CONJUGACY OF CARTAN SUBALGEBRAS

ISHAN LEVY

1. Introduction

A **Cartan subalgebra** $\mathfrak{h}$ of a (finite-dimensional) Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is its own normalizer. In the case that $\mathfrak{g}$ is semi-simple, it is abelian, and is the Lie algebra analog of the maximal torus of a Lie group. It plays a central role in the structure theory of semi-simple Lie algebras, as the weights of $\operatorname{ad} h$ acting on g, called the roots, give the root system classifying the semi-simple Lie algebra.

It follows from the definition that a Cartan subalgebra is maximal as a nilpotent subalgebra.

Lemma 1.1. *If $\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{h} \subset \mathfrak{h}'$ is a nilpotent subalgebra, then $\mathfrak{h} = \mathfrak{h}'$*

Proof. Let $\mathfrak{h}'^i$ be the lower central series, For sufficiently large i, $\mathfrak{h}'^i = 0$ so that $\mathfrak{h}'^i \subset \mathfrak{h}$. Now if $\mathfrak{h}'^i \subset \mathfrak{h}$ then since $[\mathfrak{h}'^{i-1}, \mathfrak{h}] \subset [\mathfrak{h}'^{i-1}, \mathfrak{h}'] = \mathfrak{h}'^i$, and $\mathfrak{h}$ is its own normalizer, $\mathfrak{h}^{i-1} \subset \mathfrak{h}$. Thus inductively we obtain $\mathfrak{h}' = \mathfrak{h}'^0 \subset \mathfrak{h}$. □

There are many choices of Cartan subalgebras, so one way to know that data derived from the Cartan subalgebra is actually an invariant of the Lie algebra would be to know that for any two Cartan subalgebras, there is an automorphism of the Lie algebra taking one to the other. Indeed, this is true when the underlying field is algebraically closed and of characteristic 0, and the automorphism can be thought of as coming from conjugation in the associated algebraic group.

2. Construction of Cartan subalgebras

A first step in understanding Cartan subalgebras is to understand how to construct them. It is instructive to look at the case of sl_n. A Cartan subalgebra of sl_n is given by taking the diagonal matrices. One can observe that this Cartan subalgebra can essentially be recovered from a generic element inside it. Namely, given a diagonal matrix $x \in sl_n$ with all distinct eigenvalues (this is a generic condition), then the Cartan subalgebra can be described as exactly those elements of sl_n commuting with x, as such matrices preserve the eigenspaces of x, which are all 1-dimensional, hence must be diagonal.

Cartan observed that the same process can be used to construct Cartan subalgebras in any Lie algebra. The genericity condition on x is captured in the notion of a **regular element**. Consider the characteristic polynomial $\det(\lambda - \mathrm{ad})$ of the adjoint representation of $\mathfrak{g}$. This is a polynomial $c(\mathfrak{g}, \lambda)$ in $\mathfrak{g}$ and λ of the form $\lambda^d - tr(\mathrm{ad}\, x)\lambda^{d-1} + ... \pm \det(\mathrm{ad}\, x)$. Let c_i be the i^{th} coefficient in the λ-variable: this is a polynomial of degree $d - i$ in $\mathfrak{g}$, since the determinant is a homogeneous polynomial of degree d. $c_0 \equiv 0$ as x is always in the kernel of $\mathrm{ad}\, x$. The **rank** r of $\mathfrak{g}$ is the smallest r such that $c_r \neq 0$.

Definition 2.1. *A regular element of* $\mathfrak{g}$ *is an* $x \in \mathfrak{g}$ *such that* $c_r(x) \neq 0$.

Since c_r is a nonzero polynomial, the set of regular elements is a non-empty Zariski-open set. Assuming that the field k we are working over is infinite, there must then be a regular element x defined over k. Now given a regular element, we can consider the (generalized) eigenspace decomposition $\oplus_{\alpha \in \bar{k}} \mathfrak{g}_\alpha$ with respect to $\mathrm{ad}\, x$. Let $\mathfrak{h}$ be the 0-eigenspace.

