PROOF OF COBORDISM HYPOTHESIS

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1. Introduction

The cobordism hypothesis gives a classification of extended topological field theories on manifolds with certain structures.

Fix a space X with a rank n vector bundle ξ. Let $\mathrm{Bord}_n^{X,\xi}$ denote the symmetric monoidal (∞, n) category, where i-morphisms for $i \le n$ are manifolds M^i with corners with an (X, ξ) structure. This means we think of the manifold as a bordism between manifolds between $i-1$-dimensional manifolds, and the (X, ξ) structure is a map $M^i \to X$ and an identification of $\mathbb{R}^{n-i}$ plus the tangent bundle of M with ξ. For $i > n$, the morphisms are diffeomorphisms, isotopies between diffeomorphisms, etc. We can think of X as the classifying space BG of a (topological) groupoid, so that lifting a morphism from $BO(n)$ to $BG = X$ can be thought of as giving a G structure on a rank n bundle.

Let $\tilde{X}$ be the frame bundle on X coming from the map to $BO(n)$. The cobordism hypothesis is:

Theorem 1.1 (Baez-Dolan, Hopkins-Lurie)**.** *$Fun^{\otimes}(\mathrm{Bord}_n^{X,\xi}, C) = \mathrm{Map}_{O(n)}(\tilde{X}, C^{\cong})$, where C is a symmetric monoidal (∞, n)-category with duals, and $C^{\cong}$ is the groupoid of invertible maps.*

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The isomorphism can be thought of as being given by evaluation at the space of connected 0-dimensional manifolds equipped with a trivialization of the tangent space stabilized to dimension n (which is parameterized by $\tilde{X}$). The $O(n)$ action on $C^{\cong}$ comes from the case when X is a point.

A special case of this result is the result due to Galatius-Madsen-Tillmann-Weiss which describes invertible field theories. Just as before we have a map $BG \to BO(n)$, and we consider the Thom spectrum MTG of the negative of the canonical bundle.

Theorem 1.2 (GMTW). *$Fun^{\otimes}(\mathrm{Bord}_n^{BG}, X) = \mathrm{Map}(\Sigma^n MTG, X)$, where X is a spectrum.*

Note that we can take $n \to \infty$ to get $\mathrm{Bord}_\infty^{X,\xi}$, an (∞, ∞) category of bordisms. This is a category that is stable in two ways: We have already stabilized with respect to dimension, meaning that we only care about the stable tangent bundles of manifolds. We have also stabilized with respect to embedding dimension, we can think of the manifolds involved as embedded in $\mathbb{R}^\infty$. This is reflected on the categorical side by having the category be symmetric monoidal.

Note that given a compatible family of maps to $BO(n)$, we can let n go to ∞, to describe the geometric realization of the ∞, ∞-bordism category as a Thom spectrum where the group has been stabilized eg MTO, MTU. This looks similar to the result of Pontryagin-Thom, but is subtly different: the definition of bordism category for field theories involves manifolds with structures on their stable tangent bundles, but the definition of standard cobordism spectra like MO and MU involves manifolds with structures on their stable normal bundles. In many cases these two coincide: $MO = MTO, MSO = MTSO, MU = MTU, MSpin = MTSpin$, but not always: $MPin_+ = MTPin_-$.

There is also a way to destabilize the cobordism hypthesis with respect to embedding dimension. This yields the tangle hypothesis, which is essentially the same as the cobordism hypothesis, except all manifolds involved are embedded, and the categories aren't symmetric monoidal, but rather k-fold monoidal (ie have k deloopings).

2. Outline

Here is an outline of the steps of the proof:

- First we reformulate the cobordism hypothesis in an inductive way. Let X, ξ be a space with a principle $O(n)$-bundle ξ as in the construction of the bordism category. Let X_0 be the sphere bundle of associated to the bundle, with a map f to X. $f^*\xi$ splits canonically as $\mathbb{R} \oplus \xi_0$, where ξ_0 is the bundle of vectors orthogonal to the vector on the sphere. Adding the trivial copy of $\mathbb{R}$ gives an inclusion $\mathrm{Bord}_{n-1}^{X_0,\xi_0} \to \mathrm{Bord}_n^{X,\xi}$. We will reformulate the universal property of $\mathrm{Bord}_n^{X,\xi}$ in terms of how to extend functors along this inclusion. The universal property will say that we need to specify what the field theory does on a disk of dimension n with a nondegeneracy condition on the morphism corresponding to the disk.

