GROUP EXTENSIONS

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How can we classify extensions of a group G by H?

This problem can be understood from the point of view of homotopy theory by interpreting our groups as automorphisms of some type of object, so that BG becomes the groupoid of objects of that type. Then giving an extension $1 \to H \to E \to G \to 1$ is the same as giving a pointed fibre sequence $BE \to BG$, with fibre BH. Such fibrations correspond to pointed maps $BG \to B\operatorname{Aut}(BH)$, giving the classification. A split extension is the same as a pointed fibre sequence $BE \to BG$ with a section, which corresponds to a pointed map from $BG \to B\operatorname{Aut}_*(BH)$ where Aut_* denotes automorphisms as a pointed space.

For example, extensions of $\mathbb{Z}$ by H are the same as pointed maps $B\mathbb{Z} \to B\operatorname{Aut}(BH)$ which is just $\operatorname{Aut}(BH)$. $\pi_0(\operatorname{Aut}(BH))$ is the same as the outer automorphisms $\operatorname{Out}(H)$, giving the equivalence classes of such extensions.

The fibre of the map $\operatorname{Aut}(BH) \to \operatorname{Out}(H)$ is the space of automorphisms of BH identifiable with the identity, which is the same as $B\operatorname{Aut}_{\operatorname{Aut}(BH)} 1$. Delooping the group homorphism $\operatorname{Aut}(BH) \to \operatorname{Out}(H)$, we see there is a fibre sequence $B^2\operatorname{Aut}_{\operatorname{Aut}(BH)} 1 \to B\operatorname{Aut}(BH) \to B\operatorname{Out}(H)$.

Conjugation by an element of H is a natural transformation of the identity on BH, giving a homomorphism $f : H \to \operatorname{Aut}(BH)$. The center $Z(H)$ is the fixed points of this action, i.e a path out of a fixed point x, $f : x \to a \in BH$ and an identification of the action of H on f with the trivial action. For a given path, we can identify such identifications with automorphisms of the identity on BH, so $BZ(H) = B\operatorname{Aut}_{\operatorname{Aut}(BH)} 1$.

Thus, there is a fibre sequence $B^2Z(H) \to B\operatorname{Aut}(BH) \to B\operatorname{Out}(H)$. Now suppose G, H are 1-groups (A weaker condition on G might also do). Given a central extension, or a pointed map $BG \to B\operatorname{Aut}(BH)$, the action of G on H is the induced action on the loop spaces of the base points. That action is trivial iff the delooped map is trivial when 1-truncated. But the 1-truncation is $B\operatorname{Out}(H)$, so from the fibre sequence we see that central extensions correspond to pointed maps $BG \to B^2Z(H) = K(H, 2)$. In particular, equivalence classes of central extension correspond to $H^2(BG; H)$.