DIMENSION AND MODULI

ISHAN LEVY

1. Various notions of dimension

Here we will think about irreducible varieties over an algebraically closed field k, although many arguments will work in more general settings.

We will start with a fundamental result in classical algebraic geometry that is useful in proving results about dimension.

Lemma 1.1 (Noether Normalization)**.** *Over an infinite field k, the algebra $k[x_1, \dots, x_n]/f$ for irreducible f is integral over $k[x_1 - \alpha_1 x_n, \dots, x_1 - \alpha_{n-1}x_n]$ for appropriate (generic) $\alpha_i \in k$.*

Proof. f might not be monic in x_n, but we can force it to be monic in x_n by doing a linear change of basis. Namely if we change x_i to $x_i + \alpha_i x_n$, the condition that the x_n term is nonzero for the highest homogeneous part of the polynomial is satisfied for some α_i as the field is infinite, and the condition that it is 0 is a single nonzero polynomial in the α_i. □

Exercise 1.1.1. *Show that this theorem doesn't hold for finite fields. Can you find a salvage?*

This statement has the following nice geometric interpretation: if there is an affine variety V sitting inside some $\mathbb{A}^n$, and we project it to a generic hyperplane, we will get a proper map onto its image, which will be a closed subset of $\mathbb{A}^{n-1}$. For example, consider the hyperbola $xy = 1$ in $\mathbb{A}^2$. If we project onto the x axis, the projection map will not be proper (the image isn't even closed), as 0 is missing. If we tilt the hyperbola slightly our map will become proper, and 0 will no longer be missing in the projection.

The following is a simple exercise in commutative algebra:

Exercise 1.1.2. *If $R \subset S$ is a finite integral extension, and $\mathfrak{p} \subset R$ is a prime ideal, then there are finitely many primes $\mathfrak{q}_i \subset S$ whose restriction to R is $\mathfrak{p}$.*

Since maximal ideals of the coordinate rings of varieties are the points, the fibres of a map of varieties given by an integral extension are finite.

Thus we should expect the dimension of the two varieties to be the same. By applying the lemma finitely many times, we get that the coordinate ring of any affine variety is integral over some $k[x_1, \dots, x_n]$. We call n the **dimension** of the variety. Note that this is equivalent to the transcendence degree of the fraction field over k by Lemma 1.1. If a variety is not irreducible, its dimension is the maximum of the dimension of each irreducible component.

Here is another characterization of dimension, called the **Krull dimension**, which is actually defined for any ring.

Theorem 1.2. *The dimension is 1 less than the length of the maximal chain of primes (Krull dimension) in $k[V]$.*

Proof. By Lemma 1.1, and the fact from commutative algebra that integral extensions preserve the length of a maximal chain, it suffices to show that $\dim(k[x_1, \dots, x_n]) = n$. It is easy to construct a chain of length $n+1$. Now to see that there is no longer chain, first note that any irreducible subset of codimension 1 is given by a principle ideal generated by an irreducible f since $k[x_1, \dots, x_n]$ is a UFD. Then by using Noether normalization on $k[x_1, \dots, x_n]/(f)$, induction, and the fact that integral extensions preserve the length of a maximal chain, it follows that there is no longer chain. □

The following is a special case of a result from commutative algebra called Krull's principle ideal theorem.

Theorem 1.3. *If $k[V]$ is the coordinate ring of an affine variety, and $0 \neq f \in k[V]$, then the variety cut out by f is codimension at most 1.*

Using this, the next thing we can observe about dimension is that it is local. This is already suggested by the fact that it only depends on $k(V)$, not $k[V]$, as we can invert elements of $k[V]$, effectively removing a proper closed subset from our variety, yet keep the dimension the same. Given an affine variety $k[V]$, and a maximal ideal $\mathfrak{m}$, we can consider $\mathfrak{m}/\mathfrak{m}^2$, the **cotangent space** of V at the point $\mathfrak{m}$. The way to see that this is the right definition is the following: suppose V is embedded in some affine space $\mathbb{A}^n$, so that we can think of functions on it as polynomials. Say $0 \in V$ and V is given by polynomials $v_1, \dots, v_n$. At 0 we can take the equations $\sum_i \partial_i v_j x_i = 0$, giving some linear subspace, called the **tangent space** at 0 (this can be done for any point). Now given a polynomial function, $df = \sum_i \partial_i f x_i$ is a linear form on the tangent space, that is clearly well defined on $k[V]$. It only depends on f up to a scalar and if $f \in \mathfrak{m}$, it vanishes iff $f \in \mathfrak{m}^2$, so defines an isomorphism between the dual of the tangent space and $\mathfrak{m}/\mathfrak{m}^2$.

