LIE ALGEBRAS

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All Lie algebras are finite dimensional over a field k.

1. Nilpotent and Solvable Lie Algebras

Lemma 1.1. *Every solvable Lie algebra contains an ideal of codimension* 1.

Proof. Since it is solvable, it has a proper abelian quotient, which has a quotient of dimension 1. □

Lemma 1.2. *If A is nilpotent on a vector space V, it has a nontrivial kernel, and* $\operatorname{ad} A$ *is nilpotent in* $\mathfrak{gl}_V$.

Proof. The first claim is clear, and the second follows since $\operatorname{ad} A = L_A - R_A$, which are commuting nilpotent operators on $\mathfrak{gl}_V$. □

Theorem 1.3 (Engel's Theorem)**.** *If V is a representation of a Lie algebra $\mathfrak{g}$ by nilpotent matricies, then there is a v such that $\mathfrak{g}v = 0$.*

Proof. Let $\mathfrak{h}$ be a codimension 1 ideal. By induction, there is a nontrivial subspace such that $\mathfrak{h}$ acts trivially. Since $\mathfrak{h}$ contains $[\mathfrak{g}, \mathfrak{g}]$, $\ker \mathfrak{h}$ is an invariant subspace since $hgv = [h, g]v + ghv = 0$ if $h \in \mathfrak{h}, g \in \mathfrak{g}, v \in \ker \mathfrak{h}$. Any $a \notin h$ has an element it kills $v \in \ker \mathfrak{h}$ since it acts nilpotently. Since $ka + \mathfrak{h} = \mathfrak{g}$, we see $\mathfrak{g}v = 0$. □

Corollary 1.4. *After a change of basis, a subalgebra of $\mathfrak{gl}_n$ consisting of nilpotent matricies is an algebra of strictly upper triangular matricies.*

Proof. Apply Engel's theorem to inductivly construct a flag such that the action decreases filtration. □

Corollary 1.5. *A Lie algebra is nilpotent iff for every $a \in \mathfrak{g}$,* $\operatorname{ad} a$ *is nilpotent.*

Proof. Since the lower central series eventually vanishes, $\operatorname{ad} a$ is nilpotent for a nilpotent Lie algebra. Conversely, by Engel's theorem, if $\operatorname{ad} a$ is nilpotent, then $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, so $\mathfrak{g}$ is too. □

We can classify 2-step nilpotent Lie algebras $\mathfrak{g}$. They are completely determined by the bracket $\mathfrak{g}/Z(\mathfrak{g}) \to Z(\mathfrak{g})$. This is an alternating form, and is nondegenerate because $Z(\mathfrak{g})$ is the center.

Proposition 1.6. *2-step nilpotent Lie algebras correspond to vector spaces V, W with a nondegenerate alternating pairing $V \otimes V \to W$.*

The idea in Engel's Theorem can be upgraded to nonzero weights when the characteristic is sufficiently large.

Lemma 1.7 (Lie's Lemma)**.** *If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, $V_\lambda^{\mathfrak{h}}$ a weight space of $\mathfrak{h}$ in a f.d representation V of $\mathfrak{g}$, with $\operatorname{char} k > V$, then it is $\mathfrak{g}$-invariant.*

Proof. For $v \in V_\lambda^{mh}, g \in \mathfrak{g}, h \in \mathfrak{h}$ we need to show that $hgv = \lambda(h)gv$. But it is $[h, g]v + ghv = [h, g]v + \lambda(h)gv$, so it is equivalent to show $[h, g]v = 0$.

Define W_m as the span of $g^i v$ for $i \leq m$. By induction, we can see that W_m is $\mathfrak{h}$-invariant and $hW_m/W_{m-1} = \lambda(h)W_m/W_{m-1}$, since $hg^i v = \sum g^j[h, g]g^{i-1-j}v + g^i hv$.

For sufficiently large m, W_m stabilizes, so is g-invariant. Then $\operatorname{tr}([g, h]) = 0$, but it is upper triangular with eigenvalue $\lambda([g, h])$. By our assumption on characteristic, we are done. ☐

Now we can copy the proof of Engel's theorem to get a result about solvable Lie algebras.

Theorem 1.8 (Lie's Theorem)**.** *If V is a finite dimensional representation over $\mathfrak{g}$, a solvable Lie algebra, and $\operatorname{char} k > \dim V$ and k is algebraically closed, then there is a nonzero weight space.*

Proof. By induction, a codimension 1 ideal of $\mathfrak{g}$ has a nonzero weight space W. By Lie's lemma, it is an invariant subspace. Any $a \notin h$ has an eigenvector v in W since k is algebraically closed. Since $ka + \mathfrak{h} = \mathfrak{g}$, we see $\mathfrak{g}$ acts by scalars on v. ☐

Corollary 1.9. *After a change of basis, a solvable subalgebra of $\mathfrak{gl}_n$ over an algebraically closed field $k, \operatorname{char} k > n$ is an algebra of upper triangular matricies.*

Proof. Apply Lie's theorem to inductivly construct a flag such that the action preserves filtration. ☐

Corollary 1.10. *If $\operatorname{char} k > \dim \mathfrak{g}/Z(\mathfrak{g})$ and $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. By extending scalars, we can assume k is algebraically closed. Then $\operatorname{ad} \mathfrak{g}$ acts by upper triangular matricies by Lie's theorem, so $\operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ acts by strictly upper triangular matricies, so by Engel's theorem, we are done. ☐

Now we will study generalized weight space decompositions.

Let k be algebraically closed. Via the Jordan decomposition any operator A on a finite dimensional vector space V decomposes as $A_s + A_n$ where A_s is semisimple, A_n is nilpotent, and they commute.

Lemma 1.11. *If A, B commute, they preserve eachother's generalized eigenspaces, and their Jordan decompositions commute with eachother.*

Proof. For the first statement, the support of the $k[A, B]$-module V is a finite set of points so it splits as a sum of modules supported at a point, which corresponds to finding simultaneous generalized eigenspaces of A, B. The semi-simple parts act diagonally on each component of the sum, so A, A_s commute with B, B_s, so we are done. □

Lemma 1.12. *The Jordan decomposition is the unique decomposition into a commuting semisimple and nilpotent endomorphism.*

Proof. Suppose $A = A_s + A_n = A'_s + A'_n$, where the former is the usual decomposition. Then since A'_s commutes with A, it commutes with A_s, and hence A_n. Since $A'_n = A - A'_s$, it also commutes with everything else. Then $A_s - A'_s = A_n - A'_n$ is a semisimple and nilpotent so it is 0. □

Corollary 1.13. *In* $\mathfrak{gl}_V$*, if* $x = x_s + x_n$ *is a Jordan decomposition, then* $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ *is the Jordan decomposition.*

Proof. Observe that $\mathrm{ad}(x_s)$ is semisimple and ad_n is nilpotent and use the characterization. □

We would like a theory of generalized eigenspaces that works for representations of nilpotent Lie algebra, rather than just for endomorphisms.

Lemma 1.14. *In an associative unital algebra* U *over* k*, if* $a, b \in U$ *and* $\lambda, \alpha \in k$*, then*

$$(a - \alpha - \lambda)^N b = \sum_0^N \binom{N}{j} (\mathrm{ad}\, a - \alpha)^j b (a - \lambda)^{N-j}$$

Proof. $L_a - \alpha - \lambda = (\mathrm{ad}\, a - \alpha) + (R_a - \lambda)$ where L, R denote right and left multiplication. Since these summands commute, the lemma follows from the binomial theorem. □

To avoid writing a more precise assumption, from now on assume $\mathrm{char}\, k = 0$ and k is algebraically closed. Some statements will hold without the algebraically closed assumption via extension of scalars.

Theorem 1.15. *Let* $\mathfrak{h}$ *be a nilpotent subalgebra of* $\mathfrak{g}$ *and* V *be a f.d representation of* $\mathfrak{g}$*. Then* V *decompose into generalized weight spaces* $V^{\mathfrak{h}}_\lambda$ *and* $\mathfrak{g}^{\mathfrak{h}}_\alpha V^{\mathfrak{h}}_\lambda \subset V^{\mathfrak{h}}_{\lambda+\alpha}$.

Proof. The very last statement is obtained by having both sides of the equation in the previous lemma act on $v \in V$, with U the universal enveloping algebra $U(\mathfrak{g})$, $a \in \mathfrak{h}, b \in \mathfrak{g}$.

To obtain the decomposition, first note that for a 1-dimensional Lie algebra it is just the generalized eigenspace decomposition. If there is some $a \in \mathfrak{h}$ with more than one eigenvalue, first note that $\mathfrak{h} = \mathfrak{h}^a_0$ because a acts nilpotently. Then by the last statement of the theorem, $\mathfrak{h}$ preserves the generalized eigenspaces of a, so by induction on $\dim V$ we are done.

In the case that everything has one eigenvalue, Lie's theorem shows that $\mathfrak{h}$ acts by upper triangular matricies where the diagonals are constant, so it is a weight space. □

For a Cartan subalgebra, we drop the superscript $\mathfrak{h}$ in the notation.

2. Semisimple Lie Algebras

Given a representation V of a Lie Algebra $\mathfrak{g}$, the **trace form** is $(a,b)_V = \operatorname{tr}_V ab$. One easily checks it is symmetric $(a,b) = (b,a)$ and invariant $([a,b],c) = (a,[b,c])$. The trace form on the adjoint representation is called the **Killing form**, denoted κ.

Lemma 2.1. *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\kappa_{\mathfrak{g}}|_{\mathfrak{a}} = \kappa_{\mathfrak{a}}$.*

Proof. $\operatorname{ad} b \operatorname{ad} a, a \in \mathfrak{a}$ has image in $\mathfrak{a}$, so its trace is equal to its trace on $\mathfrak{a}$. □

Weights are related to roots by rational multiples on certain parts of the Cartan subalgebra.

Lemma 2.2 (Cartan's Lemma)**.** *Let $e \in \mathfrak{g}_a, f \in \mathfrak{g}_{-a}, [e,f] = h$, and suppose that $V_\lambda \neq 0$. Then $\lambda(h) = r\alpha(h)$, where $r \in \mathbb{Q}$ doesn't depend on h.*

Proof. Consider $W = \oplus_n V_{\lambda+n\alpha}$, upon which e, f, h act. h's trace is 0 on this as it is a commutator. The trace is $\sum_n (\lambda + n\alpha)(h) \dim V_{\lambda+n\alpha} = 0$, giving the relation. □

Theorem 2.3 (Cartan's Criterion)**.** *For a subalgebra of $\mathfrak{gl}_V$, the following are equivalent:*

(1) $(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}])_V = 0$
(2) $([\mathfrak{g},\mathfrak{g}], [\mathfrak{g},\mathfrak{g}])_V = 0$
(3) $\mathfrak{g}$ *is solvable.*

Proof. (1) $\implies$ (2) is clear. (3) $\implies$ (1) follows from Lie's theorem, since the product of two elements of that form is strictly uppertriangular.

(2) $\implies$ (3): If not, then there is a point in the derived series giving a subalgebra $\mathfrak{g}'$ such that $[\mathfrak{g}',\mathfrak{g}'] = \mathfrak{g}'$. If $\mathfrak{h}$ is its Cartan subalgebra, then $\mathfrak{h} = \sum_i [\mathfrak{g}'_\alpha, \mathfrak{g}'_{-\alpha}]$. Choose an $h = [e,f]$, and observe $0 = (h,h)_V = \operatorname{tr} h^2 = \sum_\lambda \lambda(h)^2 \dim V_\lambda$. By the previous lemma, this is a nonnegative rational multiple of $\alpha(h)$, so we must have $\lambda(h) = 0$ and $V = V_0$. But then everything acts nilpotently, so by Engel's theorem $\mathfrak{g}'$ is nilpotent, a contradiction. □

Corollary 2.4. $\mathfrak{g}$ *is solvable iff* $\kappa(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$.

