AN ALTERNATE COMPUTATION OF THE STABLE HOMOLOGY OF DIHEDRAL GROUP HURWITZ SPACES

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ABSTRACT. We give an alternate proof of our result computing the stable homology of dihedral group Hurwitz spaces. This proof employs more elementary methods, instead of higher algebra.

CONTENTS

1. INTRODUCTION

Let G be a group, $c \subset G$ be a conjugacy class, and $n \in \mathbb{Z}_{\geq 0}$. Let $\mathrm{Hur}_n^{G,c}$ denote the Hurwitz space over the complex numbers which parameterizes branched G-covers of the disk, with a marked basepoint over the boundary, branched at n points, where the inertia type of each branch point lies in a the conjugacy class $c \subset G$. In algebraic topology, these Hurwitz spaces can be described as the homotopy quotient c^n/B_n, where B_n is the braid group on n strands.

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Let Conf_n denote the configuration space of n unordered, distinct points in the interior of the disc, and unbranched over the boundary. There is a natural map $\mathrm{Hur}_n^{G,c} \to \mathrm{Conf}_n$ which sends a branched cover of the disc $X \to D$ to its branch locus in the interior of the disc.

Theorem 1.0.1. *Choose an odd prime ℓ and use G to denote the dihedral group of order 2ℓ, $G := \mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. Let $c \subset G$ denote the conjugacy class of order 2 elements. There are constants I and J depending only on G so that for $n > iI + J$ and any connected component $Z \subset \mathrm{Hur}_n^{G,c}$, the map $H_i(Z, \mathbb{Q}) \to H_i(\mathrm{Conf}_n, \mathbb{Q})$ is an isomorphism.*

This is an immediate consequence of [EVW16, Theorem 6.1], which proves the homology stabilizes in a linear range, and Theorem 5.0.3, which computes the stable value of these homology groups.

Our primary motivation for proving Theorem 1.0.1 is for its application to the computation of the moments predicted by the number theoretic Cohen-Lenstra heuristics over function fields. See [LL24] for more on this application.

Remark 1.0.2. We originally came up with a more elementary argument for Theorem 1.0.1 along the lines in this paper, which guided us to later devise the argument presented in [LL24]. In some sense, these two arguments follow very similar trajectories, although this may not be evident on first blush. Our main result Theorem 1.0.1 is a special case of the results of [LL24], where we study the stable homology of $G = \mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, instead of the more general case of $H \rtimes \mathbb{Z}/2\mathbb{Z}$, for H an abelian group of odd order.

We chose to write that version there, because it seemed conceptually simpler and also generalized more easily to the case when $G = H \rtimes \mathbb{Z}/2\mathbb{Z}$, for H an arbitrary odd order abelian group. However, we still thought it would be nice to record an argument along the lines of our original one, especially for those interested in the subject and not familiar with higher algebra.

We also wanted to write this note to emphasize that our computation of the stable homology is not an abstruse result in topology. If readers unfamiliar with higher algebra only read [LL24], it is possible they could view it that way. We wrote this exposition to emphasize that our computation of the stable homology of Hurwitz spaces is an accessible result in linear algebra.

Remark 1.0.3. With significant additional work, we were able to generalize many parts of the argument presented here to the case that $G = H \rtimes \mathbb{Z}/2\mathbb{Z}$, for H an arbitrary odd order abelian group. However, we did not carefully work out all the details. In particular, we did not work out the generalization of §4. We believe it would be interesting to do so.

1.1. **Proof outline.** To prove Theorem 1.0.1, we first study what the stabilization map looks like more concretely. By combining various ideas in the literature, namely the group completion theorem and the interaction of the stabilization map with boundary monodromy, we are able to show that the homology in the kernel of the map to configuration space decomposes as a direct sum of subspaces, each of which is stabilized via multiplication by a single element of the group. This reduction is carried out in §5. The crux of the matter is therefore to prove Proposition 4.0.2, showing that any element of the stable cohomology of $\mathrm{CHur}_n^{G,c}$ not pulled back from the cohomology of configuration space, stabilized via multiplication by a single $g \in c$, is trivial. To approach this, we start by studying the stable cohomology in terms of Fox-Neuwirth-Fuks cells. Since we know the stabilization map is obtained via a single group element, we can write down an explicit form for such a cohomology class stabilized by a single group element. If we are able to show this class is cohomologous to one in a sufficiently simple form, we will be able to conclude using exactness of the stable $\mathcal{K}$-complex, a complex introduced in [EVW16, Theorem 4.2], which was the key to showing that the homologies of these Hurwitz spaces stabilize. In order to massage the element into the desired form, we extend the study of the $\mathcal{K}$-complex and introduce a new object of study, which we call the "two-sided $\mathcal{K}$-complex," generalizing the $\mathcal{K}$-complex. We compute the cohomology of a variant of a particular two-sided $\mathcal{K}$-complex in Proposition 3.4.10, via an explicit chain homotopy. This allows us to massage x into the desired form, and hence conclude the proof.

1.2. **Outline of paper.** We introduce notation for Hurwitz spaces in §2. In §3, we compute the homology of a particular complex, which we call a two-sided $\mathcal{K}$-complex. In §4, we use this computation of the homology of the two-sided $\mathcal{K}$-complex to compute the part of the stable homology of dihedral Hurwitz spaces stabilized by a single element $g \in c$. Finally, in §5, we use this computation to prove Theorem 1.0.1, computing the stable homology of dihedral group Hurwitz spaces.

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2. NOTATION FOR HURWITZ SPACES

Notation 2.0.1. We let G denote the dihedral group of order 2ℓ for an odd prime ℓ and let $c \subset G$ denote the conjugacy class of order 2 elements. As mentioned in the introduction, we use $\mathrm{Hur}_n^{G,c}$, to denote the Hurwitz space over the complex numbers which branched G-covers of the disk, with a marked basepoint over the boundary, branched at n points, where the inertia type of each branch point lies in the conjugacy class $c \subset G$. This has a model as the homotopy quotient c^n/B_n, where B_n is generated by the standard elements $\sigma_1, \ldots, \sigma_{n-1}$ where σ_i twists strands i and $i+1$, and the action is given by $\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n)$.

Notation 2.0.2. Continuing with notation as in Notation 2.0.1, we can identify the connected components of the pointed Hurwitz space $\mathrm{Hur}_{n,\mathbb{C}}^{G,c}$ with orbits of the B_n action on c^n. Under this identification, $x_1, \ldots, x_n \in c$ we use $[x_1] \cdots [x_n]$ to denote the connected component of $\mathrm{Hur}_{n,\mathbb{C}}^{G,c}$ corresponding to the B_n orbit of the tuple $(x_1, \ldots, x_n)$.

Notation 2.0.3. Fix G, c as in Notation 2.0.1. For $\beta \in \{0, 1\}$, we let $H_{i,\beta}^{G,c}$ denote the vector space $H_i(\mathrm{CHur}_n^{G,c}, \mathbb{Q})$ for $n \equiv \beta \pmod 2$ sufficiently large. We note this vector space is independent of n once n is sufficiently large by [EVW16, Theorem 6.1] (with input from [EL23, Proposition A.3.1]). Here, the isomorphism between $H_i(\mathrm{CHur}_n^{G,c}, \mathbb{Q})$ and $H_{i+2}(\mathrm{CHur}_n^{G,c}, \mathbb{Q})$ is given by the stabilization operator $\sum_{g \in c} [g]^2$, where $[g]$ corresponds to right multiplication by a generator of $H_0(\mathrm{Hur}_1^{G,c}, \mathbb{Q})$ with monodromy g.

In particular, we use $H_{i,\beta}^{\mathrm{id},\mathrm{id}}$ to denote the stable rational homology of configuration space on n points, which is well known to be 1-dimensional if $i = 0$ or 1 and 0 dimensional otherwise; one may deduce this from the computation of integral homology of ordered configuration space in [Ad69] by rationalizing and taking S_n invariants.

Throughout this paper, it will be crucial to understand the connected components of our Hurwitz spaces. For our dihedral groups of order 2ℓ, the stable components turn out to be uniquely determined by their boundary monodromy, as we explain next.

