ISOSPECTRAL MANIFOLDS/NUMBER FIELDS VIA GALOIS THEORY

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1. Number fields

Two number fields are **arithmetically equivalent** if they have the same Dedekind zeta function $\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$. Note from the zeta function, we can recover the number of ideals of a particular norm. Indeed, for any convergent Dirichlet series $\sum_i \frac{a_n}{n^s}$, the smallest nonzero a_j can be recovered by noticing that the function asymptotically behaves like $\frac{a_j}{j^s}$, and after subtracting this term off and repeating, all the terms can be recovered. Using the data of how many ideals there are of norm p^k for an rational prime p, we can recover the (unordered) residual degrees $f_1, \dots, f_k$ of the factors of all the primes above any rational prime p. This information, is called the **splitting type of** p. This recovers information such as which primes split completely over $\mathbb{Q}$. Thus we can see for example that Galois extensions over $\mathbb{Q}$ are determined by their zeta functions, and the degree over $\mathbb{Q}$ is an invariant of the zeta function.

To study further the problem of determining when $\zeta_K(s) = \zeta'_K(s)$ we will let N be a common Galois extension over $\mathbb{Q}$ of both of them with Galois group G, such that H, H' are the Galois groups over K, K. Now we can give an interpretation of splitting type in terms of Galois theory. In particular, suppose that $f_1, \dots, f_k$ is the splitting type of some unramified prime p, and D is the decomposition group of some prime over p. Then the action of G on the primes above p is the same as the action on right D cosets. Thus the sizes of the double cosets Ht_iD of H, D correspond to the f_i via $f_i|H| = |Ht_iD|$ possibly after rearrangement this. The information of the sizes of the double cosets will be called the **double coset type of (H,D)**. Since by Frobenius density theorem every cyclic subgroup is the decomposition group of infinitely many primes, we have that the double coset type of (H, D) is an invariant of the zeta function for any cyclic subgroup D.

The following lemma is purely group theoretic:

Lemma 1.1. *Two subgroups H, H' of a finite group G have the same double coset types for any cyclic subgroup D iff $|H \cap [\alpha]| = |H' \cap [\alpha]|$ for any conjugacy class $[\alpha]$.*

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It shows that two arithmetically equivalent number fields satisfy the lemma above for their corresponding subgroups. In fact, the converse is true.

To see this, note that for any unramified prime p, we can run the argument backwards to see that the number of ideals of norm that are powers of p is the same for both K, K'. Thus the Euler products of $\zeta_K(s), \zeta'_K(s)$ differ only possibly by the ramified primes. Now we will deal with the ramification.

Let C_i be a decomposition group of the real place of $\mathbb{Q}$ (generated by complex conjugation). The double cosets of H, H' with C then correspond to the real and complex places of K, K' (in particular one of size $2|H|$ is a complex embedding and one of size $|H|$ is a real embedding). Thus we can recover the numbers r_1, r_2 of real and complex places from the double coset type of this subgroup.

Now to see that $\zeta_K(s) = \zeta'_K(s)$, we will examine their functional equations.To see that they can't be different, we will use the functional equation for the Dedekind zeta function. Now recall that if $G_1 = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), G_2 = (2\pi)^{1-s}\Gamma(s)$, then $Z_K(s) = G_1(s)^{r_1}G_2(s)^{r_2}\zeta_K(s)$ has simple poles at $0, 1$, is holomorphic elsewhere, and satisfies the functional equation $Z_K(s) = |D_K|^{\frac{1}{2}-s}Z_K(1-s)$. Taking the quotient for Z_K, Z'_K, we get $\frac{\zeta_K(s)}{\zeta_{K'}(s)} = |\frac{D_K}{D_{K'}}|^{\frac{1}{2}-s}\frac{\zeta_K(s)}{\zeta_{K'}(s)}$.

But $\frac{\zeta_K(s)}{\zeta_{K'}(s)}$ is of the form $\prod_1^n(1-\frac{1}{a_i^s})\prod_1^m(1-\frac{1}{b_j^s})^{-1}$ since there are finitely many ramified primes, and the only such function that can satisfy such a functional equation is 1. Indeed, if we can't cancel any of the terms in the products, choose c to be the largest among the a_i or b_i, we can observe from the functional equation that $1 - \frac{2k\pi i}{\log c}$ is either a pole or zero of $\frac{\zeta_K(s)}{\zeta_{K'}(s)}$, which is clearly impossible. As a consequence, we also get the absolute values of the discriminants are the same. We have proven:

Theorem 1.2. *Let K, K' be number fields, N a common Galois extension over $\mathbb{Q}$, and H, H' the subgroups corresponding to each. Then K, K' are arithmetically equivalent iff $|H \cap [a]| = |H' \cap [a]|$ for all conjugacy classes.*

As an example, the Galois group of the polynomial $x^8 - 3$ is $\mathrm{Hol}(\mathbb{Z}/8\mathbb{Z}) = \mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/8\mathbb{Z}^\times$. The subgroups $\langle 0\rangle \times \mathbb{Z}/8\mathbb{Z}^\times$ and the subgroup $\langle(4, \pm 3)\rangle$ satisfy the condition we want, but are not conjugate, so the corresponding subfields are distinct but arithmetically equivalent.

