REPRESENTATION THEORY

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How this should be organized:
check:A short section on topological groups
A short section on Lie groups and their Lie algebras
PoincareBirkhoffWitt and its consequences
Maybe say something about locally compact groups and invariant measures?
Start doing rep theory.
Geometry of Lie groups?

1. Topological Groups

Definition 1.1. *A **topological group** is a group in the category of topological spaces.*

Lemma 1.2. *If the identity is closed for a topological group G, G is Hausdorff. In particular a $T1$ topological group is $T2$.*

Proof. Take the preimage of the identity on the multiplication map, and compose with $1_G \times (-)_G^{-1}$ to get that the diagonal of G is closed. □

We define G^o to be the connected component of the identity of G.

Lemma 1.3. *G^o is a normal subgroup of G.*

Proof. Conjugation by any element takes G^o to another component that contains the identity, so it must be an isomorphism on G^o. □

The quotient G/G^o is naturally identified with $\pi_0(G)$, hence gives it a group structure.

Lemma 1.4. *$\Pi_1(G)$ is commutative.*

Proof. Let $*$ be composition in Π_1 and $\times$ be multiplication in the group. If a, b are paths, then $a * b = (a * 1) \times (1 * b) \simeq (1 * a) \times (b * 1) = b * a$. □

Lemma 1.5. *If $G' \to G$ is a connected covering map, after fixing a lift of the identity, there is a unique group structure on G' such that the map is a homomorphism.*

Proof. This follows from the criterion for lifting maps to covers. □

Proposition 1.6. *Let $G' \to G$ be a connected covering map that is a homomorphism. The group of deck transformations can be identified with the kernel.*

Proof. A deck transformation is determined by where it sends the identity, which must be in the kernel. □

In particular, note that every cover is normal, giving another proof that π_1 is abelian in the case that universal covers exist.

Lemma 1.7. *If $H \subset G$ is a connected subgroup of a topological group, and G/H is connected, then G is connected.*

Proof. We'd like to show $G^o = G$. To do this, note it contains H as H is connected, and thus it passes to the quotient so its image must be G/H, and so $G^o = G$. □

Lemma 1.8. *An open subgroup H of a topological group G is closed.*

Proof. Let U be a neighborhood of the identity contained in H, and U_g be the neighborhood translated by g. Now g is a limit point of $H \implies H \ni h \subset U_g \implies g^{-1}h \in U \implies g \in H$. □

We will briefly consider locally compact Hausdorff groups, and will come back to them when considering representations.

Theorem 1.9. *Given a locally compact Hausdorff group G, there is a unique Borel measure μ on G that is invariant under left multiplication, outer regular on Borel sets, inner regular on open sets, and finite on compact sets. It is unique up to multiplication by a scalar. The completed measure is called the **Haar** measure.*

Proof. Given subsets C, D, define $[C : D]$ to be the smallest number of left translates of D covering C. Note that $[C : D][D : E] \geq [C : E]$, $[\bigcup C_i : E] \leq \Sigma[C_i : E]$, and if $D \subset E$, $[C : D] \geq [C : E]$.

Fix a compact set A containing a nontrivial neighborhood of the identity.

Given a compact set C, define $\mu(C)$ to be $\lim_{i\to\infty} \frac{[C:U_i]}{[A:U_i]}$. Here the U_i are carefully chosen as a local base around the identity so that the limit exists for all compact C. To do this, note $[C : A]$ is an upper bound of the limit, which is finite by our choice of A. Now if we index the compact sets $C_\alpha \in I$, then we get a function taking an open set U to $(\frac{[C_\alpha:U]}{[A:U]})_\alpha$. By Tychanoff's theorem, we can extract the U_i we want.

It is not hard to see that for two disjoint compact sets C_1, C_2, $\mu(C_1 \cup C_2) = \mu(C_1) + \mu(C_2)$. Suppose there were arbitrarily large i such that some translate of U_i intersected both C_1, C_2. Choose the U_i to have compact closure so that the centers of these U_i converge to some point p. Then consider any translate of some U_i centered at p. We can choose a neighborhood of p small enough such that for all $n > N$, U_n centered around that point is entirely contained in U_i (i.e such that $U_n \times U_n$ is inside U_i). But we can also find a point in this neighborhood with some $U_n, n > N$ that lies inside the translate of U_i, which is not possible. Thus for large enough i, a translate of U_i can intersect only one of C_1, C_2, and so we get a content on compact sets, which extends to the Borel measure we want (just like the Lebesgue measure). It is nontrivial as $\mu(A) = 1$.

