REPRESENTATION THEORY

ISHAN LEVY

1. Topological Groups

Definition 1.1. *A **topological group** is a group in the category of topological spaces.*

Given a subgroup $H \subset G$ the coset space G/H makes sense. We will see below that it makes sense to restrict to the case when G is Hausdorff and H is closed.

Lemma 1.2. *Let G_1 denote the closure of the identity element. G_1 is a closed normal subgroup of G and the quotient G/G_1 is a Hausdorff topological group.*

Proof. Conjugation is an automorphism, and G_1 is the smallest closed subgroup containing the identity, so conjugation must fix it. Thus G_1 is normal, and the quotient is T_1.

In the quotient, take the preimage of the identity on the multiplication map, and compose with $1_G \times (-)_G^{-1}$ to get that the diagonal of G is closed, showing that it is actually Hausdorff. □

Corollary 1.3. *The category of topological groups is equivalent to the category of Hausdorff topological groups G equipped with a group extension $E \to G$ of ordinary groups.*

Proof. If G_1 is as in the previous lemma, note that $G_1 \to G \to G_1$ is a canonical realization of G as the extension of G_1, which is Hausdorff, by G_1, which is indiscrete. It is not hard to see that this gives an equivalence. □

Lemma 1.4. *If $G' \to G$ is a connected covering map, after fixing a lift of the identity, there is a unique group structure on G' such that the map is a homomorphism.*

Proof. This follows from the criterion for lifting maps to covers. □

Proposition 1.5. *Let $G' \to G$ be a connected covering map that is a homomorphism. The group of deck transformations can be identified with the kernel, and is abelian. Further more π_1 is abelian.*

Proof. A deck transformation is determined by where it sends the identity, which must be in the kernel. Conjugation of π_1 can be identified in the cover by multiplication by an element. Since it fixes the identity, it must be the identity, so π_1 is abelian. □

Lemma 1.6. *If $H \subset G$ is a connected subgroup of a topological group, and G/H is connected, then G is connected.*

Proof. We'd like to show $G^o = G$, where G^o is the connected component of the identity. To do this, note it contains H as H is connected, and thus it passes to the quotient so its image must be G/H, and so $G^o = G$. □

Lemma 1.7. *An open subgroup H of a topological group G is closed.*

Proof. Let U be a neighborhood of the identity contained in H, and U_g be the neighborhood translated by g. Now g is a limit point of $H \implies H \ni h \subset U_g \implies g^{-1}h \in U \implies g \in H$. $\square$

We will briefly consider locally compact Hausdorff groups, and will come back to them when considering representations.

Theorem 1.8. *Given a locally compact Hausdorff group G, there is a unique Borel measure μ on G that is invariant under left multiplication, outer regular on Borel sets, inner regular on open sets, and finite on compact sets. It is unique up to multiplication by a scalar. The completed measure is called the **Haar** measure.*

Proof. Given subsets C, D, define $[C : D]$ to be the smallest number of left translates of D covering C. Note that $[C : D][D : E] \geq [C : E]$, $[\bigcup C_i : E] \leq \Sigma[C_i : E]$, and if $D \subset E$, $[C : D] \geq [C : E]$.

Fix a compact set A containing a nontrivial neighborhood of the identity.

Given a compact set C, define $\mu(C)$ to be $\lim_{i\to\infty} \frac{[C:U_i]}{[A:U_i]}$. Here the U_i are carefully chosen as a local base around the identity so that the limit exists for all compact C. To do this, note $[C : A]$ is an upper bound of the limit, which is finite by our choice of A. Now if we index the compact sets $C_\alpha \in I$, then we get a function taking an open set U to $(\frac{[C_\alpha:U]}{[A:U]})_\alpha$. By Tychanoff's theorem, we can extract the U_i we want.

It is not hard to see that for two disjoint compact sets C_1, C_2, $\mu(C_1 \cup C_2) = \mu(C_1) + \mu(C_2)$. Suppose there were arbitrarily large i such that some translate of U_i intersected both C_1, C_2. Choose the U_i to have compact closure so that the centers of these U_i converge to some point p. Then consider any translate of some U_i centered at p. We can choose a neighborhood of p small enough such that for all $n > N$, U_n centered around that point is entirely contained in U_i (i.e such that $U_n \times U_n$ is inside U_i). But we can also find a point in this neighborhood with some $U_n, n > N$ that lies inside the translate of U_i, which is not possible. Thus for large enough i, a translate of U_i can intersect only one of C_1, C_2, and so we get a content on compact sets, which extends to the Borel measure we want (just like the Lebesgue measure). It is nontrivial as $\mu(A) = 1$.