2. Riemannian Manifolds

Two compact Riemannian manifolds M, M' are **isospectral** if they have the same eigenvalues of their (Hodge) Laplacian $\Delta_M, \Delta_{M'}$. Here we will think of Δ_M as acting on p-forms for some p. Examples of isospectral manifolds have been known for a long time, such as Milnor's famous example of two isospectral flat tori, but many of the ways of producing these manifolds do not work in a lot of generality. Here is a method that yields a result highly analogous to the way of producing arithmetically equivalent number fields, and that can even produce topologically different isospectral manifolds.

Namely, given a Riemannian manifold M, we will find a condition for two covering spaces $M_1, M_2 \to M$ to be isospectral. An amazing feature of this construction is that the condition is topological, so that M_1 and M_2 will be isospectral regardless of the metric on M. Of course it is possible that for some metrics, M_1 and M_2 will also be isometric, but as long as M_1, M_2 are topologically different covers, generic metrics on M will have them not be isometric.

The first fact that we will use is that the spectrum of a Riemannian manifold M is entirely encoded in its analytic zeta function $\zeta_M(s) = \sum_i \lambda_i^{-s}$, so we will try to understand how this function with respect to covers. To do this, consider the inverse of $d + d^*$ on V_M, from its image to the orthogonal complement of the kernel. As an endomorphism of $L^2_M(\bigwedge^* TM)$ it is self adjoint and compact by Rellich compactness, so its square $(\Delta_M)^{-1}$ is trace class. It is diagonalizable so we can consider $(\Delta_M)^{-s}$ for $\mathrm{Re}(s) > 1$, which is also trace class, and moreover has trace equal to $\zeta_M(s)$ when restricted to p-forms.

Now let $\tilde{M}$ a finite Galois cover whose group of deck transformations is G, and let $M' \to M$ be a cover corresponding to a subgroup $H \subset G$. We can identify ${V_{\tilde{M}}}^H$ with $V_{M'}$ by pulling back functions and scaling by $|H|^{-\frac{1}{2}}$ to preserve the inner product, where $(-)^H$ means the subspace fixed by H. Thus using the formula for the projection to the fixed subspace,

$$\zeta_{M'}(s) = \mathrm{tr}(\Delta_{M'}^{-s}) = \mathrm{tr}({\Delta_{\tilde{M}}}^{-s}|{V_{\tilde{M}}}^H)$$

$$= \frac{1}{|H|} \sum_{h \in H} \mathrm{tr}({\Delta_{\tilde{M}}}^{-s} h) = \frac{1}{|H|} \sum_{[g] \in G} |[g] \cap H| \, \mathrm{tr}({\Delta_{\tilde{M}}}^{-s} g)$$

where $[g] \in G$ are the conjugacy classes. Note that this formula only depends on $|[g] \cap H|$ for each $[g] \in G$. Thus we have proven:

Theorem 2.1. *Let H_1, H_2 be subgroups of the fundamental group of a compact Riemannian manifold M such that $|[g] \cap H_1| = |[g] \cap H_2|$ for each $[g] \in G$. Then the covers corresponding to H_1 and H_2 are isospectral.*

The same example as for number fields from before works. Recall that given a finite presentation of a group G, via surgery one can construct $n \geq 4$ dimensional manifold with fundamental group G. Thus we get a manifold M with fundamental group $G = \mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/8\mathbb{Z}^\times$ and a generic metric, and looking at the corresponding covers gives genuinely different isospectral manifolds.

3. Questions

We have essentially the same result in what might seem like two very different contexts. The first result made use of a functional equation of a zeta function, and the latter used a projection formula for the trace of an operator, since we had an interpretation of the zeta function as the trace of an operator.

Why is there an analogy in these two situations? Geometrically, a number field behaves a lot like an algebraic curve. Indeed, there are differential operators on compact Riemann surfaces whose zeta functions are analogous with Artin L-functions on arithmetic curves. So this leads to the question: is the Dedekind zeta function (or more generally an Artin L-function) the trace of an operator? This question is known as the Hilbert-Polya conjecture, and has led to a lot of research. Alain Connes has apparently such an operator on a non-commutative space, and claims that the Riemann hypothesis can be reformulated in terms of proving a trace formula for this operator.