# TOOLS OF UNSTABLE HOMOTOPY THEORY

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The functor $\Omega$ gives an equivalence between pointed connected spaces and group-like $\mathbb{E}_1$-algebras in spaces ($\infty$-groups), so we can approach our study of spaces by studying groups. A nilpotent discrete group is effectively studied via its Lie algebra, and the higher homotopy groups in $\Omega X$ should be thought of as a nilpotent thickening of the group $\pi_1 X$. Thus if $X$ is a nilpotent space, we can expect the data of $X$ to be largely captured by Lie algebra data associated to $\Omega X$.

## 1. Whitehead and Samelson products

A basic incarnation of this is the fact the associated graded of the lower central series of the homotopy groups of $\Omega X$ forms a Lie algebra[1]. The origin of this Lie algebra structure is exactly the same as for discrete groups: it comes from the commutator map.

Given a pointed space $V$, we can take homotopy groups with coefficients in $V$, i.e $\pi_i^V(X) = \pi_i(X^V) = [\Sigma^i V, X]$. In what follows, one may assume $W, V = S^0$ to obtain statements about ordinary homotopy groups. Given $x \in \pi_k^V(\Omega X), y \in \pi_n^W(\Omega X)$, the commutator map $c(x,y) = xyx^{-1}y^{-1}$ gives a map[2],

$$\Sigma^k\Sigma^n(V \wedge W) \xrightarrow{\sim} \Sigma^k V \wedge \Sigma^n W \to \Omega X \wedge \Omega X \xrightarrow{c} \Omega X$$

which we denote $\langle x, y\rangle$, and is called the **Samelson product**. The universal case of the Samelson product is when $X = \Sigma(\Sigma^k V \vee \Sigma^n W)$, where it amounts to a map $\Sigma^k\Sigma^n(V \wedge W) \to \Omega\Sigma(\Sigma^k V \vee \Sigma^n W)$.

This is adjoint to a map $\Sigma\Sigma^k\Sigma^n(V \wedge W) \to \Sigma(\Sigma^k V \vee \Sigma^n W)$, and one sees that the composite

$$\Sigma\Sigma^k\Sigma^n(V \wedge W) \to \Sigma(\Sigma^k V \vee \Sigma^n W) \to \Sigma^{k+1} V \times \Sigma^{n+1} W$$

is canonically null, since the adjoint map is null since $\Sigma^k V$ and $\Sigma^n W$ canonically commute in $\Omega\Sigma(\Sigma^k V) \times \Omega\Sigma(\Sigma^n W)$.

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[1] if $\pi_1$ acts trivially on $\pi_n$, the lower central series is trivial, and so the Lie algebra structure is just on $\pi_*\Omega X$.

[2] It is written as $\Sigma^k\Sigma^n$ rather than $\Sigma^{k+n}$ to indicate the sign.

This nulhomotopy in fact gives rise to a cofibre sequence. To check this, it suffices to observe that we can assume $n, k = 0$, and $V, W$ are finite sets, since the sequence commutes with sifted colimits. Then this amounts to the fact that the product $(\vee_1^l S^1) \times (\vee_1^m S^1)$ is obtained from $(\vee_1^l S^1) \vee (\vee_1^m S^1)$ by attaching 2-cells killing all commutators between the two parts.

The Whitehead product is up to a sign, adjoint to the Samelson product. Given maps $x \in \pi_{k+1}^V(X), y \in \pi_{n+1}^W(X)$, the Whitehead product is the operation denoted $[x, y] \in \pi_{k+n+1}^{V\wedge W}(X)$ corepresented by the map

$$\Sigma^k \Sigma \Sigma^n (V \wedge W) \xrightarrow{\sim} \Sigma \Sigma^k \Sigma^n (V \wedge W) \to \Sigma(\Sigma^k V \vee \Sigma^n W)$$

where the second map is adjoint to the Samelson product.

