2ND ORDER ELLIPTIC PDE

ISHAN LEVY

1. Introduction

Here we will consider 2nd order linear elliptic PDE, which are PDE that behave similarly to the standard Laplacian Δ in Euclidean space. They can be interpreted as generalizing what the Laplacian does: solutions are equilibria of the flow of some quantity such as a chemical. The extra generality comes from the fact that now other factors such as the geometry of the ambient space, transport with in the space, and creation/depletion of the quantity are considered. The Laplacian itself makes sense for a Riemannian manifold, in which case it is the divergence of the differential of a function. The kinds of spaces that much the discussion here easily extends to are those with finite volume, although for simplicity we will always work with a bounded open subset $U \subset \mathbb{R}^n$.

An elliptic 2nd order operator is a partial differential operator of the form (Einstein notation is used throughout) $Lu = -(a_{ij}u_{x_i})x_j + b_iu_{x_i} + cu$, where a_{ij}, b_i, c are functions, and (a_{ij}) is positive definite at each point. We will usually want L to be uniformly elliptic, meaning that a_{ij} is uniformly positive definite, or satisfies $a_{ij}\xi_i\xi_j \geq C|\xi|^2$ on U for some $C > 0$. We will also assume $a_{ij} = a_{ji}$.

2. Existence and uniqueness

In this section we will assume $a_{ij}, b_i, c \in L^\infty(U)$.

It can be hard in general to find explicit solutions to $Lu = f$ (one can try to find Green's functions for example), but using some functional analysis it is possible to show weak solutions exist and are sometimes unique. Namely, we will recast solving the PDE $Lu = f$, $u = 0$ on ∂U as follows: If $Lu = f$, then for any test function $v \in C_0^\infty(U)$, we can integrate by parts to get

$$B[u, v] := \int_U a_{ij}u_{x_i}v_{x_j} + b_iu_{x_i}v + cuv = \int_U Luv = \int_U fv = \langle f, v\rangle$$

Note that $B[u, v]$ defined above is a bilinear form that makes sense for $u, v \in H_0^1(U)$.

Date: 1/7/2019.

Definition 2.1. *A weak solution of $Lu = f, f \in H^{-1}(U)$, $u \in H_0^1(U)$ is an element u such that for all v, $B[u,v] = \langle f, v\rangle$.*

Note that this formulation of the problem is more symmetric in u and v. Namely, if we similarly define a weak solution for the bilinear form $B^*[u,v] = B[v,u]$, and the b_i are $C^1(\bar{U})$, then u will be a weak solution of $L^*u = f$, where L^* is the adjoint elliptic operator $L^*u = -(a_{ij}u_{x_j})_{x_i} - b_j u_{x_i} + (c - b_{i,x_i})u$.

The first tool from functional analysis that will give solutions is the Lax-Milgram Theorem, which can be viewed as an asymmetric generalization of the Riesz representation theorem.

Theorem 2.2 (Lax-Milgram)**.** *Let B be a bilinear form on a Hilbert space H that satisfies $B[u,v] \leq C\|u\|\|v\|$, $B[u,u] \geq c\|u\|^2$. Then $u \to B[u,-]$ defines an isomorphism of H and H^*.*

Unfortunately, our B will not always satisfy exactly the conditions of the above theorem. But if we define $B_\mu[u,v] = B[u,v] + \mu\langle u,v\rangle$, then for sufficiently large μ, $B_\mu[u,v]$ will satisfy the Lax-Milgram theorem. Note uniform ellipticity is needed to get $B_\mu[u,u] \geq c\|u\|^2$. A solution for B_μ will be a weak solution for $L + \mu I$.

Theorem 2.3. *There is a $\gamma \geq 0$ such that for each $\mu \geq \gamma$, and each $f \in H^{-1}(U)$, there is a unique weak solution of $L_\mu = L + \mu u = f$ in $H_0^1(U)$.*

In the case that $b_i \equiv 0, c \geq 0$, then γ is 0, so there is a unique solution.

To an extent we can fix the addition of the μ using the Fredholm theory for compact operators. For sufficiently large μ, L_μ is an isomorphism, so we can consider L_μ^{-1} as a map from $L^2(U) \to L^2(U)$. As we have passed through the inclusion $H_0^1(U) \to L^2(U)$, which is compact by the Rellich-Kondrachov compactness theorem, L_μ^{-1} is compact. Finally note that if $f \in L^2(U)$, u is a weak solution of $Lu = f$ iff $L_\mu(u) = \mu u + f$ iff $u - \mu L_\mu^{-1}(u) = f$. But $K := \mu L_\mu^{-1}$ is compact, so the Fredholm alternative gives the following theorem:

