

MORSE THEORY ON LOOP SPACES

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1. SETUP AND CRITICAL POINTS

A natural object of study in algebraic topology is the space of loops ΩX of a space. The functor Ω is adjoint to the suspension Σ , giving nice properties such as $\pi_n \Omega X = \pi_{n+1} X$. Instead of studying the loop space, we may instead like to study the space of paths between two points on X , which is homotopy equivalent to the loop space if the points lie in the same path component. Our goal is to use Morse theory to learn about a CW-structure on the loop space ΩM of a smooth manifold M . To begin with, instead of considering the entire loop space, we will consider $\Omega^* M$, the space (to be topologized later) of piecewise smooth paths from p to q , two fixed points on M .

We should think about $\Omega^* M$ as an “infinite dimensional manifold”, but to avoid doing Morse theory on it directly we will ultimately use instead a finite dimensional approximation of it. Our Morse function will be the energy function $E(\gamma) = \int \|\gamma'\|^2$. Analogously to the finite dimensional case, the tangent space T_γ of a path γ will be the space of vector fields on γ that vanish at the end points, as these correspond to the derivatives of variations.

The first thing we need to do is figure out what the critical points of E are. To do this, we need to compute the derivative of the E :

Lemma 1.1 (First Variation Formula). *Let t_i be a subdivision on which the vector field $W \in T_\gamma$ is smooth, and let $\Delta_i \gamma'$ denote the jump of γ' at t_i . Then $E'(W) = -2 \sum_i \langle W, \Delta_i \gamma' \rangle - 2 \int_0^1 \langle W, \gamma''(t) \rangle$*

Proof. Let $\gamma_u(t)$ be a variation that corresponds to W . Then $E'(W) = \frac{d}{du} E(\gamma_u(t)) = \int_0^1 \frac{d}{du} \langle \frac{d}{dt} \gamma_u(t), \frac{d}{dt} \gamma_u(t) \rangle$

$$\begin{aligned} &= 2 \int_0^1 \langle \frac{D}{du} \frac{d}{dt} \gamma_u(t), \frac{d}{dt} \gamma_u(t) \rangle = 2 \sum_i \int_{t_i}^{t_{i+1}} \langle \frac{D}{dt} \frac{d}{du} \gamma_u(t), \frac{d}{dt} \gamma_u(t) \rangle \\ &= -2 \sum_i \langle \frac{d}{du} \gamma_u(t), \Delta_i \frac{d}{dt} \gamma_u(t) \rangle - 2 \int_0^1 \langle \frac{d}{du} \gamma_u(t), \frac{D}{dt} \frac{d}{dt} \gamma_u(t) \rangle \end{aligned}$$

which yields the formula setting $u = 0$. □

Corollary 1.2. *The critical points of E are the geodesics.*

Proof. By Lemma 1.1, geodesics are critical points. Conversely if γ is critical, choosing W in the direction of γ' but vanishing at the discontinuities, we γ is piecewise geodesic, but then by choosing W in the direction of discontinuities of γ' , we see indeed there are no jumps in γ' so by uniqueness of solutions to ODEs, γ is a geodesic. □

2. THE HESSIAN AND ITS NULL SPACE

Next we would like to make a notion of the Hessian at a critical point, and study its null space and index. Then, analogously to the finite dimensional case, if $\gamma_{u,w}(t)$ is a 2-parameter variation, with $\frac{d}{du}\gamma_{u,w}(t) = W_2$, $\frac{d}{dw}\gamma_{u,w}(t) = W_1$ at $(u, w) = (0, 0)$, define the Hessian as $H(\gamma_{u,w}(t)) = H(W_1, W_2) = \frac{d}{du}\frac{d}{dw}E(\gamma_{u,w}(t))$. This is clearly symmetric bilinear, and to see that it is well defined, we can compute a formula in terms of W_1 and W_2 .

