NATURAL TRANSFORMATIONS, DUALITY, & EQUIVALENCES

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1. What is a natural transformation?

Let's introduce the final essential categorical concept, the natural transformation. This is an extremely important concept, I believe Mac Lane has said that he defined the notion of category so that he could make precise a functor, and he defined a functor to make precise the notion of a natural transformation.

Definition 1.1. *Given two functors F and G from C to D, a* ***natural transformation*** *η from F to G is for each object x of C, an arrow η_x from Fx to Gx such that the following diagram commutes for all x, y, f:*

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"] & Fy \arrow[d, "\eta_y"] \\ Gx \arrow[r, "Gf"] & Gy \end{tikzcd}$$

We write $\eta : F \Rightarrow G$ to denote a natural transformation.

Natural transformation is a wonderful way of formalizing an intuitive sense of natural. For example, if V is a vector space over a field F, there is a dual vector space V^* which is the vector space of linear maps from V to F. Perhaps you know that if V is finite dimensional, it is isomorphic to its dual. However these aren't canonically isomorphic: in order to make an isomorphism, you have to **choose** a basis and then identify them. Natural transformation makes precise when this is canonical. For example, if Vect_F is the category of F-vector spaces, then $(-)^*$, the dual, is a contravariant functor from Vect_F to itself. On arrows, $(-)^*$ does the same thing as the Hom functor $C(-, F)$. We can compose $(-)^*$ with itself to get the covariant double dual functor $(-)^{**}$. If f is a map from V to W, then the double dual makes a map from V^{**} to W^{**} as follows: given a map g that takes maps h from V to F to F, we get the map $f^{**}(g)$ that takes maps $k : V \to F$ to $g(k \circ f)$. We can define a natural transformation η from 1_{Vect_F} to the double dual $(-)^{**}$: given $v \in V$, we send it to the element of V^{**} that takes an element of V^*, and evaluates it at v. This is an isomorphism if the vector space is finite dimension, and note that

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it is canonical: there was no need to make any choices. Then, we should expect this collection of maps $\eta_V, V \in \mathrm{Vect}_F$ to be a natural transformation. And indeed it is, as one can check by following an element around the diagram that we want to commute:

$$\begin{tikzcd} V \arrow[r, "1_{\mathrm{Vect}_F} f = f"] \arrow[d, "\eta_V"] & W \arrow[d, "\eta_W"] \\ V^{**} \arrow[r, "f^{**}"] & W^{**} \end{tikzcd}$$

Lets follow around an element $v \in V$:

$$\begin{tikzcd} v \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\eta_V"] & f(v) \arrow[d, mapsto, "\eta_W"] \\ h : h(g) = g(v) \arrow[r, mapsto, "f^{**}"] & k : k(g) = h(g \circ f), k : k(g) = g(f(v)) \end{tikzcd}$$

Another example is the abelianization. Given a group G, we can define a subgroup called the commutator subgroup $[G, G] = \{aba^{-1}b^{-1} | a, b \in G\}$. The abelianization of G is the group $G/[G, G]$. This is a functor as if $f : G \to H$ is a homomorphism, we can compose with the projection $H \to H/[H, H]$ to get a map $G \to H/[H, H]$. $[G, G]$ is in the kernel of this map, so we get then a map $G/[G, G] \to H/[H, H]$. This is the map that the abelianization sends f to. Now the projection $\pi_G : G \to G/[G, G]$ is a natural transformation as the diagram below commutes (by definition):

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[d] & G/[G,G] \arrow[d] \\ H \arrow[r, "\pi_H"] & H/[H,H] \end{tikzcd}$$

As a third example, consider the category ω which is the poset category of $\mathbb{N}$ with the usual ordering. Consider a diagram consisting of a sequence of sets S_n with injective maps from $S_n \to S_{n+1}$. This can be thought of as a sequence of sets increasing in size (each containing the previous). Recall that diagrams are just functors, and in this case, ω is the category for which this is a functor (we can call this functor F. Let $\widehat{\cup S_i}$ be the constant functor taking ω to $\cup S_i$, and all the arrows to the identity. Then consider the natural transformation $\eta : F \Rightarrow \widehat{\cup S_i}$ that sends each S_i with the subset it corresponds to in the union. I leave this as an easy exercise to check that this is a natural transformation (draw it!). This kind of natural transformation is called a **cocone** (this will be discussed in more depth when we do (co)limits).

