SIMPLICIAL

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Contents

1. Introduction

Simplicial things come up in mathematics a lot. There are at least three related places of origin. The first is homotopy theory. Simplicial sets give a way of modeling the homotopy theory of topological spaces, and are often a convenient model for higher categories. The second is in homological algebra. There is the Dold Kan correspondence which identifies homotopy theories of chain complexes and simplicial objects, which suggests that simplicial objects can be useful for applying homotopical methods in nonabelian settings. The third place it arises is when studying monoids. Namely, the simplex category classifies monoids in a tensor category, meaning that tensor functors from Δ coincide with monoid objects. This can be used in the construction of canonical resolutions from (co)monads, and in the formalism of monoidal ∞-categories.

The following are sources for this document:

- Goerss, Jardine - *Simplicial Homotopy Theory*
- Riehl - *Categorical Homotopy Theory*
- Lurie - *Higher Topos Theory*
- Lawson - *Localization of Enriched Categories and Cubical Sets*
- Jardine - *Categorical Homotopy Theory*
- Dwyer, Kan - *Simplicial Localization of Categories*

2. Technical Things

Definition 2.1. *A category C is presentable if it satisfies:*

(1) C is cocomplete
(2) C is generated under colimits by a set of objects
(3) Objects in C are small (equivalently the generators are small)
(4) C is locally small

Definition 2.2. *A category C is κ-accessible if it satisfies:*

(1) C has κ-filtered colimits
(2) C is generated under κ-filtered colimits by a set of objects
(3) Objects in C are κ-small (equivalently the generators are small)
(4) C is locally small

For regular cardinals $\tau \gg \kappa$ if $\tau_0 < \tau, \kappa_0 < \kappa \implies \tau_0^{\kappa_0} < \tau$

Lemma 2.3. *A κ-accessible category has a small set of κ-compact objects.*

Proof. Let S be the generating set of κ-compact objects, and let X be a κ-compact object. Presenting X as a κ-filtered colimit of objects in S, we learn that 1_X must factor through one of the objects of S. Thus, X is a retract of an object of S. but any object of S has a set of subobjects since the S-Yoneda embedding is faithful, so there is a set of such X. $\square$

Lemma 2.4. *Δ^{op} is sifted.*

Proof. Δ can be thought of as the full subcategory of categories generated by digraphs that are finite paths. Thus we need to show the category of objects over $\Delta^n \times \Delta^m$ is connected. But given an object in this over category, we can adjoin a terminal object, which is sent to the terminal object of $\Delta^n \times \Delta^m$, But then any two such maps over $\Delta^n \times \Delta^m$ are connected via the inclusion of the terminal object. $\square$

There is an involutions of Δ given by reversing the arrows of the partial orders. The action of this on simplicial objects is called the **opposite**.

Let Δ_n be the subcategory of Δ of ordinals of size $\leq n$, and call a functor $\Delta_n^{op} \to C$ a n-truncated simplicial object in C. $(\mathrm{Set}^{\Delta^{\mathrm{op}}})_n$ will denote the category of n-truncated simplicial sets. Let C be complete and cocomplete. The left and right Kan extensions $C^{\Delta_n^{op}} \to C^{\Delta^{op}}$ of the inclusion $i_n : \Delta_n \to \Delta^{op}$ are given by $(i_n)_! X = \mathrm{colim}_{n \geq k \to m} X_k, (i_n)_* X = \lim_{m \to k \leq n} X_k$. $(i_n)^*(i_n)_* = 1 = (i_n)^*(i_n)_!$.

The other unit/counits are called the **skeleta/coskeleta**. $(i_n)_!(i_n)^* \to 1$ is the n-skeleton sk_n, and $1 \to (i_n)_*(i_n)$ is the n-coskeleton cosk_n.

Let S, J be classes of arrows.

Definition 2.5. *J satisfies the 2 out of 3 property with respect to S if in Δ^2-shaped diagram where the edges are in S, if two of the arrows are in J, the third is.*

Definition 2.6. *J satisfies the 3 out of 4 property with respect to S if in a $\Delta^1 \times \Delta^1$-shaped diagram where the boundary edges are in S, if three of the arrows on the boundary are in J, the fourth is.*

Definition 2.7. *J satisfies the 2 out of 6 property with respect to S if in a Δ^3-shaped diagram where the boundary edges are in S, if the arrows $0 \to 2, 1 \to 3$ are in J, the rest are.*

If S is all the arrows, we supress it from the terminology. In this case, 2 out of 6 implies 2 out of 3 which implies 3 out of 4. 2 out of 6 is satisfied by isomorphisms.

Definition 2.8. *A **weakly saturated class** of morphisms is one containing isomorphism that is closed under transfinite composition, coproducts retracts, and pushouts.*

Definition 2.9. *The class of **(left, right, inner) anodyne extensions** is the weakly saturated class of morphisms generated from the inclusions of (left, right, inner) horns $\Lambda_i^n \to \Delta^n$.*

The adjective n-**trivial** will mean that we add $\partial\Delta^m \to \Delta^m, m < n$ to the generators of the saturated set. When $n = \infty$, we can just say trivial, and when $n = -2$, this doesn't mean anything.

Definition 2.10. *A map f has the **right lifing property** with respect to a morphism g denoted $f \oslash g$, if we can always find lifts in the diagrams*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ V \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

If L is a class of morphisms in C, define $L^{\oslash}$ to be the class of morphisms $\{f|L \oslash f\}$. Dually define ${}^{\oslash}J$. ${}^{\oslash}J$ is always weakly saturated.

Lemma 2.11. ${}^{\oslash}(J^{\oslash})^{\oslash} = J^{\oslash}$

Proof. By definition ${}^{\oslash}(J^{\oslash}) \oslash J^{\oslash}$, showing $J^{\oslash} \subset {}^{\oslash}(J^{\oslash})^{\oslash}$. But $J \oslash {}^{\oslash}(J^{\oslash})^{\oslash}$ since $J \subset {}^{\oslash}(J^{\oslash})$ giving the other inclusion (this works for any binary relation). □

${}^{\oslash}(J^{\oslash})$ is like a saturation of J, and often agrees with the weak saturation of J.

Lemma 2.12. *$f \oslash f$ iff f is an isomorphism.*

Proof. Find a lift for the square $1 \times f$ to get an inverse. □

Lemma 2.13. *Let $F \dashv G$ be an adjunction. Then $FX \oslash Y$ iff $X \oslash GY$.*

Proof. This is immediate from the definition. □

Definition 2.14. *A map is a* ***(left, right, inner) n-connected fibration*** *if it satisfies the right lifting property with respect to n-trivial (left, right, inner) fibrations.*

A fibrant simplicial set is sometimes called a **Kan complex**, and an inner fibrant simplicial set is sometimes called an **∞-category** or **$(\infty, 1)$-categor**y or **quasicategory**.

As $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ is a presheaf category, we can get natural adjunctions by left Kan extension whenever we have a cosimplicial object in some cocomplete category C. For example, the cosimiplicial category where Δ^n gets sent to the category generated by n morphisms in a row (also denoted Δ^n) gives the nerve functor with adjoint the homotopy category. The standard simplices in Top or CGHaus give geometric realization $|\cdot|$ and the singular set S. Another example is a cosimplicial object in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ given by taking an ordered set to the nerve its poset of subset. This extends to a functor $\mathrm{sd} : \mathrm{Set}^{\Delta^{\mathrm{op}}} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$, which is essentially barycentric subdivision, and its right adjoint is denoted Ex. The study of this functor is due to Kan I believe. There is a natural map $h : \mathrm{sd}\, X \to X$ called the last vertex map that sends a vertex in $\mathrm{sd}\, \Delta^n$ to the last vertex of the subset it corresponds to. The adjoint of this is a map $X \to \mathrm{Ex}\, X$.

Proposition 2.15. *Geometric realization of simplicial sets preserves finite limits in* CGHaus*, the category of compactly generated spaces.*

Proof. First let's think about products. We can reduce to the case of simplices as follows. Let K, L be simplicial sets. Since products distribute over colimits, we have

$$|K \times L| = |\int^m K_m \cdot \Delta^m \times \int^n L_n \cdot \Delta^n| = |\int^{m,n} K_m \cdot L_n \cdot \Delta^m \times \Delta^n|$$

$$= \int^{m,n} K_m \cdot L_n \cdot |\Delta^m \times \Delta^n| = \int^{m,n} K_m \cdot L_n \cdot |\Delta^m| \times |\Delta^n| = \int^m K_m \cdot |\Delta^m| \int^n L_n \cdot |\Delta^n|$$

$$= |K| \times |L|$$

Now the natural map $|\Delta^n \times \Delta^m| \to |\Delta^n| \times |\Delta^m|$ is a continuous map of compact Hausdorff spaces, so it a homeomorphism iff it is a bijection. Suppose we have an element of $|\Delta^n| \times |\Delta^m|$, given by a sequence of numbers c_i, d_j, which are the components for the standard embedding of a simplex. Define $c'_i = \sum_{k \leq i} c_i$ and similarly for d'_j, and order the set consisting of c'_i, d'_j. Now define a simplex on vertices by going through the ordering and everytime you run into a number that is one of the c'_i, increment the index of the Δ^n vertex, and similarly for the d'_i and the Δ^m vertex. It is easy to see that this is the unique nondegenerate simplex hitting c_i, d_j, and it hits it in a unique point by taking the linear combination that is the incremental differences of the ordered set of numbers.

Equalizers are easier. First observe that an inclusion of simplicial sets induces an inclusion of a closed set upon realizing. This implies that the comparison map is an inclusion. For surjectivity one has to observe that for any point equalized in the realization, the nondegenerate simplex whose interior it is in is equalized, so it is in the image. □

Lemma 2.16. *There is a homeomorphism $|\operatorname{sd} \Delta^n| \to |\Delta^n|$, which realizes $|\operatorname{sd} \Delta^n|$ as the barycentric subdivision of $|\Delta^n|$. Moreover, this is naturally homotopic to $|h|$ where h is the last vertex map.*

Proof. A zero simplex of $\operatorname{sd} \Delta^n$ is given by a subset of the vertices v_i of Δ^n, which we can send to the barycenter of that set in $|\Delta^n|$. We can then extend to $\operatorname{sd} \Delta^n$ by linear interpolation. Rewrite $\sum_i \alpha_i v_i = \sum_j t_j X_j$, where the t_j are in increasing order and X_j is a sum of v_is, so we have just regrouped the v_is with the same coefficient into one X_j. Define $N_j = \sum_{k \geq j} (n_j + 1)$ where n_j is the number of v_i in X_j.

Then $\sum_j t_j X_j = \sum_j (t_j - t_{j-1}) N_j (\frac{1}{N_j} \sum_{k \geq j} X_k)$. $(\frac{1}{N_j} \sum_{k \geq j} X_k)$ is a barycenter, and $\sum_j (t_j - t_{j-1}) N_j = 1$ so we have written (uniquely) any element of Δ^n as an element of $|\operatorname{sd} X|$. □

2.1. Generators of anodyne morphisms.

Lemma 2.17 (Generators of n-trivial inclusions)**.** *The n-trivial maps are those inclusions $X \to Y$, where Y/X has nondegenerate cells in dimensions $\leq n$. .In particular the saturation of the inclusions $\partial \Delta^n \subset \Delta^n$ is all inclusions of simplicial sets.*

Proof. It is easy to see the sets described are saturated. Given any inclusion $X \subset Y$, let $Y_n(X)$ denote the preimage of the n-skelaton of Y/X as a pointed simplicial set. Then since $Y_{-1}(X) = X, Y_n(X) \subset Y_{n+1}(X), \cup_i Y_i(X) = Y$, it suffices to show that the inclusion $Y_{n-1}(X) \to Y_n(X)$ is generated by $\partial \Delta^n \subset \Delta^n$. But there is a pushout diagram

$$\begin{tikzcd} \cup_\alpha \partial \Delta^n_\alpha \arrow[r] \arrow[d] & Y_{n-1}(X) \arrow[d] \\ \cup_\alpha \Delta^n_\alpha \arrow[r] & Y_n(X) \end{tikzcd}$$

where alpha runs over all nondegenerate n-cells in Y/X. □

Note: the inclusions $\partial \Delta^n \times \Delta^m \cup \Delta^n \times \partial \Delta^m \subset \Delta^n \times \Delta^m$ should also generate inclusions.

Proposition 2.18 (Generators of left anodyne extensions)**.** *The three classes of maps below generate the left anodyne extensions:*

(1) $\Lambda^n_i \subset \Delta^n$, $0 \leq i < n$

(2) $\Delta^1 \times \partial\Delta^n \cup \Lambda^1_1 \times \Delta^n \subset \Delta^1 \times \Delta^n$
(3) $\Delta^1 \times X \cup \Lambda^1_1 \times Y \subset \Delta^1 \times Y$ *for all* $X \subset Y$

Proof. (2) $\iff$ (3): This follows from Lemma 2.17 after observing that the class of inclusions $X \to Y$ for which (3) is true is saturated.

(2) $\implies$ (1): Let X_n be $\Delta^1 \times \partial\Delta^n \cup \Lambda^1_1 \times \Delta^n$, and consider the map $X_n \to \Lambda^{n+1}_i$ induced by the endomorphism of X_n determined on vertices by sending $(1, j)$ to $(0, j)$ for $j \neq i$. The pushout along this map of the inclusion $X \to \Delta^1 \times \Delta^n$ contains Δ^{n+1} as a retract relative to Λ^{n+1}_i.

(1) $\implies$ (2): First notice that the $n-1$ skeleta agree, so there is only need to add $n+1$-simplices. Let σ_i denote the $n+1$ simplex of $\Delta^1 \times \Delta^n$ where the Δ^1 component starts being 1 on the i^{th} vertex. Then in decreasing order of i, extend along Λ^n_i to σ_i to build up $\Delta^1 \times \Delta^n$ out of $\Delta^1 \times \partial\Delta^n \cup \Lambda^1_1 \times \Delta^n$ □

Corollary 2.19 (Generators of anodyne extensions)**.** *The three classes of maps below generate the anodyne extensions:*

(1) $\Lambda^n_i \subset \Delta^n$
(2) $\Delta^1 \times \partial\Delta^n \cup \Lambda^1_i \times \Delta^n \subset \Delta^1 \times \Delta^n$
(3) $\Delta^1 \times X \cup \Lambda^1_i \times Y \subset \Delta^1 \times Y$ *for all* $X \subset Y$

Proposition 2.20 (Generators of inner anodyne extensions)**.** *The classes of maps below generate the inner anodyne extensions:*

(1) $\Lambda^n_i \subset \Delta^n$, $0 < i < n$
(2) $\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \subset \Delta^2 \times \Delta^n$
(3) $\Delta^2 \times X \cup \Lambda^2_1 \times Y \subset \Delta^2 \times Y$ *for all* $X \subset Y$

Proof. (1) $\implies$ (2): First observe that the $n-1$-skeleton agrees with that of $\Delta^2 \times \Delta^n$. Let $\sigma'_i, i > 0$ be the nondegenerate $n+1$-simplices projecting onto $s_i\Delta^n$, that switches from 0 to 2 in Δ^2 at the i^{th} vertex. Let $\sigma_{i,0}$ be the nondegenerate $n+2$-simplex projecting to $s_i s_i \Delta^n$, and let $\sigma_{i,1}, i > 0$ be the nondegenerate $n+2$-simplex projecting to $s_i s_{i-1}\Delta^n$. Then in descending order of i, we can attach σ'_i, then $\sigma_{i,0}$, then $\sigma_{i,1}$ until we have build up $\Delta^2 \times \Delta^n$.

(2) $\iff$ (3): This follows from Lemma 2.17 after observing that the class of inclusions $X \to Y$ for which (3) is true is saturated.

(2) $\implies$ (1): Fix $0 < i < n$, and consider the endomorphism of $\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \subset \Delta^2 \times \Delta^n$, with image consisting of the vertices $(0, j), j < i$, $(1, i-1)$, and $(2, j)$, for $j \geq i$, where all the other vertices are sent to $(1, i-1)$. The image is Λ^n_i, and the pushout of the inclusion into $\Delta^2 \times \Delta^n$ along this endomorphism retracts onto the inclusion $\Lambda^n_i \to \Delta^n$. □

Corollary 2.21. *A simplicial set is an* ∞*-category iff the map* $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$ *is a trivial fibration.*

Proof. This follows from Lemma 2.20. □

Corollary 2.22. *If* $K \subset L$ *is (left, right, inner) anodyne and* $Y \subset X$*, then* $K \times X \cup L \times Y \to L \times X$ *is (left, right, inner) anodyne.*

Proof. Let us do the proof for left anodyne extensions, the proofs in other cases are similar. Since the inclusions $K \subset L$ for which the lemma is true are saturated, we reduce to the case when it is $\Delta^1 \times \partial\Delta^n \cup \Lambda^1_1 \times \Delta^n \subset \Delta^1 \times \Delta^n$. But now there is a commutative square:

$$\begin{array}{ccc} (\Delta^1 \times \partial\Delta^n \cup \Lambda^1_1 \times \Delta^n) \times X \cup \Delta^1 \times \Delta^n \times Y & \longrightarrow & \Delta^1 \times \Delta^n \times X \\ \| & & \| \\ \Delta^1 \times (\Delta^n \times Y \cup \partial\Delta^n \times X) \cup \Lambda^1_1 \times (\Delta^n \times X) & \longrightarrow & \Delta^1 \times (\Delta^n \times X) \end{array}$$

The lower map by (3) of Corollary 2.19 is anodyne. $\square$

2.2. **Enriched Categories.** A **closed monoidal category** is one where there are internal homs that are right adjoint to tensoring. This makes the category self-enriched. A V-enriched category with a (left) lax monoidal functor to C has an underlying C-enriched category, and in this case of self enrichment, the underlying category is the original, and one can use an enriched Yoneda lemma to upgrade the internal hom adjunction to an enriched adjunction. In a V enriched category C we can use an underline such as $\underline{\mathrm{Hom}}(a,b)$ or $\underline{C}(a,b)$ to denote V-homs. We say **cartesian closed** if the monoidal structure comes from products. The presheaf category of a category C is always a cartesian closed symmetric monoidal category by the formula $\mathrm{Map}(X,Y)_Z = \mathrm{Hom}(X \times Z, Y)$, where $Z \in C$. In particular this works in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$. Moreover, there is a natural 2-equivalence between $\mathrm{Set}^{C^{op}}$-enriched categories and functors $C^{op} \to \mathrm{Cat}$. The internal hom is also denoted Y^X.

For every monoidal cateogory V, the functor $X \mapsto \mathrm{Hom}(1,X)$ is right lax monoidal, so there is an underlying category. We will use $\mathrm{Hom}(a,b)$ to talk about the morphisms in the underlying category.

Given a complete and cocomplete cartesial closed symmetric monoidal category C we can form a symmetric monoidal category C_+ of based objects with a map from the terminal object. The monoidal structure is the smash product, and the functor of adding a disjoint basepoint is strongly symmetric monoidal.

Now suppose C, D are V-enriched, where V is concrete, and suppose that we have an adjunction $F : C \leftrightharpoons D : G$ on the underlying categories. We would like a situation where this can be upgraded to an enriched adjunction.

Definition 2.23. *Let $\underline{C}$ be V-enriched. $\underline{C}$ is* ***tensored*** *if there is an isomorphism natural in all 3 variables: $\underline{C}(a \otimes b, c) \cong \underline{V}(a, \underline{C}(b,c))$.*

In other words, there is an enriched left adjoint of $\underline{C}(b, -)$ for every b.

Definition 2.24. *Let $\underline{C}$ be V-enriched. $\underline{C}$ is* ***cotensored*** *if there is an isomorphism natural in all 3 variables: $\underline{C}(b, \hom(a,c)) \cong \underline{V}(a, \underline{C}(b,c))$.*

In other words, there is an enriched right adjoint of $\underline{C}(-, c)$ for every b.

Sometimes people also say copower and power instead of tensor and cotensor.

Definition 2.25. *A* ***two-variable adjunction*** *is a triple of functors $F : M \times N \to P$, $G : M^{op} \times P \to N$, $H : N^{op} \times P \to M$ so that there are natural isomorphisms $\mathrm{Hom}(F(a,b),c) = \mathrm{Hom}(b, G(a,c)) = \mathrm{Hom}(a, H(b,c))$.*

We can think of this as a family of adjunctions parameterized by a (or symmetrically in b, c). If $\underline{C}$ is tensored and cotensored, there is a two-variable enriched adjunction $\underline{C}(b, \hom(a,c)) \cong \underline{C}(a \otimes b, c)$. Moreover $\otimes$ is unital and associative, essentially giving an action of $\underline{V}$ on $\underline{C}$.

Construction 2.25.1. [Liebniz Construction] Suppose that $F : M \times N \to P$ is a functor, and P has pushouts. Then the **pushout-product** denoted $\hat{F}$ is a functor on the arrow categories defined on arrows i, j by the diagram below.

$$\begin{tikzcd}
F(a,b) \arrow[r, "F(i,1)"'] \arrow[d, "F(j,1)"] & F(a',b) \arrow[d] \arrow[ddr, "F(j,1)"] & \\
F(a,b') \arrow[r] \arrow[drr, "F(i,1)"'] & X_{ij} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\hat{F}(i,j)"] & \\
& & F(a',b'
\end{tikzcd}$$

The pushout-product of the initial map to a with a map $b \to c$ is just the map $F(a,b) \to F(a,c)$. Suppose that there is a two-variable adjunction F, G, H between M, N, P, and suppose that M has pushouts and N, P have pullbacks. Then Construction 2.25.1 applied to F, G^{op}, H^{op} give an induced 2-variable adjunction $\hat{F}, \hat{G}, \hat{H}$.

Lemma 2.26. *If F, G, H are a two-variable adjunction, then $\hat{F}(f,g) \oslash h$ iff $g \oslash \hat{G}(f,h)$ iff $f \oslash H(g,h)$.*

Proof. This follows from construction and Lemma 2.13. $\square$

Lemma 2.27. *Suppose that C, D are tensored and cotensored over V, and there is an adjunction $F : C \leftrightarrows D : G$ of functors on the underlying categories. Then the following data determine each other:*

(1) An enrichment of the functors and the adjunction
(2) An enrichment of F and natural isomorphisms $F(v \otimes m) \cong v \otimes F(m)$
(3) An enrichment of G and natural isomorphisms $G(m^v) \cong G(m)^v$.

Moreover, in this case, G preserves the cotensoring and F preserves the tensoring.

Proof. It suffices to show (1) $\iff$ (2) as (3) is dual to (2). The point is to use the Yoneda lemma. For clarity, $\underline{F}$ will denote enriched functors. If the adjunction is enriched, then $\underline{D}(v \otimes \underline{F}m, n) = \underline{V}(v, \underline{D}(Fm, n)) = \underline{V}(v, \underline{C}(m, \underline{G}(n))) = \underline{C}(v \otimes m, \underline{G}(n)) = \underline{D}(\underline{F}(v \otimes m), n)$. Conversely, if (2) is satisfied, and U denoted the forgetful functor on $\underline{V}$, then we can produce the counit via $V(c, \underline{C}(Ga, Ga)) = C(c \otimes Ga, Ga) = D(F(c \otimes Ga), a) = D(c \otimes FGa, a) = V(c, \underline{D}(FGa, a))$ and the unit via $V(a, \underline{D}(Fb, Fb)) = D(a \otimes Fb, Fb) = D(F(a \otimes b), Fb) = D(a \otimes b, GFb) = V(a, \underline{D}(b, GFb))$. $\square$

We can transport enrichments and the property of being cotensored or tensored over V to V' provided we have an adjunction $F : V \leftrightarrows V' : G$ where the left adjoint is strongly monoidal. Moreover the V' enrichments on V, V' are compatible with the adjunction in this case.

Lemma 2.28. $C^{D^{op}]}$ *is enriched over* $\mathrm{Set}^{D^{op}}$ *and is tensored if C is cocomplete and cotensored if C is additionally complete.*

Proof. If $A \in \mathrm{Set}^{D^{op}}, B \in C^{D^{op}}, d \in D$, define $(A \otimes B)_d = A_d \cdot B_d$, where $\cdot$ is the copower, and observe that it is the tensoring, if the enrichment is given by $\underline{\mathrm{Hom}}(B, C)_d = \mathrm{Hom}(d \otimes B, C)$. If C is complete and $A = \mathrm{colim}_J\, d_j$, then define B^A by $\lim_J B^{d_j}$, where B^{d_j} is given by the power $B_d^{d_j} = \prod_{\mathrm{Hom}(d, d_j)} B$. This is a right adjoint of $A \otimes (-)$. $\square$

2.3. **Basics of Simplicial Sets.**

Corollary 2.29. *Let $K \subset L$ be an inclusion, and $X \to Y$ a (left, right, inner) fibration. Then the natural map* $\mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$ *is a (left, right, inner) fibration.*

Proof. By definition $A \to B$ has the left lifting property with respect to the map above iff $A \times L \cup B \times K \subset B \times L$ has the left lifting property with respect to $X \to Y$. Thus this follows from Corollary 2.22. □

Corollary 2.30. *Let $K \subset L$ be a (left, right, inner) anodyne extension, and $X \to Y$ a (left, right, inner) fibration. Then the natural map* $\mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$ *is a trivial fibration.*

Proof. This has the same proof as Corollary 2.29. □

Lemma 2.31. *For a (left, right) fibrant simplicial set, simplicial homotopy of verticies is an equivalence relations.*

Proof. If g is a homotopy $x \to y$, then $(g, 1_y, \cdot) = \Lambda^2_0$ can be filled into Δ^2, giving a homotopy $y \to x$. $s_0 x$ gives reflexivity, and if $f : x \to y$ and $g : y \to z$ are homotopies, then by extending $(g, \cdot, f)$ to Δ^2, we get a homotopy $x \to z$. □

Corollary 2.32. *Let $X \to Y$ be a (left, right) fibration, and let $K \to L$ be a cofibration. Then homotopies of maps from L to X covering Y relative to K is an equivalence relation.*

Proof. This follows from Lemma 2.31 and Corollary 2.29. □

Define $\pi_0(X)$ to be the connected components of X.
Simplicial sets generalize categories.

Lemma 2.33. *The nerve is fully faithful on 1-categories. It's essential image is*

(1) 2-coskelatal ∞-categories
(2) Simplicial sets with unique lifts from inner horns
(3) Simplicial sets satisfying the Segal condition $X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ from the inclusion of the spine.

Proof. First we will show that the first two descriptions of the essential image are the same. Suppowe we have a 2-coskeletal ∞-category. Then we immediately get that there is a unique lift for inner horns when $n > 3$, since the inclusions of those horns are identities on the 2-skelaton. If we have two composites of $g \circ f$, σ, σ' lifting both $\Lambda^3_1 = (s_0 g, \cdot, \sigma, s_1 f)$ and $\partial\Delta^3 = (s_0 g, \sigma, \sigma', s_1 f)$ to Δ^3, we see by uniqueness of the lift on Λ^3_1 that $\sigma = \sigma'$.

Conversely given a simplicial set with unique lifts on inner horns, it is certainly an ∞-category, so it suffices to show it is 2-coskelatal. To do this, it suffices to show that for $n > 2$, we can always extend maps along $\partial\Delta^n \subset \Delta^n$. Uniqueness will follow from the fact that it extends an inner horn. This can be done inductivly on n. Since $n > 2$, we can restrict to two different inner horns and extend to Δ^n. Then the uniqueness of lifts for smaller n will show that the two extensions agree on the boundary, and are in particular extensions on $\partial\Delta^n$.

Now given such a simplicial set, rebuild the category in the obvious way. Namely, the verticies are objects, the 1-simplices arrows (with the identity as the degenerate ones). The unique lifting on inner 2 horns gives a unique composition, the 3-skelaton gives associativity, and the 2-coskelativity says that maps between such simplicial sets are determined by preserving 1-categorical structure.

Being a category implies the Segal condition, which in turn implies unique lifting of inner horns. □

Lemma 2.34. *A category is a Kan complex iff it is a groupoid.*

Proof. n-horns for $n > 2$ can always be extended for a category, and 1-horns can be extended for any simplicial set. Moreover, we see easily see that extending Λ^2_2 and Λ^2_0 is equivalent to every map having (left and right) inverses. Of course, if every map has a left inverse, it is already a groupoid by associativity. □

There is an explicit description of the homotopy 1-category of an ∞-category. Namely, we call the 0-simplices objects, 1-simplices arrows. Given simplicial sets X, Y, the join $X \star Y$ is the simplicial set where its n-simplices start in X and end in Y.

Let x, y be objects in an ∞-category. Then the **left mapping space** from $\mathrm{LMap}(x, y)$ is the simplicial set whose n-simplices are the maps $\Delta^n \star \Delta^0$ sending Δ^n to x and Δ^0 to y.

Lemma 2.35. $\mathrm{LMap}(x, y)$ *is right fibrant.*

Proof. Extending $\Lambda^n_i \to \mathrm{LMap}(x, y)$ to Δ^n is the same as extending $\Lambda^{n+1}_i = \Lambda^n_i \star \Delta^0 \to X$ to $\Delta^{n+1} = \Delta^n \star \Delta^0$, which is possible when $0 < i \leq n$. □

Being left fibrant will be later shown to be equivalent to being a Kan complex. Two maps are **left homotopic** if they are the same in $\pi_0(\mathrm{LMap}(x, y))$. This is an equivalence relation. They are **homotopic** if there is a map $\Delta^1 \times \Delta^1$ restricting to identities on $\partial\Delta^1 \times \Delta^1$ and f, g on $\Delta^1 \times \partial\Delta^1$.

Lemma 2.36. *Two maps in an ∞-category are left homotopic iff they are right homotopic iff they are homotopic.*

Proof. Let σ be a left homotopy from $f \to g$, meaning $\partial\sigma = (g, f, 1)$. Then extend the horn $(\sigma, s_0 g, \cdot, s_0 s_0 d_1 f)$ to Δ^3, d_2 of which is a right homotopy. Left homotopy clearly implies homotopy by having one of the 2-cells be degenerate. Conversely, if g, f are homotopic, they are both either left or right homotopic to the third map in the homotopy, and so by transitivity an the fact that left and right homotopies agree, g, f are left homotopic. □

Lemma 2.37 (Homotopy 1-category of an ∞-category)**.** *For an ∞-category C, the homotopy 1-category is given by the same objects, where $\pi_0(\mathrm{LMap}(x, y))$ is the homotopy classes of maps from x to y.*

Proof. A map to a 1-category certainly factors through $\pi_0 \mathrm{LMap}(x, y)$. We will show we can build a 1-category out of $\pi_0(\mathrm{LMap}(x, y))$ as the Hom sets. By Lemma 2.36 we can use any notion of homotopy. Since $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^1_2)$ is a trivial Kan fibration by Corollary 2.30, any two composites are homotopic, so there is a well-defined composite up to homotopy. For associativity, if f realizes a composite $a \circ b$, g realizes $a \circ (b \circ c)$, f' realizes $b \circ c$, then consider the map from Λ^3_2 given by $(f, g, \cdot, f')$. Extending to Δ^{n+1} and taking d_{n+1}, we see that $a \circ (b \circ c)$ is a realized as a composite of $(a \circ b) \circ c$. Any map to a 1-category then factors uniquely through this category, so it is $h(C)$. □

We define the homotopy 1-category of spaces, denoted $h(\text{Space})$ to be the category of Kan complexes and homotopy classes of maps between them. This will later be shown to be the homotopy 1-category of a suitable ∞-category of spaces, Space. Similarly we can define $h(\text{Space}_*)$ using pointed Kan complexes.

There is a homotopy category of an ∞-category that is enriched over $h(\text{Space})$, which is a better one.

Lemma 2.38. *$f : X \to Y$, $g : X' \to Y'$ are inner fibrations iff $f \star g$ is as well.*

Proof. If we are trying to lift the inclusion $\Lambda^n_i \to \Delta^n$ along $f \star g$, if Λ^n_i lands in either Y or Y', then this lifts iff f, g are inner fibrations. Otherwise, we decompose Δ^n along where it splits between Y and Y' as Δ^i and Δ^{n-i-1}, and the map $\Delta^i \star \Delta^{n-i-1}$ gives the desired lift. □

We define the **left cone** of a simplicial set K to be $\Delta^0 \star K$, denoted $K^{\triangleright}$, and similarly $K^{\triangleleft} = K \star \Delta^0$ is the **right cone**. The **cone point** is the Δ^0 inside of it.

Lemma 2.39 (Overcategories)**.** *Let $p : K \to S$ be a map of simplicial sets. Then There is a simplicial set $S_{/p}$ with the universal property* $\mathrm{Hom}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(Y, S_{/p}) = \mathrm{Hom}_p(Y \star K, S)$*.* Hom_p *denotes maps extending p.*

Proof. One can define $S_{/p}$ by $(S_{/p})_n = \mathrm{Hom}_p(\Delta^n \star K, S)$ and note that $\star$ commutes with colimits in each variable to deduce the universal property. □

Dually the under category is defined by the same universal property except with $K \star Y$ instead of $Y \star K$.

Lemma 2.40. *Suppose $f : A_0 \subset A, g : B_0 \subset B$. and either f is right anodyne or g is left anodyne. Then the inclusion $(A_0 \star B) \coprod_{A_0 \star B_0} (A \star B_0) \subset A \star B$ is inner anodyne.*

Proof. Assume f is right anodyne, as the other case is dual. Then the class of g, f such that the lemma is true is saturated, so it suffices to assume f is $\Lambda^n_i \to \Delta^n$ and g is $\partial\Delta^m \to \Delta^m$. Then h is just the inclusion of $\Lambda^{n+m+1}_i \subset \Delta^{n+m+1}$. □

The same proof yields

Lemma 2.41. *Suppose $f : A_0 \subset A, g : B_0 \subset B$. and f is left anodyne. Then the inclusion $(A_0 \star B) \coprod_{A_0 \star B_0} (A \star B_0) \subset A \star B$ is left anodyne.*

Proposition 2.42. *Suppose we are given a diagram of simplicial sets $A \subset B \xrightarrow{p} X \xrightarrow{q} S$ where q is an inner fibration. Let $r = q \circ p$, and p_0, q_0, r_0 be the restrictions to A of p, q, r. Then the induced map $X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$ is a left fibration. If in addition, q is a left fibration, then the map $X_{/p} \to X_{/p_0} \times_{S_{/r_0}} S_{/r}$ is as well.*

Proof. By the universal properties of the overcategories, the lifting problems are equivalent to maps given in Lemma 2.40 and Lemma 2.41. □

Corollary 2.43. *The map $X_{p/} \to X_{p_0/}$ is a left fibration.*

Proof. Let $S = \Delta^0$ in Proposition 2.42. □

Similarly, we can obtain

Proposition 2.44. *Suppose we are given a diagram of simplicial sets $A \subset B \xrightarrow{p} X \xrightarrow{q} S$ where either $A \subset B$ is right anodyne and q is an inner fibration or $A \subset B$ is anodyne and q is a left fibration. Let $r = q \circ p$, and p_0, q_0, r_0 be the restrictions to A of p, q, r. Then the induced map $X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$ is a trivial fibration.*

2.4. **Left fibration characterization of Kan fibrations.** Kan fibrations are left fibrations where the pushforward map on fibres are equivalences. This takes some effort to prove (can be viewed as a coherence type result), and as a special case includes the fact that left fibrant simplicial sets are Kan complexes. The main result is Proposition 2.53.

Left anodyne maps can be given by 'left deformation retracts':

Lemma 2.45. *Let $p : X \to S$ be a map with a section s and a fibrewise homotopy h from $s \circ p$ to 1. Then s is left anodyne.*

Proof. There is a retraction

$$\begin{tikzcd} S \times \Lambda^1_0 \arrow[r] \arrow[d] & (S \times \Delta^1) \times_{S \times \Lambda^1_1} (X \times \Lambda^1_1) \arrow[r] \arrow[d] & S \arrow[d] \\ X \times \Lambda^1_0 \arrow[r] & X \times \Delta^1 \arrow[r] & X \end{tikzcd}$$

where the middle vertical map is left anodyne by Corollary 2.22. □

The following lemma is a 1-categorical version of the type of coherence we are looking for. By using it in families, we will build our way to the result.

Lemma 2.46. *If $p : C \to D$ is a left fibration of ∞-categories and f is a morphism in C such that pf is an equivalence in D, then f is an equivalence. Furthermore, given an equivalence $x \to p(y)$, there is a lift to an equivalence $\tilde{x} \to p(y)$.*

Proof. If pf has a left inverse g, we can lift g to a left inverse $\tilde{g}$ in C of f. But g has a left inverse in D, f', which we can lift to see that $\tilde{g}$ is left invertible. But C is an ∞-category, so g being left invertible is equivalent to f being right invertible.

If $a : x \to p(y)$ is an equivalence, we can choose a homotopy inverse b and a lift of b $\tilde{b}$ from y to an object $\tilde{x}$. Then we can lift a 2-simplex exhibiting a as a left inverse of b to obtain the desired lift of a, which is an equivalence by the first part. □

The following consequence is quite important.

Proposition 2.47. *Let C be an ∞-category and f a morphism. f is an equivalence iff for every $n \geq 2$ and every map $f_0 : \Lambda^n_0 \to C$ that sends $\Lambda^{\{0,1\}}$ to f, there exists an extension to Δ^n.*

Proof. The lifting problem is equivalent to a lifting problem for the diagram

$$\begin{tikzcd} \Lambda^1_1 \arrow[r] \arrow[d] & C_{/\Delta^{n-2}} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & C_{/\partial\Delta^{n-2}} \end{tikzcd}$$

Because f is an equivalence and $C_{\partial\Delta^{n-2}} \to C$ is a left fibration by Corollary 2.43, the morphism in the diagram is an equivalence in $C_{\partial\Delta^{n-2}}$ by Lemma 2.46. By the same lemma, we can produce the desired lift since the right vertical map is a right fibration.

For the converse, if the extensions exist, then f has a left inverse by extending $(\cdot, 1, f)$ along $\Lambda^2_0 \subset \Delta^2$, giving a 2-simplex a with $d_0 a = f^{-1}$. By extending $(\cdot, s_0 f, s_1 f^{-1}, a)$ along $\Lambda^3_0 \subset \Delta^2$, we obtain a right inverse of f. ☐

Corollary 2.48. *TFAE for an ∞-category C:*

(1) hC is a groupoid
(2) C is a Kan complex
(3) C is left fibrant

This is the starting point for an analogous statement for left fibrations, namely the Grothendieck construction/straightening unstraightening equivalence.

Corollary 2.49. *The subcomplex of an ∞-category consisting of simplices whose edges are equivalences is a Kan complex, and is the universal Kan complex mapping into the ∞-category.*

We will need to construct the inverse on the 1-categorical level. Let $X \to S$ be a left fibration, and $f : s \to t$ an arrow in S. Then if X_s i the fibre of s, by creating a lift as in the diagram

$$\begin{tikzcd} \Lambda^1_1 \times X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \times X_s \arrow[r, "f \circ \pi_1"] \arrow[ur, dashed] & S \end{tikzcd}$$

we obtain a map from $X_s \to X_t$. Moreover the map

$$\mathrm{Map}(\Delta^1 \times X_s, X) \to \mathrm{Map}(\Lambda^1_1 \times X_s, X) \times_{\mathrm{Map}(\Lambda^1_1 \times X_s, S)} \mathrm{Map}(\Delta^1 \times X_s, S)$$

is a trivial fibration by Corollary 2.30, so in particular, there is a unique lift in the homotopy category of spaces, which will be denoted $f_!$.

Lemma 2.50. *Given a left fibration $X \to S$, the assignment $s \to X_s$, $f \to f_!$ defines a functor $hS \to h(\mathrm{Space})$.*

Proof. Because $X \to S$ is an inner fibration, composition lifts along it, showing that $f_!$ preserves composition. ☐

Lemma 2.51. *Suppose $p : S \to T$ is a left fibration and T is a Kan complex. Then p is a Kan fibration.*

Proof. S is also a Kan complex, since S is left fibrant. We need to show that for any anodyne inclusion $A \subset B$, the map $S^B \to S^A \times_{T^A} T^B$ is surjective on vertices. But it is a homotopy equivalence and a left fibration by Corollary 2.29, so is surjective since the functor from Lemma 2.50 must land in nonempty spaces. ☐

Lemma 2.52. *Suppose $p : S \to T$ is a left fibration and for each $t \in T$, S_t is contractible. Then p is a trivial Kan fibration.*

Proof. We would like to show the inclusion $\partial\Delta^n \subset \Delta^n$ has the left lifting property with respect to p. When $n = 0$, the lifting property clearly holds. By pulling back along the simplex we are trying to lift, we can assume T is Δ^n, in particular, S, T are ∞-categories. Choose a homotopy h from the inclusion $\partial\Delta^n \subset T$ to the terminal object of T, and create a lift h' in S. We would like to extend h' from $\partial\Delta^n \times \Delta^1$ to $\Delta^n \times \Delta^1$. We can extend $\partial\Delta^n \times 1$ to Δ^n by using the fact that the fibres are contractible. The rest of the cells that need to be added are inner horns (see Lemma 2.18) except for the last cell. But this can be filled in since its restriction to the initial edge lies in a fibre, so we can use Proposition 2.47. □

We come to the main result.

Proposition 2.53. *TFAE for a left fibration $p : S \to T$:*

(1) p is a Kan fibration
(2) For every edge $f \in T$, the map $f_!$ is an equivalence.

