K-THEORY AND THE INDEX THEOREM

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1. Introduction and Statement

Given an elliptic operator between (complex) vector bundles V, W on a closed manifold M, the Atiyah-Singer index theorem expresses its index in terms of topological invariants. One natural way to formulate and prove this theorem, which will be sketched here, is via K-theory and pseudo-differential operators. The topological index that we construct will factor through K-theory via a symbol map as shown in the diagram below:

$$\begin{tikzcd}
\text{elliptic operators} \arrow[r, "\sigma"] \arrow[dr, "\text{index}"'] & K(TM) \arrow[d, "t_{\text{ind}}"] \\
& \mathbb{Z}
\end{tikzcd}$$

Here $K(TM)$ is the compactly supported K-theory defined as the reduced K-group of the one point compactification. The map σ map is called the symbol map and is obtained from the principal symbol of the elliptic operator. The map t_{ind} is called the topological index. Once these maps have been defined, the theorem states:

Theorem 1.1 (Atiyah-Singer index theorem)**.** *The diagram above commutes.*

A cohomological form of the index theorem can be obtained by applying the Chern character, the ring isomorphism between rational K-theory and rational cohomology (with compact supports). The Todd class enters in the statement as a correction factor since the Thom isomorphisms of K-theory and cohomology don't exactly commute with the Chern character. A cohomological statement obtained from applying the Chern character is:

Theorem 1.2. *If D is an elliptic operator on a compact n-manifold M, then* $\operatorname{index} D = (-1)^n(ch(\sigma(D)) \smile td(TM \otimes \mathbb{C}))[TM]$

There is a natural way to extend the discussion from elliptic operators to elliptic complexes. Namely, let a complex of partial differential operators be operators $\Gamma E_0 \to \Gamma E_1 \to \cdots \to \Gamma E_n$ that form a chain complex where ΓE_i are the smooth sections of some vector bundles E_i. If b_i is the rank of the i^{th} homology group of this complex, then the index is $\sum_i (-1)^i b_i$. Associated to this complex is an associated complex $0 \to \pi^* E_0 \to \pi^* E_1 \to \cdots \to \pi^* E_n \to 0$, where π is the projection $TM \to M$, given by taking the principle symbol of the operators. The original complex is said to be **elliptic** if this associated complex is exact outside the zero section. In the case that there are only E_0, E_1, this agrees with the notion of an elliptic operator from E_0 to E_1. σ will be defined for elliptic complexes, and the index theorem will hence hold for elliptic complexes.

However, this doesn't really add more generality. Given an elliptic complex $\Gamma E_i \xrightarrow{d} \Gamma E_{i+1}$, we can consider the map $\Gamma \oplus_{j=2i} E_j \to \Gamma \oplus_{j=2i+1} E_j$ given by $d + d^*$ where d^* is the adjoint of d with respect to some hermitian inner product on each E_i. This map is still an elliptic partial differential operator, has the same index as the complex and will have the same value in $K(TM)$. Thus the index theorem only needs to be proved for elliptic operators.

To define our symbol map, we will use the fact that the K group for a locally compact space can be defined the following way. Take the semi-group of finite complexes of vector bundles on M $0 \to E_1 \to \cdots \to E_n \to 0$ such that set on which the complex is not exact is compact. Identify complexes that are homotopic, meaning there is a complex on $M \times [0, 1]$ restricting to each at the end, which also is not exact with compact support. Then taking the quotient by the complexes that are exact, the result is $K(M)$. The product structure in $K(M)$ is given by the tensor product of complexes. In fact, we can fix any length and consider only complexes of that length, and the end result will still be $K(M)$. The construction in the last paragraph using a hermitian inner product to turn complexes of arbitrary length into complexes of length 1 preserves the element of $K(M)$. The associated symbol complex of an elliptic complex ΓE^* on M is only not exact at the 0-section which is compact, so it defines an element $\sigma(\Gamma E^*) \in K(TM)$.

