# NON-EMBEDDING RESULTS VIA S-DUALITY

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What is the minimal dimension such that $\mathbb{CP}^2$ or $\mathbb{HP}^2$ can be embedded in Euclidean space? It turns out one gets the optimal answer by answering the corresponding stable problem, namely what is the minimal dimension such that a space with the stable homotopy type of $\mathbb{CP}^2$ or $\mathbb{HP}^2$ can be embedded in Euclidean space?

We can answer a generalization of this question as follows (due to Hilton and Spanier):

**Theorem 0.1.** *Let $f$ be a map $S^{m-1} \to S^n$, and let $C_f$ denote the cofibre. Suppose that $f$ cannot be stably desuspended. Then the minimum dimension embedding of $C_f$ is $m+n+1$.*

In particular, $\mathbb{CP}^2$ and $\mathbb{HP}^2$ are cofibres of Hopf maps, which cannot stably be desuspended (a stable desuspension is another map of spheres of lower dimension agreeing stably with $f$). Note that $C_f$ can be embedded in $S^{m+n+1}$ because the mapping cylinder of $f$ embeds into the join of $S^{m-1}$ and $S^n$, which is $S^{m+n}$, and the cylinder end can be coned off in $S^{m+n+1}$. So it suffices to show one cannot embed into anything smaller.

First we consider the simplest cases. If $m < n+1$, then $f$ is trivial, so, we must have $n = 0$, in which case the assertion is obvious. If $m = n+1$, then $n = 1$. Then, one can use Alexander duality to observe that were there an embedding into $S^3$, then the complement would have zero dimensional homology that is not free.

Thus we can assume that $n > 1, m > n+1$.

**Lemma 0.2.** *If $n > 1, m > n+1$, $f$ can be stably desuspended iff $C_f$ can be stably desuspended.*

*Proof.* Clearly the stable homotopy type of $C_f$ depends only on that of $f$, proving one direction. On the other hand, if $C_f$ can be stably desuspended to a space $X$, a homology decomposition of $X$ will be the cofibre of a map $g$ between spheres. After suspending enough, these will be of the same dimension, and since the map between the middle skeleton has to extend to a homotopy equivalence between the two spaces, the attaching maps differ by a unit (i.e an integer multiple), so $f$ can be stably desuspended. $\square$

The essential input of working stably is the following observation: the Spanier-Whitehead (S) dual of the cofibre of $C_f$ (denoted $DC_f$) is $\Sigma^{-m-n}C_{\pm f}$, where the sign (unimportant) I think is $(-1)^{mn}$. To see this, the S dual of a map $f$ between spheres is $\pm f$, suspended to have the right degrees. We have a cofibre sequence, $f : S^{m-1} \to S^n \to C_f$, which taking duals gives a fibre=cofbre sequence $DC_f \to S^{-n} \to S^{1-m}$. Rearranging this shows that $DC_f = \Sigma^{-m-n}C_{\pm f}$. Now the complement of $C_f$ inside $S^{m+n}$ would be $\Sigma^{m+n-1}DC_f = \Sigma^{-1}C_f$! This completes the proof via the lemma.

The same argument gives the slightly stronger version:

**Theorem 0.3.** *Let $f$ be a map $S^{m-1} \to S^n$, and let $C_f$ denote the cofibre. Let $k$ be the maximum number of times that $f$ can be stably desuspended. Then the minimum dimension embedding of $C_f$ is $m + n + 1 - k$.*