Proof. For $1 \implies 2$, since p is a left and right fibration, every morphism f induces a covariant $f_!$ and contravariant f^*, which are inverse to each other in the homotopy category of spaces.

For $2 \implies 1$, it suffices to show that p is a right fibration. It suffices to show that the map $q : S^{\Delta^1} \to S^{\Lambda^1_0} \times_{T^{\Lambda^1_0}} T^{\Delta^1}$ is a trivial Kan fibration. We already know it is a left fibration, so by Lemma 2.52, we just need to show that the fibres are contractible. For an edge $f : t \to t'$ in T, we can pullback q along the map $S_{t'} \to S^{\Lambda^1_0} \times_{T^{\Lambda^1_0}} T^{\Delta^1}$ given by picking f , and it will suffice to show the fibres of the pullback $X \to S_{t'}$ are contractible for each f. But the natural map $X \to S_t$ is a trivial Kan fibration since p is a left fibration. In particular X is a Kan complex. By Lemma 2.51, $X \to S_{t'}$ is a Kan fibration. The map $f_!$ is obtained by taking a section of the map $X \to S_t$ and composing with the map $X \to S_{t'}$, so in particular we learn that the map $X \to S_{t'}$ is an equivalence. Since it is an equivalence and a Kan fibration, it is a trivial Kan fibration. □

Corollary 2.54. *Let f be a left fibration, and g a trivial Kan fibration. Then $g \circ f$ is a trivial Kan fibration iff f is.*

Proof. If f is, $g \circ f$ is by composition. Conversely, if $g \circ f$ is, by Proposition 2.53 it suffices to show that the fibres of f are contractible. Given $x \in \operatorname{cod} f$, the map $g \circ f^{-1}(x) \to g^{-1}(g(x))$ are is a left Kan fibration of contractible Kan complexes. It follows from Lemma 2.51 that it is a Kan fibration, and since both are contractible, the fibres are too, including $f^{-1}(x)$. □

2.5. Cartesian Morphisms.

Definition 2.55. *Let $p : X \to S$ be an inner fibration. Let $x \to y$ be an edge in X. f is p-**Cartesian** if the map $X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$ is a trivial Kan fibration.*

In other words, giving a map $a \to x$ is essentially the data of giving a map $a \to y$ and a compaitble map $p(a) \to p(x)$.

Lemma 2.56. *(1) Every edge of an isomorphism of simplicial sets is Cartesian.*
(2) If $p' : X' \to S'$ is a pullback of $p : X \to S$, a morphism in X' is p'-Cartesian iff its image in X is p-Cartesian.
(3) If $p : X \to Y$ and $q : Y \to Z$ are inner fibrations and f is an edge of X such that $p(f)$ is q-Cartesian, f is p-Cartesian iff it is $p \circ q$-Cartesian.

Proof. (1), (2) follow from the definition. To see (3), consider the diagram

$$\begin{tikzcd}
X_{/f} \arrow[rr] \arrow[dr, "h"] & & X_{/x} \times_{Z_{/(q\circ p)(x)}} Z_{/(q\circ p)(f)} \\
& X_{/x} \times_{Y_{/p(x)}} Y_{/p(f)} \arrow[ur, "g"] &
\end{tikzcd}$$

The map g is a pullback of the map that is a trivial fibration since $p(f)$ is a q-Cartesian edge. By Proposition 2.42, h is a right fibration, so by Corollary 2.54 it follows that h is a trivial Kan fibration iff $g \circ h$ is. $\square$

Remark 2.56.1. An edge being p-Cartesian is equivalent to lifts always existing in diagrams of the form

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\
\Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

In particular in Proposition 2.47, the condition on f being an equivalence is exactly that it is p-Cartesian for p the projection to a point.

Proposition 2.57. *Let $p : C \to D$ be an inner fibration between ∞-categories. A morphism $f \in C$ is an equivalence iff pf is an equivalence and f is p-Cartesian.*

Proof. Let $q : D \to *$ be the projection to a point. $p(f)$ being an equivalence is equivalent by Proposition 2.47 to f being $q \circ p$-Cartesian. Then the result follows from Lemma 2.56. $\square$

Proposition 2.58. *Suppose $p : C \to D$ is an inner fibration between ∞-categories, there is a composite $f \circ g = h$ from $x \to y \to z$, and g is p-Cartesian. Then f is p-Cartesian iff h is.*

Proof. We are trying to show that one of the right fibrations (Proposition 2.42) $i_0 : C_{/f} \to C_{/y} \times_{D_{/p(y)}} D_{/p(f)}, i_1 : C_{/h} \to C_{/z} \times_{D_{/p(z)}} D_{/p(h)}$s a trivial Kan fibration iff the other is. Being a trivial Kan fibration is equivalent to the fibres being contractible. For simplicial subsets $A \subset B \subset \Delta^2$, if we define $X_B = C_{/\sigma|B} \times_{D_{/\sigma|B}} D_{/\sigma}$, then there is a natural map $j_{A,B} : X_B \to X_A$, which is a right fibration as it is a pullback of a map that is a right fibration by Proposition 2.42.

The map $j_{\Delta^{\{2\}},\Delta^{\{0,2\}}}$ is base changed from i_0 by the map $D_{/\sigma} \to D_{/p(f)}$ which is surjective on objects, and similarly $j_{\Delta^{\{2\}},\Delta^{\{0,2\}}}$ is base changed from i_1. Thus it suffices to show that $j_{\Delta^{\{2\}},\Delta^{\{0,2\}}}$ has contractible fibres iff $j_{\Delta^{\{2\}},\Delta^{\{0,1\}}}$ does.

By Proposition 2.44, $j_{A,B}$ is a trivial Kan fibration for any left anodyne inclusion. The inclusion $\Delta^{\{2\}} \to \Delta^2$ factors as both $\Delta^{\{2\}} \to \Delta^{\{0,2\}} \to \Delta^2$ and $\Delta^{\{2\}} \to \Delta^{\{1,2\}} \to \Delta^{\Lambda^2_1} \to \Delta^2$, so by Corollary 2.54, it follows that $j_{\Delta^{\{2\}},\Delta^{\{0,2\}}}$ is a trivial fibration iff $j_{\Delta^{\{1,2\}},\Lambda^2_1}$ is. $j_{\Delta^{\{1,2\}},\Lambda^2_1}$ is a pullback of $j_{\Delta^{\{2\}},\Delta^{\{0,1\}}}$ by a map that is surjective on vertices (in fact, a trivial fibration) since the inclusion $\{1\} \to \Delta^{\{1,2\}}$ is left anodyne. Thus the result we want holds. $\square$

The dual notion of Cartesian is coCartesian. The following is the essential point of being coCartesian. Its relation with the definition can be compared to Proposition 2.18.

Proposition 2.59. *Let $p : Y \to S$ be an inner fibration of simplicial sets and let $e : \Delta^1 \to Y$ be an edge. e is p-coCartesian iff the indicated lifts exist in all diagrams of the form*

$$\begin{tikzcd}
\{0\} \times \Delta^1 \arrow[d] \arrow[dr, "e"] & \\
(\Delta^n \times \{0\}) \coprod_{\partial\Delta^n \times \{0\}} (\partial\Delta^n \times \Delta^1) \arrow[r] \arrow[d] & Y \arrow[d] \\
\Delta^n \times \Delta^1 \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

Proof. For one direction, observe that the inclusion above is inner anodyne except for the attachment of one cell, for which the extension exists because f is coCartesian.

Conversely, if the lifting property holds, then first we can pull back p along the map $\Delta^n \to S$ to reduce to the case where $S = \Delta^n$, so that Y is an ∞-category. We are then trying to lift

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d] \arrow[dr, "e"] & \\
\Lambda^n_0 \arrow[r] \arrow[d] & Y \arrow[d] \\
\Delta^n \arrow[r, equal] \arrow[ur, dashed] & \Delta^n
\end{tikzcd}$$

By the proof of Proposition 2.18, there is a retraction of the inclusion $\Lambda^n_0 \times \Delta^1 \coprod_{\Lambda^n_0 \times \{0\}} \Delta^n \times \{0\}$ to the inclusion $\Lambda^n_0 \to \Delta^n$. Letting $K = \Delta^{\{1,2,\ldots n\}}$, via the retraction, we can restrict to $\partial K \times \Delta^1 \coprod_{\partial K \times \{0\}} K \times \{0\}$. $\{1\} \times \Delta^1$ is send to e, so it follows that we can extend to $K \times \Delta^1$. Then we will have a map on $(\Delta^n \times \{0\}) \coprod_{\partial\Delta^n \times \{0\}} (\partial\Delta^n \times \Delta^1)$ where the first edge is degenerate and in particular coCartesian, so we can extend to $\Delta^n \times \Delta^1$, and then restrict to Δ^n to get our desired lift. $\square$

Proposition 2.60. *Let $p : Y \to S$ be an inner fibration of simplicial sets and let $e : \Delta^1 \to Y$ be a p-Cartesian edge. Then the indicated lifts exist in all diagrams of the form*

$$\begin{tikzcd}
\{m\} \times \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "e"] & \\
(\Delta^m \times \Lambda^n_n) \coprod_{\partial\Delta^m \times \Lambda^n_n} (\partial\Delta^m \times \Delta^n) \arrow[r] \arrow[d] & Y \arrow[d] \\
\Delta^m \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

Proof. Similarly to the proofs Proposition 2.59 and Proposition 2.18, there is a canonical way to build the inclusion from filling in horns. The only horn which is not inner is the last one, which we can fill because e is p-Cartesian. $\square$

Definition 2.61. *Suppose $p : X \to S$ is an inner fibration and $e : \Delta^1 \to X$ an edge. Then e is a **locally p-Cartesian** edge if it is Cartesian with respect to the pullbback of p along $p(e)$.*

The benefit of this definition is that given a pullback diagram, an edge is locally p-Cartesian iff its image is.

2.6. **Minimal Inner Fibrations.** The purpose of this section is to produce a theory of minimal inner fibrations generalizing that in Section 3.1. Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where p is an inner fibration, we say that $f, f' : B \to X$ are homotopic relative to A over S if they are equivalent in the ∞-category that is the fibre of the map $X^B \to X^A \times_{S^A} S^B$.

Lemma 2.62 (HOWTO PROVE THIS? MAYBE USE CHARACTERIZATION OF EQUIVALENCES)**.** *[MAYBE IT IS A COCARTESIAN FIBRATION OR SOMETHING] A map in a fibre of $X^B \to X^A \times_{S^A} S^B$ for an inner fibration $X \to S$ and an inclusion $A \to B$ is an equivalence iff it is a pointwise equivalence for each $b \in B$ in each fibre of $X \to S$.*

Proof. content... ☐

Remark 2.62.1. The composite of two minimal inner fibrations is again a minimal inner fibration, and the pullback of a minimal fibration is too. Any functor of ordinary categories is a minimal inner fibration on nerves. In particular Δ^n is minimal inner fibrant. If f is an inner fibration and g is a minimal inner fibration, then $g \circ f$ is minimal iff f is.

Remark 2.62.2. $p : X \to S$ is a minimal fibration iff for every simplex of S, the fibre is a minimal ∞-category.

Lemma 2.63. *Let C be a minimal ∞-category and let $f : C \to C$ be a functor that is homotopic to the identity. Then f is an isomorphism.*

Proof. Choose a homotopy $h : 1 \to f$. By induction on n, we prove that f is an injection on simplices. Let σ, σ' be n-simplices such that $f(\sigma) = f(\sigma')$. By induction we know that f agrees on the boundary of the simplices.

By pasting the homotopies from σ, σ' to $f(\sigma)$ and the homotopy on the boundary together, we produce the diagram below and use Proposition 2.60 to find a lift.

$$\begin{tikzcd} (\Delta^2 \times \partial\Delta^n) \coprod_{\Lambda^2_2 \times \partial\Delta^n} (\Lambda^2_2 \times \Delta^n) \arrow[r] \arrow[d] & C \\ \Delta^2 \times \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

We thus obtain a homotopy relative to the boundary from σ to σ', showing they agree.

To see surjectivity, choose an n-simplex σ in C. By the inductive hypothesis and injectivity, we can assume that $\partial\sigma$ is in the image of f. By Proposition 2.59, we can extend the homotopy h backwards along σ to obtain a simplex σ'. The claim is that $f(\sigma') = \sigma$ to see this, we can use Proposition 2.60 to produce a lift in the diagram

$$\begin{tikzcd} (\Delta^2 \times \partial\Delta^n)\coprod_{\Lambda^2_0\times\partial\Delta^n}(\Lambda^2_0\times\Delta^n) \arrow[r] \arrow[d] & C \\ \Delta^2\times\Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

where one of the edges is $h_{|\sigma'}$ and the other is our homotopy from σ to σ'. This produces a relative homotopy from σ to $f(\sigma')$, so indeed σ is in the image. □

Proposition 2.64. *Let $p : X \to S$ be an inner fibration. Then there exists a retraction $X \to X$ onto a simplicial subset $X' \subset X$ such that*

(1) $p_{|X'}$ is a minimal inner fibration.
(2) $p \circ r = p$
(3) r is homotopic to S

2.7. **Trees.** Let C be a presentable category and S a set of morphisms. The weak saturation of S is obtained from S via pushouts, transfinite compositions, and retracts. Here we prove that after possibly enlarging S, the weak saturation is generated by a small set of morphisms just from pushouts and transfinite compositions.

The following is a generalization of a transfinite chain of morphisms.

Definition 2.65. *Let C be a presentable category and S be a collection of morphisms in C. An S**-tree in** C is the following data:*

*(1) An object $X \in C$ called the **root**.*
*(2) A partially ordered set A which is **well-founded** (ie any nonempty set has a minimal element)*
(3) A diagram $A \to C_{X/}$, which will be denoted $\alpha \mapsto Y_\alpha$.
(4) for each $\alpha \in A$, a pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ \lim_{\beta<\alpha} Y_\alpha \arrow[r] & Y_\alpha \end{tikzcd}$$

where $f \in S$.

We will often use just $\{Y_a\}_{a\in A}$ to denote an S-tree. Suppose that $B \subset A$ is a downward closed subset. Then $\{Y_a\}_{a\in B}$ is an S-tree. We can also let $B_\alpha = B \cup \{\beta \in A | \beta \leq \alpha\}$, and $\{(Y_{B_\alpha})_{\alpha\in A-B}\}$ is an S-tree.

Y_B denotes the colimit $\lim_{\alpha\in B} Y_\alpha$ in $C_{/X}$. In particular $Y_\phi = X$.

Let κ be a regular cardinal. An S-tree is κ-good if the diagram consists of κ-compact objects and for each $\alpha \in A$, the set $\{\beta \in A | \beta < \alpha\}$ is κ-small.

Given an S-tree for X, we can pushout along a map $f : X \to X'$ to get an S-tree for X' called the **associated** S**-tree**. Note that S-trees naturally arise from transfinite sequences of morphisms $X \to Y_0 \to Y_1 \to \dots$.

Lemma 2.66. *Given a presentable category C, S a set of morphisms, and $\{Y_\alpha\}_{a\in A}$ an S-tree. then for any $A'' \subset A' \subset A$ downward closed, the map $Y_{A''} \to Y_{A'}$ is a morphism in the weak saturation of S.*

Proof. It suffices by the earlier observations to assume that $A'' = \phi$ and $A = A'$. Write A as the union of a transfinite sequence of downward closed subsets obtained by adding one minimal element not already there at a time. The colimit of each of these is obtained from the previous one by a pushout along a morphism in S, so the map is a transfinite composition of these. $\square$

We can modify our S-trees to become κ-good under reasonable hypotheses. The following lemma is the whole reason we need the notion of an S-tree: it is not true that the modification is totally ordered if the original tree is.

Lemma 2.67. *Let C be a presentable caegory, κ a regular cardinal, and let S be a collection of morphisms between κ-compact objects. Suppose that $\{Y_\alpha\}_{\alpha \in A}$ is an S-tree in C. Then there exists the following:*

(1) A new ordering A' refining the order on A (ie there is a map $A' \to A$)
(2) A κ-good S-tree $\{Y_\alpha\}_{\alpha \in A'}$ having the same root as before.
(3) A natural transformation from the dagrams for A' to that for A.
(4) For any subset $B \subset A$, the map $Y'_B \to Y_B$ is an isomorphism where Y_B denotes the colimit for the new S-tree.

Proof. Write A as a transfinite union of posets indexed on some ordinal so that at limit ordinals, you take the union, and on successor ordinals you add a minimal element not already in the set. We will construct the new ordering satisfying the properties one at a time in compatible ways for each of these. At limit ordinals there is nothing to do. At a successor ordinal, suppose that we adjoin a map $f \in S$ as in the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ Y_B & \longrightarrow & Y_\alpha \end{array}$$

Then the map $C \to Y_B$ factors through some κ-small subdiagram Y'_B which agrees with the colimit already constructed so far. Enlarge B' to be downward closed, and then modify the partial ordering on α so that $\beta \leq \alpha$ iff $\beta \in B'$. $\square$

Lemma 2.68. *Let C be a presentable category, κ an uncountable regular cardinal, and S a collection of morphisms in C. Let $\{Y_\alpha\}_{\alpha \in A}$ be a κ-good S-tree with root X and let $T_A : Y_A \to T_A$ be an idempotent endomorphism of Y_A in the category $C_{X/}$. Let B_0 be an arbitrary κ-small subset of A. Then there is a κ-small downward closed enlargement B of B_0 and an idempotent endomorphism T_B of Y_B such that the following diagram commutes:*

$$\begin{array}{ccccc} X & \longrightarrow & Y_B & \longrightarrow & Y_A \\ \| & & \downarrow{\scriptstyle T_B} & & \downarrow{\scriptstyle T_A} \\ X & \longrightarrow & Y_B & \longrightarrow & Y_A \end{array}$$

Proof. Assume WLOG that B_0 is downward closed. We will inductively construct κ-small downward closed subsets B_i of A and morphisms $Y_{B_i} \to Y_{B_{i+1}}$ extending compatible with T_A. Taking the union of the B_i, we will be done. To construct the B_i, note that the map T_A composed with the inclusion of $Y_{B_{i-1}}$ factors through a κ-small subset since $Y_{B_{i-1}}$ is κ-small.

WLOG we can assume this is downward closed and contains B_{i-1}, and call it B_i. This gives the desired map. $\square$

Lemma 2.69. *Let C be a presentable category, κ an uncountable regular cardinal, and S a collection of morphisms in C. Let $\{Y_\alpha\}_{\alpha \in A}$ be a κ-good S-tree with root X, B a κ-small downward closed subset of A and let T_A, T_B be compatible idempotent endomorphisms of Y_A, Y_B in the category $C_{X/}$. Let C_0 be an arbitrary κ-small subset of A. Then there is a κ-small downward closed enlargement C of C_0 and a idempotent endomorphisms T_C and $T_{C\cap B}$ of $Y_C, Y_{C\cap B}$ compatible with each other and the endomorphisms for B and A.*

Proof. WLOG, C_0 is downward closed. We will construct sequences of κ-small downward closed subsets $C_i \subset A, i \geq 0$ and $D_i \subset B, i > 0$ along with idempotent endomorphisms on T_{C_i} and T_{D_i} such that

(1) D_i contains $B \cap C_{i-1}$ and C_i contains D_i.
(2) T_{C_i} is compatible with T_A and T_{D_i} is compatible with T_B.
(3) The diagrams below commute.

$$\begin{tikzcd}
Y_{D_{i-2}} \arrow[r] \arrow[d, "T_{D_{i-2}}"] & Y_{D_{i-1}} \arrow[d, "T_{D_{i-1}}"] & Y_{C_{i-2}} \arrow[r] \arrow[d, "T_{C_{i-2}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] & Y_{D_{i-1}} \arrow[r] \arrow[d, "T_{D_{i-1}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] \\
Y_{D_{i-2}} \arrow[d, "T_{D_{i-2}}"] & Y_{D_{i-1}} \arrow[d, "T_{D_{i-1}}"] & Y_{C_{i-2}} \arrow[d, "T_{C_{i-2}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] & Y_{D_{i-1}} \arrow[d, "T_{D_{i-1}}"] & Y_{C_{i-1}} \arrow[d, "T_{C_{i-1}}"] \\
Y_{D_i} \arrow[r, equal] & Y_{D_i} & Y_{C_i} \arrow[r, equal] & Y_{C_i} & Y_{D_i} \arrow[r] & Y_{C_i}
\end{tikzcd}$$

These are constructed by induction. By compactness, the third condition follows from the second if C_i, D_i are chosed large enough. By Lemma 2.68 we can then construct the desired C_i. Taking the union over i, we are done. $\square$

Lemma 2.70. *Let A be a κ-accessible presentible category. Let $f : C \to D$ be a morophism between κ-compact objects of A, let $g : X \to Y$ be a pushout of f along some morphism, and let $g' : X' \to Y'$ be a retract of g in the category of morphisms of A. Then there exists a morphism $f' : C' \to D'$ with the following properties:*

(1) C', D' are κ-compact.
(2) g' is a pushout of f'.
(3) f' is in the weakly saturated class of morphisms generated by f.

Proof. By pushing out g along the map $X \to X'$ in the retraction from g to g', we can assume that in that retraction, X is just identified with X'. In particular, the retraction is given by an idempotent e on Y with image Y'.

Now X is a κ-filtered colimit of X_λ, and since C is compact, C factors through some X_λ, and taking all the things that X_λ maps to, we can refine this as a filtered colimit in $A_{/C}$. Then Y is a filtered colimit of pushouts of X_λ and D along C. Then the composition of $D \to Y$ with e factors through some mape $D \xrightarrow{j} X_\lambda \cup_C D$ since D is κ-compact. After possibly enlarging λ, by compactness of C, the map $j \circ f$ agrees with the map canonical map $C \to X_\lambda \cup_C D = Y_\lambda$. Thus j and id_X yield a map e' from Y_λ to itself. By possibly enlarging λ, we can assume that e' is idempotent and that e' is compatible with the idempotent on Y. Let Y'_λ be the image of e' (this can is the colimit of the endomorphism e'). We have a canonical map $f' : X_\lambda \to Y'_\lambda$, which is a retract of the map $X_\lambda \to Y_\lambda$, which is a pushout of f. Thus (1) and (3) are satisfied. To check (2), we observe that the pushout square for $X \cup_{X_\lambda} Y_\lambda \cong Y$ retracts onto the analogous square for Y'_λ and Y', showing that g' is a pushout of f'. $\square$

Lemma 2.71. *Let C be a presentable category, κ a regular cardinal such that C is κ-accessible, and $S = \{f_s | C_s \to D_s\}$ a collection of morphisms of C such that each C_s is κ-compact. Let $\{Y_\alpha\}_{\alpha\in A}$ be an S-tree in C with root X and suppose that A is κ-small. Then $\{Y_\alpha\}_{\alpha\in A}$ is isomorphic as an S-tree to one of the form $\{Y'_\alpha \cup_{X'} X\}_{\alpha\in A}$ where X is κ-compact.*

Proof. If the conclusion holds, we will say that the S-tree is pushed out from X'. Write X as a κ-filtered colimit of κ-compact $X_i, i \in I$. Choose a way to write A as a union of $A(\lambda), \lambda < \beta$, where each $A(\lambda)$ is obtained form the ones before it by adding a minimal element not already there if λ is a successor, and by taking the union if λ is a limit ordinal.

We will inductively construct a (not strictly) increasing transfinite sequence of elements $\{i_\lambda \in I\}_{\lambda\geq\beta}$ such that $\{Y_\alpha\}_{\alpha\in A(\gamma)}$ is pushed out from X_{i_γ} in a way compatible for all γ. Then we will be done since I is κ-filtered, there will be some X_i from which the original S-tree is pushed out from.

At limit ordinals, we can just take the limit. At a successor ordinal, we know $A(\gamma + 1)$ is obtained from $A(\gamma)$ by adding on some element a_γ. If B is all the things less than γ, we have a pushout diagram

$$\begin{tikzcd} C_s \arrow[r, "f_s"] \arrow[d, "g"] & D_s \arrow[d] \\ Y_B \arrow[r] & Y_\alpha. \end{tikzcd}$$

We have assumed that Y_B is a pushout $Y^\gamma_B \cup_{X_i} X$ (where $\{Y^\gamma_\alpha\}_{\alpha\in A(\gamma)}$ is the S-tree in the inductive hypothesis. Using κ-compactness of C_s, g factors as through some $Y^\gamma_B \cup_{X_i} X_j$. Let $i_{\gamma+1} = j$, and define $\{Y^{\gamma+1}_\alpha\}_{\alpha\in A(\gamma)}$ by having $Y^{\gamma+1}$ being the pushout of f_s along $C_s \to Y^\gamma_B \cup_{X_i} X_j$. $\square$

Proposition 2.72. *Let C be a presentable κ-accessible category, κ an uncountable regular cardinal, and $\overline{S}$ a weakly saturated class of morphisms in C generated by S, the subcollection of morphisms between κ-compact objects.*

Then for every morphism $f : X \to Y$ in $\overline{S}$, there exists a transfinite sequence of objects in $C_{/X}$ $\{Z_\lambda\}_{\lambda<\beta}$ such that each is obtained as a pushout along a morphism in S from the limit of the ones before it, and Y is the last one.

Proof. By the small object argument, there is such a sequence $\{Y_\alpha\}_{\alpha<\beta}$ with β κ-small such that Y is a retract of the last one Y_β. We can view this as an S-tree with root X. By Lemma 2.67 we can replace this with a κ-good S-tree A' with the same colimits at every step. Choose an idempotent map e on Y_β with image Y. We will define a transfinite sequence $B(\gamma)$ indexed by some ordinals less than β and compatible systems of idempotent maps $T_{B(\gamma)}$ on $Y_{B(\gamma)}$. On limit ordinals, we can define the limit $T_{B(\gamma)}$ to be the union of the $T_{B(\gamma)}$. If we reach the ordinal β with some $B(\gamma)$, we can stop the construction. At limit ordinals $B(\gamma)$ is defined by taking the colimit, and at successors, it is defined by taking a minimal element of $A' - B(\gamma)$, applying Lemma 2.69. and defining the successor idempotent as the glued together idempotent for each piece. By Lemma 2.71, we learn that $Y_{B(\gamma)}$ is obtained from the colimit over all the things before it by a pushout of a morphism in S. Define Z_λ to be the image of the idempotent $T_{B(\gamma)}$. The colimit of Z_λ is Y, and by Lemma 2.70 each successive Z_λ is obtained from the previous by pushing out along a morphim in S. □

Corollary 2.73. *Under the hypotheses of Proposition 2.72, there is a κ-good S-tree $\{Y_a\}_{a\in A}$ such that $Y_A \cong Y$ in $C_{/X}$.*

3. Kan Fibrations

Apparently the following proposition, due to Moore, convinced Milnor that simplicial sets are the right thing.

Proposition 3.1. *A surjective homomorphism $G \to H$ of simplicial groups is a Kan fibration.*

Proof. Suppose we have a diagram

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & G \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & H \end{array}$$

Since $G \to H$ is surjective, we can lift Δ^n (without making the diagram commute), and divide by the lift, reducing to the case where $\Delta^n \to H$ is the identity. Thus we can assume H is trivial, so that it amounts to showing that a simplicial group G is fibrant.

Let $f_j, j \neq i$ denote the boundary components making up Λ_i^n. We will show by induction on $j \neq i$ that there is a simplex agreeing with Λ_i^n on the first j boundary components. For the induction step, we can divide by the simplex in the induction hypothesis to reduce to the case that f_k is the identity for $k < j$.

Consider $s_j f_j$. $d_j s_j f_j = f_j$, and $i \neq k < j$ we have $d_k s_j f_j = s_j d_k f_j = s_j 1 = 1$. □

Definition 3.2. *Let X be a Kan complex. Then $\pi_n(X, x), n \geq 1$ is the set of homotopy classes of maps from Δ^n relative to $\partial\Delta^n$ being sent to x.*

Given two elements $[a], [b]$ of $\pi_n(X, Y, x)$ we can multiply them to get $[a] \star [b]$ as follows: consider λ_n^{n+1} given by $(x, x \ldots, x, a, \cdot, b)$. By filling in the horn, d_n of the resulting simplex is the composite. By the homotopy extension property the homotopy class of the composite is only dependent on the homotopy class of a, b. Moreover, one can easily see that there are inverses and identities, and that $\pi_n(X, Y)$ with the operation $\star$ is functorial.

Lemma 3.3. *$\star$ makes π_n into a group for $n \geq 1$.*

Proof. It remains to check associativity. If $f = \Delta^{n+1}$ realizes a composite $a \star b$, g realizes $a \star (b \star c)$, f' realizes $b \star c$, then consider the map from Λ_{n+1}^{n+2} given by $(x, x, \ldots, f, g, \cdot, f')$. Extending to Δ^{n+1} and taking d_{n+1}, we see that $a \star (b \star c)$ is a realized as a composite of $(a \star b) \star c$. □

Note that with another definition of π_n one could also use the homotopy category to prove associativity.

Proposition 3.4. *Given a Kan fibration $X \to Y$ where Y is fibrant, and F is the fibre, there is a natural long exact sequence*

$$\pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F) \to \pi_{n-1}(X)$$

at any basepoint.

Proof. The maps $\pi_n(F) \to \pi_n(X) \to \pi_n(Y)$ are the obvious ones. The composite is obviously zero, and if something is in the kernel, then by the homotopy lifting property, the class is equivalent to something from F.

The boundary map $\partial : \pi_n(Y) \to \pi_{n-1}(F)$ is defined by taking a class $\alpha : \Delta^n \to Y$, and extending the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "{(\cdot,x\ldots,x)}"] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "\alpha"] \arrow[ur, dashed] & Y \end{tikzcd}$$

This is well-defined up to homotopy by Corollary 2.30. Something is in its kernel iff there is a lift of α gives a class in $\pi_n(X)$, showing that $\pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F)$ is exact.

Clearly the composite $\pi_{n+1}(Y) \to \pi_n(F) \to \pi_n(X)$ is trivial. If a class α is in the kernel $\pi_n(F) \to \pi_n(X)$, then there is a homotopy in X relative to the boundary from the trivial map. This homotopy factors through Δ^{n+1} giving a map whose boundary is $(x, x, \ldots, \alpha)$. But this projects to a class in Y such that ∂ of it is α. $\square$

The **free path space** of X is $\mathrm{Map}(\Delta^1, X)$. The **path space** of X, x, denoted $P(X)$ or $P_x(X)$ is the pullback (or fibre) of the fibration $\mathrm{Map}(\Delta^1, X) \to \mathrm{Map}(\Delta^0, X)$ given by d_0 along the inclusion of $x = \mathrm{Map}(\Delta^0, \Delta^0)$. $P(X)$ is trivial fibrant since it is the fibre of a trivial fibration.

By identifying $P(X)$ with the pullback of the fibration $\mathrm{Map}(\Delta^1, X) \to \mathrm{Map}(\partial\Delta^1, X)$ along the inclusion of $\mathrm{Map}(\Delta^0, X)$, we see that the projection $P(X) \to X$ is a fibration. Its fibre F is called the **loop space** $\Omega(X)$ or $\Omega_x(X)$ of X, x.

The fibration $\Omega_x(X) \to P_x(X) \to X$ gives an identification $\pi_n(\Omega_x(X)) = \pi_{n+1}(X)$.

For example, if G is a discrete group, $\Omega_x BG = G$, so we see $\pi_1 = G$, and the rest of the homotopy groups vanish.

Lemma 3.5. *Suppose that X is a Kan complex and has a homotopy unital multiplication. Then $\pi_i(X, x)$ is abelian for $i \geq 1$. Moreover, if it sends the basepoint x, x to x, the multiplication map agrees with the multiplication on $\pi_n(X, x)$.*

Proof. We can force the multiplication $\cdot$ to preserve the basepoint x via the homotopy extension property. Now if $\cdot$ is homotopy unital and basepoint preserving, we are done by functoriality of π_n and the Hilton-Eckmann argument: $b \cdot a = (1 \star b) \cdot (a \star 1) = (1 \cdot a) \star (b \cdot 1) = a \star b = (a \cdot 1) \star (1 \cdot b) = (a \star 1) \cdot (1 \star b) = a \cdot b$. $\square$

Lemma 3.6. *$\pi_i(X, x)$ is abelian for $i > 1$. $\Omega_x(X)$ has a unital multiplication such that the identification $\pi_n(\Omega_x(X)) \cong \pi_{n+1}(X)$ gives $\pi_0(\Omega_x(X))$ the group structure coming from $\pi_1(X)$, which is also the group structure from* $\mathrm{Hom}(x, x)$ *in* $\Pi_{\leq 1}(X)$.

Proof. Consider the trivial Kan fibration $\mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Lambda^2_1, X)$. on $\Omega_x(X) \times \cdot \cup \cdot \times \Omega_x(X)$ sits inside of $\mathrm{Map}(\Lambda^2_1, X)$ as pairs for which one map is the identity, moreover, there is a natural section on this subset given by the degeneracies. Extend this to a section of the whole fibration, and then consider the composite $\Omega_x(X)^2 \to \mathrm{Map}(\Lambda^2_1, X) \to \mathrm{Map}(\Delta^2, X) \to \mathrm{Map}(\Delta^1, X)$ where the last map is the restriction to d_1. This sends two maps to a composite, and so factors through a unital multiplication. By Lemma 3.5 $\pi_i(X, x)$ is abelian for $i > 1$, and the addition on π_i is given by the multiplication map, which is composition. For $i = 0$, one can directly identify everything. $\square$

Proposition 3.7. *Let $f : X \to Y$ be a fibration, and Y fibrant. Then f induces a surjection $\pi_n(X, x) \twoheadrightarrow \pi_n(Y, f(x))$ and injection $\pi_{n-1}(X, x) \hookrightarrow \pi_{n-1}(Y, f(x))$ iff it has the right lifting property with respect to $\partial\Delta^n \to \Delta^n$.*

Proof. If we have the lifting property, then $\partial : \pi_n(Y) \to \pi_{n-1}(F)$ is 0, and $\pi_n(X, x) \twoheadrightarrow \pi_n(Y, x)$ is also clearly surjective.

For the converse, we'd like to show the homotopy extension property with respect to $\partial\Delta^n \subset \Delta^n$. By the homotopy lifting property it suffices to lift it after a homotopy of the diagram. Now there is a canonical homotopy H on Δ^n to from the constant 0-map to the identity, which moreover restricts to a homotopy on Λ^n_0. Applying this homotopy on Δ^n and extending over the projection from Λ^n_0 to $\partial\Delta^n$, we can reduce to the case where Δ^n is trivial, and $\partial^n\Delta^n$ is trivial on Λ^n_0, i.e it looks like $(f, \cdot, \cdot, \cdots)$, where $\cdot$ denotes constant. $f \in \pi_{n-1}(X)$ is in the kernel of the map to $\pi_n(X)$. Since $\pi_{n-1}(X) \to \pi_{n-1}(Y)$ is injective, this means there is a homotopy relative to the boundary H from f to $\cdot$.

Since the map $X \to Y$ is injective on π_{n-1}, we have a homotopy trivializing x. Extending again to the whole diagram, we reduce to the case where $\partial\Delta^n$ is constant. But then surjectivity of π_n guarantees a lift. $\square$

The **fundamental groupoid** $\Pi_{\leq 1}(X)$ of a Kan complex X is its homotopy 1-category.

Proposition 3.8. *If X is right fibrant, there is a functor $x \mapsto \pi_n(X, x)$ from $\Pi_{\leq 1}(X) \to \mathrm{Grp}$.*

Proof. Let f be a path $x \to y$. Consider the composite $\partial\Delta^n \times \Delta^1 \to \Delta^1 \xrightarrow{f} X$. We can extend elements of $\pi_n(X)$ along this to get a map $\pi_n(X, x) \to \pi_n(X, y)$. By Corollary 2.30, this is well-defined, and only dependent on the homotopy class of f.

Moreover, by transporting composition maps, we see it respects composition, so is a homomorphism. $\square$

Corollary 3.9. *$\pi_n(X, x)$ isn't dependent on the basepoint.*

3.1. **Minimal fibrations.**

Definition 3.10. *A **minimal fibration** is a fibration for which fibrewise homotopies of Δ^n relative to $\partial\Delta^n$ are constant.*

Lemma 3.11. *If two degenerate simplices have the same boundary they are equal.*

Proof. Let x, y be the simplices with equal boundary, such that $x = s_m z$ and $y = s_n w$ with $m \leq n$. If $m = n$, then $z = d_m x = d_n y = w$, so x=y. If $m < n$, then $z = d_m x = d_m y = d_m s_n w = s_{n-1} d_m w$, so $x = s_m s_{n-1} d_m w = s_n s_m d_m w$ so $s_m d_m w = d_n x = d_n y = w$ and both $x, y = s_n w$. $\square$

Proposition 3.12. *Any fibration fibrewise deformation retracts onto a minimal fibration.*

Proof. Let $X \to Y$ be the Kan fibration. We will inductively define the i-skelaton Z_i as a subcomplex of X such that the inclusions $X_i \subset X$ have compatible fibrewise homotopies that retract onto Z_i, and such that $Z_i \to Y_i$ is a minimal fibration. Taking the limit, we will be done. Let Z_0 be a collection of points of X containing a unique point in every homotopy class of each fibre. We can produce a fibrewise homotopy H_0 from X_0 to Z_0 by choosing paths in the fibres to Z_0. For anything already in Z_0, choose the constant path.

Now suppose we have constructed Z_{i-1}, H_{i-1}. Choose a representative simplex for each equivalence class of i-simplex having boundary in Z_{i-1} via the equivalence relation fibrewise homotopy relative to boundary. Attach these to Z_{i-1} to obtain Z_i. By construction, Z_i has a unique simplex in each fibrewise homotopy class relative to boundary in dimension $\leq i$ and also in dimension $> i$ by Lemma 3.11, so it is a minimal fibration over Y_i. For any nondegenerate i-simplex x of X, H is a fibrewise homotopy on its boundary to Z_{i-1}, so we can extend it to a fibrewise homotopy on x, and if it is already in Z_i, we can again choose the constant homotopy. If it isn't already in Z_i, by construction it will end up fibrewise homotopic relative to the boundary to a cell in Z_i, via some homotopy H'. We can 'compose' the homotopies by creating a lift in the following diagram:

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_1) \arrow[rr, "{s_1H,(H',\cdot,H)}"] \arrow[d] & & X \arrow[d] \\
\Delta^n \times \Delta^2 \arrow[r, "\pi"] \arrow[urr, dashed] & \Delta^1 \arrow[r] & Y
\end{tikzcd}$$

By taking d_1 on the Δ^2 component of the homotopy, we get an extension of H to a fibrewise homotopy from X_i to Z_i. $\square$

Lemma 3.13. *A fibrewise homotopy equivalence of minimal fibrations is an isomorphism.*

Proof. if f is a fibre homotopy equivalence and g a homotopy inverse, it suffices so show that $g \circ f$ and $f \circ g$ are isomorphisms. But they are homotopic to the identity, so it suffices to show that any fibrewise map a to a minimal fibration that is homotopic via a homotopy h to an isomorphism b is also an isomorphism.

Let's first show that a is injective. Let α be an n-simplex such that $f(\alpha) = f(\beta)$. By induction, we can assume that $\partial\alpha = \partial\beta$. Thus h on the simplices restricts to a map on $\Delta^n \times \Lambda^2_2$, that can be extended to include $\partial\Delta^n \times \Delta^2$ via s_0h. Then extending this map to $\Delta^n \times \Delta^2$ via Corollary 2.29 and Proposition 2.22 and taking the d_2 component gives a fibrewise homotopy from α to β relative to the boundary, which has to be trivial.

For surjectivity, let α be an n-simplex. We can assume $\partial\alpha$ is in the image of f by induction. Then by the homotopy extension property we can extend h on $\partial\alpha$ to a fibrewise homotopy h_1 from a simplex z to α. $z = g(z')$ since g is surjective. So h_1 and $h|_{z'}$ are homotopies from z that agree on the boundary, so paste them together and extending to include $\partial\Delta^n \times \Delta^2$ via s_1h. Then again via Corollary 2.29 and Proposition 2.22 extend to $\Delta^n \times \Delta^2$ and take d_0 to get a fibrewise homotopy from α to whatever $g(z')$ is sent to by h. $\square$

Corollary 3.14. *A minimal fibration is trivial on each simplex.*

Proof. If $f : X \to Y$ is a minimal fibration, and $x = \Delta^n$ is a simplex in X, then there is a homotopy from the constant simplex on the zero vertex of x to the inclusion. The pullbacks of f along these two maps are then fibre homotopy equivalent, but the pullback along the constant map is a product. $\square$

3.2. **Equivalence of Quillen and Serre model structures.**

Theorem 3.15. *The realization of a Kan fibration $X \to Y$ is a Serre fibration.*

Proof. By Proposition 3.12, the projection $X \to Y$ factors through Z, where the projection $Z \to Y$ is locally trivial, hence a Serre fibration after realization. Thus it suffices to show $X \to Z$ is also a Serre fibration. To do this, we will show it is a trivial fibration, and that the the realization of a trivial fibration is a Serre fibration.

The realization of a trivial fibration $f : A \to B$ is a Serre fibration since we can find a lift in the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1_A, f"'] & A \arrow[d, "f"] \\ A \times B \arrow[r, "\pi"'] \arrow[ur, dashed] & B \end{tikzcd}$$

This shows that f is a retract of the projection $A \times B \to B$ which is a Serre fibration after realizing, so it is also a Serre fibration.