- The next step is only for simplicity (rather than necessity): we will reduce to proving the cobordism hypothesis for the unoriented bordism category Bord_n. It isn't too surprising that we can do this, since the unoriented bordism category is universal in the sense that its choice of X, ξ is universal, however the actual implementation is a bit more subtle.
- Since the inclusions $\mathrm{Bord}_n \to \mathrm{Bord}_{n+1}$ are highly connected (their k morphisms agree for small k) and lots of duals exist, it is possible to capture the data of the inclusions in terms of $(\infty, 1)$-categorical data, which is nice because $(\infty, 1)$-categories are easier to work with.
- The most important step in the proof will use a version of Morse theory due to Igusa to prove the cobordism hypothesis for another category Bord_n^{ff}. In particular, we can understand bordisms in terms of handle attachments and cancellations, which will give generators and relations for Bord_n^{ff} in terms of Bord_{n-1}. The difference between Bord_n^{ff} and Bord_n is nonexistent, and is can be thought of as the irrelevence of the choice of Morse function.
- The last step of the proof is one that really shouldn't have to be there, namely proving that Bord_n^{ff} is equivalent to Bord_n. Doing this involves understanding an obstruction theory for (∞, n)-categories, knowing from some Morse theory that they are equivalent in a range of dimensions, and doing cohomological computations to show that the two agree in general. The reason this step shouldn't have to be there is that it is equivalent to a conjecture about to Igusa's Morse theory. If an independent proof of that conjecture existed, this step could be avoided. Nevertheless, this step is interesting, as one has to try to do stable homotopy theory with (∞, n)-categories.

3. Inductive formulation

Recall that there is an map $\mathrm{Bord}_{n-1}^{X_0,\xi_0} \to \mathrm{Bord}_n^{X,\xi}$, where X_0 is the unit sphere bundle of ξ. Let C be as in the cobordism hypothesis, and consider a field theory on $\mathrm{Bord}_{n-1}^{X_0,\xi_0}$ we would like to extend along $\mathrm{Bord}_n^{X,\xi}$. For each $x \in X$, the unit disk of ξ at x, D_x^n is a bordism from ϕ to S_x^{n-1}, the fibre of $X_0 \to X$ at x. The sphere S_x^{n-1} can be broken into two hemispheres, making it the composite $D_+ \circ D_-$. We say that $D_x^n : \phi \to D_+ \circ D_-$ exhibits D_+ as right adjoint to D_-. This is a condition that doesn't depend on the way we broke up S_x^{n-1} into hemispheres.

Theorem 3.1. *Let $Z_0 : Bord_{n-1}^{X_0,\xi_0} \to C$ be symmetric monoidal, and C an (∞, n)-category with duals. Then the following data are equivalent:*

- *An extension Z of Z_0 to $Bord_n^{X,\xi}$.*

- *Families of nondegenerate morphisms* $1 \to Z_0(S^{n-1}_x)$ *parameterized by* X.

Proposition 3.2. *Theorem 3.1 in dimensions* $\leq n$ *and Theorem 1.1 in dimensions* $< n$ *imply Theorem 1.1 in dimension* n.

Proof. It suffices to assume that X is a point, because Theorem 3.1 implies that the left hand side of the cobordism hypothesis sends colimits of spaces over $BO(n)$ to limits. But spaces over $BO(n)$ is generated under colimits by a point.

When X is a point, X_0, ξ is S^{n-1} with its tangent bundle. Thus we need to show that giving the data of Z_0 as well as a nondegenerate morphism $\eta : 1 \to Z_0(S^{n-1})$ is equivalent to giving an object of C.

By Theorem 1.1 in dimension $n-1$, giving the data of Z_0 is equivalent to giving an $O(n-1)$-equivariant map from $O(n)$ to $C^{\cong}$. The action comes from the fibration $O(n-1) \to O(n) \to S^{n-1}$. We can break up $O(n)$ into the fibres at the north and south poles, and everything in between. $O(n)$ also is acted on by $O(n-1)$ from the right. While this action doesn't preserve the fibres, it preserves the north and south pole, and relates fibres along each S^{n-2} slice of the rest of the sphere. From this description, we get that an $O(n-1)$-equivariant map from $O(n)$ is the same as two $O(n-1)$-equivariant maps from $O(n-1)$ (corresponding to the north pole with the inverse right action and south pole with usual action), and an $O(n-2)$-equivariant homotopy from the first to the second conjugated by the map in $O(n-1)$ reflecting along a direction.

