# CHROMATIC HOMOTOPY AND TELESCOPIC LOCALIZATION

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## 1. MU and $\mathcal{M}_{fg}$

A ring spectrum $R$ is **complex oriented** if it is a equipped with a ring map $MU \to R$. Such a map provides the cohomology theory $R^*$ with Chern classes for complex vector bundles satisfying a Whitney sum formula.

Given two line bundles $L_1, L_2 : X \to BU(1)$, there is a universal formula for the Chern class of their tensor product:

$$c_1(L_1 \otimes L_2) = F_R(c_1(L_1), c_1(L_2))$$

$F_R$ is a power series in two variables with coefficients in $R^*$, and encodes the structure of a (1-dimensional commutative) **formal group law**. A formal group law is an abelian group structure on the formal $R$-scheme $\mathrm{Spf}(R[[x]])$. More concretely, this means that $F_R$ satisfies group axioms, such as associativity: $F_R(x, F_R(y,z)) = F_R(F_R(x,y),z)$.

An important result of Quillen says that $MU$, the universal complex oriented ring spectrum, has the universal formal group law. In particular, $MU_* = L$, where $L$ is the Lazard ring, defined by the universal property $\mathrm{Hom}(L,R) = FGL(R)$, where $FGL$ is the set of formal group laws on $R$.

However, the connection between $MU$ and formal group laws doesn't stop there. Recall we have the Adams-Novikov spectral sequence:

$$E_2 = \mathrm{Ext}_{MU_*MU}(MU_*, MU_*X) \implies \pi_*X$$

The $E_2$ term can be interpreted in terms of formal groups (which are formal group schemes Zariski-locally isomorphic to a formal group law). The Ext in the spectral sequence is taken in the category of comodules over the (graded) Hopf algebroid $(MU_*, MU_*MU)$. However, this Hopf algebroid presents $\mathcal{M}_{fg}$, the moduli stack of formal groups. $\mathcal{M}_{fg}$ has a line bundle $\omega$ that is the Lie algebra of the universal formal group. Then the Adams-Novikov $E_2$ term can be reinterpreted as

$$E_2 = H^*(\mathcal{M}_{fg}; (MU_*X)_{\text{even/odd}} \otimes \omega^{\otimes *}) \implies \pi_*X$$

Where we treat the even and odd degree parts of $MU_*X$ as a quasicoherent sheaf on $\mathcal{M}_{fg}$.

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Thus via $MU$, stable homotopy is tied to formal groups.

The study of formal groups simplifies a bit when localized at a prime. $(\mathcal{M}_{fg})_{(p)}$ has a simpler presentation as a graded Hopf algebroid $(BP_*, BP_*BP)$, where $BP_*$ is the ring $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$, with $|v_i| = 2(p^i - 1)$. In fact this Hopf algebroid (as suggested by the notation) comes from a ring spectrum called $BP$. In fact, $MU_{(p)}$ decomposes into summands that are shifts of $BP$.

$(\mathcal{M}_{fg})_{(p)}$ is a well understood stack. We can draw a picture of its points $\mathrm{Spc}((\mathcal{M}_{fg})_{(p)})$

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & \infty \\ . & . & . & . & \cdots & . \end{array}$$

There is one for each natural number and a point $\infty$. One way to interpret each point is that it classified a formal group over an algebraically closed field up to isomorphism. The point $n$ corresponds to a **height $n$ formal group**. For example, when $p = 3$, a height $n$ formal group is one such that if we choose coordinates so that the formal group is defined by a power series $F$, then $F(x, F(x, x)) = ux^{3^n} + \dots$ where $u$ is a unit and $\dots$ indicates higher order terms.

The point 0 classifies a formal group in characteristic 0, and the rest of the points classify formal groups in characteristic $p$.

Another way to interpret the picture is that it classifies invariant prime ideals of $BP_*$ in the Hopf algebroid $(BP_*, BP_*BP)$. The point $n$ corresponds to the ideal $(v_0, v_1, \dots, v_{n-1})$ where $v_0 = p$.

The space also has a topology, where the open sets are the intervals from 0 to $n$. In particular, specialization increases height.

## 2. Important Cohomology Theories and Theorems

Two important families of complex oriented cohomology theories are Morava E-theory and Morava K-theory.