To see that $p : X \to Z$ is a trivial fibration, Note that it induces an isomorphism on homotopy groups since Z is a deformation retract of X. If we are trying to lift the inclusion $\partial\Delta^n \to \Delta^n = z$ over the projection $X \to Z$, observe that we have a fibrewise homotopy H from the composite $X \to Z \to X$ to the identity. We can extend this homotopy from the boundary of the simplex over Y to H', giving a simplex y in X, and a homotopy H' from z to y. $p(y)$ certainly agrees with z on the boundary, so if they are fibrewise homotopic relative to Y, they are equal by minimality.

pH' is a homotopy not relative to the boundary from z to $p(y)$, and pH restricted to y is a homotopy agreeing with pH' on the boundary but from $p(y)$ to itself. By pasting these homotopies together, we can 'compose' along the inclusion $(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_2) \subset \Delta^n \times \Delta^1$, where on $\partial\Delta^n \times \Delta^2$ we have used $s_0 pH$. Then the resulting homotopy from $p(y)$ to z on the d_2 edge will be relative to the boundary. $\square$

Proposition 3.16. *If X is a Kan complex, the homotopy groups $\pi_n(X)$ and $\pi_n(|X|)$ agree.*

Proof. There is clearly a natural map $\pi_n(X) \to \pi_n(|X|)$ that is compatible with the long exact sequence on homotopy groups. Observe that it is an isomorphism for π_0 since the realization of a simplex is connected. Now consider the fibre sequence $\Omega_x(X) \to P_x(X) \to X$. Since there is a contracting homotopy in $P_x(X)$, $|P_x(X)|$ is contractible. Then by induction using the long exact sequence to dimension shift, π_i is an isomorphism for all i. $\square$

It is also true by definition essentially that $\pi_n(X) = \pi_n(SX)$.

Definition 3.17. *A map in* CGHaus *is a* ***Serre fibration*** *if it has the right lifting property with respect to all inclusions $\Lambda^n_i \subset \Delta^n$. It is a* ***Serre cofibration*** *if it has the left lifting property with respect to all Serre fibrations.*

A map is a weak equivalence in either CGHaus or Kan complexes if it induces an isomorphims on π_n.

Lemma 3.18. *A Serre fibration is a weak equivalence iff it has the right lifting property with respect to $|\partial\Delta^n| \subset |\Delta^n$.*

Proof. This has the same proof as Proposition 3.7. $\square$

Lemma 3.19. *The realization of a cofibration is Serre cofibrant, and the singular set of a Serre fibration is a Kan fibration.*

Proof. This first statement follows from Lemma 3.18 and Lemma 2.17. The second is clear from definition. $\square$

Proposition 3.20. *If X is a Kan complex, $X \to S|X|$ is a weak equivalence to a cofibrant fibrant object and if $X \in \mathrm{CGHaus}$, $|SX| \to X$ is a weak equivalence from a cofibrant fibrant object.*

Proof. This essentially follows from Proposition 3.16. $\square$

The following lemma about Kan fibrations is useful.

Lemma 3.21. *Suppose given $X \xrightarrow{f} Y \xrightarrow{g} Z$ that $q \circ p, p$ are Kan fibrations and p is surjective. Then q is a Kan fibration.*

Proof. We would like to be able to lift extensions of a homotopy of a boundary of a simplex to the whole simplex. To do this, by surjectivity of p, choose a lift of the simplex to X. Then, we can lift the homotopy on the boundary of the simplex to X since p is a Kan fibration. Then since $q \circ p$ is a Kan fibration, we can lift the homotopy on the simplex to X, which projects to the desired homotopy on Y. $\square$

4. Model Categories

Model categories are a 1-categorical way to present a homotopy theory (∞-category) in a computable way.

Definition 4.1. *A* ***weak factorization system*** *in a category C is pair of classes of morphisms (L, R) such that every morphism factors into $g \circ f$ where $g \in R, f \in L$, and $R = L^{\oslash}$, $L = {}^{\oslash}R$.*

It follows that L is saturated, and R is cosaturated. $(\to, \cong), (\cong, \to)$ are always trivial weak factorization systems. There is a weakening of the axioms for weak factorization systems that is equivalent.

Lemma 4.2. *Two classes L, R form a weak factorization system iff they satisfy*

- *(W1) L, R are closed under retracts*
- *(W2) L, R have the left/right lifting properties with respect to each other*
- *(W3) Any morphism factors as $g \circ f$ where $g \in L$, $f \in R$.*

Proof. Clearly these are implied by the definition of a weak factorization system. Conversely, it will suffice by duality to show that if a morphism f has the left lifting property with respect to all morphisms in R, it is in L. But we can factor $f = g \circ h$ with $h \in L, g \in R$. But finding a lift in the diagram below shows f is a retract of h.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

$\square$

Definition 4.3. *A* ***homotopical*** *category is a category with a subcategory W (weak equivalences) containing all objects and satisfying 2 out of 6.*

It follows that W contains all isomorphisms. There are two trivial homotopical structures, the minimal one, which is just isomorphisms, and the maximal one, which contains all morphisms. I think the minimal one is only useful for receiving maps (derived functors), and the maximal one is useful because its homotopy category is essentially the homotopy type of the nerve. A **homotopical functor** between homotopical categories preserves weak equivalences. If D has the minimal homotopical structure, homotopical functors $C \to D$ are the same as functors $h(C) \to D$.

The **homotopy (∞-)category** of a homotopical category is its localization under weak equivalences. In otherwords, it is the universal ∞-category receiving a map from it that such that the weak equivalences are sent to equivalences. The homotopy 1-category is the 1-coskeleton of this, given by 1-categorical localization.

Definition 4.4. *A homotopical category is* ***saturated*** *if W is exactly the class of morphisms that are inverted in its homotopy 1-category.*

Definition 4.5. *A* ***(closed) model category*** *C is a category equipped with three classes of arrows:* ***cofibrations*** *($\rightarrowtail$),* ***fibrations*** *($\twoheadrightarrow$) and* ***weak equivalences*** *($\xrightarrow{\sim}$) satisfying the following axioms:*

(CM1) C has all finite colimits and colimits

(CM2) Weak equivalences satisfy 2 out of 3

(CM3) $(\overset{\sim}{\rightarrowtail}, \twoheadrightarrow), (\rightarrowtail, \overset{\sim}{\twoheadrightarrow})$ form weak factorization systems.

Note that the definition is self-dual. Observe that for a model category, any pair of classes cofibrations, fibrations, weak equivalences determines the third.

Lemma 4.6. *Given an adjunction $F : C \leftrightarrows D : G$ between homotopical categories C, D where F, G are homotopical, there is an induced adjunction $F : hC \leftrightarrows hD : hG$.*

Proof. The data of an adjunction is the same as giving the counit and unit maps that satisfy some identities. Since F, G are homotopical, they descent to functors on hC and hD, and so do the unit and counits, and the diagrams still commute. $\square$

A main step in producing model categories is in producing factorization systems. The following, Quillen's small object argument, produces functorial factorizations under general conditions from a starting class of morphisms. In particular, if a category is presentable, the conditions are satisfied.

The following definition is slightly nonstandard.

Definition 4.7. *Let κ be a regular cardinal. $x \in C$ is κ-small if the functor it corepresents preserves α-composites for any $\alpha \geq \kappa$.*

Proposition 4.8 (Small object argument). *Let J be a set of maps in C, and suppose C is cocomplete. If the codomains of elements of J are κ-small, then there is a functorial factorization making $({}^{\oslash}(J^{\oslash}), J^{\oslash})$ into the smallest weak factorization system containing J. Moreover, ${}^{\oslash}(J^{\oslash})$ is the weak saturation of J, or alternatively retracts of morphisms coming from this construction.*

Proof. Let $f : X \to Y$ be our map and let A_f be the set of all commutative squares S of the form
$$\begin{tikzcd} S_A \arrow[r, "S_x"] \arrow[d, "S_\alpha"] & X \arrow[d] \\ S_B \arrow[r, "S_y"] & Y \end{tikzcd}$$
where $S_\alpha \in J$. Now consider the pushout diagram:

$$\begin{tikzcd} \coprod_{S\in A_f} S_A \arrow[r, "\coprod S_x"] \arrow[d, "\coprod S_\alpha"] & X \arrow[d] \\ \coprod_{S\in A_f} S_B \arrow[r] & X_1 \end{tikzcd}$$

There is a canonical projection $f_1 : X_1 \to Y$ given by the S_y and f, and the construction of X_1 is functorial. Moreover, $X \to X_1$ is in ${}^{\oslash}(J^{\oslash})$, and for any diagram in A, there is a canonical lift on X_1. Now we can inductively define f_α, X_α for any limit ordinal. Namely, $X_{\alpha+1}, f_{\alpha+1} = (X_\alpha)_1, (f_\alpha)_1$. For a limit ordinal, define it to be the limit. of the X_i, f_i before it. Then $X \to X_\kappa$ is in ${}^{\oslash}(J^{\oslash})$ as it is a transfinite composite of such things, and $f_\kappa : X_\kappa \to Y$ is in $J^{\oslash}$ since the codomains are κ-small.

This verifies factorization in a functorial way. The lifting property comes from Lemma 2.11, so this is a weak factorization system, and it is clearly the minimal one containing J on the left. We have only used the operations in the weak saturation of J to construct the factorization, so by Lemma 4.2, ${}^{\oslash}(J^{\oslash})$ is the weak saturation of J. ☐

Definition 4.9. *A model category is* ***cofibrantly generated*** *if its factorization systems are generated (on the left) by a set of morphisms.*

The factorization systems of a category C form a poset, where $(L, R) \le (L', R')$ iff $L \subset L'$.

Question 4.9.1. What are all weak factorization systems in which every arrow is in either class?

Example 4.9.1. We can easily classify nontrivial weak factorization systems (L, R) in Set. Let $\hookrightarrow', \twoheadrightarrow, \phi$ denote the classes of injections from a nonempty set, surjections, and maps from the empty set respectively, where we additionally include isomorphisms.

The first observation is that for any nontrivial map from the empty set f, ${}^{\oslash}f = f^{\oslash} = \hookrightarrow' \cup \twoheadrightarrow$. So the model structures on Set are just model structures on nonempty sets (temporarily called C) with ϕ attached to either L or R. So from now on we work in C.

Suppose $f \in L$ that isn't injective. Then the map $\cdot\cdot \to \cdot$ is a pushout of f. The maps with the right lifting property with respect to this are the injections. The maps with the left lifting property with respect to injections are the surjections, giving $(\twoheadrightarrow, \hookrightarrow')$. .

If in addition there is a nonempty nontrivial injection in L, then R has to be just isomorphism.

If L has instead a map which isn't surjective, R has all surjections, and we see L has to be injection, giving $(\hookrightarrow', \twoheadrightarrow)$. Any additional maps in L would again make it everything.

Example 4.9.2. Consider a poset P as a category. $f \oslash g$ iff $\operatorname{cod} f \leq \operatorname{dom} g$ whenever $f \leq g$. We can obtain a large family of weak factorization systems $(\rightarrowtail, \twoheadrightarrow)$ by choosing a family of arrows A such that $a \leq b, a \in A \implies b \in A$, and declaring all arrows of A along with isomorphisms to be $\rightarrowtail$, and all other nonidentity arrows along with isomorphisms to be $\twoheadrightarrow$.

For example, on $\Delta^1 \times \Delta^1$, the only weak factorization system (up to automorphisms) not coming from this construction is

$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d, two heads] \arrow[dr] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

$\Delta^1 \times \Delta^1$ can be used to construct an example of factorization systems $(\rightarrowtail, \overset{\sim}{\twoheadrightarrow}) \leq (\overset{\sim}{\rightarrowtail}, \twoheadrightarrow)$ that don't come from a model structure. In particular, consider the diagram below:

$$\begin{tikzcd} \cdot \arrow[r, tail, two heads] \arrow[d, two heads, "\wr"'] \arrow[dr] & \cdot \arrow[d, two heads, "\wr"'] \\ \cdot \arrow[r, tail, "\sim"] & \cdot \end{tikzcd}$$

Another example comes from the poset Δ^2, where we use the following pair of weak factorization systems:

$$\begin{tikzcd} & \cdot \arrow[d, tail, "\wr"'] \\ \cdot \arrow[ur, tail, two heads] \arrow[r, tail, "\sim"'] & \cdot \end{tikzcd}$$

These don't come from model structures because the weak equivalences wouldn't satisfy 2 out of 3.

The last part of Example 4.9.2 shows two obstructions to two factorization systems $(\rightarrowtail, \overset{\sim}{\twoheadrightarrow}) \leq (\overset{\sim}{\rightarrowtail}, \twoheadrightarrow)$ giving rise to a model structure. These are the only obstructions, as shown below.

Proposition 4.10. *Let C have finite limits and colimits. $(\rightarrowtail, \overset{\sim}{\twoheadrightarrow}) \leq (\overset{\sim}{\rightarrowtail}, \twoheadrightarrow)$ are the factorization systems of a model structure iff $\overset{\sim}{\twoheadrightarrow} \cup \overset{\sim}{\rightarrowtail}$ satisfies 3 out of 4 with respect to $\twoheadrightarrow \cup \rightarrowtail$, and $\overset{\sim}{\twoheadrightarrow}, \overset{\sim}{\rightarrowtail}$ satisfy 2 out of 3 with respect to $\twoheadrightarrow, \rightarrowtail$ respectively.*

Proof. Those conditions are necessary for weak equivalences to satisfy 2 out of 3. In the proof of sufficiency, we declare an arrow f to be a weak equivalence if it factors as $h \circ f$, where h is a trivial fibration and f a triival cofibration. Because of 3 out of 4 and the fact that we have a factorization system, this is equivalent to f factoring as a composite of maps that are either trivial fibrations or trivial cofibrations.

First we observe that the trivial fibrations are the fibrations that are weak equivalences and dually for cofibrations. Indeed, a trivial fibration by assumption is a fibration, and the trivial factorization shows it is a weak equivalence. If f is a trivial fibration and a week equivalence, then factoring it as a trivial fibration and cofibration and lifting as in the diagram below shows that it is a retract of a trivial fibration.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, tail, "\wr"'] & \cdot \arrow[d, two heads, "f"] \\ \cdot \arrow[r, two heads, "\sim"] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

It remains then to show weak equivalences satisfy 2 out of 3.

First we can observe that any composite of weak equivalences is a weak equivalence because it can be factored a composite of trivial fibrations and cofibrations. Now suppose $h = g \circ f$ where f, h are weak equivalences. We can factor g into a fibration g' and trivial cofibration, which we can absorb into f. Then factor f, h as $f' \circ f'', h' \circ h''$, trivial cofibrations followed by a trivial fibration. Because of 2 out of 3, it suffices to show that $g \circ f'$ is a trivial fibration. But this follows from the 3 out of 4 property on the square $(g \circ f') \circ f'' = h' \circ h''$. The other part of 2 out or 3 is dual. □

Note that such a model structure is necessarily unique. In practice, we are often supplied a notion of weak equivalence satisfying two out of three. This makes it a bit easier to check something is a model category. Here are two ways in which that can be realized.

Lemma 4.11. *Let C have finite limits and colimits. Suppose we have two factorization systems $(\rightarrowtail, \overset{\sim}{\twoheadrightarrow}) \leq (\overset{\sim}{\rightarrowtail}, \twoheadrightarrow)$ and a notion $\overset{\sim}{\to}$ of weak equivalence satisfying 2 out of 3 and such that $\overset{\sim}{\twoheadrightarrow} = \overset{\sim}{\to} \cap \twoheadrightarrow$ and $\overset{\sim}{\rightarrowtail} \subset \overset{\sim}{\to}$. Then this data gives a model structure.*

Proof. It suffices to check that $\overset{\sim}{\to} \cap \rightarrowtail \subset \overset{\sim}{\rightarrowtail}$. But we can factor a weak equivalence as a triival cofibration and trivial fibration, and if it is also a cofibration lift the square of this factorization to see that it is a a retract of a trivial cofibration. □

Proposition 4.12. *Let $F : C \leftrightarrows D : U$ be an adjunction, suppose that D is bicomplete, C has a model structure cofibrantly generated by κ-small objects that are sent to κ'-small objects in D. We can try to define a model structure by having a map in D be a fibration or weak equivalence if it is after applying U. Suppose that if $f \in D$ has the left lifting property with respect to g such that Ug is a fibration, then Uf is a weak equivalence. Then we get a cofibrantly generated model structure generated by Fi where i are the generators for C.*

Proof. The condition on fibrations and weak equivalences clearly determines the model structure if it exists. We can construct our factorizations via Proposition 4.8 since the generators of the factorization systems for D are F applied to the ones for C by the adjunction, and these are sent to κ-small objects. 2 out of 3 is clear so by Lemma 4.11 and the last assumption, we are done. □

Example 4.12.1. The **Serre model structure** on the category CGHaus has the cofibrations generated by $|\Lambda^n_i| \subset |\Delta^n|$ and the trivial cofibrations are generated by $|\partial_n \Delta^n| \subset |\Delta^n|$, and a weak equivalence to be a weak homotopy equivalence. $(CM1), (CM2)$ are clearly satisfied. By Lemma 3.18, a fibration is a trivial fibration iff it is a weak equivalence. Moreover, trivial cofibration of the construction in Proposition 4.8 are weak equivalences since they are the inclusion of a deformation retract. Thus any trivial cofibration is since it is a retract of the construction. By Lemma 4.11 we are done.

Example 4.12.2. The **Quillen model structure** on $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ has cofibrations generated by $\Lambda^n_i \subset \Delta^n$, and trivial cofibrations generated by $\partial\Delta^n \subset \Delta^n$. From the Proposition 4.8, these give factorization systems where the cofibrations are anodyne extensions, and the trivial cofibrations are inclusions by Lemma 2.17. A map $X \to Y$ is a weak equivalence if it induces an isomorphism on π_n after passing to fibrant replacements. This agrees with the usual notion of weak equivalence for fibrant things, and $(CM1), (CM2)$ are satisfied. Now given a Kan fibration, by Proposition 3.7 it is an isomorphism on homotopy groups iff it is trivial. By Proposition 3.16, $X \to Y$ is a weak equivalence iff $|X| \to |Y|$ is in the Serre model structure. Moreover, the realization of a cofibration is a cofibration, so since $|\Lambda^n_i| \subset |\Delta^n|$ is a weak equivalence, and the set of cofibrations such that $|X| \to |Y|$ is an equivalence is weakly saturated, all trivial cofibrations are weak equivalences. Thus by Lemma 4.11 we are done.

Lemma 4.13. *Suppose $f \oslash g$ in a category C. Then for any functor $D \to C$ $\tilde{f} \oslash \tilde{g}$ in $C_{/D}$ if $\tilde{f}$ is a lift of f and $\tilde{g}$ is a lift of g.*

Proof. Any solution of the lifting problem in C will actually be a solution in C/D. ☐

Lemma 4.14. *Given a factorization system $(\rightarrowtail, \twoheadrightarrow)$ in a category C and a functor $D \to C$, the morphisms in $C_{/D}$ projecting to this factorization system form a factorization system.*

Proof. This is a consequence of Lemma 4.13 and the fact that any factorzation will automatically becoma a relative factorization. ☐

Proposition 4.15. *Given a model category C and a functor $D \to C$, the category $C_{/D}$ has a model structure where the forgetful functor $C_{/D} \to C$ preserves weak equivalences, cofibrations, and fibrations.*

Proof. Finite limits and colimits exist in $C_{/D}$, and weak equivalences satisfy 2 out of 3. By Lemma 4.14, we can lift the factorization systems, so we are done. ☐

4.1. **Homotopy category of a model category.** The model structure gives a handle on the associated homotopy category. For example it is possible to construct a mapping space from cofibrant objects to fibrant objects that agrees with the mapping space in the ∞-category. We will start by understanding the homotopy 1-category, denoted $h(C)$. If A is an object of C, define a **cylinder object** of A, denoted $\Delta^1 A$ (resembling $\Delta^1 \times A$) to be a factorization

$$\begin{tikzcd} A \cup A \arrow[d, tail, "i"] \arrow[dr, "\nabla"] & \\ \Delta^1 A \arrow[r, "\sim"] & A \end{tikzcd}$$

Where ∇ is the fold map. The dual notion is a **path object** of B, denoted $B\Delta^1$ (resembling B^{Δ^1}). Cylinder objects always exist, and furthermore we can always make the weak equivalence also a fibration.

Definition 4.16. *Given a cylinder object, a **left homotopy** from f to g is a commutative diagram:*

$$\begin{tikzcd} A \cup A \arrow[d, tail, "i"] \arrow[dr, "{f,g}"] & \\ \Delta^1 A \arrow[r] & B \end{tikzcd}$$

f, g are **left homotopic** if there is a left homotopy with respect to some cylinder. If f, g are left homotopic via some cylinder, hf, hg are too. There is also a dual notion of right homotopy.

Lemma 4.17. *Let A be cofibrant. Then the components of i are trivial cofibrations, and left homotopy is an equivalence relation.*

Proof. By pushing out the initial maps to the two factors A, we see that each of the inclusions into $A \cup A$ is a cofibration, so the components of i are as well. Moreover, since the map $\Delta^1 A \to A$ is a weak equivalence, by the 2 out of 3 property, so are the components of the inclusion.

Reflexivity of left homotopy comes from the canonical map $\Delta^1 A \to A$. Symmetry comes from swapping the factors. Transitivity follows by observing that we can pushout two left homotopies and two cylinders along a common map to get another homotopy and cylinder. We need A to be cofibrant so that the pushout is also a cylinder. □

Proposition 4.18. *TFAE when A is cofibrant, B is fibrant, f, g are maps $A \to B$:*

(1) f, g are left homotopic
(2) f, g are right homotopic
(3) f, g are left homotopic with respect to a fixed cylinder object.
(4) f, g are right homotopic with respect to a fixed cylinder object.

Proof. It will suffice to show by duality that if we have a right homotopy, and a cylinder object $\Delta^1 A$ for A, then f, g are left homotopic with respect to $\Delta^1 A$. Let p_0, p_1 be the components of the projection $B\Delta^1 \to B \times B$, let $\sigma : B \to B\Delta^1$ be the equivalence coming from the path structure, and let h be the right homotopy. Then the map $p_1\gamma$ in the diagram below is a left homotopy.

$$\begin{tikzcd}
A \cup A \arrow[rr, "\sigma f, h"] \arrow[d, tail, "i"] & & B\Delta^1 \arrow[d, two heads, "p_0"] \\
\Delta^1 A \arrow[r] \arrow[urr, dashed, "\gamma"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

□

We will call a map $\gamma : \Delta^1 A \to B\Delta^1$ such that the diagram in Proposition 4.18 commutes a **correspondence** between the left and right homotopies.

Lemma 4.19 (Whitehead)**.** *If $f : X \to Y$ is a weak equivalence between cofibrant fibrant objects, it is a homotopy equivalence.*

Proof. Since homotopy equivalences are closed under composition and we can factor a weak equivalence into a trivial fibration and cofibration, it suffices by duality to show it when f is a trivial cofibration. If $\cdot$ denotes the terminal object, let g denote a lift of the diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, tail, "\wr"'] & X \arrow[d, two heads] \\
Y \arrow[r] \arrow[ur, dashed, "g"] & \cdot
\end{tikzcd}$$

By construction it is a left inverse so it suffices to show $f \circ g$ is homotopic to the identity. This can be achieved by finding a lift in the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\sigma f"] \arrow[d, tail, "\wr"'] & Y\Delta^1 \arrow[d, two heads, "p"] \\
Y \arrow[r, "{fg,1}"'] \arrow[ur, dashed, "g"] & Y \times Y
\end{tikzcd}$$

□

Whitehead's theorem has a converse.

Theorem 4.20. *The homotopy (1-)category of a model category C is given by homotopy class of maps between cofibrant fibrant objects. Moreover, isomorphism in the homotopy category is the same as weak equivalence, so model categories are saturated homotopical categories.*

Proof. It is ok to just think about the full essentially surjective subcategory of the homotopy category consisting of cofibrant fibrant objects. Now suppose we have a functor $F : C \to D$ inverting weak equivalences. Since the two inclusions into a cylinder are equal in the homotopy 1-category, F factors through the category obtained by identifying homotopy classes of maps. By Lemma 4.19, weak equivalences are already isomorphism in this category, so the factorization is unique.

Now suppose that a map $f : X \to Y$ is an isomorphism in the homotopy category. After passing to cofibrant fibrant replacements using the 2 out of 3 property, we can assume that our objects are cofibrant fibrant, so that f has a homotopy inverse g. Moreover, by factoring f into a trivial cofibration and fibration and using Lemma 4.19, we can assume f is a fibration. The strategy is to show that f has the right lifting property with respect to cofibrations. Let $\Delta^1 X, \widetilde{Y}$ be path objects for X, Y, k a right homotopy from gf to the identity, and form the diagram below given a map from a cofibration i to f given by maps α, β:

$$\begin{tikzcd}
A \arrow[r, "k\alpha"] \arrow[d, tail, "i"] & X\Delta^1 \arrow[rr, "p_1"] \arrow[d, "{p_0,?}"] & & X \arrow[d, two heads, "f"] \\
B \arrow[r, "{g\beta,\sigma\beta}"] \arrow[ur, dashed] & X \times_Y Y\Delta^1 \arrow[r, "\pi_2"] & Y\Delta^1 \arrow[r, "p_1"] & Y
\end{tikzcd}$$

If we could fill in the map ? in a way that $(p_0, ?)$ is a fibration and the diagram commuted, it would have to be a trivial fibration by the 2 out of 3 property for the composition with $X \times_{\pi_2} Y \to X$, so we could make the indicated lift and be done.

To make ? we will choose a particularly nice right homotopy from gf to the identity by slightly modifying the dual construction of Proposition 4.18.

Let $h : \Delta^1 X \to X$ be a left homotopy from gf to the identity, and s the map $\Delta^1 X \to X$. Choose a path object $Y\Delta^1$ for Y. We can make a path object $X\Delta^1$ for X with a compatible fibration $\hat{f} = ?$ to $Y\Delta^1$ by factoring the map $(\Delta, \sigma_Y f) : X \to (X \times X) \times_{Y \times Y} Y\Delta^1$ as a trivial cofibration and fibration. Then create a lift Q in the diagram below.

$$\begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, tail, "i_1"] & X\Delta^1 \arrow[d, two heads, "\pi"] \\
\Delta^1 X \arrow[r, "{((h,s),\sigma Y f s)}"'] \arrow[ur, dashed, "Q"] & (X \times X) \times_{Y \times Y} Y\Delta^1
\end{tikzcd}$$

$k = i_1 Q$ is a right homotopy from gf to the identity with the extra property that $\hat{f}k = \sigma_Y f$. This shows that the strategy works. $\square$

To compute homotopy classes of maps, we don't need both X, Y to be cofibrant fibrant.

Proposition 4.21. *Let X be cofibrant and Y be fibrant objects of a model category C. Then $\mathrm{Hom}_{hC}(X,Y)$ is the homotopy classes of maps between X and Y.*

Proof. Let X be cofibrant, and Y be fibrant. We can choose a cofibrant replacement Y' with a trivial fibration to Y. The induced map on homotopy classes of maps from X is surjective since X is cofibrant. By looking at the map $X \cup X \to \Delta^1 X$ in $C_{X\cup X/}$, it is injective on homotopy classes. Making the dual argument we can reduce to the case when X, Y are cofibrant and fibrant, where we can use Theorem 4.20. $\square$

We now briefly examine the higher categorical structure on a model category. Let X be a semi-simplicial set. We will inductively define a semi-simplicial set of cylinders for A, denoted $\mathrm{Cyl}(A)$, where a map of a X-cylinder object for A is denoted XA. A **Δ^0-cylinder** is the data below.

$$\begin{tikzcd} A \arrow[d, tail, "i"] \arrow[dr, equal] & \\ \Delta^0 A \arrow[r, "\sim"] & A \end{tikzcd}$$

Suppose we have defined the Δ^{n-1}-cylinders. Then a Δ^n-cylinder will be a choice of $\partial\Delta^n$-cylinder and a diagram as below.

$$\begin{tikzcd} \partial\Delta^n A \arrow[d, tail, "i"] \arrow[dr, equal] & \\ \Delta^n A \arrow[r, "\sim"] & A \end{tikzcd}$$

Its boundary maps will be the ones coming from $\partial\Delta^n A$. Using $\mathrm{Cyl}(A)$, we can define a semi-simplicial mapping space $\mathrm{Map}_{\mathrm{Cyl}}(A,B)$ where the n-simplices are a choice of $\Delta^n A$ and a map $\Delta^n A \to B$. There is a dual notion of $\mathrm{Path}(B)$ and $\mathrm{Map}_{\mathrm{Path}}(A,B)$. We can also force these to be simplicial sets by Kan extension.

Question 4.21.1. Is $\mathrm{Map}_{\mathrm{Cyl}}(A,B)$ a semi-simplicial Kan complex? What about its simplicialization? Can one canonically identify $\mathrm{Map}_{\mathrm{Cyl}}(A,B)$ with $\mathrm{LMap}(A,B)$ in the homotopy ∞-category (even with no conditions on A, B)?

In any case, $\mathrm{Cyl}(A)$ has a very nice property that cells can be reordered. For example, there is an involution on the 1-simplices given by swapping. This suggests it's even better than a Kan complex. Maybe a semi-simplicial structure is not the right thing but rather some other test category such as a globe or cube category is better. The following lemma suggests as much.

Lemma 4.22. *Given a left homotopy $h : \Delta^1 X \to Y$ from α to β with structure map $s : \Delta^1 X \to X$, the composite homotopy $h^{-1} \circ h : \overline{\Delta^1} X \to X$ is homotopic to the constant homotopy on α via a map H, i.e there is a diagram as below:*

$$\begin{tikzcd}
\overline{\Delta^1}X \cup_{X\cup X} \overline{\Delta^1}X \arrow[r, "{s,s}"] \arrow[d, "{h^{-1}\circ h,\alpha s}"'] \arrow[dr, tail, "{j_1,j_2}"] & X \\
Y & X' \arrow[u, two heads, "\pi"', "\wr"] \arrow[l, dashed, "H"]
\end{tikzcd}$$

Proof. Choose $Y\Delta^1$ with structure map σ, and a correspondence γ between h and a right homotopy k on $Y\Delta^1$. γ glues on $\overline{\Delta^1}X \cup_{X\cup X} \overline{\Delta^1}X$ to give a map $\overline{\gamma}$, so we can lift the diagram

$$\begin{tikzcd}
\overline{\Delta^1}X \cup_{X\cup X} \overline{\Delta^1}X \arrow[r, "{\overline{\gamma},\sigma\alpha s}"] \arrow[d, tail, "{j_1,j_2}"'] & Y\Delta^1 \arrow[d, two heads, "p_0", "\wr"'] \\
X' \arrow[r, "\alpha\pi"] \arrow[ur, dashed, "K"] & Y
\end{tikzcd}$$

and the map $p_1K = H$ works. ◻

Lemma 4.23. *Suppose X, Y are cofibrant fibrant, and $f : X \to Y$ is a trivial fibration. Then f admits a section g, and for any section there is a homotopy $H : \Delta^1 X \to X$ from gf to the identity such that fH is the constant homotopy of f (i.e a fibrewise homotopy).*

Proof. f admits a section g by lifting the identity along f, and since this is an inverse in the homotopy category, there is a left homotopy $h : \Delta^1 X \to X$ from the identity to gf. Let h^{-1} denote tha homotopy in the other direction obtained from swapping factors. gfh^{-1} is a homotopy from gf to itself, and let $gfh^{-1}\circ h : \overline{\Delta^1}X \to X$ be the composite of the homotopies gfh^{-1} and h.

By Lemma 4.22 $f(gfh^{-1}\circ h) = fh^{-1}\circ fh$ is homotopic to the identity via some homotopy $H : X' \to Y$, where i_L, i_R are the inclusions $\overline{\Delta^1}X \to X'$, we can create a lift K as below so that Ki_R will be the desired fibrewise homotopy.

$$\begin{tikzcd}
\overline{\Delta^1}X \arrow[r, "gfh^{-1}\circ h"] \arrow[d, tail, "i_L"] & X \arrow[d, two heads, "f"] \\
X' \arrow[r, "H"] \arrow[ur, dashed, "K"] & Y
\end{tikzcd}$$

◻

Proposition 4.24. *Let $F : C \leftrightharpoons D : U$ be an adjunction, suppose that D is bicomplete, C has a model structure cofibrantly generated by κ-small objects that are sent to κ'-small objects in D. We can try to define a model structure by having a map in D be a fibration or weak equivalence if it is after applying U. If D has natural path objects P and a natural fibrant replacement Q, we get a cofibrantly generated model structure generated by Fi where i are the generators for C.*

Proof. By Proposition 4.12, it suffices to show if $f \in D$ has the left lifting property with respect to g such that Uf is a fibration, then Uf is a weak equivalence. Now produce a lift u of the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "f"] & QA \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dashed, "u"] & \cdot
\end{tikzcd}$$

Then construct a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, tail, "f"] & PQA \arrow[r] & PQB \arrow[d, two heads] \\ B \arrow[r] \arrow[urr, dashed, "H"] & QB \times QA \arrow[r] & QB \times QB \end{tikzcd}$$

, so that in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "f"] & QA \arrow[d, "Qf"] \\ B \arrow[r] \arrow[ur, dashed, "u"] & QB \end{tikzcd}$$

The upper triangle commutes and the lower triangle does up to right homotopy. Applying U to this square, which preserves right homotopy, we see that $UA \to UB$ is an isomorphism in the homotopy 1-category of C. $\square$

Remark 4.24.1. There is an alternate hypothesis that will make the proof of Proposition 4.24 go through, but seems not as good. Namely, we can assume U preserves κ-sequential colimits instead of the assumption about the generators being sent to small objects. Then we can use a variant of Proposition 4.8 to produce the factorization system. Namely, after countably many steps in the small object argument, since U preserves κ-sequential colimits, we will be able to lift the generating cofibrations in C.

In a model category C, under good conditions, triangles in hC can be lifted to C.

Proposition 4.25 (Lifting from the homotopy category)**.** *Let C be a model category. Suppose we have a cofibration $i : A \to B$, a morphism $f : B \to C$ and a commutative triangle*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "{[f]}"] & B \arrow[d, "\bar{g}"] \\ & C \end{tikzcd}$$

in hC. Assume moreover that A, B are cofibrant and C fibrant. Then there is a morphism g such that $g \circ i = f$ and $[g] = \bar{g}$.

Proof. Choose a homotopy $A \cup A \to \Delta^1 A \to C$ between $g'f$ and i where g' is any map whose homotopy class is $\bar{g}$. We can choose a compatible cylinder $\Delta^1 A \coprod_{A\cup A}(B \cup B) \to \Delta^1 B \to B$ by factoring the natural map to B as a cofibration and a trivial fibration.

The homotopy naturally extends to $\Delta^1 A \coprod_A B \to C$, and $\Delta^1 A \coprod_A B \to \Delta^1 B$ is a trivial cofibration, so since X is fibrant, it extends to $\Delta^1 B$, giving a homotopy between g' and a g solving the problem. $\square$

Weak equivalences are not generally preserved under pushouts. Here is a condition under which they are:

Lemma 4.26. *Let C be a model category with a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "j'"] \\ A' \arrow[r, "i'"] \arrow[ur, phantom, "\ulcorner"] & B' \end{tikzcd}$$

Suppose that A, A' are cofibrant, i is a cofibration, and j is a weak equivalence. Then j' is a weak equivalence.

Proof. We will show that j' is an equivalence in the homotopy category. Let X be an object, and consider the map $\mathrm{Hom}_{hC}(B', X) \to \mathrm{Hom}_{hC}(B, X)$. To see it is surjective, observe that given a map $f : B \to X$, we can homotope its restriction to A to something restricted from A'. By Proposition 4.25, the original map can be homotoped to exhibit f as in the image.

To see injectivity, if s, s' are two identified maps, then there is a commutative square

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "j'"] & X\Delta^1 \arrow[d] \\ B' \arrow[r, "s\times s''"] & X \times X \end{tikzcd}$$

By the surjectivity statement in the category $C_{/X\times X}$, there is a lift in this square, giving injectivity. $\square$

Definition 4.27. *A model category is* ***left proper*** *if weak equivalneces are closed under pushouts by a cofibration.*

Corollary 4.28. *A model category C where all objects are cofibrant is left proper.*

Proof. This follows immediately from Lemma 4.26. $\square$

We will now examine the notaion of a homotopy pushout square. It is a special case of homotopy colimits, but deserves special attention.

Definition 4.29. *A* ***homotopy pushout square*** *is a commutative square that receives a level-wise equivalence from one of the form*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & D \end{tikzcd}$$

where A is cofibrant. Such a diagram is a ***cofibrant pushout square***

The pushout of a homotopy pushout square is given by taking the pushout of an equivalent cofibrant pushout square.

Proposition 4.30. *In a model category C, the pushout of a homotopy pushout square is well defined up to weak equivalence.*

Proof. Suppose that there are two cofibrant pushout diagrams with a weak equivalence to a homotopy pushout square. We can find an equivalent cofibrant pushout diagram mapping to each of them in a compatible way as follows: Take something mapping compatibly into both of the top left objects of the pushout squares, replace it with an equivalence from a cofibrant object X. The same method in $C_{/X}$ yields an entire cofibrant pushout square mapping to each.

Thus it suffices to show that for any two cofibrant pushout squares, the pushout is an equivalence. Using Lemma 4.26, we can reduce to the case that we are only changing the object in the upper left corner: if we change only one of the other objects by an equivalence, that Lemma along with gluing for pushout squares implies the result.

Furthermore, we can factor the induced map on the upper left vertex into a trivial cofibration and trivial fibration. In the case of a trivial fibration, there is a section since the objects are cofibrant, so picking a section, we there is another pushout square between the diagonals of the pushout squares, and we are done using Lemma 4.26 again. Thus we can assume that the map in the upper left hand corner is a trivial cofibration, in which case it has a retraction. This induces a retraction on the pushout, showing that the pullback map is surjective on homotopy classes of maps to any space. By again mapping to $X\Delta^1$ relative to $X \times X$, we see it is injective as well on homotopy classes, so the map on pushouts is an equivalence. □

We can often detect homotopy pushout squares in non-cofibrant diagrams

Proposition 4.31. *A coCartesian square in a model category C*

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & B \arrow[d] \\ E \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a homotopy pushout square if either of the following conditions are satisfied:

- *A, B are cofibrant.*
- *C is left proper.*

Proof. Take a cofibrant replacement of the pushout square, and observe that it induces an equivalence on the pushout by either using Lemma 4.26 under the first hypothesis, or the fact that C is left proper. □

Remark 4.31.1. In fact left proper model categories are characterized as those where pushouts along cofibrations are homotopy pushouts.

As an example, we can prove the Mayer-Vietoris sequence on homotopy for a homotopy pullback square.

Proposition 4.32. *Given a homotopy pullback square of simplicial sets*

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

Given choices of basepoints, there is a natural Mayer-Vietoris long exact sequence of homotopy groups

$$\pi_n(E') \to \pi_n(E) \oplus \pi_n(B') \to \pi_n(B) \to \pi_{n-1}(E') \to \pi_{n-1}(E) \oplus \pi_{n-1}(B')$$

Proof. WLOG, we can assume that the diagram is a pullback square of fibrant objects and the maps $E' \to B$, $E \to B$ are fibrations. Then by Proposition 3.4, there are long exact sequences associated to both fibrations, and the fibres can be identified. We can view the map between the long exact sequences as a double complex with two nontrivial rows. Since the rows are exact, the total complex is exact. But the total complex is the Mayer-Vietoris long exact sequence with an extra copy of π_n of the fibres that can be removed since the fibres are just identified. □

4.2. **Combinatorial Model Categories.** Combinatorial model categories are a large class of model categories that are well behaved. Morally, they are the same as presentable ∞-categories. Pretty much every reasonable model category should be Quillen equivalent to a combinatorial one. For example, the category Top of topological spaces isn't presentable, but $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ is.

Definition 4.33. *A* ***combinatorial model category*** *C is a model category that is presentable and cofibrantly generated.*

Our goal will be to show that it is easy to produce these.

Lemma 4.34. *Let $\tau > \kappa$ be regular cardinals and let $C \xrightarrow{p} D \xleftarrow{p'} C'$ be functors that are κ-cofinal, that preserve τ-small κ-filtered colimits, and suppose that C, C', D are κ-filtered, τ-small, and admit τ-small κ-filtered colimits. Then there exist objects $X \in C, X' \in C'$ such that $p(X) \cong p'(X')$.*

Proof. Say an ordinal is even if it is a limit ordinal plus an even number, and odd otherwise. Let (κ) be the poset of ordinals of size $< \kappa$ and let $(\kappa)_0$ and $(\kappa)_1$ denote the even and odd ones respectively.

Suppose we can construct a commutative diagram

$$\begin{tikzcd} (\kappa)_0 \arrow[r] \arrow[d] & (\kappa) \arrow[d] & (\kappa)_1 \arrow[l] \arrow[d] \\ C \arrow[r, "p"] & D & C' \arrow[l, "p'"'] \end{tikzcd}$$

Then since κ_0 and κ_1 are cofinal in (κ), their colimits in D are isomorphic, but this agrees with the image of their colimits in C, C'.

