SOME COMPLEX THEOREMS IN SIMPLE ANALYSIS

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1. Holomorphic Functions

Let's recall the statements of two important theorems that we will need. Throughout the first part of the notes (about holomorphic functions), we assume that f is a holomorphic function on an open set Ω, D is an open disk. $D(z_0, r)$ indicates that this disk is centered at z_0 of radius r.

Theorem 1.1 (Cauchy's Theorem in a Disk)**.** *If $D \subset \Omega$, then*

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in D.

Theorem 1.2 (Cauchy's Integral Formulæ)**.** *f has infinitely many derivatives on Ω and for any circle $C \subset \Omega$ that is boundary of $D \subset \Omega$ we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for any z in the interior of C.

As a consequence we have the following:

Corollary 1.3 (Cauchy Inequalities)**.** *If Ω contains the closure of $D(z_0, R)$, then*

$$|f^{(n)}(z)| \leq \frac{n!|f|_C}{2\pi R^n}$$

where $|f|_C$ is the maximum modulus f attains on C.

Proof. By Theorem 1.2 we have:

$$\begin{aligned}
|f^{(n)}(z)| &= |\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta| \\
&= \frac{n!}{2\pi}|\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})Rie^{i\theta}}{(Re^{i\theta})^{n+1}}d\theta| \\
&\leq \frac{n!}{2\pi}\int_0^{2\pi} |\frac{f(z_0 + Re^{i\theta})Rie^{i\theta}}{(Re^{i\theta})^{n+1}}|d\theta \\
&= \frac{n!|f|_C R(2\pi)}{2\pi R^{n+1}} \\
&= \frac{n!|f|_C}{R^n}
\end{aligned}$$

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□

Here is another corollary of the Cauchy Integral Formulæ:

Corollary 1.4. *Holomorphic functions are analytic.*

Proof. Suppose f is holomorphic near z_0. By Theorem 1.2 we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

where C is a small circle centered around z_0, and z is inside the disk. Additionally

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}$$

and since z is inside the circle and ζ is on the boundary, so we can expand as

$$\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} = \sum_0^\infty \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

which converges uniformly for all $\zeta \in C$. Then from uniform convergence we switch the sum and integral and get

$$f(z) = \sum_0^\infty \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta (z - z_0)^n = \sum_0^\infty \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

□

We can use the Cauchy Inequalities to deduce that a non-constant entire (ie. holomorphic on $\mathbb{C}$) function is not bounded.

Corollary 1.5 (Liouville's Theorem)**.** *A bounded entire function f is constant.*

Proof. Fix z arbitrary. Let B be the bound on f. Integrate around a large circle centered at z and use the Cauchy inequalities to get $|f'(z)| \leq \frac{B}{R}$. Then letting R tend to infinity, we get $f'(z) = 0$ so f is constant by Corollary 1.4. □

Theorem 1.6. *Suppose f is holomorphic on a connected open set, and the set of zeros has a limit point, then $f = 0$.*

Proof. Take a limit point z_0 of the set of zeros of f, and write $f(z) = \sum_i^\infty a_n(z - z_0)^n$, where $a_i \neq 0$. Then

$$f(z) = a_i(z - z_0)^i(1 + g(z - z_0))$$

where g is analytic near z_0 and $g(z - z_0)$ goes to 0 as $z \to z_0$. Then take a sequence of points $\neq z_0$ converging to z_0. But by the formula above, some of these points cannot be 0 as $1 + g(z - z_0)$ is nonzero near z_0. Then f is 0 near z_0. This shows that the set of limit points of the zeros of f is open. But it's also closed and f is holomorphic on a connected set. □

Corollary 1.7 (Analytic continuation)**.** *If there are two functions f, g holomorphic on a connected open set such that they agree on a set with a limit point, then they are the same.*

Proof. Theorem 1.6 on $f - g$. □

Analytic continuation is an important principle used to extend functions to larger parts of $\mathbb{C}$ by defining it over a small region and trying to extend it. One of the ways functions can be extended is the symmetry principle and in particular the Schwarz reflection principle. Here is a converse of Goursat's Theorem that we will use to prove the symmetry principle.

