

# ORDINARY DIFFERENTIAL EQUATIONS

ISHAN LEVY

## 1. MAIN THEOREMS

We can use the contraction mapping principle to construct solutions of ODEs.

**Lemma 1.1.** *Suppose that  $M$  is a metric space, and  $T$  is a contraction endomorphism. Then there is at most one fixed point of  $T$ , and if  $M$  is nonempty and complete, the fixed point exists.*

Solutions to ODEs locally exist uniquely with whatever amount of continuity we want as long as the defining equations satisfy uniform Lipschitz conditions locally.

**Theorem 1.2.** *Let  $U$  be a compact space,  $V \subset B$  a bounded open subset of a Banach space,  $F_U$  a continuous function from  $U$  to a space of uniformly bounded continuous functions on  $V \times (\mathbb{R})$  to  $\mathbb{R}$ , satisfying a uniform Lipschitz condition with constant  $K$ , and  $I : U \rightarrow V$  the initial condition function. Then for small  $\epsilon$ , there is a unique solution that is continuous,  $x_u(t) : U \times (-\epsilon, \epsilon) \rightarrow V$  solving the differential equation  $\frac{dx_u(t)}{dt} = F_u(x_u, t)$  with initial conditions  $x_u(0) = I(u)$ .*

*Proof.* We can write the equation as  $x_u(t) = I(u) + \int_{t_0}^t F_u(x_u(t), t) dt$ . We can think of the right hand side as an operator  $\phi x$  on possible solutions  $x$ , so that the problem amounts to carefully defining the space that  $\phi$  acts on, and showing that it is a contraction. We can define  $M$  to be the space of continuous functions  $U \times (-\epsilon, \epsilon) \rightarrow V$  satisfying the initial conditions, and let the metric be induced from the sup norm. Then since  $U$  is compact and  $F_U$  is continuous, the family  $F_u$  are uniformly bounded, so that for small  $\epsilon$ , the operator  $\phi$  will indeed send elements of  $M$  to  $M$ . To see that  $\phi$  is a contraction map, observe that if  $x, x' \in M$ , then  $d(\phi x, \phi x') = \sup_t \|\int_{t_0}^t F_u(x_u(t), t) - F_u(x'_u(t), t) dt\| \leq \sup_t \int_{t_0}^t K \|x_u(t) - x'_u(t)\| \leq K\epsilon d(x, x')$ , so if  $K\epsilon < 1$ , then this is a contraction map.  $\square$

In particular this shows that solutions depend differentiably upon initial conditions if the derivatives of  $F(x, t)$  are uniformly Lipschitz: we can take  $U$  to be a small neighborhood of 0 times a time direction, and consider solving the differential equation for the difference quotients of  $F(x, t)$  in the neighborhood, and the solution will be difference quotients of the solution for  $F$  by uniqueness. Then by continuity, the difference quotients will converge to the derivative of  $F$ , and the left hand side

will converge as well, implying that the solution of the original equation is continuously differentiable. If  $F$  is smooth, then so will the solution  $x$ . Moreover if  $F$  is analytic, then  $x$  will be a uniform limit of analytic functions, so will be as well.

Global existence of solutions can only fail if the solution blows up, and tries to leave the domain.

**Corollary 1.3.** *Suppose that  $F(x, t)$  locally is Lipschitz continuous. Then there is a maximal interval  $(-a, b)$  on which a solution to  $\frac{dx}{dt} = F(x, t)$  with an initial condition can be defined, which is unique. If  $-a$  or  $b$  is finite, then the solution leaves every compact set as it approaches  $-a$  or  $-b$ .*

*Proof.* Try to define  $-a, b$  as the the largest such that the solution exists. By local uniqueness, the solution is unique on that interval. Now if the solution  $x$  is in some compact set when  $t$  is near  $b$ , then it must converge as  $t \rightarrow b$ . But then we can use the local existence theorem, which will agree with the already existing solution to see that  $x$  can be extended a bit beyond  $b$ .  $\square$

Thus given a vector field  $V$  on a manifold  $M$ , the field induces a local flow on  $M$  that may not be globally defined, but is if  $M$  is compact. If the vector field and  $M$  are smooth, the flow will be as well. Conversely if  $g_t$  is a one parameter local group of diffeomorphisms, The derivative of  $g_t$  at  $t = 0$  is a smooth vector field inducing  $g_t$  as its flow.

One can often reduce higher order equations to first order equations. For example, if we want to solve  $x^{(n)} = F(x^{(i)}; i < n, t)$ , we can turn it into a first order equation in more variables by solving for  $(x^{(i)})$  simultaneously.

The inverse function theorem can be proven using the same contraction mapping principle.

**Lemma 1.4.** *If  $f$  is  $C^1$  in a neighborhood of a point  $p$  and has an invertible derivative at  $p$ , then it is injective near  $p$ .*

*Proof.* After a change of coordinates,  $f(0) = 0, f'(0) = I$ , so that  $f(x) = f(a) + f'(a)(x-a) + o(x-a)$ . We can choose a neighborhood such that  $\|a\| < \frac{1}{4}, \|f'(a)\| > \frac{3}{4}$  and so that the error term  $\|o(x-a)\| < \frac{1}{4}\|x-a\|$ . Then we have that  $\|f(x) - f(y)\| = \|f'(y)(x-y) + o(x-y)\| \geq \frac{3}{4}\|x-y\| - \|o(x-y)\| \geq \frac{1}{2}\|x-y\|$ .  $\square$

**Theorem 1.5** (Inverse function theorem). *If  $f$  has an invertible derivative at a point, and is injective near that point it locally has an inverse, which is differentiable at the image point.*

*Proof.*  $f(x) = y = x + \epsilon(x)$  where  $\epsilon(x) = o(x)$  after a linear change of variables. To solve  $y = x + \epsilon(x)$ , we observe that  $x = y - \epsilon(x)$ , so if  $g(y) = x$ , it is the fixed point of the mapping  $\phi(g) = y - \epsilon(g)$ . But in a small neighborhood of 0,  $\|\epsilon(x)\| < \lambda < 1$ , so

in this neighborhood,  $\phi$  is a contraction mapping on the space of possible continuous inverses with the sup norm. Note that  $\phi^n(g)$  is differentiable at 0 with derivative the identity, and uniformly converges to the inverse, so the inverse is differentiable at 0.  $\square$

There is a geometric interpretation of solving an ODE (and a similar one for PDE). Let  $T$  be the times,  $S$  be the phase space, and  $J^1(S)$  be the space of 1-jets, with the projection map to  $S$ .  $J^1(S)$  comes with a canonical distribution, given by the 1-forms  $dx_i = p_i dt$ , where  $p_i$  are the cotangent bundle coordinates. We can consider the bundles  $T \times S$ ,  $T \times J^1(S)$  over  $T$ . A smooth section of the first one has a canonical lift to a section of the second, by requiring that it is tangent to the distribution. Then if  $F(p, x, t)$  is a differential equation, a solution is exactly a section of  $T \times S$  whose lift lies in the hypersurface defined by  $F(p, x, t)$ .