A DIRECT PROOF OF THE INVERTIBILITY HYPOTHESIS

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Abstract. In this note we give a direct proof of the invertibility hypothesis for simplicially enriched categories in the universal case.

If S is a monoidal model category, Lurie described conditions under which one can construct a model structure on Cat_S, the category of S-enriched categories [Lur06, Proposition A.3.2.4]. Given an S-enriched category C, there is an hS-enriched homotopy category given by passing to the homotopy category of S on mapping objects, called hC. In Lurie's model structure, the weak equivalences are given by functors $C \to D$ that induce an equivalence of hS-enriched categories $hC \to hD$. The cofibrations are generated by the maps

(1) $[1]_s \xrightarrow{[1]_f} [1]_{s'}$
(2) $\phi \to [0]_S$

Here $[1]_s$ is the category with two objects a, b with endomorphism objects the unit, $\underline{\mathrm{Hom}}(a,b) = s$, $\underline{\mathrm{Hom}}(b,a) = \phi$, and $f : s \to s'$ ranges over a generating set of cofibrations of S.

$[0]_S$ is the S-enriched category with a single object with endomorphisms the unit of S.

Lurie showed that under favorable conditions, one could get good control on the fibrations in Cat_S for this model structure. He introduced the notion of an **excellent model category** S, which is one equipped with a symmetric monoidal structure satisfying the following conditions:

(A1) S is combinatorial
(A2) Every monomorphism is a cofibration and the collection of cofibrations is stable under products
(A3) The collection of weak equivalences is stable under filtered colimits
(A4) S is a monoidal model category
(A5) S satisfies the *invertibility hypothesis*.

The first four conditions are relatively easy to verify in practice, but the invertibility hypothesis is a rather technical, saying that inverting an equivalence in an S-enriched category doesn't change it up to equivalence. More precisely, let $[1]_S^{\sim}$ denote the S enriched category that corepresents an isomorphism: that is it contains two objects, a, b, and all mapping spaces are the unit. Let $[1]_S$ be the category $[1]_{1_S}$: that is, it corepresents a morphism in an enriched category. Given $C \in \mathrm{Cat}_S$, and a morphism $f : [1]_S \to C$, we define $C[f^{-1}]$ to be the homotopy pushout of $f : [1]_S \to C$ along the natural inclusion $[1]_S \to [1]_S^{\sim}$. f is an **equivalence** if f is an isomorphism in hC. The invertibility hypothesis asks that if f is an equivalence, then $C \to C[f^{-1}]$ is an equivalence.

Lurie showed that for S an excellent model category, the fibrations to fibrant objects agree with the local fibrations: that is functors that induce fibrations on mapping spaces and are quasifibrations on homotopy categories. In particular, this characterizes the fibrant objects as being those with fibrant mapping objects.

The fact that simplicial categories satisfy the invertibility hypothesis follows from the work of Dwyer and Kan [DK80]. Lawson used a universal property of cubical sets as a monoidal model category to show the following:

Theorem 0.1 (Lawson [Law16])**.** *If C is a monoidal model category satisfying $(A1)-(A4)$, then it satisfied the invertibility hypothesis.*

His proof of Theorem 0.1 relies on a very special case of the invertibility hypothesis for simplicial categories, which can be considered the universal case. Let E be the simplicial category such that giving a functor out of E is the same as giving three morphisms f, g, h and homotopies from $g \circ f$ and $h \circ f$ to the identity. The invertibility hypothesis was reduced to the statement:

Lemma 0.2. *The category E is equivalent to $[1]_S^{\sim}$.*

Lawson proved this statement by using the work of Dwyer and Kan, but here we show that one can directly prove Lemma 0.2.

Remark 0.2.1. Lemma 0.2 looks like a univalence kind of statement, and reads "equivalence is equivalent to isomorphism".

Proof of Lemma 0.2. It suffices to show that all the mapping spaces of E are contractible. Since the two objects in E are equivalent, it actually suffices to show that any individual mapping space in E is contractible.

First consider the category R', the category such that maps out of it are the same as giving morphisms f, g and a homotopy from $g \circ f$ to the identity. The mapping spaces are easily computed: if a is the domain of f and b the codomain, then $\mathrm{Map}(a, a)$ is $J(\Delta^1)$, where J is the free monoid, and Δ^1 is the pointed interval. $\mathrm{Map}(a, b)$ and $\mathrm{Map}(b, a)$ are also $J(\Delta^1)$ since every object comes exactly from composing with f or precomposing with g from some element of $\mathrm{Map}(a, a)$. $\mathrm{Map}(b, b)$ is the disjoint union of a point (the identity map) and $J(\Delta^1)$, coming from morphisms that factor through f. Note that $J(\Delta^1)$ is contractible since Δ^1 is contractible.

This analysis of the mapping spaces shows R' is equivalent to the category R, where g is a strict left inverse to f. E is the pushout of the diagram

$$\begin{tikzcd} {[1]_S} \arrow[r, tail] \arrow[d, tail] & R' \arrow[d] \\ R' \arrow[r] & E \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By left properness, if we replace on of the R's with R, the pushout, E', will be equivalent to E. E' is the universal category where g is a strict left inverse to f and h is a homotopy right inverse to f. The endomorphisms of the codomain of f, $\mathrm{Map}_{E'}(b, b)$ are easily computed to be the monoid $\langle x, y | yx = x, y^2 = y\rangle$ with a homotopy from x to the identity adjoined. Here x is the morphism $f \circ h$ and y is $g \circ f$.

This monoid is connected, so in particular grouplike. Moreover, the monoid $\langle x, y | yx = x, y^2 = y\rangle$ satisfies the right Ore condition, so by the group completion theorem [MS76], its classifying space is equivalent to that of its group completion, which is the free group on a generator. This in turn has the same classifying space as $\langle x\rangle$.

$\mathrm{Map}(b, b)$ is the pushout in simplicial monoids

$$\begin{tikzcd}
J(* \cup *_+) \arrow[r, "{x,1}"] \arrow[d, tail] & \langle x, y | yx = x, y^2 = y \rangle \arrow[d] \\
J(\Delta^1_+) \arrow[r] & \mathrm{Map}(b, b)
\end{tikzcd}$$

By the discussion above, if we replace $\langle x, y | yx = x, y^2 = y\rangle$ with $\langle x \rangle$, the pushout will be a monoid with equivalent group completion to $\mathrm{Map}(b, b)$. But this pushout is $J(\Delta^1)$, which is contractible. $\square$

Remark 0.2.2. The proof of Lemma 0.2 can be compared with the proof of the groupoid completion result of Dywer and Kan [DK80, Proposition 9.5], which is the main ingredient in their proof of the invertibility hypothesis.

References

[DK80] William G Dwyer and Daniel M Kan. "Simplicial localizations of categories". In: *J. Pure Appl. Algebra* 17.3 (1980), pp. 267–284.

[Law16] Tyler Lawson. "Localization of enriched categories and cubical sets". In: *arXiv preprint arXiv:1602.05313* (2016).

[Lur06] Jacob Lurie. "Higher topos theory". In: *arXiv preprint math/0608040* (2006).

[MS76] Dusa McDuff and Graeme Segal. "Homology fibrations and the "group-completion" theorem". In: *Inventiones mathematicae* 31.3 (1976), pp. 279–284.