Finally we need to define t_{ind} for any compact manifold M. It will be characterized by three properties. If $\tilde{M}$ is the one point compactification, and $Y \hookrightarrow M$ is an open inclusion, there is a natural quotient map $\tilde{M} \to \tilde{Y}$ which induces an extension map $h : K(Y) \to K(M)$. The first axiom is excision, namely that if we have an open inclusion $U \to \mathbb{R}^n$, t_{ind} commutes with the extension of $TU \to T\mathbb{R}^n$.

The second axiom is naturality with respect to the inclusion of a compact M as the 0-section of a vector bundle $\pi : E \to M$. $TE = \pi^* TM \oplus \pi^* E$, and so $TE \to TM$ is isomorphic to the bundle $\pi^* E \oplus \pi^* E$, and can be given a complex structure via $\pi^* E \oplus i\pi^* E$. Now via the Thom isomorphism, we have a map from $K(TM) \to K(TE)$. t_{ind} commutes with this map.

Finally, t_{ind} must be normalized for a point as the usual identification of $K(\cdot) \cong \mathbb{Z}$ via the rank.

We can construct t_{ind} for any manifold M by taking an embedding into $\mathbb{R}^n$, using the inclusion into the normal bundle using the Thom isomorphism axiom, extending to all of $\mathbb{R}^n$ via the extension axiom, and then viewing $\mathbb{R}^n$ as the normal bundle of a point using both axioms again to get an element of $K(\cdot)$. This does not depend on the inclusion $M \to \mathbb{R}^n$, because if i_1, i_2 are two inclusions, $i_1 \oplus i_2$ is isotopic to $i_1 \oplus 0$ and $0 \oplus i_2$, and the Thom isomorphism is well-behaved with respect to stabilization of bundles. Moreover, it is easy to see that anything satisfying the axioms must be equal to t_{ind} because we the definition can be obtained from the axioms.

2. Strategy

These axioms will guide the proof of the theorem. We would like to make a map called the **analytic index** that takes an element of $K(TM)$ to "index $\circ\sigma^{-1}$". Then if we could show it satisfies the same axioms as the topological index, the proof would be complete. Unfortunately, not every element of $K(TM)$ comes from the symbol of an elliptic operator, and it's not apriori clear that two elliptic operators with the same image via σ will have the same index. The failure of surjectivity can be seen for example in the case of odd-dimensional manifolds.

If n is odd, we can apply the antipode map a on TM to $\sigma(D)$. $\sigma(D)$ is given by the principal symbol, which is a homogeneous polynomial $p(\xi)$ on each fibre of M. This means that $p(-\xi)$ is $\pm p(\xi)$. In either case, $a^*\sigma(D) = \sigma(D)$ because the negative map is homotopic to the identity by rotating around the circle. However, if we look at $\sigma(D)$ in rational K-theory, which can be identified with compactly supported rational cohomology, TM's orientation is reversed as n is odd, so $a^*\sigma(D) = -\sigma(D)$. Thus $\sigma(D)$ is 0 in rational K-theory, so must be torsion. in particular, since t_{ind} is a homomorphism to $\mathbb{Z}$, by the index theorem $\text{index}(D) = 0$. Then σ will not be surjective.

To fix the problem, we will consider pseudo-differential operators instead of partial differential operators. Given a manifold M, a **pseudo-differential operator** D of order m is an operator of the following form from compactly supported smooth sections of a bundle E to smooth sections of a bundle F with the following properties. Consider any chart $\mathbb{R}^n$ on which E and F can be trivialized to be trivial bundles of ranks l, k. For any compactly supported function f and any smooth section u of E on the chart, we must have $D(fu)(x) = \int p_f(x,\xi)\mathscr{F}(u)e^{i\langle x,\xi\rangle}d\xi)$ where $\mathscr{F}$ is the Fourier transform, and p_f is some matrix where each partial derivative of p_f is $O(\xi^{m-|a|})$ where $|a|$ is the number of times the partial derivative is taken in the ξ direction. Moreover, we should like the limit $\sigma_f(x,\xi) = \lim_{\lambda\to\infty} \frac{p_f(x,\lambda\xi)}{\lambda^m}$ to exist. When f is compactly supported but 1 in a neighborhood of x, $\sigma_f(x,\xi)$ doesn't depend on f and is defined to be $\sigma(x,\xi)$ which is called the **principal symbol**. The principal symbol similarly defines an associated complex of bundles on TM, and the operator is elliptic if this is exact outside the 0 section. This definition comes from generalizing the definition of a partial differential operator, which acts on the Fourier transform via multiplication by a polynomial. Instead, a pseudo-differential operator acts on the Fourier transform via any function with similar growth behavior.