We can construct such a diagram inductively. On limit ordinals, we can extend by taking the colimit in either C or C'. On sucessor ordinals, we can extend because we assumed that p, p' are κ-cofinal. $\square$

Lemma 4.35. *Let $\tau \gg \kappa$ be regular cardinals with $\tau > \kappa$, D a presentable category, and $C_a, D_b, a \in A, b \in B$ τ-filtered partially diagrams in D such that their colimits agree. Then for every pair of τ-small subcategories $A_0 \subset A, B_0 \subset B$, there exist τ-small κ-filtered subsets $A' \supset A_0$, $B' \supset B_0$ such that the colimits of A' and B' agree.*

Proof. Let P_A, P_B be tha poset of τ-small κ-filtered subsets of A (resp B) which contain A_0 (or B_0).Let X be the common colimits and consider the maps $P_A, P_B \to D_{/X}$ given by taking colimits. Then applying Lemma 4.34 to this situation, we are done. $\square$

Lemma 4.36. *Let κ be a regular cardinal $C_0 \subset C$ a subcategory of a presentable category closed under κ-filtered colimits. C_0 is κ-accessible iff the following condition is satisfied for all sufficiently large $\tau \gg \kappa$:*

Let $X_\alpha, \alpha \in A$ be a τ-filtered diagram of τ-compact objects in C. For every κ-filtered subcategory $B \subset A$, let X_B denote the colimit over B. Then if X_A belongs to C_0, then for every τ-small subcategory of B, there is a $C \supset B$ that is τ-small, κ-filtered, and X_C is in C_0.

Proof. Choose τ large enough so that C is τ-presentable, let C^τ be the τ-compact objects. We'll see that $C^\tau \cap C_0$ generates C_0 under κ-filtered colimits. Given an object in C_0, present it as a colimit of a filtered diagram A. Let S be the collection of τ-small κ-filtered subdiagrams of A such that the colimit is in C_0. We can then take the colimit over S of the colimits of subdiagrams of A to obtain the same object.

Conversely, suppose that C is generated under κ-filtered colimits by a small subcategory $D \subset C_0$. Choose $\tau \gg \kappa, \tau > \kappa$ such that every object of D is τ-compact. To see that the condition is satisfied, suppose that we have a diagram $X_\alpha, \alpha \in A$, with $X_A \in C_0$ and A τ-filtered. X_A is also the colimit over $Y_\beta, \beta \in B$ with B a τ-small κ-filtered diagram in D. By replacing B with the family of τ-small κ-filtered subdiagrams of B, and using the family $Y_{B'}$ for all such subsets, we can assume that B is τ-filtered. Now we can apply Lemma 4.35. $\square$

Corollary 4.37. *Given a morphism of presentable categories preserving κ-filtered colimits, the preimage of a κ-accessible subcategory is κ-accessible.*

Corollary 4.38. *Let A be a combinatorial model category, and $W, F, \subset A^{[1]}$ the categories of weak equivalences, fibrations. Then $F, F \cap W, W$ are accessible subcategories of $A^{[1]}$.*

Proof. The subcategory of $\mathrm{Set}^{[1]}$ consisting of surjections is accessible (everything is a κ-filtered colimit of finite surjections). For any morphism $i : a \to b$, the functor $A^{[1]} \to \mathrm{Set}^{[1]}$ sending $f : x \to y$ to $\mathrm{Hom}(b, x) \mapsto \mathrm{Hom}(b, y) \times_{\mathrm{Hom}(a,y)} \mathrm{Hom}(a, x)$ preserves filtered colimits if A, B are κ-compact.

F and $W \cap F$ are obtained as the preimage of surjections on products of such maps (since they are defined via lifting properties with respect to a set of maps). Thus by Corollary 4.37 they are accessible.

By Proposition 4.8, there is a functorial factorization into a trivial cofibration and fibration. W is exactly the preimage of those that factor as a trivial cofibration and trivial fibration, so is accessible too. $\square$

Lemma 4.39. *Suppose that C, W are collections of morphism in a presentable category A with the following properties:*

(1) C is weakly saturated and is generated by a small set of morphisms C_0.
(2) $C \cap W$ is weakly saturated.
(3) W is an accessible subcategory of $A^{[1]}$
(4) W satisfies 2 out of 3.

Then $C \cap W$ is generated as a weakly saturated class by a small subset.

Proof. Choose κ so that W is κ-accessible, choose $\tau \gg \kappa, \tau > \kappa$ such that Lemma 4.36 holds, C is τ-accessible, and everything in C_0 is τ-compact. By Lemma 2.3, we can replace C_0 with $C \cap A^\tau$.

Let $S = C_0 \cap W$. We will show that S generates $C \cap W$ as a weakly saturated class of morphisms. Let $\overline{S}$ be the weak saturation of S. Suppose that $f : X \to Y$ is a morphism in $C \cap W$. By Corollary 2.73, there is a τ-good C_0-tree $\{Y_\alpha\}_{\alpha \in A}$ with root X such that Y is isomorphic to Y_A as objects in $C_{X/}$. Say that a subset B of A is *good* if it is downward-closed and the canonical map $X \to Y_B$ is in W. We would like to prove that A is good.

The set of good A is closed under transfinite union.

Suppose that $B, B' \subset A$ are such that $B, B', B \cap B'$ are good. Then $B \cup B'$ is good. To see this, we consider the pushout

$$\begin{tikzcd} Y_{B\cap B'} \arrow[r] \arrow[d] & Y_B \arrow[d] \\ Y_{B'} \arrow[r] & Y_{B\cup B'} \end{tikzcd}$$

The morphisms are all in C, and the upper horizontal map is in W by 2 out of 3. Since $C \cap W$ is closed under pushouts, the lower horizontal map is in W too, so by 2 out of 3 again, the map $X \to Y_{B'} \to Y_{B' \cup B}$ is in W.

Next, observe that if $B_0 \subset A$ is τ-small, then there is a τ-small $B \supset B_0$ such that Y_{B_0} is in C_0, and so that B is good.

Next we will prove:

- Let A' be a good subset of A and let $B_0 \subset A$ be τ-small. Then there exists a τ-small subset $B_0 \subset B \subset A$ with $B, B \cap A'$ good.

Start by setting $B'_0 = A \cap B_0$. We will inductively construct increasing sequences of τ-small subsets B_i, B'_i as follows: Choose B_{i+1} to be a τ-small good subset of A containing $B_i \cup B'_i$ and B'_{i+1} to be a τ-small good subset of A containing $B_{i+1} \cap A'$. Taking the union over i, we will have proven the desired result.

Now there is no obstruction to making a good set larger, so A is good by Zorn's Lemma. □

Here is the desired converse:

Corollary 4.40. *Let A be a presentable category and let W and C be classes of morphisms in A with the following properties:*

(1) C is weakly saturated and generated by a small set.
(2) $C \cap W$ is weakly saturated.
(3) W is accessible.
(4) W has the 2 out of 3 property.
(5) $W \subset C^{\oslash}$.

Then $C, W, (C \cap W)^{\oslash}$ give rise to a combinatorial model structure on A.

Proof. A presentable category has finite limits and colimits. To see we get a model structure, we need to check that $(C, (C \cap W)^{\oslash})$ and $(C \cap W, C^{\oslash})$ are weak factorization systems, which follows from Proposition 4.8. To see it is combinatorial, we just observe that by Lemma 4.39, $C \cap W$ is generated by a small set in addition to C. □

Our goal now is to get a more useable version of Corollary 4.40.

Definition 4.41. *Let A be a presentable category. A class W of morphisms is **perfect** if it satisfies the following conditions:*

(1) Every isomorphism belongs to W, and W satisfies 2 out of 3.
(2) W is stable under filtered colimits.
(3) There exists a small set $W_0 \subset W$ such that every morphism in W is a filtered colimit of morphisms in W_0.

The following is a corollary of Corollary 4.37.

Lemma 4.42. *For a functor between presentable categories that preserves filtered colimits, the pullback of a perfect class of morphisms is perfect.*

Proposition 4.43. *Let A be a presentable category. Suppose that W is a perfect class of morphisms and C_0 is a small set of morphisms such that*

(1) For any pair of coCartesian squares:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"] \arrow[ur, phantom, "\ulcorner", very near end] & Y' \arrow[d] \\ X'' \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Y'' \end{tikzcd}$$

where $f \in C_0$ and $g \in W$, then $g' \in W$.

(2) $W \supset C_0^{\oslash}$.

Then W and the saturation of C_0 determine a combinatorial left proper model structure.

Proof. First we will show left properness. First observe that (2) actually holds for f in a weakly saturated set (this uses that W is stable under filtered colimits), so it holds for C, the weak saturation of C_0.

To check the conditions of Corollary 4.40, we only need to check that $C \cap W$ is weakly saturated. It is stable under retracts since W is stable under filtered colimits. To see that $C \cap W$ is stable under pushouts, we can factor any pushout as a pushout by a trivial fibration and a cofibration as below.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, tail, "\wr"'] & X' \arrow[r, two heads, "\sim"] \arrow[d, "f'"] & X'' \arrow[d, "f''"] \\ Y \arrow[r] & Y' \arrow[r, "h'"] & Y'' \end{tikzcd}$$

We then know by left properness that $f' \in C \cap W$ and hence again by left properness $h' \in W$. By the two out of 3 property $f'' \in W$. $\square$

Remark 4.43.1. Combinatorial left proper model categories arising from Proposition 4.43 are exactly those with W closed under filtered colimits.

4.3. **Derived Functors.** A **left Quillen functor** between model categories is one that preserves (finite) colimits, cofibrations and trivial cofibrations. Dually there is a notion of right Quillen functor.

Quillen functors are useful because one can compute their derived functors.

Definition 4.44. *If $F : C \to D$ is a functor of homotopical categories, the* ***total left derived functor****, denoted $\mathbb{L}F$ is the* right *Kan extension of the diagram:*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d] & D \arrow[d] \\ h(C) \arrow[r, dashed, "\mathbb{L}F"] & h(D) \end{tikzcd}$$

To help compute derived functors, we can use a deformation.

Definition 4.45. *A* ***left deformation*** *on a homotopical category C is an endofunctor Q with a natural weak equivalence q from Q to the identity.*

Let C_Q denote the full subcategory of objects in the image of Q, also called a **left deformation retract**. A left deformation induces an equivalence between $h(C)$ and $h(C_Q)$. If we have a functorial map from a cofibrant fibrant factorization in a model category, that is a left defomation.

Definition 4.46. *a **left deformation** on a functor $F : C \to D$ is a left deformation Q on C such that F is homotopical on C_Q.*

F is **left deformable** if it admits a left deformation. Any left deformable functor has a maximal subcategory on which it is homotopical (not proven here but uses 2 out of 6).

Proposition 4.47. *If $F : C \to D$ between homotopical categories has a left deformation Q, q, then FQ induces a left derived functor of F.*

Proof. Let h_C denote the localization functor of C. Since F is homotopical on C_Q, $h_D FQ$ does descend to a functor from $h(C)$. $h_D Fq$ gives the natural transformation we want. Instead of proving the universal property in $h(D)^C$, it suffices to prove it in the equivalent full subcategory of $h(D)^C$ consisting of homotopical functors. Now suppose that $G \in h(D)^C$ is homotopical and we have a natural transfromation $\eta : G \to h_D F$. $Gq : GQ \to G$ is a natural transformation that is a natural isomorphism since G is homotopical. Thus by naturality of η, η factors as $h_D Fq \circ \eta Q \circ (Gq)^{-1}$. Uniqueness follows from the fact that $h_D Fq$ is a natural isomorphism since q is a weak equivalence. □

Morally here is how one can think about a deformation. When you invert weak equivalences, you really get an ∞-category, and we can expect left Kan extensions to be the 1-categorical shadow of computing (homotopy) colimits in the ∞-category, as is the formula for Kan extensions when enough colimits exist. The deformation allows you to 1-categorically change your object to one that is homotopically good, so that the ∞-categorical colimit agrees with the usual one.

Remark 4.47.1. Maybe it's worth mentioning that FQ is really an absolute Kan extension, i.e it is preserved by post composition with all functors (including the representable ones, making it a pointwise Kan extension).

There is really a pseudo 2-functor taking a homotopical category C to $h(C)$, a left deformable functor to its total left derived functor, and a natural transformation of such functors to the derived natural transformation.

Lemma 4.48 (Ken Brown's Lemma)**.** *Let $F : C \to D$ be a functor between model categories that sends trivial cofibrations between cofibrant objects to weak equivalences. Then F is homotopical on the category of cofibrant objects.*

Proof. Let $f : X \to Y$ be a weak equivalence of cofibrant objects. Factor $X \coprod Y \xrightarrow{f,1} Y$ as a cofibration and trivial cofibration, and consider the diagram below.

$$\begin{tikzcd}
\phi \arrow[rr, tail] \arrow[dd, tail] & & X \arrow[dl, tail] \arrow[dd, tail, "j"] \arrow[dddr, "f"] & \\
& X \coprod Y \arrow[dr, tail] & & \\
Y \arrow[ur, tail] \arrow[rr, tail, "i"] \arrow[drrr, equal] & & Z \arrow[dr, two heads, "\sim"] & \\
& & & Y
\end{tikzcd}$$

By 2 out of 3, i, j are weak equivalences, so they get sent to weak equivalences. But then since the map $Z \to Y$ is a retraction of i, it is also sent to a weak equivalence by 2 out of 3. Then f is too by 2 out of 3. $\square$

Corollary 4.49. *If C has a functorial cofibrant replacement, then a left Quillen functor is left deformable. In any case, the total left derived functor exists, and can be computed by taking a cofibrant replacement.*

Proof. The first fact follows from 4.48 and second just requires the additional observation that the inclusion of the full subcategory of cofibrant objects induces an equivalence on the homotopy category. $\square$

An adjunction is called a **Quillen adjunction** if the left adjoint is left Quillen and similarly for the right adjoint.

Lemma 4.50. *TFAE for an adjunction $F \dashv G$*

(1) F is left Quillen
(2) G is right Quillen
(3) F preserves cofibrations and G preserves fibrations
(4) F preserves trivial cofibrations and G preserves trivial fibrations.

Proof. The preserving limits and colimits is satisfied because of the adjunction. The equivalences then follow because of Lemma 2.13. $\square$

A Quillen adjunction induces an adjunction on the homotopy 1-categories. The same can be said of an adjoint pair of deformable functors between homotopical categories.

A Quillen adjunction is a **Quillen equivalence** if its derived functors are an equivalence.

Lemma 4.51. *A Quillen adjunction $F : C \leftrightarrows D : G$ is a Quillen equivalence iff when $c \in C$ is cofibrant and $d \in D$ is fibrant, then $Fc \to d$ is a weak equivalence iff the adjoint map $c \to Gd$ is.*

Proof. Suppose $F \dashv G$ is a Quillen equivalence. Then for c cofibrant and d fibrant, F, G agree with the derived functors, so combined with the fact that weak equivalences are the isomorphism in the homotopy category, the condition follows. Conversely if the condition holds, then the unit map $c \to GFc$ is an isomorphism in $h(C)$ since G is given by taking a fibrant replacement and applying G, so by the condition it is equivalent to Fc being isomorphic or weakly equivalent to the fibrant replacement. $\square$

For example S and $|\cdot|$ give a Quillen equivalence between the Serre and Quillen model structures.

Simplicial model categories are ones that have a compatible simplicial enrichment and tensoring/cotensoring. Many natural examples of model categories are simplicial model categories.

Definition 4.52. *A **left Quillen bifunctor** $\otimes : C \times D \to E$ is a map of model categories preserving colimits in both variables, and such that the Construction 2.25.1 $\hat{\otimes}$ sends pairs of cofibrations to cofibrations that are trivial iff either arrow is.*

Lemma 4.53. *Left Quillen bifunctors preserve cofibrant objects and are homotopical on the subcategories of cofibrant objects.*

Proof. These claims follow from the condition on $\hat{\otimes}$ and Lemma 4.48. $\square$

Lemma 4.54. *If* $\otimes, \{,\}, \mathrm{hom}$ *form a two variable adjunction between model categories, then one is a Quillen bifunctor iff the rest are.*

Proof. This follows from Lemma 2.26. $\square$

Definition 4.55. *Given a natural transformation* $\tau : F \to F'$ *between two functors* F, F', *the total derived natural transformation is the natural transformation* $L\tau$ *given by cofibrant replacing and then doing the natural transformation.*

The derived natural transformation is functorial.

Lemma 4.56. *Suppose that some diagram of left Quillen functors commutes. Then the corresponding diagram of derived functors can be made to commute.*

Proof. We can replace all the model categories with the subcategory of cofibrant objects. The left Quillen functors are still defined since they take cofibrations to cofibrations and the initial object to the initial object. Now there is no deriving necessary. $\square$

Lemma 4.57. *Suppose that in a model category* C, *weak equivalences to fibrant objects are stable along pullbacks by a fibration. Then* C *is right proper.*

Proof. Produce a diagram

$$\begin{tikzcd} & X \arrow[r, "\sim"] \arrow[d, two heads] & RX \arrow[d, two heads] \\ Z \arrow[r, "\sim"] & Y \arrow[r, "\sim"] & RY \end{tikzcd}$$

by choosing a fibrant replacement for Y and factoring $X \to Y \to RY$. The map $Y \times_{RY} RX \to RX$ is an equivalence by hypothesis, so $X \to Y \times_{RY} RX$ is an equivalence by 2 out of 3. Pulling back along $Z \to Y$ is a right Quillen functor, so it preserves weak equivalences between fibrant objects. Applying this to $X \to Y \times_{RY} RX$, we get the desired result. $\square$

Example 4.57.1. The Quillen model structure on simplicial sets is left and right proper, (or just proper). Left proper is clear. If $X \to Y$ is an equivalence, and we pull it back via a fibration, it is an equivalence by using the 5-lemma and comparing the long exact sequence on homotopy groups.

4.4. **Enriched and Simplicial model categories.** Our goal in this section is to study model categories that are enriched over others.

Definition 4.58. *A* ***monoidal model category*** *is a closed monoidal category* S *with a model structure satisfying:*

(1) The tensor product $\otimes : S \times S \to S$ *is a left Quillen bifunctor.*
(2) The unit $1 \in S$ *is cofibrant.*

Example 4.58.1. By Lemma 2.29 and Lemma 2.30, the Quillen model structure on $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ is a monoidal model category with respect to the cartesian product.

Example 4.58.2. Consider the monoidal category FinSet with the Cartesian product, isomorphisms and maps from the empty set as cofibration, and isomorphisms weak equivalences. Every cocomplete model category is enriched over this in a canonical way, since iterated coproducts of an object with itself give the tensoring/cotensoring.

Definition 4.59. *Let S be a monoidal model category. An S-enriched model category is an S-enriched model category A equipped with a model structure so that:*

(1) A is tensored and cotensored over S.
(2) $A \otimes S \to S$ is a left Quillen bifunctor.

When S is $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ with the Quillen model structure, this is the definition of a **simplicial model category**.

It follows from Example 4.58.2 that anything proven about model categories over a monoidal model category is a strict generalization about things that are true for model categories.

In some cases we can detect simplicial model structures with less effort than expected.

Proposition 4.60. *Let C be a model category enriched over $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ with every object cofibrant. Suppose that weak equivalences are closed under filtered colimits. Then the following are equivalent to the simplicial model category conditions:*

(1) A is tensored and cotensored over $\mathrm{Set}^{\Delta^{\mathrm{op}}}$.
(2) For any cofibration in $K \to L$ in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ and a cofibration $c \to d$ in C, the map $c \otimes L \cup_{c\otimes K} d \otimes K \to d \otimes L$ is a cofibration.
(3) For every $n \geq 0$ and every object $c \in C$, the natural map $c \otimes \Delta^n \to c \otimes \Delta^0$ is an equivalence.

Proof. Suppose that C is a simplicial model category. Clearly (1) and (2) are satisfied, as well as 3, since the map $\Delta^n \to \Delta^0$ admits a section which is a trivial fibration.

Now suppose the conditions are satisfied. Since every object is cofibrant, C is left proper, and moreover the functor $\otimes X : C \to C$ is homotopical.

We will first see that the tensoring bifunctor $C \times \mathrm{Set}^{\Delta^{\mathrm{op}}} \to C$ preserves weak equivalences in each variable. Fix $c \in C$ and suppose that $K \to K'$ is a weak equivalence in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$. Factor it as a trivial fibration and trivial cofibration. The trivial fibration has a section, which is a trivial cofibration, (since it is injective) so we can reduce to the case of $K \to K'$ being a trivial cofibration. The collection of all such $K \to K'$ such that $c \otimes K \to c \otimes K'$ is a trivial cofibration is weakly saturated, so we reduce to showing it for the horn inclusions $\Lambda^n_i \to \Delta^n$. The inclusion of a vertex $\{v\} \to \Lambda^n_i$ is a pushout of horn inclusions of dimension $< n$, so by induction and the 2 out of 3 property it suffices to show it for $\{v\} \to \Delta^n$ which was assumed.

Next fix K, and we will see that $c \to c \otimes K$ preserves weak equivalences. Let $f : c \to c'$ be a weak equivalence. Since weak equivalences are closed under filtered colimits, it suffices to show this for K finite. We will then prove it by induction on the subcomplex of K. Choose a subcomplex L of K for which the result is satisfied, and a nondegenerate cell σ we would like to adjoin. $f \otimes \partial\sigma, f \otimes L$ are equivalences by the inductive hypothesis, and $f \otimes \sigma$ is an equivalence by assumption.

Because our model structure is left proper, using assumption (2) we get that $D \otimes (-)$ applied to the diagram adjoining σ to L is a homotopy pushout for any D. Thus since each of the components is a weak equivalence, the pushout is a weak equivalence.

Suppose now that $i : c \to c'$ and $j : S \to S'$ are cofibrations in C and $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ respectively. We would like to prove that the induced map $i \wedge j : (c \otimes S') \cup_{(c\otimes S} (c' \otimes S')$ is an equivalence if either i, j are equivalences. But this follows from applying 2 out of 3 and left properness to the diagram

$$\begin{tikzcd}
c \otimes S \arrow[r, tail] \arrow[d, tail] & c' \otimes S \arrow[d, tail] & \\
c \otimes S' \arrow[r, tail] & (c \otimes S') \cup_{c \otimes S} (c' \otimes S') \arrow[r] & c' \otimes S'
\end{tikzcd}$$

□

With a simplicial model category, we have nice cylinders. For example, Δ^1 is a cylinder object for Δ^0 in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$, and tensoring with A gives a cylinder object for A when A is cofibrant. Thus if A is cofibrant and B fibrant, then homotopy of maps $A \to B$ agrees with equivalence in $\underline{\mathrm{Hom}}(A, B)$.

Suppose that we have a Quillen adjunction $F \dashv G$ between two S-enriched model categories C, D, and G is an S-enriched functor. Then for every $X \in C, Y \in D, s \in S$, there is a canonical map $\mathrm{Hom}(s \otimes X, GY) \to \mathrm{Hom}(s \otimes FX, Y)$, which applied to the unit of the adjunction gives a family of maps $\beta_{X,s} : s \otimes FX \to F(s \otimes X)$. In Lemma 2.27, it was determined that $\beta_{X,s}$ is an isomorphism iff the adjunction is enriched.

Remark 4.60.1. Suppose that S is $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ with the Quillen model structure. Then $\beta_{X,s}$ is an equivalence. To see this, we know it for $s = \Delta^0$, and observe that the set of simplicial sets for which it is true is closed under weak equivalences, coproducts, and homotopy pushouts.

We will now consider the situation when $\beta_{X,s}$ is a weak equivalence.

Proposition 4.61. *Let C, D be S-enriched model structures, and $F \dashv G$ a Quillen adjunction between the underlying model categories with G enriched. Assume that every object of C is cofibrant and that the map $\beta_{X,s}$ is a weak equivalence fore every s, X cofibrant. Then TFAE:*

(1) $F \dashv G$ is a Quillen equivalence.
(2) The restriction of G determines a weak equivalence of S [REQUIRES SOMETHING AHEAD:((((]

Lemma 4.62. $(W2)$ *for the factorization systems is implied by the rest of the axioms for a simplicial model category. Moreover, if the object in the upper left corner is cofibrant, the lift is unique up to relative homotopy.*

Proof. A lift in a diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

is the same as a lift of a 0-simplex in the map $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(B,Y)} \mathrm{Hom}(X, Y)$. But this is a trivial fibration if either of the vertical maps is trivial by the fact that the two-variable adjunction with $\otimes$ is Quillen, so there is a unique lift up homotopy. □

We can reduce the amount of work needed to check something is a simiplicial model category by observing that we can check the condition of $\otimes, \{,\}$ or hom being a Quillen bifunctor on generating cofibrations or generating fibrations.

Lemma 4.63. *If A, B are cofibrant, $f : A \to B$ is an equivalence in a simplicial model category iff $\underline{\mathrm{Hom}}(B, C) \to \underline{\mathrm{Hom}}(A, C)$ is a weak equivalence for every fibrant object C.*

Proof. It is clear that if f is an equivalence, then the other map is an equivalence. Conversely, if the other map is an equivalence, since $\underline{\mathrm{Hom}}(B,C) \to \underline{\mathrm{Hom}}(A,C)$ is a fibration, it is a trivial fibration, and in particular bijective on π_0. Thus f is an isomorphism in the homotopy category. □

Many categories of simplicial objects (eg: models of Lawvere theories) have natural simplicial model structures. These often come from inducing model structures through an adjunction

Proposition 4.64. *Let $F : C \leftrightarrows D : U$ be an enriched adjunction of categories tensored and cotensored over $\mathrm{Set}^{\Delta^{\mathrm{op}}}$, suppose that D is bicomplete, and C has a simplicial model structure cofibrantly generated by κ-small objects that are sent to κ'-small objects in D. We can try to define a model structure by having a map in D be a fibration or weak equivalence if it is after applying U. Suppose that if $f \in D$ has the left lifting property with respect to g such that Uf is a fibration, then Uf is a weak equivalence.*

Then D has a cofibrantly generated simplicial model structure where the fibrations and weak equivalences are determined by applying U, the cofibrations are generated by U applied to generators for D.

Proof. By Proposition 4.12 a model structure, and we only need to show that it is simplicial.

To do this, we can identify the map

$$U \hom_D(L,X) \to U(\hom_{C^{\Delta^{op}}}(K,X) \times_{\hom_{D(K,Y)}} \hom_D(L,Y))$$

with

$$\hom_C(L,UX) \to \hom_C(K,UX) \times_{\hom_{C(K,UY)}} \hom_C(L,UY)$$

since the adjunction is enriched and U preserves pullbacks. If $K \to L$ is a cofibration and $X \to Y$ is a fibration, one of which is trivial then this map is a fibration, since C is a simplicial model category. □

Lemma 4.65. *If D has a functorial fibrant replacement Q, then the last condition of Proposition 4.64 is satisfied.*

Proof. We have a functorial path object given by $\hom(\Delta^1, -)$, and Q is a functorial fibrant replacement, so by Proposition 4.24 we are done. □

The following corollary is probably bad because the last assumption is unnecessary. Or maybe not because the other version requires the objects to be "small".

Corollary 4.66. *Let $F : \mathrm{Set} \leftrightarrows C : U$ be an adjunction of categories, suppose that C is bicomplete, F sends 1 to a κ-small object in C. We can try to define a model structure on $C^{\Delta^{op}}$ by having a map in $C^{\Delta^{op}}$ be a fibration or weak equivalence if it is after applying U. Suppose that if $f \in C^{\Delta^{op}}$ has the left lifting property with respect to g such that Uf is a fibration, then Uf is a weak equivalence.*

Then $C^{\Delta^{op}}$ has a cofibrantly generated simplicial model structure where the fibrations and weak equivalences are determined by applying U, the cofibrations are generated by U applied to generators for D.

Proof. $C^{\Delta^{op}}$ is tensored and cotensored by Lemma 2.28. The adjunction $F : C^{\Delta^{op}} \leftrightarrows \mathrm{Set}^{\Delta^{\mathrm{op}}} : U$ enriches over $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ by Lemma 2.27 since $F(X \times K) \cong F(X) \otimes K$. F sends the generators of cofibrations/trivial cofibrations to κ-small objects after possibly enlarging κ since Δ^{op} is a small category, so by Proposition 4.64 we are done. □

Example 4.66.1. Any category for which the forgetful functor factors through groups satisfies Lemma 4.65 with Q the identity by Proposition 3.1. This includes simplicial groups, rings, Lie algebras, modules.

I think you can replace small in the theorem below by κ-small if you use a large ordinal power of Ex rather than Ex^∞.

Theorem 4.67. *Let C be complete and cocomplete, and let Z_i be a set of small objects of C, and regard them as constant simplicial objects. Then there is a cofibrantly generated simplicial model structure on $C^{\Delta^{op}}$ such that a map $A \to B$ is a weak equivalence or fibration iff the maps $\underline{\mathrm{Hom}}_{C^{\Delta^{op}}}(Z_i, A) \to \underline{\mathrm{Hom}}_{C^{\Delta^{op}}}(Z_i, B)$ is one.*

Proof. Use Ex to produce a natural fibrant replacement...[INCOMPLETE] □

Example 4.67.1. Simplicial objects of models of any Lawvere theory will have the forgetful functor corepresented by a small object, making Theorem 4.67 apply.

Example 4.67.2. Let $\mathbb{F}$ be a field. A coalgebra over $\mathbb{F}$ is locally finite dimensional, so the finite dimensional coalgebras are a set of small generators. Thus we can apply Theorem 4.67 on the set of finite dimensional coalgebras.

v

5. Some Important Model Structures

5.1. **Model structure on S-enriched categories.** Let S be a monoidal model category and let Cat_S be tha category of (small) S-enriched categories. We will introduce a model structure on Cat_S.

Since S is monoidal, we obtain a monoidal structure on hS by inverting weak equivalences. Given an S-enriched category C, there is an associated hS-enriched category which will be called hC.

We say that $F : C \to C'$ in Cat_S is a *weak equivalence* if the induced functor $hC \to hC'$ is an equivalence of hS-enriched categories. In other words:

- For every $X, Y \in C$, the map $\mathrm{Map}_C(X, Y) \to \mathrm{Map}(FX, FY)$ is an equivalence.
- Every object $Y \in hC'$ is equivalent to some $F(X)$.

Suppose that the weak equivalences in S are stable under filtered colimits. Then the same is true for Cat_S. Moreover if S is also combinatorial, then the weak equivalences of Cat_S are generated under filtered colimit by a set. To see this, they are clearly generated by weak equivalences between categories with finitely many objects and κ-compect mapping spaces, which is a small set. Thus in this case the weak equivalences are perfect.

If A is an object of S, let $[1]_A$ denote the S-enriched category having two objects X and Y with the only non identity morphisms being $A = \mathrm{Map}_{[1]_A}(X, Y)$. Let $[0]_S$ be the terminal S-category.

Let C_0 denote the collection of morphims in S of the following types:

(1) The inclusion $\phi \to [0]_S$.
(2) The induced maps $[1]_A \to [1]_{A'}$ where $A \to A'$ ranges over a set of generators for the weakly saturated class of cofibrations in S.

Proposition 5.1. *Let S be a combinatorial monoidal model category. Assume that every object of S is cofibrant and that the collection of weak equivalences in S is stable under filtered colimits. Then there exists a left proper combinatorial model structure on* Cat_S *with the weak equivalences as defined above and the cofibrations generated by C_0.*

Proof. It suffices to verify the conditions of Proposition 4.43. Under the hypotheses, we have already W is perfect. Suppose f has the right lifting property with respect to everything in C_0. Then because of $\phi \to [0]_S$, the map is essentially surjective, and because of $[1]_A \to [1]_{A'}$, the map on mapping spaces are triival fibrations, in particular weak equivalences.

Now we must verify the last condition, namely that weak equivalences are stable under being pushed out by pushouts of C_0. So suppose that $F : C \to D$ is a weak equivalence, and $G : C \to C'$ is a pushout of a morphism in C_0. We would like to show that $F' : C' \to D' = D \cup_C C'$ is a weak equivalence.

First consider the case that G is a pushout of a map $\emptyset \to [0]_S$. In this case, F' is just obtained by adding adjoining an object with no nonidentity morphisms, so it is clear that it is an equivalence.

Now suppose G is the pushout of a map $[1]_A \to [1]_T$. First we will construct the pushout, C'. C is equipped with a map $h : S \to \mathrm{Map}(x, y)$, and C' is the universal S-enriched category with an extension of h to T.

We define the objects of C' are that of C and the map $C \to C'$ is the identity on objects. Fix objects $w, z \in C'$. Let $M_C^k, k \geq 0$ be the objects in S given by $\mathrm{Map}_C(y, z) \otimes T \otimes (\mathrm{Map}_C(y, x) \otimes T)^{\otimes n-2} \otimes \mathrm{Map}_C(w, x)$ for $n \geq 2$, $M_C^1 = \mathrm{Map}_C(y, z) \otimes T \otimes \mathrm{Map}_C(w, x)$, $M_C^0 = \mathrm{Map}_C(w, z)$. M_C^k can be thought of as morphisms in $w \to z$ in C' with a factorization through an object of S k times.

We can identify morphisms in the different M_C^i that should be the same in C'. Namely, this means taking a colimit along various restrictions from T to A of the composition maps, and we define $\mathrm{Map}_{C'}(w, z)$ to be the result. It is straightforward to verify that this is the pushout category. Note that the object $\mathrm{Map}_{C'}(w, z)$ comes with a natural filtration via the images of the M_C^is. The same construction gives a model for D', where we would like to show that $C' \to D'$ is an equivalence. To see this it suffices to check that the map is an equivalence on mapping spaces. Since weak equivalences behave nicely wrt filtered colimits, it suffices to check this on the filtered pieces.

This follows from induction on the filtered piece. Let N_C^i be object mapping to M_C^i consisting of things in M_C^i except one morphism in A instead of T. N_C^I can be constructed as a pushout of cofibrations, which maps into M_C^i as a cofibration because the tensor product is a left Quillen bifunctor. For $i = 0$, it is an equivalence by assumption. For $i > 0$, it is a pushout of the $(i-1)^{th}$ piece and M_C^i along N_C^i. This is a homotopy pushout since S is left proper, so it suffices to check by the inductive hypothesis that $M_C^i \to M_D^i$ and $N_C^i \to N_D^i$ are equivalences. The first follows from Lemma 4.53 since our objects are cofibrant, and the second follows from the same Lemma along with a construction of N_C^i as a homotopy pushout. □

Some remarks about the result above: The model structures are functorial in S as follows: Suppose that $S \to S'$ is a monoidal left Quillen functor between model categories satisfying the Proposition. Then it induces a Quillen adjunction between Cat_S and $\mathrm{Cat}_{S'}$. If f is a Quilen equivalence, then so is the induced map. Furthermore, it follows from the proof that in the model structure, a cofibration induces a cofibration on mapping spaces.

Next we will examine the fibrations in this model structure.

Definition 5.2. *A **quasi-fibration** $f : C \to D$ is a functor such that for every and every isomorphism $f(X) \to f(Y)$, there is an isomorphism $X \to \overline{Y}$ lifting it.*

We will show later that Cat admits a model structure where the fibrations are quasi-fibrations and the weak equivalences are equivalences.

Definition 5.3. *A morphism in an S-enriched category C is an **equivalence** if it is an isomorphism in hC.*

Definition 5.4. *An S-enriched category is **locally fibrant** if for every pair of objects $X, Y \in C$, $\mathrm{Map}(X, Y)$ is fibrant in S. A functor $F : C \to C'$ is a **local fibration** if*

(1) For every pair $X, Y \in C$, $\mathrm{Map}(X, Y) \to \mathrm{Map}(FX, FY)$ is a fibration.
(2) The induced map $hC \to hC'$ is a quasi-fibration.

Remark 5.4.1. Suppose that condition (1) is satisfied. If C' is locally fibrant, then every isomorphism $[f] : F(X) \to Y$ in hC' can be represented by an equivalence $f : F(X) \to Y$ because of Proposition 4.21. Let $\overline{Y}$ be an object of C with $F(\overline{Y}) = Y$. Then by Proposition 4.25, if C' is locally fibrant, (2) is equivalent to the condition that for every equivalence $F(X) \to Y$ in C', there is an equivalence $\overline{f} : X \to \overline{Y}$ lifting it.

Let $[1]_S^{\sim}$ be the S-enriched category with two objects that are isomorphic and have endomorphisms 1_S. In otherwords, $\mathrm{Hom}(X, Y)$ is always 1_S.

Definition 5.5. *Let S be a monoidal model category satisfying the hypotheses of Proposition 5.1. S satisfies the **invertibility hypothesis** if the following condistion is satisfied:*

Let $i : [1]_S \to C$ be a cofibration of S-enriched categories classifying a morphisms $f \in C$ which is invertible in the homotopy category hC, and form a pushout

$$\begin{tikzcd} {[1]_S} \arrow[r, "i"] \arrow[d] & C \arrow[d, "j"] \\ {[1]_S^{\sim}} \arrow[r] & C\langle f^{-1}\rangle \end{tikzcd}$$

Then j is an equivalence of S-enriched categories.

Remark 5.5.1. Since Cat_S is left proper, we can take a trivial cofibration $C \to C'$ where C' is a fibrant S-enriched category. By the 3 out of 4 property, and the fact that trivial cofibrations are stable under pushout, in checking that the invertibility hypothesis holds, it suffices to assume that C is fibrant.

Remark 5.5.2. Also note that in the definition of the invertibiltiy hypothesis, since i is a cofibration and the model structure is left proper, the pushout is a homotopy pushout. More generally we can define $C[f^{-1}]$ to be the homotopy pushout along a morphism f of the map $[1]_S \to [1]_S^{\sim}$. The invertibility hypothesis is then equivalent to the statement that if f is an equivalence, then the map $C \to C[f^{-1}]$ is an equivalence. Since Cat_S is left proper, this can be computed by factoring the map $[1]_S \to [1]_S^{\sim}$ as a cofibration and a trivial fibration $[1]_S \to E \to [1]_S^{\sim}$, and taking the pushout of the first map.

Remark 5.5.3. Suppose that C is fibrant S-enriched, and containes an equivalence f. We can observe that the homotopy pushout map $C \to C[f^{-1}]$ is a trivial cofibration, so it admits a retraction. This induces a map $E \to C$. Viewing E as a cylinder for $[0]_S$, this map is a homotopy between the maps given by X and Y.

More generally, given a fibration $C \to D$ of S-enriched categories and $f : X \to Y$ a morphism in C lifting an identity in D, the functors from $[0]_S$ to C classifying X and Y are homotopic.

Definition 5.6. *A model category S is **excellent** if it has a symmetric monoidal structure and satisfies the following conditions:*

- *(A1) S is combinatorial*
- *(A2) Every monomorphism is a cofibration and the collection of cofibrations is stable under products*
- *(A3) The collection of weak equivalences is stable under filtered colimits*
- *(A4) S is a monoidal model category*
- *(A5) S satisfies the invertibility hypothesis.*

$(A5)$ is a consequence of the other axioms, which is a result of Tyler Lawson. For now, it will suffice to observe that a presentable closed monoidal category with the trivial model structure is excellent.

Let S be an excellent model category. Cat_S is naturally cotensored over S: for $K \in S$ we can define C^K to have the same objects but maps $\mathrm{Map}_C(X,Y)^K$. Cat_S is not tensored over S, but given a function $\varphi : C \to D$ only defined on objects, there is an object $\mathrm{Map}^{\phi}_{\mathrm{Cat}_S}(C,D)$ such that

$$\mathrm{Hom}_S(K, \mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(C,D)) = \mathrm{Hom}^{\varphi}_{\mathrm{Cat}_S}(C, D^K)$$

where Hom^{φ} denotes functors that agree with φ on the object level. This object satisfies the same property that you would expect for a tensoring except with φs everywhere.

Lemma 5.7. *Let S be an excellent model category. Fix a diagram in* Cat_S

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "F"] & C' \arrow[d, "F'"] \\ D \arrow[r, "u'"] & D' \end{tikzcd}$$

Assume that for every pair of objects $X, Y \in C$, the diagram

$$\begin{tikzcd} \mathrm{Map}(X,Y) \arrow[r, two heads] \arrow[d] & \mathrm{Map}(FX,FY) \arrow[d] \\ \mathrm{Map}(uX,uY) \arrow[r, two heads] & \mathrm{Map}(u'FX,u'FY) \end{tikzcd}$$

is a homotopy pullback square involving fibrant objects of S and the horizontal arrows are fibrations.

Let $G : A \to B$ be a functor between S-enriched categories which is a transfinite composition of pushouts of generating cofibrations in Cat_S *of the form $[1]_S \to [1]_{S'}$ and let φ be a function from the set of objects of B to that of C.*

Then the diagram

$$\begin{tikzcd}
\mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(B,C) \arrow[r, two heads] \arrow[d] & \mathrm{Map}^{F\varphi}_{\mathrm{Cat}_S}(B,D) \times_{\mathrm{Map}^{F\varphi}_{\mathrm{Cat}_S}(A,D)} \mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(A,C) \arrow[d] \\
\mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(B,C') \arrow[r, two heads] & Map^{u'F\varphi}_{\mathrm{Cat}_S}(B,D') \times_{\mathrm{Map}^{u'F\varphi}_{\mathrm{Cat}_S}(A,D')} \mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(A,C')
\end{tikzcd}$$

is a homotopy pullback square between fibrant object of S, and the horizontal arrow are fibrations.

