

K-THEORY AND FREDHOLM OPERATORS

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1. INTRODUCTION

Given a compact space X , we can consider the commutative monoid of complex vector bundles on X under the Whitney sum operation. We can force it to become a group in the universal way, giving a group called $K(X)$. This is a functor, so we can ask whether it is representable. If it is, the representing object should be a group up to homotopy. But in this case, it is represented by the space of Fredholm operators \mathcal{F} on Hilbert space H . In particular the composition operation is a commutative group operation on \mathcal{F} up to homotopy.

$K(X)$ for a point is just \mathbb{Z} , so in particular the components of the space of Fredholm operators should correspond to the integers, and this function from Fredholm operators to \mathbb{Z} is exactly the index of an operator. Hence in general we will want to construct a natural isomorphism $\text{index} : [-, \mathcal{F}] \rightarrow K(-)$.

To do this, first suppose that we have a map $f : X \rightarrow \mathcal{F}$, and we fix a point $x \in X$. We should note that the kernel of $f(x) = f_x$ varies continuously in X , in the sense that if V is a subspace on which f_x is injective, then nearby points will also be injective on V . In particular, as X is compact, we can find a V of finite codimension in H such that f_x is injective on V for all $x \in X$. Then $f_x(V)$ is a subbundle of $X \times H$, and we can consider $H/f_X(V)$, the bundle where the fibre of x is $H/f_x(V)$. If V is codimension k , we define $\text{index}_V(f) = k - [H/f_X(V)]$. To see this is well defined, we only need to observe that if we choose a codimension n subspace $U \subset V$ then $H/f_X(U) \cong f_X(U)^\perp \cong f_X(V)^\perp \oplus f_X(U^\perp \cap V) \cong H/f_X(V) \oplus \mathbb{C}^n \times X$. We thus have $k - [H/f_X(V)] = k + n - [H/f_X(U)]$, so for any V, W , $\text{index}_V(f) = \text{index}_{V \cap W}(f) = \text{index}_W(f)$. Moreover index is clearly natural.

Homotopy invariance comes from the commutative diagram:

$$\begin{array}{ccc}
[X, \mathcal{F}] & \xrightarrow{\text{index}(f_0)} & K(X) \\
(1_X \times 0)^* \uparrow & & \uparrow (1_X \times 0)^* \\
[X \times I, \mathcal{F}] & \xrightarrow{\text{index}(f_t)} & K(X \times I) \\
(1_X \times 1)^* \downarrow & & \downarrow (1_X \times 1)^* \\
[X, \mathcal{F}] & \xrightarrow{\text{index}(f_1)} & K(X)
\end{array}$$

The vertical maps are isomorphisms and their composites are the identity on $[X, \mathcal{F}]$ and $K(X)$, so $\text{index}(f_0) = \text{index}(f_1)$. To see index is a homomorphism, let $f, g : X \rightarrow \mathcal{F}$. Then choose U, V to be finite codimension subspaces on which f, g are respectively injective and such that $g_X(V) \subset U$, so that fg is injective on V . Then there is an exact sequence $0 \rightarrow U/g_X(V) \rightarrow H/f_X(V) \rightarrow H/f_X(U) \rightarrow 0$, so $\text{index}(fg) = \text{codim } V - [H/f_X(V)] = \text{codim } V - [U/g_X(V)] - [H/f_X(U)] = \text{codim } U + \text{codim } V - [H/g_X(U)] - [H/f_X(U)] = \text{index}(f) + \text{index}(g)$.

For surjectivity of the index, since it is a homomorphism we need every vector bundle V to be in the image, as well as $n \in \mathbb{Z}$. The latter can be done by sending X to a single operator of index n . To see V is in the image, find a vector bundle W such that $V \oplus W = \mathbb{C}^n \times X$ and let π_V, π_W be the projection maps. Now send x to the operator on $H \otimes \mathbb{C}^n \cong H$ sending $e_i \otimes v$ to $e_{i+1} \otimes \pi_W(v) + e_i \otimes \pi_V(v)$. The index is $-W$ which is V up to a trivial bundle.

Finally let's examine the kernel of the index. If something is in the kernel, there must be a finite codimension subspace $U \subset H$ with $H/f_X(U)$ trivial. Let e_1, \dots, e_n be a basis of U^\perp and let s_1, \dots, s_n be trivializing sections of $f_X(U)^\perp$. Consider the homotopy that has $f_{x,t}(e_i) = f_x(e_i)(1-t) + s_i t$. It homotopes f to something that is an isomorphism on H . Thus we have an exact sequence $[X, \text{GL}(H)] \rightarrow [X, \mathcal{F}] \rightarrow K(X) \rightarrow 0$. However, by Kuiper's theorem, $\text{GL}(H)$ is contractible, so the index is a natural isomorphism.