Finally, consider the determinant of a (invertible) matrix, $\det^n$. I claim this is a natural transformation. Consider the two functors from CRing to Grp: one taking

K to $GL_n(K)$, and the other taking it to K^* (check that these are functors). Then $\det_K^n$ is for each element of CRing a map from $GL_n(K)$ to K^* sending a linear transformation to its determinant. The diagram is the same as always:

$$\begin{tikzcd} GL_nF \arrow[r, "\det_F^n"] \arrow[d, "GL_nf"] & F^* \arrow[d, "f^*"] \\ GL_nK \arrow[r, "\det_K^n"] & K^* \end{tikzcd}$$

Given two categories C, D, we can form the **product category**, $C \times D$ where the objects are pairs of objects, the arrows are pairs of arrows, and composition is defined as usual.

Now consider the contravariant powerset $\mathrm{Set}(-, 2)$ (2 is a set with two elements, we can view this functor as $2^{(-)}$). As an exercise, try to find all the natural transformations from this functor to itself (this will come up again in a later lecture).

Definition 1.2. *Suppose F, G, H are functors in $\mathrm{Cat}(C, D)$. Then if $\eta : F \Rightarrow G$ and $\nu : G \Rightarrow H$ are natural transformations, then we can form the **vertical composite**, $\nu \cdot \eta$, a natural transformation from F to H, defined by $\nu \cdot \eta_a = \nu_a \circ \eta_a$.*

We can check this is a natural transformation via the following diagram:

$$\begin{tikzcd} Fa \arrow[r, "\eta_a"] \arrow[d, "Ff"] & Ga \arrow[r, "\nu_a"] \arrow[d, "Gf"] & Ha \arrow[d, "Hf"] \\ Fb \arrow[r, "\eta_b"] & Gb \arrow[r, "\nu_b"] & Hb \end{tikzcd}$$

This turns $\mathrm{Cat}(C, D)$ into a category, which we call the functor category. We can write this as D^C. An isomorphism in $\mathrm{Cat}(C, D)$ is called a natural isomorphism. Alternatively, it is a natural transformation η where each η_a is an isomorphism.

I use the word vertical composite, because there is also a horizontal composite. It can be seen as follows:

Given the diagram below, we would like to create a natural transformation $\nu\eta : F' \circ F \Rightarrow G' \circ G$ sometimes written $\nu \circ \eta$.

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & D \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\nu"] & E \end{tikzcd}$$

We can do this by considering the following diagram:

$$\begin{tikzcd}
F'Fa \arrow[r, "F'\eta_a"] \arrow[d, "\nu_{Fa}"] & F'Ga \arrow[d, "\nu_{Ga}"] \\
G'Fa \arrow[r, "G'\eta_a"] & G'Ga
\end{tikzcd}$$

This commutes as ν is natural for η_a. This suggests the following definition:

Definition 1.3. *Suppose F, G, F', G', η, ν are as above, we can form the **horizontal composite** $\nu\eta : F' \circ F \to G' \circ G$ so that $(\nu\eta)_a = \nu_{Ga} \circ F'\eta_a = G'\eta_a \circ \nu_{Fa}$.*

It remains to check this is a natural transformation, but this should be obvious if you draw the appropriate diagram (for a natural transformation). If $F : C \to D$, $G, H : D \to E$ are functors and $\eta : G \Rightarrow H$ a natural transformation then the natural transformation ηF denotes the horizontal composite $\eta 1_F$, and similarly if $J : E \to X$ is a functor, then $J\eta$ denotes $1_J\eta$.

Horizontal composites and vertical composites are related through the interchange law, which says $(\tau\eta) \cdot (\tau'\eta') = (\tau \cdot \tau')(\eta \cdot \eta')$. It can be described as the diagram below:

$$\begin{tikzcd}
C \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\eta'"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\eta"] & D
\end{tikzcd}
\circ
\begin{tikzcd}
D \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\tau'"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\tau"] & E
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\eta'"] & D \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\tau'"] & E \\[-2ex]
 & \bullet & \\[-2ex]
C \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\eta"] & D \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\tau"] & E
\end{tikzcd}$$

We can prove it using the diagram below. Let $\eta' : F \Rightarrow G, \eta : G \Rightarrow H, \tau' : F' \Rightarrow G', \tau : G' \Rightarrow H'$ in the figure above.

$$\begin{tikzcd}
 & F'Ga \arrow[r, "F'\eta_a"] \arrow[dd, equal] & F'Ha \arrow[r, "\tau'_{Ha}"] & G'Ha \arrow[dr, "\tau_{Ha}"] \arrow[dd, equal] & \\
F'Fa \arrow[ur, "F'\eta'_a"] \arrow[dr, "F'\eta'_a"] & & & & H'Ha \\
 & F'Ga \arrow[r, "\tau'_{Ga}"] & G'Ga \arrow[r, "G'\eta_a"] & G'Ha \arrow[ur, "\tau_{Ha}"] &
\end{tikzcd}$$

The path on the top is the natural transformation $(\tau \cdot \tau')(\eta \cdot \eta')$, and the path on the bottom is $(\tau\eta) \cdot (\tau'\eta')$. The middle rectangle commutes as τ' is a natural transformation.

