COMPLEX ANALYSIS

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1. Basics of holomorphic functions

The basic objects of complex analysis are the complex manifold and functions between them. The simplest case is when the manifolds are 1-dimensional, in which case they are called Riemann surfaces. The first important observation (known to Gauss) is that for a holomorphic differential 1-form, integrating along a curve only depends on the homology class of the curve. This is known as Cauchy's theorem.

Theorem 1.1 (Cauchy's theorem)**.** *The integral of a holomorphic differential form along a 1-chain only depends on the homology class.*

Proof. If two curves γ, γ' are homologous via a surface M, then by Stokes's theorem $\int_\gamma f dz - \int_{\gamma'} f dz = \int_M df dz = 0$, as f is holomorphic, so $\frac{df}{d\bar{z}} = 0$ and hence $df dz = 0$. □

Now given a holomorphic differential form ω on a Riemann surface C and a point p, we can think of points γ in the universal cover $\tilde{C}$ as homotopy classes of paths starting at p. Then $\gamma \to \int_\gamma \omega$ gives a function on $\tilde{C}$ called the **primitive** of ω.

Theorem 1.2. *$g = \int_\gamma \omega$ is holomorphic.*

Proof. This needs to be checked locally. $g(z+h) - g(z)$ is the integral of the path from z to $z+h$, so we see that the derivative at z is f, where $\omega = f(z)dz$. □

For example, if ω is $\frac{1}{z}dz$, then the resulting function is $\log(z)$. For any simply connected region of the universal cover of $\mathbb{C}^\times$, this descends to a function on an open set of $\mathbb{C}^\times$.

Cauchy's theorem let's us compute integrals of meromorphic differential forms in terms of residues. The residue of a differential form fdz at a point p is a_{-1} where f locally is given by $\sum_{-m}^{-1} a_k(z-p)^k + H$, where H is holomorphic. Amazingly enough, this gives an analytic proof of the algebraic fact that a_{-1} is an invariant of the coordinate system.

Theorem 1.3 (Residue formula)**.** *If fdz is a meromorphic differential form, and γ is a counterclockwise oriented curve bounding a region X, then $\int_\gamma fdz = 2\pi i \sum_{p \in X} \mathrm{res}_p(fdz)$.*

Proof. By Cauchy's theorem, this reduces to a local computation around each pole. Then note that the integral for $\frac{1}{z^k}dz$ is 0 for $k > 1$ and $2\pi i$ for $k = 1$, and since locally the form differs from a holomorphic form by a finite linear combination of these, we are done. $\square$

Theorem 1.4 (Cauchy integral formulas)**.** *A holomorphic function is analytic, and its n^{th} derivative at a point p is given by $n! \int \frac{f(z)}{2\pi i(z-p)^{n+1}}dz$ where the integral is over a small circle around p. The radius of convergence for the power series for f at a point is the largest it could possibly be.*

Proof. The case $n = 0$ follows from the residue formula, and by differentiating inside the integral, we get the formula for all n. To see analyticity, suppose that f is analytic in the neighborhood of p of radius $< r$, then let z be inside this circle. Now integrating over the circle, we compute $f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)}{\xi - z}d\xi = f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)}{\xi-p}\frac{1}{1-\frac{z-p}{\xi-p}}d\xi = \frac{1}{2\pi i}\int \sum_{n\geq 0}\frac{f(\xi)}{(\xi-p)^{n+1}}(z-p)^n d\xi = \frac{1}{2\pi i}\sum_{n\geq 0}\int \frac{f(\xi)}{(\xi-p)^{n+1}}d\xi(z-p)^n = \sum_{n\geq 0}\frac{f^{(n)}(p)}{n!}(z-p)^n$. $\square$

Corollary 1.5 (Cauchy inequalities)**.** $|f^{(n)}(z)| \leq \frac{n!}{R^{n+1}}\sup_{\gamma_R}|f(z)|$, *where γ_R is the circle of radius R around z.*

Proof. Try to bound the integral formulas. $\square$

The integral formulas have an interpretation via Fourier series. Let f be a holomorphic function on the unit disc.

