FILTERED AND SIFTED

ISHAN LEVY

A functor $f : C \to D$ is **final** if the natural map $\operatorname{colim} g \circ f \to \operatorname{colim} g$ is an isomorphism for any functor for which it exists.

Lemma 0.1. *f is final iff for any $d \in D$, the category d/f is nonempty and connected. Moreover, to test finality it suffices to check on corepresentable functors.*

Proof. We will show that the category of cones from D-shaped diagrams is equivalent to those from C-shaped diagrams via the natural restriction iff f is final.

Suppose that d/f is nonempty and connected. We can construct an inverse as follows. Given a cone from C to x, its extension to D will be defined on $d \in D$ by choosing some map to an element of c and then composeing with the map to x. Connectedness of d/f shows that this is well defined, functorial, and it is easily checked to be an inverse.

Now suppose that f is final. Consider the functor $D \to \mathrm{Set}$ given by $\mathrm{Hom}(d, -)$. Its colimit is the set of connected components of d/D, which is a singleton. Its composite with f is the functor associated to the category of elements of d/f. The colimit of this is the connected components of d/f so we see it is connected and not empty. $\square$

As an example, for the poset of natural numbers $\mathbb{N}$, the inclusion of any infinite subset is final.

Definition 0.2. *A category J is **filtered** iff for any $a, b \in J$, there is a map to a common c, and for any maps $a \to b, c$ there are maps $b, c \to d$ so that the two composites $a \to d$ agree.*

The natural numbers as a poset is filtered.

Definition 0.3. *A category J is **sifted** if for any $a, b \in J$, the category under a, b is connected.*

Any filtered category is sifted, Δ^{op} is sifted, and the category consisting of two maps $a \to b$ with a common section is sifted (colimits of the last one are called reflexive coequalizers).

Lemma 0.4. *J is sifted iff the inclusions $\Delta : J \to J^n$ are final for all n. It suffices to take $n = 2$.*

Proof. Since composites and products of final functors are final, it suffices to think about $n = 2$. Then via the characterization of final functors, we are done. □

Lemma 0.5. *J is filtered iff the inclusions $\Delta : J \to J^G$ are final for all finite diagrams G. It suffices to take $G = \cdot \leftarrow \cdot \to \cdot$ and ϕ or $G = \cdot\cdot$ and $\cdot \rightrightarrows \cdot$*

Proof. By induction, for any finite family of arrows f_i in J, there are mapsg_i so that the composites $g_i \circ f_i$ only depend on the codomain. Applying this to all the arrows in some $F \in J^G$, we see that F/Δ is nonempty. Applying this to the arrows in any two natural transformation to a constant functor shows that F/Δ is connected.

Conversely, if Δ is final for $\cdot\cdot$ and $\cdot \rightrightarrows \cdot$, one sees from the first J is sifted, implying the first condition of filteredness. The second condition is implied since given $a \to b, c$, we can first find a map from b, c to a common object so WLOG $b = c$. Then the condition on $\cdot \rightrightarrows \cdot$ gives the second property of filteredness. Alternatively, if the condition is satisfied for ϕ and $\cdot \leftarrow \cdot \to \cdot$, the first condition is satisfied since ϕ shows that J is connected. We can induct on the 'distance' between two objects using the second graph to show that the first condition is satisfied. The second condition follows immediately from the second graph. □

Lemma 0.6. *J is sifted iff J-shaped colimits commute with finite products in* Set.

Proof. The comparison map for the commutativity of finite products and colimits factors as

$$\mathrm{colim}_J \lim_I D \to \mathrm{colim}_{J^I} \lim_I D' \to \lim_I \mathrm{colim}_J D$$

Where the first map is induced by the diagonal functor, and the second is the comparison map for distributivity of finite products. The second map is always an isomorphism in Set, and the first is always an isomorphism iff J is sifted. □

Lemma 0.7. *J is filtered iff J-shaped colimits commute with finite limits in* Set.

Proof. By the results on sifted colimits, it suffices to show that for a sifted category, commutativity with an empty limit and pullbacks is equivalent to being filtered. First suppose the commutativity is satisfied. The commutativity with the empty limit shows that the undercategory for $a \in J$ is connected. Given a span $c \leftarrow a \to b$, we can consider the corresponding diagram of corepresentable functors, $J \times (\cdot \to \cdot \leftarrow \cdot) \to \mathrm{Set}$. The commutativity here says that the pullback of the connected components of the under categories of a, b, c is the same as the connected component of the undercategory for the diagram $c \leftarrow a \to b$. Now we can induct on the distance between two objects in the undercategory of a to show that the under category of the span is nonempty.

Conversely, since the undercategory of a is connected, J commutes with the empty limit. It is easy to see that equalizers (finite limits) distribute over filtered colimits in Set, so that combined with the fact that $J \to J^I$ is final when J is filtered and I finite gives that filtered limits commute with finite limits. $\square$

There should be more conceptual proofs of these things...

The Yoneda embedding is the universal cocompletion with respect to colimits.

Ind is the universal cocompletion with respect to filtered colimits.

Sind is the universal cocompletion with respect to sifted colimits.

Many large categories are Ind of their compact objects

Categories of models of algebraic theories are Sind of the opposite of the Lawvere category.