Lemma 2.1 (Second Variation Formula). *Let $\gamma_{u,w}(t)$ be a 2-parameter variation as above, and let t_i be the discontinuities of $\frac{D}{dt}W_2$. Then $H(W_1, W_2) = -2 \sum_i \langle W_1, \Delta_i \frac{D}{dt}W_2 \rangle - \int_0^1 \langle W_1, \frac{D^2}{dt^2}W_2 + R(\gamma', W_2, \gamma') \rangle$.*

Proof. From Lemma 1.1 using the fact that γ is geodesic, at $w = 0$ we get:

$$H(W_1, W_2) = -2 \sum_i \left\langle \frac{d}{dw}\gamma_{u,0}, \Delta_i \frac{D}{du} \frac{d}{dt}\gamma_{u,0} \right\rangle - 2 \int_0^1 \left\langle \frac{d}{dw}\gamma_{u,0}, \frac{D}{du} \frac{D}{dt} \frac{d}{dt}\gamma_{u,0}(t) \right\rangle$$

which after rearranging, and commuting the $\frac{D}{du}$ and $\frac{D}{dt}$ we get the formula. \square

Now a **Jacobi field** on a geodesic γ' is a vector field satisfying the second order linear differential equation called the Jacobi equation, $\frac{D^2}{dt^2}J + R(\gamma', J, \gamma') = 0$. As it is second order linear, the space of solutions is $2n$ dimensional. The proof of Corollary 1.2 gives:

Corollary 2.2. *The Jacobi fields are exactly the null space of H .*

We say that p, q on γ are **conjugate** of multiplicity $\nu > 0$ if ν is the dimension of the space of Jacobi fields vanishing at p, q . Thus γ is nondegenerate if the endpoints are nonconjugate. We can characterize Jacobi fields in another way: namely they arise from variations through geodesics.

Proposition 2.3. *Jacobi fields are exactly those arising from variations through geodesics.*

Proof. Given a variation through geodesics $\gamma_u(t)$, $0 = \frac{D}{du} \frac{D}{dt} \frac{d}{dt}\gamma_u(t) = \frac{D^2}{dt^2} \frac{d}{du}\gamma_u(t) + R(\frac{d}{dt}\gamma_u(t), \frac{d}{du}\gamma_u(t), \frac{d}{dt}\gamma_u(t))$ which shows the Jacobi equation is satisfied. Now to show all Jacobi fields arise as such, it suffices to do this on an arbitrarily small piece of γ , and then note that the variations of geodesics can be locally uniquely extended by compactness of $[0, 1]$ and that the resulting extension will yield the unique extension of the Jacobi equation by uniqueness of solutions to ODEs. Then in a small enough neighborhood of $\gamma(0)$ containing $\gamma(\epsilon)$, all geodesics are minimal, so in particular, we can take the variation that takes the unique geodesics between two paths touching $\gamma(0)$ and $\gamma(\epsilon)$ with prescribed derivatives at $\gamma(0)$ and $\gamma(\epsilon)$. This yields $2n$ linearly independent solutions of the Jacobi equation so it must yield all of them. \square

We then easily get the next proposition that guarantees the existence of nonconjugate points:

Proposition 2.4. *The nullity of $d\exp_p(v)$ is the multiplicity of conjugacy of p and $\exp_p(v)$ along the geodesic $\exp_p(v)$.*

Proof. Given a curve $\beta(u)$ at v , we can consider the variation through geodesics $\exp_p(t\beta(u))$. By restricting to an interval where geodesics are minimal, we see that this yields all Jacobi fields that vanish at p by choosing different $\beta'(0)$. In particular, the Jacobi field vanishes at the ends iff $d\exp_p(\beta'(0)) = 0$, so we get the result. \square

Corollary 2.5. *There are points nonconjugate along any geodesic.*

Proof. This follows from the previous proposition and Sard's Theorem. \square

From Proposition 2.4, we can see that the multiplicity of conjugacy between two points on a geodesic can be at most $n - 1$. Indeed by choosing $\beta'(0)$ pointing away from the origin, the resulting variation will have nonzero derivative at one end point. Alternatively one can directly see that the corresponding vector field $t\gamma'(t)$ indeed satisfies the Jacobi equation but doesn't vanish at the second endpoint.

