

# K-THEORY AND FREDHOLM OPERATORS

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## 1. INTRODUCTION

Given a compact space  $X$ , we can consider the commutative monoid of complex vector bundles on  $X$  under the Whitney sum operation. We can force it to become a group in the universal way, giving a group called  $K(X)$ . This is a functor, so we can ask whether it is representable. If it is, the representing object should be a group up to homotopy. But in this case, it is represented by the space of Fredholm operators  $\mathcal{F}$  on Hilbert space  $H$ . In particular the composition operation is a commutative group operation on  $\mathcal{F}$  up to homotopy.

$K(X)$  for a point is just  $\mathbb{Z}$ , so in particular the components of the space of Fredholm operators should correspond to the integers, and this function from Fredholm operators to  $\mathbb{Z}$  is exactly the index of an operator. Hence in general we will want to construct a natural isomorphism  $\text{index} : [-, \mathcal{F}] \rightarrow K(-)$ .

To do this, first suppose that we have a map  $f : X \rightarrow \mathcal{F}$ , and we fix a point  $x \in X$ . We should note that the kernel of  $f(x) = f_x$  varies continuously in  $X$ , in the sense that if  $V$  is a subspace on which  $f_x$  is injective, then nearby points will also be injective on  $V$ . In particular, as  $X$  is compact, we can find a  $V$  of finite codimension in  $H$  such that  $f_x$  is injective on  $V$  for all  $x \in X$ . Then  $f_x(V)$  is a subbundle of  $X \times H$ , and we can consider  $H/f_X(V)$ , the bundle where the fibre of  $x$  is  $H/f_x(V)$ . If  $V$  is codimension  $k$ , we define  $\text{index}_V(f) = k - [H/f_X(V)]$ . To see this is well defined, we only need to observe that if we choose a codimension  $n$  subspace  $U \subset V$  then  $H/f_X(U) \cong f_X(U)^\perp \cong f_X(V)^\perp \oplus f_X(U^\perp \cap V) \cong H/f_X(V) \oplus \mathbb{C}^n \times X$ . We thus have  $k - [H/f_X(V)] = k + n - [H/f_X(U)]$ , so for any  $V, W$ ,  $\text{index}_V(f) = \text{index}_{V \cap W}(f) = \text{index}_W(f)$ . Moreover index is clearly natural.

Homotopy invariance comes from the commutative diagram:

$$\begin{array}{ccc}
 [X, \mathcal{F}] & \xrightarrow{\text{index}(f_0)} & K(X) \\
 (1_X \times 0)^* \uparrow & & \uparrow (1_X \times 0)^* \\
 [X \times I, \mathcal{F}] & \xrightarrow{\text{index}(f_t)} & K(X \times I) \\
 (1_X \times 1)^* \downarrow & & \downarrow (1_X \times 1)^* \\
 [X, \mathcal{F}] & \xrightarrow{\text{index}(f_1)} & K(X)
 \end{array}$$

The vertical maps are isomorphisms and their composites are the identity on  $[X, \mathcal{F}]$  and  $K(X)$ , so  $\text{index}(f_0) = \text{index}(f_1)$ . To see  $\text{index}$  is a homomorphism, let  $f, g : X \rightarrow \mathcal{F}$ . Then choose  $U, V$  to be finite codimension subspaces on which  $f, g$  are respectively injective and such that  $g_X(V) \subset U$ , so that  $fg$  is injective on  $V$ . Then there is an exact sequence  $0 \rightarrow U/g_X(V) \rightarrow H/f_X(V) \rightarrow H/f_X(U) \rightarrow 0$ , so  $\text{index}(fg) = \text{codim } V - [H/f_X(V)] = \text{codim } V - [U/g_X(V)] - [H/f_X(U)] = \text{codim } U + \text{codim } V - [H/g_X(U)] - [H/f_X(U)] = \text{index}(f) + \text{index}(g)$ .

For surjectivity of the index, since it is a homomorphism we need every vector bundle  $V$  to be in the image, as well as  $n \in \mathbb{Z}$ . The latter can be done by sending  $X$  to a single operator of index  $n$ . To see  $V$  is in the image, find a vector bundle  $W$  such that  $V \oplus W = \mathbb{C}^n \times X$  and let  $\pi_V, \pi_W$  be the projection maps. Now send  $x$  to the operator on  $H \otimes \mathbb{C}^n \cong H$  sending  $e_i \otimes v$  to  $e_{i+1} \otimes \pi_W(v) + e_i \otimes \pi_V(v)$ . The index is  $-W$  which is  $V$  up to a trivial bundle.

Finally let's examine the kernel of the index. If something is in the kernel, there must be a finite codimension subspace  $U \subset H$  with  $H/f_X(U)$  trivial. Let  $e_1, \dots, e_n$  be a basis of  $U^\perp$  and let  $s_1, \dots, s_n$  be trivializing sections of  $f_X(U)^\perp$ . Consider the homotopy that has  $f_{x,t}(e_i) = f_x(e_i)(1-t) + s_i t$ . It homotopes  $f$  to something that is an isomorphism on  $H$ . Thus we have an exact sequence  $[X, \text{GL}(H)] \rightarrow [X, \mathcal{F}] \rightarrow K(X) \rightarrow 0$ . However, by Kuiper's theorem,  $\text{GL}(H)$  is contractible, so the index is a natural isomorphism.