On the level of homotopy classes, we have that $[x, y]$ is adjoint to $(-1)^{x+1}\langle x', y'\rangle$. We claim that the Whitehead product satisfies Lie algebra identities:

**Lemma 1.1.** *The Whitehead product satisfies for $x \in \pi_i^U(X), y \in \pi_j^V(X), z \in \pi_k^W(X)$ for $i, j, k \geq 2$:*

*(1) It is bilinear.*
*(2) $[x, y] = (-1)^{|xy|}[y, x]$*
*(3) $[[x, y], z](-1)^{|xz|} + [[y, z], x](-1)^{|yx|} + [[z, x], y](-1)^{|zy|} = 0$*

*Proof.* We will check the most interesting relation, (3). On the level of Samelson products, it corresponds to the identity:

- $\langle x, \langle y, z\rangle\rangle + \langle y, \langle z, x\rangle\rangle(-1)^{|x(y+z)|} + \langle z, \langle x, y\rangle\rangle(-1)^{|z(x+y)|} = 0$

This follows from the fact that in any group, $[x, [y, z]][y, [z, x]][z, [x, y]] = 1$ modulo commutators of length 4. Iterated Samelson products of length 4 in 3 variables are null, since they are factor through a map smashed with a diagonal map of the form $S^k \xrightarrow{\Delta} S^n \wedge S^n$, which is null since since $n \geq 1$. Thus the desired relation holds. □

When $i, j, k$ are allowed to be 1, the relations are still satisfied, but only in the associated graded of the lower central series.

Note from the proof of Lemma 1.1 that there does not seem to be a canonical homotopy for the Jacobi identity.

## 2. Hilton–Milnor and James Splitting

It turns out that iterated Whitehead/Samelson products, along with the homotopy groups of spheres, account for all $n$-ary natural operations on homotopy groups with coefficients in some spaces $X_1 \ldots X_n$. The precise result along these lines is the Hilton–Milnor theorem, which describes a decomposition of the free $\mathbb{E}_1$-group on a wedge sum of connected spaces.

**Theorem 2.1** (Hilton–Milnor)**.** *Let $n \geq 1$, and $X_i$ connected. There is a natural equivalence $\prod_{w_I} \Omega\Sigma X^{\wedge w_I} \xrightarrow{\sim} \Omega\Sigma \vee_1^n X_i$, where $w_I$ is a basis for the free Lie algebra[3] on $X_i, 1 \leq i \leq n$.*

[3]Note that the free Lie algebra appearing Theorem 2.1 is alternating rather than antisymmetric, in that $[x, x] = 0$. This does not mean however that $[x, x] = 0$ for Whitehead products, since it simply appears as a higher homotopy group of a sphere.

The map in the theorem is given by iterated Samelson products determined by the basis element of the free Lie algebra. The projections in the other direction, which are dependent on the ordering of the basis, are called the **Hilton–Hopf invariants**.

This result can be thought of as reflecting the nilpotence of $\Omega\Sigma X$ as follows: A discrete group $X$ has a lower central series with associated graded a Lie algebra. If $X$ is nilpotent, we can choose lifts of the associated graded and get an equivalence (of sets) of $X$ with the product of the terms in the associated graded. The associated graded of a free group is the free Lie algebra, but it is not true that a free group on more than one generator is nilpotent. But nevertheless, as long the space over which one takes the free group is connected, the product map from the associated graded is an equivalence, because the higher filtration terms become highly connected.

Two approachs to proving Theorem 2.1 are to either use Mather's second cube theorem (see for example [**sanathpeter**]), or to use facts about free groups, and geometrically realize these to say something about free $\mathbb{E}_1$-algebras. Ultimately these are proofs doing the same thing but the former is more axiomatic and so works in a bit more generality.

$\Omega\Sigma$ commutes with geometric realizations, so whatever we say about it reduces to the case of discrete sets, in which case it coincides with the free group functor. It is a result of Milnor [**milnorfk**] that the free simplicial group on $X$, denoted $FX$, is a model for $\Omega\Sigma X$. He used this model to prove the results below, but all one really needs is that the functor $\Omega\Sigma$ preserves geometric realizations (which is model independent, and follows from Milnor's result).

