EULER'S DESCENT

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Which odd primes are of the form x^2+ny^2? One approach to solve this problem for small n taken by Euler is to split it into two problems, the reciprocity step, or finding when $p|x^2+ny^2, p\nmid x, y$ and the descent step, to take the number that p divides and replace it with p itself. For the reciprocity step, $p|x^2+ny^2 \iff (\frac{-n}{p}) = 1$, and this can be characterized via Jacobi reciprocity. Supposing we have $p|N = x^2+ny^2$, we can always choose $|x|, |y| < p/2, (x, y) = 1$, so that $N < (1+n)(\frac{p^2}{4})$. If $n \leq 3$, all other factors of N have to be smaller than p.

Now the main lemma is that if $N \in \mathbb{Z}$ and $q \in \mathrm{Spec}(\mathbb{Z})$ are of the form a^2+nb^2 for relatively prime a, b, then N/q is too. From this, we can either continue to get smaller N that p divides until $p = N$, or find a smaller prime not of the form a^2+nb^2. By Fermat descent this is impossible (for n=3 one has to be make sure that the descending sequence of primes is odd).

So we are left to prove the main lemma, which holds for any n.

Lemma 0.1. *If $N \in \mathbb{Z}$ is of the form a^2+nb^2 and $q \in \mathrm{Spec}(\mathbb{Z})$ is of the form x^2+ny^2 and $q|N, q\nmid n$, then $N/q = c^2+nd^2$. Moreover, if $(a, b) = 1$, $(c, d) = 1$.*

Proof. The key to proving this will be two view it as a partial converse to the fact that

$$(x^2+ny^2)(c^2+nd^2) = (xc-nyd)^2+n(xd+yc)^2$$

To get the result, we will try to reconstruct c, d from the data a, b, x, y, q, N, using the fact that q is prime. To use this fact, we will need q to divide something that factors, namely, $Nx^2-qa^2 = n(xb-ya)(xb+ya)$. By possibly changing the sign of a, we can assume $q|xb-ya$. But note that $xb-ya = dq$ in the equation above, so we can call define d according to the main equation above. Similarly, $q|Nx^2-nqb^2 = (ax+nby)(ax-nby)$ so after a change of sign $q|ax+nby = qc$, so we can recover c. To check that this is sufficient, we calculate

$$q(c^2+nd^2) = \frac{(ax+nby)^2+n(xb-ya)^2}{q} = \frac{Nq}{q} = N$$

Finally if we can show that actually $a = xc-nyd, b = xd+yc$, it will be shown that $(c, d) = 1$ if $(a, b) = 1$. To do this, we can recover c in a different way. Namely we can try to show that $x|a+nyd = xc$, and since $(x, y) = 1$, this is equivalent to

$x|ay + ny^2d = bx - dx^2$, so we can define c this way. Then the main equation still holds. □

Remark: the condition $q \nmid n$ has to do with ramification in the ring $\mathbb{Z}[\sqrt{-n}]$.

Remark: We should not expect in general for it to be the case that $p|a^2+nb^2 \implies p = a^2+nb^2$. For example, $2|1^2+5*1^2$, but 2 is not of the desired form. This can be explained by the fact that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. Indeed $\mathbb{Z}[i], \mathbb{Z}[\sqrt{-2}], \mathbb{Z}[\sqrt{-3}]$ are UFDs away from the prime 2 so, from a more modern point of view we can see why this argument only works for small n.