For uniqueness, we rely on the fact that a σ-compact Radon measure is uniquely determined by its integral on continuous compactly supported functions. Our group may not be σ-compact, but G_o is by results in the previous section, and uniqueness for G_o is good enough.

Let f, g be continuous and compactly supported, and fix $\epsilon > 0$. Choose a neighborhood of the identity U_ϵ with compact closure, $U_\epsilon = U_\epsilon^{-1}$ such that in each left translate of U_ϵ, f and g vary by at most $\frac{\epsilon}{2}$. By Urysohn's Lemma, let ϕ_ϵ be a continuous function with support in U_ϵ and $\int \phi_\epsilon d\mu = 1$. Then $\int f(xy)\phi_\epsilon(y)d\mu(y) = f(x) + O(\epsilon)$ uniformly in x.

Integrating this over $d\mu'(x)$, we get

$$\int \left(\int f(xy)\phi_\epsilon(y)d\mu(y) \right) d\mu'(x) = \int f d\mu' + O(\epsilon)$$

But this is also

$$= \int \left(\int f(y)\phi_\epsilon(x^{-1}y)d\mu'(x) \right) d\mu(y) = \int \left(\int \phi_\epsilon(x^{-1})d\mu'(x) \right) f(y)d\mu(y)$$

$$= \int \phi_\epsilon(x^{-1}) d\mu'(x) \int f d\mu = c_\epsilon \int f d\mu$$

.

Doing the same for a function g and multiplying shows that $(\int f d\mu' + O(\epsilon)) c_\epsilon \int g d\mu = (\int g d\mu' + O(\epsilon)) c_\epsilon \int f d\mu$, and rearranging this gives $\int g d\mu \int f d\mu' - \int f d\mu \int g d\mu' = O(\epsilon)$, so we can let $\epsilon \to 0$, showing that the two integrals will always agree up to a scalar. □

Theorem 1.9. *For a compact group G, a left Haar measure is a right Haar measure.*

Proof. Observe that the measure $f \to \int f(yx) d\mu(x)$ is also left invariant, hence is a constant multiple of μ. We can do this for every μ giving a continuous homomorphism $G \to \mathbb{R}$. By compactness and positivity the image must be trivial. □

2. Lie groups and Lie algebras

Definition 2.1. *A Lie group is a group in the category of smooth manifolds.*

It is equally possible to work with complex Lie groups, where we work instead in the category of complex manifolds, but we will stick to real Lie groups here.

Examples include **matrix Lie groups**, or Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$. Examples of matrix Lie groups include $\mathrm{SL}_n(\mathbb{F})$, or matrices with determinant 1, $\mathrm{SO}(n)$, or orthogonal matrices with determinant 1, $\mathrm{SU}(n)$, or unitary matrices with determinant 1. The last two can be interpreted also in terms of automorphisms preserving a metric on a vector space.

Lemma 2.2. $\mathrm{GL}_n(\mathbb{R})^+, \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{F}), \mathrm{SO}(n), \mathrm{SU}(n)$ *are connected.*

Proof. $\mathrm{GL}_n(\mathbb{R})^+$ and $\mathrm{GL}_n(\mathbb{C})$ are clearly connected if $\mathrm{SL}_n(\mathbb{F})$ is as each element is path connected to one of determinant 1. Now using a continuous (slightly modified) version of Graham-Schmidt, $\mathrm{SL}_n(\mathbb{F})$ is connected if $\mathrm{SO}(n), \mathrm{SU}(n)$ are, but these are as we can use Lemma 1.6, induction, and the fibrations $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$, and $SO(n-1) \hookrightarrow SO(n) \to S^{n-1}$. □

Lemma 2.3. *The tangent bundle of a Lie group is trivial.*

Proof. We can take a frame at the identity and translate it around to get a trivialization of the tangent bundle. □

Theorem 2.4. *A Lie group's Haar measure comes from integrating a volume form.*

Proof. Translate any nonzero element of the bundle of volume forms at the identity around to get a non-vanishing translation invariant volume form. □

The **Lie algebra** associated a Lie group G (often denoted $\mathfrak{g}$) is the linear space of left-invariant vector fields, along with the Lie bracket on vector fields. Note that elements of the Lie algebra also correspond to tangent vectors. We will see that a Lie group is highly determined by Lie algebra.

