UNIVERSALS AND LIMITS

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1. Universals

An important idea in category theory is that of a universal property. This is the idea that instead of describing an object directly, you can describe a property that it satisfies in relation to other objects, that uniquely characterize it (up to isomorphism). For example, we have already seen that initial objects are uniquely characterized by the fact that there is a unique arrow to every other object. In fact, universals and limits are special cases of this, in appropriate categories. If not made explicit at any point, I leave it as an exercise to describe the category in which any definitions I give are initial or terminal (and hence unique up to isomorphism in the appropriate sense).

Definition 1.1. *Let $F : C \to D$ be a functor, $d \in D$. A **universal arrow** from d to F is an arrow $f : d \to Fc$ for some $c \in C$, such that for any other arrow $g : d \to Fb$, there is a unique arrow $h : c \to b$ such that the diagram below commutes:*

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[dr, "g"'] & Fc \arrow[d, dashed, "\exists! Fh"] \\ & Fb \end{tikzcd}$$

Corollary 1.2. *Universal arrows are unique up to isomorphism (in the appropriate sense).*

Proof. Universal arrows are initial objects in an appropriate category (find it). □

The dual concept is a universal arrow from F to d. Here are some examples of universal arrows to and from functors. Let U be the forgetful functor from Grp to Set. Then given a set X, we can construct FX, the free group on X. Then the function $f_x : X \to UFX$ sending $x \in X$ to the word consisting of x in UFX is a universal arrow from X to U. Similarly, given a group G, we can consider the functor F that takes a set to the free group on this set. Then the homomorphism g_G sending $FUG \to G$ by sending a word on the elements of G to the corresponding product in

Date: 8/3/2017.

G is a universal arrow from F to U. In this example, it is not a coincidence that we have these universal arrows. In fact these universal arrows give an adjunction, which will be the topic of the next lecture.

Here is a similar example: Let Dom_m be the category with objects integral domains, and arrows monos (ie injections), and let Fld be the category of fields. Then consider the forgetful functor $D : \mathrm{Fld} \to \mathrm{Dom}_m$. We have for any domain R, the field of fractions $\mathrm{Frac}(R)$. The embedding $h_R : R \to \mathrm{Frac}(R)$ is a universal arrow to this functor.

Universal arrows can be characterized also as the following:

Proposition 1.3. *Let $F : C \to D$ be a functor. A universal arrow $f : d \to Fc$ from d to F is given exactly by a natural bijection $\eta : D(d, F-) \cong C(c, -)$*

Proof. Given a universal arrow, there is unique map corresponding to any map from d to Fb, namely F applied to a map from c to b. This correspondence of arrows gives the natural isomorphism, and I leave you to check that this is indeed natural (it is essentially the diagram I use to prove the converse).

Conversely, given a natural isomorphism η as above, we can look at the map corresponding to $1_c \in C(c, c)$, $\eta_c^{-1}(1_c)$. This is an arrow $f : d \to Fc$, and is the universal arrow we want. By naturality of η, we get that this is a universal arrow:

$$\begin{tikzcd} c \arrow[d, "g"] \\ b \end{tikzcd} \quad \begin{tikzcd} D(d,Fc) \arrow[r, "Fg\circ(-)"] \arrow[d, "\eta_c"] & D(d,Fb) \arrow[d, "\eta_b"] \\ C(c,c) \arrow[r, "g\circ(-)"] & C(c,b) \end{tikzcd} \quad \begin{tikzcd} \eta_c^{-1}(1_c) \arrow[r, mapsto, "Fg\circ(-)"] \arrow[d, mapsto, "\eta_c"] & Fg\circ\eta_c^{-1}(1_c) = \eta_c^{-1}(g) \\ 1_c \arrow[r, mapsto, "g\circ(-)"] & g \arrow[u, mapsto, "\eta_b^{-1}"'] \end{tikzcd}$$

□

There is also the similar notion of universal element.

Definition 1.4. *Let $F : C \to \mathrm{Set}$ be a functor. A* ***universal element*** *is a universal arrow from $*$, the set with 1 object, to F (this can be thought of as an element of Fa for some $a \in C$).*

An example is the following. Let S be a set, E an equivalence relation on S. We have a projection $\pi : S \to S/E$. Then let H be the functor taking X to the set of functions from S to X such that are constant on equivalence classes. Then π (or more accurately the map from $*$ to $\pi \in HX$) is a universal element of this functor (check this).