**Lemma 2.2.** *There is a split exact sequence of groups* $1 \to \Omega\Sigma(B \vee B \wedge \Omega\Sigma A) \to \Omega\Sigma(A \vee B) \to \Omega\Sigma A \to 1$.

*Proof.* The kernel of the projection map is the free group generated by $b \in B$ and $[b, w]$, where $w$ is a nontrivial word in $FA$. The sequence obviously splits. □

**Lemma 2.3.** *There is an equivalence of groups* $\Omega\Sigma(B \wedge \Omega\Sigma A) \simeq \Omega\Sigma(B \wedge A \vee B \wedge A \wedge \Omega\Sigma A)$.

*Proof.* The free group generated by $[b, w]$ for $w$ a nontrivial word in $A$ is also freely generated by $[b, a]$ and $[[b, a], w]$ for $w$ running over nontrivial words in $A$. □

The lemma above, as a statement at the level of groups, can be delooped to get the James splitting $\Sigma(B \wedge \Omega\Sigma A) \simeq \Sigma(B \wedge A \vee B \wedge A \wedge \Omega\Sigma A)$.

**Proposition 2.4.** *Let $A$ be connected. Then there is a split exact sequence of $\mathbb{E}_1$-algebras* $1 \to \Omega\Sigma(\vee_0^\infty B \wedge A^{\wedge i}) \to \Omega\Sigma(A \vee B) \to \Omega\Sigma A \to 1$

*Proof.* Apply Lemma 2.2 and Lemma 2.3 repeatedly, using the fact that $A^{\wedge n} \wedge \Omega\Sigma A$ becomes highly connected. □

Similarly, the James splitting can be iterated to obtain:

**Theorem 2.5.** *(James) If $X$ is connected, there is an equivalence* $\vee_0^\infty \Sigma X^{\wedge i} \simeq \Sigma\Omega\Sigma X$.

Theorem 2.1 is a consequence of Proposition 2.4: one simply repeatedly applies the proposition, observing that the remaining factors become more and more highly connected. Tracing through the construction, we find that the equivalence is given by the product of iterated Samelson products where the order of the product is chosen by the order in which we have realized the equivalence.

The James splitting is very useful. For example, adjoint to the projection of $\Sigma\Omega\Sigma X$ onto the factors are the James-Hopf maps $H_n : \Omega\Sigma X \to \Omega\Sigma X^{\wedge n}$. It is these maps that are traditionally used in the construction of EHP sequences.

## 3. The EHP Sequence

The EHP sequence is a fibre sequence due to James at the prime 2 and Toda at odd primes, which relates the homotopy groups of different spheres. One can also view it as giving an understanding of the spectral sequence of the tower $\Omega^n S^n$ approximating $QS^0$.

At the prime 2, we will identify the fibre of the second James-Hopf map $H_2 : \Omega\Sigma S^n \to \Omega\Sigma S^{2n}$ with $S^n$.

**Theorem 3.1.** *There is a 2-local fibre sequence $S^n \xrightarrow{E} \Omega\Sigma S^n \xrightarrow{H} \Omega\Sigma S^{2n}$ where $H$ is the James-Hopf map $H_2$. For $n$ odd, one need not 2-localize.*

*Proof.* First, we claim that $H$ is an isomorphism on $H_{2n}$. To see this, $\Sigma H$ factors as $\Sigma\Omega\Sigma S^n \to \Sigma S^{2n} \to \Sigma\Omega\Sigma S^{2n}$, where the first map is the projection from the James splitting, and the second is the counit. Both of these maps are isomorphisms on $H_{2n+1}$.

