

NATURAL TRANSFORMATIONS, DUALITY, & EQUIVALENCES

ISHAN LEVY

1. WHAT IS A NATURAL TRANSFORMATION?

Let's introduce the final essential categorical concept, the natural transformation. This is an extremely important concept, I believe Mac Lane has said that he defined the notion of category so that he could make precise a functor, and he defined a functor to make precise the notion of a natural transformation.

Definition 1.1. *Given two functors F and G from C to D , a **natural transformation** η from F to G is for each object x of C , an arrow η_x from Fx to Gx such that the following diagram commutes for all x, y, f :*

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow \eta_x & & \downarrow \eta_y \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

We write $\eta : F \Rightarrow G$ to denote a natural transformation.

Natural transformation is a wonderful way of formalizing an intuitive sense of natural. For example, if V is a vector space over a field F , there is a dual vector space V^* which is the vector space of linear maps from V to F . Perhaps you know that if V is finite dimensional, it is isomorphic to its dual. However these aren't canonically isomorphic: in order to make an isomorphism, you have to **choose** a basis and then identify them. Natural transformation makes precise when this is canonical. For example, if Vect_F is the category of F -vector spaces, then $(-)^*$, the dual, is a contravariant functor from Vect_F to itself. On arrows, $(-)^*$ does the same thing as the Hom functor $C(-, F)$. We can compose $(-)^*$ with itself to get the covariant double dual functor $(-)^{**}$. If f is a map from V to W , then the double dual makes a map from V^{**} to W^{**} as follows: given a map g that takes maps h from V to F to F , we get the map $f^{**}(g)$ that takes maps $k : V \rightarrow F$ to $g(k \circ f)$. We can define a natural transformation η from 1_{Vect_F} to the double dual $(-)^{**}$: given $v \in V$, we send it to the element of V^{**} that takes an element of V^* , and evaluates it at v . This is an isomorphism if the vector space is finite dimension, and note that

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it is canonical: there was no need to make any choices. Then, we should expect this collection of maps $\eta_V, V \in \text{Vect}_F$ to be a natural transformation. And indeed it is, as one can check by following an element around the diagram that we want to commute:

$$\begin{array}{ccc} V & \xrightarrow{1_{\text{Vect}_F} f = f} & W \\ \downarrow \eta_V & & \downarrow \eta_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

Lets follow around an element $v \in V$:

$$\begin{array}{ccc} v & \xrightarrow{f} & f(v) \\ \downarrow \eta_V & & \downarrow \eta_W \\ h : h(g) = g(v) & \xrightarrow{f^{**}} & k : k(g) = h(g \circ f), k : k(g) = g(f(v)) \end{array}$$

Another example is the abelianization. Given a group G , we can define a subgroup called the commutator subgroup $[G, G] = \{aba^{-1}b^{-1} | a, b \in G\}$. The abelianization of G is the group $G/[G, G]$. This is a functor as if $f : G \rightarrow H$ is a homomorphism, we can compose with the projection $H \rightarrow H/[H, H]$ to get a map $G \rightarrow H/[H, H]$. $[G, G]$ is in the kernel of this map, so we get then a map $G/[G, G] \rightarrow H/[H, H]$. This is the map that the abelianization sends f to. Now the projection $\pi_G : G \rightarrow G/[G, G]$ is a natural transformation as the diagram below commutes (by definition):

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & G/[G, G] \\ \downarrow & & \downarrow \\ H & \xrightarrow{\pi_H} & H/[H, H] \end{array}$$

As a third example, consider the category ω which is the poset category of \mathbb{N} with the usual ordering. Consider a diagram consisting of a sequence of sets S_n with injective maps from $S_n \rightarrow S_{n+1}$. This can be thought of as a sequence of sets increasing in size (each containing the previous). Recall that diagrams are just functors, and in this case, ω is the category for which this is a functor (we can call this functor F). Let $\widehat{\cup S_i}$ be the constant functor taking ω to $\cup S_i$, and all the arrows to the identity. Then consider the natural transformation $\eta : F \Rightarrow \widehat{\cup S_i}$ that sends each S_i with the subset it corresponds to in the union. I leave this as an easy exercise to check that this is a natural transformation (draw it!). This kind of natural transformation is called a **cocone** (this will be discussed in more depth when we do (co)limits).