Note that a universal arrow is also a special case of universal element: a universal arrow from $d \in D$ to a functor $F : C \to D$ is just a universal element of the functor $D(d, F-)$.

2. Limits and colimits

Consider the Cartesian product in Set. It is a construction that takes a collection of sets X_i, and yields a set $\Pi_i X_i$. Whenever we have functions $f_i : Z \to X_i$, we get a function $\Pi_i f : Z \to \Pi_i X_i$ that on the X_i component is f_i. This property can be used to characterize the Cartesian product up to isomorphism, and something that satisfies this property is called a product:

Definition 2.1. *Given an indexed family X_i of objects in a category C, the* ***product*** *of these objects is an object $\Pi_i X_i$ with a collection of projections $\pi_i : \Pi_i X_i \to X_i$ such that whenever there is another object Z and a family of maps $f_i : Z \to X_i$ for each i, then there is a unique map $\Pi_i f_i : Z \to \Pi_i X_i$ such that the diagram below commutes:*

$$\begin{tikzcd} & Z \arrow[dl, "f_i"'] \arrow[d, dashed, "\Pi_i f_i"] \arrow[dr, "f_j"] & \\ X_i & \Pi_i X_i \arrow[l, "\pi_i"] \arrow[r, "\pi_j"'] & X_j \end{tikzcd}$$

The dual is a coproduct, denoted by $\coprod_i X_i$. When there are only two indexed objects, we can write $X \times Y$ for the product of X and Y, and $X \coprod Y$ for their coproduct. Convince yourself that in Set, the product is the Cartesian product, and the coproduct is the disjoint union. These notions aren't just familiar in Set, however. For example, in Ring, the product is the direct product, and the coproduct is the tensor product (over $\mathbb{Z}$). In Grp, the product is the direct product, and the coproduct is the free product. In a poset, the product is the infimum, and the coproduct is the supremum. Verify that these have the property required.

Now consider another construction. An equivalence relation E on X is a subset of $X \times X$. we then get two projections π_1 and π_2 from E to X, because $X \times X$ is the product of X with itself. More concretely, π_1 is the function that takes $(x, y) \in E$ to $x \in X$ and π_2 takes it to y. Then we get a quotient map $q : X \to X/E$. X/E satisfies the following universal property: Suppose we have a map f from X to a set S that is constant on equivalence classes (This can be described as having a map $e : E \to S$ such that $f \circ \pi_1 = f \circ \pi_2 = e$). Then this uniquely determines a compatible map $X/E \to S$. This idea is captured in the notion of a coequalizer:

Definition 2.2. *Suppose we have two arrows $f_1, f_2 : X \to Y$ in the category C. The* ***coequalizer*** *of this diagram is an arrow $q : Y \to Q$ so that $q \circ f_1 = q \circ f_2$, such that whenever we have an Q' and a map $q' : Y \to Q'$ such that $q' \circ f_1 = q' \circ f_2$, then there* *is a unique map from $Q \to Q'$ such that the diagram below commutes:*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dashed] \\ & & Q' \end{tikzcd}$$

The dual notion is equalizer. X/E as described above is the coequalizer in Set. Given two arrows $f, g : X \to Y$ in Set, the equalizer is the subset of X consisting of elements x such that $f(x) = g(x)$. I leave it to you to figure out the general description of a coequalizer in Set. In Ab, the equalizer of f and g is the kernel of $f - g$ (or more precisely the inclusion arrow of the kernel). The coequalizer is the cokernel of $f - g$ (ie. $Y/im(f - g)$).

example of a coequalizer in use (eg RP2)

We can also consider the following construction. Given two spaces X and Y, we can glue these together along another space Z. For example if X and Y are closed disks, we can glue them together along their boundary, Z (which is a circle) to get S^2, the 2-sphere. This construction has a very nice property, namely that whenever you have a continuous function from the two disks X, Y into a space that agree on the boundary, we get a map from S^2 into the space. This is because S^2 (along with the maps embedding X and Y into it) is a pushout:

Definition 2.3. *Let there be two maps $f : Z \to X$ and $g : Z \to Y$. Then the pushout is a pair of maps $f' : X \to W$ and $g' : Y \to W$ such that $g' \circ g = f' \circ f$ and if there are another pair of maps $j : X \to V$ and $k : Y \to V$, there is a unique map $W \to V$ such that the diagram below commutes:*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "g'"] \arrow[ddr, bend left, "k"] & \\ X \arrow[r, "f'"'] \arrow[drr, bend right, "j"'] & W \arrow[dr, dashed] & \\ & & V \end{tikzcd}$$

The dual notion is a pullback. The pushout in Top corresponds to glueing as described earlier, the pullback is the fibre product. The pushout in Ring corresponds to the tensor product, and the pushout is the fibre product of rings. Here is an easy lemma about pushouts:

Lemma 2.4. *A pushout as in Definition 2.3 is just a coproduct when Z is the terminal object.*

Proof. Just check that in this case W with the maps g', f' satisfies the universal property of the coproduct. I leave this checking as an exercise. $\square$

Dually, a pullback of the initial object is a product.

By this lemma, the Chinese remainder theorem is just the statement that when I and J are comaximal ideals in R, then the pullback as in the diagram below (sometimes denoted $R/I \times_{R/(I+J)} R/J$) is isomorphic to R/IJ (all the maps are projections):

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R/IJ \arrow[r] \arrow[d] & R/I \arrow[d] \\
& R/J \arrow[r] & R/(I+J)
\end{tikzcd}$$

Viewing these rings as affine varieties (or schemes if you prefer), this says that the ring of functions of two disjoint subvarieties is the product of the ring of functions for each component.

All of the before mentioned constructions (limits, equalizers, pullbacks) are important cases of a concept called a limit.

Definition 2.5. *Let J be a (small) category, and $F : J \to C$ a functor. Let $\Delta : C \to C^J$ be the diagonal functor, taking $c \in C$ to $\hat{c}$, the constant functor to c. Then a* ***limit*** *of a functor $F \in C^J$ is a universal arrow ν_F from Δ to F. We say that the object that Δ sends to $\mathrm{dom}(\nu_F)$ is the* ***limiting object*** *(sometimes also called limit) of F and sometimes is denoted $\lim(F)$.*

Corollary 2.6. *Limits are unique up to isomorphism.*

Corollary 2.7. $\mathrm{Nat}(\hat{-}, F) \cong C(-, \lim(F))$

Proof. Apply Proposition 1.3. $\square$

The dual concept is **colimit**. For example, if J is a discrete category (ie with no non-identity arrows), a limit of a functor from it is just a product. If J is the category below (two objects and two non-identity arrows):

$$\cdot \rightrightarrows \cdot$$

then the limit of a functor from it is an equalizer. If J is the category below (three objects and two non-identity arrows):

$$\cdot \longleftarrow \cdot \longrightarrow \cdot$$

then the limit of a functor from it is a pullback.

A limit can alternatively be described directly as follows: given a diagram in a category, a **cone** is an object L with a collection of maps to every diagram in the category such that the maps to the diagram commute with the arrows in the diagram (in other words every triangle consisting of an arrow in the diagram and the arrows from L to the domain and codomain of this arrow commutes). A morphism of cones from L and M is an arrow from L to M that commute with the arrows to the diagram. Then a limit is just a terminal object in the cone category. Below is a drawing of two cones and a morphism f between them, where the diagram consists of a single arrow from A to B:

$$\begin{tikzcd}
A \arrow[d] & M \arrow[l] \arrow[dl] \\
B & L \arrow[l] \arrow[u, "f"'] \arrow[ul]
\end{tikzcd}$$

I suggest the reader look at the diagrams for products, coequalizers, and pushouts above and notice how these are indeed special types of limits and colimits.

Definition 2.8. *We say a category C **has limits of shape** J if every functor from $J \to C$ has a limit.*

Definition 2.9. *If C has limits of shape J for every (small) category J, then we say J is **(small) complete**.*

Sometimes people also use “finite complete” to mean having all limits for any finite category.