Theorem 2.4. *Either there is a unique weak solution to $Lu = f$ on U, $u = 0$ on ∂U for each $f \in L^2(U)$, or there is a finite and nonzero dimensional space of weak solutions to the homogeneous problem $Lu = 0$ on U, $u = 0$ on ∂U. Moreover in the second case, the dimension of solutions to the homogeneous problem is the same as the dimension of solutions to the homogeneous problem of the adjoint operator L^*. There is a solution to $Lu = f$ iff f is orthogonal to all weak solutions to $L^*v = 0$ on U, $v = 0$ on ∂U.*

Moreover, note that an eigenvector of K with eigenvalue $\frac{\mu}{\mu+\lambda}$ is an eigenvector of L with eigenvalue λ. Thus the general knowledge of the spectrum of compact operators gives:

Theorem 2.5. *There is a set $\{\lambda_i\}$, that is either finite, or an increasing sequence, such that $Lu = \lambda u + f$, $f \in L^2(U)$ has a unique weak solution iff λ is not one of the λ_i, and if $\{\lambda_i\}$ is infinite, the λ_i tend to infinity.*

3. Regularity

Now that we can know weak solutions exist, we can ask how regular they are. For motivation, suppose $u \in H_0^2(U)$ is a weak solution to $-\Delta u = f$, and by approximation and integrating by parts twice, we can obtain the identity $\int_U f^2 = \int_U \Delta u^2 = \int_U u_{x_i x_i} u_{x_j x_j} = \int_U u_{x_i x_j} u_{x_j x_i} = \int_U |D^2 u|^2$, so in particular we see that the H^2 norm of u can be bounded by a constant times the sum of the L^2 norms of u and f. This tells use that if $f \in L^2(U), u \in H^1(U)$, we might still expect u to be in $H^2(U)$ and have similar estimates on its norm. Moreover, since $-\Delta D^\alpha u = D^\alpha$ for each multi-index α (this can be interpreted in the sense of distributions), we can expect $u \in H^{m+2}(U)$ if $f \in H^m(U)$.

The way to show this is to put uniform norm bounds on the difference quotients $D_k^h Du$ for small h, as then the difference quotients will have a subsequence weakly converging to a weak second derivative. Working with difference quotients in general involves having some room to take the difference quotient, so we must first restrict to the interior of U. The following theorem is local so we can only require that L be elliptic rather than uniformly elliptic and it is possible to remove the restriction that U is bounded.

Theorem 3.1. *Suppose $a_{ij} \in C^1(U), b_i, c \in L^\infty_{loc}(U), f \in L^2(U)$, and suppose $u \in H^1(U)$ is a weak solution of $Lu = f$. Then $u \in H^2_{loc}(U)$ and for each open V with $\bar{V} \subset U$, there is a constant only depending on V, U, L such that $\|u\|_{H^2(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)})$.*

In particular, note $Lu = f$ in $L^2(U)$ if we view L as a map $H^2(U) \to L^2(U)$. By inducting, we can obtain the following result:

Theorem 3.2 (Local regularity)**.** *Suppose $a_{ij}, b_i, c \in C^{m+1}(U), f \in H^m(U)$, and suppose $u \in H^1(U)$ is a weak solution of $Lu = f$. Then $u \in H^{m+2}_{loc}(U)$ and for each open V with $\bar{V} \subset U$, there is a constant only depending on V, U, L, m such that $\|u\|_{H^{m+2}(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)})$.*

From the Sobolev inequalities, it follows that if a_{ij}, b_i, c, f are smooth, then the weak solution u is too.

Finally, we can ask whether in the above theorem $u \in H^{m+2}(U)$. We can achieve this if we add assumptions about the behavior of u at the boundary and assumptions about the regularity of the boundary. We can for example assume u vanishes on the boundary and ∂U is C^{m+2}. The way to prove this is to prove Theorem 3.1 in the local model $B(0,1) \cap \mathbb{R}^n_{x_n \geq 0}$. We can still use difference quotients in all directions except the x_n direction to get similar bounds. For the x_n direction, since the PDE $Lu = f$ is uniformly elliptic, we also have $a_{nn}u_{x_nx_n} = Lu - f + a_{nn}u_{x_nx_n}$, and the right hand side is not dependent on $u_{x_nx_n}$, and $a_{nn} \geq \epsilon > 0$ for some ϵ so we can bound $u_{x_nx_n}$ in terms of the other second derivatives. If the boundary is C^{m+2}, we can locally do a change of variables that preserves the uniform ellipticity of the PDE and use compactness to reduce to this model case. Inducting as before we get the following result:

Theorem 3.3 (Global regularity)**.** *Suppose $a_{ij}, b_i, c \in C^{m+1}(\bar{U}), f \in H^m(U)$, ∂U is C^{m+2}, and suppose $u \in H^1_0(U)$ is a weak solution of $Lu = f$. Then $u \in H^{m+2}(U)$ and there is a constant only depending on U, L, m such that $\|u\|_{H^{m+2}(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)})$.*

If a_{ij}, f, b_i, c are in $C^\infty(\bar{U})$ and ∂U is C^∞, then the above result and the Sobolev inequalities give that $u \in C^\infty(\bar{U})$.