Definition 2.5. *The radical of* $\mathfrak{g}$, $\operatorname{rad}(\mathfrak{g})$ *is the largest solvable ideal.*

Definition 2.6. $\mathfrak{g}$ *is semisimple if* $\mathrm{rad}(\mathfrak{g}) = 0$.

Equivalently, there are no nontrivial abelian ideals. Note that a simple Lie algebra is clearly semisimple.

Corollary 2.7. *If* $\mathfrak{g}$ *is semisimple the trace form of any or every faithful representation* V *is nondegenerate.*

Proof. By Cartan's criterion, the nullspace of $(a, b)_V$ is a solvable ideal, so is 0. □

Corollary 2.8. $\mathfrak{g}$ *is semisimple iff its Killing form is nondegenerate.*

Proof. If $\mathfrak{g}$ is not semisimple, it has a nontrivial abelian ideal, which is in the kernel of κ. □

Proposition 2.9. *The Killing form of a semisimple Lie algebra* $\mathfrak{g}$ *is nondegenerate when restricted to any ideal* $\mathfrak{a}$*. A Lie algebra splits as a sum of simple Lie algebras.*

Proof. $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is a solvable ideal by Cartan's criterion, so is 0. Thus $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ as a Lie algebra. Iterating gives the last assertion. □

Example 2.9.1. $\mathfrak{gl}_n$ is $kI \oplus \mathfrak{sl}_n$ where $\mathfrak{sl}_n$ is semisimple.

Proof. Let E_{ij} be the matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We can compute $\mathrm{ad}(E_{ij})E_{kl} = E_{il}\delta_{jk} - E_{kj}\delta_{il}$. Thus $\mathrm{ad}(E_{mn})\,\mathrm{ad}(E_{ij})E_{kl} = E_{ml}\delta_{jk}\delta_{ni} - E_{mj}\delta_{il}\delta_{nk} - E_{in}\delta_{jk}\delta_{lm} + E_{kn}\delta_{il}\delta_{jm}$, so taking trace, we get $2n\delta_{ni}\delta_{mj} - 2\delta_{mn}\delta_{ij}$. By linearity, the Killing form is $(a, b) = 2n \operatorname{tr} ab - 2 \operatorname{tr} a \operatorname{tr} b$. From this we can see that it is nondegenerate on $\mathfrak{sl}_n$, and null on kI. □

Theorem 2.10. *Given a faithful irreducible representation of* $\mathfrak{g}$ *into* V *either* $\mathfrak{g}$ *is semisimple, or it is* kI *summed with something semisimple.*

Proof. If it isn't semisimple, by Lie's theorem, the radical has an nonzero weight space, which is invariant by Lie's lemma. By irreducibility, it must be all of V. Since the representation is faithful, the radical must be kI. □

Definition 2.11. *A Lie algebra is* ***reductive*** *if its adjoint representation is completely reducible.*

Lemma 2.12. *A Lie algebra is reductive iff it is a sum of a semisimple Lie algebra and an abelian one.*

Proof. The irreducible summands of a reductive Lie algebra are ideals, so the Lie algebra splits into a sum of Lie algebras where the adjoint representation is both completely reducible and irreducible. If the center of such an algebra is trivial, by the above theorem, it is semisimple (in fact simple). Otherwise, since the center is a summand of the adjoint representation, it is everything, and the algebra is abelian. The other direction has already been proven. □

We will now see that semisimple Lie algebras are the ones having a semisimple representation theory. To see it is a necessary condition, observe that the adjoint representation must be irreducible so it is reductive, and it can't have an abelian part since abelian Lie algebras aren't semisimple.

Recall that representations of $\mathfrak{g}$ are the same as representations of $U(\mathfrak{g})$. $U(\mathfrak{g})$ is a filtered cocommutative Hopf algebra, where the coproduct is $x \mapsto 1 \otimes x + x \otimes 1$, and the antipode is $x \mapsto -x$ on $\mathfrak{g} \subset U(\mathfrak{g})$. It is the algebra of differential operators on the associated formal group. U of a free Lie algebra is a tensor algebra, $U(\mathfrak{g} \oplus \mathfrak{h}) = U(\mathfrak{g}) \otimes_k U(\mathfrak{h})$, and $U(k) = k[x]$.

The argument that representations are semisimple is similar to the argument for compact Lie group. There, via averaging over a bi-invariant metric, one makes any representation unitary, after which one can produce orthogonal complements to subrepresentations. Here the bi-invariant metric becomes the Killing form, and it hard to average, but we still have an infinitesmal version of averaging, namely the Laplacian operator, which here is called the Casimir element.

Any symmetric invariant form B on $\mathfrak{g}$ can be thought of as a map of $\mathfrak{g}$-modules $\mathfrak{g} \otimes \mathfrak{g} \to k$. Nondegeneracy means the adjoint $\mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism. Via this identification, the identity $\mathfrak{g} \to \mathfrak{g}$ is adjoint to a map $k \to \mathfrak{g} \otimes \mathfrak{g}$, the image of 1 being the **Casimir element** L_B. Alternatively it is adjoint to the map B. Note that symmetry of the bilinear form implies that L_B is invariant under swapping the two terms. We can also view L in $U(\mathfrak{g})$ via the multiplication map $\mathfrak{g} \otimes \mathfrak{g} \to U(\mathfrak{g})$. It is central in $U(\mathfrak{g})$ since the map $k \to \mathfrak{g} \otimes \mathfrak{g}$ is a map of $\mathfrak{g}$-modules.

Lemma 2.13. *The category of f.d representations of $\mathfrak{g}$ is semisimple iff $\mathrm{Ext}^1(k, V) = 0$ for all irreducible V, where k is the tensor unit.*

Proof. We can identify $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ with $\mathrm{Hom}_{\mathfrak{g}}(k, W \otimes V^*)$. $(-) \otimes V^*$ is exact and preserves injectives, so we can derive the isomorphism to Ext. Semisimplicity is equivalent to $\mathrm{Ext}^1(V, W) = 0$. So we only need $\mathrm{Ext}^1(k, V) = 0$ for all V, but the long exact sequence on Ext shows it suffices to prove it for irreducible V. □

Theorem 2.14 (Weyl Complete Reducability)**.** *For a semisimple $\mathfrak{g}$, any finite-dimensional representation of $U(\mathfrak{g})$ is semisimple.*

Proof. We need to show that a short exact sequence $0 \to V \to W \to k \to 0$ splits when V is irreducible. WLOG V is faithful by passing to a quotient of $\mathfrak{g}$. L_V is 0 on k, and $\mathrm{tr}_{L_V}(V) = \dim \mathfrak{g} \neq 0$. So L_V is nonzero, and thus an isomorphism by Schur's Lemma. An element in the kernel of L_W gives a splitting. □

A remark about the proof: we should expect k to be the only irreducible representation for which L acts as 0 since every irrep embeds in L^2, where L is literally the Laplacian, and the only harmonic functions are constant.

This has an interpretation in terms of derivations. A **derivation** $d : \mathfrak{g} \to M$ is a map such that $d[a,b] = adb - bda$ It is the same as a 1-cocycle in the Chevalley-Eilenberg complex (i.e the de Rham complex) of M. A 1-boundary is an **inner derivation** or a map of the form $dx = xa, a \in M$. We can write these groups as $\mathrm{Der}(\mathfrak{g}, M)$ and $\mathrm{Ider}(\mathfrak{g}, M)$.

Corollary 2.15. *Every derivation of a semisimple Lie into a finite-dimensional module is inner.*

A special case of this is the Lie algebra of derivations of $\mathfrak{g}$ into $\mathfrak{g}$. Here is an alternate proof of the fact in that case:

Proposition 2.16. *If $\mathfrak{g}$ is semisimple, the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is an isomorphism.*

Proof. The restriction of the tautological representation $\mathrm{Der}(\mathfrak{g})$ to $\mathfrak{g}$ is the adjoint representation. Since κ is nondegenerate, $\mathrm{Der}(\mathfrak{g})$ splits as $\mathrm{ad}\,\mathfrak{g} \oplus \mathrm{ad}\,\mathfrak{g}^{\perp}$. If $D \in \mathrm{ad}\,\mathfrak{g}^{\perp}$ then $0 = [D, \mathrm{ad}\,a] = \mathrm{ad}\,Da$, so $Da \in Z(\mathfrak{g}) = 0$ showing $D = 0$. □

Definition 2.17. *An* ***abstract Jordan decomposition*** *of a is a decomposition $a = a_s + a_n$ where $[a_s, a_n] = 0$ and $\mathrm{ad}\,a_s, \mathrm{ad}\,a_n$ are semisimple and nilpotent.*

They are unique iff $Z(\mathfrak{g}) = 0$, because that is the kernel of map ad, and we know Jordan decompositions to be unique in $\mathfrak{gl}_V$.

Lemma 2.18. *Abstract Jordan decompositions exist for reductive Lie algebras.*

Proof. It suffices to assume $\mathfrak{g}$ semisimple. $A = \mathrm{ad}\,g$ has a Jordan decomposition as an endomorphism, so it suffices to show that A_s is a derivation. By linearity it suffices to do check this for elements in a root space x, y, with eigenvalues λ, μ for A. Indeed, $A_s[x,y] = (\mu + \lambda)[x,y] = [\lambda x, y] + [x, \mu y] = [A_s x, y] + [x, A_s y]$. □

Now we can better understand the Killing forms of a semisimple Lie algebra.

Theorem 2.19. *Let κ be the killing form on $\mathfrak{g}$, a semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra.*

(1) $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$.
(2) $\kappa_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is nondegenerate.
(3) $\mathfrak{h}$ is abelian
(4) $\mathfrak{h}$ consists of semisimple elements

Proof. (1): This is immediate from Theorem 1.15 by looking at how ad acts on the root spaces.

(2): This follows from (1) and the fact that κ is nondegenerate.

(3): Cartan's criterion shows that $\kappa(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$, but along with (2) on $\mathfrak{g}_0 = \mathfrak{h}$ this means $[\mathfrak{h}, \mathfrak{h}] = 0$.

(4): $\mathfrak{h}_s \subset \mathfrak{h}$ because of how $\mathrm{ad}\,\mathfrak{h}_s$ behaves, but note that by (3), $\kappa(h, h') = \kappa(h_s, h')$ for $h, h' \in \mathfrak{h}$. Thus by (2) on $\mathfrak{g}_0 = \mathfrak{h}$, $h = h_s$. □

3. The Universal Enveloping Algebra

We would in general like to understand the structure of $U(\mathfrak{g})$.

To do this, we will use a general tool due to George Bergman for finding canonical forms for elements of associative k-algebras.

Say that we have a set X, and we consider the tensor algebra $T_R(X)$ on X over a commutative ring R. Say we also have some relations $\sigma \in S$ of the form $W_\sigma = f_\sigma$, where W is a word in X. For any other words A, B, we can consider $r_{A\sigma B}$, the R-linear endomorphism on $T_R(X)$ replacing $AW_\sigma B$ with $Af_\sigma B$. We call applying this to an element of $T_R(X)$ a **reduction**. If all reductions on an element are trivial, then that element is **irreducible**. The submodule of irreducible elements is called $T_R(X)_{irr}$. We say that an element is **reduction finite** if for any infinite sequence of reductions, only finitely many act nontrivially. The reduction finite elements form an R-submodule of $T_R(X)$ called $T_R(X)_{fin}$. A sequence of reductions is **final** if it results in an irreducible element. An element a is **reduction unique** if it is reduction finite and any final sequence results in the same irreducible element. The unique result will be denoted $r(a)$.