By Theorem 1.1, this is equivalent to two functors $Z_- \to C, Z_+ \to C^{op}$ out of Bord^*_{n-1}, where C^{op} is the opposite on the level of $n-1$-morphisms, along with an isomorphism of their restrictions to $\mathrm{Bord}^{S^{n-2}}_{n-2}$ (the underlying $(\infty, n-2)$-categories of C and C^{op} agree).

By Theorem 3.1 in dimension $n-1$, this is equivalent to giving one functor $Z' : \mathrm{Bord}^{S^{n-2}}_{n-2} \to C$ and a nondegenerate morphism $f : 1 \to Z'(S^{n-2})$ and $g : 1 \to Z'(S^{n-2})$ in C and C^{op} respectively. Thus we can think of g as a morphism $Z'(S^{n-2}) \to 1$. But the composite $g \circ f$ is exactly the $Z_0(S^{n-1})$, and so the data of η and g are redundant since they exhibit f as a right adjoint of g, and right adjoints are essentially unique. Thus we are left with just the data of f and Z'.

Via Theorem 3.1 again, this is equivalent to the data of functor $Z_- : \mathrm{Bord}^*_{n-1} \to C$, which by Theorem 1.1 in dimension $n-1$ is equivalent to just an object of C. □

4. Reduction to Unoriented Case

$\mathrm{Bord}_n = \mathrm{Bord}^{BO(n)}_n$ will be the unoriented bordism category.

The right hand side of the cobordism hypothesis is $\mathrm{Map}_{O(n)}(\tilde{X}, C^{\cong})$, which is the same as $\mathrm{Map}_{O(n)}(EO(n), (C^{\cong})^{\tilde{X}}$. So naively, we could try to reduce to the unoriented case by replacing C by a category $C^{X,\xi}$ such that $(C^{X,\xi})^{\cong} = (C^{\cong})^{\tilde{X}}$, and such that $Fun^{\otimes}(\mathrm{Bord}_n^{X,\xi}, C) = Fun^{\otimes}(\mathrm{Bord}_n, C^{X,\xi})$, and the general case would follow immediately.

This strategy doesn't work exactly, but a relative version of it does: There is a fibration of symmetric monoidal (∞, n)-categories $Fam_n(C) \to Fam_n = Fam_n(*)$ and a map $(X, \xi) : Fun^{\otimes}(\mathrm{Bord}_n, Fam_n)$ such that lifts of (X, ξ) to $Fam_n(C)$ are the same as elements of $Fun^{\otimes}(\mathrm{Bord}_n^{(X,\xi)}, C)$. This is summarized in the below diagram.

$$\begin{tikzcd} Bord_n \arrow[r, dashed] \arrow[dr, "{(X,\xi)}"'] & Fam_n(C) \arrow[d] \\ & Fam_n(*) \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} Bord_n^{X,\xi} \arrow[r, dashed] \arrow[dr] & C \arrow[d] \\ & * \end{tikzcd}$$

The objects category $Fam_n(C)$ are C-values local systems of functors from a space. n-morphisms are n-fold correspondences between such local systems (the correspondences make sure that $Fam_n(C)$ has duals), and the symmetric monoidal structure is the product.

In particular $Fam_n(*)$ is the category whose objects are topological spaces and n-morphisms are n-fold correspondences between topological spaces. We can produce the functor (X, ξ) by sending a k-morphism M^k to the classifying space for (X, ξ)-structures on M^k.

From this correspondence, it should not be surprising that one can use the cobordism hypothesis for Bord_n to deduce the inductive form of it for $\mathrm{Bord}_n^{X,\xi}$.

