RIEMANNIAN GEOMETRY

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1. Basic Things

Convention 1.0.1. $R_{XY} = [\nabla_Y, \nabla_X]$ if $[X, Y] = 0$.

Definition 1.1. *The **energy** $R(\gamma)$ of a smooth curve γ is $\int_\gamma \|\dot{\gamma}(t)\|^2$.*

Lemma 1.2 (First Variation Formula)**.** *Let V be a variational field of a curve γ defined on $[0, 1]$ that is piecewise smooth on the subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$.*

$$\frac{1}{2}dE(V) = \sum_i \langle V(t_i), \Delta\gamma'(t_i)\rangle - \int \langle V, \gamma''\rangle$$

Proof. Let γ_s be a variation. Then we can calculate:

$$\frac{1}{2}\frac{d}{ds}\int \langle \gamma'_s, \gamma'_s\rangle = \int \langle \frac{D}{ds}\frac{d}{dt}\gamma_s, \gamma'_s\rangle = \int \langle \frac{D}{dt}V, \gamma'\rangle$$
$$= \int \frac{d}{dt}\langle V, \gamma'\rangle - \int \langle V, \gamma''\rangle$$

And finally, use the fundamental theorem of calculus on the first term. □

Lemma 1.3 (Second Variation Formula)**.** *Let V, W be endpoint-fixing variations of a geodesic γ that are piecewise smooth (and continuous). Then if H is the Hessian of the energy functional, $\frac{1}{2}H(V, W) = \sum_i \langle V, \Delta \frac{D}{dt} W(t_i)\rangle + \int \langle R_{\gamma', W}\gamma' - W'', V\rangle$*

Proof. Let γ be a two parameter variation in the variables s_1, s_2. Then we can calculate:

$$\frac{d}{ds_2}\int \langle \frac{D}{dt}\frac{d}{ds_1}\gamma, \gamma'\rangle = \int \langle \frac{D}{ds_2}\frac{D}{dt}\frac{d}{ds_1}\gamma, \gamma'\rangle + \int \langle \frac{D}{dt}\frac{d}{ds_1}\gamma, \frac{D}{ds_2}\gamma'\rangle$$

For the first term, we get

$$= \int \langle \frac{D}{dt}\frac{D}{ds_2}\frac{d}{ds_1}\gamma, \gamma'\rangle + \int \langle R_{W,\gamma'}V, \gamma'\rangle = \int \frac{d}{dt}\langle \frac{D}{ds_2}\frac{d}{ds_1}\gamma, \gamma'\rangle + \int \langle R_{\gamma, W}\gamma', V\rangle$$

Now we can calculate

$$\int \frac{d}{dt}\langle \frac{D}{ds_2}\frac{d}{ds_1}\gamma, \gamma'\rangle = \int \frac{d}{dt}\frac{d}{ds_2}\langle \frac{d}{ds_1}\gamma, \gamma'\rangle - \int \frac{d}{dt}\langle V, \frac{D}{dt}W\rangle$$

The first part is 0 because the variations fix the ends, and the second is the sum in the answer. Finally we calculate

$$\int \langle \frac{D}{dt}\frac{d}{ds_1}\gamma, \frac{D}{ds_2}\gamma' \rangle = \int \frac{d}{dt}\langle V\gamma, \frac{D}{ds_2}\gamma' \rangle - \int \langle V, W'' \rangle$$

Where the first term is 0 again. ☐

Lemma 1.4. $\operatorname{Hess}(u)(X,Y) = \langle \nabla_Y \nabla u, X \rangle = YXu - \nabla_X(Y)u$

Lemma 1.5 (Bochner's Formula)**.** $\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess} u|^2 + \nabla u(\Delta u) + \operatorname{Ric}(\nabla u, \nabla u)$.

Proof. In normal coordinates of a point p the ∂_i commute with each other and have $\nabla_{\partial_i}\partial_j(p) = 0$. Then we can compute:

$$\frac{1}{2}\Delta|\nabla u|^2 = \partial_i \langle \nabla_{\partial_i}\nabla u, \nabla u\rangle = \partial_i \operatorname{Hess}(u)(\partial_i, \nabla u) = \partial_i \langle \nabla_{\nabla u}\nabla u, \partial_i \rangle$$
$$= \langle \nabla_{\partial_i}\nabla_{\nabla u}\nabla u, \partial_i\rangle = \langle \nabla_{\nabla u}\nabla_{\partial_i}\nabla u, \partial_i \rangle + \langle R_{\nabla u, \partial_i}\nabla u, \partial_i\rangle + \langle \nabla_{[\partial_i, \nabla u]}\nabla u, \partial_i\rangle$$

The second term is $\operatorname{Ric}(\nabla u, \nabla u)$.

