P}^1$ given by $[\frac{x-z}{z} : \frac{y}{z}]$. This can also be written as $[\frac{xz-z^2}{x^2+y^2} : \frac{yz}{x^2+y^2}]$, so is actually regular everywhere on the conic. We showed before that there is an inverse, which also happens to be everywhere regular, so this map gives an isomorphism of this plane conic with $\mathbb{P}^1$.

For projective curves in $\mathbb{P}^2$, we can say something about the intersection between two curves. Namely,

Proposition 3.2. *If F, G are homogeneous in 3 variables of degree m, n, then the number of intersections of the varieties in $\mathbb{P}^2$ defined by F, G counted with multiplicity is mn, if the intersection is dimension 0.*

The multiplicity of an intersection point can be defined as the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[x, y]/(f, g)$, where f, g are dehomogenized equations for F, G on an affine chart.

One basic operation we would like to be able to do with projective varieties is take their product. We can do this with affine varieties very easily: The product of $V \in \mathbb{C}^n$ and $U \in \mathbb{C}^m$ as sets is a subvariety of $\mathbb{C}^{n+m}$. It is a little bit harder to do this for projective varieties. It suffices however to do this for $\mathbb{P}^n \times \mathbb{P}^m$, as if it comes with projections to $\mathbb{P}^n$ and $\mathbb{P}^m$, and $A \subset \mathbb{P}^n, B \subset \mathbb{P}^m$ are closed sets, then the intersection of their preimages under the product is the product $A \times B$. To construct the product, we can imagine it as embedded in $\mathbb{P}^{(n+1)(m+1)-1}$. Namely if $x_0, \ldots, x_n$ were the coordinates of $\mathbb{P}^n$, and $z_0, \ldots, z_m$ were the coordinates of $\mathbb{P}^m$, then the coordinates of $\mathbb{P}^{(n+1)(m+1)-1}$ can be thought of as $w_{ij}, 0 \leq i \leq n, 0 \leq j \leq m$. Then the equations defining $\mathbb{P}^n \times \mathbb{P}^m$ are $w_{ij}w_{kl} = w_{il}w_{kj}$. This is called the **Segre embedding** of $\mathbb{P}^n \times \mathbb{P}^m$. If a point in $\mathbb{P}^n \times \mathbb{P}^m$ looks like $[x_0 : \cdots : x_n; z_0 : \cdots : z_m]$, then the embedding is given by $w_{ij} = x_i z_j$.

Exercise 3.2.1. *Find the projection maps for the Segre embedding.*

For example, $\mathbb{P}^1 \times \mathbb{P}^1$ is a hypersurface in $\mathbb{P}^3$ given by the equation $w_{00}w_{11} = w_{01}w_{10}$. Note that this is a quadratic surface, the surface analog of a conic.

Exercise 3.2.2. *Show that $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$ (Hint: project from a point as in the case of a conic). Are they isomorphic?*