Corollary 1.6. *The Fourier coefficients for f restricted to the unit circle are the coefficients in the power series for f at 0.*

Proof. Try to explicitly calculate using the integral formulas. Note that the 0^{th} coefficient is the mean value property. $\square$

Corollary 1.7 (Riemann's theorem on removable singularities)**.** *Suppose that f is bounded and holomorphic and a deleted neighborhood of a point p. Then f can be extended holomorphically to p.*

Proof. Simply define f at p via the integral formula, or alternatively, note that we can extend $(z-p)^2 f$ holomorphically to be 0 at p, and then divide by the factor we put in. $\square$

The unit disk and the complex plane are not isomorphic Riemann surfaces:

Theorem 1.8 (Liouville's theorem)**.** *A bounded holomorphic function is constant.*

Proof. Apply the Cauchy inequalities. $\square$

Here is a converse to Cauchy's theorem.

Theorem 1.9 (Morera's theorem). *If f is continuous and $\int_T f(z)dz = 0$ for all triangles, then f is holomorphic.*

Proof. Note that we can locally define a holomorphic primitive of f using this fact, and it is analytic, so its derivative, f, is as well. □

Lemma 1.10. *If $f_i \to f$ uniformly on compact sets, then f is holomorphic, and the derivatives of f_i also converge to the derivatives of f.*

Proof. By Morera's theorem, the limit is holomorphic. Apply the uniform convergence to the integral formulas for the derivatives of a holomorphic function to obtain the second part. □

Lemma 1.11. *If $f(z, s)$ is continuous and holomorphic in the variable z, then $\int_{[0,1]} f(z, s)$ is holomorphic, and its derivatives are $\int_{[0,1]} f^{(n)}(z, s)$.*

Proof. Either observe by Fubini's theorem that Morera's theorem is satisfied, or note that the integral is a uniform limit of Riemann sums. □

Theorem 1.12 (Rigidity). *If f is holomorphic on a connected region and vanishes on a set of points with a limit point, then $f \equiv 0$.*

Proof. Consider the series expansion of f around the limit point p, and note that for a small enough disk around p, f cannot vanish, as it behaves like the first non-vanishing term. Thus the set of points on which f is locally identically 0 is open, but it is also closed so we are done. □

Corollary 1.13 (Analytic continuation). *If f, g are defined on regions that intersect on a connected region, and the set of points on which they agree have a limit point, then $f \equiv g$.*

Proof. By rigidity, $f - g$ is 0 on their region of common definition. □

Theorem 1.14 (Local behaviour). *Every nonconstant holomorphic function f after a holomorphic change of variables locally looks like $z \mapsto z^k$ for some $k > 1$.*

Proof. WLOG we are looking at f near 0 and $f(0) = 0$. Then f has a power series expansion, so for some k, $\frac{f}{z^k}$ has a power series expansion starting with a nonzero constant c. Since $\frac{f}{z^k}$ is continuous, for a small enough neighborhood around 0, $\frac{f}{z^k}$ is the k^{th} power of a holomorphic function g since the map $z \mapsto z^k$ is a covering map away from 0. Thus $f = (gz)^k$ locally, and gz locally has an inverse. □

Corollary 1.15 (Open mapping theorem). *A nonconstant holomorphic function is an open map.*

Proof. It locally looks like $z \mapsto z^k$, which is an open map. $\square$

Corollary 1.16 (Fundamental Theorem of Algebra)**.** *Any nonconstant polynomial has a root.*

Proof. A nonconstant polynomial p is a holomorphic function from $\mathbb{P}^1 \to \mathbb{P}^1$ so it has closed image, but it also is an open map, so must be surjective. Alternatively if there were no roots, $\frac{1}{p}$ would be an entire bounded holomorphic function, so would be constant. $\square$

Corollary 1.17 (Maximum modulus principle)**.** *For a nonconstant holomorphic function f, $|f(z)|$ cannot attain a maximum on the interior of a set.*