Lemma 2.0.4. *Let H be an odd order abelian group, and $G := H \rtimes \mathbb{Z}/2\mathbb{Z}$, with the generator of $\mathbb{Z}/2\mathbb{Z}$ acting by inversion. We let $c \subset G$ denote the conjugacy class of order 2 elements. There is a map $c^n \to G$ given by sending $(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$. This map induces a map $\pi_0(\mathrm{Hur}_n^{G,c}) \to G$. We let $\mathrm{CHur}_n^{G,c,g}$ denote those connected components of $\mathrm{CHur}_n^{G,c}$ mapping to g under the above map. For*

n sufficiently large, there are $\# \wedge^2 H$ connected components of $\mathrm{CHur}_n^{G,c,g}$ when $n \bmod 2$ agrees with the image of g in $\mathbb{Z}/2\mathbb{Z}$ under the projection $G \to G/H \simeq \mathbb{Z}/2\mathbb{Z}$ and 0 components otherwise. In particular, if $H \simeq \mathbb{Z}/\ell\mathbb{Z}$, for ℓ an odd prime, there is at most one such component.

Remark 2.0.5. The only case of Lemma 2.0.4 we will use is the case $H = \mathbb{Z}/\ell\mathbb{Z}$. It is not too difficult to show by hand that when $H = \mathbb{Z}/\ell\mathbb{Z}$, there is a unique component of $\mathrm{CHur}_n^{G,c,g}$ when $n \bmod 2$ agrees with the image of g in $\mathbb{Z}/2\mathbb{Z}$ and 0 components otherwise. However, we have opted to prove the statement in the above generality, as it appears not to have been completely spelled out in the literature.

Proof. The final statement follows from the first statement because when $H = \mathbb{Z}/\ell\mathbb{Z}$, $\wedge^2 H$ is the trivial group.

To prove the first statement, there is a certain finite abelian group $H_2(G, c)$ defined in [Woo21, Definition, p. 3], and which we will recall the definition of in the next paragraph, with a map $S_c \to G$. It follows from [Woo21, Theorem 3.1, Theorem 2.5, and the Definition on p. 3] that $H_2(G, c)$ satisfies the following property: the number of irreducible components of $\# \mathrm{CHur}_n^{G,c,g}$ is identified with $\# \ker(S_c \to G)$, when the image of g in the abelianization of G agrees with the image of n in the abelianization $\mathbb{Z}/2\mathbb{Z}$, and there are no such components otherwise.

In our present situation, we claim the finite group $H_2(G, c)$ is identified with the usual group homology $H_2(H, \mathbb{Z})$. This was outlined in [EVW12, 9.3.2] and is also closely related to the proof of [SW23, Theorem 3.1]. We now recapitulate the argument. Let $\langle x, y\rangle$ denote the image of the canonical generator $[(1,0)|(0,1)] - [(0,1)|(1,0)] \in H_2(\mathbb{Z}^2, \mathbb{Z})$ under the map $H_2(\mathbb{Z}^2, \mathbb{Z}) \to H_2(G, \mathbb{Z})$, induced by the map $\mathbb{Z}^2 \to G$ sending the first generator to x and the second generator to y. By definition the group $H_2(G, c)$ is the quotient of $H_2(G, \mathbb{Z})$ by all classes $\langle x, y\rangle$ for $x \in c$ such that x and y commute.

We next show that the quotient map $H_2(G, \mathbb{Z}) \to H_2(G, c)$ is an isomorphism. Let $\langle x\rangle$ denote the subgroup of G generated by $x \in c$, isomorphic to $\mathbb{Z}/2\mathbb{Z}$. A direct computation shows that the only elements of G commuting with x are $\{x, \mathrm{id}\}$, and in either case, we have $\langle x, y\rangle$ lies in the image of $H_2(\langle x\rangle, \mathbb{Z}) \to H_2(G, \mathbb{Z})$. For A any abelian group, $H_2(A, \mathbb{Z}) = \wedge^2 A$, so $H_2(\langle x\rangle, \mathbb{Z}) \simeq \wedge^2(\mathbb{Z}/2\mathbb{Z})$ is the trivial group, and hence $H_2(G, \mathbb{Z}) \simeq H_2(G, c)$.

It is a standard group cohomology fact that $H_2(H, \mathbb{Z}) \simeq \wedge^2 H$. Hence, to conclude the proof, it suffices to show $H_2(H, \mathbb{Z}) \simeq H_2(G, \mathbb{Z})$. Indeed, this follows from the spectral sequence associated to the exact sequence $H \to G \to \mathbb{Z}/2\mathbb{Z}$, as we now explain. We may observe that $H_0(\mathbb{Z}/2\mathbb{Z}, H_1(H, \mathbb{Z}))$ is the coinvariants of the odd order group H by the inversion action, and hence vanishes. Additionally, $H_1(\mathbb{Z}/2\mathbb{Z}, H_1(H,\mathbb{Z})) = H_2(\mathbb{Z}/2\mathbb{Z}, H_0(H,\mathbb{Z})) = 0$ as both groups are 2-torsion but are also H-modules, and hence must vanish. Therefore, the spectral sequence yields an isomorphism $H_0(\mathbb{Z}/2\mathbb{Z}, H_2(H,\mathbb{Z})) \simeq H_2(G,\mathbb{Z})$. Since $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $H_2(H,\mathbb{Z})$, we find $H_2(H,\mathbb{Z}) \simeq H_2(G,\mathbb{Z})$. $\square$

Definition 2.0.6. We assume k is a field of characteristic 0. For G a dihedral group of order 2ℓ with ℓ odd, $g \in G$, and $\phi : G \to \mathbb{Z}/2\mathbb{Z}$ the surjection given by quotienting by $\mathbb{Z}/\ell\mathbb{Z}$, we explained in Lemma 2.0.4 why there is a bijection between components of $\mathrm{CHur}_n^{G,c}$ and elements g with $\phi(g) \equiv n \bmod 2$ for n sufficiently large.

We let $\mathrm{CHur}_n^{G,c,g}$ denote the union of those connected components of $\mathrm{CHur}_n^{G,c}$ mapping to g under the map described in Lemma 2.0.4. We say such components have *boundary monodromy* g. When n is sufficiently large, it follows from Lemma 2.0.4 that $\mathrm{CHur}_n^{G,c,g}$ is either empty or connected. We let $H_i^{G,c,g}$ denote the subspace of the stable homology $H_{i,\beta}^{G,c}$, which we can identify with the subspace $H_i(\mathrm{CHur}_n^{G,c,g}, k) \subset H_i(\mathrm{CHur}_n^{G,c}, k)$ for $n \equiv \beta \bmod 2$ sufficiently large. If $x \in H_{i,\phi(g)}^{G,c}$, we say x has *boundary monodromy* $g \in G$ if x lies in $H_i^{G,c,g}$.

3. Cohomology of the two-sided $\mathcal{K}$ complex

In this section, we introduce the two-sided $\mathcal{K}$-complex associated to a pair of modules and compute its homology for a particular pair of modules. In §3.1, we first recall the usual (1-sided) $\mathcal{K}$-complex introduced in [EVW16] and refined in [RW20]. We then define the two-sided $\mathcal{K}$-complex associated to a pair of modules in §3.2. In §3.3, we produce a simple nullhomotopy on certain two-sided $\mathcal{K}$-complexes. We then proceed to compute the homology of another two-sided $\mathcal{K}$-complex in §3.4. The main result from this section we will use in the future is Proposition 3.4.10. This is essentially equivalent to Theorem 3.4.3, which computes the homology of a certain two-sided $\mathcal{K}$-complex. Theorem 3.4.3, which is a bit easier to state than Proposition 3.4.10, will help motivate our proof of Proposition 3.4.10. The key to both of these results is an explicit nullhomotopy of a large subcomplex of the two-sided $\mathcal{K}$-complex given in Lemma 3.4.9.