Indeed we can see in the case of S^n , that two antipodal points are conjugate of multiplicity $n - 1$, as the space of minimal geodesics is a copy of S^{n-1} inside the tangent space. In fact one can exploit this fact to prove the Freudenthal Suspension Theorem.

3. INDEX THEOREM

Now given a critical point, we would also like to be able to compute its index. Here γ is a critical point from p to q . We have the following theorem:

Theorem 3.1 (Index Theorem). *The index of γ is the number of points conjugate to p counted with multiplicity. Moreover this index is finite.*

Proof. First we will approximate the tangent space T_γ by a finite dimensional subspace. Namely, choose a subdivision of γ , t_i , such that γ restricted to each subinterval is in a neighborhood of minimal geodesics. Then we will split T_γ as $T_{t_i}^\gamma \oplus T'$, where T_{t_i} consists of piecewise Jacobi fields on each interval, and T' consists of vector fields that vanish at the t_i . Indeed given tangent vectors at each of the t_i , there is a unique piecewise Jacobi field with those prescribed tangent vectors since the geodesic is minimal, so T_γ does split as such, and we also have $T_{t_i}^\gamma \cong \bigoplus_1^{n-1} TM_{\gamma(t_i)}$ as a result.

Since γ is a minimal geodesic on each subinterval, and variations corresponding to elements of T' can be chosen to fix the endpoints, γ is a minimum of each of E on each of these variations, so H is positive semidefinite on T' . Now if $H(W, W) = 0$ for $W \in T_{t_i}$, then W is in the null space of H by Lemma 2.1, so is a Jacobi field, but then it must be 0. Thus H is positive definite on T' so we may focus our attention on T_{t_i} . In particular, we already get that the index is finite.

Define $\lambda(t)$ to be the index of γ restricted to the subinterval $[0, t]$. It is clear that $\lambda(t)$ is monotonically increasing as negative definite subspaces can be extended as 0 to further along the curve. Moreover, near 0, since γ is minimal, $\lambda(t) = 0$ as well. Now fixing a point τ and choosing a subdivision with $t_i < \tau < t_{i+1}$, note that $T_{t_i}^{\gamma|_{[0, t]}} \cong \bigoplus_1^i TM_{\gamma(t_i)}$ is not dependant on t for $t_i < t < t_{i+1}$. Then H varies continuously on $T_{t_i}^{\gamma|_{[0, t]}}$ near τ . Then we get that $\lambda(\tau - \epsilon) \geq \lambda(\tau)$ for small ϵ , but by monotonicity, this must be an equality. Indeed since we also know the null space is dimension ν , we also get $\lambda(\tau + \epsilon) \leq \lambda(\tau) + \nu$, and to complete the proof it suffices to prove the reverse inequality.

Now let $Y_1, \dots, Y_{\lambda(t)}$ be orthonormal and linearly independent such that H is negative definite on them, and extend them to $t + \epsilon$ by setting them to 0. Let W_1, \dots, W_ν be linearly independent Jacobi fields that vanish at t , and extend them as well to $t + \epsilon$ as 0. Then choose X_i such that $\langle X_i(t), W_j(t) \rangle = \frac{1}{2}\delta_{ij}$. Looking at the Lemma 2.1 we get $H(W_i, X_j) = \delta_{ij}$, $H(W_i, Y_j) = 0$. We can write H then as a matrix on the span of $Y_1, \dots, Y_{\lambda(t)}, cX_1 -$

$c^{-1}W_1, \dots, cX_\nu - c^{-1}W_\nu$ as

$$\begin{pmatrix} -I_{\lambda(t)} & cB \\ cB^t & -I_\nu + c^2A \end{pmatrix}$$

which is certainly negative definite for small c . \square

4. APPROXIMATING THE LOOP SPACE

Now we would like to use these results to learn about the topology of the loop space ΩM . First we would like to say that ΩM is homotopy equivalent to our space of piecewise smooth paths Ω^*M . To do this, first topologize Ω^*M with the metric $d(\gamma_1, \gamma_2) = \sup_t \rho(\gamma_1(t), \gamma_2(t)) + \int_0^1 \|\gamma'_1 - \gamma'_2\|^2$ where the second term is to ensure that E is continuous, and ρ denotes the metric from the Riemannian metric.