For uniqueness, we rely on the fact that a σ-compact Radon measure is uniquely determined by its integral on continuous compactly supported functions. Our group may not be σ-compact, but G_o is by results in the previous section, and uniqueness for G_o is good enough.

Let f, g be continuous and compactly supported, and fix $\epsilon > 0$. Choose a neighborhood of the identity U_ϵ with compact closure, $U_\epsilon = U_\epsilon^{-1}$ such that in each left translate of U_ϵ, f and g vary by at most $\frac{\epsilon}{2}$. By Urysohn's Lemma, let ϕ_ϵ be a continuous function with support in U_ϵ and $\int \phi_\epsilon d\mu = 1$. Then $\int f(xy)\phi_\epsilon(y)d\mu(y) = f(x) + O(\epsilon)$ uniformly in x.

Integrating this over $d\mu'(x)$, we get

$$\int \left(\int f(xy)\phi_\epsilon(y)d\mu(y) \right) d\mu'(x) = \int f d\mu' + O(\epsilon)$$

But this is also

$$= \int \left(\int f(y)\phi_\epsilon(x^{-1}y)d\mu'(x) \right) d\mu(y) = \int \left(\int \phi_\epsilon(x^{-1})d\mu'(x) \right) f(y)d\mu(y)$$

$$= \int \phi_\epsilon(x^{-1})d\mu'(x) \int f d\mu = c_\epsilon \int f d\mu$$

.

Doing the same for a function g and multiplying shows that $(\int f d\mu' + O(\epsilon))c_\epsilon \int g d\mu = (\int g d\mu' + O(\epsilon))c_\epsilon \int f d\mu$, and rearranging this gives $\int g d\mu \int f d\mu' - \int f d\mu \int g d\mu' = O(\epsilon)$, so we can let $\epsilon \to 0$, showing that the two integrals will always agree up to a scalar. □

Theorem 1.10. *For a compact group G, a left Haar measure is a right Haar measure.*

Proof. Observe that the measure $f \to \int f(yx)d\mu(x)$ is also left invariant, hence is a constant multiple of μ. We can do this for every μ giving a continuous homomorphism $G \to \mathbb{R}$. By compactness and positivity the image must be trivial. □

2. Lie groups and Lie algebras

Definition 2.1. *A Lie group is a group in the category of smooth manifolds.*

It is equally possible to work with complex Lie groups, where we work instead in the category of complex manifolds, but we will stick to real Lie groups here.

Examples include **matrix Lie groups**, or Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$. Examples of matrix Lie groups include $\mathrm{SL}_n(\mathbb{F})$, or matrices with determinant 1, $\mathrm{SO}(n)$, or orthogonal matrices with determinant 1, $\mathrm{SU}(n)$, or unitary matrices with determinant 1. The last two can be interpreted also in terms of automorphisms preserving a metric on a vector space.

Lemma 2.2. $\mathrm{GL}_n(\mathbb{R})^+, \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{F}), \mathrm{SO}(n), \mathrm{SU}(n)$ *are connected.*

Proof. $\mathrm{GL}_n(\mathbb{R})^+$ and $\mathrm{GL}_n(\mathbb{C})$ are clearly connected if $\mathrm{SL}_n(\mathbb{F})$ is as each element is path connected to one of determinant 1. Now using a continuous (slightly modified) version of Graham-Schmidt, $\mathrm{SL}_n(\mathbb{F})$ is connected if $\mathrm{SO}(n), \mathrm{SU}(n)$ are, but these are as we can use the previous lemma, induction, and the fibrations $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$, and $SO(n-1) \hookrightarrow SO(n) \to S^{n-1}$. ☐

Theorem 2.3. *The tangent bundle of a Lie group is trivial.*

Proof. We can take a frame at the identity and translate it around to get a trivialization of the tangent bundle. ☐

Theorem 2.4. *A Lie group's Haar measure comes from integrating a volume form.*

Proof. Translate any nonzero element of the bundle of volume forms at the identity around to get a non-vanishing translation invariant volume form. ☐

The **Lie algebra** associated a Lie group G (often denoted $\mathfrak{g}$) is the linear space of left-invariant vector fields, along with the Lie bracket on vector fields. Note that elements of the Lie algebra also correspond to tangent vectors. We will see that a Lie group is highly determined by Lie algebra.