We can compute

$$|\operatorname{Hess} u|^2 = \operatorname{Hess} u(\partial_i, \partial_j)\langle \nabla_{\partial_i}\nabla u, \partial_j\rangle = \operatorname{Hess} u(\partial_i, \nabla_{\partial_i}\nabla u)$$
$$= \langle \nabla_{[\partial_i, \nabla u]}\nabla u, \partial_i \rangle - \langle \nabla_{\partial_i}\nabla u, \nabla_{\nabla u}\partial_i\rangle$$

And also

$$\nabla u(\Delta u) = \nabla u \langle \nabla_{\partial_i}\nabla u, \partial_i\rangle = \langle \nabla_{\nabla u}\nabla_{\partial_i}\nabla u, \partial_i\rangle + \langle \nabla_{\partial_i}\nabla u, \nabla_{\nabla u}\partial_i\rangle$$

So these add up to the first and third term. ☐

2. Theorems

Let M be a connected Riemannian manifold.

Definition 2.1. *For a continuous function from an interval to a metric space, the **length** is the supremum of the sum of the distances for partitions of the interval.*

Proposition 2.2. *A length minimizing path $x \to y$ in M is a geodesic up to reparameterization.*

Proof. Locally, geodesics are length minimizing, essentially by Gauss's Lemma. Since lengths are additive under subdivision, we see by subdividing that the path has to coincide locally with minimizing geodesics. ☐

The essence of Hopf-Rinow is in the following proposition, which can be seen as a refinement of it:

Proposition 2.3. *If $\exp_p$ is defined in a ball of radius r for a point p, then any point of M of distance r from p is connected to p by a minimal geodesic.*

Proof. First the proposition will be proved for $B = \exp_p(B_r)$, which is compact. For a point q in the interior, choose a sequence of arclength parameterized piecewise geodesics in B from p to q converging to one of minimal length. By Arzela Ascoli, these converge to a path of minimal length, which is a minimal geodesic. By passing to limits, this is true even not in the interior.

Now let q be any point of distance r from p in M, and consider a sequence of paths in M from p to q, whose lengths converge to the distance. WLOG, they can be assumed to be minimal geodesics until the first point they leave M. Their initial segment is then given by exponentiating something in the ball of radius r, so these converge to some point q' by compactness. But then the distance from q to $\exp(q')$ is 0 by additivity of lengths, so $q \in \exp_p(B_r)$. $\square$

Theorem 2.4 (Hopf-Rinow)**.** *TFAE for a (connected) Riemannian manifold M:*

(1) M is geodesically complete
(2) $\exp_p$ is completely defined for some p
(3) Closed and bounded sets in M are compact
(4) M is complete as a metric space.

Moreover these imply (5)*: Any two points are connected by a minimal geodesic.*

Proof. (1) $\implies$ (2) is trivial. (2) $\implies$ (5): Follows from the proposition. We thus see that M is exhausted by the compact sets $\exp_p(B_r)$, showing (2) $\implies$ (3). (3) $\implies$ (4) is trivial. (4) $\implies$ (1): The obstruction to extending a geodesic is it being defined on an interval $[0, b)$ and it not convergint at b. $\square$

From now one, M will usually be complete.

Definition 2.5. *A map between Riemannian manifolds is* ***expanding*** *if the derivative doesn't decrease the length of tangent vectors.*

Lemma 2.6. *If $f : M \to N$ is an expanding map, M is complete, then f is a covering map.*

Proof. It suffices to show the smooth path lifting property. But the obstruction to lifting a path is the lift going off to infinity, which would give it infinite length since M is complete. but an expanding map cannot decrease the length of a curve so this isn't possible. $\square$

Proposition 2.7. *A point p in M with no conjugate points (also called a pole) has the exponential map a covering map.*

Proof. The exponential map is étale so we can give the tangent space at p the pullback metric. This is complete by Hopf Rinow since the lines going through the origin are still complete geodesics. Then the exponential map with this metric is expanding so we are done by the previous lemma. $\square$

Theorem 2.8 (Cartan-Hadamard)**.** *If M has nonpositive sectional curvature, the exponential map at any point p is a covering map.*

Proof. Using the Jacobi equation we will first see that there are no conjugate points to p. The point is that if J is a Jacobi field, then

$$\langle J, J\rangle''(t) = 2\langle J, J'\rangle' = 2\langle J, J''\rangle + 2\langle J', J'\rangle$$

The second term is nonnegative, and the first term is twice $-\langle J, R_{\gamma',J}\gamma'\rangle$ which is nonnegative by hypothesis. Then since for small t there are no conjugate points, this shows that it is true globally, so the exponential map is étale. By the previous proposition we are done. ☐