The $n^{th}$ Morava E-theory, denoted $E_n$, is an $\mathbb{E}_\infty$-ring spectrum that depends on a choice of perfect field $k$ and formal group law on $k$ of height $n$. However, none of the choices will matter for anything said here about it. Its coefficient ring is $(E_n)_* = W(k)[[v_1, \dots, v_{n-1}]][\beta^{\pm 1}]$ where $|\beta| = 2$, and $W(k)$ denotes the Witt vectors of $k$, and its formal group law is the universal deformation of the formal group law on $k$, which was studied by Lubin and Tate.

One of its important properties is that $(E_n)_*(X) = 0$ if and only if $BP_*(X)$ is supported at height $\geq n+1$ on $\mathcal{M}_{fg}$. Thus it detects information from height 0 to height $n$.

The $n^{th}$ Morava K-theory, denoted $K(n)$, is an $\mathbb{E}_1$-ring spectrum with coefficient group $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$. Once again, there are different versions of it, but the different versions will not be relevant here. $K(n)$ can be constructed from $BP$ by iteratively taking cofibres by $v_i$ for $i \neq n$ and inverting $v_n$. $K(\infty)$ is just $H\mathbb{F}_p$. An important property of $K(n)$ is that it is a **field**, since its homotopy groups are a graded field. This means that any module over $K(n)$ is free.

$K(n)$ detects information at height $n$. For example, $(E_n)_*(X) = 0 \iff (\oplus_0^n K(n))_* X = 0$.

Next, we turn to some fundamental results in chromatic homotopy theory. The first is the nilpotence theorem, due to Devinatz, Hopkins, and Smith.

**Theorem 2.1** (Nilpotence Theorem v1). *Let $R$ be a ring spectrum, and $\alpha \in \pi_n(R)$ be an element sent to* 0 *in $MU_n(R)$. Then $\alpha$ is nilpotent.*

This says that $MU$ is able to detect nilpotence of rings. An equivalent version of the theorem is

**Theorem 2.2** (Nilpotence Theorem v2). *Let $f : X \to Y$ be a map of finite spectra such that $f \otimes MU$ is 0. Then $f^{\otimes n} : X^{\otimes n} \to Y^{\otimes n}$ is null for $n \gg 0$.*

This formulation emphasizes the fact that $MU$ detects nilpotence phenomena for finite spectra. When working $p$-locally, $MU_* f = 0$ iff $BP_* f = 0$ iff $K(n)_* f = 0$ for all $n$.

**Definition 2.3.** *A finite complex/spectrum $X$ is **type** $n$ if $K(i)_*(X) = 0$ for all $i < n$ and $K(n)_* X \neq 0$.*

**Remark 2.3.1.** Every nonzero $p$-local finite spectrum is type $n$ for some $n$. This is because for $i >> 0$, the $K(i)$-based Atiyah-Hirzebruch spectral sequence degenerates for degree reasons for a fixed finite spectrum $X$, so $K(i)_*(X) = (H\mathbb{F}_p)_*(X) \otimes_{(H\mathbb{F}_p)_*} K(i)_* \neq 0$.

**Remark 2.3.2.** For a finite spectrum $X$, $K(m)_* X = 0 \implies K(m-1)_*(X) = 0$. This is essentially because its $MU$-homology is a *coherent* sheaf over $\mathcal{M}_{fg}$, so has closed support. This shows that for a type $n$ spectrum $X$, $K(m)_*(X) \neq 0$ for $m \geq n$.

**Definition 2.4.** *Let $C$ be a stable $\infty$-category. A **thick subcategory** $C' \to C$ is a stable subcategory closed under retracts.*

**Example 2.4.1.** Let $\mathrm{Sp}^{\omega}_{(p)}$ be the category of $p$-local finite spectra, and let $\mathrm{Sp}_{\geq n}$ be the category of type $\geq n$ spectra. Then $\mathrm{Sp}_{\geq n} \to \mathrm{Sp}^{\omega}_{(p)}$ is a thick subcategory.

It turns out these are all the examples. This is the content of the following result, which is a corollary of the nilpotence theorem, due to Hopkins and Smith.