Proof. The set of morphisms $A \to B$ such that the conclusion holds is weakly saturated, so it suffices to check it for $[1]_S \to [1]_{S'}$. if ϕ has images X, Y the square above becomes

$$\begin{tikzcd}
\mathrm{Map}_C(X,Y)^{S'} \arrow[r] \arrow[d] & \mathrm{Map}_D(FX,FY)^S \times_{\mathrm{Map}_D(FX,FY)^S} \mathrm{Map}_C(X,Y)^{S'} \arrow[d] \\
\mathrm{Map}_{C'}(uX,uY)^{S'} \arrow[r] & \mathrm{Map}_{D'}(u'FX,u'FY)^S \times_{\mathrm{Map}_{D'}(u'FX,u'FY)^S} \mathrm{Map}_{C'}(uX,uY)^{S'}
\end{tikzcd}$$

which follows from our assumptions. $\square$

Theorem 5.8. *Let S be an excellent model category. Then a map $C \to D$ where D is fibrant is a local fibration iff it is a fibration.*

Proof. Suppose that $F : C \to D$ is a fibration. Then it has the right lifting property with respect to $[1]_S \to [1]_{S'}$ so it follows that for $X, Y \in C$, the map $\mathrm{Map}_C(X,Y) \to \mathrm{Map}_D(X,Y)$ is a fibration. In particular, C is locally fibrant.

It suffices then to verify the condition in Remark 5.4.1. Suppose that $f : FX \to Y$ is an equivalence in D. Let $E, D[f^{-1}]$ be as in Remark 5.5.2. Since S satisfies the invertibility hypothesis, $D \to D[f^{-1}]$ is a trivial cofibration. D is fibrant, so this map admits a retraction, which induced a map $r : E \to D$. Consider the lifting problem below,

$$\begin{tikzcd}
{[0]_S} \arrow[r, "X"] \arrow[d, tail, "\wr"'] & C \arrow[d, two heads, "F"] \\
E \arrow[r, "r"] \arrow[ur, dashed] & D
\end{tikzcd}$$

a lift exists, giving the desired lift of f.

Now assume that F is a local fibration. Factor F as $C \xrightarrow{u} C' \xrightarrow{F'} D$ where u is a weak equivalence and F' is a fibration. We will prove:

- Suppose we are given a commutative diagram in Cat_S.

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, tail, "G"] & C \arrow[d, "F"] \\
B \arrow[r, "v'"] & D
\end{tikzcd}$$

  If there is a lift from B to C' making everything commute, then there exists a lift in the diagram above from B to C.

Since F' is a fibration, this will prove that F has the right lifting property with respect to all trivial cofibrations, completing the proof. First, to prove •, it suffices to assume that G is a transfinite pushout of generating cofibrations, since G is a retract of such a cofibration. We can reorder this transfinite pushout to factor G into $A \xrightarrow{G_0} B' \xrightarrow{G_1} B$ where G_0 is obtained by adjoining objects $\{B_i\}_{i\in I}$ and G_1 from pushouts of $[1]_S \to [1]_{S'}$. Since u is an equivalence, there exist objects $\{C_i\}_{i\in I}$ and equivalences $f_i : uC_i \to \alpha B_i$. Since F is a local fibration, we can lift these to equivalences $f'_i : C_i \to C''_i$ in C.

Since $C' \to D$ is a fibration and D is fibrant, it follows that C' is fibrant. Since f' is an equivalence, the induced map $\mathrm{Map}(uC''_i, C'_i) \to \mathrm{Map}(uC_i, C'_i)$ is an equivalence. Since both of these are fibrant, they induce an isomorphism on homotopy classes from the unit, so it follows that we can choose morphisms $f''_i : uC''_i \to C'_i$ in C' such that composing with uf'_i gives something homotopic to f_i. It follows that $F'(f''_i)$ is homotopic to the identity. Then since $\mathrm{Map}(uC''_i, C'_i) \to \mathrm{Map}(FC''_i, F'C'_i)$ is a fibration, we can lift this homotopy to modify f''_i within its homotopy class to lift the identity. Thus by replacing C_i with C''_i we can assume that each of the maps f_i projects to the identity in D.

Define α'_0 to be the functor $B'_0 \to C'$ given by sending $a \mapsto \alpha \circ G_1(A)$ if $a \in A$ and uC_i otherwise. By Remark 5.5.3, α_0 is homotopic to $\alpha \circ G_1$ in $(\mathrm{Cat}_S)_{/D}$. Applying Proposition 4.25, we can replace α by a map α_0. But then the map determined by C_i give a lift $B' \to C$, so we can replace A with B'. We have thus reduced to the case that the map $A \to B$ is a transfinite composition of pushouts of morphisms $[1]_S \to [1]_{S'}$. Let φ be the map from the objects of B to the objects of C determined by α. Applying Lemma 5.7, we obtain a homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(B,C) \arrow[r] \arrow[d] & \mathrm{Map}^{F\varphi}_{\mathrm{Cat}_S}(B,D) \times_{\mathrm{Map}^{F\varphi}_{\mathrm{Cat}_S}(A,D)} \mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(A,C) \arrow[d] \\
\mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(B,C') \arrow[r] & Map^{u'F\varphi}_{\mathrm{Cat}_S}(B,D) \times_{\mathrm{Map}^{u'F\varphi}_{\mathrm{Cat}_S}(A,D)} \mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(A,C')
\end{tikzcd}$$

with the horizontal maps fibrations, and all the objects fibrant. Thus we have a weak equivalence

$$\mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(B,C) \to M = \mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(B,C') \times_{\mathrm{Map}^{F\varphi}_{\mathrm{Cat}_S}(B,D)} \mathrm{Map}^{\varphi}_{\mathrm{Cat}_S}(A,C)$$

of fibrations over $N = Map^{u'F\varphi}_{\mathrm{Cat}_S}(B,D) \times_{\mathrm{Map}^{u'F\varphi}_{\mathrm{Cat}_S}(A,D)} \mathrm{Map}^{u\varphi}_{\mathrm{Cat}_S}(A,C')$. The map α is given by a morphism $1_S \to M$ lifting the map v', uv' to N. Thus by applying Proposition 4.25, we can produce a lift to N, which gives the desired map. □

Example 5.8.1. Let S be the category of sets with the trivial model structure. Then we have exhibited a left proper combinatorial model structure on Cat. Cofibrations are functors that are injective on objects, weak equivalences are equivalences, and fibrations are **isofibrations**, meaning that we can always lift isomorphisms. Note that every object is cofibrant fibrant.

5.2. **Diagram model structures and homotopy limits.** Here we will explain how to put model structures on functor categories $\mathrm{Fun}(C,A) = A^C$ where A is an S-enriched model category, and C is a small S-category. This will yield homotopy limits and colimits.

Definition 5.9. *A morphism in A^C is an* ***injective cofibration*** *if it is levelwise a cofibration, and a* ***projective fibration*** *if it is levelwise a fibration, and a weak equivalence if it is a levelwise weak equivalence.*

Similarly, injective fibrations and projective cofibrations are defined by the appropriate lifting properties, and weak equivalences are defined pointwise. A^C is enriched, and pointwise tensored and cotensored over S.

Lemma 5.10. *Let A be a presentable category which is enriched, tensored and cotensored over a presentable category S, and let C be a small S-enriched category. Let $\overline{M}$ be a weakly saturated set of morphisms of A generated by a set. Let $\tilde{M}$ be the collection of morphisms $F \to G$ in A^C such that for every $c \in C$, the map $F(c) \to G(c)$ is in $\overline{M}$. Then $\tilde{M}$ is generated by a small set of morphisms.*

Proof. The proof strategy is analogous to that of Proposition 2.72.

Choose a regular uncountable cardinal κ satisfying

(i) C has fewer than κ objects.
(ii) Let $X, Y \in C$ and let $K = \mathrm{Map}_C(X, Y)$. Then the functor $x \mapsto x^K$ preserves κ-filtered colimits. This implies that κ-compact objects are stable with respect to $\otimes K$.
(iii) A is κ-accessible. Then A^C is also κ-accessible, and the κ-compact objects are those which are pointwise κ-compact.
(iv) $\overline{M}$ is generated by the subset M_0 consisting of maps between κ-compact objects.

Let M be the collection of morphisms in A^C that are pointwise in M_0. We would like to show that the saturation of M is all morphisms f that are pointwise in $\overline{M}$. Fix such a morphism $f : F \to G$. Corollary 2.73 implies for each $c \in C$ that there is a κ-good M_0-tree $\{Y(c)_\alpha\}_{\alpha \in A(c)}$ with root $F(c)$ with colimit $f(c)$.

Define a **slice** to be the following data:

(1) For each object $c \in C$, a downward-closed subset $B(c) \subset A(c)$.
(2) For each object $c \in C$, a morphism

$$\eta_c : \coprod_{c' \in C} Y(c')_{B(c')} \otimes \mathrm{Map}_A(c', c) \to Y(c)_{B(c)}$$

making the following diagrams commutative:

$$\begin{tikzcd}[column sep=-2em]
 & Y(c'')_{B(c'')} \otimes \mathrm{Map}_A(c', c) \otimes \mathrm{Map}(c', c) \arrow[dl, "\eta_{c'}"] \arrow[dr, "\circ"] & \\
Y(c')_{B(c')} \otimes \mathrm{Map}_A(c', c) \arrow[dr, "\eta_{c''}"] & & Y(c'')_{B(c'')} \otimes \mathrm{Map}_A(c'', c) \arrow[dl, "\eta_c"] \\
 & Y(c)_{B(c)} &
\end{tikzcd}$$

$$\begin{tikzcd}
F(c') \otimes \mathrm{Map}_A(c', c) \arrow[r] \arrow[d] & F(c) \arrow[d] \\
Y(c')_{B(c')} \otimes \mathrm{Map}_A(c', c) \arrow[r, "\eta_C"] \arrow[d] & Y(c)_{B(c)} \arrow[d] \\
G(c') \otimes \mathrm{Map}_A(c', c) \arrow[r] & G(c)
\end{tikzcd}$$

Note that (2) is the data making $c \mapsto Y(c)_{B(c)}$ into an S-enriched functor lying between F and G in A^C.

Lemma 5.11. *Suppose we are given a collection of κ-small subsets $\{B_0(c) \subset A(c)\}_{c\in C}$. Then there exists a slice $\{B(c), \eta_C\}_{c\in C}$ such that each $B(c)$ is a κ-small subset of $A(c)$ containing $B_0(c)$.*

Proof. WLOG, each $B_0(c)$ is downward closed. Note by (ii) that $Y(c')_{B_0(c')} \otimes \mathrm{Map}_A(c', c)$ is κ-compact, so that the map $Y(c')_{B_0(c')} \otimes \mathrm{Map}_A(c', c) \to G(c)$. factors through some map $Y(c')_{B_0(c')} \otimes \mathrm{Map}_A(c', c) \to Y(c)_{B_1(c)}$ where $B_1(c)$ is downward closed, κ-small, and contains $B_0(c)$. This way, we can inductively construct B_i and $(\eta_c)_i$ and use compactness again to ensure that condition (2) of a slice holds (after some shift of indices). Taking the union over i, since κ is uncountable, we obtain the desired slice. (See the proof of Lemma 2.68.) $\square$

Lemma 5.12. *Let $M' = \{(A'(c), \theta_c)\}_{c\in C}$ be a slice and let $\{B_0(c) \subset A(c)\}_{c\in C}$ be a collection of κ-small subsets of $A(c)$. Then there exists a pair of slices $N = \{(B(c), \eta_c)\}_{c\in C}, N' = \{(B(c) \cap A'(c), \eta'_c)\}$ where $B(c)$ is κ-small and N' is compatible with both N and M'.*

This proof is essentially that of Lemma 2.69. Namely, by repeatedly using compactness, one inductively constructs N_i and N'_i that satisfy the desired conditions (up to a shift of indices). Then the union over i will give the desired slices.

Lemma 5.13. *Suppose that $f : F \to G$ has the property that for each $c \in C$, there exists a pushout diagram*

$$\begin{tikzcd} X_c \arrow[r, "g_c"] \arrow[d] & Y_c \arrow[d] \\ F(c) \arrow[r, "f(c)"] & G(c) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $g_c \in M_0$. Then f is a pushout of a morphism in M.

Proof. This one is analogous to Lemma 2.71. Write F as the colimit of a κ-filtered poset of κ-compact objects $\{F_\lambda\}_{\lambda\in P}$. $X_C \to F(c)$ factors through some $F_[\lambda](c)$. C has fewer than κ objects so we can choose λ uniformly in c. We can replace P with the poset of things greater than λ.

For each $c \in C$, the composite map

$$\coprod_{c'\in C} Y_{c'} \otimes \mathrm{Map}_A(c', c) \to \coprod_{c'\in C} G(c') \otimes \mathrm{Map}_A(c', c) \to G(c)$$

factors through some $F_{\lambda'(c)}(c) \coprod_{X_c} Y_c$. since $G(c)$ is the colimit of $\{F_\lambda \cup_{X_c} Y_c\}_{\lambda\in P}$. Once again we can choose λ' uniformly and WLOG $\lambda' = \lambda$. By compactness again, we can assume that the diagrams below compute.

$$\begin{array}{ccc} X_{c'} \otimes \mathrm{Map}_A(c', c) & \longrightarrow & F_\lambda(c) \\ \downarrow & & \downarrow \\ Y_{c'} \otimes \mathrm{Map}_A(c', c) & \longrightarrow & F_\lambda(c) \cup_{X_c} Y_c \end{array}$$

$$\begin{array}{ccc} Y_{c''} \otimes \mathrm{Map}_A(c'', c') \otimes \mathrm{Map}_a(c', c) & \longrightarrow & Y_{c''} \otimes \mathrm{Map}_A(c'', c) \\ \downarrow & & \downarrow \\ (F_\lambda(c') \coprod_{X_{c'}} Y_{c'}) \otimes \mathrm{Map}_A(c', c) & \longrightarrow & F_\lambda(c) \coprod_{X_c} Y_c \end{array}$$

Then we can define $G_\lambda(c) = F_\lambda(c) \coprod_{X_c} Y_c$ and it follows that there is a pushout diagram

$$\begin{tikzcd} F_\lambda \arrow[r, "f_\lambda"] \arrow[d] & G_\lambda \arrow[d] \\ F \arrow[r, "f"] \arrow[ur, phantom, "\ulcorner", very near end] & G \end{tikzcd}$$

and $f_\lambda \in M$. $\square$

Now we are ready to complete the proof. Lemmas 5.11, 5.12 allow us to inductively construct a transfinite sequence of compatible slices $\{M(\gamma) = \{(B(\gamma)(c), \eta(\gamma)_c)\}_{c \in C}\}$. Let $G(\lambda)$ be the enriched functor corresponding to $M(\lambda)$, so the union of $G(\lambda)$ is G. It suffices to show that $\lim_{\gamma' < \gamma} G(\gamma') \to G(\gamma)$ is in M.

Pointwise, $f_\gamma(c)$ is the map $Y(c)_{B'(\gamma)(c)} \to Y(c)_{B(\gamma)(c)}$. Since $B(\gamma)(c) - B'(\gamma)(c)$ is κ-small, Lemma 2.66, 2.71, it followsw that $f_\gamma(c)$ is the pushout of a morphism in M_0. Then the result follows from Lemma 5.13. $\square$

Proposition 5.14. *Let S be an excellent model category and A a combintorial S-enriched category. Then there exists two combinatorial model structures on the category A^C:*

(1) The projective model structure, with weak equivalences and fibrations pointwise defined.

(2) The injective model structure, with weak equivalences and cofibrations pointwise defined.

Proof. First we consider the projective model structure. For an object $c \in C$ and $a \in A$, define $F_a^c : C \to A$ by $F_a^c(c') = A \otimes \mathrm{Map}_c(C, C')$. For any functor G, $\mathrm{Hom}(F_a^c, G) = \mathrm{Hom}(a, G(c))$ by the Yoneda lemma. It then follows that a morphism is a (not) trivial projective fibration iff it has the right lifting property with respect to $F_a^c \to F_{a'}^c$ where $a \to a'$ ranges over a generating set of (trivial) cofibrations.

The small object argument Proposition 4.8 shows that the factorization systems we desire exist and that the (trivial) cofibrations are generated by maps of the form $F_a^c \to F_{a'}^c$. The conditions of Lemma 4.11 are easily verified, so we obtain a model structure that is combinatorial.

For the injective model structure, we appeal to Corollary 4.40. We will check hypotheses $(1) - (5)$.

(1) Injective cofibrations are generated by a small set by Lemma 5.10, and are weakly saturated since cofibrations in A are.

(2) Trivial injective cofibrations are also clearly weakly saturated, since trivial cofibrations in A are weakly saturated.
(3) It is true that weak equivalences in $\mathrm{Fun}(C, A)$ are accessible. Indeed weak equivalences are an accessible subcategory of $A^{[1]}$ and it is generally true that accessible subcategories are stable under exponentiation in this way.
(4) 2 out of 3 for weak equivalences is clear.
(5) If f has the right lifting property with respect to injective cofibrations, it has the right lifting property with respect to constant functors, so each morphism in f is a trivial fibration: in particular a weak equivalence.

□

Remark 5.14.1. If A is left proper or right proper, then so is A^C since everything is pointwise. It also follows from the proof that a projective cofibration is an injective cofibration, and that an injective fibration is a projective fibration. It isn't hard to see that these model structures are S-enriched as well.

The following result is worth noting.

Proposition 5.15. *Let S be an excellent model category, C a small S-enriched category, and $F : A \leftrightarrows B : G$ an S-enriched Quillen adjunction between combinatorial S-enriched model categories. The composition with F and G determines another S-enriched Quillen adjunction $F : A^C \leftrightarrows B^C : G$ with respect to both the projective and injective model structures. If the original adjunction was a Quillen equivalence, then so is this one.*

Proof. This is trivial, since everything is computed pointwise. □

Note also that the identity functor gives a Quillen equivalence between the projective and injective model structures.

Now we consider functoriality in the variable C. If $C \to C'$ is a functor between small categories, then the composition yields a pullback functor $f^* : A^{C'} \to A^C$, which preserves all limits and colimits. If A admits all limits and colimits (which is true for a presentable category), then there is a right adjoint f_* and left adjoint $f_!$.

Proposition 5.16. *Let A be a combinatorial S-enriched model category over an excellent model category S, and let $f : C \to C'$ be a functor between small categories. Then*
(1) The pair $(f_!, f^*)$ determines a Quillen adjunction between the projective model structures on $A^C, A^{C'}$.
(2) The pair (f_*, f^*) determines a Quillen adjunction between the injective model structures on $A^C, A^{C'}$.

Proof. This is clear because f^* preserves all limits and colimits, as well as weak equivalences, fibrations, and cofibrations. □

The left derived functor of $f_!$ is called the **(homotopy) left Kan extension** and the right derived functor of f_* is called the **(homotopy) right Kan extension**.

This gives the following notion of a right Kan extension:

Definition 5.17. *Given a functor G with a map $\eta : G \to f_*F$, η* ***exhibits*** *G* ***as th homotopy right Kan extension of*** *F if for some injectively fibrant replacement of F', the composite $G \to f_*F \to f_*F'$ is an equivalence.*

Definition 5.18. *A **homotopy limit** is a right Kan extension along the map $C \to *$.*

We can reduce homotopy Kan extensions to homotopy limits.

Lemma 5.19. *Let A be a combinatorial model category and let $g : C' \to C$ be a functor exhibiting C' as cofibred in sets over C. Then the pullback functor $g^* : A^C \to A^{C'}$ preserves injective fibrations.*

Proof. It is equivalent to show that the left adjoint $g_!$ preserves injective trivial fibrations. To see this, we can observe that $g_!$ is pointwise a coproduct over the fibres, since C' is cofibred in sets over C. Coproducts preserve trivial cofibrations, so we are done. $\square$

Proposition 5.20. *Let A be a combinatorial model category, let $f : C \to D$ be a functor between small categories, and let $F : C \to A, G : D \to A$ be diagrams. A natural transformation $\alpha : f^*(G) \to F$ exhibits G as a homotopy right Kan extension iff for each $d \in D$, α exhibits $G(d)$ as a homotopy limit of the composite diagram $F_{d/} : C \times_D D_{d/} \to C \xrightarrow{F} A$.*

Proof. WLOG F is injectively fibrant. α is a homotopy right Kan extension iff the map $G(d) \to \lim F_{d/}$ is an equivalence by the formula for right Kan extensions. Then since the map $C \times_D D_{d/} \to C$ is cofibred in sets, by Lemma 5.19 the homotopy limit is the ordinary limit of $F_{d/}$. $\square$

The following is the model categorical version of right adjoints preserve right Kan extensions (limits).

Proposition 5.21. *Right adjoints preserve homotopy right Kan extensions.*

Proof. Simply observe that the preservation of ordinary right Kan extensions is a commutative diagram of right Quillen functors. Then the result follows from Lemma 4.56 $\square$

There is a dual theory of homotopy colimits.

Example 5.21.1. Let A be a combinatorial model category and consider a diagram

$$X' \xleftarrow{f} X \xrightarrow{g} X''$$

. This is injectively cofibrant iff X is cofibrant, and f, g are cofibrations. Indeed supposing those conditions, one can show the lifing property by first lifting X and then lifting X'', X' using the fact that f, g are cofibrations.

Conversely, by choosing diagrams to test against, the conditions are necessary. Thus the notion of homotopy pushout in Definition 4.29 agrees with this notion.

We can prove as a consequence Quillen's Theorem A.

Theorem 5.22. *Let $f : C \to D$ be a functor between categories such that for each $d \in D$, the category $C_{/d}$ is contractible. Then $N(C) \to N(D)$ is an equivalence.*

Proof. It is easy to identify $N(C)$ with the colimit of $N(C_{/d})$ for $d \in D$ since the nerve preserves colimits and C is clearly the colimit of $C_{/d}$. It suffices then to show that the diagram $D \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ given by $N(C_{/d})$ is a projective cofibration, since homotopy colimits are well defined up to equivalence. Let F_i be the i-skeleton of the functor $N(C_{/d})$. it suffices to show that $F_i \to F_{i+1}$ is a projective fibration.

From the proof of Proposition 5.14, the generating cofibrations of the projective model structure on $(\mathrm{Set}^{\Delta^{\mathrm{op}}})^D$ are $F^d_{\partial\Delta^n} \to F^d_{\Delta^n}$ as d, n varies. For every n-composable non-identity morphisms in C, there is a map $F^d_{\partial\Delta^n} \to F_{n-1}$. Pushing out all of these maps along $F^d_{\Delta^n}$, one obtains F_n. □

5.3. **Reedy model structures.** There is a useful model structure between the projective and injective model structures on diagram categories when the diagram is nice.

If $f \in g^{\oslash}$ and the lifts are unique, f is said to be **right orthogonal** to g. Recall that a replete subcategory is one that is closed under isomorphisms, and contains all isomorphisms between its objects. In otherwords, the inclusion is an isofibration.

Lemma 5.23. *Let J be a category and L, R a class of morphisms such that every morphism factors as something in L and then something in R. The following conditions are equivalent:*

(1) L and R are exactly the classes of morphisms orthogonal with respect to each other (with R on the right).
(2) The factorization is unique up to unique isomorphism and L, R contain isomorphisms and are subcategories.
(3) L and R are replete subcategories of the arrow category and R is right orthogonal to L.

Proof. (1) $\implies$ (2): Given two factorizations, we can lift each of them with respect to eachother, giving a unique isomorphism between them. L, R are clearly subcategories.

(2) $\implies$ (3): L, R are clearly replete. To see that R is right orthogonal to M, factor the horizontal arrows in the diagram you are trying to lift, and by uniqueness of the factorization, we get the desired lift.

(3) $\implies$ (1): Suppose that f is right orthogonal to L. Then create a lift in the diagram

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "h"] \arrow[ur, dashed, "i"] & \cdot \end{tikzcd}$$

where we have factor f into g in L and h in R. We can observed that $g \circ i$ and the identity are both solutions of the lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "g"] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "h"] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

so they are equal. It follows that $f \in R$ since it is isomorphic to $h \in R$. □

An (orthogonal) **factorization system** on a category J is a pair of subcategories satisfying the conditions of Lemma 5.23.

Definition 5.24. *A **Reedy category** is a small category J with a factorization system J^L, J^R satisfying*

- *Every isomorphism in J is the identity.*
- *Write $X \le_0 Y$ if there is either a map $X \to Y \in J^R$ or a map $Y \to X \in J^L$. $X <_0 Y$ if $X \le_0 Y$ and $X \ne Y$. Then there are no infinite descending chains $\cdots <_0 X_2 <_0 X_1 <_0 X_0$.*

Remark 5.24.1. $<_0$ is not generally transitive. Define $<$ to be its transitive closure. This is a well-founded partial order on the objects of J.

The point of a Reedy category is that functors out of it can be built inductively as we will see. This lets us for example produce a convenient model structure on functors out of it.

Example 5.24.1. The category Δ of simplices is a Reedy category with respect to the factorization system Δ^L, Δ^R where L are surjections and R are injections.

Note if J is a Reedy category, then so is J^{op} (with the obvious Reedy structure).

Definition 5.25. *Let J be a Reedy category and C a category with small limits and colimits, and $X : J \to C$ a functor. For every object $j \in J$, we define the **latching object** $L_j(X)$ to be the colimit $\operatorname{colim}_{j' \in J^R_{/j}, j' \neq j} X(j')$. The **matching object** $M_j(X)$ is the limit $\lim_{j' \in J^L_{j'/}, j' \neq j} X(j')$. There are canonical maps $L_j(X) \to X(j) \to M_j(X)$.*

Example 5.25.1. Let X be a simplicial set and regard Δ^{op} as a Reedy category as in Example 5.24.1. Then $L_{[n]}X$ is the collection of degenerate n-simplices of X. Given a map $f : X \to Y$ of simplicial sets, if the map $L_{[n]}(Y) \coprod_{L_{[n]}(X)} X_n \to Y_n$ is an monomorphism for each n, then $X \to Y$ is a monomorphism. Indeed, it says that the nondegenerate simplices of X get sent to nondegenerate simplices of Y. To see the converse, suppose f is a monomorphism. If $f(\sigma)$ is degenerate for some simplex σ, Then $f(\sigma) = f(\alpha^*(\sigma))$ where $\alpha : [n] \to [n]$ is a nonidentity map, so it follows that $\sigma = \alpha^*(\sigma)$.

Remark 5.25.1. Let $J \to C$ be a functor from a Reedy category to a bicomplete category. $M_j(X)$ can be identified with $\lim_{j' \in S} X(j')$ where S is any full subcategory of $J_{/j}$ such that

(1) Every non-isomorphism in J^R is in S.
(2) If $f : j' \to j$ is in S, then $j \not\leq j'$.

This is because these conditions imply that S is final in the category M_j is defined as the limit over.

Definition 5.26. *Let J be a Reedy category. A **good filtration** of J is a transfinite sequence $\{J_\beta\}_{\beta < \alpha}$ of full subcategories whose union is J, and such that each successor category is obtained by adding a minimal element with respect to $<$ not already there in the union of the ones before it.*

Every Reedy category has a good filtration essentially by definition.

6. Π_1 and Simplicial abelian groups

6.1. **Fundamental Groupoid.** For a simplicial set X, there are atleast three natural ways to make sense of the fundamental groupoid. The most obvious is homotopy classes of pointed maps from S^1 to X, which is the same as $\pi_1|X|$. Another model is to consider the path category P_*X with objects vertices and morphisms 1-simplices with relations given by the 2-simplices. $G(P_*X)$, the free groupoid generated by this category. Yet another model is $G(X_{\Delta/})$, where $X_{\Delta/}$ is the category of simplices of X.

Proposition 6.1. *For a simpliciel set X, there is a natural equivalence $\Pi_1|X| \simeq G(X_{\Delta/}) \simeq GP_*X$.*

Proof. Note that there is a natural *isomorphism* $GP_*S|X| \cong \Pi_1|X|$. Since the map $X \mapsto S|X|$ is an equivalence, for one of the equivalences, it suffices to show that GP_* takes weak equivalences to equivalences. Any weak equivalence factors as a trivial cofibration and a trivial fibration, and the trivial fibration will have a section, which is a trivial cofibration, so it suffices to show that GP_* takes trivial cofibrations to weak equivalences. Every trivial cofibration is a retract of a transfinite pushout of horn inclusions and equivalences of groupoids are closed under filtered colimits and retracts, so it suffices to show that a pushout of a horn inclusion is sent to an equivalence in GP_*.

For a pushout of an inclusion $\Lambda^n_i \to \Delta^n$, the induced map on GP_* is an isomorphism if $n \geq 2$. If $n = 1$, we are just adjoining an equivalent object, which is clearly equivalent. Thus GP_*X and Π_*X are equivalent.

There is a natural functor $GP_*X \to G(X_{\Delta/})$ given by sending $h : f \to g$ to the map $(d^0)^{-1}d^1$ below:

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^1"] \arrow[dr, "f"'] & \Delta^1 \arrow[d, dashed, "h"] & \Delta^0 \arrow[l, "d^0"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

There is also a functor in the other direction: it sends a simplex to its last vertex. It is easy to see that one composite is the identity, and the other is naturally isomorphic to the identity. $\square$

There is an adjunction $\mathrm{Fun}(P_*X, C) \cong \mathrm{Hom}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(X, BC)$.

Corollary 6.2. *If C is a category, $\pi_1 BC$ is equivalent to GC.*

Proof. $P_*BC = C$ as a category, so this follows from Proposition 6.1. $\square$

We note that the map $S|X| \to BP_*S|X| \to BGP_*S|X|$ induces an isomorphism on π_1. It is really the first Postnikov section.

Recall from Example 5.8.1 that Cat has a natural model structure. Gpd inherits a model structure as a reflective subcategory by Proposition 4.12. The fibrations are isofibrations, weak equivalences are equivalences, and cofibrations are maps that are injective on objects.

Let Π_1 be the functor $X \mapsto GP_*(X)$.

Lemma 6.3. $\Pi_1 : \mathrm{Set}^{\Delta^{\mathrm{op}}} \to \mathrm{Gpd}$ *is a left Quillen functor.*

Proof. We have $\mathrm{Hom}_{\mathrm{Gpd}}(GP_*(X), Y) \cong \mathrm{Hom}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(X, BY)$, so it is a left adjoint. Weak equivalences are sent to weak equivalences by the proof of Proposition 6.1, and cofibrations are sent to cofibrations. $\square$

The fact that this preserved homotopy pushouts is essentially Van Kampen's theorem. Indeed, suppose we are given a homotopy pushout diagram of simplicial sets

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C \cup_A B \end{tikzcd}$$

where i, j are injective on Π_0. Then we can compute $\Pi_1(C \cup_A B)$ as follows: first choose a cofibrant replacement of the diagram, apply Π_1, then replace the diagram again so that each connected component has only one object. The pushout is then a pushout of groups, so on each connected component, we get a formula for $\pi_1(C \cup_A B)$ in terms of a pushout of groups.

Note that there should be (and is) an n-truncated version of this result: the n-truncation functor on the ∞-category of spaces to n-truncated spaces = n-groupoids is a left adjoint, so preserves colimits. I think if you want a model category of n-truncated spaces, there should be a model structure where you add in the inclusions $\partial\Delta^m \to \Delta^m$ for $m > n$ to the trivial cofibrations.

6.2. **Local Systems and Covering Spaces.** Given a simplicial set X (considered as a space), a **local system** on X valued in a category C is a

6.3. **Simpicial objects in Abelian Categories.** Let B be an abelian category and A be a simplicial object.

Define CA to be the chain complex with $CA_n = A_n$ and differential $\sum_0^n (-1)^i d_i$. This is sometimes called the **Moore complex**.

Define NA to be the subcomplex with $NA_n = \cap_0^{n-1} \ker(d_i) \subset A_n$. This is sometimes called the **normalized chain complex**.

Define DA be the sum of the images of the degeneracies as a subcomplex of CA. It is a subcomplex because $d_i s_i = d_{i+1} s_i = 1$, so after applying the differential, the terms without degeneracies will cancel.

In an abelian category, an element will mean a morphism from some (unspecified) object. Alternatively, it can mean an actual element by the Freyd-Mitchell embedding theorem.

Lemma 6.4. *$CA = NA \oplus DA$.*

Proof. Clearly $NA \cap DA = 0$. It suffices to show then that every element x of CA is the sum of something in NA and something in DA. By downward induction we can assume that $d_i x = 0$ for $i > j$. Then $x + s_j d_j x$ is a modification by something in DA that has $d_j = 0$ in addition. □

Proposition 6.5. *The complex DA is acyclic, and the maps $NA \to A \to A/DA \cong NA$ are naturally homotopy equivalences.*

Proof. There is a filtration D_pA of DA where we consider the images of the first p degeneracies. At each level, the filtration is finite, so it suffices to show that the associated graded complexes $D_pA/D_{p-1}A$ are acyclic.

If we have an element $s_p x$, we can compute $ds_p^2 x - s_{p-1} d s_p x = (-1)^{p-1} s_p x$, so $s_p x$ is a boundary.

We can define a complementary filtration $(N_iA)_n = \cap_0^{\min(n-1,i)} \ker(d_i)$. The maps $f_j : N_jA_n \to N_{j+1}A_n$ sending $x \mapsto x$ if $n \leq j+1$ and $x \mapsto x - s_{i+1} d_{j+1}(x)$ otherwise give a retraction $N_j \to N_{j+1}$.

We can define $t_j : N_jA_n \to N_jA_{n+1}$ by $t_k(x) = (-1)^j s_{j+1}$ if $n \geq j+1$ and 0 otherwise, which exhibits f_j is a deformation retraction. □

Note that the proof actually showed that the identity map of $D_pA/D_{p-1}A$ is nulhomotopic.

Now we explain how to recover A from NA. Given a nonnegatively graded chain complex C, we can form $\Gamma(C)$, a simplicial object defined by $\Gamma(C)_n = \oplus_{n \twoheadrightarrow k} N_k$. Given a map $m \to n$, we use the map $\oplus_{n \twoheadrightarrow k} N_k \to \oplus_{m \twoheadrightarrow k} N_k$ that on $n \twoheadrightarrow k$ is the map $N_k \xrightarrow{d^*} N_l$ where the map $l \to k$ is the mono in epi-mono factorization of the composite $m \to n \to k$.

Theorem 6.6 (Dold-Kan correspondence). *$\Gamma : \mathrm{Ch}_{\geq 0} B \to B^{\Delta^{op}}$ is an equivalence of categories with inverse N.*

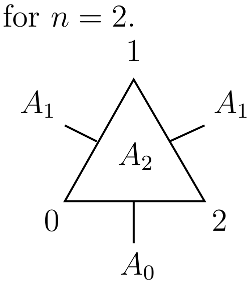
Proof. It is easy to see that $N\Gamma$ is the identity. We can construct a map $\Gamma NA \to A$ where for $n \hookrightarrow k$ we sent the corresponding NA_k component of ΓNA to A_n via the corresponding degeneracies. We will show by induction on n that in degree n it is an isomorphism. For $n = 0$, there is nothing to do. Suppose that it is and isomoprhism for $i < n$. Then the map is surjective, since it contains all degeneracies and also NA_n.

We will show that the map $\oplus_{n \twoheadrightarrow k} NA_k \to A_n$ is monic by induction on n. Every map $\sigma : n \twoheadrightarrow k$ has a section d_σ sending an element to the largest element in its preimage. We can form a partial ordering on surjective ordinal maps $n \twoheadrightarrow k$ by declaring $\sigma \geq \tau$ if $d_\sigma(i) \geq d_\tau(i)$ for all i. If $\tau \circ d_\sigma = 1$, then $\tau \geq \sigma$.

Suppose that (x_σ) is in the kernel, where x_σ is the component corresponding to $n \twoheadrightarrow k$. If we fix a $k < n$, then choose a maximal σ for which x_σ is nonzero. Then choosing the section d_σ and pulling back, the component corresponding to the identity is x_σ, so is 0 by the inductive hypothesis. Thus we can assume that the only possibly nonzero coefficient is NA_n, but NA_n embeds into A_n, so we are done. $\square$

Let $K(X, n)$ be the simplicial object corresponding to the chain complex that is X in degree n and zero elsewhere.

One can draw a picture of the Dold-Kan correspondence. It says that every simplicial object has its n-simplices look like a sum of normalized terms on every way to generate Δ^n, so we can draw a picture of an n-simplex with degeneracies labeled by which edges they contract. Below is shown the result for $n = 2$.



The following Lemma is obvious, since $D_n \subset A_n$ corresponds to the inclusion $\Lambda^n_n \to A_n$.

Lemma 6.7. *D_n is a cokernel of $\bigoplus_{0 \leq i < j \leq n-1} A_{n-2} \to \bigoplus_{0^{n-1}} A_{n-1}$, where the map is the difference of s_i and s_{j-1}.*

Lemma 6.8. *There is a natural isomorphism $\pi_n(A, 0) \cong H_n(NA)$.*

Proof. This is essentially the definition of $\pi_n(A, 0)$, only with a different operation. However, the Eckmann-Hilton argument shows that the two operations agree. $\square$

Theorem 6.9. *If B is a presentable abelian category with enough projectives, there is a right proper combinatorial simplicial model structure on $B^{\Delta^{op}}$ that has equivalences things inducing isomorphism on π_*, fibrations are maps that on the associated normalized chain complex are* *surjective in positive degree, and trivial fibrations are maps that are surjective with acyclic kernel. The generating cofibrations, trivial cofibrations are given by $M[\partial\Delta^n] \to M[\Delta^n]$ and $M[\Lambda^n_i] \to M[\Delta^n]$ respectively where M ranges over a set of projective generators.*

Proof. Choose a small set X_i of κ-compact projective generators, and let B' be the category consisting of them and their endomorphisms. Then the restricted Yoneda embedding gives an adjunction to $(\mathrm{Ab}^{B'})^{\Delta^{op}}$, which is a product of module categories. Now we can apply Proposition 4.64, where we use the model structure on R-modules in Example 4.66.1. The model structure is right proper since every object is fibrant.

We need only check that in $B^{\Delta^{op}}$ where B is a category of R-modules, the fibrations are as claimed, as the statement about acyclic fibrations is clear. Lemma 3.1 says that surjections are fibrations. Being able to lift $\Lambda^n_n \to \Delta^n$ implies that the map on normalized chain complex is surjective for $n \geq 1$. Conversely, suppose that $NX_i \to NY_i$ is surjective for $i \geq 1$. The map $K(\pi_0 X, 0) \to K(\pi_0 Y, 0)$ is a fibration since it is a map of discrete sets. Pulling back along the map $Y \to K(\pi_0, Y)$, we get the map $K(\pi_0 X, 0) \times_{K(\pi_0 Y, 0)} Y \to Y$ is a fibration, and since the map $X \to K(\pi_0 X, 0) \times_{K(\pi_0 Y, 0)} Y$ is surjective, so is the composite $X \to Y$. □

Proposition 6.10. *On the level of chain complexes, the model structure in Theorem 6.9 has as cofibrations the monomorphisms with cokernel degree-wise projective, and are generated by the maps $M[n] \to M\langle n+1\rangle$ for $n \geq -1$, where $M[n]$ is the chain complex M in degree n and otherwise 0 and $M\langle n+1\rangle$ is the chain complex that is M in degrees $n, n+1$ and 0 otherwise with differential the identity. Note that $M[-1] = 0$ and $M\langle 0\rangle = M[0]$.*

Furthermore, the trivial cofibrations are generated by $0 \to M\langle n\rangle$ for $n \geq 1$.

Proof. If $X \to Y$ is a cofibration, the pushout along $X \to 0$ is the cokernel Y/X, so the cokernel is cofibrant. Suppose that a cofibrant has the left lifting property with respect to trivial fibrations. Then in particular it has the left lifting property with respect to any surjection $M\langle m+1\rangle \to N\langle m+1\rangle$, showing that each term is projective.

Conversely, suppose $f : X \to Y$ is injective with a cokernel that is projective. We will show we can produce a lift in the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Z \arrow[d, two heads, "w", "\wr"'] \\ Y \arrow[r, "h"] & W \end{tikzcd}$$

First we will produce a lift on the n^{th} level of the chain complex. We can choose a level-wise splitting s of the projection $Y \to X/Y$, and then use projectivity to find a lift of hs along w, giving the desired lift.

Next, we need to show that we can modify these lifts to give a chain map that lifts. To do this, suppose that we have modified the maps fo agree on the j^{th} level for $j < i$. Then there are two different maps $(X/Y)_i \to Z_{i-1}$ that agree after projection to W_{i-1}, so they lift to N_{i-1}, where N is the kernel of $Z \to W$. Moreover, it is easy to see that it is a cocycle, so by acyclicity of N and projectivity of $(X/Y)_i$, it lifts to a map. $(X/Y)_i \to N_i$. We can then modify the map on the i^{th} level by this to make it compatible with the previous ones.

First we'll see that $0 \to M\langle n\rangle$ generate the trivial cofibrations. This is essentially immediate: $M\langle n\rangle$ corepresents maps from M to the n^{th} part of the chain complex. Then since M are projective generators, this is equivalent to surjectivity.

