ADJOINTS

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1. What is an Adjoint?

The concept of adjoint was one that took quite a while to develop correctly. There was a time where people (such as Bourbaki) had come up with similar concepts, but in the wrong level of generality (indeed Bourbaki's was too general). It was in 1958 that Kan introduced adjoints, which were exactly the concept needed: Once they became better known, people realized that this was an essential concept: it appeared everywhere in mathematics.

Here is an example of an adjoint: recall the two categories Fld (fields), and Dom_m (domains with injections as arrows). We have $U : \mathrm{Fld} \to \mathrm{Dom}_m$ the forgetful functor and $\mathrm{Frac} : \mathrm{Dom}_m \to \mathrm{Fld}$, sending a domain to its field of fractions. Recall last time that the inclusion of R into $U\,\mathrm{Frac}(R)$ is a universal arrow from R to Frac. Similarly, the isomorphism $K \cong \mathrm{Frac}(UK)$ is a universal arrow from K to Frac. The fact that these two universal arrows exist is due to an adjunction. More precisely, the first universal arrow $R \to U\,\mathrm{Frac}(R)$ gives $\mathrm{Dom}_m(R, UK) \cong \mathrm{Fld}(\mathrm{Frac}(R), K)$, which is natural in K. Note that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, "f"] & U\,\mathrm{Frac}(R) \arrow[d, dashed, "U\,\mathrm{Frac}(f)"] \\ S \arrow[r, "i_S"] & U\,\mathrm{Frac}(S) \end{tikzcd}$$

Note that i is not just a universal arrow, but also by this diagram a natural transformation from 1_{Dom_m} to $U\,\mathrm{Frac}$. Then, from the second universal arrow $K \to \mathrm{Frac}(UK)$ we get the same natural bijection $\mathrm{Dom}_m(R, UK) \cong \mathrm{Fld}(\mathrm{Frac}(R), K)$, except now it is natural in R. The fact that these bijections are the same shows that we end up with a bijection that is both natural in R and K. This is an adjunction, and is the familiar universal property of the field of fractions. More generally, an adjunction is:

Definition 1.1. *Let C, D be categories and $F : C \to D$, $G : D \to C$ be functors. An* ***adjunction*** *is a bijection between the functors $D(Fa, b)$ and $C(a, Gb)$ that is natural in a and b (ie it is a natural isomorphism when viewed as a functor in a and b). We say F is* ***left adjoint*** *to G and write $F \dashv G$.*

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If we have an adjunction, we can set $b = Fa$ to get $D(Fa, Fa) \cong C(a, GFa)$. This gives us a universal arrow ϵ_a to GF for each $a \in C$, which is a natural transformation $1_C \to GF$ called the **unit**. Similarly setting $a = Gb$, we get $D(FGb, b) \cong C(Gb, Gb)$, which is a universal arrow η_b from FG to b for each $b \in D$, giving a natural transformation $FG \to 1_D$ called the **counit**. For example, the unit of the adjunction Frac $\dashv U$ is just the collection of universal arrows described in the last lecture, namely η_R includes R into its field of fractions. All of the information of an adjunction is contained within the unit or counit, which is made precise by the following theorem.

Theorem 1.2. *An adjunction $F \dashv G$ is completely determined by the functors F, G, and the unit. Dually it is determined by the counit. More precisely, the unit must be a natural transformation from 1_C to GF such that at every component it is universal to G.*

Proof. Suppose we have F, G, and the unit η. The fact that η consists of universal arrows gives a natural bijection $\nu : D(Fa, b) \cong C(a, Gb)$ that is natural in b. To get naturality in a, we have that η is a natural transformation, so we want

$$\begin{tikzcd}
D(Fa, b) \arrow[r, "\circ Ff"] \arrow[d, "\nu"] & D(Fc, b) \arrow[d, "\nu"] \\
C(a, Gb) \arrow[r, "\circ f"] & C(c, Gb)
\end{tikzcd}$$

which commutes due to the following diagram:

$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[d, "f"] & GFc \arrow[d, "GFf"] & Fc \arrow[d, "Ff"] \\
a \arrow[r, "\eta_a"] \arrow[dr, "\nu_g"'] & GFa \arrow[d, dashed, "Gg"] & Fa \arrow[d, "g"] \\
& Gb & b
\end{tikzcd}$$

$\square$

In the case of the field of fractions adjunction, the counit is an isomorphism. This is not a coincidence. Note that $U : \text{Fld} \to \text{Dom}_m$ is a fully faithful functor, ie. Fld is a full subcategory of Dom_m. we call a subcategory **reflective** if the inclusion has a left adjoint (the reflection). There is a general fact that a full reflective subcategory has the counit an isomorphism (this is a nice exercise). Another example of this is the abelianization, which is the left adjoint of the inclusion of Ab into Grp.