Next, we claim that atleast 2-locally, $H_2$ is an isomorphism in cohomology in all degrees where it is nonzero. For $n = 2k$ even, we know that the generator $y_{4k}$ of $H^*(\Omega\Sigma S^{4k})$ is sent to $x_{2k}^{(2)}$, the second divided power of the class in degree $4n$ of $H^*(\Omega\Sigma S^{2k})$. It follows that $y_{4k}^{(l)}$ is sent to $x_{2k}^{(2l)}\frac{(2l)!}{2^l l!}$, which is a unit multiple of the generaor $x_{2k}^{(2l)}$ in that degree. It follows from the Serre spectral sequence that inclusion of the fibre on homology agrees with $E$, but then the map must just be $E$.

For $n = 2k + 1$, the divided power generator gets sent to a divided power generator, and so again one sees via the Serre spectral sequence that the fibre is $S^n$, but this time integrally. $\square$

The $P$ in the EHP sequence is for Whitehead *p*roduct, and refers to the associated map $P : \Omega^2\Sigma S^{2n} \to S^n$ on the bottom cell is the Whitehead square of the identity, which is easy to see from the definition of the Whitehead product.

The fibre sequence $S^n \xrightarrow{E} \Omega\Sigma S^n \xrightarrow{H} \Omega\Sigma S^{2n}$ splits at odd primes for $n$ odd: the map $\Omega[i_{n+1}, i_{n+1}]$ gives a splitting up to isomorphism. One can multiply the section and the inclusion of the fibre to obtain:

**Corollary 3.2** (Serre)**.** *After inverting 2, there is an equivalence $S^{2k-1} \times \Omega S^{4k-1} \simeq \Omega S^{2k}$.*

The EHP sequence for $n$ odd doesn't split at the prime 2 unless there is an element of Hopf invariant 1.

Let $J_i X$ be the $i$th term in the James filtration on the free monoid on $X$. The odd prime extension of the EHP sequence is below.

**Theorem 3.3.** *There are $p$-local fibre sequences:*

$$J_{p-1}S^{2n} \to \Omega\Sigma S^{2n} \xrightarrow{H_p} \Omega\Sigma S^{2np}$$

$$S^{2n-1} \to \Omega J_{p-1}S^{2n} \to \Omega\Sigma S^{2np-2}$$

## 4. James' torsion bound

A fundamental phenomena that is different between stable and unstable homotopy groups of spheres is that the unstable groups for a fixed group have a uniform torsion bound at each prime. The first result in this direction was due to James [**james1957suspension**], in the last of a series of three papers on the EHP sequence.

**Theorem 4.1.** *(James) There is a 2-local factorization*

$$\begin{tikzcd} \Omega^3 S^{2n+1} \arrow[rr, "4"] \arrow[dr, dashed] & & \Omega^3 S^{2n+1} \\ & \Omega S^{2n-1} \arrow[ur, "E^2"] & \end{tikzcd}$$

By induction, we obtain

**Corollary 4.2.** *There is a factorization*

$$\begin{tikzcd} \Omega^{2n+1} S^{2n+1} \arrow[rr, "4^n"] \arrow[dr, dashed] & & \Omega^{2n+1} S^{2n+1} \\ & \Omega S^1 = \mathbb{Z} \arrow[ur, "E^{2n}"] & \end{tikzcd}$$

In particular $4^n$ kills the 2-primary torsion in the homotopy groups of $S^{2n+1}$. Note also that by the EHP sequence, this provides torsion bounds on the homotopy groups of $S^{2n}$ as well. At odd primes, the work of Cohen–Moore–Neisendorfer shows that $p^n$ kills the $p$-primary torsion for $S^{2n+1}$.