To see that the maps $M[n] \to M\langle n+1\rangle$ generate the cofibrations, suppose $X \to Y$ has the right lifting property with respect to each of them. This means that whenever there is a map from M to the cocycles of X which is a boundary in X, it is a boundary in Y. This shows that the map $\pi_* X \to \pi_* Y$ is injective. The pushout of $M[n] \to M\langle n+1\rangle$ along the map $M[n] \to 0$ is the map $0 \to M[n+1]$, so it follows that our map is surjective on π_*, and since we also have the right lifting property with respect to the composite $0 \to M[n] \to M\langle n+1\rangle$, it is a fibration. □

Proposition 6.11. *Let B be a projectively generated presentable abelian category, X a cofibrant simplcial object, and A another simplicial object. Two maps $X \to A$ are homotopic iff the corresponding maps on chain complexes are chain homotopic.*

Proof. It follows essentially from definition that a homotopy is the same data as a chain homotopy on the Moore complexes, so this follows from Proposition 6.5, since the Moore complexes are homotopy equivalent to the normalized complexes. □

Corollary 6.12. *Let R be a ring, and A a simplicial R-module, and X a simplicial set. Two maps $X \to A$ are homotopic iff the corresponding maps $R[X] \to A$ are chain homotopic.*

Proof. This follows from Proposition 6.11 and the fact that $R[X \times \Delta^1] = R[X] \otimes \Delta^1$. □

Lemma 6.13. *Let B be an abelian category, and X, Y simplicial objects over B. A homotopy $h : X \otimes \Delta^1 \to Y$ induces a chain homotopy of the associated chain complexes.*

Proof. Let $\theta_j : n \to 1$ be the map of posets with $\theta_j(i) = 0$ iff $i \leq j$. Given a map h, we define maps $h_j : A_n \to B_{n+1}$ by $h(s_j(a) \otimes \theta_j)$. Then the alternating sum $s = \sum_0^n (-1)^i h_i : A_n \to B_{n+1}$ is a chain homotopy. □

Lemma 6.14. *Let M a projective object in a projectively generated presentable abelian category, and X a simplicial set. The functor $X \mapsto M[X]$ preserves weak equivalences.*

Proof. As usual it suffices to show trivial cofibrations are sent to weak equivalences, as a trivial fibration has a section that is a trivial cofibration. But the right adjoint preserves fibrations, so this functor preserves trivial cofibrations. □

Theorem 6.15. *There is a natural isomorphism $H^n(X; M) \cong [X, K(B, n)]$ for any R-module M.*

Proof. $H^n(X; M)$ by definition is chain homotopy classes of maps from $R[X]$ to $M[n]$, which by Corollary 6.12 is the same as $[R[X].M[n]]$. By Lemma 4.6 and Lemma 6.14 and the fact that $R[X]$ is cofibrant, it follows that there is an isomorphism $[R[X], M[n]] = [X, K(M, n)]$. □

Proposition 6.16. *Let R be a hereditary ring (eg: a Dedekind domain). Then every non-negatively graded chain complex C of R modules is equivalent to $\oplus H_n(C)[n] = \prod H_n(C)[n]$. In particular, the underlying simplicial set under the Dold-Kan correspondence is equivalent to $\prod K(H_n(C), n)$.*

Proof. Choose surjections $F_n \mapsto Z_n(C)$ where F_n is free, and $Z_n(C)$ is the cycles of C. The kernel of the map $F_n \to Z_n(C) \to H_n(C)$, K_n, is projective, and so since the map $K_n \to Z_n(C)$ factors through $B_n(C)$, it lifts to C_{n+1}. Thus we have a chain map from a complex of the form $0 \to K_n \to F_n \to 0$ to C that induces an isomorphism on H_n. Moreover, this comples is clearly quasi-isomorphic to $0 \to H_n(C) \to 0$. Taking the sum of these, we get the first claim. The second claim follows from the fact that these are all fibrant simplicial sets, so the quasi-isomorphisms are equivalences of simplicial sets. $\square$

Given an R-module A, define $WK(A,n)$ to be the simplicial set corresponding to $A\langle n+1\rangle$. There is a fibre sequence:

$$K(A,n) \to WK(A,n) \to K(A,n+1)$$

induced from the exact sequence of chain complexes

$$0 \to A[n] \to A\langle n+1\rangle \to A[n+1] \to 0$$

Lemma 6.17. *$K(A,n)$ is a minimal Kan complex and $WK(A,n) \to K(A,n+1)$ is a minimal Kan fibration.*

Proof. Suppose x is a simplex homotopic to y relative to the boundary. We would like to show that x is y. By subtracting, we can assume y is 0, but this reduces us to thinking about the fibre, which is $K(A,n)$. If x is an r-simplex whose boundary is 0, then x is a normalized r-chain. There are no nonzero normalized r-chains unless $r = n$, so we can assume that $r = n$. If z is homotopic to 0 relative to the boundary, it represents the homotopy class 0, so must be 0. $\square$

7. Bisimplicial Sets

A bisimplicial object is a simplical simplicial object.

Example 7.0.1. We can consider $\Delta^{k,l}$, the bisimplicial set given by the simplicial sets $\cup_{n\to k}\Delta^l$. This is the functor corepresented in $\Delta \times \Delta$ by (k,l).

Example 7.0.2. Given a monoidal category, the **external product** of two simplicial objects $X\tilde{\otimes}Y$ is the one given by $(X\tilde{\otimes}Y)(n,m) = X_n \otimes Y_m$. For simplicial sets, the product is a monoidal product, and in this case we denote the external product $X\tilde{\times}Y$. The **diagonal** simplicial object of a bisimplicial object X is $X(n,n)$.

We define the **vertical simplicial object** of $X(n,m)$ to be $X_n = X(n,*)$. A morphism $\theta : m \to n$ gives rise to a diagram of simplicial sets

$$\begin{tikzcd} X_n \times \Delta^m \arrow[r, "1\times\theta_*"] \arrow[d, "\theta^*\times 1"] & X_n \times \Delta^n \\ X_m \times \Delta^m & \end{tikzcd}$$

There is a map $\gamma_n : X_n \times \Delta^n \to d(X)$ given by sending $x, r \mapsto r^*(x)$.

Lemma 7.1. *The map $\coprod_{m\to n} X_n \times \Delta^m \rightrightarrows \coprod_n X_n \times \Delta^n \xrightarrow{\gamma} d(X)$ is a coequalizer diagram.*

Proof. On r-simplices, this coequalizer computes $\mathrm{colim}_{\Delta^{op}/[r]}(X_n)_r$, which is just $(X_r)_r$ since there is a terminal object. $\square$

Lemma 7.1 gives a natural filtration on $d(X)$ coming from the image of $\coprod_{n\leq p} X_n \times \Delta^n$.
Moreover, the degenerate part of X_{p+1} with respect to the horizonal simplicial structure is filtered by $s_{[r]}X_p$ which is the union of s_iX_p for $i \leq r$.

Proposition 7.2. *Let $X \to Y$ be a map of bisimplicial sets that is a weak equivalence on the vertical simplicial sets. Then $d(X) \to d(Y)$ is a weak equivalence.*

Proof. There are pushout diagrams

$$\begin{array}{ccc} s_{[r]}X_{p-1} & \xrightarrow{s_{r+1}} & s_{[r]}X_p \\ \downarrow & & \downarrow \\ X_p & \xrightarrow{s_{r+1}} & s_{[r+1]}X_p \end{array} \qquad \begin{array}{ccc} (s_{[p]}X_p \times \Delta^{p+1}) \cup (X_{p+1} \times \partial\Delta^{p+1}) & \longrightarrow & d(X)^{(p)} \\ \downarrow & & \downarrow \\ X_{p+1} \times \Delta^{p+1} & \longrightarrow & d(X)^{(p+1)} \end{array}$$

which are homotopy pushouts since the indicated arrows are injections. Thus by induction it follows that the map $d(X)^{(p)} \to d(Y)^{(p)}$ is an equivalence for all p, and taking the union, we are done. $\square$

Let D be a category with coproducts, and C any small category. Given a functor $f : C \to D$, there is a simplicial object $E_C f$ whose n-simplices are $\coprod_{\sigma:[n]\to C} f(\sigma(0))$ called the **translation object**.

Given a composite $C' \xrightarrow{g} C \xrightarrow{f} D$, there is a natural map $E_{gf} \to E_f$. Moreover, given a natural transformation $f \to g$, there is an associated map $E_f \to E_g$.

Finally, observe that the simplicial object corresponding to the composite $C \times C' \xrightarrow{\pi} C \xrightarrow{f} D$ is $E_f \otimes BC'$ where $\otimes$ is the usual tensoring of simplicial objects by the simplicial set BC'.

Definition 7.3. *We define the **diagonal homotopy colimit** of $f : C \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ to be $d(E_C f)$ viewed as a bisimplicial set (this is nonstandard). We will later show this is equivalent to the usual homotopy colimit, and can be thought of as a bar resolution of the functor f.*

In this case, the n-simplices of $E_C f$ are the nerve of a category $E_C f_n$ whose objects are an object $c \in C$ and an n-simplex in $f(c)$. This category is called the **translation category**.

Example 7.3.1. Let G be a group acting on a simplicial set X. Then the homotopy colimit in this sense agrees with the Borel construction $EG \times_G X$. This sends the equivalence class $[e \to g_1 \to g_2g_1 \to \dots, x]$ to $x \to g_1x \to \dots$.

7.1. **Bisimplicial objects in an abelian category.**

Lemma 7.4. *Let D be an abelian category, and let $f : I \to D$ be a functor such that I has a specified terminal object t. Then there is a canonical homotopy equivalence $Ef \cong K(A_t, 0)$, given on n-simplices on each term by f applied to the unique map to t.*

Proof. The identity of I is homotopic to the constant map t, so applying this to the translation object construction, we get that Ef is simplicial homotopic to $K(A_t, 0) \otimes BI \cong K(A_t, 0)$ since I is contractible. $\square$

By the Dold-Kan correspondence, bisimplicial objects in an abelian category are the same as chain complexes of chain complex $C_{p,q}$ with a horizontal and vertical differential ∂_h and ∂_v that commute. By changing ∂_v to $(-1)^p\partial_v$, they anti commute and form a double complex. This gives an equivalence between first quadrant double complexes and bisimplicial objects.

Filtering a double complex in the horizontal direction gives a spectral sequence whose E_2-term is $H_p(H_qC_{**})$, converging to $H_{p+q}(\mathrm{Tot}(C_{**}))$ with a finite filtration in each degree. It follows that the homology of $\mathrm{Tot}(C_{**})$ is only dependent on the weak equivalence classes of the chain complexes in the vertical direction. There is also a diagonal simplicial chain complex $d(C)$ corresponding to the diagonal simplicial set. It is more convenient to work with the Moore complexes, since then $d(C)$ is just given by C_{nn} with the Moore differential $\sum_i (-1)^i \partial_i^h \partial_i^v$.

Proposition 7.5 (Eilenberg-Zilber)**.** *For a bisimplicial object, there is a natural homotopy equivalence $d(C) \cong \mathrm{Tot}(C)$.*

Proof. It suffices to work with the Moore complexes by Proposition 6.5. There is a map $f_{p,q} : C_{n,n} \to C_{p,q}$ where $p+q=n$ given by $\partial_{p+1}^h \dots \partial_n^h \partial_0^v \dots \partial_0^v$. Consider the **Alexander-Whitney map** given by $\sum_{p+q=n} f_{p,q} : dC_n \to \mathrm{Tot}(C)_n$. This gives a chain map $\mathrm{Tot}(C_{**}) \to D(C_{**})$. The **Eilenberg-Zilber map** in the other direction is given from $C_{p,q} \to C_{n,n}$ by $\sum_\mu (-1)^\mu s^h_{\mu(n)} \dots s^h_{\mu(p+1)} s^v_{\mu(p)} \dots s^v_{\mu(1)}$ where μ ranges over (p,q) shuffles. One can check that the Eilenberg-Zilber map composed with the Alexander-Whitney map is the identity. An explicit homotopy for the other composite to the identity is given by the **Shih operator** map $C_{m,m} \to C_{m+1,m+1}$ that in degree 0 is 0 and in positive degree is (signs and/or indices may be wrong):

$$\sum_{\substack{0 \le q \le m-1 \\ 0 \le p \le m-q-1 \\ \mu \in \{(p+1,q) \text{ shuffles}\}}} (-1)^{m'+1+\mu} s^h_{\mu(n)} \dots s^h_{\mu(p+2)} \partial^h_{m-q+1} \dots \partial^h_m s^v_{\mu(p+1)} \dots s^v_{\mu(1)} \partial^v_{m'} \dots \partial^v_{m-q-1}$$

where $m' = m-p-q$. There are more conceptual arguments for this, but it is interesting to know there is an explicit universal formula for the chain homotopy. □

Lemma 7.6. *A map $A \to B$ of bisimplicial objects in an abelian category D that is a quasi-isomorphism on vertical simplicial objects induces a weak equivalence $f_* : d(A) \to d(B)$ on the diagonal complexes.*

Proof. There is a bisimplicial abelian group which in vertical degree m is the translation object $EA(*,m)$ associated to the functor $A(*,m) : (\Delta_{/[m]})^{op} \to D$. Since $\Delta^{op}_{[m]}$ has a terminal object, there is a canonical homotopy equivalence by Lemma 7.4 from $EA(*,m) \to K(A(m.m),0)$. By letting m vary, we get a canonical weak equivalence $\mathrm{Tot}\, EA(*,*) \to d(A)$.

In horizontal degree k, the simplicial object is $\oplus_{f:[k]\to\Delta^{op}} \Delta^{f(0)} \otimes A(f(k),*)$, so it follows that this is invariant under weak equivalences, so the total complex, and hence $d(A)$, is too. □

Define the **horizontal normalization** $(N_h A)_n$ to be the simplicial chain complex whose n-simplices are $N_h A_n := NA(*,n)$. There is the Postnikov filtration $F_p(N_h A)_i$, defined as the chain complex obtained by truncating levelwise, so that it is concentrated in degrees $0 \le i \le p$. This is really the same filtration as the one used to define the spectral sequence of a double complex.

Lemma 7.7. *Let A be a bisimplicial object in an abelian category D. The natural map $\pi_n(N_h A)_p \to N(\pi_n A_*)_p$ is an isomorphism.*

Proof. We will inductively show that the natural map $\pi_n(N_h^j A)_p \to N^j(\pi_n A_*)_p$ is an isomorphism where

$$N_h^j A_p = \cap_0^j \ker(d_i^h) \subset A_p$$

When $j = 0$, we can use the fact that there is a split short exact sequence:

$$0 \to N_h^0(A_p) \to A_p \xrightarrow{d_0^h} A_{p-1} \to 0$$

split by s_0^h. We also obtain that $\pi_n(N_h^0 A)_p \to \pi_n A_p$ is monic. Suppose this holds for j. There is a pullback diagram

$$\begin{tikzcd} N_h^{j+1} A_p \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \ker(d_{j+1}^h) \arrow[d] \\ N_h^j(A_p) \arrow[r] & A_p \end{tikzcd}$$

and we can form the pushout $N_h^j A_p + \ker(d_{j+1}^h)$. The pullback square allows us to see that the map $N_h^{j+1} A_p \to N_h^j(A_p)$ is split by the map $1 - s_{j+1}^h d_{j+1}^h$. Thus it follows that we get a split pushout of objects in D

$$\begin{tikzcd} \pi_n N_h^{j+1} A_p \arrow[r] \arrow[d, hook] & \pi_n \ker(d_{j+1}^h) \arrow[d] \\ \pi_n N_h^j(A_p) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \pi_n(N_h^j A_p + \ker(d_{j+1}^h)) \end{tikzcd}$$

$\ker(d_{j+1}^h) \to \pi_n(N_h^j A_p + \ker(d_{j+1}^h))$ is split monic, and the map $\pi_n N_h^j(A_p) \to]\pi_n(A_p)$ by assumption is monic, so the map from the pushout is too, meaning that the sequence below is exact, giving the result.

$$0 \to \pi_n N_h^{j+1} A_p \to \pi_n N_h^j(A_p) \oplus \pi_n \ker(d_{j+1}^h) \to \pi_n A_p$$

□

Corollary 7.8. *There is a spectral sequence with E_2-term $\pi_p(\pi_q A_*)$, converging to $\pi_{p+q}(d(A))$ arising from the Postnikov filtration $F_p(N_h(A))$ on $N_h(A)$.*

Proof. $\Gamma F_p(N_h A)$ is a filtered bisimplicial object. We can apply the diagonal d to it to obtain a filtered chain complex.

The associated spectral sequence has E_1-term $\pi_{p+q} d(\Gamma N_h(A)[p])$ Thiere are natural isomorphism $\pi_{p+q} d(\Gamma N_h(A)[p]) \cong \pi_q d(\Gamma N_h(A)) \cong \pi_q \Gamma N_h(A)$, and the chain complex $\pi_* \Gamma N_h(A)$ by Lemma 7.7 can be identified with the complex $N(\pi_q(A))$. Thus the E_2-term is as stated. □

Lemma 7.9. *Let D be an abelian category with enough projectives. and $f : I \to D$ is a functor with I small. Then there is a natural isomorphism $\pi_n(Ef, 0) \cong L_n(\operatorname{colim} f)$.*

Proof. The groups $\pi_n(EA, 0)$ are the homology groups of the Moore complex of $H_n Ef$. $H_0 Ef$ is the cokernel $\oplus_{i \to j} f_i \xrightarrow{d_0 - d_1} \oplus_i f_i$ which is colim f by the formula for colimits. $\pi_n(EA, 0)$ is a δ-functor, so it suffices to check it is coeffacable. To see this, given a projective M and $i \in I$, define $F_i(M)_j = \oplus_{i \to j} M$. We can produce surjections from sums these to anything, and $EF_i M = M \otimes B(I_{/i})$ which has a terminal object, so it is contractible. □

Corollary 7.10. *Let $X : I \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ be a functor, and M a module over a ring R. There is a strongly convergent spectral sequence*

$$E_2^{p,q} = L_q(\operatorname{colim}_I) H_q(X, A) \to H_{p+q}(\lim X, A)$$

where $\lim X$ *is the diagonal homotopy colimit.*

7.2. **Model structures.** There are three closed model structures on bisimplicial sets:

- The *Bousfield-Kan structure*, which is another name for the projective model structure on $\mathrm{Fun}(\Delta^{op}, \mathrm{Set}^{\Delta^{\mathrm{op}}})$.
- The *Reedy structure*, which in this case just agrees with the injective model structure on $\mathrm{Fun}(\Delta^{op}, \mathrm{Set}^{\Delta^{\mathrm{op}}})$.
- The *Moerdijk structure*, in which a fibration or weak equivalence is a weak equivalence iff it induces an equivalence on diagonal simplicial sets.

For an n-truncated simplicial object there is a canonical comparison map $\varphi : (i_n)_! Z \to (i_n)_* Z$. For each compsite $[k] \xrightarrow{\gamma} [n+1] \xrightarrow{\theta} [m]$ with $k, m \leq n$ we get a commutative diagram

$$\begin{tikzcd} Z_m \arrow[r, "(\theta\gamma)^*"] \arrow[d, "\theta^*"] & Z_k \\ (i_n)_! Z_{n+1} \arrow[r, "\varphi"] & (i_n)_* Z_{n+1} \arrow[u, "\gamma^*"'] \end{tikzcd}$$

Lemma 7.11. *Giving an* $n+1$*-truncated simplicial object is the same as giving an* n*-truncated simplicial object* Z *along with an object* Z_{n+1} *and a factorization*

$$\begin{tikzcd} (i_n)_! Z_{n+1} \arrow[r] \arrow[dr, "\varphi"] & Z_{n+1} \arrow[d] \\ & (i_n)_* Z_{n+1} \end{tikzcd}$$

Proof. The horizontal map can be viewed as the degeneracies, and the vertical map can be viewed as the attaching maps. It is easy to check that this is equivalent to an $n+1$-truncated simplicial object by using the standard presentation of the simplex category. I think this is really just a property of Reedy categories. $\square$

We define the n^{th} **matching object** $M_n X$ of a simplicial object to be $(\mathrm{cosk}_{n-1} X)_n$, and dually, $L_n X = (\mathrm{sk}_{n-1} X)_n$ is the n^{th} **latching object**. A way to remember it is that the n^{th} latching object is the degenerate n-simplces. There are natural maps $L_n X \to X_n \to M_n X$.

Let C be a model category. Say that a map $X \to Y$ of simplicial objects is a **Reedy (trivial) fibration** iff the maps $X_n \to Y_n \times_{M_n Y} M_n X$ are (trivial) fibrations for $n \geq 0$.

Lemma 7.12. *The Reedy (trivial) fibrations on bisimplicial sets are injective (trivial) fibrations.*

Proof. We would like to show for example that a Reedy fibration $X \to Y$ has the right lifting property with respect to an injective trivial cofibration $U \to V$. We can show inductively that there exist compatible lifts on the n-truncations. In the inductive step, we are trying to lift

$$\begin{tikzcd} \mathrm{sk}_n V_{n+1} \cup_{\mathrm{sk}_n U_{n+1}} U_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ V_{n+1} \arrow[r] & Y_{n+1} \times_{\mathrm{cosk}_n Y_{n+1}} \mathrm{cosk}_n X_{n+1} \end{tikzcd}$$

The left vertical map is a cofibration, and is a trivial cofibration by the proof of Proposition 7.2 since $U \mapsto \mathrm{sk}_n U_{n+1}$ preserves weak equivalences and the model structure is left proper, so we can use the 2 out of 3 property on $U_{n+1} \to U_{n+1} \cup_{\mathrm{sk}_n U_{n+1}} \mathrm{sk}_n V_{n+1} \to V_{n+1}$. Thus the desired lift exists. A similar argument works for trivial Reedy fibrations. $\square$

Lemma 7.13. *Injective (trivial) fibrations are Reedy (trivial) fibrations.*

Proof. The left adjoint of $X \mapsto X_n$ is $K \mapsto F_n(K) := \Delta^n \tilde{\otimes} K$. A map $K \to \mathrm{cosk}_{n-1} X_n$ is the same as a map $i^*_{n-1} F_n(K) \to i^*_{n-1} X$, which is the same as a map $\mathrm{sk}_{n-1} F_n(K) \to X$. Thus lifting the diagram

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & X_n \arrow[d] \\
\Delta^m \arrow[r] & Y_n \times_{\mathrm{cosk}_{n-1} Y_n} \mathrm{cosk}_{n-1} X_n
\end{tikzcd}$$

is the same as lifting

$$\begin{tikzcd}
F_n(\partial\Delta^m) \cup_{\mathrm{sk}_{n-1} F_n(\partial\Delta^m)} \mathrm{sk}_{n-1} F_n\Delta^m \arrow[r] \arrow[d] & X \arrow[d] \\
F_n(\Delta^m) \arrow[r] & Y
\end{tikzcd}$$

which we can do for an injective trivial fibration. The argument for fibrations is the same. $\square$

Next, we study the Moerdijk model structure. The fibrations and weak equivalences are those that are after applying the diagonal. The diagonal functor $d = d^*$ from bisimplicial sets to simplicial sets has a left adjoint $d_!$ and a right adjoint d_*

Lemma 7.14. *The inclusion $d_!(\Lambda^n_k) \to d_!(\Delta^n)$ is a diagonal weak equivalence, and the map d_* sends fibrations to diagonal fibrations.*

Proof. $d_!(\Delta^n) = \Delta^{n,n}$, and $d^* d_!(\Delta^n) = \Delta^n \times \Delta^n$, so it is contractible. There is a natural transformation $d_! X \to * \tilde{\times} X$ determined by the maps $\Delta^{n,n} \to * \tilde{\times} \Delta^n$ that forget the left simplex.

This natural transformation for Λ^n_k is the projection

$$\coprod_{\beta \in \Lambda^n_k} C_\beta \to \coprod_{\beta \in \Lambda^n_k} *$$

where C_β is the subcomplex of Λ^n_k generated by faces containing β. Each C_β is contractible since it is a cone over the k^{th} vertex. Thus by Proposition 7.2, the map $d^* d_!(\Lambda^n_k) \to d^*(* \tilde{\times} \Lambda^n_k) = \Lambda^n_k$ is a weak equivalence, so $d^* d_!(\Lambda^n_k)$ is contractible too.

It follows that d_* sends fibrations to diagonal fibrations by Lemma 2.13. $\square$

Proposition 7.15. *Diagonal fibrations and diagonal weak equivalences form a combinatorial model structure called the Moerdijk model structure on bisimplicial sets.*

Proof. We would like to apply Proposition 4.12 for the adjunction between d^* and $d_!$. To do this, we only need to check that any map with the left lifting property with respect to diagonal fibrations is a diagonal equivalence. But by Lemma 7.14, it has the left lifting property with respect to $d_*(f)$ where f is a Kan fibration, so d applied to it has the left lifting property with respect to all Kan fibrations, and so is a trivial cofibration. $\square$

7.3. **Bousfield-Friedlander.** There is a construction $M_K(X)_p = \mathrm{Hom}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(K, X(*,p))$ generalizing $M_n(X)_p = M_{\partial\Delta^n}(X)_p$.

Given a bisimplicial set X, we can form $\pi_n(X)$, the simplicial set with m-simplices $\coprod_{x\in X(m,0)} \pi_n(X_m, x)$. This is equipped with a natural map to $X(*,0)$, and is a group object in ${\mathrm{Set}^{\Delta^{\mathrm{op}}}}_{/X(*,0)}$ for $m \geq 1$. A vertex v in $M_K(X)$ is a map $v : K \to X(*,0)$, so we can define $M_K(\pi_m X, v)$ to be $\mathrm{Hom}_{{\mathrm{Set}^{\Delta^{\mathrm{op}}}}_{/X(*,0)}}(v, \pi_m(X))$, which is a group for $m \geq 1$.

$M_K(\pi_m X, v)$ is a contravariant functor in the pair K, v, and it sends colimits to limits since it is given by Hom out of v. Note that $M_{\Delta^n}(\pi_m X, v) = \pi_m(X_n, v)$. Let dv be the restriction of an n-simplex v to some (unspecified) horn Λ^n_k. It follows that there is an equalizer diagram

$$M_{\Lambda^n_k}(\pi_m X, dv) \to \Pi_{i\neq k}\pi_m(X_{n-1}, d_i v) \rightrightarrows \Pi_{i<j;i,j\neq k}\pi_m(X_{n-1}, d_i d_j v)$$

A projectively fibrant bisimplicial set X is said to satisfy the **π_*-Kan condition** if the maps $d : \pi_m(X_n, v) \to M_{\lambda^n_k}(\pi_m(X), dv)$ are surjective group homomorphisms for $m \geq 1$. This is equivalent to the maps $\pi_m(X) \to X(*,0)$ being Kan fibrations for all $m \geq 1$.

Let $\pi_0(Y)$ for a bisimplicial set be the simplicial set defined by $\pi_0(Y)_n = \pi_0(Y_n)$.

Lemma 7.16. *(1) A projectively fibrant bisimplicial set X satisfies the π_*-Kan condition if all of the vertical simplicial sets X_n are path connected.*

(2) If $f : X \to Y$ is a vertical weak equivalence of projectively fibrant bisimplicial sets, then X satisfies the π_*-Kan condition iff Y does.

Proof. To show (1), we observe that we can choose a path from our base point in $X(n,0)$ to a degenerate one coming from $X(0,0)$. This induces a change of basepoint isomorphism, so we only need to check the π_*-Kan condition at degenerate basepoints $s(y)$. But the assignment $n \mapsto \pi_m(X_n, s(y))$ is a simplicial group, hence a Kan complex.

To see (2), observe that $X \to Y$ is a vertical weak equivalence iff $\pi_0 X \to \pi_0 Y$ is an isomorphism, and the diagrams

$$\begin{tikzcd}
\pi_m X \arrow[r] \arrow[d] & \pi_m Y \arrow[d] \\
X(*,0) \arrow[r] & Y(*,0)
\end{tikzcd}$$

are pullbacks for $m \geq 1$. Kan fibrations are stable under pullback, so if Y satisfies π_*-Kan, X does too.

Conversely suppose that the π_*-Kan condition holds for X. To check it for Y, we can do a change of basepoints to check it at basepoints in the image of X. But then it follows from the π_*-Kan condition for Y since the homotopy groups agree. $\square$

We say that an arbitrary bisimplicial set satisfies the π_*-Kan condition iff a projectively fibrant replacement does.

$X \mapsto M_K X$ is right adjoint to $Y \mapsto K\tilde{\times}Y$. Thus since $K\tilde{\times}\Delta^n \cup_{K\tilde{\times}\Lambda^n_k} L\tilde{\times}\Lambda^n_k \to L\tilde{\times}\Delta^n$ is a Reedy trivial cofibration for an inclusion $K \to L$, it follows that $M_L X \to M_L Y \times_{M_K Y} M_K X$ is a Kan fibration when $X \to Y$ is a Reedy fibration.

Lemma 7.17. *Suppose X is a Reedy fibrant bisimplicial set satisfying the π_*-Kan condition. Given a vertex $x \in X(n,*)$, there are isomorphisms*

$$\pi_m(M_{\Lambda^n_k} X, dx) \cong M_{\Lambda^n_k}(\pi_m X, dx)$$

$$\pi_0(M_{\Lambda^n_k}X) \cong M_{\Lambda^n_k}(\pi_0 X)$$

Proof. Let $\Delta^n\langle s_0, \ldots, s_r\rangle$ be the subcomplex of Δ^n generated by the faces $d_{s_0} \ldots d_{s_r}$, where $0 \leq s_0 < s_1 \cdots < s_r \leq n$. We can use $M_n^{(s_0,\ldots,s_r)}$ to denote $M_{\Delta^n\langle s_0,\ldots,s_r\rangle}$.

There is a pushout diagram

$$\begin{tikzcd}
\Delta^{n-1}\langle s_0, \ldots, s_{r-1}\rangle \arrow[r] \arrow[d] & \Delta^n\langle s_0, \ldots, s_{r-1}\rangle \arrow[d] \\
\Delta^{n-1} \arrow[r, "d_{s_r}"] \arrow[ur, phantom, "\ulcorner", very near end] & \Delta^n\langle s_0, \ldots, s_r\rangle
\end{tikzcd}$$

inducing a pullback square

$$\begin{tikzcd}
M_n^{(s_0,\ldots,s_r)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{n-1} \arrow[d, "d_{s_r}"] \\
M_n^{(s_0,\ldots,s_{r-1})} \arrow[r] & M_{n-1}^{(s_0,\ldots,s_{r-1})}
\end{tikzcd}$$

Choose a sequence s_i with $s_i \neq k$. The map d_{s_r} is a fibration since X is Reedy fibrant, so this is a homotopy pullback square. We can assume by induction that the map

$$\pi_m(M_n^{(s_0,\ldots,s_{r-1})}X, dx) \to M_n^{(s_0,\ldots,s_{r-1})}(\pi_m X, dx)$$

is an isomorphism, and similarly for $M_{n-1}^{(s_0,\ldots,s_{r-1})}$. The map $\pi_m(X_{n-1}, dx) \to M_{n-1}^{(s_0,\ldots,s_{r-1})}(\pi_m X, dx)$ is a surjection since X satisfies the π_*-Kan condition, so the square is a pullback, showing that the result holds for $s_0, \ldots, s_r$. A similar argument works for π_0, only one has to observe that the inclusion of the fibres of the vertical maps in the pullback square is injective on π_0 since the maps are surjective on π_1.

$$\begin{tikzcd}
\pi_*(M_n^{(s_0,\ldots,s_r)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \pi_*(X_{n-1}) \arrow[d, "d_{s_r}"] \\
\pi_*(M_n^{(s_0,\ldots,s_{r-1})}) \arrow[r] & \pi_*(M_{n-1}^{(s_0,\ldots,s_{r-1})})
\end{tikzcd}$$

is a pullback square from Proposition 4.32. $\square$

Lemma 7.18. *Suppose that X and Y are Reedy fibrant bisimplicial sets which satisfy the π_*-Kan condition and that the bisimplicial set map $f : X \to Y$ is a Reedy fibration. Suppose further that the induced simplicial set map of vertical path components $\pi_0 X \to \pi_0 Y$ is a Kan fibration. Then the map f is a horizontal pointwise Kan fibration.*

Proof. Being a horizontal pointwise Kan fibration is equivalent to the maps $X_n \to Y_n \times_{M_{\Lambda^n_k}Y} M_{\Lambda^n_k}X$ being surjective simplicial set maps. But since f is a Reedy fibration, this map is a Kan fibration, so it suffices to show it is surjective on π_0. By Lemma 7.17, this is equivalent to the map

$$\pi_0 X_n \to \pi_0 Y_n \times_{M_{\Lambda^n_k}\pi_0 Y} M_{\Lambda^n_k}\pi_0 X$$

being surjective, which is true since $\pi_0 X \to \pi_0 Y$ is a Kan fibration. $\square$

Lemma 7.19. *Suppose that $f : X \to Y$ is a Reedy fibration and a horizontal pointwise Kan fibration. Then f is a diagonal fibration.*

Proof. The cofibration $d^*\Lambda^n_k \to d^*\Delta^n = \Delta^{n,n}$ factors as a composite $d^*\Lambda^n_k \subset \Lambda^n_k \tilde{\times} \Delta^n \subset \Delta^n \tilde{\times} \Delta^n$. By the proof of Lemma 7.14, The first map is a vertical Reedy cofibration, and the second has the left lifting property with respect to horizontal pointwise fibrations. $\square$

A map of bisimplicial sets is pointwise homotopy cartesian if it is homotopy cartesian in each degree.

Theorem 7.20 (Bousfield-Friedlander)**.** *Given a pointwise homotopy cartesian square of bisimplicial sets*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & W \end{array}$$

such that Y, W satisfy the π_*-Kan condition and $\pi_0 Y \to \pi_0 W$ is a Kan fibration. Then the square

$$\begin{array}{ccc} d(X) & \longrightarrow & d(Y) \\ \big\downarrow & & \big\downarrow \\ d(Z) & \longrightarrow & d(W) \end{array}$$

is homotopy cartesian.

Proof. WLOG, we can assume that the map $p : Y \to W$ is a Reedy fibration and W is Reedy fibrant. By Lemma 7.18 and Lemma 7.19, p is then a diagonal fibration. Thus the square

$$\begin{array}{ccc} d(Z \times_W Y) & \longrightarrow & d(Y) \\ \big\downarrow & & \big\downarrow \\ d(Z) & \longrightarrow & d(W) \end{array}$$

is homotopy cartesian, so since the map $X \to Z \times_W Y$ is a weak equivalence, so is the square we wanted. $\square$

Corollary 7.21. *Suppose that X is a pointwise fibrant and pointwise connected bisimplicial set. Then there is a weak equivalence $d(\Omega X) \cong \Omega d(X)$.*

Proof. X and PX satisfy the π_*-Kan condition by Lemma 7.16, and $\pi_0 PX \to \pi_0 X$ is clearly a Kan fibration. Thus by Theorem 7.20, $d(\Omega X) \to d(PX) \to d(X)$ is a fibre sequence, and $d(PX)$ is contractible. $\square$

Proposition 7.22. *Suppose that X is a bisimplicial set which is pointwise k-connected. $d(X)$ is then k-connected.*

Proof. First, when $k = 0$, $\pi_0(d(X))$ is the coequalizer of the two boundary maps $\pi_0(X_1) \to \pi_0(X_0)$, so this follows. Now, we can induct on k and use Corollary 7.21 to get the result. $\square$

7.4. **Theorem B and Group Completion.** Given a map $f : E \to B$ of simplicial sets, there is a functor $f^{-1} : \Delta/B \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ where Δ/B is the category of simplices of B, taking a simplex to its fibre product under f.

B is the colimit in $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ of its simplices, and pulling back along f is a left adjoint, so we can identify $\lim_{\Delta/B} f^{-1} = E$.

Lemma 7.23. *Ef^{-1} is naturally equivalent to the constant bisimplicial simplicial set E.*

Proof. Let Ef^{-1}_m denote the category of m-simplices in f^{-1}. Because $\lim_{\Delta/B} f^{-1} = E$, the path components of this agree with E_m. Given an m-simplex $x \in E_m$, let $Ef^{-1}_{m,x}$ denote the path component corresponding to x. x viewed as an object of this category is a terminal object. It follows that the map $Ef^{-1}_m \to E_m$ is a weak equivalence of simplicial sets, so that $Ef^{-1} \to E$ is a weak equivalence, where E is a constant simplicial set. ☐

Corollary 7.24. *B is naturally equivalent to the homotopy colimit or diagonal homotopy colimit of the constant functor $B_{\Delta/} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ taking value $*$, which is equivalent to the nerve of $B_{\Delta/}$.*

Proof. By applying Lemma 7.23 to the identity functor, we see that the constant bisimplicial object B is equivalent to $E1_B^{-1}$. 1_B^{-1} is equivalent to the constant functor taking value $*$. Note that the homotopy colimit and diagonal agree on the constant bisimplicial object B, and so applying both of these functors to the zig-zag of equivalences gives the desired result. ☐

Given a map $f : E \to B$ and a projective object M in a projectively generated abelian category, by Lemma 6.14 $M[Ef^{-1}] \to M[E]$ is a weak equivalence.

There is thus a convergent spectral sequence with E_2-term $\pi_p\pi_q(M[Ef^{-1}])$ converging to $H_{p+q}(E; M)$. Now suppose that f is a Kan fibration. Then given a path σ in B, the induced map f_* on the fibres is an equivalence. Suppose $\pi_1(B)$ acts trivially on $H_q(F; M)$. Then for any $\sigma \in B_n$, $\pi_q(M[f^{-1}\sigma])$ is naturally isomorphic to $H_q(F; Ef^{-1})$. It follows that the E_2-term of the spectral sequence in this case is the same as $H_p(EB_{/\Delta}; H_q(F; M))$. But by Corollary 7.24, this is just $H_p(B; H_q(F; M))$.

Construction 7.24.1 (Serre Spectral Sequence)**.** Consider a fibration $E \to B$ with $\Pi_1(B)$ acting trivially on $H_q(F; M)$, where M is a projective object in an abelian category. There is a natural first quadrant spectral sequence with E_2-term $H_p(B; H_q(F, M))$ converging to $H_{p+q}(E; M)$.

Of course, this is weaker than the most general form of the Serre spectral sequence.

An important ingredient in the proof of Theorem B is the following lemma.

Lemma 7.25. *Suppose that $X : I \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ is a functor taking values in weak equivalences. Then for each $j \in I$, there is a homotopy cartesian diagram*

$$\begin{tikzcd} X(j) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\ * \arrow[r] & BI \end{tikzcd}$$

where Y is the diagonal homotopy colimit of X.

Proof. Note that applying the diagonal homotopy colimit to the natural transformation $X \to *$, there is a (not homotopy) pullback diagram as above. To see it is a homotopy pullback diagram, we will find a factorization $* \to U \to BI$ such that the first map is a trivial cofibration, the second is a fibration, and the map $X(j) \to U \times_{BI} Y$ is a weak equivalence. Pullbacks preserve colimits, which are stable under filtered colimits, so it suffices to check by the small object argument that $\Lambda^n_k \times_B IY \to \Delta^n \times_{BI} Y$ is an equivalence for any simplex σ in BI.

$\Delta^n \times_{BI} Y$ can be identified with the diagonal homotopy colimit of the composite functor $\Delta^n \xrightarrow{\sigma} I \xrightarrow{X} S$. The map $\Lambda^n_k \times_B IY \to \Delta^n \times_{BI} Y$ is the diagonal of the map between bisimplicial sets

$$\coprod_{k_0 \to \cdots \to k_r \in \Lambda^n_r} X\sigma(k_0) \to \coprod_{k_0 \to \cdots \to k_r \in \Delta^n} X\sigma(k_0)$$

But by the assumption on X, the natural transformation from $\Delta^n \to \{0\} \to \Delta^n$ to the identity induces an equivalence on the bisimplicial sets in the map above to those where $\sigma(k_0)$ is replaced with $\sigma(0)$. But the induced map between the diagonals of these bisimplicial sets is the map $\Lambda^n_k \times X\sigma(0) \to \Delta^n \times \sigma(0)$, which is an equivalence.

□

Theorem 7.26 (Theorem B). *Suppose $F : C \to D$ is a functor between small categories such that for every morphism $\alpha : y \to y'$ of D, the induced simplicial set map $\alpha^* : NF_{y/} \to NF_{y'/}$ is a weak equivalence. Then for every object, the diagram*

$$\begin{array}{ccc} NF_{y/} & \longrightarrow & NC \\ \downarrow & & \downarrow \\ ND_{y/} & \longrightarrow & ND \end{array}$$

is homotopy cartesian.

Proof. The assignment $y \mapsto NF_{y/}$ is a functor $D \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$, determining a bisimplicial set with (n,m) simplices $\coprod_{y_n \to \cdots \to y_0} (NF_{y_0/})_m$. In otherwords, it is strings of arrows

$$y_n \to \cdots \to y_0 \to F(x_0) \to \cdots \to F(x_m)$$

Forgetting $F(x_0) \to \cdots \to F(x_m)$ gives a diagram of bisimplicial sets

$$\begin{array}{ccc} \coprod_{y_n \to \cdots \to y_0} (NF_{y_0/}) & \longrightarrow & NC \\ \downarrow & & \downarrow \\ \coprod_{y_n \to \cdots \to y_0} (ND_{y_0/}) & \longrightarrow & ND \end{array}$$

In the horizontal direction the top horizontal maps above are given by

$$\coprod_{x_0 \to \cdots \to x_m} (D_{F(x_0)/})^{op} \to \coprod_{x_0 \to \cdots \to x_m} *$$

which is clearly an horizontal equivalence. The bottom horizontal map is a horizontal equivalence for the same reason.

