K-THEORY AND FREDHOLM OPERATORS

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1. Introduction

Given a compact space X, we can consider the commutative monoid of complex vector bundles on X under the Whitney sum operation. We can force it to become a group in the universal way, giving a group called $K(X)$. This is a functor, so we can ask whether it is representable. If it is, the representing object should be a group up to homotopy. But in this case, it is represented by the space of Fredholm operators $\mathscr{F}$ on Hilbert space H. In particular the composition operation is a commutative group operation on $\mathscr{F}$ up to homotopy.

$K(X)$ for a point is just $\mathbb{Z}$, so in particular the components of the space of Fredholm operators should correspond to the integers, and this function from Fredholm operators to $\mathbb{Z}$ is exactly the index of an operator. Hence in general we will want to construct a natural isomorphism $\mathrm{index} : [-, \mathscr{F}] \to K(-)$.

To do this, first suppose that we have a map $f : X \to \mathscr{F}$, and we fix a point $x \in X$. We should note that the kernel of $f(x) = f_x$ varies continuously in X, in the sense that if V is a subspace on which f_x is injective, then nearby points will also be injective on V. In particular, as X is compact, we can find a V of finite codimension in H such that f_x is injective on V for all $x \in X$. Then $f_x(V)$ is a subbundle of $X \times H$, and we can consider $H/f_X(V)$, the bundle where the fibre of x is $H/f_x(V)$. If V is codimension k, we define $\mathrm{index}_V(f) = k - [H/f_X(V)]$. To see this is well defined, we only need to observe that if we choose a codimension n subspace $U \subset V$ then $H/f_X(U) \cong f_X(U)^{\perp} \cong f_X(V)^{\perp} \oplus f_X(U^{\perp} \cap V) \cong H/f_X(V) \oplus \mathbb{C}^n \times X$. We thus have $k - [H/f_X(V)] = k + n - [H/f_X(U)]$, so for any V, W, $\mathrm{index}_V(f) = \mathrm{index}_{V \cap W}(f) = \mathrm{index}_W(f)$. Moreover index is clearly natural.

Homotopy invariance comes from the commutative diagram:

$$\begin{tikzcd}
{[X,\mathscr{F}]} \arrow[r, "\mathrm{index}(f_0)"] & K(X) \\
{[X\times I,\mathscr{F}]} \arrow[u, "(1_X\times 0)^*"] \arrow[d, "(1_X\times 1)^*"'] \arrow[r, "\mathrm{index}(f_t)"] & K(X\times I) \arrow[u, "(1_X\times 0)^*"'] \arrow[d, "(1_X\times 1)^*"] \\
{[X,\mathscr{F}]} \arrow[r, "\mathrm{index}(f_1)"] & K(X)
\end{tikzcd}$$

The vertical maps are isomorphisms and their composites are the identity on $[X,\mathscr{F}]$ and $K(X)$, so $\mathrm{index}(f_0) = \mathrm{index}(f_1)$. To see index is a homomorphism, let $f, g : X \to \mathscr{F}$. Then choose U, V to be finite codimension subspaces on which f, g are respectively injective and such that $g_X(V) \subset U$, so that fg is injective on V. Then there is an exact sequence $0 \to U/g_X(V) \to H/fg_X(V) \to H/f_X(U) \to 0$, so $\mathrm{index}(fg) = \operatorname{codim} V - [H/fg_X(V)] = \operatorname{codim} V - [U/g_X(V)] - [H/f_X(U)] = \operatorname{codim} U + \operatorname{codim} V - [H/g_X(U)] - [H/f_X(U)] = \mathrm{index}(f) + \mathrm{index}(g)$.

For surjectivity of the index, since it is a homomorphism we need every vector bundle V to be in the image, as well as $n \in \mathbb{Z}$. The latter can be done by sending X to a single operator of index n. To see V is in the image, find a vector bundle W such that $V \oplus W = \mathbb{C}^n \times X$ and let π_V, π_W be the projection maps. Now send x to the operator on $H \otimes \mathbb{C}^n \cong H$ sending $e_i \otimes v$ to $e_{i+1} \otimes \pi_W(v) + e_i \otimes \pi_V(v)$. The index is $-W$ which is V up to a trivial bundle.

Finally let's examine the kernel of the index. If something is in the kernel, there must be a finite codimension subspace $U \subset H$ with $H/f_X(U)$ trivial. Let $e_1, \dots e_n$ be a basis of $U^\perp$ and let $s_1, \dots s_n$ be trivializing sections of $f_X(U)^\perp$. Consider the homotopy that has $f_{x,t}(e_i) = f_x(e_i)(1-t) + s_i t$. It homotopes f to something that is an isomorphism on H. Thus we have an exact sequence $[X, \mathrm{GL}(H)] \to [X,\mathscr{F}] \to K(X) \to 0$. However, by Kuiper's theorem, $\mathrm{GL}(H)$ is contractible, so the index is a natural isomorphism.