Theorem 2.2 (Construction of Cartan subalgebras)**.** *Using notation as above,* $\mathfrak{h}$ *is the unique Cartan subalgebra containing* x*.*

Proof. Let V be the sum of the nonzero eigenspaces. Let $\mathfrak{h}_V$ be the subset of $y \in \mathfrak{h}$ such that $\mathrm{ad}\, y$ is invertible on V, and let $\mathfrak{h}_N$ be the subset of $\mathfrak{h}$ such that $\mathrm{ad}\,|_h$ is not nilpotent. Being in $\mathfrak{h}_N$ is a Zariski open condition, since its complement is the vanishing locus of the $c_i^h, i < d$, where c_i^h are the c_i for $\mathfrak{h}$. Similarly being in $\mathfrak{h}_V$ is an open condition since invertability is an open condition on the determinant of ad restricted to V, which is a polynomial. If $\mathfrak{h}$ were not nilpotent, $\mathfrak{h}_N$ would be nonempty, and since $x \in \mathfrak{h}_V$, $\exists y \in \mathfrak{h}_V \cap \mathfrak{h}_N$ (y might be defined over a finite extension of k). But then the 0-eigenspace of $\mathrm{ad}\, y$ is strictly smaller than the 0-eigenspace of $\mathrm{ad}\, x$, which is a contradictio since x is regular. Thus $\mathfrak{h}$ is nilpotent by Engel's theorem. Now suppose that y normalizes $\mathfrak{h}$. Then since $\mathrm{ad}\, y(x) = -\mathrm{ad}\, x(y) \in \mathfrak{h}$, but since $\mathrm{ad}\, x|_V$ is invertible, the V component of y is 0, so $y \in \mathfrak{h}$.

To see uniqueness, observer that if x is in some $\mathfrak{h}'$ that is nilpotent, $\mathrm{ad}\, x$ acts nilpotently, so $\mathfrak{h}' \subset \mathfrak{h}$. Thus if $\mathfrak{h}'$ is a Cartan subalgebra, by maximality, $\mathfrak{h}' = \mathfrak{h}$. □

Note that the dimension of the Cartan subalgebra produced this way is exactly the rank of $\mathfrak{g}$.

3. Conjugacy of Cartan subalgebras

To show that all Cartan subalgebras are equivalent, we need to assume that k is algebraically closed and characteristic 0 (or the characteristic is at least sufficiently large with respect to the dimension of the Lie algebra), which we will assume from now on. We can consider the root space decomposition $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ with respect to a Cartan subalgebra $\mathfrak{h}$, which is the weight space decomposition of $\operatorname{ad}\mathfrak{h}$ on $\mathfrak{g}$. Since $\operatorname{ad}\mathfrak{g}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\beta+\alpha}$, ad applied to any element of a nonzero root space is a nilpotent derivation of $\mathfrak{g}$. Thus its exponential is an automorphism of $\mathfrak{g}$, that can be thought of as conjugation by the element in the associated algebraic group. The idea to show that any two Cartan subalgebras are conjugate is to use these automorphisms to show that all these automorphisms can conjugate $\mathfrak{h}$ to contain any of a Zariski open set of points in $\mathfrak{g}$. Since $\mathfrak{h}$ is an arbitrary Cartan subalgebra, and since regular elements are open, we can sent two Cartan subalgebras to ones that contain the same regular element, but then that Cartan subalgebra must be the one obtained from that regular element, hence the two original Cartan subalgebras can be related via an automorphism of $\mathfrak{g}$.

To realize this idea, let $x_1, \dots, x_n$ be a basis of the sum of the nonzero root spaces of $\mathfrak{h}$, such that each one is in some root space. Indeed, the zero simultaneous eigenspace of $\mathfrak{h}$ will always be itself, as it will normalize $\mathfrak{h}$. Then $e^{\operatorname{ad} x_i}$ is an automorphism of $\mathfrak{g}$, so we can consider the **conjugation map** $\mathfrak{g} = \bigoplus_1^n kx_i \oplus \mathfrak{h} \to \mathfrak{g}$ sending t_i, h to $\prod_1^n e^{\operatorname{ad} t_i x_i} h$ (the product is ordered as the exponentials don't generally commute). Since each x_i is nilpotent, this map is actually a polynomial map.

