

# K-THEORY AND THE INDEX THEOREM

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## 1. INTRODUCTION AND STATEMENT

Given an elliptic operator between (complex) vector bundles  $V, W$  on a closed manifold  $M$ , the Atiyah-Singer index theorem expresses its index in terms of topological invariants. One natural way to formulate and prove this theorem, which will be sketched here, is via K-theory and pseudo-differential operators. The topological index that we construct will factor through K-theory via a symbol map as shown in the diagram below:

$$\begin{array}{ccc} \text{elliptic operators} & \xrightarrow{\sigma} & K(TM) \\ & \searrow \text{index} & \downarrow t_{\text{ind}} \\ & & \mathbb{Z} \end{array}$$

Here  $K(TM)$  is the compactly supported  $K$ -theory defined as the reduced  $K$ -group of the one point compactification. The map  $\sigma$  map is called the symbol map and is obtained from the principal symbol of the elliptic operator. The map  $t_{\text{ind}}$  is called the topological index. Once these maps have been defined, the theorem states:

**Theorem 1.1** (Atiyah-Singer index theorem). *The diagram above commutes.*

A cohomological form of the index theorem can be obtained by applying the Chern character, the ring isomorphism between rational K-theory and rational cohomology (with compact supports). The Todd class enters in the statement as a correction factor since the Thom isomorphisms of  $K$ -theory and cohomology don't exactly commute with the Chern character. A cohomological statement obtained from applying the Chern character is:

**Theorem 1.2.** *If  $D$  is an elliptic operator on a compact  $n$ -manifold  $M$ , then  $\text{index } D = (-1)^n(ch(\sigma(D)) \smile td(TM \otimes \mathbb{C}))[TM]$*

There is a natural way to extend the discussion from elliptic operators to elliptic complexes. Namely, let a complex of partial differential operators be operators  $\Gamma E_0 \rightarrow \Gamma E_1 \rightarrow \cdots \rightarrow \Gamma E_n$  that form a chain complex where  $\Gamma E_i$  are the smooth sections of some vector bundles  $E_i$ . If  $b_i$  is the rank of the  $i^{\text{th}}$  homology group of this complex, then the index is  $\sum_i (-1)^i b_i$ . Associated to this complex is an associated

complex  $0 \rightarrow \pi^* E_0 \rightarrow \pi^* E_1 \rightarrow \cdots \rightarrow \pi^* E_n \rightarrow 0$ , where  $\pi$  is the projection  $TM \rightarrow M$ , given by taking the principle symbol of the operators. The original complex is said to be **elliptic** if this associated complex is exact outside the zero section. In the case that there are only  $E_0, E_1$ , this agrees with the notion of an elliptic operator from  $E_0$  to  $E_1$ .  $\sigma$  will be defined for elliptic complexes, and the index theorem will hence hold for elliptic complexes.

However, this doesn't really add more generality. Given an elliptic complex  $\Gamma E_i \xrightarrow{d} \Gamma E_{i+1}$ , we can consider the map  $\Gamma \oplus_{j=2i} E_j \rightarrow \Gamma \oplus_{j=2i+1} E_j$  given by  $d + d^*$  where  $d^*$  is the adjoint of  $d$  with respect to some hermitian inner product on each  $E_i$ . This map is still an elliptic partial differential operator, has the same index as the complex and will have the same value in  $K(TM)$ . Thus the index theorem only needs to be proved for elliptic operators.

To define our symbol map, we will use the fact that the K group for a locally compact space can be defined the following way. Take the semi-group of finite complexes of vector bundles on  $M$   $0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow 0$  such that set on which the complex is not exact is compact. Identify complexes that are homotopic, meaning there is a complex on  $M \times [0, 1]$  restricting to each at the end, which also is not exact with compact support. Then taking the quotient by the complexes that are exact, the result is  $K(M)$ . The product structure in  $K(M)$  is given by the tensor product of complexes. In fact, we can fix any length and consider only complexes of that length, and the end result will still be  $K(M)$ . The construction in the last paragraph using a hermitian inner product to turn complexes of arbitrary length into complexes of length 1 preserves the element of  $K(M)$ . The associated symbol complex of an elliptic complex  $\Gamma E^*$  on  $M$  is only not exact at the 0-section which is compact, so it defines an element  $\sigma(\Gamma E^*) \in K(TM)$ .