Elliptic pseudo-differential operators, like elliptic operators, are Fredholm. An example of an elliptic pseudo-differential operator is the operator A_{S^1} on the trivial line bundle of S^1 sending $e^{ik\theta}$ to itself when k is negative and to $e^{(i+1)k\theta}$ when k is nonnegative. The symbol is $e^{i\theta}$ when ξ is positive and 1 when it is negative. It has index -1 which we saw is impossible for an elliptic partial differential operator.

Any element of $K(TM)$ comes from an elliptic pseudo-differential operator. To see this, an element of $K(TM)$ can be obtained from some bundle map $f : E \to F$ which is an isomorphism away from some compact set L. Choose a metric for which the unit disk bundle of TM/M contains L. The composite $TM \xrightarrow{\pi} M \xrightarrow{0} TM$ is homotopic to the identity, so if E_0, F_0 are the pullbacks of E, F along this map, they are isomorphic to E, F. We can produce a map $f_m : E_0 \to F_0$ by extending the map from E to F as a homogeneous function of degree m, meaning, that $f_m(\lambda v) = \lambda^m f_m(v), v \in TM$. It is easy to see that f_m is homotopic to f. Moreover, if f is an isomorphism it can be made to be constant, so that the corresponding map f_m will be constant on the unit sphere bundle. Conversely if the f_m is constant on the unit sphere bundle, then the homotopy $f_m(v,t) = \|v\|^{tm} f_m(\frac{v}{\|v\|})$ will take it to an isomorphism. Thus we have obtained the following lemma:

Lemma 2.1. *$K(TM)$ can be taken to be the monoid of homotopy classes of homogeneous degree m maps of bundles on TM $E \to F$ modulo those which are constant on the unit sphere bundle.*

Note that these homogeneous maps may not be continuous at the origin for $m \le 0$. Thus given an element of $K(TM)$ represent it via some homogeneous map $f_m : E \to F$, choose coordinates, a trivialization of TX, E, F, and a smooth function φ that is 0 in a neighborhood of the zero section of TM (where the map may not be continuous) and 1 far away from this section. The pseudo-differential operator defined locally via $Du(x) = \int \varphi(\xi) f_m(x,\xi) \mathscr{F}(u) e^{i\langle x,\xi\rangle} d\xi$ will then be an order m and have f_m as its symbol.

3. Proof

In order to have a well-defined analytic index, we need the following fact to be true:

Lemma 3.1. *If D, E are pseudo-differential operators on a compact M and $\sigma(D) = \sigma(E)$, then* $\operatorname{index}(D) = \operatorname{index}(E)$.

Proof. First, suppose D, E have the same degree. If they have the same symbol, the linear homotopy $tD + (1-t)E$ preserves the symbol, so provides a homotopy through elliptic operators between D and E. Since elliptic operators are Fredholm and index is locally constant, $\text{index}(D) = \text{index}(E)$. Now if D, E have symbols that are homotopic as homogeneous complexes, we can build a homotopy of operators as in the proof of surjectivity of σ between two operators with the same symbols as D, E showing again that they have the same index. If D, E have symbols differing by a map that is constant on the unit sphere bundle, it must be the case that $m = 0$ as the map must be an isomorphism of bundles and homogeneous. But then if α is the function on the unit sphere bundle, D, E differ by the operator $Pf = \alpha f$ which has index 0. Finally suppose that D, E have the same restriction to the unit sphere bundle, but have different orders. Then we can consider $\sigma(D)/\sigma(E)$ which is the identity on the unit sphere bundle, so is self-adjoint, and so there is a self adjoint elliptic R with $\sigma(D) = \sigma(E)\sigma(R)$. But R is self-adjoint, so $0 = \text{index}(R) = \text{index}(D) - \text{index}(E)$. □