We would like to show the image contains a nonempty Zariski open set, but in order to do this, we will need an algebro-geometric lemma that works as a kind of inverse function theorem:

Lemma 3.1. *Given a morphism $\mathbb{A}^n \to \mathbb{A}^n$ such that the Jacobian at a point is invertible, then the image contains an open set.*

Proof. Suppose that the morphism is given by the polynomials $f_1, \dots, f_n$. WLOG, suppose that the Jacobian at 0, i.e. $(\partial_j f_i)$ is invertible. If the f_i satisfy a polynomial relation $F(f_1, \dots, f_n) = 0$, then $\sum_i \partial_{f_i} F \partial_j f_i = 0$ for all j, so $\partial_{f_i} F$ is in the kernel of the Jacobian of the f_i. Since the Jacobian is invertible at a point, it is invertible on a nonempty Zariski open set, and since $(\partial_{f_i} F)$ is nonzero on a Zariski open set, it follows that $(\partial_{f_i} F)$ must be identically zero in order to always be in the kernel of the Jacobian. But each $\partial_{f_i} F$ is of smaller degree, so if we choose F to be of minimal degree, then we get a contradiction.

Thus the f_i are algebraically independent, so the morphism gives an inclusion of function fields $k(f_1, \dots, f_n) \hookrightarrow k(x_1, \dots, x_n)$. This extension is finite, since both are finitely generated, and have the same transcendence degree. Thus after inverting finitely many elements of $k[f_1, \dots, f_n]$, which is the same as inverting their product, g, we get that $k[f_1, \dots, f_n, g^{-1}] \hookrightarrow k[x_1, \dots, x_n, g^{-1}]$ is finite. A finite morphism is surjective, so our morphism surjects onto the open set that is the complement of the zero locus of g. $\square$

Proposition 3.2. *The image of the conjugation map $\mathfrak{g} \to \mathfrak{g}$ contains a nonempty Zariski open set.*

Proof. If we can find a point where the Jacobian of this function is invertible, then by the previous lemma, the image contains a Zariski open set. Since the domain and codomain of the map have the same dimension, it will suffice to show that the Jacobian is injective. To do this, we will compute the Jacobian of the function at a point $a \in \mathfrak{h}$, in the direction b_i, h. Indeed, it is $\frac{d}{dt}_{t=0} \prod_1^n e^{tb_i \operatorname{ad} x_i}(a+th) = \frac{d}{dt}_{t=0} \prod_1^n (1+tb_i \operatorname{ad} x_i)(a+th) = \sum_i b_i[x_i, a] + h$. This is guaranteed to be 0 for all nonzero choices of b_i, h as long as $[x_i, a] \neq 0$ for each i, since the x_i are generalized eigenvectors of $\operatorname{ad} a$. For each nonzero weight space, the elements of $\mathfrak{h}$ with 0 eigenvalue on that weight space is a hyperplane, and since finitely many hyperplanes cannot cover the whole space, there is a point a satisfying the conditions we want. $\square$

Theorem 3.3. *Any two Cartan subalgebras are conjugate via the group of automorphisms of $\mathfrak{g}$ generated by $e^{\operatorname{ad} b}$ for $\operatorname{ad} b$ nilpotent.*

Proof. If $\mathfrak{h}, \mathfrak{h}'$ are two Cartan subalgebras, according to the previous proposition, the image of conjugation of each contains a nonempty Zariski open set. Intersecting these two open sets with the open set of regular elements, we still get a nonempty open set, so there is an x regular, such that both $\mathfrak{h}, \mathfrak{h}'$ can be conjugated to contain x. But in Theorem 2.2 it was shown that there is a unique Cartan subalgebra containing any regular element, so both $\mathfrak{h}$ can be conjugated to the same Cartan subalgebra. $\square$

4. References

This is based on notes for Victor Kac's class Introduction to Lie Algebras.
http://math.mit.edu/classes/18.745/classnotes.html.