Lemma 2.5. *Elements of $\mathfrak{g}$ define global flows on G.*

Proof. If at a point p our flow is defined on $(-\epsilon, \epsilon)$, we can translate our parallel curve $\frac{\epsilon}{2}$ in either direction to extend the flow further by the fact that the flow is left invariant. □

Now observe that our Lie algebra can also be described as $\mathrm{Hom}(\mathbb{R}, G)$. Namely, for each left-invariant vector field $V \in \mathfrak{g}$, there is a unique homomorphism f_V from $\mathbb{R}$ whose derivative is that tangent vector. As it is a homomorphism, it must be an integral curve to V. Conversely, integral curves to V at the origin are easily seen to be homomorphisms from left invariance of the flow and uniqueness of integral curves.

The **exponential map** is the map from $\mathfrak{g} \to G$ defined by $\exp(V) = e^V = f_V(1)$. It is a smooth map, and we can compute its derivative at the origin as $\frac{d}{dt}_{t=0} f_{tV}(1) = \frac{d}{dt}_{t=0} f_V(t) = V$. Thus its derivative is an isomorphism on the tangent space, so it is a local diffeomorphism at the identity.

Proposition 2.6. *The image of the exponential map generates G_o.*

Proof. The subgroup it generates is contained in G_o as it is connected, and contains an open neighborhood of the identity by the inverse function theorem, so it is open. It is thus all of G_o by Lemma 1.7. □

With the Lie algebra construction we can produce a Lie functor from the category of Lie groups to Lie algebras. Given a homomorphism $G \to G'$, we get an associated map $\mathfrak{g} \to \mathfrak{g}'$ just by the fact that the Lie algebra is given by $\mathrm{Hom}(\mathbb{R}, -)$ (alternatively it is just the derivative at the identity). It is linear as the derivative is, and to check that it preserves the Lie bracket, note that it preserves the flow, so that $f([X,Y]_e) = f(\frac{d}{dt}_{t=0}(dF_X(-t))(Y_{F_X(t).e})) = \frac{d}{dt}_{t=0}(dF_{f(X)}(-t))(Y_{F_{f(X)}(t).e}) = [f(X), f(Y)]_e$.

Note that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R})$, all $n \times n$ matrices. The exponential map is given by $\exp(X) = \sum_0^\infty \frac{X^n}{n!}$. Given a subgroup G of $\mathrm{GL}_n(\mathbb{R})$, the map $\mathrm{Hom}(\mathbb{R}, G) \to \mathrm{Hom}(\mathbb{R}, \mathrm{GL}_n(\mathbb{R}))$ is an inclusion, so the Lie algebra is exactly those matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

Lemma 2.7. $\det(e^X) = e^{\mathrm{tr}(X)}$, $e^{X^*} = (e^X)^*$, $e^{X^\top} = (e^X)^\top$ $e^X e^Y = e^{X+Y}$ *when* $[X,Y] = 0$, *and* $Y^{-1} e^X Y = e^{Y^{-1}XY}$.

Proof. All the results immediately follow from looking at power series except for the first. For that, note that $\det e^{tX}$ is a homomorphism from the additive group $\mathbb{R}X$ to $\mathbb{R}^\times$, and so is $e^{\mathrm{tr}(tX)}$, so we just need to verify their derivatives are the same at the 0. The derivative of the latter is clearly $\mathrm{tr}(tX)$, and to compute the former's derivative, the determinant is given by a sum over permutations of the matrix. The diagonal entries of e^{tX} look like $1 + tX_{ii} + O(t^2)$, and the off diagonals look like $tX_{ij} + O(t^2)$, so when taking the determinant and approaching 0, the only linear part is $\sum_i tX_{ii}$, so the derivative at 0 is the trace. □

Using the above lemma, we can easily describe the Lie algebras for our other matrix lie groups. For example, the Lie algebra of $\mathrm{SU}(n)$ (denoted $\mathfrak{su}(n)$) consists of traceless skew-hermitian matrices. This allows for efficient computation of the dimension of a Lie group, since it is the same as the dimension of its Lie algebra.

Theorem 2.8 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a Lie algebra over R where the underlying module is free. Choose a well-ordered basis of $\mathfrak{g}$, $x_\alpha, \alpha \in I$. Then $x_{a_1}^{e_1} \dots x_{a_n}^{e_n}$ for x_{a_1} in increasing order, over all possible a_i and e_i form a basis of $U(\mathfrak{g})$.*

Theorem 2.9 (Ado's Theorem)**.**

Lemma 2.10. *For any Lie group G,* $\exp(X+Y) = \lim_{n\to\infty}(\exp(\frac{X}{n})\exp(\frac{Y}{n}))^n$.

