PONTRYAGIN CONSTRUCTION

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1. Introduction

The Pontryagin construction is a way of relating framed submanifolds to homotopy classes of maps to a sphere. Here a **framed submanifold** is a submanifold with a trivialization of the normal frame bundle. Now we don't want to consider all framed submanifolds, but rather mod out by an equivalence relation called **cobordism**. We say that $N, N' \subset M$ are cobordant manifolds if $N \times [0,\epsilon] \cup N' \times [1-\epsilon, 1]$ can be extended in the interval $M \times [\epsilon, 1-\epsilon]$ to a submanifold of $M \times [0,1]$ with boundary $N \cup N'$. In particular we would like to consider framed submanifolds up to framed cobordism, where we require the extension to be framed. I will use $\simeq$ to denote homotopic maps and $\sim$ to denote framed cobordant submanifolds.

Throughout, we will assume that M is compact, $f : M \to S^p$ (S^p oriented) a smooth map, y a regular value, we naturally get a framed submanifold by looking at $N_f = f^{-1}(y)$, and $f_{|N_f}$ induces a bundle map on N_f's normal bundle and T_y, trivializing it using a positively oriented basis of T_yS^p.

We would like to prove:

Theorem 1.1. *N_f is well defined up to framed cobordism class, and only depends on the homotopy class of f. Moreover, $f \mapsto N_f$ gives a bijection between framed compact cobordism classes of codimension p and $[M, S^p]$.*

2. Well defined

We will begin by showing the first statement. First note that the cobordism class doesn't depend on the basis we chose for f, only the orientation, since $\mathrm{GL}_n(\mathbb{R})^+$ is connected (This can be proven by using row/column operations carefully or using Graham-Schmidt to reduce to showing $\mathrm{SO}(n)$ is connected, which is done by using induction and the fibration $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \to S^{n-1}$). Then given two choices of frames on N_f, they are pullbacks of two different elements of the tangent frame bundle of y, so by choosing a smooth path on $\mathrm{GL}_n(\mathbb{R})^+$ that is constant on $[0,\epsilon] \cup [1-\epsilon, 1]$, we have framed $y \times [0,1]$ in $S^p \times [0,1]$, and by considering the natural map induced by f from $M \times [0,1] \to S^p \times [0,1]$, this framing of $y \times [0,1]$ induces a cobordism

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between the two frames of N_f. Thus we will ignore the particular frame chosen at y from now on.

We would like to show the cobordism class is well defined up to homotopy. Given a homotopy, we would like to take the preimage of y on the homotopy to get a cobordism. Unfortunately y is not necessarily a regular value of the homotopy. To fix this, the following lemma:

Lemma 2.1. *The cobordism class of $f^{-1}(z)$ is constant for z in a neighborhood of y.*

Proof. The set of critical points is compact as M is, hence there is a convex neighborhood of y consisting of regular values. Now choosing a family r_t of smooth rotations of the sphere that takes y to z, and is constant on $[0,\epsilon] \cup [1-\epsilon, 1]$. Then consider the map $r \circ f : M \times [0,1] \to S^p \times [0,1] \to S^p$. y is regular for $r \circ f$, so we get a cobordism between $f^{-1}(y)$ and $f^{-1}(z)$. $\square$

Theorem 2.2. *The cobordism class is well defined, and is only dependant on homotopy class.*

Proof. First note that if $f \simeq g$, then we can assume the homotopy is constant on $[0,\epsilon] \cup [1-\epsilon, 1]$, and choose z a regular value of the homotopy satisfying the conditions of the previous lemma for f and g so that $f^{-1}(y) \sim f^{-1}(z) \sim g^{-1}(z) \sim g^{-1}(y)$. Now if z is another regular value, and r a rotation sending z to y, $r \circ f \simeq f$ so $f^{-1}(y) \sim (r \circ f)^{-1}(y) = f^{-1}(z)$. $\square$

3. Surjectivity

We would now like to show that for any framed submanifold N, we can produce a map f with $N_f \sim N$.