Proof. This follows as f is an open map. $\square$

Theorem 1.18. *The meromorphic functions on $\mathbb{P}^1$ are the rational functions.*

Proof. Note that such a meromorphic function f has finitely many zeros and poles. after multiplying by a polynomial, we can remove all the poles in $\mathbb{C}$. But then we have have at worst a pole at infinity, so $|f(z)|$ grows at most like $c|z|^k$ where k is the order of the pole. But then by the Cauchy inequalities, some derivative of f is 0, so f is a polynomial. $\square$

Theorem 1.19 (Schwarz reflection principle)**.** *Suppose that f is holomorphic on a region D^+ in the upper half plane and extends continuously to a real-valued function $\partial D \cap \mathbb{R}$. Then if D^- is D reflected along $\mathbb{R}$, then f extends to a holomorphic function on the interior of $D^+ \cup D^-$.*

Proof. Define f on D^- by $f(\bar{z}) = \overline{f(z)}$, and note that it is holomorphic on the interior, since it is still given by a power series around every point. Moreover, by Morera's theorem it is holomorphic at the boundary connecting D^+, D^-. $\square$

Theorem 1.20 (Argument principle)**.** *For f meromorphic, $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}dz$, integrated over a closed curve bounding a region X is the number of zeros minus the number of poles of f in the region, counted with multiplicity.*

Proof. First note that the formula in the statement is a homomorphism from meromorphic functions to $\mathbb{R}$. Then note that for a non-vanishing holomorphic function, it is 0, and that for $(z - z_0)^k$, where z_0 is in X, it is k. $\square$

The following theorem can be used to give an alternate proof of the open mapping theorem.

Theorem 1.21 (Rouché's theorem)**.** *If, $|f| > |g|$ are holomorphic on the interior of an open set, then $f, f + g$ have the same number of zeros.*

Proof. Let $f_t = f + tg$, and note that the argument principle gives a continuous function from the number of zeros to $\mathbb{Z}$, so it must be constant. $\square$

Theorem 1.22 (Runge's approximation theorem)**.** *If f is holomorphic on an open set U and K is compact in U, then f can be uniformly approximated on K by meromorphic functions, and if K^c is connected, it can be uniformly approximated by polynomials.*

K is contained in finitely many squares D_i that are part of a grid and inside U. $f(z)$ can be given by $\frac{1}{2\pi i}\sum_i \int_{D_i} \frac{f(\zeta)}{\zeta - z} d\zeta$, and each integral can be uniformly approximated by Riemann sums, which are rational functions. Finally, consider the set S of z_0 such that $\frac{1}{z-z_0}$ can be uniformly approximated by polynomials. Using the geometric series, since K is bounded, sufficiently large $|z_0|$ are in S. Now clearly S is closed, and to see S is open, simply note that $\frac{1}{z-z_1} = \frac{1}{z-z_0}\frac{1}{1-\frac{z_1-z_0}{z-z_0}}$, and so for z_1 close to z_0, the geometric series again gives a uniform approximation.

2. Entire functions

The number of zeroes of an entire function is related to the growth of the function.

Theorem 2.1 (Jensen's Formula)**.** *Let f be holomorphic near the disk of radius R, and nonzero at 0. Then if $z_1, \dots, z_k$ are the zeros of f, $\log|f(0)| = \sum \log(\frac{|z_i|}{R}) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta$.*

Proof. Note that if the formula holds for f, g, then it holds for fg. Thus we only need to show it for $z - z_0$ where z_0 lies in the disk, and non-vanishing holomorphic functions. For the latter, note that $\log|f(z)|$ is the real part of the log of f, so the formula follows from the mean value property. For $f = z - z_0$, we have $\frac{1}{2\pi}\int \log|Re^{i\theta} - z_0| d\theta - \log R = \frac{1}{2\pi}\int \log|e^{i\theta} - \frac{z_0}{R}| d\theta = \frac{1}{2\pi}\int \log|1 - \frac{z_0}{R}e^{i\theta}| d\theta$ by a change of variables $\theta \to -\theta$, which by the mean value property is $\mathrm{Re}(\log(1 - \frac{z_0}{R}z))|_{z=0} = 0$. $\square$

Given a holomorphic function f not vanishing at 0, let $n_f(r)$ be the number of zeros with modulus $\leq R$. Then Jensen's formula gives

Corollary 2.2. $\int_0^R \frac{n_f(r)}{r} dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|$.