By Nakayama's lemma, the dimension of the cotangent space is also the minimal number of generators of the ideal $\mathfrak{m}$ in the local ring $k[V]_{\mathfrak{m}}$ (recall we invert all elements that don't vanish at $\mathfrak{m}$). A local ring is called **regular** if the number of generators of the ring is its Krull dimension. We see below that regular is a way of making the notion of smoothness of a point intrinsic.

Theorem 1.4. *The points of a variety $V \subset \mathbb{A}^n$ with a regular local rings are the smooth ones.*

Proof. Let the kernel of $k[\mathbb{A}^n] \to k[V]$ be $(f_1, \dots, f_m)$. Now the rank of the Jacobian is exactly describing the common kernel of the elements df_i, and since the generate the ideal, the rank is the codimension of the tangent space of V in $\mathbb{A}^n$. Then the point is smooth iff this rank is then the codimension of the variety iff the dimension of the cotangent space is the dimension of the variety iff the point is regular. □

Theorem 1.5. *The dimension of the local ring of any irreducible variety at a maximal ideal (point) is the dimension of the variety.*

Proof. Intersect generic hyperplanes with the variety, so that the irreducible component at that point keeps getting smaller by 1 dimension by Theorem 1.3. We can invert elements so as to ignore the other components, and so we will eventually get an isolated point. Each irreducible component will correspond to a prime in the local ring 1 dimension smaller than the next. □

Combining the last two theorems, we get for example that at smooth points, the dimension of the cotangent space is the dimension of the variety.

2. Lines on surfaces

Let's use the notion of dimension to consider lines on surfaces.

We will need the following fact about projective varieties.

Proposition 2.1. *The image of a projective variety under a morphism is closed.*

Morally this is true because projective varieties are closed subsets of $\mathbb{P}^n$, hence are compact.

We will also use the following basic result about dimension:

Theorem 2.2. *Let $f : X \to Y$ be a surjective map of irreducible projective varieties. Then $\dim f^{-1}(Y) \geq \dim X - \dim Y$ for any fibre F of any point in Y. Moreover, the set of points for which equality holds is dense and open.*

Proof. For the first statement, note that on an affine neighborhood of Y after inverting some elements, any point is defined by $r = \dim(Y)$ equations, $g_1, \dots g_r$. Then the fibre is given by the zero sets of $f^*(g_i)$, which by Theorem 1.3 is dimension $\geq \dim X - \dim Y$.

For the second, we locally have an inclusion $k[U] \hookrightarrow k[V]$. Let $v_1, \dots, v_{n-m} \in k[V]$ be a transcendence base over $k[U]$, and let $w_1, \dots, w_r$ generate the extension of algebras, so that the polynomials $F_i(w_i, v_j)$ are (not monic) minimal polynomials over $k[U][v_j]$. Now the points where the dimension of the fibre jumps are exactly the points where the F_i become identically 0 polynomials, as then the w_i are algebraically independent in the fibre. This condition shows that the points where equality doesn't hold is a proper closed set. □

Corollary 2.3. *The subset of points in Y for a map $X \to Y$ of projective varieties for which the dimension of the fibre is at least k is closed.*

Proof. The set on which the dimension of the fibre is higher than the difference in dimensions is a closed subset of lower dimension by the previous theorem, so by only looking at that set of points and using induction, we are done. □

Theorem 2.4. *If $f : X \to Y$ is a surjective map of projective varieties, Y is irreducible, and all the fibres are irreducible, then X is irreducible.*

Proof. Let X_i be the components of X. Let U be an open subset of Y for which the dimension of the fibres on each X_i is generic (i.e. empty if f doesn't map X_i surjectively onto Y. By irreducibility of the fibres, each fibre is contained in at least one X_i, and so for some X_i, say X_0, the set on which the fibre is contained in X_0 is dense. Then f restricted to X_0 is surjective, and the fibres are at least the same dimension of the fibres on X, so by irreducibility, they must be the same, and $X_0 = X$. □

If the fibres are irreducible and same dimension Y irreducible, then X is irreducible. proof: One irreducible component must hit everything, then apply the previous theorem on that, and observe that everything that is missing must have smaller dimension.

We will try to answer the question: on a general hypersurface in $\mathbb{P}^3$, how many lines are there?

To do this we will consider moduli spaces, or varieties that parameterize the objects we are looking at. For example, we would like to make a variety where the points are the lines in $\mathbb{P}^3$.