3.1. **Review of the usual $\mathcal{K}$-complex.**

Fix a field k of arbitrary characteristic and let $A := \oplus_{n\geq 0} C_\bullet(\mathrm{Hur}_n^{G,c}, k)$ denote the algebra of singular k-chains associated to the Hurwitz spaces parameterized by G and c. We use $R := \oplus_{n\geq 0} H_0(\mathrm{Hur}_n^{G,c}, k)$ to denote the ring of components of Hurwitz spaces.

Note that A has a grading given by the index n parameterizing the number of branch points.

We say M is a *discrete* graded A module if M is a chain complex concentrated in degree 0, which is a module for A. (In other words, M is just what one usually thinks of as an R module.) The $\mathcal{K}$-complex, $\mathcal{K}(M)$, introduced in [EVW16, 4.1] and also [RW20, p. 16], is the chain complex

$$(3.1) \qquad \cdots \to k\{c\}^{\otimes n} \otimes M \to k\{c\}^{\otimes(n-1)} \otimes M \to \cdots \to k\{c\} \otimes M \to M.$$

The boundary maps in (3.1) are given by

(3.2)

$$d(g_1 \otimes \cdots \otimes g_n \otimes [m]) = \sum_{i=1}^{n} (-1)^i g_1 \otimes \cdots g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n \otimes (g_n^{-1} \cdots g_{i+1}^{-1} g_i g_{i+1} \cdots g_n) \cdot [m].$$

Remark 3.1.1. Suppose $A \to R := \oplus_{n \geq 0} H_0(\mathrm{Hur}_n^{G,c}, k)$ denotes the quotient map. If M is an R module, Ellenberg Venkatesh and Westerland introduced the $\mathcal{K}$-complex $\mathcal{K}(M)$ associated to such an M [EVW16, 4.1]. As mentioned in the last paragraph of [RW20, p. 16], $\mathcal{K}(M)$ is the Koszul complex for computing the homology of the derived tensor product $k \otimes_A^{\mathbb{L}} M$.

3.2. **Definition the two-sided $\mathcal{K}$-complex.** Generalizing the $\mathcal{K}$-complex described in § 3.1, we next introduce the two-sided $\mathcal{K}$-complex. Still letting $A := \oplus_{n \geq 0} C_\bullet(\mathrm{Hur}_n^{G,c}, k)$ as in § 3.1, we now let M be a discrete right A module and N be a discrete left A module.

Definition 3.2.1. Define the *two-sided $\mathcal{K}$-complex*, $\mathcal{K}(M, A, N)$ to be the double complex with (i, j) term given by $\oplus_{\alpha+\beta=j} M_\alpha \otimes k\{c\}^i \otimes N_\beta$. The the total differential on this double complex is the sum of a "rightward" differential d_{r} and a "leftward" differential d_{l} which are given as follows. The rightward differential is given by

(3.3)

$$d_{\mathrm{r}}([m] \otimes g_1 \otimes \cdots \otimes g_n \otimes [\omega]) = \sum_{i=1}^{n} (-1)^i [m] \otimes g_1 \otimes \cdots g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n \otimes (g_n^{-1} \cdots g_{i+1}^{-1} g_i g_{i+1} \cdots g_n) \cdot [\omega],$$

while the leftward differential is given by

(3.4)

$$d_{\mathrm{l}}([m] \otimes g_1 \otimes \cdots \otimes g_n \otimes [\omega]) = \sum_{i=1}^{n} (-1)^i [m] \cdot g_i \otimes (g_i^{-1} g_1 g_i) \otimes \cdots (g_i^{-1} g_{i-1} g_i) \otimes g_{i+1} \otimes \cdots \otimes g_n \otimes [\omega].$$

It may be helpful to refer to Figure 1 for a visualization of a particular summand of a two-sided $\mathcal{K}$ complex.

The next remarks are not needed in what follows, but may serve as some motivation for our above definition.

Remark 3.2.2. There is an alternate, more abstract definition of the two sided $\mathcal{K}$ complex, $\mathcal{K}(M, A, N)$. Namely, it is equivalent to the derived tensor product $M \otimes^{\mathbb{L}}_A N$. To see this, note that $M \otimes^{\mathbb{L}}_A N$ has a double filtration, induced by filtering both M and N by their gradings. Taking the associated graded with respect to both filtrations, we get $(k \otimes^{\mathbb{L}}_A k) \otimes_k (M \otimes_k N)$. Similarly to §3.1, running the spectral sequence in each filtration direction, we find that the spectral sequences collapses at the E_2-page because of the discreteness of M and N. We thus obtain a double complex whose underlying bigraded vector space has homology agreeing with $(k \otimes^{\mathbb{L}}_A k) \otimes_k (M \otimes_k N)$ and whose total complex is quasi-isomorphic to $M \otimes^{\mathbb{L}}_A N$. Concretely, one can show that the E^1 page of this spectral sequence can be viewed as a double complex whose (i, j)th term agrees with $\mathcal{K}(M, A, N)_{i,j}$. Moreover, the differential $\mathcal{K}(M, N)_{i,j}$ to $\mathcal{K}(M, N)_{i-1,j}$ can be shown to agree with the differential on this E^1 page of this spectral sequence (see [RW20, Theorem 6.2 and p. 16]).

Remark 3.2.3. In the case that M and N are modules coming from the action of the Hurwitz space on a set, the two-sided $\mathcal{K}$-complex can be viewed as a cellular chain complex for the spaces in [LL24, Theorem A.4.9].

3.3. **A nullhomotopy for two-sided $\mathcal{K}$ complexes.** We next show that certain types of two-sided $\mathcal{K}$ complexes are exact. The proof of the following lemma is inspired by [EVW16, Lemma 4.11]. One can also deduce this by using [LL24, Lemma 4.3.1].

Lemma 3.3.1. *Let $A := \oplus_{n \geq 0} C_\bullet(\mathrm{Hur}^{G,c}_n, k)$ Suppose M is a left A module and N is a right A module. Assume there is some $h \in c$ so that either*

(1) $[h]$ acts invertibly on M and by 0 on N, or
(2) $[h]$ acts invertibly on N and by 0 on M.

Then, there is a chain homotopy between the identity map on $\mathcal{K}(M, A, N)$ and the 0 map on $\mathcal{K}(M, A, N)$. In particular, $\mathcal{K}(M, A, N)$ is exact.

Proof. We just explain the second case that $[h]$ acts invertibly on N and by 0 on M, as the first case is analogous. We now give an explicit chain homotopy between the identity and 0.

We start with a warm up computation. Define $S^0_{n,j}$ by

$$S^0_{n,j} : \mathcal{K}(M, A, N) \to \mathcal{K}(M, A, N)$$
$$[m] \otimes g_1 \otimes \cdots \otimes g_n \otimes [\omega] \mapsto (-1)^{n+1}[m] \otimes g_1 \otimes \cdots \otimes g_n \otimes h \otimes [h]^{-1} \cdot [\omega].$$

Note that the definition of this chain map uses that $[h]$ acts invertibly on N. Using that $[g_1][g_1^{-1}g_2g_1] = [g_2][g_1]$, a routine computation similar (but easier than) the computation in Lemma 3.4.9 verifies that

$$\begin{aligned}&(dS^0_{n,j} + S^0_{n-1,j}d)([m] \otimes g_1 \otimes \cdots \otimes g_n \otimes [\omega]) \\ &= [m] \otimes g_1 \otimes \cdots \otimes g_n \otimes [\omega] + [m][h] \otimes h^{-1}g_1h \otimes \cdots \otimes h^{-1}g_nh \otimes [h]^{-1}[\omega].\end{aligned}$$

Now, using that $[m] \cdot [h] = 0$, the map $S^0_{n,j}$ gives a chain homotopy between the identity and the 0 map. This implies that $\mathcal{K}(M, A, N)$ is exact. $\square$

Notation 3.3.2. For $g \in c$, let $k[g]$ be the graded A-bimodule that is k in each natural number degree, such that left and right multiplication by $[g]$ acts by shifting. Moreover, for any $g \neq h \in c$, $[h]$ acts by 0. We use $k[g, g^{-1}]$ to denote the analogous graded A modules with integer gradings. (So right and left multiplication by $[g]$ shifts the grading while $[h]$ acts by 0 for $h \neq g$.)

