

# NATURAL TRANSFORMATIONS, DUALITY, & EQUIVALENCES

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## 1. WHAT IS A NATURAL TRANSFORMATION?

Let's introduce the final essential categorical concept, the natural transformation. This is an extremely important concept, I believe Mac Lane has said that he defined the notion of category so that he could make precise a functor, and he defined a functor to make precise the notion of a natural transformation.

**Definition 1.1.** *Given two functors  $F$  and  $G$  from  $C$  to  $D$ , a **natural transformation**  $\eta$  from  $F$  to  $G$  is for each object  $x$  of  $C$ , an arrow  $\eta_x$  from  $Fx$  to  $Gx$  such that the following diagram commutes for all  $x, y, f$ :*

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ \downarrow \eta_x & & \downarrow \eta_y \\ Gx & \xrightarrow{Gf} & Gy \end{array}$$

We write  $\eta : F \Rightarrow G$  to denote a natural transformation.

Natural transformation is a wonderful way of formalizing an intuitive sense of natural. For example, if  $V$  is a vector space over a field  $F$ , there is a dual vector space  $V^*$  which is the vector space of linear maps from  $V$  to  $F$ . Perhaps you know that if  $V$  is finite dimensional, it is isomorphic to its dual. However these aren't canonically isomorphic: in order to make an isomorphism, you have to **choose** a basis and then identify them. Natural transformation makes precise when this is canonical. For example, if  $\text{Vect}_F$  is the category of  $F$ -vector spaces, then  $(-)^*$ , the dual, is a contravariant functor from  $\text{Vect}_F$  to itself. On arrows,  $(-)^*$  does the same thing as the Hom functor  $C(-, F)$ . We can compose  $(-)^*$  with itself to get the covariant double dual functor  $(-)^{**}$ . If  $f$  is a map from  $V$  to  $W$ , then the double dual makes a map from  $V^{**}$  to  $W^{**}$  as follows: given a map  $g$  that takes maps  $h$  from  $V$  to  $F$  to  $F$ , we get the map  $f^{**}(g)$  that takes maps  $k : V \rightarrow F$  to  $g(k \circ f)$ . We can define a natural transformation  $\eta$  from  $1_{\text{Vect}_F}$  to the double dual  $(-)^{**}$ : given  $v \in V$ , we send it to the element of  $V^{**}$  that takes an element of  $V^*$ , and evaluates it at  $v$ . This is an isomorphism if the vector space is finite dimension, and note that it is canonical: there was no need to make any choices. Then, we should expect this collection of maps  $\eta_V, V \in \text{Vect}_F$  to be a natural transformation. And indeed it is, as one can check by following an element around the diagram that we want to commute:

$$\begin{array}{ccc} V & \xrightarrow{1_{\text{Vect}_F} f = f} & W \\ \downarrow \eta_V & & \downarrow \eta_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

Lets follow around an element  $v \in V$ :

$$\begin{array}{ccc}
 v & \xrightarrow{f} & f(v) \\
 \downarrow \eta_V & & \downarrow \eta_W \\
 h : h(g) = g(v) & \xrightarrow{f^{**}} & k : k(g) = h(g \circ f), k : k(g) = g(f(v))
 \end{array}$$

Another example is the abelianization. Given a group  $G$ , we can define a subgroup called the commutator subgroup  $[G, G] = \{aba^{-1}b^{-1} | a, b \in G\}$ . The abelianization of  $G$  is the group  $G/[G, G]$ . This is a functor as if  $f : G \rightarrow H$  is a homomorphism, we can compose with the projection  $H \rightarrow H/[H, H]$  to get a map  $G \rightarrow H/[H, H]$ .  $[G, G]$  is in the kernel of this map, so we get then a map  $G/[G, G] \rightarrow H/[H, H]$ . This is the map that the abelianization sends  $f$  to. Now the projection  $\pi_G : G \rightarrow G/[G, G]$  is a natural transformation as the diagram below commutes (by definition):

$$\begin{array}{ccc}
 G & \xrightarrow{\pi_G} & G/[G, G] \\
 \downarrow & & \downarrow \\
 H & \xrightarrow{\pi_H} & H/[H, H]
 \end{array}$$