5. Unfolding of Higher Categories

The next step is to reinterpret the data of the categories Bord_n in terms of $(\infty, 1)$-categories. All categories and functors here are symmstric monoidal unless otherwise specified. We have currently reduced the cobordism hypothesis to a statement about the sequence $\mathrm{Bord}_1 \to \mathrm{Bord}_2 \to \mathrm{Bord}_3 \to \dots$. To see that the data might be describable in terms of $(\infty, 1)$-categories, we can observe that the inclusion of Bord_n into Bord_{n+1} is that it is n-connected, which is a notion analogous to the corresponding notion for spaces. In this case it amounts to the observation that the k-morphisms for $k \le n$ are the same in each category. Thus there aren't as many levels of non-invertible morphisms where they differ, so this data should be able to be captured using a lower level of category.

The sequence Bord_n is an example of what is called a **skeletal sequence** of categories, which is a way of formalizing the fact that certain adjoints exist and that each inclusion is sufficiently connected. We can replace Bord_n with a sequence of symmetric monoidal $(\infty, 1)$ categories $\mathrm{Cob}_{\partial}^{un}(n)$ that form a **categorical chain complex**: In otherwords there are symmetric monoidal coCartesian fibrations $d : \mathrm{Cob}_{\partial}^{un}(n) \to \mathrm{Cob}_{\partial}^{un}(n-1)$ and isomorphisms of d^2 with the constant map to the unit in a coherent way. $\mathrm{Cob}^{un}_{\partial}(n)$ is the category of manifolds with boundary of dimension $n-1$ with boundary with maps given by the space of cobordisms (not necessarily trivial on the boundary). The map between them is given by taking the boundary.

Let's see how this replacement works for $n = 1$. $\mathrm{Bord}_1 = \mathrm{Cob}^{un}_{\partial}(1)$. The map $\mathrm{Bord}_1 \to \mathrm{Bord}_2$ is essentially surjective, and for any X, Y in Bord_2, the category $\mathrm{Map}_{\mathrm{Bord}_2}(X, Y)$ is equivalent to $\mathrm{Map}_{\mathrm{Bord}_2}(1, Y \otimes X^{\vee})$. This means that to recover the map to Bord_2 we just need to remember $Y \mapsto \mathrm{Map}_{\mathrm{Bord}_2}(1, Y)$ as a lax monoidal functor $\mathrm{Bord}_1 \to \mathrm{Cat}_{(\infty,1)}$. But by a version of the Grothendieck construction, this is equivalent to a coCartesion fibration to Bord_1, and one can show this gives exactly something equivalent to $\mathrm{Cob}^{un}_{\partial}(2)$. This example can be spiced up to give a general correspondence between skeletal sequences and categorical chain complexes (for which the definitions have only been sketched here).

A similar construction for (∞, n)-categories gives the following:

Proposition 5.1. *The following data are equivalent for a symmetric monoidal $(\infty, n-1)$ category with duals:*

- *$n-2$-connected symmetric monoidal functors $B_{n-1} \to B_n$, B_n a symmetric monoidal (∞, n) category*
- *Lax symmetric monoidal functors $\Omega^{n-2}B_{n-1} \to Cat_{\infty,1}$*
- *Symmetric monoidal coCartesian fibrations $C \to \Omega^{n-2}B_{n-1}$.*

6. The Index Filtration

$\Omega^{n-2}\,\mathrm{Bord}_{n-1}$ is the category $\mathrm{Cob}^{un}_{t}(n-1)$ of closed unoriented manifolds of dimension $n-2$ and bordisms between them. The map $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n$ corresponds via the above proposition to the map $\mathrm{Cob}^{un}_{t}(n-1)$ sending an $n-2$ manifold M to the category $B(M)$, where the objects are $n-1$-manifolds equipped with a diffeomorphism of their boundary with M, and the maps are bordisms that are trivial on the boundary between such manifolds. To analyse this category, we will use Morse theory.

Let's recall the basic idea of Morse theory: Suppose you have a compact manifold N and you choose a smooth Morse (i.e generic) function f from N to the real numbers $\mathbb{R}$. Morse means that its derivative vanishes at isolated points such that the Hessian is nondegenerate. We can consider the process of building up N from its descending manifolds N^d_r which is the preimage under f of $(-\infty, r]$. The diffeomorphism type of N^d_r is empty for small r and N for large r, and only changes when a critical point of f happens (because apart from that f is a proper submersion). By the nondegeneracy hypothesis, the diffeomorphism type changes at the critical points in a very predictable way, namely by handle attachments.