Note there are natural graded bimodule maps $R \to k[g] \to k[g, g^{-1}]$, where the first map is given by quotienting out by elements that are not multiples of g.

Lemma 3.3.3. *The complex $\mathcal{K}(k, A, k[g, g^{-1}])$ is exact. In particular, the complex $\mathcal{K}(M)$ associated to $M := k[g, g^{-1}]$, as defined in* (3.1)*, is exact.*

Proof. Since g acts by 0 on k and invertibly on $k[g, g^{-1}]$, it follows that $\mathcal{K}(k, A, k[g, g^{-1}])$ is exact. Since $\mathcal{K}(k, A, k[g, g^{-1}])$ is a sum of $\mathbb{Z}$ many shifted copies of $\mathcal{K}(M)$, so that, in particular, $\mathcal{K}(M)$ is a summand of $\mathcal{K}(k, A, k[g, g^{-1}])$, we obtain $\mathcal{K}(M)$ is exact. $\square$

3.4. **A particular two-sided $\mathcal{K}$-complex.** In this subsection, we investigate the homology of a particular two-sided $\mathcal{K}$ complex. This two-sided $\mathcal{K}$ complex will appear in the homology of Hurwitz spaces, and understanding its homology is the crucial step in understanding the stable homology of Hurwitz spaces.

Notation 3.4.1. We fix a group G and a conjugacy class $c \subset G$ generating G and a field k. We assume that the for any $h \in c$, the centralizer of h in c is precisely h. We let $A := \oplus_{n \geq 0} C_\bullet(\mathrm{Hur}^{G,c}_n, k)$ denote the algebra of singular k-chains associated to the Hurwitz spaces parameterized by G and c. We let $B := C_*(\mathrm{Conf}_n, k)$ denote the corresponding algebra of singular k-chains on configuration space.

Remark 3.4.2. We note that this condition on the centralizer in Notation 3.4.1 will hold for dihedral groups, with c the conjugacy class of involutions.

$$
\begin{tikzcd}[column sep=small]
 & k\{c^0\} & & k\{c^0\} & & k\{c^0\} & & k\{c^0\} \\
\cdots \arrow[ur] & & k\{c^1\} \arrow[ul] \arrow[ur] & & k\{c^1\} \arrow[ul] \arrow[ur] & & k\{c^1\} \arrow[ul] \arrow[ur] & \\
 & k\{c^2\} \arrow[ul] \arrow[ur] & & k\{c^2\} \arrow[ul] \arrow[ur] & & k\{c^2\} \arrow[ul] \arrow[ur] & & \\
\cdots \arrow[ur] & & k\{c^3\} \arrow[ul] \arrow[ur] & & k\{c^3\} \arrow[ul] \arrow[ur] & & & \\
 & k\{c^4\} \arrow[ul] \arrow[ur] & & k\{c^4\} \arrow[ul] \arrow[ur] & & & & \\
\cdots \arrow[ur] & & k\{c^5\} \arrow[ul] \arrow[ur] & & & & & \\
 & k\{c^6\} \arrow[ul] \arrow[ur] & & & & & & \\
\cdots \arrow[ur] & & & & & & &
\end{tikzcd}
$$

FIGURE 1. The complex above depicts the summand $C_{\bullet,\bullet}$ (defined in Remark 3.4.5) of two sided $\mathcal{K}$-complex, $\mathcal{K}(k[g], A, k[g, g^{-1}])$.

Our goal in this section is to compute the homology of $\mathcal{K}(k[g], A, k[g, g^{-1}])$. With notation as in Notation 3.4.1, There is a map $B \to A$ induced by the inclusion $\mathrm{Conf}_n \to \mathrm{CHur}_n^{G,c}$ with image the component of Hurwitz space parameterizing covers whose monodromy is g at every branch point. (These will be disconnected covers, unless G is already a cyclic group.)

Theorem 3.4.3. *The map $\mathcal{K}(k[g], B, k[g, g^{-1}]) \to \mathcal{K}(k[g], A, k[g, g^{-1}])$, induced by the above map $B \to A$, is an isomorphism.*

We will prove Theorem 3.4.3 in §3.4.11. The following is a direct consequence of the spectral sequence described in §3.2.

Lemma 3.4.4. *There is a collection of double complexes $C^z_{\bullet,\bullet}$, indexed by $z \in \mathbb{Z}$ so that the sum of the associated total complex, $\bigoplus_{z \in \mathbb{Z}} C^z_{\bullet,\bullet}$, viewed as an object in the derived category, is isomorphic to $\mathcal{K}(k[g], A, k[g, g^{-1}])$. Specifically, $C^z_{i,j} = k\{c\}^{z-i-j}$ and there are differentials $C^z_{i,j} \to C^z_{i,j+1}$ and $C^z_{i,j} \to C^z_{i+1,j}$ given by the maps $d_{\mathrm{r}}, d_{\mathrm{l}}$ of* (3.3) *and* (3.4) *associated to the modules $M = k[g], N = k[g, g^{-1}]$ from Notation 3.3.2.*

Remark 3.4.5. It follows immediately from the description of Lemma 3.4.4 that $C^z_{s,u}$ can be identified with those terms of the form $g^s \otimes k\{c\}^t \otimes g^u$ where $s+t+u = z$ and $s \geq 0$. Additionally, $C^z_{i,j} \simeq C^{z'}_{i,j+z-z'}$, compatibly with the differentials, for any z, z', as follows from the definition. Hence, it follows that these two complexes are isomorphic, up to a shift. To simplify notation a bit, we write $C_{\bullet,\bullet} := C^0_{\bullet,\bullet}$. See Figure 1 for a picture of this complex.

Remark 3.4.6. Let $\delta_{h,h'}$ be 1 if $h = h'$ and 0 otherwise, Using the definition of the differentials for the two-sided $\mathcal{K}$ complex given in Definition 3.2.1, we can describe the differentials on $C_{\bullet,\bullet}$ as

$$d_{\mathrm{r}}([g^\alpha] \otimes g_1 \otimes \cdots \otimes g_n \otimes [g^\beta]) = \sum_{i=1}^{n} \delta_{g, g_n^{-1} \cdots g_{i+1}^{-1} g_i g_{i+1} \cdots g_n} (-1)^i [g^\alpha] \otimes g_1 \otimes \cdots g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n \otimes [g^{\beta+1}]. \tag{3.5}$$

and

$$d_{\mathrm{l}}([g^\alpha] \otimes g_1 \otimes \cdots \otimes g_n \otimes [g^\beta]) = \sum_{i=1}^{n} \delta_{g, g_i} (-1)^i [g^{\alpha+1}] \otimes (g_i^{-1} g_1 g_i) \otimes \cdots (g_i^{-1} g_{i-1} g_i) \otimes g_{i+1} \otimes \cdots \otimes g_n \otimes [g^\beta]. \tag{3.6}$$

Notation 3.4.7. Let $C'_{\bullet,\bullet}$ denote the subcomplex of $C_{\bullet,\bullet}$ which is spanned by all basis elements $(g_1, \ldots, g_n) \in c^n$ with some $g_i \neq g$. In particular, $\dim C'_{i,j} = |c^{i+j}| - 1$. Let T_n denote the total complex $\oplus_{i+j=n} C'_{i,j}$ associated to $C'_{i,j}$.

Consider a basis element v in $C'_{i,j}$ of the form

$$(x_1, \ldots, x_s, h, \underbrace{g, \ldots, g}_{t \text{ times}}) \tag{3.7}$$

with $h \neq g$. Define a filtration $F^i_n \subset T_n$ spanned by those v as above with $t \leq i$. Consider the map $\sigma_n : T_n \to T_{n+1}$ sending

$$(x_1, \ldots, x_s, h, \underbrace{g, \ldots, g}_{t \text{ times}}) \mapsto (-1)^n \cdot (x_1, \ldots, x_s, hgh^{-1}, h, \underbrace{g, \ldots, g}_{t \text{ times}}).$$

Remark 3.4.8. In [LL24] we use a geometric map which corresponds algebraically to the chain homotopy

$$(x_1, \ldots, x_s, h, \underbrace{g, \ldots, g}_{t \text{ times}}) \mapsto (-1)^n \cdot (g x_1 g^{-1}, \ldots, g x_s g^{-1}, g, h, \underbrace{g, \ldots, g}_{t \text{ times}}).$$

Similarly to Lemma 3.4.9, one can also verify this defines a chain homotopy between an isomorphism and 0. This alternate chain homotopy was pointed out to us by Andrea Bianchi.