Finally we need to define  $t_{\text{ind}}$  for any compact manifold  $M$ . It will be characterized by three properties. If  $\tilde{M}$  is the one point compactification, and  $Y \hookrightarrow M$  is an open inclusion, there is a natural quotient map  $\tilde{M} \rightarrow \tilde{Y}$  which induces an extension map  $h : K(Y) \rightarrow K(M)$ . The first axiom is excision, namely that if we have an open inclusion  $U \rightarrow \mathbb{R}^n$ ,  $t_{\text{ind}}$  commutes with the extension of  $TU \rightarrow T\mathbb{R}^n$ .

The second axiom is naturality with respect to the inclusion of a compact  $M$  as the 0-section of a vector bundle  $\pi : E \rightarrow M$ .  $TE = \pi^* TM \oplus \pi^* E$ , and so  $TE \rightarrow TM$  is isomorphic to the bundle  $\pi^* E \oplus \pi^* E$ , and can be given a complex structure via  $\pi^* E \oplus i\pi^* E$ . Now via the Thom isomorphism, we have a map from  $K(TM) \rightarrow K(TE)$ .  $t_{\text{ind}}$  commutes with this map.

Finally,  $t_{\text{ind}}$  must be normalized for a point as the usual identification of  $K(\cdot) \cong \mathbb{Z}$  via the rank.

We can construct  $t_{\text{ind}}$  for any manifold  $M$  by taking an embedding into  $\mathbb{R}^n$ , using the inclusion into the normal bundle using the Thom isomorphism axiom, extending

to all of  $\mathbb{R}^n$  via the extension axiom, and then viewing  $\mathbb{R}^n$  as the normal bundle of a point using both axioms again to get an element of  $K(\cdot)$ . This does not depend on the inclusion  $M \rightarrow \mathbb{R}^n$ , because if  $i_1, i_2$  are two inclusions,  $i_1 \oplus i_2$  is isotopic to  $i_1 \oplus 0$  and  $0 \oplus i_2$ , and the Thom isomorphism is well-behaved with respect to stabilization of bundles. Moreover, it is easy to see that anything satisfying the axioms must be equal to  $t_{\text{ind}}$  because the definition can be obtained from the axioms.

## 2. STRATEGY

These axioms will guide the proof of the theorem. We would like to make a map called the **analytic index** that takes an element of  $K(TM)$  to “index  $\circ \sigma^{-1}$ ”. Then if we could show it satisfies the same axioms as the topological index, the proof would be complete. Unfortunately, not every element of  $K(TM)$  comes from the symbol of an elliptic operator, and it’s not apriori clear that two elliptic operators with the same image via  $\sigma$  will have the same index. The failure of surjectivity can be seen for example in the case of odd-dimensional manifolds.

If  $n$  is odd, we can apply the antipode map  $a$  on  $TM$  to  $\sigma(D)$ .  $\sigma(D)$  is given by the principal symbol, which is a homogeneous polynomial  $p(\xi)$  on each fibre of  $M$ . This means that  $p(-\xi)$  is  $\pm p(\xi)$ . In either case,  $a^*\sigma(D) = \sigma(D)$  because the negative map is homotopic to the identity by rotating around the circle. However, if we look at  $\sigma(D)$  in rational K-theory, which can be identified with compactly supported rational cohomology,  $TM$ ’s orientation is reversed as  $n$  is odd, so  $a^*\sigma(D) = -\sigma(D)$ . Thus  $\sigma(D)$  is 0 in rational K-theory, so must be torsion. in particular, since  $t_{\text{ind}}$  is a homomorphism to  $\mathbb{Z}$ , by the index theorem  $\text{index}(D) = 0$ . Then  $\sigma$  will not be surjective.

To fix the problem, we will consider pseudo-differential operators instead of partial differential operators. Given a manifold  $M$ , a **pseudo-differential operator**  $D$  of order  $m$  is an operator of the following form from compactly supported smooth sections of a bundle  $E$  to smooth sections of a bundle  $F$  with the following properties. Consider any chart  $\mathbb{R}^n$  on which  $E$  and  $F$  can be trivialized to be trivial bundles of ranks  $l, k$ . For any compactly supported function  $f$  and any smooth section  $u$  of  $E$  on the chart, we must have  $D(fu)(x) = \int p_f(x, \xi) \mathcal{F}(u) e^{i\langle x, \xi \rangle} d\xi$  where  $\mathcal{F}$  is the Fourier transform, and  $p_f$  is some matrix where each partial derivative of  $p_f$  is  $O(\xi^{m-|a|})$  where  $|a|$  is the number of times the partial derivative is taken in the  $\xi$  direction. Moreover, we should like the limit  $\sigma_f(x, \xi) = \lim_{\lambda \rightarrow \infty} \frac{p_f(x, \lambda \xi)}{\lambda^m}$  to exist. When  $f$  is compactly supported but 1 in a neighborhood of  $x$ ,  $\sigma_f(x, \xi)$  doesn’t depend on  $f$  and is defined to be  $\sigma(x, \xi)$  which is called the **principal symbol**. The principal symbol similarly defines an associated complex of bundles on  $TM$ , and the operator is elliptic if this is exact outside the 0 section. This definition comes from generalizing the definition of a partial differential operator, which acts on the Fourier transform