Proposition 4.1. *The inclusion $i : \Omega^*M \rightarrow \Omega M$ is a homotopy equivalence.*

Proof. We cover M by geodesically convex neighborhoods, i.e. neighborhood such that any two points are connected by a unique minimal geodesic that lies entirely inside the neighborhood. Now let ΩM_i be the collection of paths such that the $[\frac{k}{i}, \frac{k+1}{i}]$ interval lies entirely inside one of these neighborhoods, and let $\Omega^*M_i = i^{-1}\Omega M_i$. Then Ω^*M is the homotopy direct limit of the Ω^*M_i , and likewise for ΩM , so it suffices to show that the inclusion $i : \Omega^*M_i \rightarrow \Omega M_i$ is a homotopy equivalence. To show this, our homotopy inverse j will take a path and return the unique geodesic that agrees on each $\frac{k}{i}$. $i \circ j$ is homotopic to the identity as given a path, we can continuously deform it to $i \circ j$ in a natural way: namely at the point $\frac{k}{i} < r < \frac{k+1}{i}$ let $H_r(\gamma)(t)$ be $\gamma(t)$ for $t > r$, on $[\frac{l}{i}, \frac{l+1}{i}]$ and on $[\frac{k}{i}, r]$, the unique minimal geodesic between those points. A similar construction shows that $j \circ i$ is homotopic to the identity. \square

Thus we can use Ω^*M to study the loop space. However, we would like to approximate Ω^*M further, so that we can do Morse theory on it. Namely, let $\Omega^c = E^{-1}[0, c]$, and $\text{Int } \Omega^c = E^{-1}(0, c)$.

Theorem 4.2. *Assume M complete. $\text{Int } \Omega^c$ can be approximated by a finite dimensional manifold B in a natural way. E will be a smooth function on B , The tangent space of B will be the T_{t_i} , and the critical points and Hessian, will be as before. Moreover $\text{Int } \Omega^c$ deformation retracts onto B .*

Proof. The metric ball of radius \sqrt{c} is compact as M is complete, and paths in $\text{Int } \Omega^c$ lie there by the Cauchy Schwarz inequality. Thus it has a positive injectivity radius ϵ , and we can make a subdivision t_i such that $t_{i+1} - t_i < \frac{\epsilon^2}{c}$. By the Cauchy Schwarz inequality again, we get that every element of $\text{Int } \Omega^c$ is length $< \epsilon$ in each interval $[t_i, t_{i+1}]$. Then we can let B be the subspace of piecewise geodesics on this subdivision. It is naturally a manifold as each path in B is determined by its values on the t_i , so B embeds as an open subset of $M \times \dots \times M$. Now we immediately see then that its critical points are unbroken geodesics, and that its tangent space at these critical points is exactly the T_{t_i} from the proof of the Index Theorem. A similar construction to Proposition 4.1 shows that B is a deformation retract of $\text{Int } \Omega^c$. \square

The previous theorem implies that we can do our usual Morse theory on the finite dimensional approximation to get results about the full loop space. As usual, we get:

Theorem 4.3 (Fundamental Theorem). *Let p, q be nonconjugate along any geodesic, and M complete. Then ΩM has the homotopy type of a CW-complex with a λ -cell in each dimension for each geodesic with index λ .*

In the case of S^n , any two distinct non-antipodal points are nonconjugate, and we can see that there is a unique geodesic of index $k(n - 1)$ for each k .

Corollary 4.4. *ΩS^n has the homotopy type of a CW-complex with one $k(n - 1)$ -cell for each $k \geq 0$.*