Thus we can define the analytic index $a_{\text{ind}} : K(TM) \to \mathbb{Z}$ to be $\text{index} \circ \sigma^{-1}$, given by taking an elliptic pseudo-differential operator whose symbol is the element in $K(TM)$ and taking its index.

Now let's start verifying that the axioms of the topological index hold for the analytic index, beginning with proving the theorem for a point. An elliptic operator in this case is just a linear map $f : V \to W$ of finite dimensional vector spaces. The symbol is the element of $K(\cdot)$ associate to the complex formed by this map, which is just $\dim V - \dim W = \text{index}\, V$.

If we have an open inclusion $U \hookrightarrow \mathbb{R}^n$, we would like the natural extension homomorphism $K(TU) \to K(T\mathbb{R}^n)$ to commute with a_{ind}. To see this, an element of $K(TU)$ is given by a homogeneous map $a : E \to F$, which must be homogeneous of degree 0 because U is not compact. After homotopy, we can take a to be the identity outside some compact set (via some trivializations at infinity). Then we can extend the map as the identity on $T\mathbb{R}^n$. If P is an operator representing this extension, $P|U$ represents the original element. If $Pf = 0$, since P is the identity outside of U, f must be supported in U. Thus the kernels of $P, P|U$ have the same dimension, and the same is true for the adjoints, so the indexes agree.

It remains to show that a_{ind} commutes with the Thom isomorphism. Recall that in K-theory the Thom isomorphism $K(M) \to K(V)$ is given by $u \mapsto \pi^*(u) \cdot \lambda_V$, where λ_V is the Thom class in $K(V)$ defined by complex given by the exterior algebra $\Lambda(V)$ where the maps are given by $(v, w) \mapsto (v, v \wedge w)$.

To prove that the analytic index commutes with the Thom isomorphism, we will use a product formula that will now be explained. In particular we would like a statement of the form $a_{\text{ind}}(\pi^* x \cdot \lambda_V) = a_{\text{ind}} x \cdot a_{\text{ind}} \lambda_V = a_{\text{ind}} x$. Let $V \to M$ be a vector bundle with a metric reducing its group to $O(n)$. If $P \to M$ is the associated principal bundle, $P \times_{O(n)} \mathbb{R}^n = V$, and so $TV = P \times_{O(n)} T\mathbb{R}^n \oplus \pi^* TM$ giving a multiplication map $K(TM) \otimes K(P \times_{O(n)} T\mathbb{R}^n) \to K(TV)$. Compose with the map $K_{O(n)}(T\mathbb{R}^n) \to K_{O(n)}(P \times T\mathbb{R}^n) = K(P \times_{O(n)} T\mathbb{R}^n)$ we get the multiplication map we want$K(TM) \otimes K_{O(n)}(T\mathbb{R}^n) \to K(TV)$, where $K_{O(n)}$ is the equivariant K group.

Most of the discussion so far without significant change transfers over to equivariant K theory. In particular, there is an analytic index $a_{\text{ind}}^{O(n)} : K_{O(n)}(T\mathbb{R}^n) \to K_{O(n)}(\cdot)$ defined the same way as usual (the index is now lies in the representation ring of $O(n)$ which is naturally isomorphic to $K_{O(n)}(\cdot)$). We would like to show the that $a_{\text{ind}}(a)a_{\text{ind}}^{O(n)}(b) = a_{\text{ind}}(ab)$ provided that $a_{\text{ind}}^{O(n)}(b) \in K_{O(n)}(\cdot)$ is a multiple of the trivial representation 1. In fact, this holds in a slightly more general setting where $O(n)$ is some Lie group H, P is any principal bundle, $V = P \times_H F$ where we have an H-action on F, the analog of $\mathbb{R}^n$. We have in general a multiplication map $K(TM) \otimes K_H(TF) \to K(TV) = K(P \times_H F)$ that is defined in the same way.