Lemma 3.4.9. *The map $\sigma_\bullet : T_\bullet \to T_{\bullet+1}$ defines a chain homotopy between an isomorphism and* 0*. Hence, the complex $T_\bullet^n$ is exact.*

Proof. If $\sigma_\bullet$ defines a chain homotopy between an isomorphism and 0, this implies 0 acts the same on cohomology as an isomorphism, and so $T_\bullet^n$ is exact.

We now show $\sigma_\bullet$ defines a chain homotopy between an isomorphism and 0. Explicitly we wish to show $\sigma_{n-1}d_n + d_{n+1}\sigma_n$ induces an isomorphism $T_n \to T_n$. To prove this, we will first show that this map preserves the filtration F_n^i, and then that it induces isomorphisms on the associated graded of the filtration F_n^i / F_n^{i-1}.

First, let us verify the map $\sigma_{n-1}d_n + d_{n+1}\sigma_n$ preserves the filtration F_n^i. Note that the map σ_j sends F_j^i to F_{j+1}^i by construction. Therefore, to show $\sigma_{n-1}d_n + d_{n+1}\sigma_n$ preserves the filtration, it is enough to show d_j sends F_j^i to F_{j-1}^i. Moreover, since the differential d is a sum of $d_\mathrm{l} + d_\mathrm{r}$, it is enough to show that d_l and d_r separately preserve the filtration. Using the definition of these differentials appearing in (3.5) and (3.6), we see each such differential is a sum of terms associated to each of the n entries of v, so it is enough to show each of these terms lies in F_{j-1}^i. First, let us analyze the terms in d_l. For the terms associated to $x_1, \ldots, x_s$ for v as in (3.7), these terms do not alter the t entries g on the right, and so the filtration is preserved for these terms. For the term corresponding to h, (in position $s+1$,) since $h \neq g$, this term must be 0. Finally, each of the terms associated to one of the rightmost t entries g lie in the filtration F_{j-1}^{i-1}, which indeed lies in F_{j-1}^i. To conclude, we show d_r sends F_j^i to F_{j-1}^i. For each of the terms associated one of the t rightmost g's, this sends F_j^i to F_{j-1}^{i-1}, which indeed lies in F_{j-1}^i. The term associated to h vanishes because the conjugate of h by any power of g is not g since $h \neq g$. Finally, we note that the term associated to one of the $x_1, \ldots, x_s$, for x_i as in (3.7) lies in F_{j-1}^i because the term is either 0 or of the form $\pm(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s, h, g, \ldots, g)$.

Having shown the filtration is preserved, we conclude by checking that the map induced by $\sigma_{n-1}d_n + d_{n+1}\sigma_n$ on F_n^t / F_n^{t-1} is a multiple of the identity. In other words, we assume there are precisely t elements g to the right of h, for h the rightmost term not equal to g. Via the analysis above, the terms associated to the right most $t+1$ terms in d_l and d_r both vanish, and hence we only need analyze the remaining terms. First, we will compute the map induced by $\sigma_{n-1}d_n = \sigma_{n-1}d_{\mathrm{r}} + \sigma_{n-1}d_{\mathrm{l}}$. Using the definition of the differential from (3.6),

(3.8)
$$\sigma_{n-1}d_{\mathrm{l}}(v) = (-1)^{n-1}\sum_{i=1}^{s}\delta_{g,x_i}(-1)^i\left(x_i^{-1}x_1x_i,\ldots,x_i^{-1}x_{i-1}x_i,x_{i+1},\ldots,x_s,hgh^{-1},h,\underbrace{g,\ldots,g}_{t\text{ times}}\right).$$

Similarly, using the definition of the right differential from (3.5),

(3.9)
$$\sigma_{n-1}d_{\mathrm{r}}(v) = (-1)^{n-1}\sum_{i=1}^{s}\delta_{g,(x_{i+1}\cdots x_shg^t)^{-1}x_i(x_{i+1}\cdots x_shg^t)}(-1)^i$$

(3.10)
$$\left(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_s,hgh^{-1},h,\underbrace{g,\ldots,g}_{t\text{ times}}\right).$$

Next, we compute the map induced by $d_{n+1}\sigma_n = d_{\mathrm{r}}\sigma_n + d_{\mathrm{l}}\sigma_n$. Again, we have two computations, which are nearly the same as (3.8) and (3.9). First,

$$\sigma_n(v) = \left(x_1,\ldots,x_s,hgh^{-1},h,\underbrace{g,\ldots,g}_{t\text{ times}}\right).$$

Therefore,

(3.11)
$$d_{\mathrm{l}}\sigma_n(v) = (-1)^n\sum_{i=1}^{s}\delta_{g,x_i}(-1)^i\left(x_i^{-1}x_1x_i,\ldots,x_i^{-1}x_{i-1}x_i,x_{i+1},\ldots,x_s,hgh^{-1},h,\underbrace{g,\ldots,g}_{t\text{ times}}\right).$$

Note here that a priori there could have been an additional term associated to hgh^{-1}, but this vanishes because $\delta_{g,hgh^{-1}} = 0$ since $hgh^{-1} \neq g$, using the assumption from (3.4.1) that g is its own centralizer and $h \neq g$. We see from (3.8) and (3.11) that

$$d_{\mathrm{l}}\sigma_n + \sigma_{n-1}d_{\mathrm{l}} = 0, \tag{3.12}$$

since they have opposite signs.

Finally, we compute

$$d_{\mathrm{r}}\sigma_n(v) = (-1)^n \sum_{i=1}^{s} \delta_{g,(x_{i+1}\cdots x_s(hgh^{-1})hg^t)^{-1}x_i(x_{i+1}\cdots x_s(hgh^{-1})hg^t)}(-1)^i$$
$$\left(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_s,hgh^{-1},h,\underbrace{g,\ldots,g}_{t\text{ times}}\right) \tag{3.13}$$
$$+(-1)^n(-1)^{s+1}(x_1,\ldots,x_s,h,\underbrace{g,\ldots,g}_{t\text{ times}}).$$

Now, we claim the terms in the sum over i in (3.13) precisely cancels with the corresponding terms in (3.9). To see this, we need only observe that

$$\delta_{g,(x_{i+1}\cdots x_s(hgh^{-1})hg^t)^{-1}x_i(x_{i+1}\cdots x_s(hgh^{-1})hg^t)} = \delta_{g,(x_{i+1}\cdots x_shg^t)^{-1}x_i(x_{i+1}\cdots x_shg^t)}$$

In other words, we wish to show

$$(x_{i+1}\cdots x_s(hgh^{-1})hg^t)^{-1}x_i(x_{i+1}\cdots x_s(hgh^{-1})hg^t) = g$$

if and only if

$$(x_{i+1}\cdots x_shg^t)^{-1}x_i(x_{i+1}\cdots x_shg^t) = g.$$

Conjugating both sides by g^{t+1} in the first equation, we obtain it is equivalent to

$$(x_{i+1}\cdots x_sh)^{-1}x_i(x_{i+1}\cdots x_sh) = g.$$

This is also equivalent to the second equation conjugated by g^t. Hence, the two are equivalent, as claimed. This implies

$$(d_{\mathrm{r}}\sigma_n+\sigma_{n-1}d_{\mathrm{r}})(v) = (-1)^n(-1)^{s+1}(x_1,\ldots,x_s,h,\underbrace{g,\ldots,g}_{t\text{ times}}) = (-1)^{n+s+1}v. \tag{3.14}$$

Therefore, all in all, adding (3.12) and (3.14), we find that the map induced by $\sigma_{n-1}d_n + d_{n+1}\sigma_n$ is given by $v \mapsto (-1)^{n+s+1}(v)$, and therefore is an isomorphism, as we wished to show. $\square$

The key result we will need from this section is the following consequence of the fact that $T^n_\bullet$ is exact.