Lemma 3.1. *The reduction unique elements form a submodule denoted $T_R(X)_{un}$, and $r : T_R(X)_{un} \to T_R(X)_{irr}$ is R-linear.*

Proof. Suppose $a, b \in T_R(X)_{un}, k \in R$. $a + kb$ is reduction finite, and if we have two reduction sequences of it to an irreducible, we can consider doing the same sequence on a, b and extend them to one that makes each irreducible (note that this won't change the result, only the sequence). Then by uniqueness for a, b we see that the two reductions are the same, and that r is R-linear. □

Lemma 3.2. *Let $a, b, c \in T_R(X)$ have the property that if A, B, C are nonzero monomials in a, b, c, then ABC is reduction unique. Then if $r_\Sigma(b)$ is the result of some finite reductions on b, then $ar_\Sigma(b)c$ is reduction unique.*

Proof. Note in particular that the hypotheses imply that if abc is reduction unique, and that if the conclusion holds, $r(ar_\Sigma(b)c) = r(abc)$. By linearity we only need to show this when a, b, c are monomials, and when $r_\Sigma(b)$ is a single reduction $r_{d\sigma e}$. But $ar_{d\sigma e}(b)c = r_{ad\sigma ec}(abc)$, so this follows since abc is reduction unique. □

We say that a **overlap ambiguity** is a pair $\sigma, \sigma' \in S$ and a triple A, B, C of nonempty words such that $AB = W_\sigma, BC = W'_\sigma$ (they overlap). It is **resolvable** if there is are sequences of reductions r, r' such that $r \circ r_\sigma(ABC) = r' \circ r_{\sigma'}(ABC)$. An **inclusion ambiguity** is the same, except when $W_\sigma = B, W'_\sigma = ABC$, and has the same conditions for resolvability (except A, C can be empty).

Say that a **compatible partial ordering** $\leq$ on words of X is one such that $A \leq B \implies CAD \leq CBD$ and such that the monomials of the f_σ are smaller than the monomials of W_σ. Let I be the two-sided ideal generated by the relations in S, and I_A be the R-submodule of $T_R(X)$ generated by $B\sigma C < A, \sigma \in S$ (every monomial is smaller than A). An ambiguity is **resolvable relative to** $\leq$ if $r_\sigma(ABC) - r'_\sigma(ABC) \in I_{ABC}$. Note that this is an easy condition to check.

Finally we arrive at this theorem, which can be considered a Diamond Lemma for rings:

Theorem 3.3. *Suppose we have $S, X, \leq, I$ as above where $\leq$ is compatible with S and satisfies the descending chain condition. Then the following are equivalent:*

1. *Every ambiguity is resolvable.*
2. *Every ambiguity is resolvable with respect to $\leq$.*
3. *Every element is reduction unique.*
4. *The natural quotient identifies the submodule spanned by irreducible monomials with $T_R(X)/I$, which is $T_R(X)_{irr}$.*

Proof. Clearly (3) $\implies$ (1) $\implies$ (2). If (3) is true, r defines a projection onto $T_R(X)_{irr}$. The kernel is contained in I by definition of r and contains I, as for any AB, $r(A\sigma B) = 0$ by Lemma 3.2. Thus as an R-module, $T_R(X) = I \oplus T_R(X)_{irr}$, giving (4). Conversely if (4) is true, Then since reductions are equal in the quotient, they must be unique.

Thus it suffices to prove (2) $\implies$ (3), and by linearity, we need only show this for a monomial A. From the descending chain condition, we can assume that any smaller monomial is reduction unique. Now suppose that we have two reductions of a monomial A.

If there is no ambiguity, they commute, so we can create two more reductions to an irreducible where the first two steps are the first step of these reductions but in different orders. They will give the same element, and by induction will show that the two original reductions are also the same.

If there is ambiguity, then since it is resolvable relative to $\leq$ by induction the difference can be resolved to 0, so by linearity the two resolutions must agree. □

Theorem 3.4 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a Lie algebra over R where the underlying module is free. Choose a well-ordered basis of $\mathfrak{g}$, $x_\alpha, \alpha \in I$. Then $x_{a_1}^{e_1} \dots x_{a_n}^{e_n}$ for x_{a_1} in increasing order, over all possible a_i and e_i form a basis of $U(\mathfrak{g})$.*

Proof. Construct an ordering on monomials in the basis where monomials of smaller degree are smaller, and if two monomials have the same degree, we compare them lexicographically using the order on our basis. This is a well-ordering on monomials, and a sufficient set of relations for $U(\mathfrak{g})$ can be written as $xy = yx + [x, y]$ for $x > y$ in the basis. The order is compatible with this, and we can check that every ambiguity is resolvable with respect to our order. Here, the only nontrivial kind of ambiguity that can occur is an overlap ambiguity, when we have something of the form $A(yxz+[x,y]z)B = AxyzB = A(xzy+x[y,z])B$. The difference is:

$$A(yxz+[x,y]z-(xzy+x[y,z]))B = A(yzx+y[x,z]+z[x,y]+[[x,y],z]$$
$$-(zxy+[x,z]y+[y,z]x+[x,[y,z]]))B = A(zyx+[y,z]x+y[x,z]+z[x,y]+[[x,y],z]$$
$$-(zyx+z[x,y]+y[x,z]+[[x,z],y]+[y,z]x+[x,[y,z]])B$$
$$= A([[x,y],z]-[[x,z],y]-[x,[y,z]])B = 0$$

Where at the last step, we have used the Jacobi identity. This shows that the ambiguity is resolvable under the ordering, so that by Theorem 3.3 we are done. □

Note that as a corollary, any Lie algebra $\mathfrak{g}$ (not necessarily finite-dimensional) is faithfully represented in $U(\mathfrak{g})$.

4. $\mathfrak{sl}_2 k$

The structure theory of semisimple Lie algebras is centered around the behavior of $\mathfrak{sl}_2 k$. Before moving on let's study it more carefully.

Let's consider a root system for $\mathfrak{sl}_2$.

It has a basis $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and can be presented via the relations $[e,f]=h, [h,e]=2e, [h,f]=-2f$. kh is a Cartan subalgebra, and e,f generate the root spaces.

We can try to classify finite-dimensional irreducible $\mathfrak{sl}_2$-modules. There is a standard one, given on a vector space V of dimension 2. $\operatorname{Sym}^n V$ is a representation of dimension $n+1$.

Proposition 4.1. $\operatorname{Sym}^n V$ *is a complete list of f.d irreducible* $\mathfrak{sl}_2$*-representations.*

Proof. Let v be a nonzero element of V in the kernel of e. v, fv form a basis of V. Note that e acts nilpotently with 1-dimensional kernel $v^{\otimes n}$. Thus any subrepresentation contains that vector, and since it generates the rest of the space by powers of f, $\operatorname{Sym}^n V$ is irreducible.

Now let W be an any f.d irreducible representation. It decomposes into eigenspaces W_λ for h, and $eW_\lambda \subset W_{\lambda+2}, fW_\lambda = W_{\lambda-2}$. We see then by irreducibility $\oplus_n W_{\lambda+2n} = W$, where the nonzero terms occur in a segment, and are one-dimensional as $[e,f] = h$. Since $\operatorname{tr} h$ is 0, we must have the highest and lowest weights be opposite in signs, so λ is an integer n. This is enough to see that λ uniquely characterizes the irrep, so it is $\operatorname{Sym}^n V$. □

By this theorem, we can understand the isomorphism class of any finite-dimensional $\mathfrak{sl}_2$-module by looking at how h acts on the kernel of e. Namely, every eigenvalue of n corresponds to a copy of $\operatorname{Sym}^n V$.

Corollary 4.2. *Let W be a finite dimensional representation of $\mathfrak{sl}_2$, and v a vector such that $ev = 0$. Then $hv = \lambda v$, $\lambda \in \mathbb{Z}_{\geq 0}$. λ is the largest k such that $f^k v \neq 0$. Moreover, $v, fv, \dots f^\lambda v$ are linearly independent, $hf^k v = (\lambda - 2k)f^k v$ and $ef^k v = n(\lambda - n + 1)f^{k-1}v$.*

Corollary 4.3. *A f.d representation of $\mathfrak{sl}_2$ is irreducible iff it satisfies any of the following conditions:*

- *The kernel of e is 1-dimensional.*
- *The kernel of f is 1-dimensional.*
- *The eigenspaces of h are 1-dimensional.*

Given a finite-dimensional $\mathfrak{sl}_2$-module V, let $\tau = \exp(e)\exp(f)\exp(-e)$.

Lemma 4.4. *τ swaps the $-n$ and n h-eigenspaces of any f.d. $\mathfrak{sl}_2$-module W.*

Proof. Since W is a sum of summands of tensor products of the standard representation, it suffices to verify for the standard representation. Here τ is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. □

Definition 4.5. *An $\mathfrak{sl}_2$-**triple** is a homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$.*

For any semisimple Lie algebra, we will get an $\mathfrak{sl}_2$-triple for each root, so we will use the representation theory of $\mathfrak{sl}_2$ to show that the roots form a root system.

5. Root Systems

First we will need to show that we get a root system from a semisimple Lie algebra. Let $\mathfrak{g}$ be a fixed semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, Δ the set of roots. κ induces a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$, and we will use κ to denote this as well. Let $(-)^*$ be the isomorphism between $\mathfrak{g}$ and its dual induced by κ.

Lemma 5.1 ($\mathfrak{sl}_2$-triples)**.** *If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then $[e, f] = \kappa(e, f)\alpha^*$. Moreover, if $\alpha \in \Delta, \kappa(\alpha, \alpha) \neq 0$.*

Proof. $\kappa(h, [e, f]) = \kappa(\operatorname{ad} h(e), f) = \alpha(h)\kappa(e, f) = \kappa(h, \alpha^*\kappa(e, f))$, giving the first result since κ is nondegenerate on $\mathfrak{h}$. For the second, choose e, f so that $\kappa(e, f) = 1$. If $\kappa(\alpha, \alpha) = \kappa(\alpha, [e, f]) = \alpha([e, f])$ is 0, then α^*, e, f form a subalgebra presented by the relations $[e, f] = \alpha^*, [\alpha^*, e] = [\alpha^*, f] = 0$. This is a nilpotent subalgebra, so by Lie's theorem, $\alpha^* = [e, f]$ must act nilpotently on $\mathfrak{g}$. but it acts semisimply so $\operatorname{ad}\alpha^* = 0$, a contradiction. □

Thus given α, we can pick e_α, f_α so that $\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(\alpha,\alpha)}$. Then $[e_\alpha, f_\alpha] = h_\alpha = \frac{2\alpha^*}{\kappa(\alpha,\alpha)}$. Then $e_\alpha, f_\alpha, h_\alpha$ give an $\mathfrak{sl}_2$-triple associated with α.

$\mathfrak{sl}_2$-triples associated with α are unique up to scaling $f \mapsto \lambda f, e \mapsto \lambda^{-1}e$. This follows from the next theorem.

For convenience, define $\langle\alpha|\beta\rangle$ as $2\frac{\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)}$. This notation is nonstandard, but $|$ is being used to remind us that it is not symmetric.

Theorem 5.2. *(1)* $\dim \mathfrak{g}_\alpha = 1, \alpha \in \Delta$.
(2) If $\alpha, \beta \in \Delta$*, then* $\beta + r\alpha, r \in k$ *is a connected string where* r *are integers going from* $-p$ *to* q *where* $p - q = \langle\alpha|\beta\rangle$
(3) If $\alpha, \beta, \alpha+\beta \in \Delta$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
(4) $\alpha \in \Delta \implies r\alpha \in \Delta$ *iff* $r = \pm 1$.

Proof. (1): If e, e' are nonzero in $\mathfrak{g}_\alpha$, rescale them and choose an f in $\mathfrak{g}_{-\alpha}$, so that $e, f, [e,f]$ and $e', f, [e', f]$ extend to an $\mathfrak{sl}_2$-triple. But $[e,f] = \frac{2\alpha^*}{\kappa(\alpha,\alpha)} = [e', f]$, so the representation theory of $\mathfrak{sl}_2$ tells us $e = e'$.