The **Grassmannian** $G(k,n)$ will be a variety parameterizing the subspaces of dimension k inside an n-dimensional vector space (it is also denoted $G(k,V)$ for a vector space V). In particular, $G(k,n) \cong G(n-k,n)$ and $G(1,n) = \mathbb{P}^n$. To realize it as a projective variety, we note that a k-plane in V can be thought of as a line in $\bigwedge^k V$, by wedging a basis together. However not every element of $\bigwedge^k V$ comes from a k-plane, so the Grassmannian will be a closed subset of $\mathbb{P}(\bigwedge^k V)$. For an explicit description of how to describe the Grassmannian, we have Plücker coordinates which will not be discussed here.

For example, lines in $\mathbb{P}^3$ are the same as 2-planes in a 4-dimensional vector space, so are parameterized by $G(2,4)$, which is a hypersurface in $\mathbb{P}^5$. Note that every Grassmannian comes with a tautological bundle, namely the subset of $G(k,n) \times \mathbb{A}^n$, where the fibres of the projection to $G(k,n)$ are all the points in the corresponding plane. We can projectivize this bundle to get a closed subset of $G(k,n) \times \mathbb{P}^{n-1}$.

The other moduli space we will need will parameterize hypersurfaces in $\mathbb{P}^n$ of degree m. Since hypersurfaces are given by a homogeneous polynomial up to a scalar multiple, we need to parameterize homogeneous polynomials of degree m in n variables up to scalar multiples. But these form an affine space of dimension $\binom{n+m}{m}$, so our moduli space will be $\mathbb{P}^{\binom{n+m}{m}-1}$. We also have something like the tautological bundle, which is a subset of $\mathbb{P}^n \times \mathbb{P}^{\binom{n+m}{m}-1}$, where the fibres on the projection to the moduli space are just the hypersurfaces corresponding to the point.

Now let's try to understand lines in the general hypersurface of degree m in $\mathbb{P}^3$. Lines are parameterized by $G(2,4)$, and hypersurfaces are parameterized by $\mathbb{P}^{\binom{3+m}{m}-1} = \mathbb{P}^{c_m}$.

Now we can consider the subset Γ_m of $\mathbb{P}^{c_m} \times G(2,4)$ consisting of pairs where the line is contained in the hypersurface. We can pullback the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$ to get a subset of $\mathbb{P}^3 \times G(2,4) \times \mathbb{P}^3 \times \mathbb{P}^{c_m}$ consisting of points where the line intersects the hypersurface. Now by projecting to $G(2,4) \times \mathbb{P}^{c_m}$ and applying Corollary 2.3, and then taking the image under the projection, we see that Γ_m is closed. Now we can project Γ_m to $\mathbb{P}^{c_m}$. If a line is given by $l_1 = l_2 = 0$, the fibres are exactly the forms of the form $l_1 G + l_2 H$. These forms form a linear subspace of dimension $2\binom{m+2}{3} - \binom{m+1}{3} = c_m - (m+1)$. The fibres are all irreducible, so Γ_m is too, and is dimension $c_m - (m+1) + 4 = c_m - m + 3$.

We can now consider the other projection from Γ_m to $\mathbb{P}^{c_m}$. If $m > 3$, then the map cannot be surjective as the dimension is too small, so the image is a proper irreducible closed subset, and so we get:

Proposition 2.5. *The generic surface of degree > 3 has no lines.*

Now we can consider surfaces of degree ≤ 3. Hypersurfaces of degree 1 and 2 are uninteresting: they are the same up to a linear change of coordinates (in characteristic not 2). So we can just consider $x_0x_1 = x_2x_3$ (this is a Segre embedded $\mathbb{P}^1 \times \mathbb{P}^1$). In the affine chart $x_3 \neq 0$, this is $x_0x_1 = x_3$. A line is parameterized by $x_i = a_it + b_i$. One of x_0, x_1 must be a constant, so the line is just determined by that constant, and is 1-dimensional. For a cubic surface, consider $x_0x_1x_2 = x_3^3$. In the chart $x_3 \neq 0$, there are no affine lines as $x_0x_1x_2 = 1$ has no solutions as parameterized lines. Thus all the lines are at infinity, which is given by $x_0x_1x_2 = 0$, so this cubic has 3 lines. By Theorem 2.2, the image is c_3-dimensional, so the map is surjective, and the generic fibre is finite. We get the following:

Proposition 2.6. *The generic cubic surface has finitely many lines.*

With calculation, one can say more. Namely, the projection from Γ_3 gives a finite field extension of $k(\mathbb{P}^{c_3})$, and the degree of this extension is 27, which tells us that in fact the generic cubic has exactly 27 lines. In fact, any smooth cubic surface has 27 lines, but to prove this requires different techniques.