Proposition 3.4.10. *For $x \in \mathbb{Z}_{\leq 0}$, let $D^x_{\bullet,\bullet}$ denote the subcomplex of $C'_{\bullet,\bullet}$ (defined in Notation 3.4.7) which is equal to $C'_{i,j}$ if $i \geq 0$ and $j \geq x$ and is 0 otherwise. (Pictorially, $D^x_{\bullet,\bullet}$ makes a cone shape above $C'_{0,x} \simeq k\{c^{-x}\}$ in the picture Figure 1.) Then, $D^x_{\bullet,\bullet}$ is exact except possibly at position $(0,x)$.*

Proof. By Lemma 3.4.9, $T^n_\bullet$ is exact, or equivalently $C'_{\bullet,\bullet}$ is exact. Define the complex $J_\bullet$ as follows: Let $J_{-i} := \ker(C_{0,-i} \to C_{1,-i})$. Define the differential $J_{-i} \to J_{-i+1}$ to be given by d_{r} by viewing $J_{-i} \subset C_{0,-i}$.

We claim $J_\bullet \simeq C'_{\bullet,\bullet}$. We use that $C'_{\bullet,\bullet}$ has a filtration whose ith term is $D^{-i}_{\bullet,\bullet}$. The associated graded of this filtration is given by the complex $C'_{\bullet,-i}$. (This corresponds to a sequence of vector spaces pointing diagonally up and left in Figure 1, starting with $k\{c^i\}$.) Since $C'_{\bullet,-i}$ is a truncated summand of the usual $\mathcal{K}$ complex, it is exact except at $C'_{0,-i}$ by Lemma 3.3.3. Moreover, the homology of $C'_{\bullet,-i}$ is precisely J_{-i}. This implies $J_\bullet \to C'_{\bullet,\bullet}$ is a quasi-isomorphism, since it is a map inducing a quasi-isomorphism on each associated graded part of the filtration $D^{-i}_{\bullet,\bullet}$ of $C'_{\bullet,\bullet}$. The same argument moreover shows that the subcomplex $J_{\geq x}$, consisting of J_t for $t \geq x$ and 0 for $t < x$, is quasi-isomorphic to the subcomplex $D^x_{\bullet,\bullet}$.

Note that, $J_\bullet$ is exact because it is quasi-isomorphic to $C'_{\bullet,\bullet}$, which is an exact complex, by Lemma 3.4.9. It follows that $J_{\geq x}$ has only a single nonzero homology group, which occurs in degree x. Hence, the same is true of $D^x_{\bullet,\bullet}$. Concretely, this means that $D^x_{\bullet,\bullet}$ is exact except possibly at position $(0, x)$. $\square$

Combining what we have done so far, we deduce Theorem 3.4.3.

3.4.11. *Proof of Theorem 3.4.3.*

Proof. Observe that the map $\mathrm{Conf} \to \mathrm{Hur}^{G,c}$ in Theorem 3.4.3 is induced by functoriality of the Hurwitz space construction for $\mathrm{Hur}^{\langle g\rangle,\{g\}} \to \mathrm{Hur}^{G,c}$. Applying this functoriality with Lemma 3.4.4 and Remark 3.4.5, we can identify $\mathcal{K}(k[g], B, k[g, g^{-1}]) \to \mathcal{K}(k[g], A, k[g, g^{-1}])$ as quasi-isomorphic to the inclusion of a summand of $\oplus_z C^z_{\bullet,\bullet}$, where the cokernel of this inclusion is isomorphic to a sum over z of complexes isomorphic to shifts of $T_\bullet$. The result then follows from exactness of $T^n_\bullet$, proven in Lemma 3.4.9. $\square$

4. Computing the cohomology stabilized by a single monodromy

In this section, we compute the stable homology of Hurwitz space stabilized by a single element $g \in c$, and killed by all other elements of c. The main result of this section is Proposition 4.0.8, which has a somewhat elaborate proof which we break into steps. The aim of this proposition is to show we can show that a cocycle well into the stable range can be put into a particularly simple form, and we accomplish by using our computation of the homology of the two-sided $\mathcal{K}$-complex from the previous section. We recognize this argument is a bit involved, so we run through the computation for the stable first cohomology in Example 4.0.16. It may be helpful to read this before going through the more general argument.

We fix G, c, k as in Notation 3.4.1. We now introduce the Fox-Neuwirth/Fuks cell complex which is a cell complex computing the cohomology of Hurwitz space. Define $\mathrm{W}_n^{G,c,i}$ to be the free k-vector space spanned by tuples of $n-i$ words in c whose total length is n. That is, a basis element of $\mathrm{W}_n^{G,c,i}$ is of the form

$$w_1 \otimes \cdots \otimes w_{n-i} \tag{4.1}$$

where w_j is a word of length v_j such that $\sum_{j=1}^{n-i} v_j = n$. This chain complex has a differential

$$\delta(w_1 \otimes \cdots \otimes w_{n-i}) = \sum_{j=1}^{n-i-1} (-1)^{j+1}(w_1, \cdots, w_{j-1}, \mathrm{sh}(w_j, w_{j+1}), w_{j+2}, \ldots, w_{n-i})$$

where $\mathrm{sh}(w_j, w_{j+1})$ is the shuffle product defined explicitly as follows: Suppose $w = g_1 \cdots g_s$ and $w' = h_1 \cdots h_t$. Then $\mathrm{sh}(w, w')$ is the sum of words of length $s+t$

$$\mathrm{sh}(w, w') = \sum_{\sigma \in S_{s,t}} \mathrm{sgn}(\sigma) \, \mathrm{sh}_\sigma(g_1, \ldots, g_s, h_1, \ldots, h_t),$$

with the notation above defined as follows: We let $S_{s,t} \subset S_{s+t}$ denote the subset of permutations of $s+t$ elements which preserves the relative order of the first s elements and preserves the relative order of the last t elements. We let $\mathrm{sgn}(\sigma)$ denote the sign of σ when viewed as an element of S_{s+t}. Finally, $\mathrm{sh}_\sigma(g_1, \ldots, g_s, h_1, \ldots, h_t)$, denotes the word of length $s+t$ whose $\sigma(i)$th letter is h_{i-s} for $i > s$ and is $\alpha_i^{-1} g_i \alpha_i$ for $i < s$, where $\alpha_i = h_1 \cdots h_v$, for v is the largest positive integer satisfying $\sigma(s+v) < \sigma(i)$. By [ETW17, Theorem 3.3], we can identify the cohomology of this cochain complex with the cohomology of the Hurwitz space $\mathrm{Hur}_n^{G,c}$, using the isomorphism between the homology of the 1-point compactification of a space and the cohomology of that space. We note that [ETW17, Theorem 3.3] is stated in the language of local systems on configuration space, and if $f : \mathrm{Hur}_n^{G,c} \to \mathrm{Conf}_n$ denotes the finite covering space, we use the isomorphism $H^i(\mathrm{Hur}_n^{G,c}, \mathbb{Q}) \simeq H^i(\mathrm{Conf}_n, f_* \mathbb{Q})$.

Remark 4.0.1. Suppose we start with an element $z \in \mathrm{W}_n^{G,c,i}$ which we write as $\sum_{j=1}^{f} c_j w_1^j \otimes \cdots \otimes w_{n-i}^j$. It follows from the definition above, that multiplication by the dual of the element $[g]$ corresponds to sending z to $\sum_{j | w_{n-i}^j = g} c_j w_1^j \otimes \cdots \otimes w_{n-i-1}^j$. In other words, costabilization by $[g]$ picks out all terms whose last word is the length 1 word equal to g. Indeed, this is dual to the map on homology which sends a tensor of words (in the dual basis to that described above) to that same tensor with an additional g tacked on at the end.

The next proposition is the main result of this section. It will enable us to run the inductive step and is really the crux of our argument. The proof will be given later in §4.0.15.