Theorem 2.5. *Elements of* $\mathfrak{g}$ *define global flows on* G.

Proof. If at a point p our flow is defined on $(-\epsilon, \epsilon)$, we can translate our parallel curve $\frac{\epsilon}{2}$ in either direction to extend the flow further by the fact that the flow is left invariant. ☐

Now observe that our Lie algebra can also be described as $\mathrm{Hom}(\mathbb{R}, G)$. Namely, for each left-invariant vector field $V \in \mathfrak{g}$, there is a unique homomorphism f_V from $\mathbb{R}$ whose derivative is that tangent vector. As it is a homomorphism, it must be an integral curve to V. Conversely, integral curves to V at the origin are easily seen to be homomorphisms from left invariance of the flow and uniqueness of integral curves.

The **exponential map** is the map from $\mathfrak{g} \to G$ defined by $\exp(V) = e^V = f_V(1)$. It is a smooth map, and we can compute its derivative at the origin as $\frac{d}{dt}_{t=0} f_{tV}(1) = \frac{d}{dt}_{t=0} f_V(t) = V$. Thus its derivative is an isomorphism on the tangent space, so it is a local diffeomorphism at the identity.

Proposition 2.6. *The image of the exponential map generates G_o.*

Proof. The subgroup it generates is contained in G_o as it is connected, and contains an open neighborhood of the identity by the inverse function theorem, so it is open. It is thus all of G_o by Lemma 1.8. $\square$

With the Lie algebra construction we can produce a Lie functor from the category of Lie groups to Lie algebras. Given a homomorphism $G \to G'$, we get an associated map $\mathfrak{g} \to \mathfrak{g}'$ just by the fact that the Lie algebra is given by $\mathrm{Hom}(\mathbb{R}, -)$ (alternatively it is just the derivative at the identity). It is linear as the derivative is, and to check that it preserves the Lie bracket, note that it preserves the flow, so that $f([X,Y]_e) = f(\frac{d}{dt}_{t=0}(dF_X(-t))(Y_{F_X(t).e})) = \frac{d}{dt}_{t=0}(dF_{f(X)}(-t))(Y_{F_{f(X)}(t).e}) = [f(X), f(Y)]_e$.

Note that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R})$, all $n \times n$ matrices. The exponential map is given by $\exp(X) = \sum_0^\infty \frac{X^n}{n!}$. Given a subgroup G of $\mathrm{GL}_n(\mathbb{R})$, the map $\mathrm{Hom}(\mathbb{R}, G) \to \mathrm{Hom}(\mathbb{R}, \mathrm{GL}_n(\mathbb{R}))$ is an inclusion, so the Lie algebra is exactly those matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

Lemma 2.7. $\det(e^X) = e^{\mathrm{tr}(X)}, e^{X^*} = (e^X)^*, e^{X^\top} = (e^X)^\top$ $e^X e^Y = e^{X+Y}$ *when* $[X,Y] = 0$, *and* $Y^{-1}e^X Y = e^{Y^{-1}XY}$.

Proof. All the results immediately follow from looking at power series except for the first. For that, note that $\det e^{tX}$ is a homomorphism from the additive group $\mathbb{R}X$ to $\mathbb{R}^\times$, and so is $e^{\mathrm{tr}(tX)}$, so we just need to verify their derivatives are the same at the 0. The derivative of the latter is clearly $\mathrm{tr}(tX)$, and to compute the former's derivative, the determinant is given by a sum over permutations of the matrix. The diagonal entries of e^{tX} look like $1 + tX_{ii} + O(t^2)$, and the off diagonals look like $tX_{ij} + O(t^2)$, so when taking the determinant and approaching 0, the only linear part is $\sum_i tX_{ii}$, so the derivative at 0 is the trace. $\square$

Using the above lemma, we can easily describe the Lie algebras for our other matrix lie groups. For example, the Lie algebra of $\mathrm{SU}(n)$ (denoted $\mathfrak{su}(n)$) consists of traceless skew-hermitian matrices. This allows for efficient computation of the dimension of a Lie group, since it is the same as the dimension of its Lie algebra.

3. The Universal Enveloping Algebra and Ado's Theorem

A general Lie algebra $\mathfrak{g}$ over k is a k-vector space with a alternating bilinear bracket $[\cdot,\cdot]$ satisfying the Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$.

