CHERN-WEIL AND GAUSS-BONNET

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Given a manifold, Chern-Weil theory says that we can obtain characteristic classes by applying invariant polynomials on the curvature of a connection. We will see here an explicit proof (without using the Chern-Weil homomorphism) of the Gauss-Bonnet theorem for vector bundles, which is an example of the phenomenon.

Let $E \to M$ be a rank 2p real vector bundle with a metric and a metric connection ∇ , and let Ω_E be its curvature 2-form. Then we can take the Pfaffian $\operatorname{Pf}(\Omega_E)$ multiplied by a normalizing constant $(\frac{-1}{2\pi})^p$ of the curvature to get a d-form, whose cohomology class we should interpret by Chern-Weil theory as a characteristic class of the bundle. Indeed, we can call this class the geometric Euler class $(e_g(E))$, and we can prove that it indeed coincides with the topological Euler class $(e_t(E))$. This can be viewed as a generalization of Gauss-Bonnet:

Theorem 0.1 (Gauss-Bonnet). Given an even dimensional Riemannian manifold M^{2p} , if Ω is the curvature, then $\int_{M} \left(\frac{-1}{2\pi}\right)^{p} \operatorname{Pf}(\Omega) = \chi(M)$.

In the case that the bundle is the tangent bundle, and the metric is a Riemannian metric, this becomes the Gauss-Bonnet theorem. Indeed, the Euler class integrates to the Euler characteristic, and the geometric Euler class is an integral of the Pfaffian of the Riemann curvature tensor (up to a constant).

The first thing to note is that the geometric Euler class is natural. It is easy to check that it commutes with pullbacks, and that $e_g(E_1 \oplus E_2) = e_g(E_1) \wedge e_g(E_2)$ (Note: here the notation is abused since e_g seems to depend on the connection). Then by the splitting principle, it suffices to show that $e_g = e_t$ for oriented plane bundles, for which we can more explicitly calculate.

For a plane bundle $E \xrightarrow{\pi} M$, let the connection be given in local neighborhood U_{α} by the skew-symmetric matrix of 1-forms $(\theta_{\alpha})_{i}^{j} = \omega_{\alpha}$. The curvature $\Omega_{\alpha} = d\omega_{\alpha} - \omega_{\alpha} \wedge \omega_{\alpha}$ is given by the matrix $\begin{pmatrix} (\theta_{\alpha})_{1}^{2} \wedge (\theta_{\alpha})_{1}^{2} & d(\theta_{\alpha})_{1}^{2} \\ -d(\theta_{\alpha})_{1}^{2} & (\theta_{\alpha})_{1}^{2} \wedge (\theta_{\alpha})_{1}^{2} \end{pmatrix}$ so that the Pfaffian is $d(\theta_{\alpha})_{1}^{2}$.

Now suppose we have a partition of unity γ_{α} subordinate to the choice of local coordinate cover U_{α} , and let $g_{\alpha\beta}$ be the transition functions with values in SO(2) that define the vector bundle. Then by identifying SO(2) = $\mathbb{R}/2\pi\mathbb{Z}$, we can think of the $g_{\alpha\beta}$ as the angle the transition function rotates counterclockwise. By one construction (eg. in Bott and Tu's book) e_t is given by $\frac{-1}{2\pi}\sum_{\beta}d\gamma_{\beta}dg_{\alpha\beta}$. If r_{α}, r'_{α}

make up the local frame in U_{α} , since the connection is a metric connection, we have that $dr_{\alpha} = (\theta_{\alpha})_{1}^{2} r'_{\alpha}$ (here we view the connection as on the frame bundle).

On the bundle since $g_{\alpha\beta}$ is the transition function, we have $d\pi^*r_{\alpha} = (\pi^*dr_{\beta} + \pi^*g_{\alpha\beta})\pi^*r'_{\alpha}$. By injectivity of π^* we obtain $dr_{\alpha} = dr_{\beta} + dg_{\alpha\beta}r'_{\alpha}$. Thus we must have $dg_{\alpha\beta} = (\theta_{\alpha})_1^2 - (\theta_{\beta})_1^2$.

Then we have:

$$\frac{-1}{2\pi} \sum_{\beta} d(\gamma_{\beta} dg_{\alpha\beta}) = \frac{-1}{2\pi} \sum_{\beta} d(\gamma_{\beta} ((\theta_{\alpha})_{1}^{2} - (\theta_{\beta})_{1}^{2})) = \frac{-1}{2\pi} d(\theta_{\alpha})_{1}^{2} + \frac{1}{2\pi} d(\sum_{\beta} \gamma_{\beta} (\theta_{\beta})_{1}^{2})$$

The second resulting term defines a global form which is clearly exact, and we get that e_t is cohomologous to $-\frac{1}{2\pi}d(\theta_{\alpha})_1^2$, which is exactly e_g .