Proposition 4.0.2. *With notation for the stable homology groups as in Definition 2.0.6 and Notation 2.0.3, consider an element $z \in \ker(H_i^{G,c,g} \to H_{i,\beta}^{\mathrm{id},\mathrm{id}})$ (where* id *denotes the trivial group/element). Assume that for any $j < i$ and any $g \in G$, $H_j^{G,c,g} \to H_{j,\beta}^{\mathrm{id},\mathrm{id}}$ is an isomorphism. Assume that $z[g]^w[h] = 0$ for any $w \geq 0$ and $h \neq g$. Then $z = 0$.*

Before reading the rest of this section, we recommend the reader jump to Example 4.0.16, which runs through the special case of Proposition 4.0.2 where $i = 1$.

The next lemma shows that in order to prove Proposition 4.0.2, it is enough to show that z is cohomologous to a cocycle in a form ending in g; the argument for this reduction is given in §4.0.15.

Lemma 4.0.3. *Suppose $x \in H^i(\mathrm{CHur}_{n+1}^{G,c}, k)$ is represented by a cocycle of the form $y \otimes g$ for some $y \in \mathrm{W}_n^{G,c,i}$. Then, y is a cocycle and it is cohomologous to $z \otimes g$ for some $z \in \mathrm{W}_{n-1}^{G,c,i}$.*

Proof. We can write $y = \sum_{j=0}^{i} \sum_{\tau} s_j^{\tau} \otimes t_j^{\tau}$ for $s_j^{\tau} \in \mathrm{W}_{n-j-1}^{G,c,i-j}$ ranging over a basis of this vector space (as τ varies for j fixed) and $t_j^{\tau} \in \mathrm{W}_{j+1}^{G,c,j}$ is a linear combination of words in c of length j.

By assumption, $y \otimes g$ is a cycle, and hence vanishes under the shuffling coboundary map δ. In particular, we claim this implies $t_j^{\tau} \otimes g \in \mathrm{W}_{j+2}^{G,c,j}$ lies in the kernel of the coboundary map δ for each j with $0 \leq j \leq i$ and each τ. Indeed, this can be seen by expanding the image of $y \otimes g$ under the coboundary map and noting that the sum of all terms whose component in $\mathrm{W}_{n-j-1}^{G,c,i-j}$ equal to s_j^{τ} is precisely $\delta(t_j^{\tau} \otimes g)$. By Lemma 3.3.3, we find that for each j, $t_j^{\tau} = \delta(r_j^{\tau} \otimes g)$, for some $r_j^{\tau} \in \mathrm{W}_j^{G,c,j-1}$. Indeed this is because the $\mathcal{K}$-complex, $\mathcal{K}(k, A, k[g, g^{-1}])$ is exactly dual to the complex whose j^{th} cohomology group vanishing shows that there is no obstruction to finding such an r_j^{τ}. Now, consider the element $\sum_{j=0}^{i} \sum_{\tau} s_j^{\tau} \otimes r_j^{\tau} \otimes g \in \mathrm{W}_n^{G,c,i-1}$. Applying the coboundary map δ to this, and using that $\delta(r_j^{\tau} \otimes g) = t_j^{\tau}$, we obtain

$$\delta(\sum_{j=0}^{i}\sum_{\tau} s_j^\tau \otimes r_j^\tau \otimes g) = \sum_{j=0}^{i}\sum_{\tau} \delta(s_j^\tau \otimes r_j^\tau) \otimes g + (-1)^{n-i} \sum_{j=0}^{i}\sum_{\tau} s_j^\tau \otimes t_j^\tau$$
$$= \sum_{j=0}^{i}\sum_{\tau} \delta(s_j^\tau \otimes r_j^\tau) \otimes g + (-1)^{n-i} y.$$

This shows that y is cohomologous to $(-1)^{n-i+1} \sum_{j=0}^{i} \sum_\tau \delta(s_j^\tau \otimes r_j^\tau) \otimes g$, so we conclude by taking $z = (-1)^{n-i+1} \sum_{j=0}^{i} \sum_\tau \delta(s_j^\tau \otimes r_j^\tau)$. □

As mentioned prior to Lemma 4.0.3 our next goal will be to show that z is cohomologous to a cocycle in a form ending in g. We now begin preparations to accomplish this in Proposition 4.0.8.

Notation 4.0.4. We will use the notation $\mathrm{CW}_n^{G,c,i}$ for the subspace of $\mathrm{W}_n^{G,c,i}$ spanned by those basis elements such that the union of g with those elements of c appearing in that basis element generate all of G. More precisely, $\mathrm{CW}_n^{G,c,i}$ is generated by tensors of words of the form $w_1 \otimes \cdots \otimes w_{n-i}$ with $w_j = g_{j,1} \cdots g_{j,v_j}$ and the $g_{s,t}$ for $1 \leq s \leq n-i$ and $1 \leq t \leq v_s$ all together with g generate G. Note that $\mathrm{CW}_n^{G,c,\bullet}$ forms a subcomplex of the chain complex $\mathrm{W}_n^{G,c,\bullet}$. We will use $\mathrm{Z}_n^{G,c,i}$ for the subspace of $\mathrm{W}_n^{G,c,i}$ consisting of cocycles, $\mathrm{B}_n^{G,c,i}$ for the subspace of $\mathrm{W}_n^{G,c,i}$ consisting of coboundaries, and $\mathrm{H}_n^{G,c,i} := \mathrm{Z}_n^{G,c,i} \,/\, \mathrm{B}_n^{G,c,i}$.

Notation 4.0.5. Suppose we are in the situation of Proposition 4.0.2, In particular, $z[g]^j[h] = 0$ for any $j \geq 0$ and $h \neq g$. Choose some sufficiently large n so that we may represent x by a class $z \in \mathrm{CW}_n^{G,c,i}$. (How large we have to take n will be determined in the proof of Proposition 4.0.8.)

As previously mentioned, our aim will be to prove Proposition 4.0.2, which amounts to showing $z = 0$, after modification by a coboundary. The next lemma translates the hypothesis that each $[g]^j[h]$, for $h \in c - g$ and $j \geq 0$, kills z to a concrete description of the form of z.

Lemma 4.0.6. *With notation as in Notation 4.0.4 and Notation 4.0.5, for any fixed m, and every $s < m$, any cocycle $z \in \mathrm{W}_n^{G,c,i}$ is cohomologous to an element whose projection onto $\mathrm{W}_{n-s}^{G,c,i} \otimes \underbrace{k\{c\} \otimes \cdots \otimes k\{c\}}_{s \text{ times}}$ is of the form $z_s \otimes \underbrace{g \otimes \cdots \otimes g}_{s \text{ times}}$.*

Proof. We first claim that, after modifying z by a coboundary, we may assume $z[g]^s[h]$ for any fixed value of s. To see this, note that for any $h \neq g$, we know $z[g]^s[h]$ is a coboundary by Notation 4.0.5. If $z[g]^s[h] = \delta(x)$, we then find that $z - \delta(x \otimes h \otimes \underbrace{g \otimes \cdots \otimes g}_{s \text{ times}}) = 0$. Therefore, we may modify z by a coboundary to assume that $z[g]^s[h] = 0$ for all $h \neq g$ and any fixed value of s.

Applying the above with $s = 1$ shows the lemma statement holds for $s = 1$. By induction on s, assuming this holds for $s - 1$ we may assume the projection onto $\mathrm{W}^{G,c,i}_{n-(s-1)} \otimes \underbrace{k\{c\} \otimes \cdots \otimes k\{c\}}_{s-1 \text{ times}}$ is of the form $z_{s-1} \otimes \underbrace{g \otimes \cdots \otimes g}_{s \text{ times}}$. Then, again modifying z by a coboundary, we may assume $[g]^s[h]$ acts by 0 on z. We find z_{s-1} has projection onto $\mathrm{W}^{G,c,i}_{n-s} \otimes k\{c\}$ with $k\{c\}$ term in the span of g. Hence, the projection onto $\mathrm{W}^{G,c,i}_{n-s} \otimes \underbrace{k\{c\} \otimes \cdots \otimes k\{c\}}_{s \text{ times}}$ is of the form $z_s \otimes \underbrace{g \otimes \cdots \otimes g}_{s \text{ times}}$. □

We next wish to show that z is cohomologous to a cochain in a form to which we can apply Lemma 4.0.6. The next lemma puts a serious constraint on what z can look like.

