

ORDINARY DIFFERENTIAL EQUATIONS

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1. MAIN THEOREMS

We can use the contraction mapping principle to construct solutions of ODEs.

Lemma 1.1. *Suppose that M is a metric space, and T is a contraction endomorphism. Then there is at most one fixed point of T , and if M is nonempty and complete, the fixed point exists.*

Solutions to ODEs locally exist uniquely with whatever amount of continuity we want as long as the defining equations satisfy uniform Lipschitz conditions locally.

Theorem 1.2. *Let U be a compact space, $V \subset B$ a bounded open subset of a Banach space, F_U a continuous function from U to a space of uniformly bounded continuous functions on $V \times (\mathbb{R})$ to \mathbb{R}), satisfying a uniform Lipschitz condition with constant K , and $I : U \rightarrow V$ the initial condition function. Then for small ϵ , there is a unique solution that is continuous, $x_u(t) : U \times (-\epsilon, \epsilon) \rightarrow V$ solving the differential equation $\frac{dx_u(t)}{dt} = F_u(x_u, t)$ with initial conditions $x_u(0) = I(u)$.*

Proof. We can write the equation as $x_u(t) = I(u) + \int_{t_0}^t F_u(x_u(t), t) dt$. We can think of the right hand side as an operator ϕx on possible solutions x , so that the problem amounts to carefully defining the space that ϕ acts on, and showing that it is a contraction. We can define M to be the space of continuous functions $U \times (-\epsilon, \epsilon) \rightarrow V$ satisfying the initial conditions, and let the metric be induced from the sup norm. Then since U is compact and F_U is continuous, the family F_u are uniformly bounded, so that for small ϵ , the operator ϕ will indeed send elements of M to M . To see that ϕ is a contraction map, observe that if $x, x' \in M$, then $d(\phi x, \phi x') = \sup_t \|\int_{t_0}^t F_u(x_u(t), t) - F_u(x'_u(t), t) dt\| \leq \sup_t \int_{t_0}^t K \|x_u(t) - x'_u(t)\| dt \leq K\epsilon d(x, x')$, so if $K\epsilon < 1$, then this is a contraction map. \square

In particular this shows that solutions depend differentiably upon initial conditions if the derivatives of $F(x, t)$ are uniformly Lipschitz: we can take U to be a small neighborhood of 0 times a time direction, and consider solving the differential equation for the difference quotients of $F(x, t)$ in the neighborhood, and the solution will be difference quotients of the solution for F by uniqueness. Then by continuity, the difference quotients will converge to the derivative of F , and the left hand side

will converge as well, implying that the solution of the original equation is continuously differentiable. If F is smooth, then so will the solution x . Moreover if F is analytic, then x will be a uniform limit of analytic functions, so will be as well.

Global existence of solutions can only fail if the solution blows up, and tries to leave the domain.

Corollary 1.3. *Suppose that $F(x, t)$ locally is Lipschitz continuous. Then there is a maximal interval $(-a, b)$ on which a solution to $\frac{dx}{dt} = F(x, t)$ with an initial condition can be defined, which is unique. If $-a$ or b is finite, then the solution leaves every compact set as it approaches $-a$ or $-b$.*

Proof. Try to define $-a, b$ as the the largest such that the solution exists. By local uniqueness, the solution is unique on that interval. Now if the solution x is in some compact set when t is near b , then it must converge as $t \rightarrow b$. But then we can use the local existence theorem, which will agree with the already existing solution to see that x can be extended a bit beyond b . \square

Thus given a vector field V on a manifold M , the field induces a local flow on M that may not be globally defined, but is if M is compact. If the vector field and M are smooth, the flow will be as well. Conversely if g_t is a one parameter local group of diffeomorphisms, The derivative of g_t at $t = 0$ is a smooth vector field inducing g_t as its flow.

One can often reduce higher order equations to first order equations. For example, if we want to solve $x^{(n)} = F(x^{(i)}; i < n, t)$, we can turn it into a first order equation in more variables by solving for $(x^{(i)})$ simultaneously.

The inverse function theorem can be proven using the same contraction mapping principle.

Lemma 1.4. *If f is C^1 in a neighborhood of a point p and has an invertible derivative at p , then it is injective near p .*

Proof. After a change of coordinates, $f(0) = 0, f'(0) = I$, so that $f(x) = f(a) + f'(a)(x-a) + o(x-a)$. We can choose a neighborhood such that $\|a\| < \frac{1}{4}, \|f'(a)\| > \frac{3}{4}$ and so that the error term $\|o(x-a)\| < \frac{1}{4}\|x-a\|$. Then we have that $\|f(x) - f(y)\| = \|f'(y)(x-y) + o(x-y)\| \geq \frac{3}{4}\|x-y\| - \|o(x-y)\| \geq \frac{1}{2}\|x-y\|$. \square

Theorem 1.5 (Inverse function theorem). *If f has an invertible derivative at a point, and is injective near that point it locally has an inverse, which is differentiable at the image point.*

Proof. $f(x) = y = x + \epsilon(x)$ where $\epsilon(x) = o(x)$ after a linear change of variables. To solve $y = x + \epsilon(x)$, we observe that $x = y - \epsilon(x)$, so if $g(y) = x$, it is the fixed point of the mapping $\phi(g) = y - \epsilon(g)$. But in a small neighborhood of 0, $\|\epsilon(x)\| < \lambda < 1$, so

in this neighborhood, ϕ is a contraction mapping on the space of possible continuous inverses with the sup norm. Note that $\phi^n(g)$ is differentiable at 0 with derivative the identity, and uniformly converges to the inverse, so the inverse is differentiable at 0. \square

There is a geometric interpretation of solving an ODE (and a similar one for PDE). Let T be the times, S be the phase space, and $J^1(S)$ be the space of 1-jets, with the projection map to S . $J^1(S)$ comes with a canonical distribution, given by the 1-forms $dx_i = p_i dt$, where p_i are the cotangent bundle coordinates. We can consider the bundles $T \times S$, $T \times J^1(S)$ over T . A smooth section of the first one has a canonical lift to a section of the second, by requiring that it is tangent to the distribution. Then if $F(p, x, t)$ is a differential equation, a solution is exactly a section of $T \times S$ whose lift lies in the hypersurface defined by $F(p, x, t)$.