There is a natural Lie algebra structure on the endomorphisms of a vector space $\mathrm{End}(V)$ given by $[a,b] = ab - ba$. A Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}(V)$ is called a representation of $\mathfrak{g}$ into V. In order to reduce proving things about general Lie groups to matrix Lie groups, we will prove that a finite dimensional Lie algebra has a faithful finite dimensional representation, so that we can think of it as a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. More generally than looking at representations into $\mathrm{End}(V)$, we can look at representations of $\mathfrak{g}$ into some associative algebra such that each element is sent to multiplication by an element of the algebra, and such that $[a, b]$ is sent to multiplication by $ab - ba$ in the algebra. There is a universal such algebra called the **universal enveloping algebra**.

Definition 3.1. *The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a universal element of the functor taking an algebra to its set of representations.*

In other words, representations into an algebra correspond to algebra homomorphisms from the universal enveloping algebra.

To construct the universal enveloping algebra $U(\mathfrak{g})$, take the tensor algebra generated by $\mathfrak{g}$, and mod out by the two-sided ideal generated by $[a, b] = ab - ba$. The natural map from $\mathfrak{g}$ is the representation. If $\mathfrak{g}$ has a faithful representation, $U(\mathfrak{g})$ will also be faithful by the universal property. Thus we would like to understand the structure of $U(\mathfrak{g})$.

To do this, we will use a general tool due to George Bergman for finding canonical forms for elements of associative k-algebras.

Say that we have a set X, and we consider the tensor algebra $T_R(X)$ on X over a commutative ring R. Say we also have some relations $\sigma \in S$ of the form $W_\sigma = f_\sigma$, where W is a word in X. For any other words A, B, we can consider $r_{A\sigma B}$, the R-linear endomorphism on $T_R(X)$ replacing $AW_\sigma B$ with $Af_\sigma B$. We call applying this to an element of $T_R(X)$ a **reduction**. If all reductions on an element are trivial, then that element is **irreducible**. The submodule of irreducible elements is called $T_R(X)_{irr}$. We say that an element is **reduction finite** if for any infinite sequence of reductions, only finitely many act nontrivially. The reduction finite elements form an R-submodule of $T_R(X)$ called $T_R(X)_{fin}$. A sequence of reductions is **final** if it results in an irreducible element. An element a is **reduction unique** if it is reduction finite and any final sequence results in the same irreducible element. The unique result will be denoted $r(a)$.

Lemma 3.2. *The reduction unique elements form a submodule denoted $T_R(X)_{un}$, and $r : T_R(X)_{un} \to T_R(X)_{irr}$ is R-linear.*

Proof. Suppose $a, b \in T_R(X)_{un}, k \in R$. $a + kb$ is reduction finite, and if we have two reduction sequences of it to an irreducible, we can consider doing the same sequence on a, b and extend them to one that makes each irreducible (note that this won't change the result, only the sequence). Then by uniqueness for a, b we see that the two reductions are the same, and that r is R-linear. □

Lemma 3.3. *Let $a, b, c \in T_R(X)$ have the property that if A, B, C are nonzero monomials in a, b, c, then ABC is reduction unique. Then if $r_\Sigma(b)$ is the result of some finite reductions on b, then $ar_\Sigma(b)c$ is reduction unique.*

Proof. Note in particular that the hypotheses imply that if abc is reduction unique, and that if the conclusion holds, $r(ar_\Sigma(b)c) = r(abc)$. By linearity we only need to show this when a, b, c are monomials, and when $r_\Sigma(b)$ is a single reduction $r_{d\sigma e}$. But $ar_{d\sigma e}(b)c = r_{ad\sigma ec}(abc)$, so this follows since abc is reduction unique. $\square$

We say that a **overlap ambiguity** is a pair $\sigma, \sigma' \in S$ and a triple A, B, C of nonempty words such that $AB = W_\sigma, BC = W'_\sigma$ (they overlap). It is **resolvable** if there is are sequences of reductions r, r' such that $r \circ r_\sigma(ABC) = r' \circ r_{\sigma'}(ABC)$. An **inclusion ambiguity** is the same, except when $W_\sigma = B, W'_\sigma = ABC$, and has the same conditions for resolvability (except A, C can be empty).

