MORITA EQUIVALENCE

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Two rings are **Morita equivalent** when their category of right modules are the same. An equivalence is automatically additive, since the sum can be described categorically as $M \xrightarrow{\Delta} M \oplus M \xrightarrow{f \oplus g} M \oplus M \xrightarrow{\nabla} M$. Moreover, the notion is only interesting for non-commutative rings as the center of the ring can be recovered as the automorphisms of the identity functor. A nontrivial example is that for a division ring R, $\mathrm{End}(R^n)$ is Morita equivalent to R. We will see that in a sense all examples are of this form via an alternate characterization of Morita equivalence. A **progenerator** is a generator of the category of modules that is finitely generated and projective.

Note that this definition is purely categorical because of the lemma:

Lemma 0.1. *A projective module P is finitely generated iff* $\mathrm{Hom}(P, -)$ *commutes with direct sums.*

Proof. If it is finitely generated, it is a summand of a finitely generated free module, and Hom commutes with sums for finitely generated free modules. Conversely if $\mathrm{Hom}(P, -)$ commutes with directs sums, note that there is a surjection from a free module that splits, and the splitting factors through a finitely generated free module by assumption. □

The next thing we will need is the Eilenberg-Watts theorem, which is a kind of representability theorem:

Theorem 0.2. *The natural functor R-Mod-$S \to \mathrm{Hom}_{ccon}(\mathrm{Mod}\text{-}R, \mathrm{Mod}\text{-}S)$ given by the tensor product is an equivalence, where ccon denotes the cocontinuous functors.*

Proof. It is not hard to see that the maps is well defined. We will first see the functor is essentially surjective. Given a functor F, consider $F(R)$, which has a natural bimodule structure by using the R-module homomorphism given by left multiplication. Now there is a natural isomorphism $\oplus_i R_i \otimes F(R) \cong F(\oplus_i R_i)$ because both functors commute with direct sums. There is a natural transformation $M \otimes F(R) \to F(M)$ given by $M \otimes F(R) \cong \mathrm{Hom}(R, M) \otimes F(R) \to F(M)$ where the last map is the natural one given by applying F and evaluating. It is clearly a natural isomorphism for free modules as both functors preserve direct sums. In general, since both functors preserve cokernels and any module is a cokernel of a morphism of free modules, We

see it is a natural isomorphism. Faithfulness is clear, and fullness follows again from the fact that both functors preserve sums and cokernels, so what happens on R determines everything. $\square$

Theorem 0.3. *S is Morita equivalent to R iff there is a left R-module P that is a progenerator such that $R = \mathrm{End}^{op}(P)$.*

Proof. Since progenerator is a categorical notion, if $R\text{-}\mathrm{Mod} \xrightarrow{F} S\text{-}\mathrm{Mod}$ is an equivalence, $F(R)$ is a progenerator in S and $R \cong \mathrm{End}_R^{op}(R) \cong \mathrm{End}_S^{op}(F(R))$. Conversely let P be a progenerator and consider $\mathrm{End}_R^{op}(P)$. P is a $(R\text{-}\mathrm{End}_R^{op}(P))$ bimodule, and $\mathrm{Hom}(P, R)$ is a $(\mathrm{End}_R^{op}(P)\text{-}R)$ bimodule. If $P \otimes \mathrm{Hom}(P, R) \cong R$ and $\mathrm{Hom}(P, R) \otimes R \cong \mathrm{End}_R^{op}(P)$, then the tensor products with these bimodules will induce an equivalence.

There is a natural map $P \otimes \mathrm{Hom}(P, R) \xrightarrow{\phi} R$. By using the fact that P is a generator on the maps $R \xrightarrow{\pi, 0} R/\operatorname{im}\phi$, ϕ is surjective. Similarly there is a natural map $\mathrm{Hom}(P, R) \otimes P \xrightarrow{\varphi} \mathrm{End}_R^{op}(P)$. Note that if $P \oplus Q = R^n$, then there is a similar map $\mathrm{Hom}(P \oplus Q, R) \otimes (P \oplus Q) \to \mathrm{End}_R^{op}(P \oplus Q)$ that is an isomorphism, and by restricting to the corresponding summands, φ is an isomorphism. If $\sum_j x_j \otimes g_j$ is an element that gets sent to 1, any element $\sum_i p_i \otimes f_i \in P \otimes \mathrm{Hom}(P, R)$ is equal to $\sum_{i,j}(p_i \otimes f_i)\phi(x_j \otimes g_j) = \sum_{i,j} \phi(p_i \otimes f_i)(x_j \otimes g_j)$, so ϕ is injective. $\square$

Corollary 0.4. *R, S have equivalent categories of finitely generated modules iff R, S are Morita equivalent.*

Proof. If R, S are Morita equivalent, the equivalence is given by tensoring with a finitely generated module, so the finitely generated module categories are equivalent. Conversely, by running the proof of the theorem above for finitely generated modules, we see that if they are equivalent, there is a progenerator such that S is the (opposite) endomorphism ring, and so R, S are Morita equivalent. $\square$

Given a finite dimensional algebra A over a division ring k, A might decompose as an A module into the sum of many modules e_iA where e_i are idempotents. If the modules e_iA are distinct for maximally chosen e_i, A is said to be **basic**.

Corollary 0.5. *Any finite dimension algebra A over a division ring k is Morita equivalent to a basic one.*

Proof. Let e be the sum of the idempotents of all distinct isomorphism classes of e_iA. Then eAe is an algebra that is $\mathrm{End}(eA)$, and eA is a progenerator, so eAe is Morita equivalent to A. Moreover we have chosen eAe to be basic. $\square$