GALOIS THEORY AND RIEMANN SURFACES

ISHAN LEVY

1. Introduction

The Galois correspondence for fields gives an equivalence between finite extensions and continuous actions of the absolute Galois group. There is a similar theorem in topology which can be stated as follows: for every reasonable space, finite covering maps correspond to continuous actions of the profinite completion of the fundamental group.

These correspondences are highly analogous: the first comes from looking at the maps from the field extension into a separable closure and acting on them by the Galois group, or by looking at maps from the universal cover to the space, and acting on them by the fundamental group.

We will see that in the case of compact Riemann surfaces, this relationship is especially strong. A compact Riemann surface is determined by its field of meromorphic functions (function field), which is a transcendence degree 1 field extension of $\mathbb{C}$, and we will see that Galois theory for the function field will tell us about finite branched covers of the Riemann surface. This relationship is clarified after realizing that compact Riemann surfaces all come from some projective variety.

The first thing we will see is that branched covers are determined by their topological data.

Theorem 1.1. *Given a finite topological branched cover of a compact Riemann surface, there is a unique holomorphic structure on it making the covering holomorphic.*

Proof. After removing the fibres of the branch points this is true because the map is a local homeomorphism. In order to put holomorphic charts on those fibres, observe that near each branch point we have a finite cover of a punctured disk, which locally looks like a disk covering itself via $z \to z^k$, so we can put back in the branch points using charts that look like $z \to z^k$. Given two ways to fill in the branch points, we can make a holomorphic isomorphism from one cover to the other away from the branch points, which extend to the branch points by the Riemann extension theorem. $\square$

Any nonconstant map of compact Riemann surfaces is a finite branched cover, and any finite branched cover as we have seen comes from a unique Riemann surface, so these are the same thing (i.e we have an equivalence of categories).

Note that we also have that the topological automorphisms of a branched cover are holomorphic. Indeed, given a topological automorphism, there is both a unique holomorphic structure making the automorphism a branched cover, and making the cover holomorphic, but the first implies the second, so by uniqueness, these are the same.

Now we will use a theorem that is obtained from analysis to study maps of compact Riemann surfaces further.

Theorem 1.2 (Riemann existence theorem)**.** *Given $x_1, \dots, x_n$ on a Riemann surface C, and $a_1, \dots, a_n \in \mathbb{P}^1$, there is a meromorphic function on C taking x_i to a_i.*

It is not hard to see that this follows from the fact that any two points can be distinguished by meromorphic functions.

The meromorphic functions form a field (we don't count the constant map to infinity as meromorphic), which we denote $M(C)$. Given a nonconstant map $C_1 \to C_2$ of compact Riemann surfaces, we can pullback meromorphic functions from C_2 to C_1. Since a meromorphic function is determined locally, and there are points where the map is locally an isomorphism, we get an inclusion $M(C_2) \to M(C_1)$.

Theorem 1.3. *The degree of the inclusion $M(C_2) \to M(C_1)$ is the degree of the map $C_1 \to C_2$.*

Proof. First we will show that every meromorphic function on C_1 is the zero of a polynomial in $M(C_1)[x]$ of degree d, that of the map $C_1 \to C_2$.

If the function f is nonconstant, at every unramified point of C_2, we can take a small neighborhood, and look at the preimage, which will be d copies of that neighborhood, each with a function $f_1, \dots, f_d$. We can consider the functions $e_1, \dots, e_d$ on the neighborhood in C_2 that are the elementary symmetric functions on the f_i. Since we can glue these together locally, the e_i define a meromorphic function on the unramified points of C_2, which extend to all of C_2 by the Riemann extension theorem. Then f satisfies $x^d + \sum e_i x^{d-i} = 0$. To see this, it suffices by the identity principle to observe it locally at an unramified point, which is true because at an unramified point, the polynomial factors as $\prod_1^n (x - a_i)$, where a_i is a cyclic permutation of f.

Thus by the primitive element theorem, the degree is at most d, as every element is degree at most d. It suffices to produce an element of degree d. By the Riemann existence theorem, for an unramified point p, there is a meromorphic function f taking distinct values on the fibres. It cannot have degree less than d. Indeed, all the a_i must be roots of the minimal polynomial as they all vanish when any one of them does. $\square$

There is a converse:

Theorem 1.4. *Given a finite field extension $K/M(C_2)$, there is a map of Riemann surfaces $C_1 \to C_2$ such that the induced field extension is exactly $K/M(C_2)$.*

Proof. Take a primitive element of the field extension, f with minimal polynomial p. f is relatively prime to its derivative, so $fa + f'b = 1$. Away from the poles of the coefficients of a, b, f and f' cannot have common zeros, so have distinct roots. Remove these points and those where $f' = 0$. Now we can produce a sheaf on the complement of these points where a section is a function on the open set satisfying the polynomial relation. This is a locally constant sheaf by the implicit function theorem, as $f' \neq 0$, and has d stalks as the roots are distinct. This gives a holomorphic cover of the complement of these points, and we can fill in the finitely many points to form a branched cover of C_2. The degree is clearly d, and moreover there is a tautological function on it that is a solution to $p(f) = 0$, namely the one that sends the stalk to its value. This is the cover we want. □

In other words, maps of compact Riemann surfaces are exactly the same as finite field extensions of the function field!

In particular, we can interpret Galois theory of the function field in terms of branched covers. Field automorphisms correspond to automorphisms of the branched cover. If we take the quotient of the Riemann surface by the action of the group of cover automorphisms, the resulting Riemann surface's function field is exactly the fixed field of the corresponding automorphism of fields.

Theorem 1.5. *Compact Riemann surfaces are equivalent to finitely generated transcendence degree* 1 $\mathbb{C}$*-algebras.*

Proof. A meromorphic function is a finite map to $\mathbb{P}^1$. By the Riemann existence theorem, nontrivial meromorphic functions exist, and so the function field is a finite extension of $M(\mathbb{P}^1) = \mathbb{C}(t)$ which is exactly a transcendence degree 1 extension of $\mathbb{C}$. If one is a little more careful, this gives an equivalence of categories. □

In fact, we can prove more.

Theorem 1.6. *Compact Riemann surfaces are the same as smooth projective curves.*

Proof. Let f be a primitive element of $M(C)$ over t, some finite map to $\mathbb{P}^1$. Then consider the map $C \to \mathbb{P}^2$ given by $(f : t : 1)$. As we saw in Theorem 1.4, we can choose the function f to distinguish fibres of the map to $\mathbb{P}^1$ generically, so the generic element of the image of this map has 1 preimage. The image is a singular variety given by the polynomial equation that f satisfies. By resolving the singularities of this variety (there are plenty of ways to do this), we can obtain a projective curve such that if we lift the map to the resolution, we get an isomorphism with our Riemann surface. In particular, note that the functions f, t are rational functions on this curve, and every rational function is meromorphic, so its field of rational functions is exactly that of C. □

Algebraically, the field of rational functions of a variety has all the information of the variety up to taking dense open sets, which in this case means removing finitely many points. This is why one should expect the field extensions to correspond to branched covers: there is an algebraic curve with the same function field such that the branched cover is an actual algebraic cover.