Say that a **compatible partial ordering** $\leq$ on words of X is one such that $A \leq B \implies CAD \leq CBD$ and such that the monomials of the f_σ are smaller than the monomials of W_σ. Let I be the two-sided ideal generated by the relations in S, and I_A be the R-submodule of $T_R(X)$ generated by $B\sigma C < A, \sigma \in S$ (every monomial is smaller than A). An ambiguity is **resolvable relative to** $\leq$ if $r_\sigma(ABC) - r'_\sigma(ABC) \in I_{ABC}$. Note that this is an easy condition to check.

Finally we arrive at this theorem, which can be considered a Diamond Lemma for rings:

Theorem 3.4. *Suppose we have $S, X, \leq, I$ as above where $\leq$ is compatible with S and satisfies the descending chain condition. Then the following are equivalent:*

1. *(1) Every ambiguity is resolvable.*
2. *(2) Every ambiguity is resolvable with respect to $\leq$.*
3. *(3) Every element is reduction unique.*
4. *(4) The natural quotient identifies the submodule spanned by irreducible monomials with $T_R(X)/I$, which is $T_R(X)_{irr}$.*

Proof. Clearly (3) $\implies$ (1) $\implies$ (2). If (3) is true, r defines a projection onto $T_R(X)_{irr}$. The kernel is contained in I by definition of r and contains I, as for any AB, $r(A\sigma B) = 0$ by Lemma 3.3. Thus as an R-module, $T_R(X) = I \oplus T_R(X)_{irr}$, giving (4). Conversely if (4) is true, Then since reductions are equal in the quotient, they must be unique.

Thus it suffices to prove (2) $\implies$ (3), and by linearity, we need only show this for a monomial A. From the descending chain condition, we can assume that any smaller monomial is reduction unique. Now suppose that we have two reductions of a monomial A.

If there is no ambiguity, they commute, so we can create two more reductions to an irreducible where the first two steps are the first step of these reductions but in different orders. They will give the same element, and by induction will show that the two original reductions are also the same.

If there is ambiguity, then since it is resolvable relative to $\leq$ by induction the difference can be resolved to 0, so by linearity the two resolutions must agree. $\square$

Theorem 3.5 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a Lie algebra over R where the underlying module is free. Choose a well-ordered basis of $\mathfrak{g}$, $x_\alpha, \alpha \in I$. Then $x_{a_1}^{e_1} \ldots x_{a_n}^{e_n}$ for x_{a_1} in increasing order, over all possible a_i and e_i form a basis of $U(\mathfrak{g})$, which is a free module. In particular, every Lie algebra has a faithful representation.*

Proof. Construct an ordering on monomials in the basis where monomials of smaller degree are smaller, and if two monomials have the same degree, we compare them lexicographically using the order on our basis. This is a well-ordering on monomials, and a sufficient set of relations for $U(\mathfrak{g})$ can be written as $xy = yx + [x, y]$ for $x > y$ in the basis. The order is compatible with this, and we can check that every ambiguity is resolvable with respect to our order. Here, the only nontrivial kind of ambiguity that can occur is an overlap ambiguity, when we have something of the form $A(yxz + [x, y]z)B = AxyzB = A(xzy + x[y, z])B$. The difference is:

$$A(yxz + [x, y]z - (xzy + x[y, z]))B = A(yzx + y[x, z] + z[x, y] + [[x, y], z]$$

$$-(zxy + [x, z]y + [y, z]x + [x, [y, z]]))B = A(zyx + [y, z]x + y[x, z] + z[x, y] + [[x, y], z]$$

$$-(zyx + z[x, y] + y[x, z] + [[x, z], y] + [y, z]x + [x, [y, z]])B$$

$$= A([[x, y], z] - [[x, z], y] - [x, [y, z]])B = 0$$

Where at the last step, we have used the Jacobi identity. This shows that the ambiguity is resolvable under the ordering, so that by Theorem 3.4 we are done. $\square$

4. Nilpotent Lie Algebras and Engel's Theorem

Other things to do: Levi's theorem that the radical of $\mathfrak{g}$ splits the Lie algebra. Lie's theorem on eigenvectors of Lie algebras

Theorem 4.1 (Ado's Theorem)**.** *Every finite dimensional Lie algebra admits a faithful finite dimensional representation.*

Proof. This is a hard theorem. $\square$

Lemma 4.2. *For any Lie group G,* $\exp(X + Y) = \lim_{n\to\infty}(\exp(\frac{X}{n})\exp(\frac{Y}{n}))^n$.

Proof. By Ado's Theorem, it suffices to prove this for $GL_n(\mathbb{R})$, in which case it follows by looking at Taylor series. $\square$

Theorem 4.3. *A continuous homomorphism of Lie groups is smooth.*

Proof. First consider a homomorphism $f : \mathbb{R} \to G$. G is locally a diffeomorphism, so for small ϵ, choose $X \in \mathfrak{g}$ such that $\exp(X) = f(\epsilon)$. Since f and exp are homomorphisms, $f(q\epsilon) = \exp(qX)$ for any rational q, and by continuity, $f(t) = \exp(\frac{tX}{\epsilon}$, so f is smooth. Now for any $\phi : G \to H$, identify the Lie algebras with $\mathrm{Hom}(\mathbb{R}, -)$, so that we get a map of Lie algebras, which is linear by the previous Lemma. Now since exp is a local diffeomorphism and commutes with ϕ, ϕ is smooth near the identity, hence everywhere. ☐

Theorem 4.4. *A homomorphism $f : G \to H$ to a connected Lie group H is a covering map iff $df : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*

Proof. If f is a covering map, then in particular it is a local diffeomorphism hence an isomorphism on Lie algebras. Conversely if df is an isomorphism, since exp commutes with f and generates H, f is surjective, and since we can find a neighborhood on which $\exp_H$ is a diffeomorphism, take its preimage, and intersect it with one for which $\exp_G$ is a diffeomorphism, we see that the kernel is discrete, so f is a covering map. ☐

Theorem 4.5 (Lie subgroup Adjunction)**.** *There is an adjunction between the category of subgroups of a Lie group and the category of subalgebras of its Lie algebra.*

Proof. This follows from the Frobenius's integrability criterion. Given a subgroup, the left cosets form a left invariant foliation, giving a left invariant subbundle of the tangent bundle that is closed under the Lie bracket, corresponding to a Lie subalgebra. Conversely, given a Lie subalgebra, its span is a subbundle satisfying Frobenius's theorem, so there is a parallel foliation. The submanifold containing the identity will be a subgroup as Something to do with Frobenius' theorem from differential geometry about foliations. ☐

The unit is an isomorphism and the counit is the connected component.

Theorem 4.6 (Lie's Third Theorem)**.** *The Lie functor is essentially surjective.*

Proof. By Ado's Theorem, any Lie subgroup ☐

Proposition 4.7. *If $df = dg$ for two maps $f, g : G \to H$ with G connected, $f = g$.*

Proof. This follows from commutativity of the exponential map with f and g and the fact that the image of the exponential map generates g. ☐

Corollary 4.8 (Lie's Second Theorem)**.** $\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ *via the Lie functor if G is simply connected.*

Proof. We may assume H connected since the image of G always lies in H^o and H^o has the same Lie algebra. By Proposition 4.7 it suffices to show surjectivity. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is an algebra morphism, consider the map $\phi \oplus 1_{\mathfrak{g}}$, and consider the subgroup A of $H \times G$ corresponding to the image. A comes with projections π_H, π_G. Note that π_G is an isomorphism of Lie algebras, so since G is simply connected and by Theorem 4.4 π_G is an isomorphism, so $\pi_H \circ \pi_G^{-1}$ is our desired map. □

Theorem 4.9 (Lie group/algebra adjunction)**.** *The Lie functor has a left adjoint.*

Proof. First note that to any Lie algebra, there is a unique simply connected Lie algebra, by Lie's second and third theorems. Then our left adjoint will take a Lie algebra and give the unique simply connected associated Lie group. The functoriality follows from Lie's second theorem, as well as the fact that this is an adjunction. □

The unit is an isomorphism and the counit is the universal cover of the connected component.

Theorem 4.10. *Regular Lie subgroups are closed Lie subgroups.*

Proof. If $H \subset G$ is regular, and $x_i \to x$ with x_i in H, then choose a cubical chart U around e, and pick □

Proposition 4.11. *If G is a compact Lie group and (π, V) a finite dimensional representation, and any G-invariant metric, $d\pi$ is skew-symmetric. In particular, there is a metric on $T_e(G)$ that is Ad-invariant, and such that ad is skew symmetric.*

Proof. $\langle e^{tX}A, e^{tX}B\rangle = \langle A, B\rangle$, so taking the derivative at 0 we are done. □

Definition 4.12. *Let G be a compact Lie group and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra of its Lie algebra. An element X of the subalgebra is* ***regular*** *if $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(X)$.*

Lemma 4.13. *Let I be a nonzero ideal of $\mathbb{R}[x_1, \ldots, x_n]$. Then $Z(I)$ is a closed measure 0 set.*

Proof. $Z(I)$ is the intersection of the zero set of each element of I, so is closed. By Fubini's Theorem it suffices to show that the intersection of $Z(I)$ with almost all hyperplanes along one direction is measure 0. To do this we can use induction. This is clearly true for $\mathbb{R}^1$, and by induction hypothesis, it suffices to show there can only be finitely many entire hyperplanes along one direction intersecting with $Z(I)$. To do this, note that the hyperplanes correspond for example to the ideals $(x_1 + c)$ for some $c \in \mathbb{R}$, and so this follows from looking at the Zariski topology of $\mathbb{R}^1$ again. □

Theorem 4.14. *The set of regular elements is an open with full measure (with respect to the Lebesgue measure for any basis).*

Proof. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{t}$. Then note that $\mathrm{ad}(X_i), \mathrm{ad}(X_j)$ commute, and that if $h_i = \ker(\mathrm{ad}(X_i))$ and $j_i = \ker(\mathrm{ad}(X_i))^{\perp}$, then note that $\mathrm{ad}(X_i)$ preserves the h_i, j_i $\mathfrak{t}$ splits as $\oplus_{g_i = j_i, h_i} \cap_1^n g_i$. Now for $c_i \neq 0$, if $l = \sum c_i X_i$, then $\ker(\mathrm{ad}(l))$ is $\cap_1^n h_i$, hence l is regular, iff it is invertible when restricted to each summand of t

that is not $\cap_1^n g_i$. We will show that the set of c_i for which $\mathrm{ad}(l)$ is invertible on each of these is open with full measure. To do this, the determinant of $\mathrm{ad}(l)$ on each of these spaces is a polynomial in the c_i, and is nonzero since some X_i is nonzero on the space. Then by Lemma 4.13 on this polynomial, we see that on a open with full measure, $\mathrm{ad}_(l)$ is invertible. Thus almost all elements are regular. Now to show the regular points are open, given a regular element X, choose a basis X_i of $\mathfrak{t}$ such that none of the coefficients of X are 0 when X is written as a combination of the X_i. Then by the same argument, X is contained in some open set of regular points. The independence of basis follows from the properties of Lebesgue measure under linear transformation. □

Theorem 4.15. *For any $X \in \mathfrak{g}$, with $\mathfrak{t}$ a Cartan subalgebra for a compact Lie group G, there is some g such that $\mathrm{Ad}(g)X \in \mathfrak{t}$.*

Proof. By Theorem 4.14, picking a regular element Y, it suffices to show that there is some g with $[\mathrm{Ad}(g)X, Y] = 0$. By Proposition 4.11, it suffices to show for all Z, $\langle [\mathrm{Ad}(g)X, Y], Z\rangle = 0$, which happens iff $\langle Y, \mathrm{ad}(Z)\,\mathrm{Ad}(g)X\rangle = 0$. G is compact, so choose g a minimum of the function $\langle Y, \mathrm{Ad}(g)X\rangle$. Then the function $\langle Y, \exp(tZ)\,\mathrm{Ad}(g)X\rangle$ has a minimum at $t = 0$ for all Z so taking the derivative, we are done. □

Theorem 4.16. *G acts transitively on the set of Cartan subalgebras (via Ad) and maximal tori (via conjugation).*

Proof. If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two Cartan subalgebras, write them as $\mathfrak{z}_{\mathfrak{g}(X_i)}$. Then by the previous theorem, choose g such that $\mathrm{Ad}(g)X_1 \in \mathfrak{t}_2$. Then $\mathrm{Ad}(g)\mathfrak{t}_1 = \{\mathrm{Ad}(g)Y | [Y, X_1] = 0\} = \{Y | [\mathrm{Ad}(g^{-1})Y, X_1] = 0\} = \{Y | [Y, \mathrm{Ad}(g)X_1] = 0\} = \mathfrak{z}_{\mathfrak{g}}(\mathrm{Ad}(g)X_1)$. Thus we have $\mathrm{Ad}(g)\mathfrak{t}_1 \supset \mathfrak{t}_2$, but my maximality they must be equal.

Now if T_i are two maximal tori, with $\mathfrak{t}_i$ the corresponding subalgebras, then if $\mathrm{Ad}(g)t_1 \supset t_2$, then □