4. Maximum principles

Like Δ, general elliptic operators L satisfy maximum principles. In this section we will assume $u \in C^2(U) \cap C(\bar{U})$. If $c \equiv 0$, then at an interior maximum of u, we have $Lu = -a_{ij}u_{x_ix_j}$. We can use the ellipticity condition to see that since the Hessian is negative definite and a_{ij} diagonalizable, $Lu \geq 0$. Thus if u is a strict subsolution of L, meaning $Lu < 0$, maxima must occur at the boundary. We can remove the strict assumption if u is uniformly elliptic by using a sequence of subsolutions of u converging to u, giving the following:

Theorem 4.1. *If $c \equiv 0, Lu \leq 0$, then $\max_{\bar{U}} u \leq \max_{\partial U} u$.*

If $c \geq 0$, and u is a subsolution, we can apply the above theorem on the open set where $u > 0$ to obtain:

Theorem 4.2. *If $c \geq 0, Lu \leq 0$, then either $u \leq 0$ or $\max_{\bar{U}} u \leq \max_{\partial U} u$.*

A stronger maximal principle states that if the maximum is obtained on the interior, then u is constant.

It follows from Hopf's lemma, which is a technical analytic computation.

Lemma 4.3 (Hopf)**.** *Assume $u \in C^2(U) \cap C^1(\bar{(U)})$ and $c \equiv 0$, and u is a subsolution to L. If there is a point $x_0 \in \partial U$ that satisfies $u(x_0) > u(x)$ for all $x \in U$, and such that u satisfies the interior ball condition, namely there is a ball B in U with x_0 on the boundary, then $\frac{\partial u}{\partial \nu}(x_0) > 0$, where ν is the outer normal to B at x_0. If $c, u(x_0) \geq 0$, the same conclusion holds.*

By applying Hopf's lemma to a point x on the interior of U that maximizes u, but in one direction is surrounded by points that don't maximize U, we get that $\frac{\partial u}{\partial \nu}(x) > 0$, by choosing a ball containing non-maximizing points of u with x on the boundary. This contradicts the fact that $Du(x) = 0$, so we get

Theorem 4.4. *If $u \in C^2(U) \cap C(\bar{U})$, U connected, $c \equiv 0$ and u is a subsolution of L that attains a maximum on its interior, then u is constant.*

The version for $c \geq 0$ is:

Theorem 4.5. *If $u \in C^2(U) \cap C(\bar{U})$, U connected, $c \geq 0$ and u is a subsolution of L that attains a nonnegative maximum on its interior, then u is constant.*

Finally just as for harmonic functions, if $Lu = 0$, then nearby points of u are comparable.

Theorem 4.6 (Harnack's inequality)**.** *If $Lu = 0, u \in C^2(U)$, then for any connected V with $\bar{V} \subset U$ we have $\sup_V u \leq C \inf_V u$ where C depends on L, V.*

5. Eigenvalues and Eigenfunctions

Suppose L is of the form $Lu = -(a_{ij}u_{x_i})_{x_j}$. Then from earlier we have that L is an isomorphism $H_0^1(U) \to H^{-1}(U)$. Moreover as from earlier we have $L^{-1} : L^2(U) \to L^2(U)$ is a compact operator, and since L is self adjoint, L^{-1} is too. Moreover, $\langle L^{-1}(u), u\rangle = B[L^{-1}(u), L^{-1}(u)] \geq 0$. Thus the spectral theory of symmetric compact operators gives the following theorem:

Theorem 5.1. *Suppose L is of the form above. Then the eigenvalues of L are real, positive and tending to infinity. Eigenvectors form an orthonormal basis, and the eigenspaces are finite dimensional.*

By regularity, the eigenvectors are all smooth functions if the coefficients of L are smooth, which we will assume.

The first (principle) eigenvalue λ_1 is very special. If w_k are eigenvalues that are an orthonormal basis of L^2, then since B satisfies the Lax-Milgram Theorem and is symmetric, w_k is also an orthonormal basis of the inner product given by B. Let $s(u) = \frac{B[u,u]}{\langle u,u\rangle}$. Then one finds that $s(u) \geq \lambda_1$ with equality holding iff u is in the λ_1 eigenspace. This condition for being in the eigenspace can then show that $\max(u, 0), \min(u, 0)$ are in the eigenspace if u is. By the strong maximal principle, u^+, u^- are strictly positive/negative or 0. In particular, u must have been either identically 0, or never 0. All together we have:

Theorem 5.2. *$\lambda_1 = \inf_{u \neq 0} s(u)$, the eigenspace of λ_1 is 1-dimensional and spanned by a positive function.*

For a non-symmetric elliptic operator, we cannot apply the spectral theory of symmetric compact operators, but nevertheless we can obtain the following similar result:

Theorem 5.3. *There is an eigenvalue $\lambda_1 \in \mathbb{R}$ such that any complex eigenvalue satisfies $\mathrm{Re}(\lambda) \geq \lambda_1$, and moreover the eigenspace of λ_1 is 1-dimensional and spanned by a positive function.*

6. Exercises

From now on all coefficients are smooth and U is bounded, has smooth boundary, and operators L are uniformly elliptic.