via multiplication by a polynomial. Instead, a pseudo-differential operator acts on the Fourier transform via any function with similar growth behavior.

Elliptic pseudo-differential operators, like elliptic operators, are Fredholm. An example of an elliptic pseudo-differential operator is the operator  $A_{S^1}$  on the trivial line bundle of  $S^1$  sending  $e^{ik\theta}$  to itself when  $k$  is negative and to  $e^{(i+1)k\theta}$  when  $k$  is nonnegative. The symbol is  $e^{i\theta}$  when  $\xi$  is positive and 1 when it is negative. It has index  $-1$  which we saw is impossible for an elliptic partial differential operator.

Any element of  $K(TM)$  comes from an elliptic pseudo-differential operator. To see this, an element of  $K(TM)$  can be obtained from some bundle map  $f : E \rightarrow F$  which is an isomorphism away from some compact set  $L$ . Choose a metric for which the unit disk bundle of  $TM/M$  contains  $L$ . The composite  $TM \xrightarrow{\pi} M \xrightarrow{0} TM$  is homotopic to the identity, so if  $E_0, F_0$  are the pullbacks of  $E, F$  along this map, they are isomorphic to  $E, F$ . We can produce a map  $f_m : E_0 \rightarrow F_0$  by extending the map from  $E$  to  $F$  as a homogeneous function of degree  $m$ , meaning, that  $f_m(\lambda v) = \lambda^m f_m(v), v \in TM$ . It is easy to see that  $f_m$  is homotopic to  $f$ . Moreover, if  $f$  is an isomorphism it can be made to be constant, so that the corresponding map  $f_m$  will be constant on the unit sphere bundle. Conversely if the  $f_m$  is constant on the unit sphere bundle, then the homotopy  $f_m(v, t) = \|v\|^{tm} f_m(\frac{v}{\|v\|})$  will take it to an isomorphism. Thus we have obtained the following lemma:

**Lemma 2.1.**  *$K(TM)$  can be taken to be the monoid of homotopy classes of homogeneous degree  $m$  maps of bundles on  $TM$   $E \rightarrow F$  modulo those which are constant on the unit sphere bundle.*

Note that these homogeneous maps may not be continuous at the origin for  $m \leq 0$ . Thus given an element of  $K(TM)$  represent it via some homogeneous map  $f_m : E \rightarrow F$ , choose coordinates, a trivialization of  $TX, E, F$ , and a smooth function  $\varphi$  that is 0 in a neighborhood of the zero section of  $TM$  (where the map may not be continuous) and 1 far away from this section. The pseudo-differential operator defined locally via  $Du(x) = \int \varphi(\xi) f_m(x, \xi) \mathcal{F}(u) e^{i\langle x, \xi \rangle} d\xi$  will then be an order  $m$  and have  $f_m$  as its symbol.

### 3. PROOF

In order to have a well-defined analytic index, we need the following fact to be true:

**Lemma 3.1.** *If  $D, E$  are pseudo-differential operators on a compact  $M$  and  $\sigma(D) = \sigma(E)$ , then  $\text{index}(D) = \text{index}(E)$ .*

