

# PONTRYAGIN CONSTRUCTION

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## 1. INTRODUCTION

The Pontryagin construction is a way of relating framed submanifolds to homotopy classes of maps to a sphere. Here a **framed submanifold** is a submanifold with a trivialization of the normal frame bundle. Now we don't want to consider all framed submanifolds, but rather mod out by an equivalence relation called **cobordism**. We say that  $N, N' \subset M$  are cobordant manifolds if  $N \times [0, \epsilon] \cup N' \times [1 - \epsilon, 1]$  can be extended in the interval  $M \times [\epsilon, 1 - \epsilon]$  to a submanifold of  $M \times [0, 1]$  with boundary  $N \cup N'$ . In particular we would like to consider framed submanifolds up to framed cobordism, where we require the extension to be framed. I will use  $\simeq$  to denote homotopic maps and  $\sim$  to denote framed cobordant submanifolds.

Throughout, we will assume that  $M$  is compact,  $f : M \rightarrow S^p$  ( $S^p$  oriented) a smooth map,  $y$  a regular value, we naturally get a framed submanifold by looking at  $N_f = f^{-1}(y)$ , and  $f|_{N_f}$  induces a bundle map on  $N_f$ 's normal bundle and  $T_y$ , trivializing it using a positively oriented basis of  $T_y S^p$ .

We would like to prove:

**Theorem 1.1.**  *$N_f$  is well defined up to framed cobordism class, and only depends on the homotopy class of  $f$ . Moreover,  $f \mapsto N_f$  gives a bijection between framed compact cobordism classes of codimension  $p$  and  $[M, S^p]$ .*

## 2. WELL DEFINED

We will begin by showing the first statement. First note that the cobordism class doesn't depend on the basis we chose for  $f$ , only the orientation, since  $\mathrm{GL}_n(\mathbb{R})^+$  is connected (This can be proven by using row/column operations carefully or using Graham-Schmidt to reduce to showing  $\mathrm{SO}(n)$  is connected, which is done by using induction and the fibration  $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \rightarrow S^{n-1}$ ). Then given two choices of frames on  $N_f$ , they are pullbacks of two different elements of the tangent frame bundle of  $y$ , so by choosing a smooth path on  $\mathrm{GL}_n(\mathbb{R})^+$  that is constant on  $[0, \epsilon] \cup [1 - \epsilon, 1]$ , we have framed  $y \times [0, 1]$  in  $S^p \times [0, 1]$ , and by considering the natural map induced by  $f$  from  $M \times [0, 1] \rightarrow S^p \times [0, 1]$ , this framing of  $y \times [0, 1]$  induces a cobordism

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between the two frames of  $N_f$ . Thus we will ignore the particular frame chosen at  $y$  from now on.

We would like to show the cobordism class is well defined up to homotopy. Given a homotopy, we would like to take the preimage of  $y$  on the homotopy to get a cobordism. Unfortunately  $y$  is not necessarily a regular value of the homotopy. To fix this, the following lemma:

**Lemma 2.1.** *The cobordism class of  $f^{-1}(z)$  is constant for  $z$  in a neighborhood of  $y$ .*

*Proof.* The set of critical points is compact as  $M$  is, hence there is a convex neighborhood of  $y$  consisting of regular values. Now choosing a family  $r_t$  of smooth rotations of the sphere that takes  $y$  to  $z$ , and is constant on  $[0, \epsilon] \cup [1 - \epsilon, 1]$ . Then consider the map  $r \circ f : M \times [0, 1] \rightarrow S^p \times [0, 1] \rightarrow S^p$ .  $y$  is regular for  $r \circ f$ , so we get a cobordism between  $f^{-1}(y)$  and  $f^{-1}(z)$ .  $\square$

**Theorem 2.2.** *The cobordism class is well defined, and is only dependant on homotopy class.*

*Proof.* First note that if  $f \simeq g$ , then we can assume the homotopy is constant on  $[0, \epsilon] \cup [1 - \epsilon, 1]$ , and choose  $z$  a regular value of the homotopy satisfying the conditions of the previous lemma for  $f$  and  $g$  so that  $f^{-1}(y) \sim f^{-1}(z) \sim g^{-1}(z) \sim g^{-1}(y)$ . Now if  $z$  is another regular value, and  $r$  a rotation sending  $z$  to  $y$ ,  $r \circ f \simeq f$  so  $f^{-1}(y) \sim (r \circ f)^{-1}(y) = f^{-1}(z)$ .  $\square$

### 3. SURJECTIVITY

We would now like to show that for any framed submanifold  $N$ , we can produce a map  $f$  with  $N_f \sim N$ .