Thus it suffices to we can replace the Cartesian diagram in the theorem statement with the left vertical map above. But $ND_{y/} \simeq *$ and $\coprod_{y_n \to \cdots \to y_0} (ND_{y_0/}) \simeq \coprod_{y_n \to \cdots \to y_0} *$ because the categories have terminal objects. Thus we are reduced to showing that the bisimplicial set diagram

$$\begin{tikzcd}
NF_{y/} \arrow[r] \arrow[d] & \coprod_{y_n\to\cdots\to y_0}(NF_{y_0})_m \arrow[d] \\
* \arrow[r] & \coprod_{y_n\to\cdots\to y_0} *
\end{tikzcd}$$

is Cartesian on diagonals, but this follows from Lemma 7.25. $\square$

8. Some Classical Homotopy Theory

8.1. **The Hurewicz Map.** Given a ring R, the R-Hurewicz map for a pointed simplicial set X is the map $\pi_n(X,*) \to \pi_n(R[X],*) \to \pi_n(R[X]/R[*]),0) = \tilde{H}_n(X;R)$. For example, for the n-sphere $\Delta^n/\partial\Delta^n$, the map given by the n-cell gets sent to the generator $1 \in \pi_n(K(R,n))$.

To analyse it the Hurewicz map, we will need the cellular formulation of the Serre spectral sequence. To get this, given a fibration $p : E \to B$, filter B by skeleta. $p^{-1}\operatorname{sk}_n B$ gives a filtration of $R[E]$. We assume again that π_1 acts trivially on homology. The Serre spectral sequence will be the associated spectral sequence of this filtration. For every boundary of a nondegenerate cell $\partial\Delta^n \to B$, the Serre spectral sequence $H_*(\partial\Delta^n; H_*F) \to H_*(p^{-1}(\partial\Delta^n))$ degenerates, because the inclusions $\Delta^{\{0\}} \to \partial\Delta^n \to \Delta^n$ splits off the fibre. Thus the E_1-term of the spectral sequence is $\tilde{H}^*(\bigvee p^{-1}\Delta/p^{-1}\partial\Delta^n)$, which is the sum over the nondegenerate n-cells of $H^*(F)[n]$. The d_1-differential is the alternating sum of the boundary maps, giving the E_2-term as $H^*(B;H^*(F))$.

The transgression in the Serre spectral sequence can be identified with the map

$$\begin{tikzcd}
 & & H_{n-1}F \arrow[d] \\
 & H_n(F_nE/F_{n-1}E) \arrow[r, "\partial"] \arrow[d] & H_{n-1}F_{n-1}E \\
H_n(\operatorname{sk}_n B) \arrow[r] \arrow[d] & H_n(\operatorname{sk}_n B/\operatorname{sk}_{n-1} B) & \\
H_nB & &
\end{tikzcd}$$

defined on the subset of elements where it makes sense.

Lemma 8.1. *If Y is n-connected for $n \geq 0$ then $\tilde{H}_i(Y) = 0$ for $i \leq n$.*

Proof. We can replace Y by a minimal Kan complex. It follows that $\tilde{H}_i(Y) = 0$ since there are no nondegenerate cells of dimension $\leq n$ other than the basepoint, so the normalized complex of $R[Y]$ vanishes in those degrees. $\square$

Lemma 8.2. *Suppose that $X \to \Omega Y$ is a map of pointed simplicial sets, where Y is n-connected for $n \geq 1$. Then for $i \leq 2n$ there is a commutative diagram*

$$\begin{tikzcd}
\tilde{H}_i(X \wedge S^1) \arrow[r, "\simeq"] \arrow[d, "\tilde{f}_*"] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\
\tilde{H}_i Y \arrow[r, "\simeq"] & \tilde{H}_{i-1}\Omega Y
\end{tikzcd}$$

where $\tilde{f}$ is the adjoint of f.

Proof. S^1 can be taken to be the simplicial circle $\Delta^1/\partial\Delta^1$. The top isomorphism is via the connecting homomorphism using the fact that $X \wedge S^1 = CX/X$, where CX is the cone on X, given by $X \wedge \Delta^1$.

To get the lower isomorphism, we use the fibration $\Omega Y \to PY \to Y$ that, and observe that the transgression has to be an isomorphism in degrees $\leq 2n$ by Lemma 8.1.

To see that the diagram is commutative, we apply homology to the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "d^1"] \arrow[d, "f"] & X \wedge \Delta^1_* \arrow[r] \arrow[d, "f_*"] & X \wedge S^1 \arrow[d, "\tilde{f}"] \\
\mathrm{Hom}_*(S^1, Y) \arrow[r, "\pi_*"] & \mathrm{Hom}_*(\Delta^1_*, Y) \arrow[r, "d_1"] & Y
\end{tikzcd}$$

to get

$$\begin{tikzcd}
\tilde{H}_i(X \wedge S^1) \arrow[d, "\tilde{f}_*"] & \tilde{H}_i(CX/X) \arrow[l, "\cong"'] \arrow[r, "\partial"] \arrow[d, "f_*"] & \tilde{H}_{i-1}X \arrow[d, "f_*"] \\
\tilde{H}_i(Y/*) & \tilde{H}_i(PY/\Omega Y) \arrow[l, "d_{1*}"] \arrow[r, "\partial"] & \tilde{H}_{i-1}\Omega Y
\end{tikzcd}$$

The bottom composite is the transgression, and the top composite is the suspension isomorphism. $\square$

Taking f to be the identity map $\Omega Y \to \Omega Y$ we obtain:

Corollary 8.3. *Let Y be n-connected for $n \geq 1$ and pointed. Then the transgression of the path space fibration for $i \leq 2n$ is given by*

$$\tilde{H}_i(Y) \xrightarrow{\epsilon_*^{-1}} \tilde{H}_i((\Omega Y) \wedge S^1) \xrightarrow{\partial} \tilde{H}_{i-1}(\Omega Y)$$

Proposition 8.4. *Suppose Y is a pointed and n-connected for $n \geq 1$. Then for $i \leq 2n$, there is a commutative diagram*

$$\begin{tikzcd}
\pi_i Y \arrow[r] \arrow[d] & \pi_{i-1}(\Omega Y) \arrow[d] \\
\tilde{H}_i Y \arrow[r] & \tilde{H}_{i-1}(\Omega Y)
\end{tikzcd}$$

Proof. This follows by applying the Hurewicz map to the path space fibration, and using Corollary 8.3 and the commutative diagram

$$
\begin{tikzcd}
\pi_i Y \arrow[rr, "\partial"] \arrow[d] & & \pi_{i-1}(\Omega Y) \arrow[d] \\
\tilde{H}_i(Y) \arrow[r, "\epsilon_*^{-1}"] & \tilde{H}_i((\Omega Y) \wedge S^1) \arrow[r, "\partial"] & \tilde{H}_{i-1}(\Omega Y)
\end{tikzcd}
$$

□

Now we will give an explicit description of the Postnikov tower. Consider the functor $Z \mapsto Z(r)$ that identifies simplices by the equivalence relation that checks whether they agree on the r-skeleton. There is clearly a natural transformation $Z \to Z(r) \to Z(r-1)$, giving the Postnikov tower. $Z(r)$ is called the r^{th} Postnikov section. Postnikov sections and the Postnikov tower also refer to any maps equivalent to these. This particular model is sometimes called the Moore-Postnikov tower.

Lemma 8.5. *The Postnikov tower is a tower of Kan complexes and Kan fibrations. If X is a Kan complex, then so is $X(r)$, and $\pi_i(X) = \pi_i(X(r))$ for $i \leq r$ and $\pi_i(X(r)) = 0$ at every basepoint.*

Proof. By Lemma 3.21 it suffices to show the statement about homotopy groups and that $X \to X(r)$ is a minimal Kan fibration. To see this is a Kan fibration, there is clearly a lift to every horn of an n-simplex for $n \leq r+1$. for $n > r+1$ any n-simplex in $X(r)$ will restrict to the horn in the right way because of the definition of $X(r)$, so the lifting property follows from surjectivity of $X \to X(r)$.

To see the claim about π_*, the vanishing result follows because π_* is given by maps from $\Delta^n/\partial\Delta^n$, which have to be trivial for $n > r$ since the boundary is that of the completely degenerate simplex. It is easy to see that $\pi_i(X) = \pi_i(X(r))$ for $i < r$. For $i = r$, we observe that the same equivalence relation on maps from $\Delta^r/\partial\Delta^r$ have been imposed since simplices have only been identified relative to the boundary. □

We can also use the notation $X_{\leq r}$ to denote $X(r)$ and $X_{\geq r+1}$ to denote the fibre of $X \to X(r)$.

Lemma 8.6. *Let X be connected. Then the Hurewicz map identifies $H_1(X;\mathbb{Z})$ with the abelianization of $\pi_1(X)$.*

Proof. We can choose a model for X having one vertex. Then the fundamental group is generated by 1-simplices modulo the relation $[\sigma] = [\sigma'][\sigma'']$ when there is a 2-simplex forcing that relation. H_1 has the same generators but with the abelianized relations. □

Lemma 8.7. *The simplicial circle S^1 is a $K(\mathbb{Z}, 1)$.*

Proof. By Proposition 6.1, its π_1 is $\mathbb{Z}$. It suffices to show $\pi_i(S^1) = 0$ for $i > 1$. To see this, consider the complex $S^1_{\geq 2}$ that consist of an integer's worth of Δ^1 such that the end of the i^{th} on is identified with the beginning of the $i+1^{th}$ one. The map $S^1_{\geq 2} \to S^1$ is a Kan fibration, and it is easy to see that $S^1_{\geq 2}$ has the homotopy type of a point, since the inclusion of any point is anodyne. The fibres are discrete, giving the desired vanishing. □

Theorem 8.8 (Hurewicz)**.** *Suppose that X is n-connected. Then the Hurewicz map $\pi_i X \to \tilde{H}_i(X;\mathbb{Z})$ is an isomorphism for $i \leq n+1$ and a surjection for $i = n+2$.*

Proof. Consider the fibre sequence $X_{\geq n+2} \to X \to X_{\leq n+1}$. By the naturality of the Hurewicz map and the Serre spectral sequence, we reduce to the case of $X_{\leq n+1}$. By Proposition 8.4 the isomorphism in dimension $n+1$ reduces to the case $n = 0$, where it follows from Lemma 8.6.

So assume X has homotopy groups only in dimension $n+1$. The map $X \to \mathbb{Z}[X] \to \mathbb{Z}[X]_{\leq n+1}$ (where the truncation is taken in simplicial abelian groups) is an equivalence, so X is WLOG $K(\pi_{n+1}(X), n+1)$, so it suffices to show $\tilde{H}_{n+2}(K(A, n+1))$ vanishes for any abelian group A. A is a filtered colimit of finitely generated abelian groups which $\tilde{H}_{n+2}$ commutes with, so it suffices to show it for $\mathbb{Z}$ and $\mathbb{Z}/p^i$. For $\mathbb{Z}/p^i$, we can reduce to $\mathbb{Z}/p$ by using the extensions $\mathbb{Z}/p \to \mathbb{Z}/p^i \to \mathbb{Z}/p^{i-1}$.

Next, we can use Proposition 8.4 to reduce to the case $n = 2$. Using the path fibre sequence $K(\mathbb{Z}, 1) \to * \to K(\mathbb{Z}, 2)$ and the Serre spectral sequence, we get the result for $\mathbb{Z}$ using Lemma 8.7. We can use the fiber sequence $K(\mathbb{Z}, 2) \xrightarrow{p} K(\mathbb{Z}, 2) \to K(\mathbb{Z}/p, 2)$ and examine the spectral sequence to get the result for $K(\mathbb{Z}/p, 2)$. $\square$

Theorem 8.9 (Freudenthal). *Suppose X is an n-connected space with $n \geq 0$. Then the map $X \to \Omega(X \wedge S^1)$ is $2n$-connected.*

Proof. For $i = 0$, we can observe that the map on π_1 factors as $\pi_1(X) \to H_1(X) = H_1(\Omega X \wedge S^1) = \pi_1(\Omega X \wedge S^1)$, so the result follows by Lemma 8.6.

For $i > 0$, we examine the triangle

$$\begin{tikzcd} X \wedge S^1 \arrow[rr, equal] \arrow[dr] & & X \wedge S^1 \\ & \Omega(X \wedge S^1) \wedge S^1 \arrow[ur] & \end{tikzcd}$$

and use Corollary 8.3 to learn that it induces an isomorphism in $\tilde{H}_i$ for $i \leq 2n+1$. Thus by examining the Serre spectral sequence for the fibre of $X \to \Omega(X \wedge S^1)$ and using Hurewicz, we learn that the fibre F is $2n$-connected. $\square$

Theorem 8.10 (Relative Hurewicz). *Suppose that $f : X \to Y$ is a map with fibre F and cofibre Y/X. Suppose that f is n-connected, X is 1-connected, and Y is connected. Then $F \wedge S^1 \to Y/X$ is $n+2$-connected.*

Proof. Observe first that Y is actually simply connected by assumption on f. Thus the Serre spectral sequence for $F \to X \to Y$ gives an exact sequence

$$H_{n+3}Y \xrightarrow{d_{n+3}} E^{0,n+2}_{n+3} \to H_{n+2}X \to H_{n+2}Y \xrightarrow{d_{n+2}} H_{n+1}F \to H_{n+1}X \to \cdots$$

From the description of the transgression, there is a comparison of exact sequences

$$\begin{tikzcd} H_{n+3}(X/F) \arrow[r, "\partial"] \arrow[d, two heads] & H_{n+2}F \arrow[r] \arrow[d, two heads] & H_{n+2} \arrow[r] \arrow[d, equal] & H_{n+2}(X/F) \arrow[r, "\partial"] \arrow[d, two heads] & \cdots \\ H_{n+3}Y \arrow[r, "d_{n+3}"] & E^{0,n+2}_{n+3} \arrow[r] & H_{n+2}(X) \arrow[r] & H_{n+2}Y \arrow[r, "d_{n+2}"] & \cdots \end{tikzcd}$$

The left hand map is surjective since every element of $H_{n+3}Y$ is transgressive. The rest of the maps $H_i(X/F) \to H_iY$ for $i < n+3$ are isomorphisms by the 5-lemma. Thus by comparing the long exact sequences on homology for $X \to X/F \to \Sigma F$ and $X \to Y \to Y/X$, we learn that the map $F \wedge S^1 \to Y/X$ is $n+2$-connected. □

Corollary 8.11. *Suppose that A is simply connected and the pair (X, A) is n-connected for $n \geq 1$. Then the Hurewicz map $\pi_i(X, A) \to \tilde{H}_i(X, A)$ is an isomorphism for $i \leq n+2$ and surjective for $i = n+2$.*

Proof. The relative homotopy groups are the homotopy groups of the homotopy fibre F, and the Hurewicz map is the composite

$$\pi_i(F) \xrightarrow{\Sigma} \pi_{i+1}\Sigma F \to \pi_i(X/A) \to \tilde{H}_i(X/A)$$

, which is an isomorphism by a combination of Theorem 8.9 and the Theorem 8.10. □

Theorem 8.12. *Let X be a pointed Kan complex and let $\alpha_i, 0 \leq i \leq n+1$ be pointed maps from $\Delta^n/\partial\Delta^n \to X$. Then there is an n-simplex such that $d_i w = \alpha_i$ iff $\sum_0^{n+1}(-1)^i[\alpha_i] = 0$ in $\pi_n(X)$.*

Proof. It is easy for $n = 1$ by definition of multiplication on π_1, so WLOG $n \geq 2$. Consider the fibre sequence $X_{\geq n} \to X \to X(n)$. F is $n-1$-connected, so we can assume by the homotopy extension property that each α_i lies in the fibre. We can also assume that w lives in the fibre, since $p(w)$ is nulhomotopy relative to the boundary, so we can choose a lift a nulhomotopy in $X(n)$ to a homotopy in X.

Thus we can assume X is $n-1$-connected. By the Hurewicz theorem, it follows that if we have such a w, then $\sum_0^{n+1}(-1)^i[\alpha_i]$ because it is dw in the Moore complex under the Hurewicz map.

Conversely, suppose that the formula is satisfied. we can form an extension in the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\alpha_1,\dots,\alpha_{n+1}"] \arrow[d] & X \\ \Delta^{n+1} \arrow[ur, "\theta"] & \end{tikzcd}$$

By examining the homology class of $d_0\theta$, we see it is the same as α_0, so by Hurewicz, it is homotopic to α_0. Thus by using the homotopy extension property, we can replace θ with the desired simplex. □

9. Models for Homotopy Types

9.1. **Test Categories.** Let X be a simplicial set. We can put a homotopical structure on the Cat by declaring $C \to D$ to be an equivalence if $BC \to BD$ is. By Corollary 7.24, the map $X \to B(X_{\Delta/})$ is an equivalence. It follows that the homotopy 1-category of Cat agrees with that of simplicial sets.

This construction can be attempted on any small category. Let A be a small category, and $X \in \mathrm{Set}^{A^{op}}$, which we will call an A-set. We can then construct the category $X_{A/}$ and declare the weak equivalences of A-sets to be those such that $B(-)_{A/}$ is a weak equivalence. Then it was asked by Grothendieck in Pursuing stacks, when does the map $i_A : X \to X_{A/}$ induce an equivalence on homotopy 1-categories?

i_A has a right adjoint i_A^*, sending C to $a \mapsto \mathrm{Hom}(A_{/a}, C)$. $i_A^* i_A(C)$ is the category with objects functors $A_{/a} \to C$ for some $a \in A$. The counit $\epsilon : i_A^* i_A(C) \to C$ of the adjunction sends such a functor F to $F(1_A)$.

Lemma 9.1. *There is an isomoprhism $\epsilon_{/c} \cong i_A^* i_A(C_{/c})$.*

Proof. An object of $i_A^* i_A(C_{/c})$ is a functor $A_{/a} \to C_{/c}$. But a functor $A_{/a} \to C_{/c}$ is the same as a functor $A_{/a} \to C$ sending 1_A to C, which is an object of $\epsilon_{/c}$. ☐

We would like to have conditions so that ϵ is a natural weak equivalence. The triangle identity will then show that the unit map is also a natural weak equivalence, and we will get the equivalence of homotopy categories.

Definition 9.2. *A **weak test category** A is a small category such that ϵ is a weak equivalence for all small categories C.*

Definition 9.3. *A functor $f : C \to D$ is **aspherical** if $B(f_{/d})$ is contractible for all $d \in D$.*

A category is aspherical if the map to the terminal category is, meaning it is contractible. From Quillen's Theorem A (Theorem 5.22), it follows that an aspherical map is a weak equivalence. We say that a map of A-sets is aspherical if after applying i_A, it is aspherical. An A-set X is itself aspherical if the map to a point is aspherical, which is the same as saying that the map $X_{A/} \to A$ is aspherical.

Lemma 9.4. *TFAE:*

(1) A is a weak test category.
(2) If D is a category with a terminal object, $i_A^*(D)$ is equivalent to a point.
(3) If C is aspherical, then $i_A^*(C)$ is equivalent to a point.

Proof. (1) $\implies$ (3) $\implies$ (2) is clear since they are just special cases. (2) $\implies$ (1) follows from Quillen's Theorem A and Lemma 9.1 ☐

Definition 9.5. *A **local test category** A is a category such that $A_{/a}$ is a weak test category for all a.*

Lemma 9.6. *TFAE:*

(1) A is a local test category.
(2) If D is a category with a terminal object, $i_A^*(D)$ is aspherical.
(3) If C is aspherical, then $i_A^*(C)$ is aspherical.

Proof. This follows from Lemma 9.4 and the fact that

$$i_A i_A^*(C)_{/a} \cong i_{A/a} i_{A/a}^*(C)$$

☐

Definition 9.7. *A **test category** is a weak test category that is also a local test category.*

Lemma 9.8. *A category is a test category iff it is a local test category and is aspherical.*

Proof. This is because aspherical maps are weak equivalences, and the 2 out of 3 property, and the characterizations in Lemma 9.6 and Lemma 9.4. ☐

Observe that $i_A(X)_{/a} = i_A(X \times a)$. This is the key to the next example.

Example 9.8.1. The simplex category Δ is a test category. We already know it is a weak test category, and to see it is a local test category, we just need to observe that $i_\Delta(X)_{/\Delta^n} = i_\Delta(X \times \Delta^n)$, so that if C is aspherical, then $i_{\Delta/\Delta^n} i_{\Delta/\Delta^n}^*(C)$ is too since it is equivalent to $i_\Delta^*(C)$.

Lemma 9.9. *Suppose that A, B are small categories and that $f : X \to Y$ is a morphism of $A \times B$-sets. If f induces weak equivalences of B-sets for each $a \in A$, then f is a weak equivalence of $A \times B$-sets.*

Proof. Consider the composite $i_{A\times B}X \xrightarrow{\pi_X} A \times B \xrightarrow{p} A$. There is a functor $\omega_a : i_B X(a, -) \to p\pi_X/_a$ sending a object $x \in X(a, b)$ to the object given by $1_a, x$. There is a functor γ_a in the other direction sending $f : a \to a', x \in X(a', b)$ in $p\pi_X/_a$ to $(f, 1)^* x$ in $X(a', b)$. $\gamma_a \omega_a$ is the identity, and there is a natural transformation $\omega_a \gamma_a \to 1$ given by $(f, 1) : (1_a, (f, 1)^*(x)) \to (f, x)$. Thus we get a canonical homotopy equivalence $i_B X(a, -) \simeq (p\pi_X)_{/a}$.

Thus by the assumption, we get weak equivalences $(p\pi_X)_{/a} \to (p\pi_X)_{/a}$ for each $a \in A$. By taking (diagonal) homotopy colimits over A, we get that $Bi_{A\times B}(X) \to Bi_{A\times B}(Y)$ is an equivalence. □

Proposition 9.10. *Suppose A is a local test category and B is a small category. Then $A \times B$ is a local test category.*

Proof. Let C be a category with a terminal object t. It suffices to show that $i *_{A\times B} C_{/(a,b)}$ is equivalent to a point. To see this, we first observe that $A \times B_{/(a,b)} = A_{/a} \times B_{/b}$.

Thus $i^*_{A_{/a}\times B_{/b}} C(a', b') = Hom((A_{/a})_{/a'} \times (B_{/b})_{/b'}, C) = \mathrm{Hom}((A_{/a})_{/a'}, C^{(B_{/b})_{/b'}})$. $C^{(B_{/b})_{/b'}}$ has a terminal object, so all the $A_{/a}$-sets are pointwise equivalent to a point, and by Lemma 9.9, we are done. □

Corollary 9.11. *Suppose A is a test category and B is an aspherical category. Then $A \times B$ is a test category.*

Proof. $A \times B$ is aspherical because A and B are, and $A \times B$ is a local test category by Proposition 9.10 □

Lemma 9.12. *Suppose that A, B are small categories and B is aspherical. Let p^* be the pullback functor from A-sets to $A \times B$ sets. then p^* reflects weak equivalences.*

Proof. $i_{A\times B} p^* X = i_A X \times B$. □

Let A be a category. Let p be the projection as in the lemma above, let i be the cocontinuous map from A-sets to $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ sending a to $B(A_{/a})$, and let i^* be its right adjoint. Let j be the cocontinuous functor from $A \times \Delta$-sets to $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ sending a, n to $B(A_{/a}) \times \Delta^n$. This has a right adjoint j^*. Let q^* be the pullback functor from $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ to $A \times \Delta$-sets.

Lemma 9.13. *Suppose A is a local test category. Then there are natural weak equivalences of $A \times \Delta$-sets*

$$p^* i^* X \to j^* X \leftarrow q^* X$$

for any simplicial set X.

Proof. There is a map $q^* X(a, *) \to j^* X(a, *)$ sending $X \to X^{B(A_{/a})}$ via the terminal object. Since this is an equivalence for each a, by Lemma 9.9 it is an equivalence.

The map $p^* i^* X \to j^* X$ in simplicial degree n is a map of A-sets $X^{B(A_{/a}} \to (X^{\Delta^n})^{B(A_{/a})}$. But this is a componentwise weak equivalence because Δ^n is contractible, so again by Lemma 9.9 it is an equivalence. □

Corollary 9.14. *The functor i^* preserves weak equivalences.*

Proof. q^* preserves weak equivalences by Lemma 9.9, so p^*i^* does too by Lemma 9.13. Since p^* reflects weak equivalences by Lemma 9.12, i^* preserves weak equivalences. $\square$

The proof of Lemma 9.13 shows:

Lemma 9.15. *Suppose A is a small category such that $i_A(a)$ has a terminal object for all a and $i^*(\Delta^1)$ is aspherical. Then there is a natural weak equivalence of $A \times \Delta$-sets*

$$p^*i^*X \to j^*X \leftarrow q^*X$$

for any simplicial set X.

Corollary 9.16. *Suppose that in addition to the assumptions of Lemma 9.15, A is aspherical, so it is a test category. Then i^* preserves and reflects weak equivalences.*

Proof. From Lemma 9.9, q^* preserves and reflects weak equivalences, so p^*i^* does too by Lemma 9.15. p^* does as well by Lemma 9.12, so i^* does too. $\square$

The following gives a way of producing local test categories.

Lemma 9.17. *(1) Suppose that A is a local test category and X is an A-set. Then i_AX is a local test category.*
(2) The category of i_AX-sets is equivalent to A-sets over X.

Proof. The second claim is straightforward, and for the first, we use the natural isomorphism $i_AX_{/a} \cong A_{/a}$ for any map $a \to X$. $\square$

Corollary 9.18. *Suppose A is a local test category and Y is an A-set. Then the functor $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ to A-sets defined by $X \mapsto Y \times i^*X$ preserves weak equivalences.*

Proof. Write $i^*_{[A]}X$ for i^*X to emphasize the dependence on A. There is an isomorphism $i^*_{[i_AY]}X \cong Y \times i^*_{[A]}X$, so the result follows from Lemma 9.17. $\square$

$i_\square(B_\square C \times \square^n)$

9.2. **Basics of Cubical Sets.** The category $\square$ has the n-cubes $\square^n$ as objects for $n \geq 0$. The maps $\square^n \to \square^m$ can be identified with functors $(\Delta^1)^n \to (\Delta^1)^m$ which are composites of coordinate projections and face inclusions.

There is a monoidal product $\otimes : \square \otimes \square \to \square$ corresponding to the isomorphism $(\Delta^1)^n \times (\Delta^1)^m = (\Delta^1)^{n+m}$. This is not symmetric monoidal.

Let $\square^{\leq 1}$ be the full subcategory of $\square^0$ and $\square^1$. Here is the universal property of $\square$:

Proposition 9.19. *Let C be a monoidal category. For any functor $F : \square^{\leq 1} \to C$, there is a unique (up to natural isomorphism) extension to a functor $\tilde{F} : \square \to C$ which is monoidal.*

Corollary 9.20. *Let C be a cocomplete monoidal category. For any functor $F : \square^{\leq 1} \to C$, there is a unique (up to natural isomorphism) extension to a cocontinuous functor $\tilde{F} : \mathrm{Set}_\square \to C$ which is lax monoidal, and monoidal if the tensor product on C preserves colimits. It has a left adjoint, the singular cubical set of the cocubical object given by $X \mapsto \mathrm{Hom}_C(F(\square^i), X)$.*

The monoidal structure on $\square$ gives rise to a Day convolution product $\otimes$ on $\mathrm{Set}_\square$. For $X, Y \in \mathrm{Set}_\square$, this is given as the left Kan extension of $X \times Y : \square^{op} \times \square^{op} \to \mathrm{Set}$ along the tensor product map. Explicitly, $X \otimes Y = \mathrm{colim}_{\square^n \to X, \square^m \to Y} \square^{n+m}$. Furthermore, $\mathrm{Set}_\square$ is closed: we can define $\hom(C, D)_n = \hom(C \otimes \square^n, D)$.

The category $\square$ has an orthogonal factorization system into epis, called degeneracy maps, and monos, called face maps.

Example 9.20.1. The objects $(\Delta^1)^n$ form a cocubical object $\square \to \mathrm{Set}^{\Delta^{op}}$. This gives a pair of adjoint functors $|\cdot| : \mathrm{Set}_\square \leftrightarrows \mathrm{Set}^{\Delta^{op}} : S$, the singular cubical set and simplcial realization. This is monoidal with respect to the product structure on $\mathrm{Set}^{\Delta^{op}}$.

The functor $\square \to \mathrm{Set}^{\Delta^{op}}$ factors through Cat. The right adjoint of the extension of this map to $\mathrm{Set}_\square$ is the cubical nerve functor $B_\square(C)$.

Similarly to simplicial sets, we can make sense of degenerate simplices and n-skeleta. An n-skeleton is left Kan extended from $\square^{\leq n}$.

Lemma 9.21. *If x, y are generate n-cells of a cubical set X whose boundaries agree, then $x = y$.*

Proof. The proof is basically that of Lemma 3.11. $\square$

Define $\partial\square^n$ the way you would expect. The following lemma then holds, analogously to Lemma 2.17.

Lemma 9.22. *The saturation of the inclusions $\partial\square^n \to \square^n$ are all inclusions.*

The analogs of the horns are $\sqcap^n_{\epsilon,i}$, which are defined by removing the ϵ, i face from $\partial\square^n$, where $\epsilon \in \{0,1\}, 0 \leq i \leq n$. $|\partial\square^n| \to |\square^n|$ is an anodyne extension since it is an inclusion and both are contractible.

Lemma 9.23. *Suppose x, y are n-cells of a cubical set X such that the induced maps of simplicial sets $x_*, y_* : |\square^n| \to |X|$ agree. Then x, y agree.*

Proof. From induction on n and Lemma 9.21, we can assume y is nondegenerate, and that the boundaries of x, y agree. We can also replace X by the subcomplex generated by the $n-1$-skeleton, x, and y.

If $x \neq y$, let X_0 be the subcomplex generated by the $n-1$-skeleton and x. X is obtained from X_0 by adjoining an n-cell. The fact that $x_* = y_*$ gives a lift in the pushout diagram

$$\begin{tikzcd}
{|\partial\square^n|} \arrow[r] \arrow[d] & {|X_0|} \arrow[d] \\
{|\square^n|} \arrow[ur, "x_*"] \arrow[r, "y_*"] & {|X|}
\end{tikzcd}$$

This shows that $|X_0| \to |X|$ must be surjective, which is a contradiction. $\square$

Corollary 9.24. *If $X \to Y$ is a map of cubical sets such that $f_* : |X| \to |Y|$ is a monomorphism, then $X \to Y$ is a monomorphism. If it is an isomorphism of cubical sets, then so is $X \to Y$.*

Proof. For the second statement, if an inclusion $X \to Y$ is not an isomorphism, it is obtained by adjoining cells, which would give nontrivial inclusions after applying $|\cdot|$. $\square$

Lemma 9.25. *Suppose that C has a terminal object. Then the functor $i_\square B_\square C \to \square$ is aspherical.*

Proof. This amounts to showing that for each n, $i_\square(B_\square C \times \square^n)$ is contractible. its objects consist of a functor $f : \square^k \to C$ and a map $\sigma : \square^k \to \square^n$.

Since C has a terminal object t, there is a natural homotopy to the constant map to the terminal object:

$$\begin{tikzcd} \square^k \arrow[d, "d_0"'] \arrow[dr, "f"] & \\ \square^{k+1} \arrow[r, "hf"] & C \\ \square^k \arrow[u, "d_1"] \arrow[ur, "t"'] & \end{tikzcd}$$

This gives a deformation retraction from $i_\square(B_\square C \times \square^n)$ to $i_\square \square^n$, which is aspherical since it has a terminal object, the identity. $\square$

Proposition 9.26. *Suppose that $i : A \to \mathrm{Cat}$ is functor on a small category. Let i^* be the left adjoint of the extension to A-sets, and suppose that*

(1) $i(a)$ has a terminal object for each a.
(2) If D has a terminal object, then i^*D is aspherical.

Then A is a local test category.

Proof. Let D be a category with a terminal object. By Lemma 9.6, it suffices to check that $i_A(i_A^* D \times a)$ is aspherical for each $a \in A$. Picking a terminal object for each $i(a)$, there is a functor $\square_{/a} \to i(a)$ sending a map $\theta : a' \to a$ to θ applied to the terminal object of $i(a')$. This induces an A-set map $i^*C \to i_A^* C$.

Suppose that h is a contracting homotopy of D, and consider the composite

$$i_A^*(D) \times B_A(\Delta^1) \xrightarrow{1\times\alpha} i_A^* D \times i_A^* \Delta^1 \cong i_A^*(D \times \Delta^1) \xrightarrow{h} i_A^*(D)$$

This provides a nulhomotopy from $i_A^*(D)$ to the point, where we use (2) to see that it is really a weak equivalence of A-sets. The nulhomotopy multiplied with an object a shows that in fact $i_A^*(D) \times a \to a$ is an equivalence, so that $i_A(i_A^* D \times a)$ is aspherical. $\square$

Corollary 9.27. *$\square$ is a test category.*

Proof. By Proposition 9.26 and Lemma 9.25, it is a local test category, and it has a terminal object, so is aspherical. $\square$

9.3. **Cisinski model structures.** Let A be a small category. Let Set_A denote the category of A-sets, and let C be a set of monomorphisms of Set_A.

Definition 9.28. *An **interval theory** is an action on the category Set_A by the monoidal category $\square$, subject to the conditions:*

(I1) $(-) \otimes \square^1$ preserves monomorphisms and filtered colimits.
(I2) For every monomorphism $i : X \to Y$ and coface $d : \square^{n-1} \to \square^n$, the square

$$\begin{tikzcd} X \otimes \square^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \otimes \square^{n-1} \arrow[d] \\ X \otimes \square^n \arrow[r] & Y \otimes \square^n \end{tikzcd}$$

is a pullback.

(I3) For $1 \le i \le n$, the square

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \otimes \square^{n-1} \arrow[d] \\ X \otimes \square^n \arrow[r] & Y \otimes \square^n \end{tikzcd}$$

is a pullback.

The interval theory makes Set_A tensored over $\mathrm{Set}_\square$. Moreover, there is a cubical function space $\hom_\square(X,Y)_n = \hom(X \otimes \square^n, Y)$, giving an adjunction $\hom(X \otimes K, Y) = \hom(K, \hom_\square(X,Y))$.

Lemma 9.29. *The map $X \otimes \partial\square^n \to X \otimes \square^n$ is an inclusion.*

Proof. By induction on the number of cells using $(I2), (I3)$, we can show that the canonical map $X \otimes \partial\square^n \to \cup_{i,\epsilon} X \otimes \square^{n-1}$ is an isomorphism ($\cup$ denotes the not disjoint union). $\square$

It follows that for any monomorphism $K \to L$, the map $X \otimes K \to X \otimes L$ is an monomorphism.

Example 9.29.1. If I is an A-presheaf equipped with a monomorphism $(d_0, d_1) : * \coprod * \to I$, then the assignment $X \otimes \square^n = X \times I^n$ gives an action that is an interval theory.

Example 9.29.2. The tensor product gives $\mathrm{Set}_\square$ the structure of an interval theory. $(I1), (I3)$ are easy, and $(I2)$ can be proven by using Corollary 9.24.

Remark 9.29.1. For any inclusion $K \subset L$ in $\mathrm{Set}_\square$ and inclusion $X \to Y$ of A-sets, the map $Y \otimes K \cup_{X \otimes K} X \otimes L \to Y \otimes K \cup X \otimes L$ is an isomorphism, where the latter is the union as subobjects of $Y \otimes L$. This essentially follows from $(I2)$ and the fact that it suffices to prove it for the inclusions $\partial\square^n \to \square^n$.

The **anodyne** $(\otimes, S)$**-cofibrations** (or just anodyne cofibrations) are the saturated class of morphisms generated by the inclusions

(A1) $(Y \otimes \square^n) \cup (a \otimes \sqcap^n_{\epsilon,i}) \to a \otimes \square^n$ for all subobjects Y of $a \in A$

(A2) $A \otimes \square^n \cup B \otimes \partial\square^n \to B \otimes \square^n$ for all monomorphisms $A \to B$ in S.

We define the **naive fibrations** to be the maps that have the left lifting property with respect to all anodyne cofibrations. A cofibration is an inclusion. Notice that since the category is presheaves on a small category, the inclusions are generated by a small set under filtered colimits.

A **naive homotopy** between two maps $X \to Y$ is a map $X \otimes \square^1 \to Y$ restricting to the two maps. If Y is a naively fibrant object, then this is an equivalence relation by using the anodyne extensions $X \otimes \sqcap^n_{\epsilon,i} \to X \otimes \square^n$.

We say that $X \to Y$ is a weak equivalence iff it induces a bijection on naive homotopy classes of maps to any naively fibrant object. Our model category on A-set valued presheaves will be defined using cofibrations and weak equivalences.

Lemma 9.30. *There is a cardinal κ such that $|X \otimes \square^n| < \lambda$ if $|X| < \lambda$ for all $\lambda > \kappa$.*

Proof. Choose an infinite cardinal β such that $|A| < \beta$. Choose κ sot hat $|A \otimes \square^n| < \kappa$ when $|A| < \beta$. Then κ works. $\square$

Lemma 9.31. *If $C \to D$ is an anodyne cofibration, then so is $C \otimes \square^1 \cup D \otimes \partial\square^1 \to D \otimes \square^1$.*

Proof. Use essentially the proof of Corollary 2.22. □

Lemma 9.32. *An anodyne cofibration is a weak equivalence.*

Proof. Suppose that $C \to D$ is an anodyne cofibration. Then it induces a surjection on maps to a naively fibrant object by the lifting property. It is an injection on homotopy classes by Lemma 9.31. □

Lemma 9.33. *For all sufficiently large cardinals κ, given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ Z \arrow[r] & Y \end{tikzcd}$$

of cofibrations of A-sets such that i is an equivalence and $|Z| < \kappa$, there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < \kappa$ and $B \cap X \to B$ is an equivalence.

Proof. Let F be the naive fibrant replacement functor coming from the small object argument. $FX \to FY$ is a weak equivalence by the 3 out of 4 property and the fact that $X \to FX$ is by Lemma 9.32. Thus it is a naive homotopy equivalence. By the homotopy extension property, we can assume that the homotopy inverse σ of Fi is a left inverse. Let h be a homotopy from $Fi \circ \sigma$ to the identity. Let κ be sufficiently large. If Z is a subobject of Y such that Z is κ-small, by Lemma 9.30 and the fact that F preserves κ-filtered colimits, the homotopy H restricted to $FZ \otimes \square^1$ factors through FZ' for some κ-small FZ'. By repeating this, we form a sequence of κ-small subobjects Z_i such that the homotopy restricted to Z_i factors through Z_{i+1}. Then $\cup Z_i$ has the property that $F(\cup Z_i \cap X) \to FZ$ is a naive homotopy equivalence, so by the 3 out of 4 property, we get the desired result. □

Lemma 9.34. *Suppose given a diagram*

$$\begin{tikzcd} C \arrow[r, "f,g"] \arrow[d, "i"] & E \\ D & \end{tikzcd}$$

that i is a cofibration and f, g are naively homotopic. Then the map $D \to D \cup_f E$ is an equivalence iff $D \to D \cup_g E$ is.

Proof. Considering the diagram

$$\begin{tikzcd} D \arrow[r, tail, "\sim"] & D \cup_C (C \otimes \square^1) \arrow[r] \arrow[d, tail, "\wr"'] & D \cup_f E \arrow[d, tail, "\wr"'] \\ & D \otimes \square^1 \arrow[r] & (D \otimes \square^1) \cup_h E \end{tikzcd}$$

where the indicated maps are anodyne, we see that the map for f is an equivalence iff the lower map is. The same is true for g, giving the result. □

Lemma 9.35. *Suppose that $i : C \to D$ is an inclusion that is an equivalence. Then the cofibration $C \otimes \square^1 \cup D \otimes \partial\square^1 \to D \otimes \square^1$ is an equivalence.*

Proof. Let F be the naive fibrant replacement functor from the small object argument. using the 3 out of 4 property, we can replace D with FD. Similarly, we can factor $C \to D$ through an anodyne map and a naive fibration that is an equivalence. But then it is a homotopy equivalence, so by Lemma 9.34, we can reduce to when $C \to D$ is an anodyne cofibration, when this follows by Lemma 9.31. $\square$

Lemma 9.36. *The inclusions which are weak equivalences are closed under pushouts.*

Proof. Suppose we are givne a pushout

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"] \arrow[dr, phantom, "\ulcorner", very near end] & C' \arrow[d, "j'"] \\ D \arrow[r] & D' \end{tikzcd}$$

with j a weak equivalence and an inclusion. Then any map from C' into an fibrant object Z can be extended to D'. On the other hand by Lemma 9.35, in the diagram

$$\begin{tikzcd} C \otimes \square^1 \cup D \otimes \partial\square^1 \arrow[r] \arrow[d] & C' \otimes \square^1 \cup D' \otimes \partial\square^1 \arrow[d] \\ D \otimes \square^1 \arrow[r] & D' \otimes \square^1 \end{tikzcd}$$

the left vertical map is an equivalence and an inclusion, so homotopies from C' to Z can be extended to D'. $\square$

Lemma 9.37. *Suppose $p : X \to Y$ is a naive fibration of naively fibrant objects. Then p is a fibration.*

Proof. Suppose then that we have a square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"] & Y \end{tikzcd}$$

$\square$

where i is a cofibration and an equivalence. Then there is a map $\theta : B \to X$ making the upper right triangle commute since i is an equivalence there is a homotopy extension property from A to B, and X is naively fibrant. The commutative square for θ is homotopic to the original diagram, and by finding a lift in the diagram

$$\begin{tikzcd} A \otimes \square^1 \cup B \arrow[r] \arrow[d] & X \arrow[d] \\ B \otimes \square^1 \arrow[r] & Y \end{tikzcd}$$

we can produced the originally desired lift.