3. Existence/Construction of Limits

How can we know if a category is complete? It turns out one has all limits if one has all (small) products and equalizers. One can think of the product as being able to provide enough maps to a small diagram, and the equalizer as being able to “thin” the product to make these maps to the diagram be compatible (yielding a cone) and have this cone be terminal. The following construction allows one to produce any (small) limit in a category given these:

Theorem 3.1. *A category is (small) complete if it has (small) products and equalizers*

Proof. The proof can be summed up in one big commutative diagram:

$$
\begin{tikzcd}
F_j & F\operatorname{cod}(f) & \\
L \arrow[u, "l_j"'] \arrow[r, "l"] & \prod_{j\in J} Fj \arrow[u, "\pi"] \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "F^*"'] \arrow[d, "\pi"] & \prod_{f\in J} F\operatorname{cod}(f) \arrow[ul, "\pi"'] \arrow[d, "\pi"] \\
K \arrow[u, dashed, "m"] \arrow[ur, "k"] \arrow[uu, bend left, "k_j"] & F\operatorname{dom}(f) \arrow[r, "Ff"] & F\operatorname{cod}(f)
\end{tikzcd}
$$

To make sense of this diagram, we'll start on the right half. J is small, so we can take the product of the codomains of Ff for each arrow f of J ($\prod_{j\in J} Fj$), and also take the product of Fj for each $j \in J$ ($\prod_{f\in J} F\operatorname{cod}(f)$). Now in order to make the two arrows Δ, and F^*, the universal property of product says we need a map to $F\operatorname{cod}(f)$ for each $\operatorname{cod}(f)$. One way to do this is to just project $\prod_{j\in J}$ to $F\operatorname{cod}(f)$, and another way is to first project to $F\operatorname{dom}(f)$ and then apply Ff. These induce the maps Δ and F^* by the universal property of products. We then take the equalizer of these two maps to get L and l. By the universal property of products, l splits into maps l_j to the diagram, and the fact that L equalizes Δ, F^* (by equalize I mean $\Delta \circ l = F^* \circ l$) amounts to these forming a cone. We must check that this cone is terminal. Suppose K is another object with a collection of maps k_j that form a cone. These give a map to the product, k, and the fact that this is a cone means that k also equalizes Δ and F^*, so this induces a map m as L is an equalizer. Why is m unique? Well any map like m that is a cone morphism (commutes with k_j, l_j must commute with k, l, hence is unique as L is an equalizer. □

Note that the dual of this is that a category is cocomplete if it has coproducts and coequalizers. This theorem immediately implies that many familiar categories, such as Cat, Set, Ring, Grp, Vect_F, and so on are complete and cocomplete. Indeed this proof gives a way of actually constructing these limits and colimits.

To end, here are some examples of limits and colimits.

In Galois theory, one studies field extensions and their symmetries. In particular, if K/k is a finite Galois (normal and separable) extension, we have a correspondence between the subgroups of its automorphism group and the intermediate fields. When the extension K is not finite but still normal and separable, one has too many subgroups so the correspondence doesn't hold. However, one may consider K to be the colimit (union in this case) of all of the finite Galois intermediate extensions of it, and in this way, we can treat its automorphism group as a colimit of the automorphism groups of the finite intermediate extensions. However, consider the category of topological groups, Grp Top, which consists of a group on a topological space such that the multiplication and negation operations are continuous (arrows are continuous group homomorphisms). We can view the finite Galois groups of the finite intermediate extensions as discrete topological groups. If we have a tower of Galois extensions, $K/M/L/k$, where M/L and L/k are finite, then there is a restriction map from $\mathrm{Gal}(M/k)$ to $\mathrm{Gal}(L/k)$. If we take the limit of all of the finite Galois intermediate extensions of K/k with these restriction maps, we get a natural topology on the automorphism group of K over k. Then infinite Galois theory says that the closed subgroups correspond to the intermediate extensions.

Let $x \in \mathbb{C}$. Given any neighborhood U of x, we can consider the ring R_U of holomorphic functions on this neighborhood. If $V \subset U$, then there is a natural (injective if U and V are connected) map from R_U to R_V, namely restriction. Then we can define the ring of germs at x to be the colimit of the diagram consisting of each R_U and the restriction maps. People study such things when they construct compact Riemann surfaces or study similar analytic/geometric things. They are especially nice as they are local rings, so provide a nice setting in which to study local properties of a function.

In the category of R-modules, R-Mod, R itself is an R-module. In this category, a finite coproduct ($\bigoplus$) is the same as a finite product ($\prod$). Recall that one thinks of an $n \times n$ square matrix as representing an endomorphism of $f : R^n \to R^n$. But then f can be thought of as a map from $\bigoplus_1^n R \to \prod_1^n R$. By the universal property of the product and coproduct, this is determined by n^2 maps from R to R, each of which is determined by where 1 is sent (recall $\mathrm{End}_R(R) \cong R$ from last lecture), which are exactly what the entries of the matrix represent.