Proof. By Ado's Theorem, it suffices to prove this for $GL_n(\mathbb{R})$, in which case it follows by looking at Taylor series. $\square$

Theorem 2.11. *A continuous homomorphism of Lie groups is smooth.*

Proof. First consider a homomorphism $f : \mathbb{R} \to G$. G is locally a diffeomorphism, so for small ϵ, choose $X \in \mathfrak{g}$ such that $\exp(X) = f(\epsilon)$. Since f and exp are homomorphisms, $f(q\epsilon) = \exp(qX)$ for any rational q, and by continuity, $f(t) = \exp(\frac{tX}{\epsilon}$, so f is smooth. Now for any $\phi : G \to H$, identify the Lie algebras with $\mathrm{Hom}(\mathbb{R}, -)$, so that we get a map of Lie algebras, which is linear by the previous Lemma. Now since exp is a local diffeomorphism and commutes with ϕ, ϕ is smooth near the identity, hence everywhere. $\square$

Theorem 2.12. *A homomorphism $f : G \to H$ to a connected Lie group H is a covering map iff $df : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.*

Proof. If f is a covering map, then in particular it is a local diffeomorphism hence an isomorphism on Lie algebras. Conversely if df is an isomorphism, since exp commutes with f and generates H, f is surjective, and since we can find a neighborhood on which $\exp_H$ is a diffeomorphism, take its preimage, and intersect it with one for which $\exp_G$ is a diffeomorphism, we see that the kernel is discrete, so f is a covering map. $\square$

Theorem 2.13 (Lie subgroup Adjunction)**.** *There is an adjunction between the category of subgroups of a Lie group and the category of subalgebras of its Lie algebra.*

Proof. This follows from the Frobenius's integrability criterion. Given a subgroup, the left cosets form a left invariant foliation, giving a left invariant subbundle of the tangent bundle that is closed under the Lie bracket, corresponding to a Lie subalgebra. Conversely, given a Lie subalgebra, its span is a subbundle satisfying Frobenius's theorem, so there is a parallel foliation. The submanifold containing the identity will be a subgroup as Something to do with Frobenius' theorem from differential geometry about foliations. $\square$

The unit is an isomorphism and the counit is the connected component.

Theorem 2.14 (Lie's Third Theorem)**.** *The Lie functor is essentially surjective.*

Proof. By Ado's Theorem, any Lie subgroup $\square$

Proposition 2.15. *If $df = dg$ for two maps $f, g : G \to H$ with G connected, $f = g$.*

Proof. This follows from commutativity of the exponential map with f and g and the fact that the image of the exponential map generates g. $\square$

Corollary 2.16 (Lie's Second Theorem)**.** $\mathrm{Hom}(G, H) \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ *via the Lie functor if G is simply connected.*

Proof. We may assume H connected since the image of G always lies in H^o and H^o has the same Lie algebra. By Proposition 2.15 it suffices to show surjectivity. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is an algebra morphism, consider the map $\phi \oplus 1_{\mathfrak{g}}$, and consider the subgroup A of $H \times G$ corresponding to the image. A comes with projections π_H, π_G. Note that π_G is an isomorphism of Lie algebras, so since G is simply connected and by Theorem 2.12 π_G is an isomorphism, so $\pi_H \circ \pi_G^{-1}$ is our desired map. $\square$

Theorem 2.17 (Lie group/algebra adjunction)**.** *The Lie functor has a left adjoint.*

Proof. First note that to any Lie algebra, there is a unique simply connected Lie algebra, by Lie's second and third theorems. Then our left adjoint will take a Lie algebra and give the unique simply connected associated Lie group. The functoriality follows from Lie's second theorem, as well as the fact that this is an adjunction. □

The unit is an isomorphism and the counit is the universal cover of the connected component.

Theorem 2.18. *Regular Lie subgroups are closed Lie subgroups.*

Proof. If $H \subset G$ is regular, and $x_i \to x$ with x_i in H, then choose a cubical chart U around e, and pick □

Proposition 2.19. *If G is a compact Lie group and (π, V) a finite dimensional representation, and any G-invariant metric, $d\pi$ is skew-symmetric. In particular, there is a metric on $T_e(G)$ that is Ad-invariant, and such that ad is skew symmetric.*

Proof. $\langle e^{tX}A, e^{tX}B\rangle = \langle A, B\rangle$, so taking the derivative at 0 we are done. □

Definition 2.20. *Let G be a compact Lie group and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra of its Lie algebra. An element X of the subalgebra is **regular** if $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(X)$.*

Lemma 2.21. *Let I be a nonzero ideal of $\mathbb{R}[x_1, \ldots, x_n]$. Then $Z(I)$ is a closed measure 0 set.*