**Lemma 3.1** (Tubular Neighborhood Theorem). *Let  $P \subset M$  be submanifold of codimension  $p$ , with  $P$  compact. Then there is a neighborhood of  $P$  diffeomorphic to the normal bundle of  $P$ , with  $P$  as the 0-section.*

*Proof.* By exponentiating the normal bundle, we get a local diffeomorphism  $P \times B_\epsilon \rightarrow M$ , and since  $B_\epsilon$  is diffeomorphic to  $\mathbb{R}^p$ , it suffices to show that for small  $\epsilon$ , this is injective. However, if  $(p_i, x_i), (q_i, y_i)$  are a sequence of points for which it is not injective with the magnitude of the  $x_i, y_i$  going to 0, by compactness of  $P \times \overline{B}_{\frac{\epsilon}{2}}$ , we can extract a convergent subsequence, which contradicts local injectivity.  $\square$

This Lemma holds for non-compact submanifolds but the proof is a bit more annoying.

**Theorem 3.2.** *The map  $f \rightarrow N_f$  is surjective.*

*Proof.* We consider a tubular neighborhood of a framed submanifold  $N$ , giving a map  $f : \mathbb{R}^p \times N \rightarrow \mathbb{R}^p$ . Now consider  $S^p = y_0 \cup \mathbb{R}^p$ , and smoothly extend  $f$  to  $M$  by setting all other values to  $y_0$ . Then  $f^{-1}(0) = N$ .  $\square$

#### 4. INJECTIVITY

We would now like to show that if we have a cobordism  $f^{-1} \sim g^{-1}$  via some framed submanifold  $P \subset M \times [0, 1]$ ,  $f \simeq g$ . To do this, given the cobordism, we would like to use the proof of surjectivity on the cobordism to yield a homotopy. However, this still leaves us to prove:

**Lemma 4.1.** *If  $f^{-1}(y) = g^{-1}(y) = N$ ,  $f \simeq g$ .*

*Proof.* If  $f, g$  agree on a neighborhood of  $N$ , then removing the neighborhood, we get a map to  $\mathbb{R}^p$  instead of  $S^p$ , which we can linearly homotopy without spoiling the overall smoothness. So it suffices to deform  $f$  to agree with  $g$  in a neighborhood of  $N$ . To do this, choose a tubular neighborhood  $N \times \mathbb{R}^p$  that misses the antipode  $y_0$  of  $y$ . Then we have maps  $F, G : N \times \mathbb{R}^p \rightarrow \mathbb{R}^p$  with  $DF|_{N \times 0} = DG|_{N \times 0}$ , and we can assume that  $DF|_{N \times 0}$  is the identity on each  $n \times \mathbb{R}^p$ . We would like to linearly deform  $f$  to match  $g$ , but we would like to avoid adding new zeroes. To do this, note by compactness of  $N$ , there is an  $\delta$  ball around 0 such that when  $F, G$  are restricted to it,  $\|DF - I\|, \|DG - I\| < \epsilon$ . Then  $\|F(n, x) - x\| \leq \|cx^2\|$  for small  $\|x\|$  by Taylor's theorem, so by multiplying by  $\|x\|$  on either side and using Cauchy Schwarz, we get  $|F(n, x) \cdot x| \geq \|x\|^2 - c\|x\|^3$  which is positive when  $\|x\|$  and  $c$  are small. Then doing the same with  $G$ , we find that  $F$  and  $G$  lie in the same half plane for small  $\|x\|$ , so that we can linearly deform  $F$  to match  $G$  locally without adding new 0s.  $\square$

**Theorem 4.2.** *If  $N_f \sim N_g$ ,  $f \simeq g$ .*

*Proof.* As in the proof of surjectivity, choose a tubular neighborhood of a cobordism and construct a homotopy  $H$  such that  $H^{-1}(y)$  is the cobordism. Now by the previous lemma,  $f \simeq H_0 \simeq H_1 \simeq g$ .  $\square$

#### 5. APPLICATIONS

The Pontryagin construction can be viewed as a generalization of degree theory, and we can see that the most trivial case of it does coincide with degree theory.

**Theorem 5.1** (Theorem of Hopf). *If  $M^n$  is compact, orientable, and connected, then  $[M^n, S^n] \cong \mathbb{Z}$ , where the isomorphism is given by degree. If  $M^n$  is non-orientable, then  $[M^n, S^n] \cong \mathbb{Z}/2\mathbb{Z}$  with the isomorphism given by degree mod 2. In particular,  $\pi_n(S^n) \cong \mathbb{Z}$ .*

*Proof.* The codimension 0 compact framed submanifolds are finite collections of points with a  $\pm 1$  orientation. Now it is clear that if  $M$  is orientable, then the cobordism class is only dependant on degree, ie. the sum of these orientations. If  $M$  is not orientable, then points with positive or negative orientation are the same up to cobordism, so degree mod 2 determines the cobordism class.  $\square$