As a third example, consider the category  $\omega$  which is the poset category of  $\mathbb{N}$  with the usual ordering. Consider a diagram consisting of a sequence of sets  $S_n$  with injective maps from  $S_n \rightarrow S_{n+1}$ . This can be thought of as a sequence of sets increasing in size (each containing the previous). Recall that diagrams are just functors, and in this case,  $\omega$  is the category for which this is a functor (we can call this functor  $F$ ). Let  $\widehat{\cup S_i}$  be the constant functor taking  $\omega$  to  $\cup S_i$ , and all the arrows to the identity. Then consider the natural transformation  $\eta : F \Rightarrow \widehat{\cup S_i}$  that sends each  $S_i$  with the subset it corresponds to in the union. I leave this as an easy exercise to check that this is a natural transformation (draw it!). This kind of natural transformation is called a **cocone** (this will be discussed in more depth when we do (co)limits).

Finally, consider the determinant of a (invertible) matrix,  $\det^n$ . I claim this is a natural transformation. Consider the two functors from  $\mathbf{CRing}$  to  $\mathbf{Grp}$ : one taking  $K$  to  $GL_n(K)$ , and the other taking it to  $K^*$  (check that these are functors). Then  $\det_K^n$  is for each element of  $\mathbf{CRing}$  a map from  $GL_n(K)$  to  $K^*$  sending a linear transformation to its determinant. The diagram is the same as always:

$$\begin{array}{ccc}
 GL_n F & \xrightarrow{\det_F^n} & F^* \\
 \downarrow GL_n f & & \downarrow f^* \\
 GL_n K & \xrightarrow{\det_K^n} & K^*
 \end{array}$$

Given two categories  $C, D$ , we can form the **product category**,  $C \times D$  where the objects are pairs of objects, the arrows are pairs of arrows, and composition is defined as usual.

Now consider the contravariant powerset  $\text{Set}(-, 2)$  (2 is a set with two elements, we can view this functor as  $2^{(-)}$ ). As an exercise, try to find all the natural transformations from this functor to itself (this will come up again in a later lecture).

**Definition 1.2.** Suppose  $F, G, H$  are functors in  $\text{Cat}(C, D)$ . Then if  $\eta : F \Rightarrow G$  and  $\nu : G \Rightarrow H$  are natural transformations, then we can form the **vertical composite**,  $\nu \cdot \eta$ , a natural transformation from  $F$  to  $H$ , defined by  $\nu \cdot \eta_a = \nu_a \circ \eta_a$ .

We can check this is a natural transformation via the following diagram:

$$\begin{array}{ccccc}
Fa & \xrightarrow{\eta_a} & Ga & \xrightarrow{\nu_a} & Ha \\
\downarrow Ff & & \downarrow Gf & & \downarrow Hf \\
Fb & \xrightarrow{\eta_b} & Gb & \xrightarrow{\nu_b} & Hb
\end{array}$$

This turns  $\text{Cat}(C, D)$  into a category, which we call the functor category. We can write this as  $D^C$ . An isomorphism in  $\text{Cat}(C, D)$  is called a natural isomorphism. Alternatively, it is a natural transformation  $\eta$  where each  $\eta_a$  is an isomorphism.

I use the word vertical composite, because there is also a horizontal composite. It can be seen as follows:

Given the diagram below, we would like to create a natural transformation  $\nu\eta : F' \circ F \Rightarrow G' \circ G$  sometimes written  $\nu \circ \eta$ .

$$\begin{array}{ccccc}
& F & & F' & \\
C & \xrightarrow{\quad} & D & \xrightarrow{\quad} & E \\
& \Downarrow \eta & & \Downarrow \nu & \\
& G & & G' &
\end{array}$$

We can do this by considering the following diagram:

$$\begin{array}{ccc}
F'Fa & \xrightarrow{F'\eta_a} & F'Ga \\
\downarrow \nu_{Fa} & & \downarrow \nu_{Ga} \\
G'Fa & \xrightarrow{G'\eta_a} & G'Ga
\end{array}$$

This commutes as  $\nu$  is natural for  $\eta_a$ . This suggests the following definition:

**Definition 1.3.** Suppose  $F, G, F', G', \eta, \nu$  are as above, we can form the **horizontal composite**  $\nu\eta : F' \circ F \rightarrow G' \circ G$  so that  $(\nu\eta)_a = \nu_{Ga} \circ F'\eta_a = G'\eta_a \circ \nu_{Fa}$ .