As an example, let f be an **entire function of finite order** ρ, where ρ is the infimum of r such that $|f(z)| \leq Ce^{A|z|^r}$.

Theorem 2.3. *If f is of finite order $\leq \rho$, then $n_f(R) = O(|R|^\rho)$. Moreover, if z_k are the zeros, then $\sum_i |z_k|^{-s}$ converges for $s > \rho$.*

Proof. WLOG, $f(0) \neq 0$. Jensen's formula gives $n_f(R) \leq 2\int_R^{2R} \frac{n_f(r)}{r} dr \leq \frac{1}{\pi}\int \log|f(2Re^{i\theta})| d\theta - \log|f(0)| = O(|R|^\rho)$. This bound generally implies convergence of the series. $\square$

Lemma 2.4. *Let $F_i(z)$ be holomorphic functions on U such that $|F_i(z) - 1| \le c_n$ for some summable sequence c_n. Then $F(z) = \prod_i F_i(z)$ converges uniformly to a holomorphic function on U that vanishes iff one of the F_i does. Moreover, if no F_i vanishes at a point p, then $\frac{F'(p)}{F(p)} = \sum_i \frac{F_i'(p)}{F_i(p)}$.*

Proof. WLOG we can assume that $|F_i(z) - 1| \le \frac{1}{2}$ as c_n is summable. Then we have $F(z) = e^{\sum_i \log F_i(z)}$. By the power series for $\log(1+z)$, $\log F_i(z) \le 2|F_i(z) - 1| \le 2c_n$, so the sum uniformly converges to a holomorphic function. Note in particular this shows that $F(z)$ doesn't vanish as it is the exponential of something finite. Moreover, $\log(F)'(z) = \sum_i \log(F_i)'(z) = \sum_i \frac{F'}{F}(z)$. □

We will now derive the product formula for $\sin(z)$.

Lemma 2.5. $\pi \cot(\pi z) = \sum_{k \in \mathbb{Z}} \frac{1}{z+k} = \frac{1}{z} + \sum_{n>0} \frac{2z}{z^2 - n^2}$.

Proof. These can be characterized by Liouville's theorem as the unique $\mathbb{Z}$-periodic meromorphic function bounded away from a simple pole at $\mathbb{Z}$ of residue 1. □

Proposition 2.6. $\sin(\pi z) = \pi z \prod_1^\infty (1 - \frac{z^2}{n^2})$.

Proof. Their logarithmic derivatives agree by the previous two lemmas, and so the two sides agree up to multiplication by a constant. By dividing by z and taking the limit as $z \to 1$, we get 1 on both sides, so they are equal. □

Given a sequence $a_n \in \mathbb{C}$ approaching ∞, we can produce a function F vanishing exactly at the a_i. WLOG, the a_i are nonzero. The function $\prod(1 - \frac{z}{a_i})$ may not converge, so we will have to correct for this. The idea is as follows: $(1 - z) = e^{\log(1-z)} = e^{-\sum \frac{z^j}{j}}$. Thus $E_k(z) = (1-z)e^{\sum_1^k \frac{z^j}{j}}$. for large k will not grow as fast, but will have the same zeros as $E_0(z) = (1 - z)$. The $E_k(z)$ are called **canonical factors**.

Theorem 2.7 (Weierstrass)**.** *If $a_i \neq 0$ tend to ∞, $\prod_i E_i(\frac{z}{a_i})$ is entire and has zeroes exactly at the a_i.*

Proof. For $|z| \le \frac{1}{2}$, $|E_i(z) - 1| = |e^{\sum_{k+1}^\infty \frac{z^j}{j}} - 1|$. The sum is bounded above by $c|z|^{k+1}$, so we get this is $\le c'|z^{k+1}|$. If we fix some r, then in the ball of radius r, eventually the $\frac{z}{a_i}$ are within $\frac{1}{2}$, so the product converges. □

Any function f can be divided by a function constructed as above to get a non-vanishing entire function, which must be of the form $e^{g(z)}$ for some entire g. This can be refined for f of finite order ρ.