A d-handle H in dimension n is a copy of $D^{n-d} \times D^d$ with an attaching map $\phi : D^{n-d} \times S^{d-1} \to N$ onto a manifold with boundary N. We can glue the handle

$H \cup_\phi N$ along ϕ and smooth the corners to obtain another smooth manifold with boundary, a process we refer to as attaching a handle. Every time we pass a critical point, our manifold changes by a handle attachment, and moreover the attaching data is determined by local data of the critical point. For example d is the index of the Hessian as a symmstric bilinear form over the reals.

The perspective we want to take is that choosing a Morse function really is a way of presenting N as a bordism. For example, by choosing a Morse function with critical points who take on distinct values we obtain the result that every bordism is a composite of handle attachments, as we can break up N into $f^{-1}([a_i, a_{i+1}])$, where only one critical point occurs within each interval. When we attach a d-handle, we do surgery on the boundary of the manifold, replacing a copy of $D^{n-d} \times S^{d-1}$ with $S^{n-d-1} \times D^d$.

A more refined version of Morse theory, called Cerf theory, explains how to move between different Morse functions f_0, f_1 via a family f_t that fail to be Morse at finitely many values in predictable ways. One can describe all the possible moves entirely at the level of handles. the two possible moves are essentially changing the time when a handle is attached, isotoping the attaching map ϕ of a handle, and cancelling or uncancelling a pair of d and $d-1$ handles whose cores intersect geometrically once.

However what we really want is even more sophisticated than Cerf theory: we want an understanding of the whole space of ways to present a bordism, not just generic paths between bordisms. To do this we use Igusa's theory of framed functions. A framed function is like a Morse function with slightly worse singularities allowed (those that occur when cancelling handles), and a identification of the negative index part of the Hessian at each critical point with a standard negative definite form. We can replace $B(M)$ with $B(M)^{ff}$ where the 1-morphisms in $B(M)^{ff}$ are equipped with framed functions. There is a natural map $B(M)^{ff}$ to $B(M)$ that forgets the structure. By the category unfolding equivalence, the assignment $M \mapsto B(M)$ corresponds to a map $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n^f f$, and moreover we get a map $\mathrm{Bord}_n^f f \to \mathrm{Bord}_n$.

The point is to use Igusa's theory to prove that the cobordism hypothesis holds for the map $\mathrm{Bord}_{n-1}\, to\, \mathrm{Bord}_n^{ff}$, and then show that Bord_n^{ff} and Bord_n agree.

Bord_n^{ff} is good because it is as if each morphism comes with a presentation in terms of handle attachments. To study it, we will filter between Bord_n and Bord_n^{ff} via a category F_k, which will unfold to the assignment $M \mapsto B_k(M)$ where $B_k(M)$ is the category of bordisms for which all ciritcal points are index $\leq k$. For $k \geq n$ this is just $B^{ff}(M)$ and for $k < 0$, there are no critical points so all the bordisms are trivial. Thus the F_k interpolate between Bord_{n-1} and Bord_n^{ff}, and are called the index filtration. It will turn out that only index 0 and index 1 are important with respect to mapping into fully dualizable categories.

Given this filtration, the main things proven about it are essentially a version of handle calculus:

- The functor $B_0 : \Omega^{n-2} \operatorname{Bord}_{n-1} \to \operatorname{Cat}_{(\infty,1)}$ is freely generated as a lax symmetric monoidal functor by the $O(n)$-equivariant morphism given by a disk in $B_k(\phi)$. Note that there is no nondegeneracy condition as well as no dualizability condition on the target. This is quite reasonable to expect, since for index 0, we can only add in disks that are disjoint to the rest of the bordism, and this is a well defined operation up to a $O(n)$ action. This implies that F_0 is freely generated as a symmetric monoidal (∞, n)-category from Bord_n by a morphism given by a disk.
- For $k > 0$, the functor $B_k : \Omega^{n-2} \operatorname{Bord}_{n-1} \to \operatorname{Cat}_{(\infty,1)}$ is generated from B_{k-1} by an $O(n-k)$-equivariant 1-morphism (handle attachment of index k) subject to one relation i.e $O(n-k)$-equivariant 2-morphism (handle cancellation between index k and $k-1$).
- The next claim is that if C is symmetric monoidal $Fun^{\otimes}(F_1, C) \to Fun^{\otimes}(F_0, C)$ is fully faithful with essential image the functors such that the morphism corresponding to the disk is nondegenerate. This quite reasonably follows from the previous, since the only relation obtained when adding a 1-handle is cancellation with 0-handles, and we can copy the proof of the $(1,1)$-categorical cobordism hypothesis in dimension 1 using 0 and 1-handles to see that this just adds in the relation that the disk is nondegenerate.
- The final claim is that if C additionally has all duals, then for $k \geq 1$, $Fun^{\otimes}(F_{k+1}, C) \to Fun^{\otimes}(F_k, C)$ is an equivalence for $k \geq 1$. This follows from the same claim as before, but the reason is more subtle, and involves thinking more carefully about how the unfolding of categories works. The point is that the handle cancellations of $k-1$ and k-handles impose another nondegeneracy condition, that is redundant when mapping to C, since duals are essentially unique. Alternatively, the rest of the handles can be thought of as being there to make the category Bord_n itself have duals.