Lemma 3.2. *For the multiplication map above, with $a \in K(TM), b \in K_H(TF)$, we have $a_{ind}(a)a_{ind}^H(b) = a_{ind}(ab)$ provided that $a_{ind}^H(b) \in K_H(\cdot)$ is a multiple of the trivial representation* 1.

Proof. Represent a, b by elliptic pseudo-differential operators A, B of order 1, whose symbol yields the right element of the K group. Now via a partition of unity on a trivial open cover, lift A, B to an pseudo-differential operators $\tilde{A}, \tilde{B}$ on V by having it locally act trivially in the direction of the fibres.

Then the operator D on V given by the matrix $\begin{pmatrix} \tilde{A} & \tilde{B}^* \\ -\tilde{B} & \tilde{A}^* \end{pmatrix}$ has a symbol representing the product ab in $K(TV)$. Now

$$DD^* = \begin{pmatrix} \tilde{A}\tilde{A}^* + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}^*\tilde{A} + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix}$$

$$D^*D = \begin{pmatrix} \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_1 \end{pmatrix}$$

so we have:

$$\begin{aligned} \operatorname{index} D &= \ker(D) - \ker(D^*) = \ker(DD^*) - \ker(D^*D) \\ &= \sum_{i=0,1} (\ker(P_i) - \ker(Q_i)). \end{aligned}$$

Now $\langle P_0 u, u \rangle = \langle \tilde{A}^* u, \tilde{A}^* u \rangle + \langle \tilde{B} u, \tilde{B} u \rangle$, so $\ker(P_0) = \ker(\tilde{A}) \cap \ker(\tilde{B}^*)$, and analogous results hold for P_1, Q_0, Q_1. $\ker(\tilde{B})$ is the sections of the vector bundle $P \times_H \ker(B) = K_B$ since $\tilde{B}$ is a lift of B. Thus A acts on this bundle via an operator C (because the action of A commutes with B), and $\sigma(C) = a[K_B]$ since the action is the tensor product action. Thus, $a_{\text{ind}}(a[K_B]) = \text{index}(C) = \ker(P_1) - \ker(Q_1)$. If we define L_B analogously as $P \times_H \text{coker}(B)$, then putting everything together we have

$$a_{\text{ind}}(ab) = \text{index}\, D = a_{\text{ind}}(a([K_B] - [L_B]))$$

By our assumption, $[K_B] - [L_B]$ is an integer $a^H_{\text{ind}}(b)$, so since a_{ind} is a homomorphism, we get the desired product formula. □

Applying this product formula, it only remains to show that in our case $a^{O(n)}_{\text{ind}}(\lambda_n) = 1$ where λ_n is the Thom class in $K_{O(n)}(T\mathbb{R}^n)$. To do this computation we can reduce to the cases $n = 1, 2$ by observing that a representation of $O(n)$ is determined by its value on all the subgroups obtained by splitting $\mathbb{R}^n$ as a sum $\bigoplus \mathbb{R} \oplus \bigoplus \mathbb{R}^2$, and considering the product $\prod O(1) \times \prod SO(2) \subset O(n)$ acting diagonally. Thus if we show that λ_i is 1 in these cases, the product formula in the lemma above will tell us that λ_n is 1 for these subgroups and hence for $O(n)$. These last two cases $i = 1, 2$ can be worked out by finding explicit homotopies between $h(\lambda_i)$ and the symbol of an operator of index 1 on S^i for $i = 1, 2$ where h is the extension homomorphism $K(T\mathbb{R}^i) \to K(TS^i)$.