# ORDINARY DIFFERENTIAL EQUATIONS

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## 1. Main Theorems

We can use the contraction mapping principle to construct solutions of ODEs.

**Lemma 1.1.** *Suppose that $M$ is a metric space, and $T$ is a contraction endomorphism. Then there is at most one fixed point of $T$, and if $M$ is nonempty and complete, the fixed point exists.*

Solutions to ODEs locally exist uniquely with whatever amount of continuity we want as long as the defining equations satisfy uniform Lipschitz conditions locally.

**Theorem 1.2.** *Let $U$ be a compact space, $V \subset B$ a bounded open subset of a Banach space, $F_U$ a continuous function from $U$ to a space of uniformly bounded continuous functions on $V \times (\mathbb{R})$ to $\mathbb{R}$ ), satisfying a uniform Lipschitz condition with constant $K$, and $I : U \to V$ the initial condition function. Then for small $\epsilon$, there is a unique solution that is continuous, $x_u(t) : U \times (-\epsilon, \epsilon) \to V$ solving the differential equation $\frac{dx_u(t)}{dt} = F_u(x_u, t)$ with initial conditions $x_u(0) = I(u)$.*

*Proof.* We can write the equation as $x_u(t) = I(u) + \int_{t_0}^t F_u(x_u(t), t)dt$. We can think of the right hand side as an operator $\phi x$ on possible solutions $x$, so that the problem amounts to carefully defining the space that $\phi$ acts on, and showing that it is a contraction. We can define $M$ to be the space of continuous functions $U \times (-\epsilon, \epsilon) \to V$ satisfying the initial conditions, and let the metric be induced from the sup norm. Then since $U$ is compact and $F_U$ is continuous, the family $F_u$ are uniformly bounded, so that for small $\epsilon$, the operator $\phi$ will indeed send elements of $M$ to $M$. To see that $\phi$ is a contraction map, observe that that if $x, x' \in M$, then $d(\phi x, \phi x') = \sup_t \| \int_{t_0}^t F_u(x_u(t), t) - F_u(x'_u(t))dt\| \leq \sup_t \int_{t_0}^t K\|x_u(t) - x'_u(t)\| \leq K\epsilon d(x, x')$, so if $K\epsilon < 1$, then this is a contraction map. , $\square$

In particular this shows that solutions depend differentiably upon initial conditions if the derivatives of $F(x, t)$ are uniformly Lipschitz: we can take $U$ to be a small neighborhood of 0 times a time direction, and consider solving the differential equation for the difference quotients of $F(x, t)$ in the neighborhood, and the solution will be difference quotients of the solution for $F$ by uniqueness. Then by continuity, the difference quotients will converge to the derivative of $F$, and the left hand side will converge as well, implying that the solution of the original equation is continuously differentiable. If $F$ is smooth, then so will the solution $x$. Moreover if $F$ is analytic, then $x$ will be a uniform limit of analytic functions, so will be as well.

Global existence of solutions can only fail if the solution blows up, and tries to leave the domain.

**Corollary 1.3.** *Suppose that $F(x,t)$ locally is Lipschitz continuous. Then there is a maximal interval $(-a,b)$ on which a solution to $\frac{dx}{dt} = F(x,t)$ with an initial condition can be defined, which is unique. If $-a$ or $b$ is finite, then the solution leaves every compact set as it approaches $-a$ or $-b$.*

*Proof.* Try to define $-a, b$ as the the largest such that the solution exists. By local uniqueness, the solution is unique on that interval. Now if the solution $x$ is in some compact set when $t$ is near $b$, then it must converge as $t \to b$. But then we can use the local existence theorem, which will agree with the already existing solution to see that $x$ can be extended a bit beyond $b$. □

Thus given a vector field $V$ on a manifold $M$, the field induces a local flow on $M$ that may not be globally defined, but is if $M$ is compact. If the vector field and $M$ are smooth, the flow will be as well. Conversely if $g_t$ is a one parameter local group of diffeomorphisms, The derivative of $g_t$ at $t = 0$ is a smooth vector field inducing $g_t$ as its flow.

One can often reduce higher order equations to first order equations. For example, if we want to solve $x^{(n)} = F(x^{(i)}; i < n, t)$, we can turn it into a first order equation in more variables by solving for $(x^{(i)})$ simultaneously.

The inverse function theorem can be proven using the same contraction mapping principle.

**Lemma 1.4.** *If $f$ is $C^1$ in a neighborhood of a point $p$ and has an invertible derivative at $p$, then it is injective near $p$.*

*Proof.* After a change of coordinates, $f(0) = 0, f'(0) = I$, so that $f(x) = f(a) + f'(a)(x-a) + o(x-a)$. We can choose a neighborhood such that $\|a\| < \frac{1}{4}$, $\|f'(a)\| > \frac{3}{4}$ and so that the error term $\|o(x-a)\| < \frac{1}{4}\|x-a\|$. Then we have that $\|f(x)-f(y)\| = \|f'(y)(x-y) + o(x-y)\| \geq \frac{3}{4}\|x-y\| - \|o(x-y)\| \geq \frac{1}{2}\|x-y\|$. □

**Theorem 1.5** (Inverse function theorem)**.** *If $f$ has an invertible derivative at a point, and is injective near that point it locally has an inverse, which is differentiable at the image point.*

*Proof.* $f(x) = y = x + \epsilon(x)$ where $\epsilon(x) = o(x)$ after a linear change of variables. To solve $y = x + \epsilon(x)$, we observe that $x = y - o(x)$, so if $g(y) = x$, it is the fixed point of the mapping $\phi(g) = y - \epsilon(g)$. But in a small neighborhood of 0, $\|\epsilon(x)\| < \lambda < 1$, so in this neighborhood, $\phi$ is a contraction mapping on the space of possible continuous inverses with the sup norm. Note that $\phi^n(g)$ is differentiable at 0 with derivative the identity, and uniformly converges to the inverse, so the inverse is differentiable at 0. $\square$

There is a geometric interpretation of solving an ODE (and a similar one for PDE). The let $T$ be the times, $S$ be the phase space, and $J^1(S)$ be the space of 1-jets, with the projection map to $S$. $J_1(S)$ comes with a canonical distribution, given by the 1-forms $dx_i = p_i dt$, where $p_i$ are the cotangent bundle coordinates. We can consider the bundles $T \times S$, $T \times J_1(S)$ over $T$. A smooth section of the first one has a canonical lift to a section of the second, by requiring that it is tangent to the distribution. Then if $F(p, x, t)$ is a differential equation, a solution is exactly a section of $T \times S$ whose lift lies in the hypersurface defined by $F(p, x, t)$.