**Theorem 2.5** (Thick subcategory Theorem). *Let $C \subset \mathrm{Sp}^{\omega}_{(p)}$ be a nonzero thick subcategory. Then $C = \mathrm{Sp}_{\geq n}$ for some $n$.*

It is true that $\mathrm{Sp}_{\geq n}$ are distinct as $n$ varies, but showing this requires a bit more work.

**Definition 2.6.** *Let $X$ be a finite complex/spectrum. A $v_n$**-self map** $v_n : \Sigma^d X \to X$ is a map that*

*(1) induces 0 on $K(m)_*$ for $m \neq n$.*
*(2) induces an isomorphism on $K(n)_*$.*

The use of $v_n$ as a name for the $v_n$-self map is slightly misleading: a more appropriate name is $v_n^k$, because when they exist, they can be chosen to induce multiplication by a power of $v_n$ on $K(n)_*$.

Using a construction due to Smith, Hopkins and Smith proved the following result:

**Theorem 2.7** (Periodicity Theorem). *Every type $n$ spectrum admits a $v_n$-self map.*

From this theorem, it is easy to see why $\mathrm{Sp}_{\geq n}$ are distinct. For example, the sphere $\mathbb{S}$ is a type 0 but not type 1 spectrum. Given a type $n$ but not type $n+1$ spectrum, we can take the cofibre of a $v_n$-self map to obtain a type $n+1$ but not type $n+2$-spectrum, thereby inductively distinguishing the categories $\mathrm{Sp}_{\geq n}$.

$v_n$-self maps are well behaved. After replacing one with a sufficiently large power, we can assume

- the $v_n$-self map induces multiplication by $v_n^i$ on $K(n)_*$.

- the $v_n$-self map is central in $\mathrm{End}_*(X) = \pi_* X \otimes DX$.

Given a map of finite type $n$-spectra $f : X \to Y$ equipped with a $v_n$-self map, we can replace the $v_n$-self maps by an iterate to make the diagram below commute:

$$\begin{tikzcd}
\Sigma^d X \arrow[r, "f"] \arrow[d, "v_n"] & \Sigma^d Y \arrow[d, "v_n"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

In this sense, $v_n$-self maps are almost functorial. Note that if we take $f$ above to be the identity, we see that $v_n$-self maps are also unique up to taking iterations.

## 3. Chromatic Localizations

The moduli stack of formal groups is filtered by the open substacks of formal groups of height $\leq n$. Chromatic localizations are a way to turn this algebraic filtration into a topological one, and their study was pioneered by Doug Ravenel. To talk about them, we will briefly review Bousfield localizations of the category Sp.

Given a spectrum $X$, there is an adjunction

$$L_X : \mathrm{Sp} \leftrightharpoons \mathrm{Sp}_X : i$$

such that

- $L_X$ inverts $X$**-equivalences**: that is morphisms $f$ such that $f \otimes X$ is an equivalence.
- $L_X$ kills (sends to 0) the $X$**-acyclic objects**, i.e those objects $Y$ such that $Y \otimes X = 0$.
- $i$ is fully faithful, so $\mathrm{Sp}_X$ is a reflective subcategory of Sp.
- The essential image of $i$ consists of $X$**-local spectra**, that is objects $Z$ such that there are no nonzero maps from $X$-acyclic objects to $Z$.

The composite $i \circ L_X$ will often be shortened to $L_X$. The unit of the adjunction gives a natural map $Y \to L_X Y$, characterized by the fact that it is an $X$-equivalence to an $X$-local object.

The construction $L_X$ doesn't depend on all of $X$ but rather on the **Bousfield class**, that is $\langle X \rangle = \{X\text{-acyclic objects}\}$.

We can often break up a Bousfield localization into smaller pieces, and glue them back together.

**Lemma 3.1.** *Suppose $L_E$ preserves $F$-acyclic objects. Then*

$$\begin{tikzcd}
L_{E \oplus F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_F X \arrow[d] \\
L_E X \arrow[r] & L_E L_F X
\end{tikzcd}$$

*is a pullback square.*

*Proof.* Let $P = L_E X \times_{L_E L_F X} L_F X$.