Theorem 1.8 (Morera's Theorem)**.** *If f is continuous on Ω and for any triangle T we have $\int_T f(z)dz = 0$ then f is holomorphic.*

Proof. By the proof of Goursat's Theorem, f has a primitive, and so the derivative of this primitive, f, is by Corollary 1.4 also holomorphic. □

Now we will prove the symmetry principle. This is a way of extending a holomorphic function.

Theorem 1.9 (Symmetry principle)**.** *Suppose that f^+ and f^- are holomorphic functions such that f^+ is holomorphic on a region of the upper half-plane, and f^- is holomorphic on the lower half-plane such that they can be extended continuously to an interval of $\mathbb{R}$ such that they agree on the boundary, then they can be extended to a holomorphic function on the union.*

Proof. For any triangle in the union of the regions, we can split it into a bunch of triangles in the region where f^+ and f^- are continuous (and possibly also the boundary). Then we can approximate the triangles with ones in the interior (where f^- and f^+ are holomorphic) and by Goursat's Theorem, the integral on these triangles is 0, so by continuity the integral on the triangles that do hit the boundary are also 0. Then any triangle's integral is 0, so the result follows from Theorem 1.8. □

As a special case we have the Schwarz reflection principle.

Theorem 1.10 (Schwarz reflection principle)**.** *Suppose that f is holomorphic on some Ω in the upper half-plane, and extends continuously to a real-valued function on an interval on the boundary, I. Then f can be extended to a holomorphic function on the region $\Omega \cup -\Omega \cup I$, where $f(\bar{z}) = \bar{f(z)}$.*

Proof. Use Theorem 1.9. □

2. Meromorphic Functions

Sometimes we don't want to require such a strong condition as holomorphicity, but would rather have a function that is "almost" holomorphic, or meromorphic. I offer two ways to think about this. One way is that a function can have three types of singularities: removable singularities, poles, and essential singularities. Removable singularities are as their name suggests removable (ie the function can be extended to fill in the hole). Essential singularities are wild, we would not like our notion of meromorphic to allow this. Poles, however, are quite reasonable and interesting singularities, hence we can allow these. Another way to think about meromorphic functions is as holomorphic functions, except these no longer land in the complex plane. Instead, we add a point at infinity to $\mathbb{C}$, and call this $\mathbb{P}^1$, the Riemann sphere. This has a complex analytic structure: it can be thought of as two open disks in $\mathbb{C}$ glued together. Then a meromorphic function is a holomorphic function from an open subset of $\mathbb{C}$ to $\mathbb{P}^1$.

First let's study singularities. A singularity is a point such that f is defined around the point but not on that point. We have a few possibilities:

Definition 2.1. *A singularity z_0 is* ***removable*** *if f is bounded near z_0. A singularity is a* ***pole of order*** *i if $(z - z_0)^i f$ is bounded near z_0 for some i. A singularity is* ***essential*** *if it is neither of the other two.*

Now we can define meromorphic:

Definition 2.2. *f is meromorphic on an open set Ω if there is a set of points $X \subset \Omega$ without a limit point such that f has removable singularities and poles on X and is holomorphic on $\Omega - X$.*

Theorem 2.3. *Removable singularities are removable.*

Proof. Suppose f is bounded near z_0 (WLOG set $z_0 = 0$). Then we have $z^2 f$ (extended to be 0 at z_0) is holomorphic near 0 but also at 0. Then f is of the form

$$\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots$$

f is bounded near zero so a_{-2} and a_{-1} are 0. □

We can characterize poles in the way that agrees with the second way of thinking about meromorphic functions:

Theorem 2.4. *A point z_0 is a pole iff $1/f$ is holomorphic near z_0 where $1/f(z_0)$ is defined to be 0.*

Proof. By Theorem 2.3 we have that f having a pole of order n at z_0 is equivalent to $f(z) = (z - z_0)^{-n} h(z)$ for some function h holomorphic near z_0 that is nonzero at z_0. Then f is meromorphic implies $1/f = (z - z_0)^n (1/h)(z)$ is holomorphic. Conversely if $1/f$ is holomorphic we can write $1/f$ in the same way, and invert it to get f in the form we want. □