As a final example, the p-adic integers, a often studied ring in number theory, can be described as a limit. Namely, consider the rings $\mathbb{Z}/(p^n)$ for each $n \in \mathbb{N}$. Then if $n < m$, there is a projection from $\mathbb{Z}/(p^m) \to \mathbb{Z}/(p^n)$. The limit of each of these rings along with these projections is the ring $\mathbb{Z}_p$ of p-adic integers. This is known as completing with respect to the ideal (p). One can do the same for $R[x]$ with the ideal (x), namely take the rings $R[x]/(x^n)$ for each $n \in \mathbb{N}$ along with similar projections, and the limit yields $R[[x]]$, the ring of formal power series.

4. Group Objects

Recall that in the Yoneda lemma lecture I said that the reason that certain representable functors lifted to other categories like Grp was because the representing objects were something called group objects. I can now make this precise using products and the Yoneda lemma. The set up is the following: We have a representable functor $C(-, a)$ that lifts to Grp as in the diagram below:

$$
\begin{tikzcd}
C \arrow[r, "F"] \arrow[dr, "C(-,a)"'] & \mathrm{Grp} \arrow[d, "U"] \\
& Set
\end{tikzcd}
$$

In order for this to happen, we need a way to "multiply arrows" to the object a. In other words, we need a natural transformation $C(-,a) \times C(-,a) \to C(-,a)$. By Corollary 2.7, $C(-,a) \times C(-,a) \cong C(-, a \times a)$ (We assume here that the limit exists). By the Yoneda lemma, such a natural transformation then is given by a map from $a \times a$ to a. We would like this map to be associative, and have an inverse and identity. Hence the following definition:

Definition 4.1. *A* ***group object*** *in a category C is an object X with a map $\mu : X \times X \to X$, a map $\nu : X \to X$, and a map $i : t \to X$ (t is the terminal object), such that we have the following diagrams commute:*

Associativity of μ:

$$
\begin{tikzcd}
X \times (X \times X) \arrow[r, "\eta_X"] \arrow[d, "1_X \times \mu"] & (X \times X) \times X \arrow[r, "\mu \times 1_X"] & X \times X \arrow[d, "\mu"] \\
X \times X \arrow[rr, "\mu"] & & X \arrow[r, equal] & X
\end{tikzcd}
$$

Here η_X is the (natural) isomorphism one generally has between $(X \times Y) \times Z$ and $X \times (Y \times Z)$ (use the universal property to define it and show it's an isomorphism).

Left and right identity:

$$
\begin{tikzcd}
t \times X \arrow[r, "\eta_X"] \arrow[dr, "\gamma_l"'] & X \times X \arrow[d, "\mu"] & X \times t \arrow[l, "1_X \times i"'] \arrow[dl, "\gamma_r"] \\
& X &
\end{tikzcd}
$$

Here I leave it to you again to define the isomorphisms γ_l, γ_r.

Left and right inverses:

$$
\begin{tikzcd}
X \times X \arrow[r, "1_X \times \nu"] \arrow[d] & X \times X \arrow[d, "\mu"] & X \times X \arrow[l, "\nu \times 1_X"'] \arrow[d] \\
t \arrow[r, "i"] & X & t \arrow[l, "i"']
\end{tikzcd}
$$

I leave it to you to prove the following result which I have hinted to (use Yoneda a lot):

Proposition 4.2. *Let C be a category with finite products. The functor $C(-,a)$ factors through the category of groups iff a is a group object.*

If C is a category, we can consider the category of group objects on C, which I will denote $\mathrm{Grp}\, C$. The objects are group objects in C and the morphisms are group object homomorphisms in C (I leave it to you to define these). For example, $\mathrm{Grp} = \mathrm{Grp}\,\mathrm{Set}$, and the category of topological groups I described earlier is $\mathrm{Grp}\,\mathrm{Top}$ (hence the notation). One similarly can define ring objects, monoid objects, lattice objects, and so on for any algebraic theory. If you are curious I recommend figuring out how the examples in the duality lecture are $(-)$ objects in their respective categories.