MULTIPLICATIVE STRUCTURES ON SPHERES

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What multiplicative structures do the spheres S^n have? Globally, $S^0 = \mathbb{F}_2, S^1 = \mathrm{B}\mathbb{Z}$ are $\mathbb{E}_\infty$-spaces and $S^3 = \Omega\mathbb{HP}^\infty$ is a $\mathbb{E}_1$-space. S^7 admits an $\mathbb{A}_2$-structure because of the octonions, but this doesn't extend to $\mathbb{A}_3$, as there is a 3-primary obstruction visible from the Steenrod algebra action on the cohomology of the hypothetical $\mathbb{OP}^3$. By Hopf invariant one, there are no more $\mathbb{A}_2$ multiplications on spheres, with the obstruction being $K(1)$-local at the prime 2.

If we work p-adically (equivalently p-locally), then more spheres can become multiplicative. Rationally, S^{2n} is not even $\mathbb{A}_2$ (the obstruction is the whitehead square of the identity), and since the obstruction persists after base-change to $\mathbb{Q}_p$, S^{2n} has no hope of being $\mathbb{A}_2$ p-adically. S^{2n-1} is an Eilenberg Mac Lane space rationally, so there is a hope of getting multiplicative structures p-adically for odd spheres.

Proposition 0.1. *For $p > 2$, the map $S^{2k-1} \to \Omega J_{p-1}(S^{2k})$ can be refined to a map of $\mathbb{A}_{p-1}$-algebras.*

Proof. The obstruction to refining an $\mathbb{A}_{i-1}$-algebra map to an $\mathbb{A}_i$-algebra map amounts to producing a lift in the diagram

$$\begin{tikzcd} S^{2ni-3} \arrow[r] \arrow[d, "\mathrm{ob}_i"'] & D^{2ni-2} \arrow[d] \arrow[dl, dashed] \\ S^{2k-1} \arrow[r] & \Omega J_{p-1}(S^{2k}) \end{tikzcd}$$

where ob_i is the obstruction to refining an $\mathbb{A}_{i-1}$ algebra structure on S^{2k-1} to an $\mathbb{A}_i$ algebra structure.

By the EHP sequence, the fibre of the lower horizontal map is $\Omega^2 S^{2pk-1}$, so we learn that there is no obstruction to producing the lift for $i < p$. $\square$

From the proof above, we see that there is a potential obstruction to an $\mathbb{A}_p$-structure on S^{2k-1}. The obstruction doesn't always cause issues:

Theorem 0.2 (Sullivan)**.** *S^{2k-1} is an $\mathbb{E}_1$-algebra if $k | p-1$.*

Proof. For the first statement, we observe that $S^{2k-1} = \Omega((B^2\mathbb{Z}_p)_{hG_k})$, where G_k is the cyclic subgroup of $\mathbb{F}_p^\times$ of order k acting on $\mathbb{Z}_p$. This can be seen from the Eilenberg-Moore/Serre spectral sequence. $\square$

Wilkerson showed that in all other odd primary cases, there is a $K(1)$-local obstruction to an $\mathbb{A}_p$-multiplication.

Theorem 0.3 (Wilkerson)**.** *If $k \nmid p-1$, then S^{2k-1} is not $\mathbb{A}_p$.*

Proof. If S^{2k-1} was $\mathbb{A}_p$, we could form the p^{th}-truncated bar construction $B_p(S^{2k-1})$, which would have cohomology ring $\mathbb{Z}_p[x]/x^{p+1}$. It follows that $K^0(B_p(S^{2k-1}))$ as a ring is isomorphic to $\mathbb{Z}_p[x]/x^{p+1}$. The Adams operations ψ^q, give an action of $\mathbb{Q}_p^\times$ on this, with the following properties:

(1) $\psi^p(x) \equiv x^p \pmod{p}$
(2) $\psi^q(x) \equiv qx \pmod{x^2}$

The second property comes from the inclusion of the first cell, which is a sphere, whose Adams operations we know.

Given a power series f, let $f[x^i]$ denote the x^i-coefficient. We show inductively for $i \leq p$ that

$$v_p(\psi^p(x)[x^i]) \geq k - (v_p(k)+1)|\{0 < j < i \text{ such that } p-1|jk\}|$$

Let m_i denote the number on the right hand side of the above equation. If we can show the claim, we are finished, since we know from (1) that the x^p coefficient is 1 mod p, but $m_p = k - (v_p(k)+1)\gcd(i, p-1)$ is positive unless $k|p-1$.

To show the claim inductively for $i+1$ assuming it for lower i, we compare the x^{i+1} coefficients of the equation $\psi^p\psi^q(x) = \psi^q\psi^p(x)$ after reducing mod p^{m_i}, using the inductive hypothesis to see that $\psi^p(x)[x^{i+1}]q^{k(i+1)} = \psi^p(x)[x^{i+1}]q^k$ mod p^{m_i}. It follows that $v_p(\psi^p(x)[x^i]) + v_p(q^{k(i+1)} - q^k) \geq m_i$. Choose q to be a topological generator of $\mathbb{Z}_p^\times$, so that $v_p(q^{k(i+1)} - q^n)$ is 0 unless $p-1|k(i+1)$ in which case it is $v_p(k)+1$. This gives the inductive step. $\square$