- $P$ is $E \oplus F$ local. Indeed, if $Z$ is $E \oplus F$ acyclic, $P^Z = 0 \otimes_0 0 = 0$.
- $X \to P$ is an $E \oplus F$ equivalence. To see it is an $E$-equivalence, after tensoring with $E$ it becomes

$$X \otimes E \xrightarrow{\sim} X \otimes E \times_{E \otimes L_F X} E \otimes L_F X$$

To see it is an $F$-equivalence, by the hypothesis on $L_E$, we learn that the natural transformation $Y \to L_E Y$ is an $F$-equivalence. Thus after tensoring with $F$, we get

$$X \otimes F \xrightarrow{\sim} X \otimes F \times_{X \otimes F} X \otimes F$$

$\square$

If $X$ is a type $n$ spectrum, We can invert a $v_n$-self map to get $X[v_n^{-1}]$, which is called the **telescope** of $X$ and denoted $T(n)$. By the almost uniqueness of $v_n$-self maps, $T(n)$ only depends on $X$. Essentially by the thick subcategory theorem, $\langle T(n) \rangle$ only depends on $n$.

There are two flavors of chromatic localizations that are studied. The first are the telescopic and finite localizations $L_{T(n)}$ and $L_n^f := L_{\oplus_0^n T(i)}$. The second are the $K(n)$ and $E_n$ localizations $L_{K(n)}$ and $L_n := L_{E(n)} = L_{\oplus_0^n K(n)}$. The hope is that we can understand stable homotopy via the towers of localizations

$$X \to \cdots \to L_n X \to L_{n-1} X \to \dots L_1 X \to L_0 X$$

(and similarly for $L_n^f$ in place of $L_n$).

The two flavors of localizations are related to each other. If $Y \otimes T(n) = 0$, then $X \otimes T(n) \otimes K(n) = 0$, but $T(n) \otimes K(n)$ is a nonzero sum of copies of $K(n)$, so $X \otimes K(n) = 0$. Thus we get factorizations of the natural maps

$$X \to L_n^f X \to L_n X \ , \ \ X \to L_{T(n)} X \to L_{K(n)} X$$

An important property of $L_n X$ is that it is colimit preserving:

**Theorem 3.2** (Smashing Theorem). $L_n X = L_n \mathbb{S} \otimes X$

The same is true of $L_n^f$, but it is easier to prove, as will now be explained.

**Lemma 3.3.** *The $L_n^f$-acyclic spectra coincide with* $\mathrm{Ind}(\mathrm{Sp}_{\geq n+1})$*: that is they are filtered colimits of type $\geq n+1$ spectra.*

*Proof.* It is easy to see that $\mathrm{Ind}(\mathrm{Sp}_{\geq n+1})$ consists of $T(n)$-acyclic spectra; we will show the reverse inclusion. First let $n = 0$, and suppose $X$ is $T(0)$-acyclic. Then there is a cofibre sequence

$$X \to p^{-1} X = X \otimes T(0) \to X \otimes \mathbb{S}/p^\infty$$

where $\mathbb{S}/p^\infty$ is the colimit of $\mathbb{S}/p^n$ over all $n$. Since $X \otimes T(0)$ vanishes, we learn that $X = \Sigma^{-1} X \otimes \mathbb{S}/p^\infty$. $X$ is a filtered colimit of finite spectra, and after tensoring with $\mathbb{S}/p^n$, this becomes a filtered colimit of type 1 spectra.

Now we can induct on $n$. For example, let $n = 1$, and assume that in addition, $X$ is $T(1)$-acyclic. Then there is a cofibre sequence

$$X \otimes \mathbb{S}/p^n \to X \otimes v_1^{-1} \mathbb{S}/p^n = X \otimes T(1) \to X \otimes \mathbb{S}/p^n, v_1^\infty$$

, so since $X \otimes T(1) = 0$, we learn that $X = \Sigma^{-2} X \otimes \mathbb{S}/p^\infty, v_1^\infty$, which is in $\mathrm{Ind}(\mathrm{Sp}_{\geq 2})$. $\square$

**Remark 3.3.1.** The argument in the above lemma shows that there is a cofibre sequence

$$\Sigma^{-1-n} \mathbb{S}/v_0^\infty, \dots v_n^\infty \to \mathbb{S} \to L_n^f \mathbb{S}$$