(2&3): Consider $M = \oplus_{r\in k}\mathfrak{g}_{\beta+r\alpha}$ as a representation of an $\mathfrak{sl}_2$-triple associated to α. $h_\alpha \mathfrak{g}_{\beta+r\alpha} = (\langle\alpha|\beta\rangle + 2r)\mathfrak{g}_\beta$ by definition of h_α, so M is irreducible by (1) and Corollary 4.3. The results then follows from the representation theory of $\mathfrak{sl}_2$.

(4): This follows from the former observation that M is an irreducible $\mathfrak{sl}_2$ module for $\beta = \alpha$ along with the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$. □

Because the (nonzero) roots are 1-dimensional, and $\mathfrak{h}$ is abelian, it follows that if $a, b \in \mathfrak{h}$, then $\kappa(a,b) = \sum_{\alpha\in\Delta} \alpha(a)\alpha(b)$.

κ restricted to the roots is really defined over $\mathbb{Q}$ and is positive definite.

Theorem 5.3. *(1)* Δ *spans* $\mathfrak{h}^*$.
(2) $\kappa(\alpha,\beta) \in \mathbb{Q}$ *for* $\alpha, \beta \in \Delta$.
(3) $\kappa|_{\mathfrak{h}^*_{\mathbb{Q}}}$ *is positive definite.*

Proof. (1): If there is some $h \in \mathfrak{h}$ such that $\kappa(\alpha^*, h) = \alpha(h) = 0$ for all $\alpha \in \Delta$, then h acts trivially on all $\mathfrak{g}_\alpha$, so must be 0.

(2&3): Let $\alpha, \beta \in \Delta$. Then the formula for κ shows $\frac{4}{\kappa(\alpha,\alpha)} = \sum_{\lambda\in\Delta}\langle\alpha|\lambda\rangle \in \mathbb{Z}$ so $\kappa(\alpha,\alpha) \in \mathbb{Q}$. Since $\frac{\langle\alpha|\beta\rangle\kappa(\alpha,\alpha)}{2} = \kappa(\alpha,\beta)$, $\kappa(\alpha,\beta) \in \mathbb{Q}$. Finally the formula for $\kappa(\alpha,\alpha)$ shows it is positive, since it is a sum of squares of rational numbers. □

We have shown that h^*, Δ is a root system, of which a definition will now be given. I think the right way to do this gives something called root datum, where you don't identify the vector space with its dual via an inner product.

Definition 5.4. *A* ***root system*** *is an vector space* V *with a conformal structure (i.e. an inner product up to scaling) and a collection of nonzero elements (roots)* Δ *spanning it, such that:*

- *If* $\alpha \in \Delta$, $n\alpha \in \Delta$ *iff* $n = \pm 1$.
- *If* $\alpha, \beta \in \Delta$*, then* $\beta + n\alpha, n \in \mathbb{Z}$ *is in* $\Delta \coprod 0$ *for* n *in a connected string from* $-p$ *to* q*, where* $p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$.

The second condition is called the string condition.

We denote $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ as $\langle\alpha|\beta\rangle$ like before.

We would like to determine when a Lie algebra is simple rather than just semisimple. Let us consider a sum of two semisimple Lie algebras $\mathfrak{g}, \mathfrak{g}'$, let $\mathfrak{h}, \mathfrak{h}'$ be their Cartan subalgebras and let Δ, Δ' be the roots. $\mathfrak{h} \oplus \mathfrak{h}'$ is a Cartan subalgebra and $\Delta \coprod \Delta'$ is the corresponding set of roots. Note that the splitting of $\mathfrak{g} \oplus \mathfrak{g}'$ can be detected entirely through the root system. Namely $\mathfrak{h}, \mathfrak{h}'$ orthogonally decompose the space, and Δ, Δ' respects the decomposition. Conversely such a decomposition clearly indicates how to decompose the Lie algebra. By the axioms of a root system, this is equivalent to finding a partition $\Delta \coprod \Delta'$ where $a + b \notin \Delta \coprod \Delta' \coprod 0$ if $a \in \Delta, b \in \Delta'$. When a nontrivial partition exists, the root system is **indecomposable**.

Lemma 5.5. $\mathfrak{g}$ *is simple iff its root system is indecomposable.*

We can easily check indecomposability as follows:

Lemma 5.6. *A root system Δ is indecomposable iff any two roots can be connected by a sequence such that for any neighboring pair in the sequence, the sum is in $\Delta \coprod 0$.*

The existence of a root system is really equivalent to being semisimple. Now we will produce many examples of semisimple Lie algebras and hence root systems. To exhibit a Lie algebra as semisimple, it really suffices to provide a root space decomposition.

Theorem 5.7. *If $\mathfrak{g}$ has a decomposition as $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, $\Delta \subset \mathfrak{h}^*$ such that*

(1) Δ spans $\mathfrak{h}^*$, and $\mathfrak{h}$ is nilpotent.

(2) $\mathfrak{g}_\alpha$ are weight spaces for the action of $\mathfrak{h}$.

(3) $\alpha(a, \mathfrak{g}_{-\alpha}) \neq 0$ for $a \in \mathfrak{g}_\alpha$.

Then $\mathfrak{g}$ is semisimple, and $\mathfrak{h}, \Delta, \mathfrak{g}_\alpha$ denote the usual things.

Proof. Consider an abelian ideal $\mathfrak{a}$. It cannot be contained in $\mathfrak{h}$, as α of any element in $\mathfrak{h}$ is nonzero for some α by (1), so $\mathfrak{g}_\alpha$ would also be in it. Consider any element $a \in \mathfrak{a} - \mathfrak{h}$. By acting on it by elements of $\mathfrak{h}$ many times, since $\mathfrak{h}$ is nilpotent and $\mathfrak{g}_\alpha$ are weight spaces for $\mathfrak{h}$, we can assume its component in $\mathfrak{h}$ is 0 and its components in the $\mathfrak{g}_\alpha$ are restricted to α supported on a line. By further acting by an element in $\mathfrak{h}$ for which α doesn't vanish, and taking linear combinations, we can assume a is supported on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Suppose its component in $\mathfrak{g}_\alpha$ is 0. Then by acting on a by an element of $\mathfrak{g}_{-\alpha}$, by (3) we can produce an element $b \in \mathfrak{a}$ supported in $\mathfrak{h} \oplus \mathfrak{g}_{-2\alpha}$ that has $\alpha \neq 0$, but this doesn't commute with a. $\square$

Now we can construct the classical simple Lie algebras and their root decompositions. We will not show they are simple since it easily follows from our criteria.

Example 5.7.1 ($\mathfrak{sl}_n$)**.** The diagonal matricies form an abelian subalgebra, with basis $e_i = E_{i,i} - E_{i+1,i+1}$ for $1 \leq i < n$. Anything that commutes with e_i commutes with all diagonal matricies preserves their eigenspaces so must be diagonal. $\operatorname{ad} e_i(E_{jk})$ was computed before, and we see that $E_{jk}, j \neq k$ are the weight spaces with weight $e_j^* - e_k^*$. The roots are then $e_j^* - e_k^*$ for $j \neq k$ both less than n.

Given a bilinear form B on a vector space V, the set of endomorphisms A such that $B(Aa, b) + B(a, Ab) = 0$ is a Lie algebra. When B is a nondegenerate symmetric bilinear form on a vector space of dimension n, this is called $\mathfrak{so}_n$, and when it is a nondegenerate alternating biliner form on a vector space of dimension $2n$, this is called $\mathfrak{sp}_{2n}$.

Example 5.7.2 ($\mathfrak{so}_n$)**.** Here $n \geq 3$. Choose a basis so that B is the matrix that is 1s on the antidiagonal. Then $\mathfrak{so}_n$ consists of all matrices that are anti symmetric with respect to flipping across the antidiagonal. A Cartan subalgebra consists of all matrices that are diagonal, a basis given by $e_i = E_{ii} - E_{n+1-i,n+1-i}$ for $1 \leq i \leq \frac{n}{2}$. The root spaces are given by $E_{ij} - E_{n+1-i,n+1-j}$, corresponding to the roots $e_i^* - e_j^*$, where e_j^* really means $-e_{n+1-j}^*$ when $j > \frac{n}{2}$ and 0 when $j = \frac{n}{2}$. When n is even, these then give $\pm(e_i^* + e_j^*), e_i^* - e_j^*$ for $i \neq j$ at most $\frac{n}{2}$, and when n is odd, they give $\pm(e_i^* + e_j^*), e_i^* - e_j^*, \pm e_i^*$ for $i \neq j$ at most $\frac{n}{2}$.

Example 5.7.3 ($\mathfrak{sp}_{2n}$)**.** Here $n \geq 1$. We can choose a basis so the bilinear form looks like 1 on the upper half of the antidiagonal and -1 on the other half. We can split any matrix in $\mathfrak{sp}_{2n}$ into four quadrants like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and if a' denotes the reflection across the antidiagonal, the conditions are that $a' = -d, b' = b, c' = c$. Again the diagonal matrices form a Cartan subalgebra, with bases $e_i = E_{ii} - E_{n+1-i,n+1-i}$. Now there are different kinds of rootspaces. One kind is $E_{ij} - E_{n+1-i,n+1-j}$ where i, j is in the upper left quadrant, which gives $e_i^* - e_j^*$. Another is $E_{ij} + E_{n+1-i,n+1-j}$ where i, j is in the upper left half of either the top right or bottom left quadrant, giving $\pm(e_i^* + e_j^*)$. The last kind is E_{ij} where i, j lies on the antidiagonal, giving $\pm 2e_i^*$.

There is another naming scheme related to Dynkin diagrams: $A_n = \mathfrak{sl}_{n+1}, B_n = \mathfrak{so}_{2n+1}, C_n = \mathfrak{sp}_{2n}, D_n = \mathfrak{so}_{2n} (n > 1)$.

The Killing forms are easy to understand up to scalar multiplication:

Lemma 5.8. *Any nontrivial invariant bilinear form on a simple Lie algebra is nondegenerate. Moreover any invariant nondegenerate bilinear form on a simple Lie algebra is unique up to scalar multiplication. In particular it must be proportional to* κ*.*

Proof. An invariant bilinear form is a map of $\mathfrak{g}$-modules $\mathfrak{g}\otimes\mathfrak{g}\to k$. $\mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{g},k)=\mathrm{Hom}(\mathfrak{g},\mathfrak{g}^*)$ which is one-dimensional since $\mathfrak{g}$ is simple. The result follows since κ is an example of a nontrivlal form. □

Thus the trace form of the standard representation is proportional to κ. Now let's study root systems in general. There is an analog of the previous lemma for root systems.

Proposition 5.9. *An indecomposable root system has a unique up to scalars inner product making it a root system. Moreover, if $(\alpha,\alpha)\in\mathbb{Q}$ for some $\alpha\in\Delta$, it is true for all (α,β).*

Proof. Given (α,α), the string condition lets us compute (α,β) whenever $\alpha+\beta\in\Delta$, which then lets us compute (β,β). Continuing this way via Lemma 5.6, we can recover the rest of the inner product showing it is determined by (α,α) and moreover in $\mathbb{Q}$ if (α,α) is. □

Thus we can really choose to work over $\mathbb{Q}$ or $\mathbb{R}$ and it makes no difference. $\mathbb{Z}\Delta$ is a finitely generated subgroup of a rational vector space, so is a lattice called the **root lattice**.

Let E_r be the lattice consisting of linear combinations a_ie_I where either $a_i\in\mathbb{Z}$ or $a_i\in\mathbb{Z}+\frac{1}{2}$ for all i and $\sum_i a_i\in 2\mathbb{Z}$.

A lattice is **even** if $(a,b)\in\mathbb{Z}$ for any a,b in it. The following is straightforward.