The conjectured optimal torsion bound at the prime 2 is the order of the restriction of the universal line bundle $\gamma$ in the diagram below, which is known to be a lower bound.

$$\begin{tikzcd} \mathbb{RP}^{2n} \arrow[rr, "\gamma"] \arrow[dr] & & \mathrm{BO} \\ & \Omega^{2n+1} S^{2n+1}\langle 2n+1\rangle \arrow[ur] & \end{tikzcd}$$

In order to prove Theorem 4.1, we will produce the following diagram

$$\begin{tikzcd} & & \Omega^3 S^{2n+1} \arrow[dl, dashed] \arrow[d, "2"] \\ & \Omega^2 S^{2n} \arrow[r, "E"] \arrow[dl, dashed] \arrow[d, "1-\Omega(-1)"] & \Omega^3 S^{2n+1} \arrow[d, "2"] \\ \Omega S^{2n-1} \arrow[r, "E"] & \Omega^2 S^{2n} \arrow[r, "E"] & \Omega^3 S^{2n+1} \end{tikzcd}$$

The square in the diagram is commutative because there are natural homotopies between $\Sigma f \circ (-1)$ and $(-1) \circ \Sigma f$ for $f$ any map between suspensions, because one can do the map $-1$ in the suspension coordinate.

So it suffices to show that the dashed arrows in the diagram above exist. By the EHP sequence it then suffices to show the proposition below

**Proposition 4.3.** *The composites*

$$\Omega^2 S^n \xrightarrow{1-\Omega(-1)} \Omega^2 S^n \xrightarrow{\Omega H} \Omega^2 S^{2n-1}$$

*and*

$$\Omega^2 S^{2n+1} \xrightarrow{2} \Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1}$$

*are null.*

*Proof.* The first statement follows from the commutative square

$$\begin{tikzcd}
\Omega\Sigma S^{n-1} \arrow[r, "H"] \arrow[d, "\Omega\Sigma -1"] & \Omega\Sigma S^{2n-2} \arrow[d, "\Omega\Sigma(-1\wedge -1)=1"] \\
\Omega\Sigma S^{n-1} \arrow[r, "H"] & \Omega\Sigma S^{2n-2}
\end{tikzcd}$$

with the additional observation that $\Omega H$ is a group homomorphism.

For the second statement, note that we have the following commutative diagram

$$\begin{tikzcd}
\Omega\Sigma S^{2n} \arrow[r, "\Omega\Sigma\mathrm{pinch}"] \arrow[dr] \arrow[rr, bend left, "0"] & \Omega\Sigma(S^{2n} \vee S^{2n}) \arrow[r, "\Omega\Sigma(1\vee -1)"] \arrow[d, "h"] & \Omega\Sigma S^{2n} \\
& \prod_w \Omega\Sigma S^{2n|w|} \arrow[ur, "f"] &
\end{tikzcd}$$

Where $h$ is the Hilton–Hopf invariants, and $f$ is the product of iterated Samelson products. In odd degrees, any iterated Samelson product of the identity of length $\geq 3$ vanishes, so the composite factors as

$$\Omega\Sigma S^{2n} \xrightarrow{1,1,h_2} \Omega\Sigma S^{2n} \times \Omega\Sigma S^{2n} \times \Omega\Sigma S^{4n} \xrightarrow{\Omega 1 \times \Omega -1 \times \Omega[1,-1]} \Omega\Sigma S^{2n}$$

It follows that $0 = 1 + \Omega(-1) + \Omega[1,-1] \circ h$. Looping so that everything is a group homomorphism, composing with $\Omega H$, and using that $\Omega H(\Omega(-1)) = \Omega H$, we get $2\Omega H = -\Omega(H \circ \Omega[1,-1] \circ h)$. We claim that $H \circ \Omega[1,-1]$ is null. $[1,-1]$ is a suspension since it is 2-torsion, and its Hopf invariant lands in a torsion-free group. Write $[1,-1] = \Sigma v$. Then we have a commutative diagram

$$\begin{tikzcd}
\Omega\Sigma S^{4n} \arrow[r, "H"] \arrow[d, "\Omega\Sigma v"] & \Omega\Sigma S^{8n} \arrow[d, "\Omega\Sigma(v\wedge v)"] \\
\Omega\Sigma S^{2n} \arrow[r, "H"] & \Omega\Sigma S^{4n}
\end{tikzcd}$$

But $\Sigma v \wedge v$ factors through $\Sigma v \wedge 1 = \Sigma^{2n} v$, and $\Sigma^2 v$ is null by the EHP sequence.

$\square$