As a final note, there is an analogy between natural transformations and homotopies.

If X and Y are topological spaces, and f and g are maps (continuous, as always) from X to Y, a homotopy from f to g is a map from $X \times [0, 1]$ to Y that at 0 restricts to f and at 1 restricts to g. The definition of a natural transformation can be presented analogously: Let 2 be the category with 2 objects, called 0 and 1 and one non identity arrow from 0 to 1 (we can say the arrow category, as this is the category that represents the diagram consisting of a generic arrow).

If C and D are categories, and F and G are functors from C to D, a natural transformation is a functor from F to G is a functor from $C \times 2$ to D that on 0 restricts to F and on 1 restricts to G.

Check that these two definitions of natural transformations are equivalent and note the similarity with homotopies. In a way, a natural transformation is categorification of homotopy.

Finally let's end with an interesting non-example. Let FinSet_g be the category of finite sets and bijections between them. Consider two functors to Set, the first, Aut, takes X to the set of bijections from X to itself, on maps, it takes $f : X \to Y$ to the function that takes $\phi : X \to X$ to $f \circ \phi \circ f^{-1} : Y \to Y$. The second, Ord, takes X to the set of total orders on X, and on maps takes f to the total order on Y induced by the bijection. These two functors send isomorphic objects to isomorphic sets, but are not naturally isomorphic: in fact, there isn't even a natural transformation between them! For, let's consider f, the nontrivial bijection from a set $\{a, b\}$ to itself. If there was a natural transformation, we would have

$$\begin{tikzcd}
\{1, (a, b)\} \arrow[r, "\mathrm{Aut}(f)"] \arrow[d, "\eta_b"] & \{1, (a, b)\} \arrow[d, "\eta_c"] \\
\{a < b, b < a\} \arrow[r, "Ff"] & \{a < b, b < a\}
\end{tikzcd}$$

$\mathrm{Aut}(f)$ is the identity, but Ff is not, so this diagram cannot commute.

The fact that this bijection is not natural has an interesting interpretation in the context of a combinatorics problem. In particular, let's count the number of trees on a set of n elements, which we'll call T_n. Let $|\cdot|$ denote cardinality of a set. Consider the product $T_n \times n \times n$, consisting of a tree on the set n, as well as a head and a tail (shown in Fig 2).

Note that since there is a unique path between any two points in a tree, we can draw an arrow from the tail to the head, yielding a total ordering on a subset of 1 to n, ie. a skeleton, as well as trees coming out of each point. Note that the skeleton and the trees coming out of each point completely determine $T_n \times n \times n$. Then as total orders are in bijections with permutations, we can consider the set of permutations with trees coming out of them, a typical example in the figure below:

These are in bijection with functions from the set of n elements to itself, as a function determines such a tree by writing where everything goes, which eventually (after applying the function enough times) determines the cycles and the trees coming out of them. Thus $T_n \times n \times n$ is in bijection with the set of functions from $\{1, ..., n\}$ to itself, which is n^n. Thus $|T_n| = n^{n-2}$ (This is known as Cayley's Theorem). Perhaps

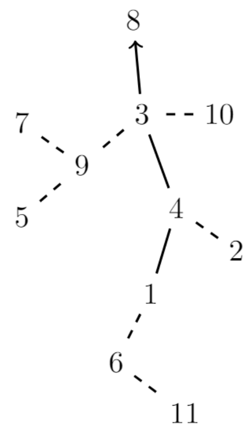

FIGURE 1. A tree, on 11 elements, with a skeleton, indicated by the bold lines, is determined by the total ordering on the skeleton and the trees coming out of each point on the skeleton.

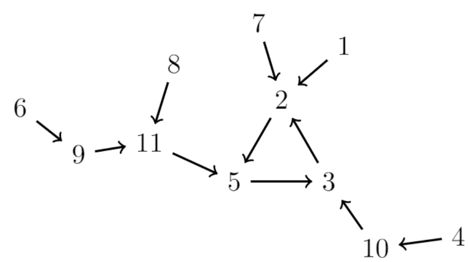

FIGURE 2. A permutation on 2, 3, and 5 with trees coming out of it.

the reason this proof does something nontrivial is because it used this bijection which was unnatural.