Since $L_n^f$ kills a category that is generated by compact objects, it preserves filtered colimits. It also preserves finite colimits, so $L_n^f$ preserves all colimits. The only colimits preserving endomorphisms of Sp are given by tensoring, so we learn

**Corollary 3.4.** $L_n^f X = L_n^f \mathbb{S} \otimes X$.

The corollary above is one way to see that $L_m^f$ preserves $\oplus_{m+1}^n T(i)$-acyclic objects. Thus we learn from Lemma 3.1:

**Corollary 3.5.** *There is a pullback diagram*

$$\begin{tikzcd}
L_n^f X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{\oplus_{m+1}^n T(i)} X \arrow[d] \\
L_m^f X \arrow[r] & L_m^f L_{\oplus_{m+1}^n T(i)} X
\end{tikzcd}$$

Note that the same is true with $K(n)$ replacing $T(n)$ and $L_n$ replacing $L_n^f$ by the smashing theorem. These pullback squares allow one to reduce the study of $L_n^f$ to the study of $L_{T(n)}$.

The exact relation between $L_{T(n)}$ and $L_{K(n)}$ is not known. It was conjectured by Ravenel that there is no difference between the two.

**Conjecture 3.6** (Telescope conjecture)**.** *The map $L_{T(n)} X \to L_{K(n)} X$ is an equivalence.*

This conjecture is known to be true for $n = 1, 0$, and many believe it to be false otherwise. Nevertheless, so long as we are concerned with rings or finite spectra, the nilpotence theorem implies that $T(n)$ and $K(n)$ behave similarly.

**Lemma 3.7.** *If $R$ is a ring spectrum, $R \otimes T(n) = 0 \iff R \otimes K(n) = 0$.*

*Proof.* Let $V_n$ be a type $n$ spectrum that is an $\mathbb{E}_1$-ring. For example, one can start with any type $n$ spectrum $X$ and replace it with its endomorphism ring $X \otimes DX$. Let $v_n$ be a central $v_n$-self map, so that $T(n) = V_n[v_n^{-1}]$ is a ring. Then

$$\begin{aligned}
& R \otimes T(n) = 0 \\
\iff & \text{the unit of } R \otimes T(n) \text{ is nilpotent} \\
\iff & \text{the unit of } R \otimes T(n) \otimes K(m) \text{ is nilpotent for all } m \\
\iff & \text{the unit of } R \otimes T(n) \otimes K(n) \text{ is nilpotent} \\
\iff & R \otimes T(n) \otimes K(n) = 0 \\
\iff & R \otimes K(n) = 0
\end{aligned}$$

Where in the second step, we use the nilpotence theorem, and in the last step we use the fact that $T(n) \otimes K(n)$ is a free $K(n)$-module. ☐

## 4. Telescopic Localization and the Bousfield Kuhn functor

Now we will look the telescopic localization functors more in depth and see their to unstable homotopy theory. We will set $n \geq 1$, and let $V_n$ denote a type n *space* with a $v_n$-self map $v_n : \Sigma^d V_n \to V_n$.

**Definition 4.1.** *For a space/spectrum $X$, the $v_n$-**periodic homotopy groups** with coefficients in $V_n$, denoted $v_n^{-1}\pi_*(X; V_n)$ are defined as $v_n^{-1}\pi_*(\mathrm{Map}_*(V_n, X))$.*

It isn't hard to see that $v_n^{-1}\pi_*(X; V_n)$ is the homotopy groups of a $d$-periodic spectrum called $\Phi_{V_n}(X)$, given by the formula $\mathrm{colim}_k \Sigma^{\infty - kd} \mathrm{Map}_*(V_n, X)$ where the colimit is uses the map $\mathrm{Map}_*(V_n, X) \to \mathrm{Map}_*(\Sigma^d V_n, X) = \Omega^d \mathrm{Map}_*(V_n, X)$ induced by $v_n$.

Note that if $X$ is a spectrum, then $\Phi_{V_n}(X)$ is the spectrum $X \otimes DV[v_n^{-1}] = X \otimes T(n)$.