Another important fact in practice is the triangle identities: $\epsilon F \circ F\eta = 1_F$ and $G\epsilon \circ \eta G = 1_G$. These are dual, so I will prove the first. ϵ and η are universal arrows, so if $\varphi : C(a, Gb) \to D(Fa, b)$ is our natural bijection, we have $\varphi(f) = G(f) \circ \eta_a$ if $f : Fa \to b$ is an arrow. Then the triangle identity is $1_{Ga} = \varphi(\epsilon_a)$ by definition $= G(\epsilon_a) \circ \eta_{Ga}$ by the formula above.

2. Adjunctions are Everywhere

Adjoints are everywhere in mathematics. Another example is as follows: We have the forgetful functor U : CRing $\to$ Set. It has a left adjoint, namely, the functor $\mathbb{Z}[-]$ taking a set X to its polynomial ring $\mathbb{Z}[X] = \mathbb{Z}[x \in X]$. The natural bijection $Set(X, UR) \cong \mathrm{CRing}(\mathbb{Z}[X], R)$ is the familiar universal property of polynomial rings. This works more generally: given a forgetful functor, we may call the left adjoint the free functor.

Let J, C be categories, and recall the diagonal functor $\Delta : C \to C^J$. If C has all colimits of shape J, the left adjoint is the functor that takes a functor F to its colimit, and similarly the right adjoint takes it to its limit (if they all exist).

A special case is this: Consider a 1-place predicate P: ie a function from a fixed set X to $\Omega = \{\perp, \top\}$. We can treat X as a **discrete** category, with only identity arrows, and treat Ω as a poset, with $\perp \leq \top$. Then Ω^X the functor category, is the poset of subsets of Ω ordered by inclusion, and can also be thought of as the collection of 1-place predicates. Then we have the diagonal functor Ω, and the right adjoint is $\forall x : P(x)$, and the left adjoint is $\exists x : P(x)$:

$$\begin{tikzcd} \Omega \arrow[r, "\Delta"] & \Omega^X \arrow[l, bend right=40, "\exists"'] \arrow[l, bend left=40, "\forall"] \end{tikzcd}$$

This is also a special case of adjoints on partial orders, called a Galois connection. In particular, we have the following:

Definition 2.1. *Let P and Q be posets, $G : P \to Q$, $F : Q \to P$ contravariant (order reversing) functors. These functors are a* ***Galois connection*** *if they satisfy $Fa \geq b$ iff $a \leq Gb$.*

Note that the triangle identities in this case amount to $FGF = F$ and $GFG = G$. The classic example of this is when a group G acts on a set S. Then if 2^{UG} is the lattice of subsets of the underlying group of G, we get a functor F from 2^{UG} to 2^S, which is just the fixed points of the subset. Conversely, we can take an element of 2^S and send it to its stabilizer. This is order reversing, and $Fa \geq b$ iff $a \leq Gb$ as the fixed points of a contain b if and only if $\forall x \in a, y \in b$, $x.y = y$ if and only if a is contained in the stabilizer of b. Note that due to the triangle identities we get an induced (contravariant) equivalence of subcategories, namely the images of F and G. In the case of Galois theory, when the Galois group acts on the field, the main theorem of Galois theory identifies these subcategories as the lattice of subgroups of the Galois group and the lattice of subfields (and in addition identifies a similar correspondence between normal subgroups and subfields).

Another interesting example of a Galois connection is Hilbert's Nullstellensatz. Here we have a correspondence between the lattice of ideals L of a polynomial ring $K[x_1, ..., x_n]$ (K algebraically closed) and the subsets of $\mathbb{A}^n$, affine space. Hilbert's Nullstellensatz says that the functor taking an ideal to its locus and the one taking a subset of affine space to the ideal of functions that are zero on it are adjoints. In particular, the unit of this adjunction is the closure in the Zariski topology, and the counit is the radical of an ideal. This example shows that identifying the counit and unit of an adjunction is an important part of understanding the adjunction.