*Proof.* First, suppose  $D, E$  have the same degree. If they have the same symbol, the linear homotopy  $tD + (1 - t)E$  preserves the symbol, so provides a homotopy

through elliptic operators between  $D$  and  $E$ . Since elliptic operators are Fredholm and index is locally constant,  $\text{index}(D) = \text{index}(E)$ . Now if  $D, E$  have symbols that are homotopic as homogeneous complexes, we can build a homotopy of operators as in the proof of surjectivity of  $\sigma$  between two operators with the same symbols as  $D, E$  showing again that they have the same index. If  $D, E$  have symbols differing by a map that is constant on the unit sphere bundle, it must be the case that  $m = 0$  as the map must be an isomorphism of bundles and homogeneous. But then if  $\alpha$  is the function on the unit sphere bundle,  $D, E$  differ by the operator  $Pf = \alpha f$  which has index 0. Finally suppose that  $D, E$  have the same restriction to the unit sphere bundle, but have different orders. Then we can consider  $\sigma(D)/\sigma(E)$  which is the identity on the unit sphere bundle, so is self-adjoint, and so there is a self adjoint elliptic  $R$  with  $\sigma(D) = \sigma(E)\sigma(R)$ . But  $R$  is self-adjoint, so  $0 = \text{index}(R) = \text{index}(D) - \text{index}(E)$ .  $\square$

Thus we can define the analytic index  $a_{\text{ind}} : K(TM) \rightarrow \mathbb{Z}$  to be  $\text{index} \circ \sigma^{-1}$ , given by taking an elliptic pseudo-differential operator whose symbol is the element in  $K(TM)$  and taking its index.

Now let's start verifying that the axioms of the topological index hold for the analytic index, beginning with proving the theorem for a point. An elliptic operator in this case is just a linear map  $f : V \rightarrow W$  of finite dimensional vector spaces. The symbol is the element of  $K(\cdot)$  associate to the complex formed by this map, which is just  $\dim V - \dim W = \text{index } V$ .

If we have an open inclusion  $U \hookrightarrow \mathbb{R}^n$ , we would like the natural extension homomorphism  $K(TU) \rightarrow K(T\mathbb{R}^n)$  to commute with  $a_{\text{ind}}$ . To see this, an element of  $K(TU)$  is given by a homogeneous map  $a : E \rightarrow F$ , which must be homogeneous of degree 0 because  $U$  is not compact. After homotopy, we can take  $a$  to be the identity outside some compact set (via some trivializations at infinity). Then we can extend the map as the identity on  $T\mathbb{R}^n$ . If  $P$  is an operator representing this extension,  $P|_U$  represents the original element. If  $Pf = 0$ , since  $P$  is the identity outside of  $U$ ,  $f$  must be supported in  $U$ . Thus the kernels of  $P, P|_U$  have the same dimension, and the same is true for the adjoints, so the indexes agree.

It remains to show that  $a_{\text{ind}}$  commutes with the Thom isomorphism. Recall that in K-theory the Thom isomorphism  $K(M) \rightarrow K(V)$  is given by  $u \mapsto \pi^*(u) \cdot \lambda_V$ , where  $\lambda_V$  is the Thom class in  $K(V)$  defined by complex given by the exterior algebra  $\Lambda(V)$  where the maps are given by  $(v, w) \mapsto (v, v \wedge w)$ .

To prove that the analytic index commutes with the Thom isomorphism, we will use a product formula that will now be explained. In particular we would like a statement of the form  $a_{\text{ind}}(\pi^*x \cdot \lambda_V) = a_{\text{ind}}x \cdot a_{\text{ind}}\lambda_V = a_{\text{ind}}x$ . Let  $V \rightarrow M$  be a vector bundle with a metric reducing its group to  $O(n)$ . If  $P \rightarrow M$  is the associated principal bundle,  $P \times_{O(n)} \mathbb{R}^n = V$ , and so  $TV = P \times_{O(n)} T\mathbb{R}^n \oplus \pi^*TM$  giving a

multiplication map  $K(TM) \otimes K(P \times_{O(n)} T\mathbb{R}^n) \rightarrow K(TV)$ . Compose with the map  $K_{O(n)}(T\mathbb{R}^n) \rightarrow K_{O(n)}(P \times T\mathbb{R}^n) = K(P \times_{O(n)} T\mathbb{R}^n)$  we get the multiplication map we want  $K(TM) \otimes K_{O(n)}(T\mathbb{R}^n) \rightarrow K(TV)$ , where  $K_{O(n)}$  is the equivariant K group.