Finally, consider the determinant of a (invertible) matrix, \det^n . I claim this is a natural transformation. Consider the two functors from \mathbf{CRing} to \mathbf{Grp} : one taking K to $GL_n(K)$, and the other taking it to K^* (check that these are functors). Then \det_K^n is for each element of \mathbf{CRing} a map from $GL_n(K)$ to K^* sending a linear transformation to its determinant. The diagram is the same as always:

$$\begin{array}{ccc} GL_n F & \xrightarrow{\det_F^n} & F^* \\ \downarrow GL_n f & & \downarrow f^* \\ GL_n K & \xrightarrow{\det_K^n} & K^* \end{array}$$

Given two categories C, D , we can form the **product category**, $C \times D$ where the objects are pairs of objects, the arrows are pairs of arrows, and composition is defined as usual.

Now consider the contravariant powerset $\mathbf{Set}(-, 2)$ (2 is a set with two elements, we can view this functor as $2^{(-)}$). As an exercise, try to find all the natural transformations from this functor to itself (this will come up again in a later lecture).

Definition 1.2. Suppose F, G, H are functors in $\mathbf{Cat}(C, D)$. Then if $\eta : F \Rightarrow G$ and $\nu : G \Rightarrow H$ are natural transformations, then we can form the **vertical composite**, $\nu \cdot \eta$, a natural transformation from F to H , defined by $\nu \cdot \eta_a = \nu_a \circ \eta_a$.

We can check this is a natural transformation via the following diagram:

$$\begin{array}{ccccc} Fa & \xrightarrow{\eta_a} & Ga & \xrightarrow{\nu_a} & Ha \\ \downarrow Ff & & \downarrow Gf & & \downarrow Hf \\ Fb & \xrightarrow{\eta_b} & Gb & \xrightarrow{\nu_b} & Hb \end{array}$$

This turns $\mathbf{Cat}(C, D)$ into a category, which we call the functor category. We can write this as D^C . An isomorphism in $\mathbf{Cat}(C, D)$ is called a natural isomorphism. Alternatively, it is a natural transformation η where each η_a is an isomorphism.

I use the word vertical composite, because there is also a horizontal composite. It can be seen as follows:

Given the diagram below, we would like to create a natural transformation $\nu\eta : F' \circ F \Rightarrow G' \circ G$ sometimes written $\nu \circ \eta$.

$$\begin{array}{ccccc} & F & & F' & \\ C & \xrightarrow{\quad} & D & \xrightarrow{\quad} & E \\ & \Downarrow \eta & & \Downarrow \nu & \\ & G & & G' & \end{array}$$

We can do this by considering the following diagram:

$$\begin{array}{ccc} F'Fa & \xrightarrow{F'\eta_a} & F'Ga \\ \downarrow \nu_{Fa} & & \downarrow \nu_{Ga} \\ G'Fa & \xrightarrow{G'\eta_a} & G'Ga \end{array}$$

This commutes as ν is natural for η_a . This suggests the following definition:

Definition 1.3. Suppose F, G, F', G', η, ν are as above, we can form the **horizontal composite** $\nu\eta : F' \circ F \rightarrow G' \circ G$ so that $(\nu\eta)_a = \nu_{Ga} \circ F'\eta_a = G'\eta_a \circ \nu_{Fa}$.

It remains to check this is a natural transformation, but this should be obvious if you draw the appropriate diagram (for a natural transformation). If $F : C \rightarrow D$, $G, H : D \rightarrow E$ are functors and $\eta : G \Rightarrow H$ a natural transformation then the natural transformation ηF denotes the horizontal composite $\eta 1_F$, and similarly if $J : E \rightarrow X$ is a functor, then $J\eta$ denotes $1_J\eta$.

Horizontal composites and vertical composites are related through the interchange law, which says $(\tau\eta) \cdot (\tau'\eta') = (\tau \cdot \tau')(\eta \cdot \eta')$. It can be described as the diagram below:

$$\begin{array}{c} \begin{array}{ccc} C & \xrightarrow{\quad} & D \\ \downarrow \eta' & & \downarrow \eta \\ C & \xrightarrow{\quad} & D \end{array} \circ \begin{array}{ccc} D & \xrightarrow{\quad} & E \\ \downarrow \tau' & & \downarrow \tau \\ D & \xrightarrow{\quad} & E \end{array} = \begin{array}{ccc} C & \xrightarrow{\quad} & D \\ \downarrow \eta' & & \downarrow \tau' \\ C & \xrightarrow{\quad} & D \end{array} \bullet \begin{array}{ccc} D & \xrightarrow{\quad} & E \\ \downarrow \tau' & & \downarrow \tau \\ D & \xrightarrow{\quad} & E \end{array}$$

We can prove it using the diagram below. Let $\eta' : F \Rightarrow G, \eta : G \Rightarrow H, \tau' : F' \Rightarrow G', \tau : G' \Rightarrow H'$ in the figure above.