Lemma 5.10. *E_r is even iff $8|r$.*

Theorem 5.11. *Given an even lattice such that Δ, the set of elements in the lattice such that $(a,a)=2$ span the vector space, Δ is a root system.*

Proof. One really only needs to check the string property. Since $0\le(\alpha-\beta,\alpha-\beta)=(\alpha,\alpha)+(\beta,\beta)-2(\alpha,\beta)$, we see that $(\alpha,\beta)\le 2$. One can calculate in each case that the string condition holds. □

Example 5.11.1 (E_8)**.** The theorem implies E_8 is a root system, which can be checked is indecomposable. It can be described as the root system consisting of $\pm(e_i+e_j),\pm e_i\mp e_j$ and things of the form $\frac{1}{2}(e_1\pm\dots e_8)$ where there are an even number of minus signs.

Example 5.11.2 (E_7)**.** We can construct another called E_7 as a sub-root system as follows: consider the vectors in E_8 orthogonal to the constant vector with entry $\frac{1}{2}$. It is still even, so a root system. These are of the form $e_i-e_j, i\ne j$ and things of the form $\frac{1}{2}(e_1\pm\dots e_8)$ where the number of minus signs is divisible by four.

Example 5.11.3 (E_6)**.** Finally, E_6 can be constructed as the root system consisting of vectors in E_7 that are orthogonal to e_7+e_8. It is still even, so a root system. It contains $e_i - e_j$ such that if either of $\{i, j\} \cap \{7, 8\}$ is $\{7, 8\}$ or ϕ, as well as things of the form $\frac{1}{2}(e_1 \pm \ldots e_8)$ where the number of minus signs is divisible by four, and e_7, e_8 have opposite signs.

There are two more exceptional root systems, which have a different origin.

Example 5.11.4 (F_4)**.** Consider the vector space generated by e_1, e_2, e_3, e_4 and consider the lattice consisting of elements of the form $\sum_i a_i e_i$ where either $a_i \in \mathbb{Z}$ or all $a_i \in \mathbb{Z}+\frac{1}{2}$. The elements of the lattice such that $(x, x) = 1, 2$ form an indecomposable root system. More explicitly, the roots are $\pm e_i, \pm(e_i + e_j), e_i - e_j, \pm\frac{1}{2}(e_1 \pm \ldots e_4)$.

Example 5.11.5 (G_2)**.** Here the lattice is the same as that for A_2, namely all elements $a_1e_1 + a_2e_2 + a_3e_3$ such that $\sum_i a_i = 0$, except we take elements with $(x, x) = 2, 6$. This is an indecomposable root system whose elements are $\pm(e_i + e_j), e_i - e_j, \pm(2e_i - e_j - e_k)$.

6. Classification

To classify root systems, we will really classify root systems with a generic linear functional, and show it doesn't depend on the functional. Suppose that we have a root system (V, Δ) and a linear functional f not vanishing on any root. Then we say a root α is **positive** if $f(\alpha) > 0$, and negative otherwise. A positive root is **simple** if it is not the sum of two positive roots. A **highest root** θ is a root such that $f(\theta)$ is maximal. Δ_+, Δ_- denote the positive and negative roots, and Π denotes the simple roots. Π is indecomposable if it can't be nontrivially partitioned into orthogonal sets.

Theorem 6.1 (Dynkin)**.**
1. *If $\alpha, \beta \in \Pi$ are distinct, then $\alpha - \beta \notin \Delta$ and $(\alpha, \beta) \leq 0$.*
2. *Every positive root is a nonnegative integral linear combination of simple roots.*
3. *If $\alpha \in \Delta_+ - \Pi$, then for some $\gamma \in \Pi$, $\alpha - \gamma \in \Pi$, and when this is true, $\alpha - \gamma \in \Delta_+$.*
4. *Π is a basis of the lattice generated by Δ.*
5. *Δ is indecomposable iff Π is.*
6. *For an indecomposable root system, there is a unique highest root θ, and $(\theta, \alpha) > 0, \alpha \in \Delta_+$.*

Proof. (1): If it were in Δ, by possibly taking its negative, we could assume it is in Δ_+. But then $\alpha - \beta + \beta = \beta$ so β isn't simple. The string condition then implies that $(\alpha, \beta) \leq 0$.

(2): Break up a positive root into smaller positive roots until they are all simple.

(3): $\alpha = \sum_i c_i\gamma_i$ where γ_i are positive simple roots, and $c_i > 0$. we have that $\sum_i c_i\langle\alpha|\gamma_i\rangle = \langle\alpha|\alpha\rangle = 2$, so $\langle\alpha|\gamma_i\rangle$ is positive for some γ_i, so $\gamma_i - \alpha \in \Delta$. If $\gamma - \alpha$ is ever in Δ_+, then since α is too, γ cannot be simple. Thus $\alpha - \gamma \in \Delta_+$.

(4): Since Δ spans the lattice, so does Δ_+, so by (2), Π does too. Suppose there is a relation $\sum c_i\gamma_i = \sum c'_j\gamma'_j$ where $c_i, c'_j \geq 0$, and γ_i, γ'_j are a disjoint set of simple roots. Then $(\sum c_i\gamma_i, \sum c'_j\gamma'_j) \leq 0$ by (1) so the coefficients are 0.

(5): If Γ is decomposable, the string condition and $(2),(3)$ implies that any element in Δ_+ is decomposable into things in exactly one of the decompositions of Γ, showing Δ is decomposable. Conversely if Δ is decomposable, as noted earlier the string condition shows the decomposition is orthogonal, implying that that the simple roots decompose orthogonally too.

(6): If θ is a highest root and α a positive root, then $\theta + \alpha \notin \Delta$ because it is larger than θ. Then the string condition implies $(\theta, \alpha) \geq 0$. By $(1),(2)$, the set of $\gamma \in \Pi$ orthogonal to θ and the set of $\gamma \in \Pi$ in the decomposition of θ are a decomposition of Π, so by indecomposability we must have $(\theta, \gamma) > 0$ for $\gamma \in \Pi$ and hence for all elements of Δ_+.

But if $\theta_1 \neq \theta_2$ are highest roots, $f(\theta_1 - \theta_2) = 0$ so it cannot be a root and $f(\theta_1 + \theta_2) > f(\theta_1)$ so it isn't a root. the string condition then says $(\theta_1, \theta_2) = 0$, but this was shown to not be possible. □

Definition 6.2. *If $\gamma_1, \ldots, \gamma_r$ are the simple roots, then the matrix A with entries $\langle\gamma_i|\gamma_i\rangle$ is called the* ***Cartan matrix****.*

Lemma 6.3. *The Cartan matrix A is an integer matrix with 2 on the diagonals, nonpositive numbers on the off-diagonals, and positive principle values. Moreover $A_{ij} = 0 \iff A_{ji} = 0$.*

Proof. The first,second, and last claim are clear, the third follows from the previous theorem, and the fourth follows from Sylvester's criterion because the Cartan matrix is the inner product matrix except with the rows rescaled. □

Definition 6.4. *If $\gamma_1, \ldots, \gamma_r, \theta$ are the simple roots, then the matrix $\tilde{A}$ with the same entries for A except including θ is called the* ***extended Cartan matrix****.*

$\tilde{A}$ clearly satisfies the same properties as A except the determinant is 0.

Definition 6.5. *An abstract Cartan matrix is one satisfying the properties of Lemma 6.3.*

A Cartan matrix is **indecomposable** iff it is not the direct sum of Cartan matrices. Clearly a root system is indecomposable iff its Cartan matrix is.

Lemma 6.6. *The only* $2x2$ *Cartan matricies (up to rearrangement) are*

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Because of this classification and the fact that every matrix is determined by its graph of $2x2$ principal submatrices, the Cartan matrix is completely encoded by a **Dynkin diagram**, where there is a vertex for every diagonal entry and an edge corresponding to each $2x2$ principal submatrix.

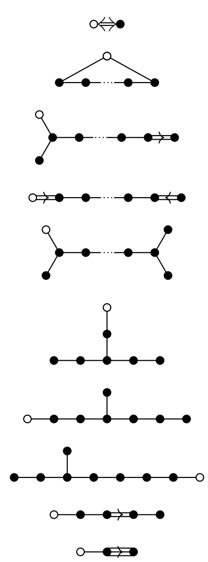
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

An edge indicates two arrows that are not orthogonal. The double and triple edges indicate that one root is longer, and they point to the shorter root.

Extended Cartan matrices gives an **extended Dynkin diagram**, where we mark the highest root as special. In the case of A_1 the extended Cartan matrix consists of all 2s, so it is exceptional, and we denote it by .

Example 6.6.1 (Extended Dynkin Diagrams)**.** Here is a list of the extended Dynkin diagrams of the thus-far constructed root systems. We will soon see that this list is complete.

A_1, A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2



Now let's see why these are as above. We will always choose f so the e_i are positive and e_i is much larger than e_{i+1}.

- A_n:, The roots are $e_i - e_j$ (we drop the $(-)^*$ in the notation). Then the simple roots are $e_i - e_{i+1}$, and the largest root is $e_1 - e_n$.
- B_n: The roots are $\pm(e_i + e_j), e_i - e_j, \pm e_i$. Then the simple roots are $e_n, e_i - e_{i+1}$, and the largest root is $e_1 + e_2$.
- C_n: The roots are $\pm(e_i + e_j), e_i - e_j, 2e_i$. The simple roots are $2e_n, e_i - e_{i+1}$, and the largest root is $2e_1$.
- D_n: The roots are $\pm(e_i + e_j), e_i - e_j$. The simple roots are $e_i - e_{i+1}, e_n + e_{n-1}$ and the largest root is $e_1 + e_2$
- E_8: The roots are $\pm(e_i + e_j), e_i - e_j$ and things of the form $\frac{1}{2}(e_1 \pm \ldots e_8)$ where there are an even number of minus signs. The simple roots are then $e_i - e_{i+1}$ for $i > 1$, $e_8 + e_7, \frac{1}{2}(e_1 + e_8 - \sum_2^6 e_i)$, and the largest root is $e_1 + e_2$.
- E_7: The roots are $e_i - e_j, i \neq j$ and things of the form $\frac{1}{2}(e_1 \pm \ldots e_8)$ where the number of minus signs is divisible by four. The simple roots are then $e_i - e_j$ for $i > 1$ and $\frac{1}{2}(\sum_{1,6,7,8} e_i - \sum_{2,3,4,5} e_i)$, and the largest root is $e_1 + e_2$.
- E_6: The roots are $e_i - e_j$ such that if either of $\{i, j\} \cap \{7, 8\}$ is $\{7, 8\}$ or ϕ, as well as things of the form $\frac{1}{2}(e_1 \pm \ldots e_8)$ where the number of minus signs is divisible by four, e_7, e_8 have opposite signs. The simple roots are $e_i - e_{i+1}$ for $i \neq 1, 7$ and $\frac{1}{2}(\sum_{1,5,6,8} e_i - \sum_{2,3,4,7} e_j)$ and the largest root is $e_1 - e_2$.
- F_4: The roots are $\pm e_i, \pm(e_i + e_j), e_i - e_j, \pm\frac{1}{2}(e_1 \pm \ldots e_4)$. The simple roots are $\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, e_2 - e_3, e_3 - e_4$, and the largest root is $e_1 + e_2$.
- G_2: The roots are $\pm(e_i + e_j), e_i - e_j, \pm(2e_i - e_j - e_k)$. The simple roots are $e_2 - e_3, e_1 - 2e_2 + e_3$, and the largest root is $2e_1 - e_2 - e_3$.

Theorem 6.7. *The examples contain a complete list of indecomposable Cartan matrices.*

Proof. Every subdiagram of a Dynkin diagram is a Dynkin diagram. Moreover, any extended Dynkin diagram is not a Dynkin diagram. Finally, for any multiple edge on a vertex of degree 1 on a Dynkin diagram, reversing an isolated set of multiple edges preserves being a Dynkin diagram since it preserves determinants of principle submatrices. Reducing an edge's multiplicity preserves being a Dynkin diagram since it increases the determinant of the principal submatrices. These observations essentially prove the theorem. Namely, the extended Dynkin diagrams constructed obstruct any other Dynkin diagrams from existing via these observations.