Exercise 6.0.1 (2)**.** *A function $u \in H_0^2(U)$ is a weak solution of $\Delta^2 u = f, u = \frac{\partial u}{\partial \nu} = 0$ iff $\int_U \Delta u \Delta v = \int_U fv$ for any $v \in H_0^2(U)$. Show that for $f \in L^2(U)$, there is a unique weak solution of $\Delta^2 u = f$.*

Proof. We will show that the bilinear form $B[u,v] = \int_U \Delta u \Delta v$ satisfies the conditions of the Lax-Milgram theorem. That theorem will then immediately imply the result.

Indeed, B is bilinear, $|B[u,v]| \leq \|\Delta u\|_{L^2(U)} \|\Delta v\|_{L^2(U)} \leq \|D^2 u\|_{L^2(U)} \|D^2 v\|_{L^2(U)} \leq \|D^2 u\|_{H^2(U)} \|D^2 v\|_{H^2(U)}$. For the other estimate, note that for any $u \in H_0^2(U)$, we have $\|u_{x_i x_i} u_{x_j x_j}\|_{L^1(U)} = \|u_{x_i x_j} u_{x_j x_i}\|_{L^1(U)}$. This is proven by approximating u by compactly supported smooth functions, for which it follows by integrating by parts twice. Thus we can bound the mixed partial derivatives of u with $\int_U (\Delta u)^2$, so we have $B[u,u] \geq C \int_U |D^2 u|^2$. By applying Poincare's inequality twice, we then have $C \int_U |D^2 u|^2 \geq C' \|u\|^2_{H^2(U)}$. □

Exercise 6.0.2 (4)**.** *Let $u \in H^1(\mathbb{R}^n)$ be a weak compactly supported solution of $-\Delta u + c(u) = f$, where $f \in L^2(\mathbb{R}^n)$, c a non-decreasing smooth function with $c(0) = 0$. Prove $u \in H^2(\mathbb{R}^n)$.*

Proof. As u is a weak solution, for each $v \in H^1(\mathbb{R}^n)$ we have $\int Du \cdot Dv = \int fv - c(u)v$. Set $v = -D^{-h} D_k^h(u)$, h small. Then the left hand side is $-\int Du D D_k^{-h} D_k^h(u) = \int |D D_k^h(u)|^2$. Note that $\int v^2 = \int (-D_k^{-h} D_k^h u)^2 \leq C \int |D_k^h Du|^2$ for small h, where C doesn't depend on h as u is compactly supported. Thus by Cauchy inequality with ϵ the right hand side is $\leq \frac{1}{2} \int |D_k^h Du|^2 + C' \int |f - c(u)|^2$. Thus $\int |D_k^h Du|^2 \leq 2C' \int |f - c(u)|^2$, and the right hand side doesn't depend on h, so u has weak second derivatives in $L^2(\mathbb{R}^n)$, and $u \in H^2(\mathbb{R}^n)$. □

Exercise 6.0.3 (7)**.** *Assume U is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions to $-\Delta u = 0$ on U, $\frac{\partial u}{\partial \nu} = 0$ on ∂U are constant.*

Proof. Here is a proof using energy methods. Integrating by parts gives $0 = \int_U u\Delta u = \int_U |Du|^2$ so $Du = 0$, and u is constant.

Here is a proof using the maximal principle. If u is not constant, it attains a maximum on the boundary. But by Hopf's Lemma, $\frac{\partial u}{\partial \nu}$ is positive at that maximum, contradicting the boundary condition. ☐

Exercise 6.0.4 (8)**.** *Let $u \in H^1(U)$ be a bounded weak solution of $Lu = -\sum(a^{ij}u_{x_i})_{x_j} = 0$. Then show that if ϕ is a convex smooth function, then $\phi(u)$ is a weak subsolution, meaning $B[\phi(u), v] \leq 0$ for all nonnegative v.*

Proof. By regularity, u is smooth and solves the PDE. Now observe that $L\phi(u) = -\phi''(u)(a_{ij}u_{x_i}u_{x_j}) \leq 0$ as a_{ij} is positive definite, and ϕ'' is positive. Thus integrating by parts gives $B[\phi(u), v] = \int_U L\phi(u)v \leq 0$ as $v \geq 0$. ☐