Lemma 2.8. *Let g be a holomorphic function such that $\operatorname{Re} g(z) \le c|z|^s$ for values of $|z|$ tending to ∞. Then g is a polynomial of degree at most s.*

Proof. Let $g(z) = \sum_i a_i z^i$. By Lemma 1.6, we have that $a_n = \frac{1}{2\pi R^n}\int_0^{2\pi} g(Re^{i\theta})e^{-in\theta}d\theta$. Conjugating the equation and adding for n and $-n$ gives $a_n = \frac{1}{\pi R^n}\int_0^{2\pi} \operatorname{Re} g(Re^{i\theta})e^{-in\theta}d\theta$ for $n > 0$ and $\operatorname{Re} a_0 = \frac{1}{\pi}\int_0^{2\pi}\operatorname{Re} g(e^{i\theta})d\theta$. Thus we have for $n > s$, $|a_n| = \frac{1}{\pi R^n}|\int_0^{2\pi}\operatorname{Re} g(Re^{i\theta})e^{-in\theta}d\theta| = \frac{1}{\pi R^n}|\int_0^{2\pi}\operatorname{Re}(cR^s - g(Re^{i\theta}))e^{-in\theta}d\theta| \leq 2CR^{s-n} - \operatorname{Re} a_0 R^{-n} \to 0$ as $R \to \infty$. $\square$

Theorem 2.9 (Hadamard factorization theorem)**.** *If f is a holomorphic function of finite order ρ and $k = \lfloor\rho\rfloor$, then $f = e^P(z)z^m\prod_i E_k(\frac{z}{a_i})$, where P is a polynomial of degree $\leq k$.*

Proof. The product $E(z) = z^m\prod_i E_k(\frac{z}{a_i})$ converges, where m is the order of the zero of f, and a_i the other zeroes, as f is finite order, so $|E_k(\frac{z}{a_i}) - 1|$ is bounded from above by $c|\frac{z}{a_i}|^{k+1}$, and the series $\frac{1}{|a_i|^{k+1}}$ converges. Thus $\frac{f}{E}$ is an entire function with no zeroes, so is $e^{g(z)}$ for some g. If we can produce lower bounds on E for arbitrarily large $|z|$, then by the lemma, g will have to be a polynomial. To do this, for $\rho < s < k+1$, we would like to show that $|\prod_k E_k(\frac{z}{a_n})| \geq e^{-C|z|^s}$ except maybe in the balls around each a_i of radius $|a_i|^{-(k+1)}$. Then since $R - \sum_{|a_i|\leq R} 2|a_i|^{-k+1} \to \infty$ as $R \to \infty$, there are are arbitrarily large R such that when $|z| = R$, it doesn't intersect any of the balls, and the bound holds. This bound will then be good enough to show that g is a polynomial of the right degree.