**Definition 4.2.** *If $f : X \to Y$ is a map of spaces or spectra, $f$ is a $v_n$**-periodic equivalence** if $\Phi_{V_n} X \to \Phi_{V_n} Y$ is an equivalence.*

Essentially by the thick subcategory theorem, the notion of $V_n$-periodic equivalence only depends on $n$.

**Lemma 4.3.** *Let $n \geq 1$, and $X$ be a spectrum. Then*

*(1) $\Phi_{V_n} X = \Phi_{V_n} \Omega^\infty X$.*
*(2) The map $\tau_{\geq k} X \to X$ is a $v_n$-periodic equivalence.*
*(3) The map $X \to X_p^\wedge$ is a $v_n$-periodic equivalence.*

*Proof.* (1): This follows from the adjunction between $\Sigma^\infty$ and $\Omega^\infty$.

(2): The fibre is bounded above, and $v_n^{-1}\pi_*(Y; V_n) = 0$ whenever $Y$ is bounded above because $|v_n| > 0$, and the homotopy groups of the mapping space are bounded above.

(3): The fibre $F \to X \to X_p^\wedge$ is $\mathbb{S}/p$-acyclic. This means that it is killed by tensoring with $\mathbb{S}/p^m$ for all $m$. But for any type $n$ spectrum, some power of $p$ acts by 0, so the same is true for $T(n)$. Thus $F \otimes \mathbb{S}/p^k \otimes T(n) = F \otimes (T(n) \oplus \Sigma T(n)) = 0$ for $k \gg 0$, so $F$ is $T(n)$-acyclic. □

The above lemma, along with the fact that $\Phi_{V_n} X = X \otimes T(n)$ implies that for $n \geq 1$, $L_{T(n)} X$ only depends on $\Omega^\infty X$ as an $\mathbb{E}_\infty$-space. A wonderful insight of Bousfield and Kuhn is that in fact it only depends on $\Omega^\infty X$ as a space!

To see this, we start by thinking about the construction taking a pair of a type $n$ space and $v_n$-self map $(V_n, v_n)$ to the functor $\Phi_{V_n} : S_* \to \mathrm{Sp}$. If we replace $v_n$ by an iterate, it is easy to see that it doesn't change $\Phi_{V_n}$, so since $v_n$-self maps are unique, the data of $v_n$ is not important in the construciton of $\Phi_{V_n}$.

Secondly, if we replace $V_n$ by $\Sigma V_n$, $\Phi_{V_n}$ changes to $\Phi_{\Sigma V_n} = \Sigma^{-1}\Phi_{V_n}$. Thus $\Phi_{V_n}$ only depends on the spectrum $\Sigma^\infty V_n$.

These observations can be souped up to construct a functor

$$\mathrm{Sp}_{\geq n} \to \mathrm{Fun}(S_*, \mathrm{Sp})$$

that sends a type $n$ spectrum $V$ to $\Phi_V$.

**Definition 4.4.** *The **Bousfield-Kuhn functor** $\Phi$ is a functor $S_* \to \mathrm{Sp}$ given by $\Phi := \lim_{V \to \mathbb{S}} \Phi_V$.*

Another way to describe it is that you right Kan extend the functor $\mathrm{Sp}_{\geq n} \to \mathrm{Fun}(S_*, \mathrm{Sp})$ along the inclusion to Sp, and evaluate on $\mathbb{S}$. An important property of $\Phi$ is that it realizes the factorization of $L_{T(n)}$ through $\Omega^\infty$ as a space:

**Proposition 4.5.** *$\Phi\Omega^\infty X = L_{T(n)} X$.*

*Proof.* We have from the definition and our previous observations $\Phi\Omega^\infty X = \lim_{V \to \mathbb{S}} \Phi_V X = \lim_{V \to \mathrm{Sp}} DV[v_n^{-1}] \otimes X$.

Each term in the limit is $T(n)$-local, so it agrees with $L_{T(n)}(DV[v^{-1}] \otimes X) = L_{T(n)}(DV \otimes X) = L_{T(n)}(X^V)$.

