BOTT PERIODICITY FOR CLIFFORD ALGEBRAS

ISHAN LEVY

For a real or complex finite dimensional vector space vector space with a nondegenerate quadratic form, let's try to identify the algebra we get as the Clifford algebra. Nondegenerate real quadratic forms are classified by rank and signature, and complex ones by rank, so we need to identify $\mathrm{Cl}(p,q)$, the real Clifford algebra where the positive definite part is p-dimensional and the negative definite part q-dimensional, and $\mathrm{Cl}(n)$, the complex Clifford algebra coming from a complex n dimensional space.

The complex case is easy to deal with. It is easy to identify $\mathrm{Cl}(0) = \mathbb{C}, \mathrm{Cl}(1) = \mathbb{C} \oplus \mathbb{C}$. $\mathrm{Cl}(2)$ is easily seen to be $\mathrm{End}(\mathbb{C}^2)$ by sending an orthonormal basis to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. These cases suffice to compute the rest of $\mathrm{Cl}(n)$. If $n \geq 2$, we there is a map from $\mathrm{Cl}(n+2)$ to $\mathrm{Cl}(n) \otimes \mathrm{Cl}(2)$ produced in the following way. Let $e_1, \ldots e_{n+2}$ be the orthonormal primitive elements of $\mathrm{Cl}(n)$. and similarly let $e'_1, \ldots e'_n$ and e'_{n+1}, e'_{n+2} be the same for $\mathrm{Cl}(n), \mathrm{Cl}(2)$ respectively. Then we can send $e_i, 1 \leq i \leq n$ to $ie'_i \otimes e'_{n+1}e'_{n+2}$ and $e_i, n < i$ to $1 \otimes e_i$. Via the universal property one sees that the relations are satisfied to extend to an algebra homomorphism. Since it is injective on generators, it is injective, and for dimension reasons, it must be an isomorphism.

Thus we inductively get that $\mathrm{Cl}(n) \cong \mathrm{End}(\mathbb{C}^{2^{\frac{n}{2}}})$ for even n and $\mathrm{End}(\mathbb{C}^{2^{\frac{n-1}{2}}}) \oplus \mathrm{End}(\mathbb{C}^{2^{\frac{n-1}{2}}})$ for odd n. There is a periodicity here with period 2: it alternates between being a matrix algebra and a sum of them. One way to think about the periodicity, is to say that $\mathrm{Cl}(n)$ is Morita equivalent to $\mathrm{Cl}(n+2)$, meaning that they have the same categories of left modules. Indeed for any division ring R, an equivalence between left modules on R and $\mathrm{End}(R^n)$ is given by the functor sending a left module M to $M \otimes_R R^n$ where R^n is a left $\mathrm{End}\,(R^n)$ module, but also a R-module via the natural homomorphism $R \to \mathrm{End}\,(R^n)$. Thus categorical properties of complex Clifford modules depend on the parity of the dimension.

Theorem 0.1. *For even n, $\mathrm{Cl}(n)$ is Morita equivalent to $\mathbb{C}$, and for odd n, $\mathrm{Cl}(n)$ is Morita equivalent to $\mathbb{C} \oplus \mathbb{C}$.*

The same kind of Bott periodicity result holds for real Clifford algebras, but it works mod 8. To prove it we can produce relations among the real Clifford algebras similarly to the way it was proven for complex Clifford algebras. If $e_1, \ldots, e_{n+2}$ is an orthonormal basis for $\mathrm{Cl}(n+2,0)$, and $e'_1,\dots,e'_n$, e'_{n+1},e'_{n+2} orthonormal bases for $\mathrm{Cl}(2,0),\mathrm{Cl}(0,n)$ respectively, then by sending $e_i, 1\le i\le n$ to $e'_i\otimes e'_{n+1}e'_{n+2}$ and $e_i, n<i$ to $1\otimes e'_i$ we get an algebra isomorphism $\mathrm{Cl}(n+2,0)\cong \mathrm{Cl}(0,n)\otimes\mathrm{Cl}(2,0)$. The exact same map also gives isomorphisms $\mathrm{Cl}(0,n+2)\cong\mathrm{Cl}(n,0)\otimes\mathrm{Cl}(0,2)$ and $\mathrm{Cl}(p+1,q+1)\cong\mathrm{Cl}(p,q)\otimes\mathrm{Cl}(1,1)$. $\mathrm{Cl}(1,1)$ is seen to be $\mathrm{End}(\mathbb{R}^2)$ by sending an orthonormal basis to $\begin{pmatrix}0&1\\-1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Since this is Morita equivalent to $\mathbb{R}$, we already see that the Morita equivalence class is only dependent on the signature $\sigma=p-q$. Moreover, $\mathrm{Cl}(0,2)$ is also $\mathrm{End}(\mathbb{R}^2)$ via $\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\mathrm{Cl}(2,0)$ is $\mathbb{H}$. Since $\mathbb{H}$ is the nontrivial element in the Brauer group of $\mathbb{R}$, the relations give $\mathrm{Cl}(n+8,0)\cong\mathrm{Cl}(n,0)\otimes\mathrm{End}(\mathbb{R}^4)\otimes\mathbb{H}\otimes\mathbb{H}=\mathrm{Cl}(n,0)\otimes\mathrm{End}(\mathbb{R}^{16})$ and similarly for $\mathrm{Cl}(0,n)$, so the Morita equivalence class for positive and negative signatures depends only on σ mod 8. To compute it, we can begin with the basic examples $\mathrm{Cl}(0,0)=\mathbb{R},\mathrm{Cl}(0,1)=\mathbb{R}\oplus\mathbb{R},\mathrm{Cl}(1,0)=\mathbb{C}$, and use the relations proven above and the facts that $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}=\mathrm{End}(\mathbb{C}^2),\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}=\mathbb{C}\oplus\mathbb{C}$. The result is the following:

Theorem 0.2. *The Morita equivalence class of* $\mathrm{Cl}(p,q)$ *depends only on* $\sigma=p-q$ *mod* 8. *It is given by:*

- $\mathbb{R},\sigma\equiv 0$
- $\mathbb{C},\sigma\equiv 1$
- $\mathbb{H},\sigma\equiv 2$
- $\mathbb{H},\oplus\mathbb{H},\sigma\equiv 3$
- $\mathbb{H},\sigma\equiv 4$
- $\mathbb{C},\sigma\equiv 5$
- $\mathbb{R},\sigma\equiv 6$
- $\mathbb{R}\oplus\mathbb{R},\sigma\equiv 7$

This can actually be used to compute what the Clifford algebra is since we know the dimension. For example, $\mathrm{Cl}(4,7)$ has Morita equivalence class $\mathbb{C}$ and dimension 2^{11} so it must be $\mathrm{End}(\mathbb{C}^{2^5})$.