Lemma 3.1 (Tubular Neighborhood Theorem)**.** *Let $P \subset M$ be submanifold of codimension p, with P compact. Then there is a neighborhood of P diffeomorphic to the normal bundle of P, with P as the* 0*-section.*

Proof. By exponentiating the normal bundle, we get a local diffeomorphism $P \times B_\epsilon \to M$, and since B_ϵ is diffeomorphic to $\mathbb{R}^p$, it suffices to show that for small ϵ, this is injective. However, if (p_i, x_i), (q_i, y_i) are a sequence of points for which it is not injective with the magnitude of the x_i, y_i going to 0, by compactness of $P \times \overline{B}_{\frac{\epsilon}{2}}$, we can extract a convergent subsequence, which contradicts local injectivity. $\square$

This Lemma holds for non-compact submanifolds but the proof is a bit more annoying.

Theorem 3.2. *The map $f \to N_f$ is surjective.*

Proof. We consider a tubular neighborhood of a framed submanifold N, giving a map $f : \mathbb{R}^p \times N \to \mathbb{R}^p$. Now consider $S^p = y_0 \cup \mathbb{R}^p$, and smoothly extend f to M by setting all other values to y_0. Then $f^{-1}(0) = N$. □

4. Injectivity

We would now like to show that if we have a cobordism $f^{-1} \sim g^{-1}$ via some framed submanifold $P \subset M \times [0, 1]$, $f \simeq g$. To do this, given the cobordism, we would like to use the proof of surjectivity on the cobordism to yield a homotopy. However, this still leaves us to prove:

Lemma 4.1. *If $f^{-1}(y) = g^{-1}(y) = N$, $f \simeq g$.*

Proof. If f, g agree on a neighborhood of N, then removing the neighborhood, we get a map to $\mathbb{R}^p$ instead of S^p, which we can linearly homotopy without spoiling the overall smoothness. So it suffices to deform f to agree with g in a neighborhood of N. To do this, choose a tubular neighborhood $N \times \mathbb{R}^p$ that misses the antipode y_0 of y. Then we have maps $F, G : N \times \mathbb{R}^p \to \mathbb{R}^p$ with $DF_{|N\times 0} = DG_{|N\times 0}$, and we can assume that $DF_{|N\times 0}$ is the identity on each $n \times \mathbb{R}^p$. We would like to linearly deform f to match g, but we would like to avoid adding new zeroes. To do this, note by compactness of N, there is an δ ball around 0 such that when F, G are restricted to it, $\|DF - I\|, \|DG - I\| < \epsilon$. Then $\|F(n, x) - x\| \leq \|cx^2\|$ for small $\|x\|$ by Taylor's theorem, so by multiplying by $\|x\|$ on either side and using Cauchy Schwarz, we get $|F(n, x) \cdot x| \geq \|x\|^2 - c\|x\|^3$ which is positive when $\|x\|$ and c are small. Then doing the same with G, we find that F and G lie in the same half plane for small $\|x\|$, so that we can linearly deform F to match G locally without adding new 0s. □

Theorem 4.2. *If $N_f \sim N_g$, $f \simeq g$.*

Proof. As in the proof of surjectivity, choose a tubular neighborhood of a cobordism and construct a homotopy H such that $H^{-1}(y)$ is the cobordism. Now by the previous lemma, $f \simeq H_0 \simeq H_1 \simeq g$. □

5. Applications

The Pontryagin construction can be viewed as a generalization of degree theory, and we can see that the most trivial case of it does coincide with degree theory.

Theorem 5.1 (Theorem of Hopf)**.** *If M^n is compact, orientable, and connected, then $[M^n, S^n] \cong \mathbb{Z}$, where the isomorphism is given by degree. If M^n is non-orientable, then $[M^n, S^n] \cong \mathbb{Z}/2\mathbb{Z}$ with the isomorphism given by degree mod 2. In particular, $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. The codimension 0 compact framed submanifolds are finite collections of points with a ± 1 orientation. Now it is clear that if M is orientable, then the cobordism class is only dependant on degree, ie. the sum of these orientations. If M is not orientable, then points with positive or negative orientation are the same up to cobordism, so degree mod 2 determines the cobordism class. $\square$