Theorem 2.9 (Bonnet-Myers)**.** *If M has $\mathrm{Ric} \geq \frac{n-1}{r^2}g$, then the diameter of the universal cover is $\leq \pi r$, and in particular it is compact.*

Proof. One can pass to the universal cover without affecting the hypotheses. One needs to show that a unit speed parameterized geodesic γ of length $l \geq \pi r$ cannot be minimal. We can wiggle the curve and see it reduces length by the second variation formula. Choose an orthonormal parallel frame of the normal bundle of γ: $E_1, \dots E_{n-1}$, and let $F_i = \sin(\frac{\pi}{l}t)E_i$. $\sum_i H(F_i, F_i) = \sum_i \int \langle R_{\gamma',F_i}\gamma' - F_i'', F_i\rangle = \int -\mathrm{Ric}(\gamma', \gamma') + \frac{(n-1)\pi^2}{l^2}\sin^2(\frac{\pi}{l}t) < 0$ by assumption on Ric, l. Thus the trace of H is negative on the subspace spanned by F_i so there is a negative eigenvalue, i.e a variation reducing length. ☐

Lemma 2.10. *An orthogonal transformation of $\mathbb{R}^n$ with determinant $(-1)^{n-1}$ has a fixed point.*

Proof. If n odd, then the characteristic polynomial being odd degree has a zero, so there is a real eigenvalue. Complex conjugate pairs contribute positive terms to the determinant, so there is an eigenvalue of 1. If n even, then again since conjugate pairs contribute positive terms, there must be an odd number of -1 eigenvalues. Since there is an even number of real eigenvalues, there is an eigenvalue of 1. ☐

Theorem 2.11 (Weinstein-Synge)**.** *If M^n is compact, oriented, and has positive sectional curvature and has a conformal self map f inducing $(-1)^n$ on $H^n(M;\mathbb{Z})$, then there is a fixed point.*

Proof. Let p minimize $d(p, f(p)) = l$, which we assume is positive. Choose a minimal geodesic γ from p to $f(p)$. Let A be the orthogonal map given by applying df to $T_p(M)$ and parallel transporting back to p along γ. Since γ is minimal, $d(\gamma(t), f(\gamma(t)) \geq l)$ which together with the triangle inequality with $\gamma(t), f(p), f(\gamma(t))$ shows that $\gamma, f(\gamma)$ glue together to form a geodesic. This implies that A is the identity on γ', so by the lemma, A has a fixed line orthogonal to γ', v. Let E_v be the parallel field corresponding to v. Now applying the second variation formula to E_v, we get $H(E_v, E_v) = \int \langle R_{\gamma', E_v}\gamma', E_v\rangle \leq 0$ because positive sectional curvature. But then for points q slightly in the direction of E_v, $d(q, f(q)) < l$, a contradiction. $\square$

Question 2.11.1. Is this true if f is a diffeomorphism?

Corollary 2.12 (Synge)**.** *Let M^n have possitive sectional curvature. If its dimension is even, it is simply connected. If it is odd, it is orientable.*

Proof. Apply the previous theorem to an appropriate cover, noting that M^n must be compact. $\square$

Theorem 2.13. *Let Δ be the Bochner Laplacian. Then for a compact M, subharmonic functions $\Delta f \geq 0$ are constant.*

Proof. By passing to a cover, we can assume M is orientable. Then for any g, via the Cartan homotopy formula we have $\int_M \Delta g \omega_{vol} = \int_M \operatorname{div} \nabla g \omega_{vol} = \int_M L_{\nabla g}\omega_{vol} = \int_M d\iota_{\nabla g}\omega_{vol} = 0$ by Stoke's theorem. Plugging in $g = f^2$ we find $0 = \int_M \Delta f^2 \omega_{vol} = \int_M (2f\Delta f + \langle \nabla f, \nabla f\rangle)\omega_{vol}$ so since f is subharmonic, we see f is constant. $\square$

Proposition 2.14. *Having $Ric \geq (n-1)\kappa g$ is equivalent to the Bochner inequality $\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \nabla u(\Delta u) + (n-1)\kappa\|\nabla u\|^2$ for all u.*

Proof. Note that the Cauchy Schwarz inequality for $I, \operatorname{Hess}(u)$ gives $|\operatorname{Hess}(u)|^2 \geq \frac{(\Delta u)^2}{n}$. This together with the Bochner formula and the bound gives the inequality. We can always find a function u so that at a given point p, $\operatorname{Hess} u(p)$ is diagonal, and $\nabla u(p)$ is arbitrary. Then the inequality with the Bochner formula gives the Ricci curvature bound. $\square$