To actually work this out, A_n obstructs cycles from existing, so all Dynkin diagrams are trees. G_2 obstructs triple edges from existing anywhere else, F_4 obstructs double edges from appearing not on a leaf. C_n obstructs double edges from appearing multiple times elsewhere, B_n obstructs branching when there is a double edge. D_n

obstructs double branching, and E_6, E_7, E_8 obstruct branches from getting too long. $\square$

Observe that the list is a bit redundant. Namely, $A_1 = B_1 = C_1$, $D_2 = A_1 \oplus A_1$, $B_2 = C_2$, $D_3 = A_3$, but the rest are not redundant. Sometimes these are called **exceptional isomorphisms**.

We will now show that this classification also classifies simple Lie algebras and indecomposable root systems as well. First we will show nothing depends on f, and that we can recover the root system from the Cartan matrix.

There is another way of thinking about root systems. Namely, reflection across the orthogonal plane to a is given by the equation $r_a(v) = v - \langle a|v\rangle a$. One then sees from the string condition that Δ is closed under reflection by any element in it (this condition is actually equivalent to it by checking what happens in dimension 2). For the root system of a Lie algebra, one can see the reflection arising from the action of the element τ of the $\mathfrak{sl}_2$-triple associated to the root. The group generated by $r_\alpha, \alpha \in \Delta$ is called the **Weyl group** and denoted $W(\Delta)$. It is finite since it acts faithfully on Δ. The fact that there are so many symmetries already suggests f doesn't really do much.

If $\gamma_1, \ldots, \gamma_r$ are the simple roots, let $s_i = r_{\gamma_i}$; these are called simple reflections. Define the **height** of a positive root α to be $\mathrm{ht}(\alpha) = \sum_i c_i$ where $\alpha = \sum_i c_i \gamma_i$.

Theorem 6.8. *(1) s_i preserves $\Delta_+ - \gamma_i$.*
(2) If $\alpha \in \Delta_+ - \Pi$, there is an i so that $\mathrm{ht}(s_i(\alpha)) < \mathrm{ht}(\alpha)$.
(3) There is a sequence of simple reflections taking any positive root to an element of Π such that at every step it is still positive.
(4) W is generated by simple reflections.

Proof. (1): Such an element looks like $\sum_j c_j \gamma_j$ where $c_j > 0$ for some $j \neq i$. γ_j are linearly independent and everything is either a strictly positive or negative linear combination of them, so the result follows.

(2): $(\alpha, \alpha) > 0$, and α is a positive sum of γ_is so $(\alpha, \gamma_i) > 0$ for some i. Then $s_i(\alpha)$ has smaller height.

(3): Follows immediately from (2).

(4): By (3) we can get between from element of Δ and an element γ_i of Π via simple reflections. Now conjugating s_i by this composite of simple reflections gives the reflection by that element. $\square$

We can consider $V - \cup_{\alpha\in\Delta} T_\alpha$ where T_α is the plane perpendicular to α. The components of this are called the **Weyl chambers**. The Weyl group acts on these chambers since $r_\alpha(T_\beta) = T_{r_\alpha(\beta)}$.

Lemma 6.9. *x such that $(x, \gamma) > 0$ for any $\gamma \in \Delta$ form a chamber.*

Proof. Since anything is a positive or negative learn combination of Π, the set described doesn't contain anything orthogonal to anything in Δ. Moreover its boundary clearly consists of things that do. □

A word in the s_i is **reduced** if it isn't equivalent to a shorter word in W. The length of a reduced work representing $w \in W$ is called its **length**, denoted l.

Lemma 6.10 (Exchange Lemma)**.** *Suppose that $s_{i_1} \dots s_{i_{t-1}}(\gamma_{i_t}) \in \Delta_-$. Then $s_{i_1} \dots s_{i_t}$ isn't reduced. In fact, it is equal to $s_{i_1} \dots s_{i_{m-1}} s_{i_{m+1}} \dots s_{i_{t-1}}$ for some m.*

Proof. The sequence $\beta_k = s_{i_k} \dots s_{i_{t-1}}(\gamma_{i_t})$ at some point switches from negative to positive, which can only happen by the previous theorem if β_{m+1} is γ_{i_m}. But then $w = s_{i_{m+1}} \dots s_{i_{t-1}}$ satisfies $w s_{i_t} w^{-1} = s_{i_m}$ which gives the result after multiplying by w on the right. □

Theorem 6.11. *W acts simply and transitively on chambers, and on possible sets of simple roots.*

Proof. To see it acts transitively on chambers, make a generic path between two chambers and everytime you cross a wall, do a reflection. By realizing the reflections on f and on Π, we see that the chambers correspond to sets of simple roots for various choices of f. To see that the action is simple, suppose that $w \in W$ fixes Π. If $w = s_{i_1} \dots s_{i_t}$, then the exchange lemma shows it isn't reduced. Thus w must be trivial. □

Corollary 6.12. *The Cartan matrix is not dependent on f and one can recover the root system from the Cartan matrix. In fact there is a bijection between indecomposable Cartan matrices and root systems.*

Proof. The Cartan matrix essentially encodes the inner product on Π which form a basis for the vector space. Thus we can recover these vectors, but by using the reflections by these vectors, we can recover everything. To see this always gives a root system, note that finiteness follows from everything lying on a lattice and being bounded. Most of the conditions are easy, and since it is closed under reflections of the generators and generated by them, it is closed under reflections of all elements. □

Example 6.12.1. A_n has Weyl group S_{n+1} coming from the permutations of the e_i.

Recall $\mathfrak{g}$ semisimple is $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha = \mathfrak{g} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$. where $\mathfrak{n}_+, \mathfrak{n}_-$ are the nilpotent subalgebra of positive/negative root spaces. We would like to find a description in terms of the Cartan matrix A_{ij}. Let $\gamma_1 \dots \gamma_r$ be the simple roots, and let e_i, f_i, h_i be $\mathfrak{sl}_2$-triples corresponding to γ_i. Note that they generate the algebra, are linearly independent, and satisfy the following relations called the **Chevalley relations**:

(1) $[h_i, h_j] = 0$

(2) $[h_i, e_j] = \langle \gamma_i | \gamma_j \rangle e_i$
(3) $[h_i, f_j] = -\langle \gamma_i | \gamma_j \rangle f_i$
(4) $[e_i, f_j] = \delta_{ij} h_i$

Note that these relations come from entries in the Cartan matrix: namely $\langle \gamma_i | \gamma_j \rangle$ is A_{ij}. Let $\tilde{\mathfrak{g}}$ denote the free Lie algebra presented by these relations. Let $\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_-$ denote the subalgebras generated by e_i and f_i respectively. When the Cartan matrix was constructed from a Lie algebra, there is a natural surjective map to $\tilde{\mathfrak{g}} \to \mathfrak{g}$. Note that the construction is symmetric in the f_i, e_i: we can swap the two and replace h_i with its negative and the same relations hold. This means we have to prove half as many things about the construction. We will still use $\gamma_i \in \Pi$ to denote the weights of the span of $\mathrm{ad}(h_i)$ on e_i.

Consider the $T(V)$ the tensor algebra on the vector space V generated by $v_1, \ldots, v_r$. This should be thought of as the universal enveloping algebra of $\tilde{\mathfrak{g}}/(\tilde{\mathfrak{n}}_+ \oplus \mathfrak{h})$. Of course this doesn't make sense since the quotient isn't by an ideal, but it is the quotient as an algebra and can probably be thought of as the invariant differential operators on the quotient formal group. We can produce an action of $\tilde{\mathfrak{g}}$ on $T(V)$ as follows: h_k sends $v_{i_1} \ldots v_{i_s}$ to $-(\sum_1^s A_{k,i_j}\rangle) v_{i_1} \ldots v_{i_s}$, f_k sends $v_{i_1} \ldots v_{i_s}$ to $v_k v_{i_1} \ldots v_{i_s}$, and e_k sends $v_{i_1} \ldots v_{i_s}$ to $\sum_1^s (\delta_{k,i_j} v_{i_1} \ldots \hat{v_{i_j}} \ldots v_{i_s})$. A straightforward calculation shows that this satisfies the Chevalley relations, so is indeed an action.

For the next lemma, it will be convenient to introduce notation. $e(s), f(s)$ denote an iterated bracket of s of the e_i, f_i respectively, and Σ to denote a linear combination. $f(0)$ or $e(0)$ will mean the h_i.

Lemma 6.13. *(1)* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{n}}_- \oplus \mathfrak{h}$
(2) $\mathfrak{n}_+, \mathfrak{n}_-$ *are free on the* e_i, f_i *and* $\tilde{\mathfrak{n}}_+ = \oplus_{\alpha \in \mathbb{Q}_+} \mathfrak{g}_\alpha, \tilde{\mathfrak{n}}_- = \oplus_{\alpha \in \mathbb{Q}_+} \mathfrak{g}_{-\alpha}$ *where* $\mathbb{Q}_+$ *is* $\mathbb{Z}_{\geq 0}\Pi - 0$ *and* α *is the weights of the* $\mathfrak{h}$ *action.*
(3) If I *is an ideal in* $\tilde{\mathfrak{g}}$ *then* $I = \mathfrak{h} \cap I \oplus \bigoplus_\alpha \mathfrak{g}_\alpha \cap I$.
(4) Maximal ideals in $\tilde{\mathfrak{g}}$ *correspond to components of the Dynkin diagram.*

Proof. (1): The action on $T(V)$ gives a Lie algebra homomorphism $\tilde{\mathfrak{g}} \to T(V)$. First note that the h_i are linearly independent since the Cartan matrix is nonsingular. The Jacobi identity (equivalent to the fact that ad is a derivation) $[h, e(s)] = [h, [e(1), e(s-1)]] = [[h, e(1)], e(s)] + [e(1), [h, e(s-1)]]$ shows via induction on s that for an iterated bracket of $e_{i_1}, \ldots, e_{i_s}$ that $\mathrm{ad}\, h_k$ has eigenvalue $\sum_1^s A_{k,i_j}$, and by symmetry the same is true for f_i with the opposite eigenvalue.

Next we can see inductively that $[f(s), e(s')]$ is a linear combination of $[f(s-s')]$ when $s \geq s' \geq 0$ via $[f(s), e(s')] = [f(s), [e(s'-1), e(1)]] = [[f(s), e(s'-1)], e(1)] + [e(s'-1), [f(s), e(1)]]$ reducing to $s' = 1$ and $[f(s), e(1)] = [[f(s-1), f(1)], e(1)] = [[f(s-1), e(1)], f(1)] + [f(s-1), [f(1), e(1)]]$ reducing to $s = 1$, where it follows from relation (4). The same statement holds when e, f are switched.

Thus $\tilde{\mathfrak{n}}_+ + \tilde{\mathfrak{n}}_- + \mathfrak{h}$ is a subalgebra, but it isn't clear that the sum is direct. However, by nondegeneracy we can find $h \in \mathfrak{h}$ such that its eigenvalue on $e(s)$ is positive, and hence on $f(s)$ is negative. Moreover $[\mathfrak{h}, \mathfrak{h}] = 0$ so the eigenspace decomposition of h separates these subalgebras.

(2): There is a commutative square

$$\begin{array}{ccc} F(f_i) & \longrightarrow & \tilde{\mathfrak{n}}_- \\ \downarrow & & \downarrow \\ \mathrm{End}(T(f_i)) & \longrightarrow & \mathrm{End}(T(V)) \end{array}$$

where the left vertical map is the canonical action on the universal enveloping algebra. Since the left arrow is injective and the bottom arrow is an isomorphism, the top arrow is injective. But it is also clearly surjective, so it is an isomorphism. Thus $\tilde{\mathfrak{n}}_+$ is free and by symmetry $\tilde{\mathfrak{n}}_-$ is too. The second statement was already proven in (1).

(3): Given an element $x \in I$, keep acting by the elements of $\mathfrak{h}$. Since $\mathfrak{h}$ acts semisimply, some linear combination of the $\mathrm{ad}(\mathfrak{h})^i x$ will be the projection onto the various eigenspaces of x.