7. Obstruction theory

We know at this point that $\operatorname{Bord}_n^{ff}$ has the right universal property, but we need to know that our choice of framed function didn't matter. It is true that given a bordism, the space of framed functions is contractible. However, this fact is actually equivalent to the fact that $\operatorname{Bord}_n^{ff} \to \operatorname{Bord}_n$ is an equivalence, and as of yet doesn't have an independent proof. Nevertheless, Igusa did show some connectivity bounds on the space of framed functions.

The reason we can show that $\operatorname{Bord}_n^{ff} \to \operatorname{Bord}_n$ is an equivalence is because of the connectivity bound that Igusa showed as well as some cohomology computations generalizing work of GMTW. The goal is to set up an obstruction theory for (∞, n) categories that generalizes the Postnikov tower, and use it to show the equivalence. To see why we would need both a connectivity bound as well as cohomology computations, we can make an analogy with the case $n = 0$: it is not true that a map of spaces that is an equivalence if it is just an equivalence on homology. One also needs to check that it is an equivalence on the fundamental groupoid, which is a connectivity statement.

Here, we will use a generalization of the Postnikov tower, namely the (m, n)-truncation of an (∞, n)-category (or the homotopy m-category). The analog of Eilenberg-Maclane spaces and cohomology will arise from the category $\mathrm{Loc}(C)$ of local systems on C, which for an (∞, n) category has an inductive definition. For example $n = 0$ it is just a functor into abelian groups, and for $n = 1$ we require a functor from $\mathrm{Map}(x, y)$ into abelian groups as well as a map local systems on $\mathrm{Map}(x, y) \times \mathrm{Map}(y, z)$ from the product of the pullbacks of the local system on $\mathrm{Map}(x, y)$ and $\mathrm{Map}(y, z)$ to the pullback of the local system on $\mathrm{Map}(x, z)$.

The definition of cohomology on a local system is also inductive, and can be thought of as the homotopy classes of sections of a fibration of (∞, n)-categories whose fibres are Eilenberg-MacLane spaces. There are a few subtleties, for example that we really would like to work with a local system compatible with the symmetric monoidal structure (since we are essentially doing stable homotopy theory for n-categories).

The resulting obstruction theory says that a map $C \to C'$ of symmetric monoidal (∞, n)-categories is an equivalence iff it is an equivalence on homotopy $(n+1, n)$-categories and induces an isomorphism on cohomology in any local system. In the case of interest, the first claim comes from a reult of Igusa on connectivity of the space of framed functions on a bordism. The cobordism hypothesis can be used to identify the relative cohomology of the pair $(\mathrm{Bord}_n, \mathrm{Bord}_{n-1})$ with a degree shifted cohomology of $BO(n)$ for appropriate coefficient systems. This result is then true for Bord_n^{ff}, so it suffices to show it is also true for Bord_n.

This relative cohomology for constant local systems is also the relative cohomology of the pair $(\Sigma^n MTO(n), \Sigma^{n-1} MTO(n-1))$. But the cofibre of these spectra is indeed $\Sigma_+^{\infty+n} BO(n)$, by cohomology calculations in the work of GMTW. The paper claims that their methods can be generalized to show that the relative cohomology agrees for arbitrary coefficient systems, which completes the proof.