To obtain the bound, first obtain from the definition of $E_k(z)$ that when $|z| \leq \frac{1}{2}$, $|E_k(z)| \geq e^{-c|z|^{k+1}}$ and $|E_k(z)| \geq |1-z|e^{-c'|z|^k}$ if $|z| \geq \frac{1}{2}$. Then split the product $|\prod_k E_k(\frac{z}{a_n})|$ into the product when $|a_n| \leq 2|z|$ and $|a_n| > 2|z|$. For the second product, we have $\prod_n |E_k(\frac{z}{a_n})| \geq e^{-c|\sum_n \frac{|z|^{k+1}}{a_n^{k+1}}} \geq e^{-c''|z|^s|\sum_n a_n^{-s}} \geq e^{-c'''|z|^s|}$ as the sum converges. For the first product, $\prod_n |E_k(\frac{z}{a_n})| \geq \prod|1 - \frac{z}{a_n}|\prod e^{-c'|\frac{z}{a_n}|^k}$. The first term is $\prod\frac{|a_n - z|}{|a_n|}$, which is $\geq \prod\frac{1}{|a_i|^{k+2}} \geq e^{-(k+2)\log n_f(2|z|)\log(2|z|)}$, and applying Theorem 2.3, this is $\geq e^{-C'|z|^{s+\epsilon}}$, where $\epsilon > 0$ is arbitrary. Note that if we had just chosen s a bit smaller, then $s+\epsilon$ can be chosen to be the original real number we wanted. Finally for $\prod e^{-c'|\frac{z}{a_n}|^k}$, note $c'\sum|\frac{z}{a_n}|^k \leq C''|z|^s\sum\frac{1}{|a_n|^s}$, so that the overall bound $|\prod_k E_k(\frac{z}{a_n})| \geq e^{-C|z|^s}$ holds outside the chosen balls. $\square$

3. The gamma function

The gamma function $\Gamma(s)$ is a analytic function that extends the factorial to a meromorphic function. It begins with integration by parts: $\int_0^\infty t^s e^{-t}dt = t^s(-e^{-t})|_0^\infty - \int_0^\infty st^{s-1} - e^{-t} = s\int_0^\infty t^{s-1}e^{-t}$. If we define $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$, then it will satisfy $\Gamma(s) = (s-1)\Gamma(s-1)$. Moreover $\Gamma(1) = 1$, and so by induction $\Gamma(n) = (n-1)!$. Note that we can split the integral into the ones on $[0,1]$ and $[1,\infty]$, and that each of these integrals converges uniformly when $\operatorname{Re} s$ is bounded above and below, so $\Gamma(s)$ is a holomorphic function on $\operatorname{Re}(s) > 0$. Moreover the equation $\Gamma(s) = (s-1)\Gamma(s-1)$ shows that it extends to a meromorphic function with simple poles at the nonnegative integers.

4. Elliptic functions

Given a lattice Γ in $\mathbb{C}$, we can consider the Riemann surface (elliptic curve) given by $E = \mathbb{C}/\Gamma$. dz gives a trivialization of the canonical bundle of E, and hence of all of its powers. Thus to give a meromorphic function or differential form it suffices to give a Γ-periodic function f, called an **elliptic** function. Let $\mathbb{C}(E)$ be the field of elliptic functions. Given an elliptic function f, let $v_p(f)$ be the order of the zero/pole at p. For a fixed nonzero f only finitely many of these can be nonzero. The facts marked RS denote that the theorem holds for some more general type of Riemann surface.

Lemma 4.1 (RS)**.** $\sum_p v_p(f) = 0$.

Proof. Use the argument principle and integrate around a fundamental domain. One one hand the sides cancel out so we get 0, and on the other hand, we get the sum of the orders of the poles and zeroes on the interior. □

The **order** of an elliptic function is the sum of the orders of the zeros.

Corollary 4.2 (RS)**.** *A holomorphic elliptic function is constant.*

Proof. This follows from Louisville's theorem and the previous lemma. □

Lemma 4.3 (RS)**.** *The sum of the residues of an elliptic function is* 0.

Proof. Integrate f around the fundamental domain. □

Corollary 4.4 (RS)**.** *An elliptic function cannot have a unique simple pole.*

Proof. The sum of the residues would be nonzero. □

Now we let $\mathbb{H}$ be the upper half plane, and consider M, the quotient of $\mathbb{H}/\operatorname{PSL}_2(\mathbb{Z})$ via fractional linear transformations. This space classifies lattices up to homothety, and hence elliptic curves, by identifying the lattice $[\tau, 1]$ with τ, where $\tau \in \mathbb{H}$. Note that $d(\frac{az+b}{cz+d}) = \frac{1}{(cz+d)^2}dz$, so that a section of ω^k on $\mathbb{H}/\operatorname{PSL}_2(\mathbb{Z})$ is given by a weakly modular function of weight $2k$. Really, we want to compactify $\mathbb{H}$ to include the point $\mathbb{P}^1_{\mathbb{Q}}$, so that a section of ω^k is a modular form of weight k.