Putting this together, we have

$$\Phi\Omega^\infty X = \lim_{V \to \mathbb{S}} L_{T(n)} X^V$$

Now I claim that $T(n)$-locally $\mathbb{S}$ is a filtered colimit of type $n$ spectra. This claim completes the proof, because it identifies $\lim_{V\to\mathbb{S}} L_{T(n)}X^V$ with $L_{T(n)}X^{\mathbb{S}} = L_{T(n)}X$.

To see the claim we recall that we had a cofibre sequence

$$\Sigma^{-n}\mathbb{S}/p^\infty, \dots, v_{n-1}^\infty \to \mathbb{S} \to L_{n-1}^f\mathbb{S}$$

$L_{n-1}^f\mathbb{S}$ is $T(n)$-acyclic since $T(n)$ is a filtered colimit of type $n$ spectra, which $L_{n-1}^f$ kills. Thus applying $L_{T(n)}$ to the cofibre sequence above, we get a formula for $L_{T(n)}\mathbb{S}$ as a filtered colimit of ($T(n)$-localizations of) type $n$ spectra. $\square$

Some other important facts about the Bousfield-Kuhn functor are:

- It inverts $v_n$-periodic equivalences, and takes values in $T(n)$-local spectra. This is indicated by the factorization below, where $S_*^{v_n}$ is the localization of pointed spaces at the $v_n$-periodic equivalences. The factored map is also denoted $\Phi$.

$$\begin{tikzcd} S_* \arrow[r, "\Phi"] \arrow[d] & \mathrm{Sp} \arrow[d] \\ S_*^{v_n} \arrow[r, "\Phi"] & \mathrm{Sp}_{T(n)} \end{tikzcd}$$

- $\Phi : S_*^{v_n} \to \mathrm{Sp}_{T(n)}$ preserves limits.

A consequence of the factorization in the proposition above is:

**Corollary 4.6.** *Let $f : X \to Y \in \mathrm{Sp}$ be a map. If $\Sigma^\infty\Omega^\infty f$ is a $T(n)$-equivalence, so is $f$.*

*Proof.* By assumption, $L_{T(n)}\Sigma^\infty\Omega^\infty f$ is an equivalence. But this is equal to $\Phi\Omega^\infty\Sigma^\infty\Omega^\infty f$, and by the triangle identity for the adjunction between $\Sigma^\infty$ and $\Omega^\infty$, the map $\Phi\Omega^\infty f$ is a retract of $\Phi\Omega^\infty\Sigma^\infty\Omega^\infty f$. Thus $\Phi\Omega^\infty f = L_{T(n)}f$ is also an equivalence. $\square$

The functor $\Sigma^\infty\Omega^\infty$ doesn't *preserve* $T(n)$-local equivalences in general.

**Example 4.6.1.** $H\mathbb{Z}$ is $T(n)$-acyclic, but $\Sigma^\infty\Omega^\infty H\mathbb{Z}$ is a sum of spheres, so is not.

Nevertheless, for sufficiently connected maps, $\Sigma^\infty\Omega^\infty$ does preserve $T(n)$-local equivalences. Here is a version of that statement for the finite localizations.

**Proposition 4.7.** *Let $n \geq 1$. There is an $m \geq 2$ such that:*

*(1) If $F$ is an $m$-connected pointed space such that $v_i^{-1}\pi_*(F; V_i) = 0$ for $0 \leq i \leq n$, then $F$ is $L_n^f$-acyclic.*

*(2) If $f : X \to Y$ is an $m$-connected map that is a $v_i$-periodic equivalence for $0 \leq i \leq n$, then $\Sigma^\infty f$ is an $L_n^f$ equivalence.*

*(3) $\Sigma^\infty\Omega^\infty$ preserbes $m$-connected $L_n^f$ equivalences.*

*Proof.* (1): Omitted. This relies on results of Bousfield on unstable localization.

(2): The fibre $F$ satisfies the hypotheses of (1). Then $f$ can be identified with $\mathrm{colim}_Y F \to \mathrm{colim}_Y *$, which is a $T(n)$ equivalence since $F$ is $L_n^f$-acyclic.

(3): Apply $\Omega^\infty$ and (2). $\square$

**Remark 4.7.1.** In fact in the above proposition, $m$ can be taken to be $n + 1$. This is a consequence of ambidexterity of the $T(n)$-local category, which was proven by Carmeli, Schlank, and Yanovski.