Another important type of adjunction is important and common enough for categories to have them to have a special name.

Definition 2.2. *Suppose C is a category with finite products, ie a terminal object (empty product) and a right adjoint $(-) \times (-)$ to the functor $\Delta : C \to C \times C$. Then C is* ***cartesian closed*** *if the functor $(-) \times b$ has a right adjoint for each $b \in C$. We write the right adjoint as $(-)^b$. Explicitly, we have $C(a \times b, c) \cong D(a, c^b)$ which is natural in a, c. We call c^b an* ***exponential*** *object.*

The counit of this adjunction is a map $e_c : c^b \times b \to c$ called the **evaluation** map. Examples of cartesian closed categories are Set, Cat, kTop (compactly generated spaces), Pos.

The last important example I will give is the tensor-hom adjunction. Let R-Mod be the category of left R-modules. Recall that $\mathrm{Hom}(R, -)$ is a functor from R-Mod to itself. The left adjoint of this is called the tensor product. Explicitly, we have $\mathrm{Hom}(S \otimes R, T) \cong \mathrm{Hom}(S, \mathrm{Hom}(R, T))$. This is an isomorphism of R-modules, and is actually natural in R, S, and T.

If we take the counit of this adjunction for vector spaces, we get an evaluation map $ev : V \otimes \mathrm{Hom}(V, W) \to W$. Letting $W = F$, we get the trace $tr : V \otimes V^* \to F$. Then, applying the adjunction to this map, we get our map $\eta : V \to V^{**}$.

Now we can also prove $V \otimes W^* \cong \mathrm{Hom}(W, V)$ when V is finite dimensional. In particular $V \otimes W^* \cong V^{**} \otimes W^* \cong \mathrm{Hom}(V^* \otimes W, F) \cong \mathrm{Hom}(W, V^{**}) \cong \mathrm{Hom}(W, V)$. When $W = V$ this gives $\mathrm{End}(V) \cong V \otimes V^*$ which shows that the map I called trace above is the usual notion of trace.

3. Theorems about adjunctions

Knowing that a functor is an adjunction is can be very useful. For example, recall the fact $\mathbb{Z}[x, y] = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[y]$. This is actually a consequence of the functor taking a set to the polynomial ring on that set being a left adjoint.

Definition 3.1. *A functor $F : C \to D$ **preserves limits** or is **continuous** if for every limiting cone $K : J \to C$, the composite $F \circ K : J \to D$ is a limiting cone. In other words, $F \lim(K) \cong \lim(FK)$ (these are isomorphisms of cones).*

We have the following useful theorem about adjoints:

Theorem 3.2 (RAPL)**.** *Right adjoints preserve limits. Dually left adjoints preserve colimits (LAPC).*

Proof. Suppose $G : C \to D$ has a left adjoint $F : D \to C$, and K is a diagram in C. Then the proof is indicated by this diagram:

$$\begin{tikzcd}
\lim(K) \arrow[d, "\pi_i"] \arrow[dr, "\pi_j" near start] & FS \arrow[l, dashed] \arrow[dl, "Fs_i" near start] \arrow[d, "Fs_j"] \\
K_i \arrow[r, "Kf"] & K_j
\end{tikzcd}
\qquad
\begin{tikzcd}
G\lim(K) \arrow[d, "G\pi_i"] \arrow[dr, "G\pi_j" near start] & S \arrow[l, dashed] \arrow[dl, "s_i" near start] \arrow[d, "s_j"] \\
GK_i \arrow[r, "GKf"] & GK_j
\end{tikzcd}$$

Our argument is as follows: Start with a limiting cone $\lim(K)$ and apply G to it. Suppose we have another cone from S in D. Then we can apply the adjunction to it to get a cone in C from FS. Now in C there is a unique arrow making the diagram commute. applying the adjunction to this arrow, we get a map from S to $G \lim(K)$ in D that commutes by naturality. For uniqueness, if there were another cone morphism from S, applying the adjunction again would by naturality yield a cone morphism from FS, which would have to be the same. Then the fact that the adjunction is a bijection means that the original cone morphism was the same as well. $\square$

As an exercise, you can try to prove that representable functors are also continuous. Indeed limits and colimits can (and were originally) be defined via the fact that representable functors are continuous.