NON-EMBEDDING RESULTS VIA S-DUALITY

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What is the minimal dimension such that $\mathbb{CP}^2$ or $\mathbb{HP}^2$ can be embedded in Euclidean space? It turns out one gets the optimal answer by answering the corresponding stable problem, namely what is the minimal dimension such that a space with the stable homotopy type of $\mathbb{CP}^2$ or $\mathbb{HP}^2$ can be embedded in Euclidean space?

We can answer a generalization of this question as follows (due to Hilton and Spanier):

Theorem 0.1. *Let f be a map $S^{m-1} \to S^n$, and let C_f denote the cofibre. Suppose that f cannot be stably desuspended. Then the minimum dimension embedding of C_f is $m+n+1$.*

In particular, $\mathbb{CP}^2$ and $\mathbb{HP}^2$ are cofibres of Hopf maps, which cannot stably be desuspended (a stable desuspension is another map of spheres of lower dimension agreeing stably with f). Note that C_f can be embedded in S^{m+n+1} because the mapping cylinder of f embeds into the join of S^{m-1} and S^n, which is S^{m+n}, and the cylinder end can be coned off in S^{m+n+1}. So it suffices to show one cannot embed into anything smaller.

First we consider the simplest cases. If $m < n+1$, then f is trivial, so, we must have $n = 0$, in which case the assertion is obvious. If $m = n+1$, then $n = 1$. Then, one can use Alexander duality to observe that were there an embedding into S^3, then the complement would have zero dimensional homology that is not free.

Thus we can assume that $n > 1, m > n+1$.

Lemma 0.2. *If $n > 1, m > n+1$, f can be stably desuspended iff C_f can be stably desuspended.*

Proof. Clearly the stable homotopy type of C_f depends only on that of f, proving one direction. On the other hand, if C_f can be stably desuspended to a space X, a homology decomposition of X will be the cofibre of a map g between spheres. After suspending enough, these will be of the same dimension, and since the map between the middle skeleton has to extend to a homotopy equivalence between the two spaces, the attaching maps differ by a unit (i.e an integer multiple), so f can be stably desuspended. $\square$

The essential input of working stably is the following observation: the Spanier-Whitehead (S) dual of the cofibre of C_f (denoted DC_f) is $\Sigma^{-m-n}C_{\pm f}$, where the sign (unimportant) I think is $(-1)^{mn}$. To see this, the S dual of a map f between spheres is $\pm f$, suspended to have the right degrees. We have a cofibre sequence, $f : S^{m-1} \to S^n \to C_f$, which taking duals gives a fibre=cofbre sequence $DC_f \to S^{-n} \to S^{1-m}$. Rearranging this shows that $DC_f = \Sigma^{-m-n}C_{\pm f}$. Now the complement of C_f inside S^{m+n} would be $\Sigma^{m+n-1}DC_f = \Sigma^{-1}C_f$! This completes the proof via the lemma.

The same argument gives the slightly stronger version:

Theorem 0.3. *Let f be a map $S^{m-1} \to S^n$, and let C_f denote the cofibre. Let k be the maximum number of times that f can be stably desuspended. Then the minimum dimension embedding of C_f is $m + n + 1 - k$.*