Proof. $Z(I)$ is the intersection of the zero set of each element of I, so is closed. By Fubini's Theorem it suffices to show that the intersection of $Z(I)$ with almost all hyperplanes along one direction is measure 0. To do this we can use induction. This is clearly true for $\mathbb{R}^1$, and by induction hypothesis, it suffices to show there can only be finitely many entire hyperplanes along one direction intersecting with $Z(I)$. To do this, note that the hyperplanes correspond for example to the ideals $(x_1 + c)$ for some $c \in \mathbb{R}$, and so this follows from looking at the Zariski topology of $\mathbb{R}^1$ again. □

Theorem 2.22. *The set of regular elements is an open with full measure (with respect to the Lebesgue measure for any basis).*

Proof. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{t}$. Then note that $\mathrm{ad}(X_i), \mathrm{ad}(X_j)$ commute, and that if $h_i = \ker(\mathrm{ad}(X_i))$ and $j_i = \ker(\mathrm{ad}(X_i))^{\perp}$, then note that $\mathrm{ad}(X_i)$ preserves the h_i, j_i $\mathfrak{t}$ splits as $\oplus_{g_i = j_i, h_i} \cap_1^n g_i$. Now for $c_i \neq 0$, if $l = \sum c_i X_i$, then $\ker(\mathrm{ad}(l))$ is $\cap_1^n h_i$, hence l is regular, iff it is invertible when restricted to each summand of t that is not $\cap_1^n g_i$. We will show that the set of c_i for which $\mathrm{ad}(l)$ is invertible on each of these is open with full measure. To do this, the determinant of $\mathrm{ad}(l)$ on each of these spaces is a polynomial in the c_i, and is nonzero since some X_i is nonzero on the space. Then by Lemma 2.21 on this polynomial, we see that on a open with full measure, $\mathrm{ad}_(l)$ is invertible. Thus almost all elements are regular. Now to show the regular points are open, given a regular element X, choose a basis X_i of $\mathfrak{t}$ such that none of the coefficients of X are 0 when X is written as a combination of the X_i. Then by the same argument, X is contained in some open set of regular points. The independence of basis follows from the properties of Lebesgue measure under linear transformation. □

Theorem 2.23. *For any $X \in \mathfrak{g}$, with $\mathfrak{t}$ a Cartan subalgebra for a compact Lie group G, there is some g such that $\mathrm{Ad}(g)X \in \mathfrak{t}$.*

Proof. By Theorem 2.22, picking a regular element Y, it suffices to show that there is some g with $[\mathrm{Ad}(g)X, Y] = 0$. By Proposition 2.19, it suffices to show for all Z, $\langle [\mathrm{Ad}(g)X, Y], Z\rangle = 0$, which happens iff $\langle Y, \mathrm{ad}(Z)\,\mathrm{Ad}(g)X\rangle = 0$. G is compact, so choose g a minimum of the function $\langle Y, \mathrm{Ad}(g)X\rangle$. Then the function $\langle Y, \exp(tZ)\,\mathrm{Ad}(g)X\rangle$ has a minimum at $t = 0$ for all Z so taking the derivative, we are done. $\square$

Theorem 2.24. *G acts transitively on the set of Cartan subalgebras (via* Ad*) and maximal tori (via conjugation).*

Proof. If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two Cartan subalgebras, write them as $\mathfrak{z}_{\mathfrak{g}(X_i)}$. Then by the previous theorem, choose g such that $\mathrm{Ad}(g)X_1 \in \mathfrak{t}_2$. Then $\mathrm{Ad}(g)\mathfrak{t}_1 = \{\mathrm{Ad}(g)Y | [Y, X_1] = 0\} = \{Y | [\mathrm{Ad}(g^{-1})Y, X_1] = 0\} = \{Y | [Y, \mathrm{Ad}(g)X_1] = 0\} = \mathfrak{z}_{\mathfrak{g}}(\mathrm{Ad}(g)X_1)$. Thus we have $\mathrm{Ad}(g)\mathfrak{t}_1 \supset \mathfrak{t}_2$, but my maximality they must be equal.

Now if T_i are two maximal tori, with $\mathfrak{t}_i$ the corresponding subalgebras, then if $\mathrm{Ad}(g)t_1 \supset t_2$, then $\square$

How this should be organized:

A short section on Lie groups and their Lie algebras

Nothing about just lie algebras, but can use theory of Lie algebras

Maybe say something about locally compact groups and invariant measures?

Start doing rep theory.

Geometry of Lie groups?

Things to add:

Maximal compact subgroups existence and uniqueness Continuous manifold topological groups are Lie groups, and their homomorphisms are Lie group homomorphisms

Lie's theorems