3. Comparison

Lemma 3.1. *If f solves $f'' = -kf$, then $\frac{f'}{f}$ solves $\rho' + \rho^2 = -k$.*

Proof. Calculation. $\square$

Let sn_k denote the solution of $f'' = -kf$ with $\operatorname{sn}_k(0) = 0, \operatorname{sn}_k'(0) = 1$

Lemma 3.2 (Riccati Comparison)**.** *If $\rho_1, \rho_2 : (0.b) \to \mathbb{R}$ are smooth functions satisfying $\limsup_{t\to 0}(\rho_2(t) - \rho_1(t)) \geq 0$ and $\rho_1' + \rho_1^2 \leq \rho_2' + \rho_2'^2$, then $\rho_1(t) \leq \rho_2(t)$. The same holds if all inequalities are strict.*

Proof. The second hypothesis implies that $e^{\int \rho_1 + \rho_2}(\rho_2 - \rho_1)$ is increasing. The first hypothesis shows it is nonnegative for arbitrarily small t, so we are done. $\square$

Definition 3.3. *A **distance function** ρ is one with $|\nabla\rho| \equiv 1$.*

For example, the distance from a submanifold is distance function on some open set. Given a distance function, II is the second fundamental form on the level sets of ρ, m is their mean curvature. Then the Bochner formula reads:

$$|II|^2 + \partial_r m + \mathrm{Ric}(\partial_r, \partial_r) = 0$$

where $\partial r = \nabla \rho$.

Applying the Cauchy Schwartz inequality and $\mathrm{Ric} \geq (n-1)\kappa g$, this becomes the Riccati inequality

$$\frac{m^2}{n-1} + \partial_r m + \kappa(n-1) \leq 0$$

If this were an equality, the solution would be $\frac{(n-1)\,\mathrm{sn}'_\kappa}{\mathrm{sn}_\kappa}$.

Proposition 3.4 (Mean Curvature Comparison)**.** *Suppose* $\mathrm{Ric} \geq (n-1)\kappa g$*,* $m(r)$ *is the mean curvature of the geodesic sphere of radius* r *around a point* p*, and* $m_\kappa(r)$ *is that for the model space. Then* $m(r) \leq m_\kappa(r)$*.*

Proof. Apply the Ricatti comparison to the inequality above. □

Remark 3.4.1. I think if you use Gauss's equation you can show that the above is an equality iff the sectional curvatures of the radial vectors are κ.

Remark 3.4.2. A stronger theorem is true, namely the inequality holds in the weak sense for the Laplcian of any increasing or decreasing radial function (doesn't have to be distance).

This gives a second proof of Bonnet-Myers.

Corollary 3.5 (Bonnet-Myers 2)**.** *If* M *has* $\mathrm{Ric} \geq \frac{n-1}{r^2} g$*, then the diameter of the universal cover is* $\leq \pi r$*, and in particular it is compact.*

Proof. Again suppose there is a minimal geodesic γ of length $l > \pi r$. The distance from one end point is a distance function near γ. The mean curvature in the model space goes to $-\infty$ as the distance goes to πr so by the comparison theorem we get a contradiction. □

Definition 3.6. *The* ***cut locus*** *at* p*, denoted* $\tilde{\mathrm{cut}}_p(M)$ *is the region of* $T_p(M)$ *consisting of all points* v *where* $\exp(v)$ *is a minimizing geodesic, but* $\exp(tv)$ *isn't for* $t > 1$*.*

The image of $\tilde{\mathrm{cut}}_p(M)$ in M is also called the cut locus and is denoted $\mathrm{cut}_p(M)$. A point in the cut locus is a **cut point**.

Lemma 3.7. *For a cut point* v*, either* $\exp(v)$ *has its first conjugate point at* $t = 1$ *or there is another geodesic of the same length going to* v*.*

Proof. Let γ be a limit of a subsequence of the curves $\exp(t_i v)$, where $t_i \to 1^+$. If $\gamma = \exp(v)$ then $\square$

Lemma 3.8. *The cut locus has measure* 0*.*

Theorem 3.9 (Rauch Comparison)**.** *Let M, M' be manifolds with Jacobi fields $X, \tilde{X}$ on geodesics $\gamma, \tilde{\gamma}$, and suppose that $X(0) = \tilde{X}(0) = 0$.*

Things to add: Cut locus stuff Bishop-Gromov volume comparison Rauch Comparison

Things to potentially include:

- "things are determined by curvature" and symmetric spaces
- Cheng's theorem on sphere
- Louiville's theorem on conformal transformations
- Construction of spaces of constant curvature
- Maybe some basic things like Killing's equation, Jacobi equation First/Second Variation formulae.