CHERN-WEIL AND GAUSS-BONNET

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Given a manifold, Chern-Weil theory says that we can obtain characteristic classes by applying invariant polynomials on the curvature of a connection. We will see here an explicit proof (without using the Chern-Weil homomorphism) of the Gauss-Bonnet theorem for vector bundles, which is an example of the phenomenon.

Let $E \to M$ be a rank $2p$ real vector bundle with a metric and a metric connection ∇, and let Ω_E be its curvature 2-form. Then we can take the Pfaffian $\mathrm{Pf}(\Omega_E)$ multiplied by a normalizing constant $(\frac{-1}{2\pi})^p$ of the curvature to get a d-form, whose cohomology class we should interpret by Chern-Weil theory as a characteristic class of the bundle. Indeed, we can call this class the geometric Euler class ($e_g(E)$), and we can prove that it indeed coincides with the topological Euler class ($e_t(E)$). This can be viewed as a generalization of Gauss-Bonnet:

Theorem 0.1 (Gauss-Bonnet)**.** *Given an even dimensional Riemannian manifold M^{2p}, if Ω is the curvature, then $\int_M (\frac{-1}{2\pi})^p \mathrm{Pf}(\Omega) = \chi(M)$.*

In the case that the bundle is the tangent bundle, and the metric is a Riemannian metric, this becomes the Gauss-Bonnet theorem. Indeed, the Euler class integrates to the Euler characteristic, and the geometric Euler class is an integral of the Pfaffian of the Riemann curvature tensor (up to a constant).

The first thing to note is that the geometric Euler class is natural. It is easy to check that it commutes with pullbacks, and that $e_g(E_1 \oplus E_2) = e_g(E_1) \wedge e_g(E_2)$ (Note: here the notation is abused since e_g seems to depend on the connection). Then by the splitting principle, it suffices to show that $e_g = e_t$ for oriented plane bundles, for which we can more explicitly calculate.

For a plane bundle $E \xrightarrow{\pi} M$, let the connection be given in local neighborhood U_α by the skew-symmetric matrix of 1-forms $(\theta_\alpha)_i^j = \omega_\alpha$. The curvature $\Omega_\alpha = d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha$ is given by the matrix $\begin{pmatrix} (\theta_\alpha)_1^2 \wedge (\theta_\alpha)_1^2 & d(\theta_\alpha)_1^2 \\ -d(\theta_\alpha)_1^2 & (\theta_\alpha)_1^2 \wedge (\theta_\alpha)_1^2 \end{pmatrix}$ so that the Pfaffian is $d(\theta_\alpha)_1^2$.

Now suppose we have a partition of unity γ_α subordinate to the choice of local coordinate cover U_α, and let $g_{\alpha\beta}$ be the transition functions with values in $\mathrm{SO}(2)$ that define the vector bundle. Then by identifying $\mathrm{SO}(2) = \mathbb{R}/2\pi\mathbb{Z}$, we can think of the $g_{\alpha\beta}$ as the angle the transition function rotates counterclockwise. By one construction (eg. in Bott and Tu's book) e_t is given by $\frac{-1}{2\pi}\sum_\beta d\gamma_\beta dg_{\alpha\beta}$. If r_α, r'_α make up the local frame in U_α, since the connection is a metric connection, we have that $dr_\alpha = (\theta_\alpha)_1^2 r'_\alpha$ (here we view the connection as on the frame bundle).

On the bundle since $g_{\alpha\beta}$ is the transition function, we have $d\pi^* r_\alpha = (\pi^* dr_\beta + \pi^* g_{\alpha\beta})\pi^* r'_\alpha$. By injectivity of π^* we obtain $dr_\alpha = dr_\beta + dg_{\alpha\beta} r'_\alpha$. Thus we must have $dg_{\alpha\beta} = (\theta_\alpha)_1^2 - (\theta_\beta)_1^2$.

Then we have:

$$\frac{-1}{2\pi}\sum_{\beta} d(\gamma_\beta dg_{\alpha\beta}) = \frac{-1}{2\pi}\sum_{\beta} d(\gamma_\beta((\theta_\alpha)^2_1 - (\theta_\beta)^2_1)) = \frac{-1}{2\pi} d(\theta_\alpha)^2_1 + \frac{1}{2\pi} d(\sum_{\beta}\gamma_\beta(\theta_\beta)^2_1)$$

The second resulting term defines a global form which is clearly exact, and we get that e_t is cohomologous to $-\frac{1}{2\pi}d(\theta_\alpha)^2_1$, which is exactly e_g.