(4): Consider any proper ideal of $\tilde{\mathfrak{g}}$. By (3) if its intersection with $\mathfrak{h}$ wasn't 0, it would have to contain all nonzero weight spaces of its intersection in $\mathfrak{h}$, which would contain atleast one f_i, e_i. Thus it would contain $[e_i, f_i] = h_i$, for which the eigenvalue doesn't vanish for all neighbors of γ_i. Continuing this way since it is indecomposable, it would have to contain the whole component. Thus the union of all ideals containing the part of $\mathfrak{h}$ orthogonal to some component of a Dynkin diagram is a maximal ideal, and moreover all maximal ideals must be contained in one of these. □

If the Cartan matrix is indecomposable, there is a unique maximal ideal, and if we already know that the Cartan matrix comes from a simple Lie algebra, the quotient has to be that simple Lie algebra. The only information this really gives us is knowing the quotient is finite-dimensional. However we will now prove that we should apriori expect the quotient to be finite dimensional.

Theorem 6.14. *Consider the intersection of all the maximal ideals of $\tilde{\mathfrak{g}}$. The quotient by this is a semisimple Lie algebra, whose Cartan subalgebra and root system are exactly the ones used to construct it.*

Proof. We can make a reduction to the case of an indecomposable Cartan matrix by observing that a splitting of the Cartan matrix splits everything involved $\mathfrak{h}, \tilde{\mathfrak{g}}, \tilde{\mathfrak{n}_+}, \tilde{\mathfrak{n}_-}$. In this case when it is simple, if we can find a proper ideal with quotient finite dimensions > 1, then the quotient is simple, and $\mathfrak{h}$ is clearly self normalizing with

$\mathfrak{g}_\alpha$ are its weight spaces, so we would be done. The quotient contains cannot be 1-dimensional since if an ideal contains any of the e_i, f_i, h_i, it must contain everything.

To prove finite-dimensionality, we first prove that when $i \neq j$ and k is arbitrary, $\operatorname{ad} e_k(\operatorname{ad} f_i)^{1-A_{ij}}(f_j) = 0$. If $k \neq i$, this is equal to $(\operatorname{ad} f_i)^{1-A_{ij}} \operatorname{ad} e_k(f_j) = (\operatorname{ad} f_i)^{1-A_{ij}}\delta_{jk}h_j = (\operatorname{ad} f_i)^{-A_{ij}}A_{ij}f_i = 0$. If $k = i$, e_i, f_i are part of an $\mathfrak{sl}_2$-triple, so it follows from representation theory of $\mathfrak{sl}_2$ that $\operatorname{ad} e_i(\operatorname{ad} f_i)^t(f_j) = t(A_{ik} - t + 1)(\operatorname{ad} f_i)^{t-1}(f_j) = 0$ by our choice of t.

Now let J_+, J_- be the ideals of $\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_-$ generated by $(\operatorname{ad} f_i)^{1-A_{ij}}(f_j)$ and $(\operatorname{ad} e_i)^{1-A_{ij}}(e_j)$. By what was just shown, these are also ideals of $\tilde{\mathfrak{g}}$. Let J be their sum. I claim $\tilde{\mathfrak{g}}/J$ is finite dimensional. Note that $\operatorname{ad} e_i, \operatorname{ad} f_i$ act locally nilpotently on this, since they do on generators and ad is a derivation. Now we still have for each i the $\mathfrak{sl}_2$-triple associated to e_i, f_i, h_i. Then $\tilde{\mathfrak{g}}/J$ is a sum of finite-dimensional modules for these $\mathfrak{sl}_2$-triples since the e_i, f_i act locally nilpotently.

Thus we can consider the action of τ_i from the representation theory of $\mathfrak{sl}_2$ which is an involution swapping positive and negative eigenspaces, implementing the Weyl group reflection of the corresponding simple root on $\mathbb{Z}\Pi$. We know that $\tilde{\mathfrak{g}}_{k\gamma_i}$ is 1-dimensional for $k = \pm 1$ and 0-dimensional for $|k| > 1$ so the same is true for anything connected to these by the Weyl group action. In particular all the roots of the corresponding root space have 1-dimensional eigenspaces. Now consider α in $\mathbb{Q}_+ \coprod -\mathbb{Q}_+$ that is not a multiple of a root. then the plane orthogonal to α isn't contained in any of the boundaries of the Weyl chambers, and we can find $\mu \in \mathbb{Z}\Pi$ orthogonal to it on the interior of a Weyl chamber. Then acting by an element of the Weyl group w we can move μ to the Weyl chamber corresponding to Π. Then $w(\mu)$ is a sum of positive multiples of the γ_i and since it is orthogonal to $w(\alpha)$, $w(\alpha)$ must contain both positive and negative terms in its decomposition as $\sum_i c_i\gamma_i$. But then $0 = \tilde{\mathfrak{g}}_{w(\alpha)} \cong \tilde{\mathfrak{g}}_\alpha$. Thus the quotient is finite-dimensional, and in fact we have proven J is the maximal ideal. □

7. Homological Methods

Lets expand on homological methods, which were first mentioned in the proof of Weyl's theorem that a semisimple Lie algebra has semisimple finite dimensional representations.

Usually Ext and Tor are the main tools of studying representations of algebras but for a Hopf algebra, we can do with a bit less. First observe that the antipode makes left and right modules canonically equivalent, so we can drop the lefts and rights.

Definition 7.1. *$H_*(\mathfrak{g}, M)$ is $\operatorname{Tor}_*^{U(\mathfrak{g})}(k, M)$, $H^*(\mathfrak{g}, M)$ is $\operatorname{Ext}^*_{U(\mathfrak{g})}(k, M)$.*

In otherwords, H_*, the homology, is the left derived functors of the coinvariants and H^*, cohomology, is the right derived functors of invariants. The $\mathfrak{g}$ will be dropped if unambiguous. The homology, cohomology should be thought of as sheaf cohomology on the associated Lie group (the sheaf being trivial for the module k). When the module is k itself, we can also refer to it as cohomology and homology of the Lie algebra. M should correspond to a right invariant trivial vector bundle such that differentiating the Ad action gives the actiono of $\mathfrak{g}$ on the fibre. Indeed this construction coincides with the construction of ad. The reason these are well equipped to replace Ext and Tor is the following:

Proposition 7.2. $\mathrm{Ext}^*(M,N) = H^*(\mathrm{Hom}(M,N))$, $\mathrm{Tor}^*(M,N) = H_*(\mathrm{Hom}(M,N)))$.

Proof. $\mathrm{Hom}_{\mathfrak{g}}(M,N) \cong \mathrm{Hom}_{\mathfrak{g}}(k, \mathrm{Hom}(M,N))$. Take right derived functors in N. On the right hand side, $\mathrm{Hom}(M,-)$ is exact and sends injectives to injectives by the tensor hom adjunction since tensor products are exact, so the edge homomorphism in the Grothendieck spectral sequence is an isomorphism.

The same argument works for Tor since $M \otimes_{\mathfrak{g}} N \cong k \otimes_{\mathfrak{g}} (M \otimes N)$. □

Since $U(\mathfrak{g}) \otimes_k U(\mathfrak{g}') = U(\mathfrak{g} \oplus \mathfrak{g}')$ we get Kunneth isomorphisms:

$$\bigoplus_{p+q=n} H_p(\mathfrak{g}, k) \otimes H_p(\mathfrak{g}', k) = H_n(\mathfrak{g} \oplus \mathfrak{g}', k)$$

$$\bigoplus_{p+q=n} H^p(\mathfrak{g}, k) \otimes H^p(\mathfrak{g}', k) = H^n(\mathfrak{g} \oplus \mathfrak{g}', k)$$

Using the diagonal $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ we get a coalgebra structure and an algebra structure on the homology and cohomology of $\mathfrak{g}$.

H_1, H^1 are easy to interpret. Let J denote the augementation ideal of $U(\mathfrak{g})$.

Lemma 7.3. $H_1(k) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$

Proof. $0 \to J \to U(\mathfrak{g}) \to k \to 0$, so taking the long exact sequence in homology, we get $H_1(k) = H_0(J) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. □

Lemma 7.4. $H^1(M) = \mathrm{Der}(M)/\,\mathrm{Ider}(M)$

Proof. Again look at the long exact sequence for $0 \to J \to U(\mathfrak{g}) \to k \to 0$ after taking Hom into M. We get $0 \to \mathrm{Hom}(k, M) \to \mathrm{Hom}(U(\mathfrak{g}), M) \to \mathrm{Hom}(J, M) \to H^1(M) \to 0$. $\mathrm{Hom}(J, M)$ is the derivations, and the image of $\mathrm{Hom}(U(\mathfrak{g})), M$ is the inner derivations. □

If M is trivial, then $\mathrm{Ider}(M)$ is trivial, and $\mathrm{Der}(M)$ is $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, M)$. Suppose we have a short exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$. The forgerful functor from $\mathfrak{g}/\mathfrak{h}$ modules to $\mathfrak{g}$ modules has left adjoint $\mathfrak{h}$-coinvariants and right adjoint $\mathfrak{h}$-invariants. Taking $\mathfrak{g}$-invariants is the composite of taking $\mathfrak{h}$-invariants and then taking $\mathfrak{g}/\mathfrak{h}$-invariants, and similarly for coinvariants so we get Grothendieck spectral sequences:

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)) \Rightarrow H^{p+q}(\mathfrak{g}, M)$$
$$E^2_{p,q} = H_p(\mathfrak{g}/\mathfrak{h}, H_q(\mathfrak{h}, M)) \Rightarrow H_{p+q}(\mathfrak{g}, M)$$

These are also called the **Hochschild-Serre spectral sequences**.

$H^2(M)$ has a simple interpretation: it is extensions of $\mathfrak{g}$ by the abelian Lie algebra M.

Theorem 7.5. $\mathrm{Ext}(\mathfrak{g}, M) = H^2(\mathfrak{g}, M)$.

Given an extension $0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$, we can look at low dimenisonal exact sequence of the spectral sequence in cohomology:

$$0 \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{e}, M) \to H^1(M, M)^{\mathfrak{g}} \to H^2(\mathfrak{g}, M) \to H^2(\mathfrak{e}, M)$$

The image of the identity in $H^1(M, M)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(M, M)$ is the class corresponding to the extension. This gives a map $\mathrm{Ext}(\mathfrak{g}, M) \xrightarrow{d^2} H^2(\mathfrak{g}, M)$. To go in the other direction, we can use a presentation of $\mathfrak{g}$.

Lemma 7.6. *For a free Lie algebra on the set X, J is a free module on the set X. Thus $H^i(M), H_i(M) = 0$ for $i > 1$.*

Proof. This is because it is the augmentation ideal of a tensor algebra. □

Now consider a presentation $0 \to \mathfrak{a} \to \mathfrak{f} \to \mathfrak{g} \to 0$, where $\mathfrak{f}$ is free, and mod out by the ideal generated by $[\mathfrak{a}, \mathfrak{a}]$ so that it is an extension of $\mathfrak{g}$ by $\mathfrak{g}_{ab}$ which is abelian. This is actually the unversal extension in that $\mathrm{Hom}(\mathfrak{a}_{ab}, M) = H^2(\mathfrak{g}, M)$ classifies extensions by M.