Lemma 4.0.7. *Keeping notation as in Notation 4.0.4 and Notation 4.0.5, for $2 \leq j \leq t \leq i + 1$, z is cohomologous to a cocycle whose projection to*

$$\oplus_{\alpha=0}^{i+1-j} \mathrm{W}^{G,c,i-j+1}_{n-j-\alpha} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \tag{4.2}$$

is zero.

Moreover, the projection onto $\mathrm{W}^{G,c,i}_{n-s} \otimes \underbrace{k\{c\} \otimes \cdots \otimes k\{c\}}_{s \text{ times}}$ is of the form $z_s \otimes \underbrace{g \otimes \cdots \otimes g}_{s \text{ times}}$ for $s \leq i + 2$.

We will prove Lemma 4.0.7 later in §4.0.14 after establishing some preliminary lemmas. Before proving this, let us see why this implies the following proposition, which lets us write z with a g on the right.

Proposition 4.0.8. *With notation as in Notation 4.0.4 and Notation 4.0.5, z is cohomologous to a cocycle of the form $y \otimes g$.*

Proof. For $t = i + 1$, after modifying z by a coboundary, we may assume it satisfies the conclusion of Lemma 4.0.7. We find z has zero projection to $\mathrm{W}^{G,c,i-j+1}_{n-j} \otimes k\{c^j\}$ for all $2 \leq j \leq i + 1$. This implies that $z \in \mathrm{W}^{G,c,i}_{n-1} \otimes k\{c\}$. The second condition of Lemma 4.0.7 with $s = 1$ implies that $z = y \otimes g$ for some $y \in \mathrm{W}^{G,c,i}_{n-1}$, as we wished to show. □

Summarizing, what we have accomplished so far, in order to prove Proposition 4.0.2, it remains to prove Lemma 4.0.7; to see exactly why this suffices, one may examine the proof of Proposition 4.0.2, given in §4.0.15. This is probably the most involved proof in the paper, and so we will require a number of sublemmas. We first show certain projections of z have parts which are cocycles.

Lemma 4.0.9. *Using notation as in Lemma 4.0.7, suppose Lemma 4.0.7 holds for $t-1$. (This is vacuous if $t=2$.) Then, for $2 \leq j \leq t$, we may modify z by a coboundary so that the conclusion of Lemma 4.0.7 still holds for $t-1$ and the projection of z onto*

$$\oplus_{\alpha=0}^{i+2-j} \mathrm{W}_{n-j-\alpha}^{G,c,i-j+1} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \tag{4.3}$$

in fact lies in

$$\oplus_{\alpha=0}^{i+2-j} \mathrm{Z}_{n-j-\alpha}^{G,c,i-j+1} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}. \tag{4.4}$$

We note that (4.3) is very similar to (4.2) except that the index on α goes up to $i+2-j$ instead of only $i+1-j$.

Proof. To see this, let us show the projection $\mathrm{pr}_j(z)$ of z onto the summand of (4.3) indexed by α lies in $\mathrm{Z}_{n-j-\alpha}^{G,c,i-j+1} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}$, after modifying z by a coboundary. If it did not, there would necessarily be a term in $\delta(\mathrm{pr}_j(z))$ of the form $\mathrm{B}_{n-j-\alpha}^{G,c,i-j+1} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}$. Hence, since $\delta(z) = 0$ so some term must cancel the above term in $\delta(\mathrm{pr}_j(z))$, there would be a term in z of the form $\mathrm{W}_{n-j-\alpha}^{G,c,i-j+1} \otimes k\{c^v\} \otimes k\{c^{j-v}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}$, (with $v > 0$ and $j - v > 0$) whose coboundary via shuffling the the words of length v and $j-v$ is nonzero. Now, if $j > 2$, we can modify z by a coboundary so that this is impossible by our hypothesis for Lemma 4.0.7 (which we are assuming holds for $t-1$) with j replaced by $j-v$, as no such terms exist. (Note that because $j - v < j$ we obtain that $i - (j-v) + 1 \geq i - j + 2$, so we obtain this when $\alpha = i + 2 - j$ as well.) Finally, if $j = 2$, we must have $v = 1$ and $j - v = 1$. Note that $\alpha \leq i + 2 - j = i$, so $\alpha + 2 \leq i + 2$. Hence, we may apply the second part of Lemma 4.0.7 for $s = \alpha + 2$, (using that s is at most $i+2$

as explained above,) such terms are of the form $z_{\alpha+2} \otimes \underbrace{g \otimes \cdots \otimes g}_{\alpha+2 \text{ times}}$, and the coboundary of $\underbrace{g \otimes \cdots \otimes g}_{\alpha+2 \text{ times}}$ vanishes. □

We next wish to show the cocycles appearing in the statement of Lemma 4.0.9 can be expressed in terms of particular representatives, pulled back from the cohomology of configuration space. We set up notation for these representatives and then show the cohomology can be expressed in terms of these representatives in Lemma 4.0.11.

Notation 4.0.10. We keep notation as in Notation 4.0.4 and Notation 4.0.5. Choose n large enough so that $n - t - 1$ is in the stable range for the $(i - t + 1)$st cohomology and such that Lemma 4.0.7 is true for $t - 1$. In particular, we are assuming via Notation 4.0.5 that $H_j^{G,c,g} \to H_{j,\beta}^{\mathrm{id},\mathrm{id}}$ is an isomorphism for any $j < i$ and any $g \in G$. Hence, the stable cohomology classes in $(H_{i-t+1,\beta}^{G,c})^\vee$ are all pulled back from configuration space.

Since configuration space corresponds to a Hurwitz space for the trivial group id, we use $(H_{i-t+1,\beta}^{\mathrm{id},\mathrm{id}})^\vee$ as notation for the stable cohomology of configuration space. For all $n > 1$ choose a sequence of cocycles $w_n \in \mathrm{Z}_n^{\mathrm{id},\mathrm{id},i-t+1}$ compatible with the costabilization map, which are 0 whenever $(H_{i-t+1,\beta}^{\mathrm{id},\mathrm{id}})^\vee = 0$, and otherwise project to a nonzero element in $(H_{i-t+1,\beta}^{\mathrm{id},\mathrm{id}})^\vee$. For the readers benefit, we point out that $(H_{i,\beta}^{\mathrm{id},\mathrm{id}})^\vee = 0$ unless $i = 0, 1$, in which case it is 1-dimensional. Fixing a value of n mod 2, if we let $f : \mathrm{CHur}_n^{G,c,g} \to \mathrm{CHur}_n^{\mathrm{id},\mathrm{id}}$ denote the projection from the fixed stable component with boundary monodromy $g \in G$, we obtain a sequence of cocycles $f^*(w_n) \in \mathrm{Z}_n^{G,c,i-t+1}$ compatible with the costabilization maps uniquely representing a spanning cocycle for $H^{l-i+1}(\mathrm{CHur}_n^{G,c,g}, k)$, for n in the stable range.

Lemma 4.0.11. *Use notation from Notation 4.0.4, Notation 4.0.5, and Notation 4.0.10. Assume Lemma 4.0.7 holds for $t - 1$. (This is vacuous if $t = 2$.) For $2 \leq j \leq t$, we can modify z by coboundaries so its projection to* (4.3) *lies in* (4.4)*. Moreover, when this projection is expanded as a of simple tensors in terms of a basis for $\mathrm{Z}_{n-j-\alpha}^{G,c,i-j+1}$, we may assume each such basis element has component in $\mathrm{Z}_{n-j-\alpha}^{G,c,i-j+1}$ equal to a multiple of the chosen representative $f^*(w_{n-j-\alpha})$.*

Proof. Since we are assuming Lemma 4.0.7 holds for $t - 1$, we can apply Lemma 4.0.9. We can then assume the projection of z to (4.3) lies in (4.4). