ORDINARY DIFFERENTIAL EQUATIONS

ISHAN LEVY

1. Main Theorems

We can use the contraction mapping principle to construct solutions of ODEs.

Lemma 1.1. *Suppose that M is a metric space, and T is a contraction endomorphism. Then there is at most one fixed point of T, and if M is nonempty and complete, the fixed point exists.*

Solutions to ODEs locally exist uniquely with whatever amount of continuity we want as long as the defining equations satisfy uniform Lipschitz conditions locally.

Theorem 1.2. *Let U be a compact space, $V \subset B$ a bounded open subset of a Banach space, F_U a continuous function from U to a space of uniformly bounded continuous functions on $V \times (\mathbb{R})$ to $\mathbb{R}$), satisfying a uniform Lipschitz condition with constant K, and $I : U \to V$ the initial condition function. Then for small ϵ, there is a unique solution that is continuous, $x_u(t) : U \times (-\epsilon, \epsilon) \to V$ solving the differential equation $\frac{dx_u(t)}{dt} = F_u(x_u, t)$ with initial conditions $x_u(0) = I(u)$.*

Proof. We can write the equation as $x_u(t) = I(u) + \int_{t_0}^t F_u(x_u(t), t)dt$. We can think of the right hand side as an operator ϕx on possible solutions x, so that the problem amounts to carefully defining the space that ϕ acts on, and showing that it is a contraction. We can define M to be the space of continuous functions $U \times (-\epsilon, \epsilon) \to V$ satisfying the initial conditions, and let the metric be induced from the sup norm. Then since U is compact and F_U is continuous, the family F_u are uniformly bounded, so that for small ϵ, the operator ϕ will indeed send elements of M to M. To see that ϕ is a contraction map, observe that that if $x, x' \in M$, then $d(\phi x, \phi x') = \sup_t \| \int_{t_0}^t F_u(x_u(t), t) - F_u(x'_u(t))dt\| \leq \sup_t \int_{t_0}^t K\|x_u(t) - x'_u(t)\| \leq K\epsilon d(x, x')$, so if $K\epsilon < 1$, then this is a contraction map. , $\square$

In particular this shows that solutions depend differentiably upon initial conditions if the derivatives of $F(x, t)$ are uniformly Lipschitz: we can take U to be a small neighborhood of 0 times a time direction, and consider solving the differential equation for the difference quotients of $F(x, t)$ in the neighborhood, and the solution will be difference quotients of the solution for F by uniqueness. Then by continuity, the difference quotients will converge to the derivative of F, and the left hand side will converge as well, implying that the solution of the original equation is continuously differentiable. If F is smooth, then so will the solution x. Moreover if F is analytic, then x will be a uniform limit of analytic functions, so will be as well.

Global existence of solutions can only fail if the solution blows up, and tries to leave the domain.

Corollary 1.3. *Suppose that $F(x,t)$ locally is Lipschitz continuous. Then there is a maximal interval $(-a,b)$ on which a solution to $\frac{dx}{dt} = F(x,t)$ with an initial condition can be defined, which is unique. If $-a$ or b is finite, then the solution leaves every compact set as it approaches $-a$ or $-b$.*

Proof. Try to define $-a, b$ as the the largest such that the solution exists. By local uniqueness, the solution is unique on that interval. Now if the solution x is in some compact set when t is near b, then it must converge as $t \to b$. But then we can use the local existence theorem, which will agree with the already existing solution to see that x can be extended a bit beyond b. $\square$

Thus given a vector field V on a manifold M, the field induces a local flow on M that may not be globally defined, but is if M is compact. If the vector field and M are smooth, the flow will be as well. Conversely if g_t is a one parameter local group of diffeomorphisms, The derivative of g_t at $t = 0$ is a smooth vector field inducing g_t as its flow.

One can often reduce higher order equations to first order equations. For example, if we want to solve $x^{(n)} = F(x^{(i)}; i < n, t)$, we can turn it into a first order equation in more variables by solving for $(x^{(i)})$ simultaneously.

The inverse function theorem can be proven using the same contraction mapping principle.

Lemma 1.4. *If f is C^1 in a neighborhood of a point p and has an invertible derivative at p, then it is injective near p.*

Proof. After a change of coordinates, $f(0) = 0, f'(0) = I$, so that $f(x) = f(a) + f'(a)(x-a)+o(x-a)$. We can choose a neighborhood such that $\|a\| < \frac{1}{4}$, $\|f'(a)\| > \frac{3}{4}$ and so that the error term $\|o(x-a)\| < \frac{1}{4}\|x-a\|$. Then we have that $\|f(x)-f(y)\| = \|f'(y)(x-y) + o(x-y)\| \geq \frac{3}{4}\|x-y\| - \|o(x-y)\| \geq \frac{1}{2}\|x-y\|$. $\square$

Theorem 1.5 (Inverse function theorem)**.** *If f has an invertible derivative at a point, and is injective near that point it locally has an inverse, which is differentiable at the image point.*

Proof. $f(x) = y = x + \epsilon(x)$ where $\epsilon(x) = o(x)$ after a linear change of variables. To solve $y = x+\epsilon(x)$, we observe that $x = y-o(x)$, so if $g(y) = x$, it is the fixed point of the mapping $\phi(g) = y - \epsilon(g)$. But in a small neighborhood of 0, $\|\epsilon(x)\| < \lambda < 1$, so in this neighborhood, ϕ is a contraction mapping on the space of possible continuous inverses with the sup norm. Note that $\phi^n(g)$ is differentiable at 0 with derivative the identity, and uniformly converges to the inverse, so the inverse is differentiable at 0. $\square$

There is a geometric interpretation of solving an ODE (and a similar one for PDE). The let T be the times, S be the phase space, and $J^1(S)$ be the space of 1-jets, with the projection map to S. $J_1(S)$ comes with a canonical distribution, given by the 1-forms $dx_i = p_i dt$, where p_i are the cotangent bundle coordinates. We can consider the bundles $T \times S$, $T \times J_1(S)$ over T. A smooth section of the first one has a canonical lift to a section of the second, by requiring that it is tangent to the distribution. Then if $F(p, x, t)$ is a differential equation, a solution is exactly a section of $T \times S$ whose lift lies in the hypersurface defined by $F(p, x, t)$.