Most of the discussion so far without significant change transfers over to equivariant K theory. In particular, there is an analytic index  $a_{\text{ind}}^{O(n)} : K_{O(n)}(T\mathbb{R}^n) \rightarrow K_{O(n)}(\cdot)$  defined the same way as usual (the index is now lies in the representation ring of  $O(n)$  which is naturally isomorphic to  $K_{O(n)}(\cdot)$ ). We would like to show the that  $a_{\text{ind}}(a)a_{\text{ind}}^{O(n)}(b) = a_{\text{ind}}(ab)$  provided that  $a_{\text{ind}}^{O(n)}(b) \in K_{O(n)}(\cdot)$  is a multiple of the trivial representation 1. In fact, this holds in a slightly more general setting where  $O(n)$  is some Lie group  $H$ ,  $P$  is any principal bundle,  $V = P \times_H F$  where we have an  $H$ -action on  $F$ , the analog of  $\mathbb{R}^n$ . We have in general a multiplication map  $K(TM) \otimes K_H(TF) \rightarrow K(TV) = K(P \times_H F)$  that is defined in the same way.

**Lemma 3.2.** *For the multiplication map above, with  $a \in K(TM)$ ,  $b \in K_H(TF)$ , we have  $a_{\text{ind}}(a)a_{\text{ind}}^H(b) = a_{\text{ind}}(ab)$  provided that  $a_{\text{ind}}^H(b) \in K_H(\cdot)$  is a multiple of the trivial representation 1.*

*Proof.* Represent  $a, b$  by elliptic pseudo-differential operators  $A, B$  of order 1, whose symbol yields the right element of the  $K$  group. Now via a partition of unity on a trivial open cover, lift  $A, B$  to an pseudo-differential operators  $\tilde{A}, \tilde{B}$  on  $V$  by having it locally act trivially in the direction of the fibres.

Then the operator  $D$  on  $V$  given by the matrix  $\begin{pmatrix} \tilde{A} & \tilde{B}^* \\ -\tilde{B} & \tilde{A}^* \end{pmatrix}$  has a symbol representing the product  $ab$  in  $K(TV)$ . Now  $DD^* = \begin{pmatrix} \tilde{A}\tilde{A}^* + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}^*\tilde{A} + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix}$ ,  $D^*D = \begin{pmatrix} \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_1 \end{pmatrix}$ , so we have:

$$\text{index } D = \ker(D) - \ker(D^*) = \ker(DD^*) - \ker(D^*D) = \sum_{i=0,1} (\ker(P_i) - \ker(Q_i)).$$

Now  $\langle P_0 u, u \rangle = \langle \tilde{A}^* u, \tilde{A}^* u \rangle + \langle \tilde{B} u, \tilde{B} u \rangle$ , so  $\ker(P_0) = \ker(\tilde{A}) \cap \ker(\tilde{B}^*)$ , and analogous results hold for  $P_1, Q_0, Q_1$ .  $\ker(\tilde{B})$  is the sections of the vector bundle  $P \times_H \ker(B) = K_B$  since  $\tilde{B}$  is a lift of  $B$ . Thus  $A$  acts on this bundle via an operator  $C$  (because the action of  $A$  commutes with  $B$ ), and  $\sigma(C) = a[K_B]$  since the action is the tensor product action. Thus,  $a_{\text{ind}}(a[K_B]) = \text{index}(C) = \ker(P_1) - \ker(Q_1)$ . If we define  $L_B$  analogously as  $P \times_H \text{coker}(B)$ , then putting everything together we have

$$a_{\text{ind}}(ab) = \text{index } D = a_{\text{ind}}(a([K_B] - [L_B]))$$

By our assumption,  $[K_B] - [L_B]$  is an integer  $a_{\text{ind}}^H(b)$ , so since  $a_{\text{ind}}$  is a homomorphism, we get the desired product formula.  $\square$

Applying this product formula, it only remains to show that in our case  $a_{\text{ind}}^{O(n)}(\lambda_n) = 1$  where  $\lambda_n$  is the Thom class in  $K_{O(n)}(T\mathbb{R}^n)$ . To do this computation we can reduce to the cases  $n = 1, 2$  by observing that a representation of  $O(n)$  is determined by its value on all the subgroups obtained by splitting  $\mathbb{R}^n$  as a sum  $\bigoplus \mathbb{R} \oplus \bigoplus \mathbb{R}^2$ , and considering the product  $\prod O(1) \times \prod SO(2) \subset O(n)$  acting diagonally. Thus if we show that  $\lambda_i$  is 1 in these cases, the product formula in the lemma above will tell us that  $\lambda_n$  is 1 for these subgroups and hence for  $O(n)$ . These last two cases  $i = 1, 2$  can be worked out by finding explicit homotopies between  $h(\lambda_i)$  and the symbol of an operator of index 1 on  $S^i$  for  $i = 1, 2$  where  $h$  is the extension homomorphism  $K(T\mathbb{R}^i) \rightarrow K(TS^i)$ .