It remains to check this is a natural transformation, but this should be obvious if you draw the appropriate diagram (for a natural transformation). If  $F : C \rightarrow D$ ,  $G, H : D \rightarrow E$  are functors and  $\eta : G \Rightarrow H$  a natural transformation then the natural transformation  $\eta F$  denotes the horizontal composite  $\eta 1_F$ , and similarly if  $J : E \rightarrow X$  is a functor, then  $J\eta$  denotes  $1_J \eta$ .

Horizontal composites and vertical composites are related through the interchange law, which says  $(\tau\eta) \cdot (\tau'\eta') = (\tau \cdot \tau')(\eta \cdot \eta')$ . It can be described as the diagram below:

$$\begin{array}{ccc}
\begin{array}{ccc}
C & \xrightarrow{\quad} & D \circ D \\
\Downarrow \eta' & & \Downarrow \tau' \\
C & \xrightarrow{\quad} & D
\end{array}
& \circ &
\begin{array}{ccc}
D & \xrightarrow{\quad} & E \\
\Downarrow \tau & & \Downarrow \tau' \\
D & \xrightarrow{\quad} & E
\end{array}
& = &
\begin{array}{ccc}
C & \xrightarrow{\quad} & D \\
\Downarrow \eta' & & \Downarrow \tau' \\
C & \xrightarrow{\quad} & D
\end{array}
\bullet
\begin{array}{ccc}
D & \xrightarrow{\quad} & E \\
\Downarrow \tau & & \Downarrow \tau' \\
D & \xrightarrow{\quad} & E
\end{array}
\end{array}$$

We can prove it using the diagram below. Let  $\eta' : F \Rightarrow G, \eta : G \Rightarrow H, \tau' : F' \Rightarrow G', \tau : G' \Rightarrow H'$  in the figure above.

$$\begin{array}{ccccccc}
& & F'Ga & \xrightarrow{F'\eta_a} & F'Ha & \xrightarrow{\tau'_{Ha}} & G'Ha \\
& \nearrow^{F'\eta'_a} & \parallel & & & & \parallel \searrow^{\tau_{Ha}} \\
F'Fa & & & & & & \\
& \searrow_{F'\eta'_a} & & & & & \\
& & F'Ga & \xrightarrow{\tau'_{Ga}} & G'Ga & \xrightarrow{G'\eta_a} & G'Ha \\
& & & & & & \parallel \nearrow_{\tau_{Ha}} \\
& & & & & & H'Ha
\end{array}$$

The path on the top is the natural transformation  $(\tau \cdot \tau')(\eta \cdot \eta')$ , and the path on the bottom is  $(\tau\eta) \cdot (\tau'\eta')$ . The middle rectangle commutes as  $\tau'$  is a natural transformation.

As a final note, there is an analogy between natural transformations and homotopies.

If  $X$  and  $Y$  are topological spaces, and  $f$  and  $g$  are maps (continuous, as always) from  $X$  to  $Y$ , a homotopy from  $f$  to  $g$  is a map from  $X \times [0, 1]$  to  $Y$  that at 0 restricts to  $f$  and at 1 restricts to  $g$ . The definition of a natural transformation can be presented analogously: Let  $2$  be the category with 2 objects, called 0 and 1 and one non identity arrow from 0 to 1 (we can say the arrow category, as this is the category that represents the diagram consisting of a generic arrow).

If  $C$  and  $D$  are categories, and  $F$  and  $G$  are functors from  $C$  to  $D$ , a natural transformation is a functor from  $F$  to  $G$  is a functor from  $C \times 2$  to  $D$  that on 0 restricts to  $F$  and on 1 restricts to  $G$ .

Check that these two definitions of natural transformations are equivalent and note the similarity with homotopies. In a way, a natural transformation is categorification of homotopy.

Finally let's end with an interesting non-example. Let  $\text{FinSet}_g$  be the category of finite sets and bijections between them. Consider two functors to  $\text{Set}$ , the first,  $\text{Aut}$ , takes  $X$  to the set of bijections from  $X$  to itself, on maps, it takes  $f : X \rightarrow Y$  to the function that takes  $\phi : X \rightarrow X$  to  $f \circ \phi \circ f^{-1} : Y \rightarrow Y$ . The second,  $\text{Ord}$ , takes  $X$  to the set of total orders on  $X$ , and on maps takes  $f$  to the total order on  $Y$  induced by the bijection. These two functors send isomorphic objects to isomorphic sets, but are not naturally isomorphic: in fact, there isn't even a natural transformation between them! For, let's consider  $f$ , the nontrivial bijection from a set  $\{a, b\}$  to itself. If there was a natural transformation, we would have

$$\begin{array}{ccc}
\{1, (a, b)\} & \xrightarrow{\text{Aut}(f)} & \{1, (a, b)\} \\
\downarrow \eta_b & & \downarrow \eta_c \\
\{a < b, b < a\} & \xrightarrow{Ff} & \{a < b, b < a\}
\end{array}$$

$\text{Aut}(f)$  is the identity, but  $Ff$  is not, so this diagram cannot commute.

The fact that this bijection is not natural has an interesting interpretation in the context of a combinatorics problem. In particular, let's count the number of trees on a set of  $n$  elements, which we'll call  $T_n$ . Let  $|\cdot|$  denote cardinality of a set. Consider the product  $T_n \times n \times n$ , consisting of a tree on the set  $n$ , as well as a head and a tail (shown in Fig 2).

Note that since there is a unique path between any two points in a tree, we can draw an arrow from the tail to the head, yielding a total ordering on a subset of 1 to  $n$ , ie. a skeleton, as well as trees coming out of each point. Note that the skeleton and the trees coming out of each point completely determine  $T_n \times n \times n$ . Then as total orders are in bijections with

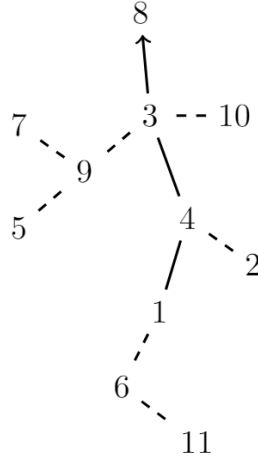


FIGURE 1. A tree, on 11 elements, with a skeleton, indicated by the bold lines, is determined by the total ordering on the skeleton and the trees coming out of each point on the skeleton.

permutations, we can consider the set of permutations with trees coming out of them, a typical example in the figure below:

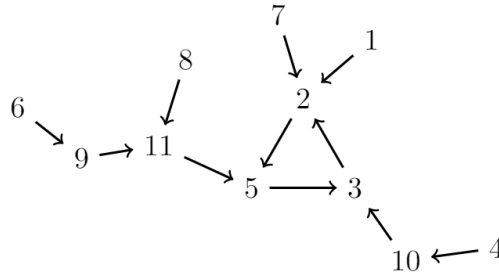


FIGURE 2. A permutation on 2, 3, and 5 with trees coming out of it.

These are in bijection with functions from the set of  $n$  elements to itself, as a function determines such a tree by writing where everything goes, which eventually (after applying the function enough times) determines the cycles and the trees coming out of them. Thus  $T_n \times n \times n$  is in bijection with the set of functions from  $\{1, \dots, n\}$  to itself, which is  $n^n$ . Thus  $|T_n| = n^{n-2}$  (This is known as Cayley's Theorem). Perhaps the reason this proof does something nontrivial is because it used this bijection which was unnatural.