$$\begin{array}{ccccccc} & & F'Ga & \xrightarrow{F'\eta_a} & F'Ha & \xrightarrow{\tau'_{Ha}} & G'Ha \\ & \nearrow F'\eta'_a & \parallel & & & & \searrow \tau_{Ha} \\ F'Fa & & & & & & \\ & \searrow F'\eta'_a & & & & & \\ & & F'Ga & \xrightarrow{\tau'_{Ga}} & G'Ga & \xrightarrow{G'\eta_a} & G'Ha \\ & & & & & & \nearrow \tau_{Ha} \\ & & & & & & H'Ha \end{array}$$

The path on the top is the natural transformation $(\tau \cdot \tau')(\eta \cdot \eta')$, and the path on the bottom is $(\tau\eta) \cdot (\tau'\eta')$. The middle rectangle commutes as τ' is a natural transformation.

As a final note, there is an analogy between natural transformations and homotopies.

If X and Y are topological spaces, and f and g are maps (continuous, as always) from X to Y , a homotopy from f to g is a map from $X \times [0, 1]$ to Y that at 0 restricts to f and at 1 restricts to g . The definition of a natural transformation can be presented analogously: Let 2 be the category with 2 objects, called 0 and 1 and one non identity arrow from 0 to 1 (we can say the arrow category, as this is the category that represents the diagram consisting of a generic arrow).

If C and D are categories, and F and G are functors from C to D , a natural transformation from F to G is a functor from $C \times 2$ to D that on 0 restricts to F and on 1 restricts to G .

Check that these two definitions of natural transformations are equivalent and note the similarity with homotopies. In a way, a natural transformation is categorification of homotopy.

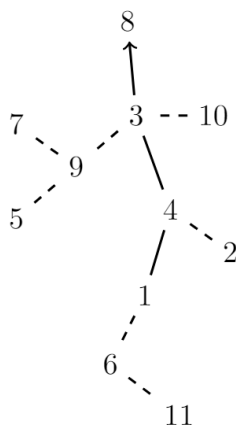
Finally let's end with an interesting non-example. Let FinSet_g be the category of finite sets and bijections between them. Consider two functors to Set , the first, Aut , takes X to the set of bijections from X to itself, on maps, it takes $f : X \rightarrow Y$ to the function that takes $\phi : X \rightarrow X$ to $f \circ \phi \circ f^{-1} : Y \rightarrow Y$. The second, Ord , takes X to the set of total orders on X , and on maps takes f to the total order on Y induced by the bijection. These two functors send isomorphic objects to isomorphic sets, but are not naturally isomorphic: in fact, there isn't even a natural transformation between them! For, let's consider f , the nontrivial bijection from a set $\{a, b\}$ to itself. If there was a natural transformation, we would have

$$\begin{array}{ccc} \{1, (a, b)\} & \xrightarrow{\text{Aut}(f)} & \{1, (a, b)\} \\ \downarrow \eta_b & & \downarrow \eta_c \\ \{a < b, b < a\} & \xrightarrow{Ff} & \{a < b, b < a\} \end{array}$$

$\text{Aut}(f)$ is the identity, but Ff is not, so this diagram cannot commute.

The fact that this bijection is not natural has an interesting interpretation in the context of a combinatorics problem. In particular, let's count the number of trees on a set of n elements, which we'll call T_n . Let $|\cdot|$ denote cardinality of a set. Consider the product $T_n \times n \times n$, consisting of a tree on the set n , as well as a head and a tail (shown in Fig 1).

Note that since there is a unique path between any two points in a tree, we can draw an arrow from the tail to the head, yielding a total ordering on a subset of 1 to n , ie. a skeleton, as well as trees coming out of each point. Note that the skeleton and the trees coming out of each point completely determine $T_n \times n \times n$. Then as total orders are in bijections with permutations, we can consider the set of permutations with trees coming out of them, a typical example in the figure below:



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graph TD
    1 --> 2
    2 --> 3
    3 --> 5
    5 --> 11
    11 --> 9
    9 --> 6
    8 --> 11
    7 --> 2
    4 --> 10
    10 --> 3

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Fig. 2: A permutation on 2,3, and 5 with trees coming out of it.

These are in bijection with functions from the set of n elements to itself, as a function determines such a tree by writing where everything goes, which eventually (after applying the function enough times) determines the cycles and the trees coming out of them. Thus $T_n \times n \times n$ is in bijection with the set of functions from $\{1, \dots, n\}$ to itself, which is n^n . Thus $|T_n| = n^{n-2}$ (This is known as Cayley's Theorem). Perhaps the reason this proof does something nontrivial is because it used this bijection which was unnatural.