Corollary 9.38. *The notions fibrant and naively fibrant coincide.*

More generally, naive fibrations and fibrations to a fibrant object coincide.

Theorem 9.39. *Suppose A is a small category with an interval theory, and S a set of morphisms in A-sets. Then the cofibrations, fibrations and weak equivalences as defined give a combinatorial left proper model structure on A-sets.*

Proof. From Lemma 9.33 and Lemma 9.36, the cofibrations that are weak equivalences are generated by a small set, and the cofibrations are clearly generated by a small set. These then give two factorization systems. By Lemma 4.11, it suffices to check that trivial fibrations are weak equivalences.

Given a map $f : X \to Y$, there is a section s. sf is homotopic to the identity because of the lifting property, so f is a homotopy equivalence. $\square$

When the model structure is defined by an interval object I, then we call it the (I, S)-model structure.

Theorem 9.40. *Let the interval theory on A-sets be defined by an interval object I. Suppose that for any diagram*

$$\begin{tikzcd} U \times_Y X \arrow[r, "f_*"] \arrow[d] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ U \arrow[r, "f"] & V \arrow[r] & Y \end{tikzcd}$$

where p is a fibration, Y fibrant, $f \in S$, the map f_* is a weak equivalence.
Then the (I, S) model structure on A-sets is proper.

Proof. Let W be the class of maps $U \to V$ satisfying the condition on f in the statement of the theorem. W is closed under transfinite composition and retracts. The cofibrations in W are closed under pushout by an arbitrary morphism, because a basechange of a pushout square of sets where one map is an injection is a still a pushout square, and trivial cofibrations are stable under cobase change. Furthermore, W satisfies a weak 2 out of 3 property: if $g \circ f$ and f are in W, then g is too. This follows from the 2 out of 3 property.

By assumption W contains S, and it is easy to see that W contains all projections $K \otimes \square^n \to K$. Next, we see that W contains all the standard inclusions $K \to K \otimes \square^1$. Indeed, Let Y be an object with a fibration to $K \otimes \square^1$ and Y' be the pullback to K. Then by creating a lift in the diagram

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "\wr"'] & Y \arrow[d, two heads] \\ Y \otimes \square^1 \arrow[r, "h"] \arrow[ur, dashed, "H"] & Z \otimes \square^1 \end{tikzcd}$$

where h is the homotopy contracting the image of Y to the copy of Z over which Y' lives, we get a map $g : Y \to Y$ that factors through Y' by the universal property of pullback. By construction, the composite $f_* \circ g$ is homotopic to the identity via H. By the universal property of pullback again, the homotopy $H_{|Y'}$ factors through Y', giving a homotopy between $f_* \circ g$ and the identity.

It follows that whether a map is in W depends only on its naive homotopy class (or rather the equivalence relation generated by naive homotopy). It follows that the maps $K \otimes \sqcap^n_{i,\epsilon} \to K \otimes \square^n$ are in W.

By considering the factorization

$$\begin{tikzcd}
a \otimes \sqcap^n_{(i,\epsilon)} \arrow[rr] \arrow[dr] & & a \otimes \square^n \\
& a \otimes \sqcap^n_{(i,\epsilon)} \cup Y \otimes \square^n \arrow[ur, "i"] &
\end{tikzcd}$$

and using the weak 2 out of 3 property and the fact that W is closed under pushouts of monomorphisms, we see that the generating cofibrations labelled i are in W. We know that $S \in W$ by assumption. By induction on n and comparing the pushout diagrams

$$\begin{tikzcd}
C \otimes \partial\square^{n-1} \arrow[r] \arrow[d] & C \otimes \sqcap^n_{(i,\epsilon)} \arrow[d] \\
C \otimes \square^{n-1} \arrow[r] & C \otimes \partial\square^n
\end{tikzcd}$$

it follows that $f \otimes \partial\square^n$ is in W for each $f \in S$.

Then, by considering the factorization

$$\begin{tikzcd}
A \otimes \square^n \arrow[rr] \arrow[dr] & & B \otimes \square^n \\
& A \otimes \square^n \cup B \otimes \partial\square^n \arrow[ur, "i"] &
\end{tikzcd}$$

and using the weak 2 out of 3 property, we see that all the generating anodyne cofibrations are in W.

Now given a fibration $p : X \to Y$ with Y fibrant, and an equivalence $Z \to Y$, we can factor $Z \to Y' \to Y$, where $Z \to Y'$ is an anodyne cofibration, and by Lemma 9.37 the map $Y' \to Y$, which is apriori a naive fibration, is actually a trivial fibration. Then the pullback of $Z \to Y$ along p is an equivalence, because we know that this is true for anodyne cofibrations and trivial fibrations. Thus by Lemma 4.57, the model structure is proper. $\square$

Example 9.40.1. Let C be a small category, and $A = C \times \Delta$. Then $\mathrm{Set}_A = \mathrm{Set}^{\Delta^{\mathrm{op}} A}$. We can take I to be the constant simplicial presheaf Δ^1, so this gives a way of constructing standard model structures on simplicial presheaves.

Example 9.40.2. The Quillen model structure on simplicial sets is an example with $A = \Delta$ and $I = \Delta^1$.

Example 9.40.3. When $A = \square$, we get a model structure on cubical sets. It will be proven later, but it turns out that the weak equivalences are those that topologically are weak equivalences, and that the cofibrations are anodyne maps in this case.

9.4. **Colimits in Cisinski model structures.**

Lemma 9.41. *Suppose A is a small category, and C is a small category with a terminal obeject. Then the A-set $i_A^*(C) \to *$ has the right lifting property with respect to all inclusions.*

Proof. It suffices to show that for any inclusion $X \to Y$ of A-sets, $C \to *$ has the right lifting property with respect to $i_A(X) \to i_A(Y)$. However, if $f : x \to y$ is a map in $i_A(Y)$ such that $y \in i_A(X)$, then $f \in i_A(X)$. Thus, we can just send anything in $i_A(Y) - i_A(X)$ to the terminal object, giving the lift. $\square$

It follows that in a Cisinski model structure, the projection $i_A^*(C) \to *$ is a trivial fibration. We say that a model structure on A-sets (or its weak equivalences) is **regular** if the map $\mathrm{hocolim}_{a\to X} a \to X$ is a weak equivalence for any X, where hocolim is the homotopy colimit in the projective model structure.

Throughout this subsection, we will view the category Set_A as coming with a Cisinski model structure for some interval theory and inclusions S.

We define the internal nerve B_hC of a category C to be $\mathrm{hocolim}_C *$.

Lemma 9.42. *Suppose that X is a set, thought of as a discrete category. Then the natural map $B_hX \to X$ is an equivalence.*

Proof. The constant diargam to $*$ is projectively cofibrant, and its colimit is X. □

Lemma 9.43. *Suppose $f : C \to D$ is a functor between small categories. There is a canonical weak equivalence* $\mathrm{hocolim}_{d\in D} B_h(f/c) \to B_h(C)$*.*

Proof. This amounts to factoring the homotopy left Kan extension from $C \to *$ through D. □

Corollary 9.44. *Suppose that $f : X \to Y$ is an A-set morphism. Then there is a canonical weak equivalence* $\mathrm{hocolim}_{a\to Y} B_h(i_A(a \times_Y X)) \to B_h(i_AX)$*.*

Proof. Apply Lemma 9.43 to the map $i_AX \to i_AY$. □

Lemma 9.45. *Suppose $f : C \to D$ and $g : D \to C$ define a homotopy equivalence of categories in the sense that there are natural transformations between the composites and the identity. Then the induced maps $B_hC \to B_hD$ are weak equivalences.*

Proof. Suppose E has a terminal object and consider the projection $\pi : C \times E \to C$. $\pi_{/c} \cong C_{/c}\times E$ for each c, which has a terminal object for each c. Thus the map $B_h(C\times E) \to B_h(C)$ is an equivalence. Thus it follows that the maps f and g induce after applying B_h are homotopy inverses, so $B_hC \to B_hD$ is a weak equivalence. □

Definition 9.46. *Say that the model structure satisfies $M1$ if the maps $a \to *$ are weak equivalences for each $a \in \mathrm{Set}_A$.*

Corollary 9.47. *Suppose that $M1$ is satisfied and the model structure is regular. Suppose that $X \to Y$ is a map of A-sets. Then the canonical map* $\mathrm{hocolim}_{a\to Y} \Delta^a \times_Y X \to X$ *is a weak equivalence.*

Proof. Apply Corollary 9.44, using regularity and $M1$. □

Corollary 9.48. *Suppose that $M1$ is satisfied and the model structure is regular. Then there are natural weak equivaelnces*

$$i_A^*(C) \xleftarrow{\sim} \mathrm{hocolim}_{a\to i_A^*(C)} a \xrightarrow{\sim} B_h(i_Ai_A^*C) \xrightarrow{\sim} B_h(C)$$

for any small category C.

Proof. The fibres of the map $\epsilon : i_Ai_A^*C \to C$ are $\epsilon/c \cong i_Ai_A^*(C/c)$, and the map $i^*(C/c) \to *$ is a weak equivalence by Lemma 9.41. This gives the last equivalence. The other two follow from regularity, $M1$, and comparing colimits. □

Recall the classical Grothendieck construction, which takes a functor $I \to \mathrm{Cat}$ to a category $\int_I F$ with objects $i \in I, x \in F(i)$. This functor naturally lives over I.

Lemma 9.49. *Let $F : C \to D$ be a functor. Then there is a natural homotopy equivalence $\int_D F_{/d} \to C$.*

Proof. There is clearly a forgetful functor $Q : \int_D F_{/d} \to C$. We construct a homotopy inverse by having $i(c)$ be the pair $c, 1_{f(c)}$. $Q \circ i = 1$ and there is a natural transformation $iQ \to 1$. □

Quite similarly, we obtain:

Lemma 9.50. *For a diagram $I \to \mathrm{Cat}$, there is a natural homotopy equivalence $g_i : F(i) \to \pi_{/i}$ where $\pi : \int_I F \to I$ is the natural map.*

Corollary 9.51. *For any diagram $F : I \to \mathrm{Cat}$, there is a weak equivalence*

$$\mathrm{hocolim}_{i\in I} B_h F(i) \to B_h(\int_I F)$$

Proof. There is a weak equivalence $\mathrm{hocolim}_{i\in I} B_h \pi_{/i} \to B_h(\int_I F)$ by Lemma 9.43. Thus by Lemma 9.45 and Lemma 9.50, it is an equivalent to the stated homotopy colimit. □

Corollary 9.52. *Suppose $F \to G$ is a natural transformation of I-diagrams of small categories such that $B_h F(i) \to B_h G(i)$ is an equivalence for each i. Then $B_h(\int_I F) \to B_h(\int_I G)$ is an equivalence.*

If $F : I \to \mathrm{Set}_A$ is a diagram, then composing with i_A gives a functor to Cat. Observe that there is an isomorphism of Grothendieck constructions $\int_I i_A(F) \cong \int_A \hom(a, F)$, where $\hom(a, F)$ is the functor $A \to \mathrm{Cat}$ sending a to the catrgory with objects a map $a \to F(i)$ for some i.

The maps $\hom(a, F) \to \lim_i F(i)(a)$ where the latter is viewed as a discrete category assemble into a map $\int_{i\in I} i_A F(i) \to i_A \operatorname{colim}_i(F(i))$.

Lemma 9.53. *The map $B_h(\int_I i_A F(i)) \to B_h(i_A(\operatorname{colim}_i(F(i))))$ is a weak equivalene if $\mathrm{hocolim}\, F \simeq \operatorname{colim} F$.*

Proof. By Corollary 9.52 and Lemma 9.42, it suffices to show that $F(i)(a) \to \operatorname{colim}_i F(i)(a)$ induces a weak equivalence $B_h(\int_i F(i)(a)) \to B_h(\operatorname{colim}_i F(i)(a)) \cong \operatorname{colim}_i F(i)(a)$ in both cases. By Lemma 9.51, there is an equivalence $\mathrm{hocolim}_I B_h F(i)(a) \to B_h(\int_i F(i)(a))$ and each $B_h(\int_i F(i)(a))$ is equivalent to the discrete A-set $F(i)(a)$ by Lemma 9.42, so this follows from the fact that hocolim = colim. □

An A-set X is **regular** if the map $\mathrm{hocolim}_{a\in X}\, a \to X$ is an equivalence.

Lemma 9.54. *If a diagram $F : I \to \mathrm{Set}_A$ is component-wise regular, and has colimit computing the homotopy colimit, then the colimit is also regular.*

Proof. By Lemma 9.53 and Corollary 9.51, the map $\mathrm{hocolim}(B_h i_A F) \to B_h i_A \operatorname{colim} F$ is an equivalence. But each of $B_h i_A F$ is computed as a colimit of its maps from A, so by exchanging the order of taking colimits, we see $\operatorname{colim} F$ is regular. □

Finally, we will construct a natural Cisinski model structure on test categories with the right weak equivalences.

Let C, A be small categories, and $\mathrm{Set}_{A\times C}$ the category of presheaves of C with values in A-sets. If A is a test category, let the pullback of $I = i_A^*(1)$ to $\mathrm{Set}_{A\times C}$ define an interval theory. Let Δ^1 denote the interval theory on $\mathrm{Set}_{\Delta\times C}$ given by Δ^1 pulled back, and let S

be a set of cofibrations of simplicial presheaves. The (Δ^1, S) model structure is the Cisinski model structure on $\mathrm{Set}_{\Delta\times C}$ coming from this data. A map $X \to Y$ is an **S-equivalence** in $\mathrm{Set}_{A\times C}$ if the induced map after applying $i_\Delta^* i_A$ is an equivalence in the (Δ^1, S)-model structure.

Lemma 9.55. *For all sufficiently large cardinals κ, given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ Z \arrow[r] & Y \end{tikzcd}$$

of inclusions of A-sets such that i is an equivalence and $|Z| < \kappa$, there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < \kappa$ and $B \cap X \to B$ is an equivalence.

Proof. There is an induced diagram of inclusions of simplicial sets

$$\begin{tikzcd} & i_\Delta^* i_A X \arrow[d, "i"] \\ i_\Delta^* i_A Z \arrow[r] & i_\Delta^* i_A Y \end{tikzcd}$$

such that i is an a weak equivalence, satisfying the conditions of Lemma 9.33. Thus we can apply that lemma to find a small subobject A_0 of $i_\Delta^* i_A Y$ satisfying its conclusion. This small subobject will be a subobject of $i_\Delta^* i_A B_0$ for some small B_0. We can then replace Z with B_0, and iteratively construct a sequence B_i, A_i. Since $i_\Delta^* i_A$ preserves filtered colimits, $\mathrm{colim}_i\, i_\Delta^* i_A B_i = \mathrm{colim}\, A_i$ shows that $\cup_i B_i$ works. □

Theorem 9.56. *Suppose that A is a test category and C a small category. Then there is a combinatorial model structure on the category $\mathrm{Set}_{A\times C}$ such that the weak equivalences are the S-equivalences, and the cofibrations are the monomorphisms. Bi_A gives a Quillen equivalence with $\mathrm{Set}_{\Delta\times C}$.*

Proof. That Bi_A gives a Quillen equivalence is clear since it is a left adjoint, and preserves monomorphisms and gives an equivalence on homotopy 1-categories. Combinatoriality follows from Lemma 9.55. So it suffices to show the existence of the model structure.

We can use the dual of Proposition 4.12 for tha adjunction coming from Bi_A: we only need to show that if $p : X \to Y$ has the right lifting property with respect to all inclusions, it is an equivalence. But the lifting property implies that p has a section σ. Moreover letting $I = i_A^*(1)$, by choosing a lift in the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "{1,\sigma p}"] \arrow[d] & X \arrow[d] \\ X \times I \arrow[r] & Y \end{tikzcd}$$

and applying Bi_A, we see that p is a homotopy equivalence. □

When A is a test category, and $C = *, S = \phi$, the model structure arising here is called the **standard model structure** for the test category.

9.5. **Weak equivalence classes of functors.**

Definition 9.57. *A* ***weak equivalence class*** W *of functors between small categories is a class such that the following hold:*

LF1 W is weakly saturated (bad terminology), meaning that:
W containes isomorphisms
W satisfies 2 out of 3.
if $i \circ r \in W$ and $r \circ i = 1$, then $r \in W$.

LF2 If C has a terminal object, then $C \to *$ is in W.

LF3 Given a commutative triangle $\alpha = \beta \circ u$, if all the functors $\alpha_{/c} \to \beta_{/c}, c \in \operatorname{cod} \beta = \operatorname{cod} \alpha$ are in W, then u is.

Grothendieck called these fundamental localizers.

Example 9.57.1. Let W_∞ denote the class of functors such that they are equivalent as simplicial sets. Then this is a weak equivalence class.

If W is a weak equivalence class and $C \times D \to C$ is a projection where D has a terminal object, then $C \times D \to C$ is in W, essentially by $LF3$. It follows that given a natural transformation $f \to g$, $f \in W$ iff g is.

Lemma 9.58. *Given a diagram of categories*

$$\begin{array}{ccc} C_0 & \xrightarrow{f} & C_2 \\ \downarrow{\scriptstyle g} & & \\ C_1 & & \end{array}$$

such that $g \in W$, then the map $C_2 \to \int_i C_i$ is in W.

Proof. By comparing with the case when $f = 1$ and using $LF3$, we can reduce to the case $f = 1$. In this case, there is a canonical map $r : \int_i C_i \to C_0 \cup_{C_1} C_2 = C_2$, which has an obvious section j. There is a zig zag of natural transformations between 1 and jr , so jr is in W. Thus by $LF1$, r, j are in W. ∎

Note that W is closed under small disjoint unions by $LF3$.

Suppose that A is a test category.

Lemma 9.59. *(1) Suppose given a diagram of A-sets*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_2 \\ \downarrow & & \\ X_1 & & \end{array}$$

where i is a monomorphism. Then the induced map $\int_i i_A X_i \to i_A(X_1 \cup_{X_0} X_2)$ is in W.

(2) Suppose given a diagram Y in A-sets index by an ordinal number α, and such that all morphisms $Y_i \to Y_j$ are monomorphisms. Then the induced map

$$\int_i i_A Y_i \to i_A(\operatorname{colim}_i Y(i))$$

is in W.

Proof. By $LF3$, it suffices to show that the diagrams of discrete categories $F(i)(a) \to \operatorname{colim}_i F(i)(a)$ is an equivalence, reducing to the case of a discrete diagram.

In the first case, the diagram is a union of diagrams of either form:

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & \phi \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \quad \begin{tikzcd} X_0 \arrow[r] \arrow[d] & * \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd}$$

for which we can apply Lemma 9.58.

In the second case, given $y \in \lim Y_i$, the category $\pi_{/y}$ has an initial object given by the smallest i such that $y \in Y_i$. Thus applying $LF3$ again, we are done. ☐

The following follows immediately from Lemma 9.58 and Lemma 9.59.

Corollary 9.60. *Suppose given a pushout diagram of A-sets*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "i"] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_1 \cup_{X_0} X_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where i is an inclusion. If the functor $i_A X_0 \to i_A X_2$ is in W, then the functor $i_A X_1 \to i_A X_1 \cup_{X_0} X_2$ is in W. Similarly with X_1, X_2 reversed.

Theorem 9.61. *Suppose W is a weak equivalence class of functors, and $X \to Y$ is a weak equivalence of simplicial sets. Then the functor $i_\Delta X \to i_\Delta Y$ is in W.*

Proof. Every weak equivalence factors as a trivial fibration which admits a section and a trivial cofibration, so if suffices to prove it for trivial cofibrations, which then reduces via retractions from anodyne maps. These are filtered colimits of pushouts of the generating anodyne inclusions $\Lambda^n_k \to \Delta^n$, so by Corollary 9.60, it suffices to show it for these inclusions.

$i_\Delta \Delta^n$ has a terminal object, so its map to the terminal object is in W. It follows that all the maps $i_\Delta \Delta^n \to i_\Delta \Delta^m$ are in W. By constructing the horns as iterated pushouts of cells, this can be shown again using Corollary 9.60 and induction. ☐

The following result is a conjecture of Grothendieck, proved by Cisinski.

Corollary 9.62. *Suppose W is a weak equivalence class and $C \to D$ is a functor between small categories such that $f_* : BC \to BD$ is a weak equivalence of simplicial sets. Then $f \in W$.*

Proof. By $LF2$ and $LF3$, the map $i_\Delta i_\Delta^* C \to C$ is in W, and so by Theorem 9.61 and the diagram

$$\begin{tikzcd} i_\Delta i_\Delta^* C \arrow[r] \arrow[d] & i_\Delta i_\Delta^* D \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the map is in W. ☐

Theorem 9.63. *Suppose that A is a test category, and M is a Cisinski model structure on A-sets satisfying $M1$ and is regular. M contains all the test weak equivalences.*

Proof. The class $F(M)$ of all functors $f : C \to D$ which induce weak equivalences on B_h is a weak equiavlence class. $LF1, LF2$ are easy, and $LF3$ follows from Lemma 9.43.

Suppose $X \to Y$ is a test weak equivalence. Then Bi_A applied to it is a weak equivalence, and since $F(M)$ is a weak equivalence class, by Corollary 9.62, $B_h i_A$ applied to it is a weak equivalence of M. By the regularity assumption, this implies the map is a weak equivalence. $\square$

Lemma 9.64. *Let A be a test category, and suppose that Y is a fibrant object in the standard model structure. Then the functor $X \mapsto X \times Y$ is homotopical.*

Proof. Let $i^* : \mathrm{Set}^{\Delta^{\mathrm{op}}} \to \mathrm{Set}_A$ be the functor defined by $i^*X(a) = X^{BA_{/a}}$, the right adjoint to i, which sends Z to Bi_AZ. The unit map $Z \to i^*Bi_AZ$ is an equivalence, so Bi_AZ preserves trivial cofibrations, and i^* preserves fibrations.

Let $j : Bi_AY \to Z$ be a trivial cofibration with Z Kan complex. Then the composite $X \times Y \to X \times i^*Bi_AY \to X \times i^*Z$ is the product of 1_X with a homotopy equivalence so we can replace Y by i^*Z. By Corollary 9.18, we can replace Z with BC for a category C.

Observe $i^*BC = i_A^*C$. Write π for the composite

$$i_A(X \times i_A^*C) \to i_A i^* C \xrightarrow{\epsilon} C$$

There are isomorphisms $\pi_{/c} \cong i_AX \times \epsilon_{/c} \cong i_AX \times i_A i_A^*(C_{/c})$ by Lemma 9.1. The functor $X \mapsto X \times i_A^*(D)$ preserves weak equivalences if D has a terminal object, and there are natural weak equivalences $\mathrm{hocolim}_C\, B(\pi_{/c}) \to Bi_A(X \times i_A^*(C))$ so the result follows.

$\square$

9.6. **Model structure on Cubical Sets.** We define the **canonical interval** model structure on cubical sets by taking the interval theory defined by the object $I = i_\square^*(1)$ and letting S be the set of vertex maps $* \to \square^n$, and using the Cisinski model structure.

Theorem 9.65. *The canonical interval model structure on cubical sets agrees with the standard model structure.*

Proof. The cofibrations agree, so it suffices to show that the weak equivalences agree. The weak equivalences of the canonical interval model structure are test weak equivalences. One way to see this is to take a fibrant replacement via anodyne maps using Corollary 9.38 which are in particular test weak equivalences, and observe that equivalences between fibrant objects are homotopy equivalences, which are test weak equivalences.

For the converse, by Theorem 9.63, it suffices to check regularity on the canonical interval model structure ($M1$ is satisfied by construction). By Lemma 9.54, it suffices to check this on $\square^n$. but $B_h i_\square \square^n$ is the homotopy colimit of a diagram with a terminal object, so it contractible, but so is equivalent $\square^n$, which is contractible by construction. $\square$

Theorem 9.66. *The standard model structure on cubical sets is proper.*

Proof. It suffices to show that the maps $* \to \square^n$ satisfy the criterion in Theorem 9.40. To see this, recall that in the proof of that theorem, the criterion was shown to be naively homotopy invariant. Any map $\square^n \to Y$ for any fibrant Y is naively nulhomotopic because $* \to \square^n$ is an anodyne inclusion. Thus we can assume that the map is null, so the relevant diagram is

$$\begin{tikzcd} F \arrow[r, "v_*"] \arrow[d] & F \otimes \square^n \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r, "v"] & \square^n \arrow[r] & * \arrow[r] & Y \end{tikzcd}$$

and v_* is an equivalence by Lemma 9.64. $\square$

Lemma 9.67. *Suppose that A is the test category $i_\square \square^k$ and M is the Cisinski model structure with the interval theory given by tensoring and $S = \phi$. then every weak equivalence of the standard model structure is a weak equivalence of M.*

Proof. We use Theorem 9.63, so it suffices to check $M1$ and regularity.

All the vertex maps $* \to \square^n \to \square^k$ are trivial cofibrations, so that all the morphisms $\square^n \to \square^m \to \square^k$ are weak equivalences. In particular, M satisfies $M1$.

As in Theorem 9.65, it suffices to check regularity on the n-cubes. This can be done by observing that there are equivalences

$$\begin{tikzcd} \operatorname{hocolim}_{\square^r \to \square^n} \square^r \arrow[r] \arrow[d, "\wr"'] & \square^n \arrow[d, "\wr"'] \\ \operatorname{hocolim}_{\square^r \to \square^n} \square^k \arrow[r, "\sim"] & \square^k \end{tikzcd}$$

$\square$

Corollary 9.68. *The standard model structure coincides with the model structure in Lemma 9.67.*

Proof. Use Lemma 9.67 to see that the weak equivalences are the same, and the cofibrations are clearly the same. $\square$

A map is an naive fibration in the model structure on cubical sets if it has the right lifting property with respect to the inclusions $\sqcap_{(i,\epsilon)} \subset \square^n$.

Lemma 9.69. *Every naive fibration $X \to \square^k$ is a fibration.*

Proof. This follows from Lemma 9.67 and Corollary 9.38. $\square$

Theorem 9.70. *Every naive fibration of cubical sets is a fibration. In particular, the anodyne maps $\sqcap^n_{(i,\epsilon)} \to \square^n$ generate the trivial cofibrations.*

Proof. First we will show that a naive fibration that is a weak equivalence is a trivial fibration. So let $V \to W$ be such a map. We can form the diagram

$$\begin{tikzcd} \square^m \times_W V \arrow[r, "\tau_*"] \arrow[d, "q_0"] & \square^n \times_W V \arrow[r] \arrow[d, "q_1"] & V \arrow[d] \\ \square^m \arrow[r, "\tau"] & \square^n \arrow[r] & W \end{tikzcd}$$

q_0, q_1 are fibrations by Lemma 9.69. By right properness, τ_* is an equivalence. It follows from Quillen's Theorem B (Theorem 7.26) that the diagram of simplicial sets

$$\begin{array}{ccc} Bi_\square(\square^n \times_W V) & \longrightarrow & Bi_\square V \\ \downarrow & & \downarrow \\ Bi_\square(\square^n) & \longrightarrow & Bi_\square W \end{array}$$

is homotopy cartesian. Since the right vertical map is an equivalence, the left vertical map is too. Thus q_1, q_0 are trivial fibrations, so $V \to W$ has the right lifting property with respect to all inclusions $\partial\square^n \to \square^n$.

Now suppose that $f : X \to Y$ is a naive fibration. Choose a fibrant replacement $Y \to RY$, and factor $X \to RY$ through an object U as an anodyne map and a fibration using Lemma 9.37. By properness (Theorem 9.66), the map $X \to Y \times_{RY} U$ is an equivalence, so it factors as an anodyne map $i : X \to W$ and a naive fibration $q : W \to Y \times_{RY} U$ that is an equivalence. q is trivial fibration by what was already proven. f is a retract of the composite $W \to Y \times_{RY} U \to Y$ so is a fibration. $\square$

Recall there is a triangulation functor $|\cdot| : \mathrm{Set}_\square \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$. Since $|\square^n|$ is contractible, the canonical map $\mathrm{hocolim}_{\square^n \to X} |\square^n| \to Bi_\square X$ is an equivalence. There is also a canonical map $\mathrm{hocolim}_{\square^n \to X} |\square^n| \to |X|$.

Theorem 9.71. $|\cdot| : \mathrm{Set}_\square \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ *is the left adjoint in a Quillen equivalence.*

Proof. It is a left adjoint, preserves monomorphisms, and trivial cofibrations, since they are generated by $\sqcap^n_{(i,\epsilon)} \to \square^n$ by Theorem 9.70, which are clearly sent to equivalences. By Theorem 9.65, cubical sets is regular, so the map $\mathrm{hocolim}_{\square^n \to X} \square^n \to X$ is an equivalence, and as a left Quillen functor, after applying $|\cdot|$, we get an equivalence too. It follows that there is a zigzag of natural equivalences

$$|X| \leftarrow \mathrm{hocolim}_{\square^n \to X} |\square^n| \to Bi_\square X$$

and we know $Bi_\square X$ induces an equivalence on homotopy categories since $\square$ is a test category, so $|X|$ does too. $\square$

10. Simplicial localization

The goal of this section is to show that the invertibility hypothesis is redundant in the definition of an excellent model category. This actually involves proving it in the case of simplicial categories.

10.1. Free Categories and Localization.

Let $O\,\mathrm{Cat}$ be the category of categories with a fixed set of objects O and morphisms maps that fix the set objects. A simplicial category with objects O is a simplicial object in $O\,\mathrm{Cat}$, and we denote this category $sO\,\mathrm{Cat}$. There is a simplicial nerve functor N taking a simplicial category to the diagonal of the bisimplicial set obtained by taking the nerve levelwise. By homotopy invariance of the diagonal, it follows that an equivalence of simplcial categories induces an equivalence on nerves.

A category C in $O\,\mathrm{Cat}$ is **free** if there exists a set of morphisms S such that every non-identity map in C is a unique composite of maps in C. There is a forgetful map from $U : \mathrm{Cat} \to \mathrm{DiGrph}$, the category of directed graphs, and the free categories are the essential image of the left adjoint. When we fix the set of objects, every free category is a coproduct of free categories on a single map. The free forgetful adjunction induces the free monad

F on $O\,\mathrm{Cat}$, taking a category to the free category on its nonidentity morphisms. This monad gives a simplicial resolution F_*C of an object in $O\,\mathrm{Cat}$, which is equipped with a map $F_*C \to C$. Similarly, given a simplicial category, one can take levelwise free resolutions and take the diagonal simplicial category.

Finally, observe that $sO\,\mathrm{Cat}$ and $O\,\mathrm{Cat}$ inherit a model structure from $\mathrm{Cat}_\Delta, \mathrm{Cat}$, since they are subcategories of the slice category under the discrete category with objects O that are closed under all the operations needed to inherit the model structure.

Lemma 10.1. *The map $F_*C \to C$ is an equivalence.*

Proof. By homotopy invariance of the diagonal, it suffices to consider C an ordinary category. For any objects X, Y, the map taking $x \in F_nC$ to $[x] \in F_{n+1}C$ is a contracting homotopy of F_*C onto C. □

Lemma 10.2. *If $\star$ denotes the free product in $sO\,\mathrm{Cat}$, then $\star$ is homotopical, and computes the homotopy coproduct.*

Proof. It suffices to show that $A \star B \to A' \star B$ is an equivalence if $A \to A'$ is. The argument in Lemma 10.1 shows that $A \star F_*B \to A \star B$ is an equivalence, so we can assume B is free in each degree. In the degreewise free case, by observing that $A \star F_*B$ is the diagonal of the bisimplicial category given by $A \star F_nB$, we can reduce to the case B is discrete and free. In this case, the map $O \to B$ is a cofibration, so since $\star$ is the pushout relative to O, it is a homotopy pushout by left properness. □

Lemma 10.3. *If C is a free category, the inclusion $\mathrm{sk}_k\, NC \to NC$ is an equivalence for $k \geq 1$.*

Proof. It's easy to see that the $k+1$-skeleton deformation retracts onto the k-skeleton, because C is free. □

Let $C \in O\,\mathrm{Cat}$ and W a subcategory of C. $C[W^{-1}]$ is the localization in the classical sense: its objects are the same, and morphisms are zig-zags

$$X \to X_1 \leftarrow X_2 \to X_3 \ldots Y$$

where the left arrows are in W, modulo the obvious equivalence relation. The map $C \to C[W^{-1}]$ universally sends W to isomorphisms. Note that this is also the pushout of categories:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & C \arrow[d] \\ W[W^{-1}] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C[W^{-1}] \end{tikzcd}$$

Where $W[W^{-1}] = GW$ is the groupoid generated by W.

Lemma 10.4. *Let $C = D \star E$. Then the inclusion $ND \cup_O NE \to ND \star E$ is an equivalence.*

Proof. For a free categories D, E, this follows from Lemma 10.3. In general, we take a free resolution and use Lemma 10.1, Lemma 10.2, and the fact that the diagonal of a bisimplicial set is homotopical. □

Lemma 10.5. *If $C = D \star W \in O\,\mathrm{Cat}$, where W is free, then the map $N(D \star W) = NC \to NC[W^{-1}]$ is an equivalence.*

Proof. By transfinite composition and Lemma 10.4, it suffices to show it for $C = W$ free on one generator. This is easy to see beacause both nerves are contractible. □

10.2. **Dwyer-Kan Localization.** Let $C \in O\,\mathrm{Cat}$ with a subcategory W. The **standard simplicial localization** is defined by $L(C, W) = F_*C[F_*W^{-1}]$. The notation will often be shortened to LC. The goal will be to show that this is a well behaved left derived functor of the localization.

The first thing to observe is that $hLC = C[W^{-1}]$, since the homotopy category is a left adjoint, so preserves colimits.

Lemma 10.6. *The simplicial localization preserves the homotopy type of the nerve.*

Proof. Using Lemma 10.5 and Lemma 10.1 we see that $NC \leftarrow NF_*C \rightarrow NLC$ gives a zig-zag of equivalences between the two. □

The following lemma is quite easy from definition:

Lemma 10.7. *A map $u : X \to Y \in C$ is in W iff it induces an isomorphism on covariant and contravariant mapping spaces.*

Proposition 10.8. *Suppose $W = C$, and NC is connected. Then LC is a simplicial groupoid, all the mapping spaces are equivalent, and the endomorphism spaces $LC(X, X)$ are simplicial groups. Moreover, the homotopy type of $LC(X, X)$ is that of the loop space of NC.*

Proof. The fact that the endomorphisms are simplicial groups is trivial. To prove the last statement, first replace NC with NLC by Lemma 10.6. Now for every k, $F_kC[F_kC^{-1}]$ is a connected groupoid, so let $UF_kC[F_kC^{-1}]$ be its universal cover, which is contractible, and has fibre $LC(X, X)_k$. Then applying Theorem 7.20, we obtain the result. □

Now we generalize to simplicial localizations of a simplicial category. Given a simplicial category $B \in sO\,\mathrm{Cat}$ and $V \subset B$ a subcategory, the standard simplicial localization is the diagonal of $F_*B[F_*V^{-1}]$, which we denote $L(B, V)$ or LB when it isn't confusing.

Again, upon taking homotopy categories, it agrees with the localization of the homotopy category. The groupoid completion of a simplicial category V is $L(V, V)$.

Lemma 10.9. *The map $V \to L(V, V)$ induces an equivalence on nerves.*

Proof. This follows because the diagonal is homotopy invariant and Lemma 10.6. □

Proposition 10.10. *A map $U \to V$ induces a weak equivalence on groupoid completions iff it induces a weak homotopy equivalence on nerves.*

Proof. WLOG we can assume U, V are connected, in which case this follows from Lemma 10.9 and the fact that by the argument in Proposition 10.8 shows that $LU(X, Y)$ has the homotopy type of the loopspace of NU. □

It follows that the $L(U, U)$ is the left derived functor of the naive level-wise localization localization functor, computed by the deformation given by diagonalizing the standard free resolution.

Proposition 10.11. *Let $V \in sO\,\mathrm{Cat}$. Then the natural map $V \to L(V, V)$ is an equivalence iff hV is a groupoid.*

Proof. The only if part is trivial since $hL(V, V)$ is a groupoid. For the if, □

10.3. **Reduction to Cubical Sets.** The strategy will be to reduce the invertibility hypothesis to the case of cubical sets via its universal property below, and then use the work of Dwyer and Kan on simplicial localization. We use the standard model structure on S-enriched categories.

Proposition 10.12. *Suppose C is a monoidal model category, $F : \square^{\leq 1} \to C$ is a functor, and $\tilde{F}$ is its cocontinuous monoidal extension to $\mathrm{Set}_\square$. Then $\tilde{F}$ is a left Quillen functor iff F expresses $F(\square^1)$ as a cylinder object for the unit of F, meaning the inclusions $F(\square^0)$ are trivial cofibrations. In particular, such a monoidal left Quillen functor exists.*

Proof. F applied to the inclusions $j_0, j_1 : \square^0 \to \square^1$ must be a trivial cofibration. The map $j_0 \cup j_1 = i : \square^0 \cup \square^0 \to \square^1$ must also be sent to a cofibration. Note that $\tilde{F}$ applied to the inclusion $\phi \to \square^0$ is a cofibration, which amounts to the unit being cofibrant, which is part of the monoidal model category axioms.

Conversely, if F satisfies those conditions, the generating (trivial) cofibrations are sent to (trivial) cofibrations because $\otimes$ is a left Quillen bifunctor, and the generating (trivial) cofibrations are built from i, j_ϵ using the pushout-product construction. $\square$

Let $\mathrm{Cat}_\square$ denote the category of categories enriched in cubical sets. Define P, H, E via the pushout squares below.

$$
\begin{tikzcd}
{[1]_\phi} \arrow[r, tail] \arrow[d, tail, "v"] & {[1]_{\square^0}} \arrow[d, tail] \\
{[1]_{\square^0}} \arrow[r, tail, "u"] \arrow[ur, phantom, "\ulcorner"] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]_{\partial\square^1}} \arrow[r] \arrow[d, tail] & P \arrow[d, tail] \\
{[1]_{\square^1}} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]_{\square^0}} \arrow[r, tail] \arrow[d, tail] & H \arrow[d, tail] \\
H \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & E
\end{tikzcd}
$$

P is defined so maps out of it are exactly a pair of morphisms in opposite direction, and H is in addition a homotopy from $v \circ u$ to the identity, and E is a choice of left and right homotopy inverse.

Lemma 10.13. *Suppose $C \in \mathrm{Cat}_\square$ is fibrant and $f : [1]_\square \to C$ is a map which becomes an isomorphism in the homotopy category. Then f extends to a map from E.*

Proof. The mapping spaces of C are all fibrant since C is fibrant. It follows that we can make the homotopies desired in the mapping spaces, extending the map to E by its universal property. $\square$

Proposition 10.14. *Let $E[f^{-1}]$ denote the localization of E at the map f (the one with the choice of homotopy inverses). Then the map $E \to E[f^{-1}] \to [1]^{\sim}_\square$ are equivalences.*

Proof. There is a monoidal left Quillen equivalence (Theorem 9.71) between $\mathrm{Set}_\square$ and $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ preserving the localization, giving a left Quillen equivalence between categories of enriched categories. Thus it suffices to show the corresponding statement for simplicial categories.

But then f is a morphism that is already invertible, so by [DWYERKAN], it follows that $E \to E[f^{-1}]$ is an equivalence. We still need to show that $E[f^{-1}] \to [1]^{\sim}_\square$ is an equivalence. To do this, since the two objects in $E[f^{-1}]$ are equivalent, it suffices to show that either of them has contractible endomorphisms. However, this mapping space from the construction of pushouts is the free monoid on the based space $\Delta^1 \vee \Delta^1$, which is contractible since $\Delta^1 \vee \Delta^1$ is.

$\square$

Corollary 10.15. *The map* $[1]_{\square^0} \to E$ *is a cofibrant replacement for* $[1]_{\square^0} \to [1]^{\sim}_{\square}$.

Theorem 10.16. *Suppose that* S *is a monoidal model category satisfying* $(A1)-(A4)$ *of Definition 5.6. Then* S *satisfies the invertibility hypothesis.*

Proof. Suppose that $f : [1]_S \to C$ is a cofibration that is an equivalence in the homotopy category. By Remark 5.5.1, we can assume that C is fibrant. By Proposition 10.12, there is a left Quillen functor $L : \mathrm{Set}_{\square} \to S$, with right adjoint R. This induces a Quillen adjunction $L : \mathrm{Cat}_{\square} \leftrightarrows \mathrm{Cat}_S : R$

Because L sends the unit to the unit, R preserves homotopy categories. Thus the image of f in the homotopy category after applying R extends to a map $E \to RC$, which is adjoint to a map $L(E) \to C$. We can assume WLOG that this map is a cofibration by modifying C.

Then we can consider the diagram of pushouts:

$$\begin{tikzcd}
{[1]_S} \arrow[r, tail] \arrow[d] & L(E) \arrow[r, tail] \arrow[d] & C \arrow[d] \\
{[1]^{\sim}_S} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & L(E[f^{-1}]) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C[f^{-1}]
\end{tikzcd}$$

The middle vertical map is an equivalence by Proposition 10.14. Thus the right vertical map is an equivalence by left properness.

□

Corollary 10.17. *A symmetric monoidal model category satisfying* $(A1)-(A4)$ *is excellent.*