We will construct a function on the universal cover of the universal elliptic curve as follows: $\wp_\tau(z) = \frac{1}{z^2} + \sum'_{w \in \Gamma_\tau}((z+w)^{-2} - w^{-2})$, where the sum is over nonzero lattice points. It isn't hard to see that this converges uniformly on set bounded away from lattice points, so defines a meromorphic function. To see this is indeed elliptic, first note its derivative $\frac{\partial \wp}{\partial z} = \wp' = \sum_{w \in \Gamma_\tau} \frac{-2}{(z+w)^3}$ is clearly odd, elliptic, and is order 3 since it has a pole at the lattice points. Thus after translation by a fixed lattice point, $\wp$ can change by some constant. But $\wp$ is even, so that constant had better be 0, and so it is elliptic. It is easy to see that the zeroes of $\wp'$ are two torsion, since the opposite terms in the sum cancel out, and these must be simple as $\wp'$ is order 3.

Proposition 4.5. *An even elliptic function is a rational function in $\wp$.*

Proof. Let f be our function. If f has a pole at a non-lattice point c, we can multiply f by a power of $\wp(z) - \wp(c)$ to remove it. We are left with a function that only has poles at the lattice point. Note that if f has a zero at a point of 2-torsion, it must have a double zero, as it is even with respect to that point as well. Thus we can subtract a constant from f until it has a zero in common with $\wp$, and then divide by $\wp$. This lowers the order, as it will cancel out zeroes and poles, so repeat this process. $\square$

Corollary 4.6. $\mathbb{C}(E) = \mathbb{C}(\wp, \wp')$ *where* $\wp'^2 = 4\prod_{x \in E[2]}(\wp(z) - \wp(x))$.

Proof. Every function is the sum of an even and odd function. An odd function times $\wp'$ is even, and so we get the first part. For the second part, note that the two functions have the same zeroes and poles, and the factor of 4 comes from looking at the expansion around 0. $\square$

By looking at the series expansion at the origin, one sees that $\mathfrak{p}_\tau(z) = \mathfrak{p}_{\tau+1}(z)$. Moreover, $\mathfrak{p}_{\frac{-1}{\tau}}(z) = \tau^2 \mathfrak{p}_\tau(\tau z)$.

Given a lattice $[b, a]$, define G_k, $k \geq 1$ to be the sum $\sum_{m \in \mathbb{Z}} \sum'_{n \in \mathbb{Z}} \frac{1}{(na+mb)^{2k}}$ where the sum runs over the nonzero lattice points. Because of the way that G_k scales, it is weakly modular of weight k when $k \geq 2$ When $k = 1$ it is still invariant under $\tau \mapsto \tau + 1$. And indeed, we can compute the q-expansion, so these are actually modular for $\mathrm{PSL}_2(\mathbb{Z})$ when $k > 1$. First, $\pi(1 - 2\sum_0^\infty q^n) = \pi\frac{q+1}{q-1} = \pi\cot(\pi\tau) = \sum \frac{1}{\tau+m}$, and so taking the $k - 1^{th}$ derivative ($k \geq 2$), we get $\sum \frac{(-2\pi i)^k}{(k-1)!} \sum_{l \geq 0} l^{k-1} q^l = \sum \frac{1}{(\tau+m)^k}$. Thus $G_k(\tau) = 2\zeta(2k) + 2\sum_1^\infty \sum_{\mathbb{Z}} \frac{1}{(n\tau+m)^{2k}} = 2\zeta(2k) + \frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!}\sum_1^\infty \sigma_{2k-1}(l)q^l$.

Moreover, looking at the definition of $\wp$, we see that $\wp(z) = \frac{1}{z^2} + \sum_1^\infty (2k + 1)G_{k+1}z^{2k}$. Moreover, comparing the negative terms of $\wp, (\wp')^2, \wp^3$ near 0, we get that $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ where $g_2 = 60G_2$, $g_3 = 140G_3$.