Given any extension $\mathfrak{e}$, we can form the diagram below since $\mathfrak{f}$ is free and M is abelian:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d] & \mathfrak{f} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{a}_{ab} \arrow[r] \arrow[d] & \mathfrak{f}/[\mathfrak{a}, \mathfrak{a}] \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathfrak{e} \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Now the map producing the class in H^2 corresponding to $\mathfrak{e}$ factors as

$$\mathrm{Hom}_{\mathfrak{g}}(M, M) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{a}_{ab}, M) = H^1(\mathfrak{a}_{ab}, M)^{\mathfrak{g}} \xrightarrow{d^2} H^2(\mathfrak{g}, M)$$

By naturality the extension class in $H^2(\mathfrak{g}, \mathfrak{a}_{ab})$ gets sent to the corresponding class in $H^2(\mathfrak{g}, M)$. Since $H^2(\mathfrak{f}, M)$ vanishes, the restriction of any class in $H^2(\mathfrak{g}, M)$ to

$\mathfrak{f}$ vanishes. From the low-dimensional exact sequence, this means that there is an element of $H^1(\mathfrak{a}, M)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{a}_{ab}, M)$ lifting $H^2(\mathfrak{g}, M)$. One can then from the semidirect product $M \ltimes \mathfrak{f}$ over $\mathfrak{a}$ to get an extension of $\mathfrak{g}$ by M realizing the class.

It suffices to show that the cohomology class determines the extension. But suppose there were two extensions $\mathfrak{e}_1, \mathfrak{e}_2$ coming from the same class. They are classified by some maps $\phi_1, \phi_2 \in H^1(\mathfrak{a}_{ab}, M)^{\mathfrak{g}}$. By assumption their image via d^2 is the same, so there is some element in $H^1(\mathfrak{f}/[a,a], M) = \mathrm{Der}(\mathfrak{f}, M)$ whose image is $\phi_1 - \phi_2$. WLOG, the map $\mathfrak{f} \to \mathfrak{e}_1, \mathfrak{e}_2$ is surjective, and it is then easy to see that $\phi_2 + (\phi_1 - \phi_2)$ descends to an isomorphism of extensions from $\mathfrak{e}_1$ to $\mathfrak{e}_2$.

Corollary 7.7. *If $H^2(\mathfrak{g}, M)$ vanishes for all M, then $\mathfrak{g}$ is free.*

Proof. Choose a presentation $\mathfrak{g} = \mathfrak{f}/\mathfrak{a}$ that is minimal in the sense that $\mathfrak{a} \subset [\mathfrak{f}, \mathfrak{f}]$. Mod out by $[\mathfrak{f}, \mathfrak{a}]$ to see that $\mathfrak{f}/[\mathfrak{f}, \mathfrak{a}] = \mathfrak{g} \oplus \mathfrak{a}/[\mathfrak{f}, \mathfrak{a}]$ since extensions by abelian things are trivial. But f is minimal, so taking the commutator of that equation, we see $\mathfrak{a}/[\mathfrak{f}, \mathfrak{a}] = 0$, but any subalgebra of a free algebra satisfying that is trivial. ☐

There is a canonical and efficient complex called the Chevalley-Eilenberg complex that computes Lie algebra homology and cohomology. It is a free resolution of k, and is essentially the de Rham complex.

Consider the chain complex where $V_n(\mathfrak{g}) = U\mathfrak{g} \otimes_k \Lambda^n \mathfrak{g}$, and $d(g \otimes w_1 \cdots \wedge w_n) = \sum_i (-1)^{i+1} u x_i \otimes w_1 \cdots \wedge \hat{w}_i \cdots \wedge w_n + \sum_{i<j} (-1)^{i+j} u \otimes [w_i, w_j] \otimes w_1 \cdots \wedge \hat{w}_i \cdots \wedge \hat{w}_j \cdots \wedge w_n$. There is an obvious augmentation $V_*(\mathfrak{g}) \to k$.

Lemma 7.8. *$V_*(\mathfrak{g})$ is a free resolution of k.*

Proof. There is a filtration on V_* coming from the tensor products of the filtrations on $U(\mathfrak{g}), \Lambda^*(\mathfrak{g})$. The associated graded by PBW is $k[\mathfrak{g}] \otimes_k \Lambda^*(\mathfrak{g})$, and differential is the Koszul differential. Thus the spectral sequence for the filtration degenerates at E^1 to the expected cohomology.

☐

Corollary 7.9. *For an $\mathfrak{g}$-module M,*

$$H_*(M) = H_*(M \otimes \Lambda^*(\mathfrak{g})), H^*(M) = H^*(\mathrm{Hom}(\Lambda^*(\mathfrak{g}), M)$$

This complex can give explicit proofs of the interpretations of H^1, H^2.

Corollary 7.10. *If $\mathfrak{g}$ has dimension n, its cohomological dimension is n.*

Proof. The Chevalley-Eilenberg complex shows it is at most n. But it also shows that $H^n(\Lambda^n \mathfrak{g}) = k \neq 0$. ☐

The Chevalley-Eilenberg complex for the Lie algebra of a Lie group can be used to show that cohomology and homology of the Lie algebra agree with homology and cohomology of a compact connected Lie group (over k). Namely, one can choose a bi-invariant metric, and use Hodge theory to show the harmonic forms are invariant and correspond to the cohomology of the Chevalley-Eilenberg complex. Using this we can see why we should expect $H^3(\mathfrak{sl}_2) = k$, since $\mathfrak{sl}_2\mathbb{C}$ is the complexification of the lie algebra of $\mathfrak{su}(2)$, and SU(2) is a 3-sphere. Indeed, the Chevalley-Eilenberg complex verifies this.

We already showed $H^1(\mathfrak{g}, M)$ vanishes for a semisimple Lie algebra, but the same is true for H^2. First we can upgrade Weyl's theorem.

Proposition 7.11. *Let M be a finite-dimensional nontrivial irreducible $\mathfrak{g}$ module in characteristic 0. Then $H^i(\mathfrak{g}, M) = H_i(\mathfrak{g}, M) = 0$ for any I.*

Proof. Over $\bar{k}$, L, the Casimir element, acts by a scalar on M and by 0 on k, hence it acts on the homology and cohomology, and the two actions coming from M and k coincide since L is central. Thus a nonzero scalar is equal to zero, so the cohomology and homology vanish. □

Thus all the interesting (finite-dimensional) cohomological information is in the trivial sheaf.

Theorem 7.12 (Second Whitehead Lemma)**.** *If $\mathfrak{g}$ is semisimple over characteristic 0 and M is finite-dimensional, then $H^2(\mathfrak{g}, M) = 0$.*

Proof. By the previous proposition, it suffices to show $H^2(\mathfrak{g})$ vanishes, which is equivalent to any saying any extension $k \to \mathfrak{e} \to \mathfrak{g}$ splits. To see that is true, simply observe that the $[\mathfrak{e}, \mathfrak{e}]$ gives a splitting since k commutes with everything and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. □

This is analogous to (and partially proves) a fact in Lie groups, that π_2 vanishes for any Lie group. Perhaps one complete the proof for compact Lie groups by proving that H_2 is the kernel of the universal central extension, so there can't be any torsion as it would give a nontrivial cover.

Corollary 7.13 (Levi Decomposition)**.** *In characteristic 0, any Lie group splits as a semidirect product $\mathfrak{a} \ltimes \mathfrak{g}$ where $\mathfrak{a}$ is solvable and $\mathfrak{g}$ is semisimple.*

Proof. Note that $\mathfrak{a}$ is really the radical, so we want to show that $\operatorname{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ splits. We can do this by induction on the length of the derived series. Mod out by $[\operatorname{rad}(\mathfrak{g}), \operatorname{rad}(\mathfrak{g})]$ to get an extension by an abelian Lie algebra, for which there is a splitting by the second Whitehead lemma. The preimage of this splitting is an ideal $\mathfrak{h}$ such that there is an extension $[\mathfrak{a}, \mathfrak{a}] \to \mathfrak{h} \to \mathfrak{g}$, at which point we can induct. □

Lemma 7.14 (Hopf)**.** *Let $\mathfrak{f}/\mathfrak{a}$ be a presentation of $\mathfrak{g}$. Then $H_2(\mathfrak{g}) = \frac{\mathfrak{a} \cap [\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{f}, \mathfrak{a}]}$.*

Proof. Consider the low term exact sequence from the homology spectral sequence for $0 \to \mathfrak{a} \to \mathfrak{f} \to \mathfrak{g} \to 0$ and the trivial module:

$$0 \to H_2(\mathfrak{g}, k) \to \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]_{\mathfrak{g}} \to \mathfrak{f}/[\mathfrak{f}, \mathfrak{f}] \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to 0$$

The second term is $\mathfrak{a}/[\mathfrak{f}, \mathfrak{a}]$, giving the result. □

Lemma 7.15. *If $\mathfrak{e}, \mathfrak{e}'$ are central extensions of $\mathfrak{g}$, and $\mathfrak{e}$ is perfect there is at most one homomorphism of extensions $\mathfrak{e} \to \mathfrak{e}'$.*

Proof. Any two homomorphisms differ by something in the kernel of $\mathfrak{e}$, and $\mathfrak{e}'$ is perfect and the kernel is central, it has to be 0. □

Theorem 7.16. *$\mathfrak{g}$ has a universal central extension iff $\mathfrak{g}$ is perfect, and it is a central extension by $H_2(\mathfrak{g}, k)$.*

Proof. Suppose $\mathfrak{g}$ has a universal central extension $\mathfrak{e}$, it must have a unique map to the trivial central extension, which implies that $\mathfrak{g}$ has to be perfect. Conversely let $\mathfrak{g}$ be perfect. Choose a presentation $\mathfrak{g} = \mathfrak{f}/\mathfrak{a}$, for $\mathfrak{f}$ free. Then $\mathfrak{h} = [\mathfrak{f}, \mathfrak{f}]/[\mathfrak{f}, \mathfrak{a}]$ will be shown to be the universal central extension. By the previous lemma, it is an extension by $H_2(\mathfrak{g})$. Moreover, one can see it is perfect. Since $\mathfrak{g}$ is perfect, any $x \in \mathfrak{f}$ decomposes as $x' + r$ where $x' \in \mathfrak{f}, r \in \mathfrak{a}$. Then choosing such a decomposition for $x = x' + r, y = x' + r$ Then $[x, y] = [x', y']$ modulo $[\mathfrak{f}, \mathfrak{a}]$. This gives uniqueness of a map to any extension.

Now let $\mathfrak{e}$ be any other central extension. There is a map from $\mathfrak{f}$ to $\mathfrak{e}$ lifting the projection to $\mathfrak{g}$, and since $\mathfrak{e}$ is central, it induces a map from $\mathfrak{h}$. □

Here is a recognition criterion for universal central extensions.

Lemma 7.17. *TFAE for a central extension $\mathfrak{e} \to \mathfrak{g}$:*

(1) $\mathfrak{e}$ is a universal central extension.
(2) Any central extension of $\mathfrak{e}$ splits in a unique way.
(3) $H_1(\mathfrak{e}) = H_2(\mathfrak{e}) = 0$.

Moreover the universal central extension is idempotent.

Proof. Consider $\mathfrak{e}'$, the universal central extension of the universal central extension, and let x be in its kernel to $\mathfrak{g}$. Since $\mathfrak{e}'$ is perfect, for any a, there is some a', a'' so that $[x, a] = [x, [a', a']]$. But by the Jacobi identity this is zero since $\mathfrak{e}', \mathfrak{e}$ are central extensions. Thus $\mathfrak{e}'$ is a central extension so it splits, and must be $\mathfrak{e}$.

(1) $\implies$ (2): Any central extension $\mathfrak{e}'$ of $\mathfrak{e}$ splits. $\mathfrak{e}'$ is thus also a central extension of $\mathfrak{g}$ and since $\mathfrak{e}$ is perfect, this splitting is unique by Lemma 7.15.

(2) $\implies$ (3): Since the a trivial k extension splits in a unique way, we see $H_1(\mathfrak{e}) = 0$. Moreover since any extension splits, the universal central extension splits, so it must be trivial, showing $H_2(\mathfrak{e}) = 0$.

(3) $\implies$ (1): Given any central extension of $\mathfrak{g}$, we can pull it back to a central extension of $\mathfrak{e}$ which splits, giving a map to the central extension. Uniqueness follows from perfectness. $\square$

8. To add

Affine Kac-Moody algebras Representation theory/Character formulas More on Cartan subalgebras Malcev theorem Ado's theorem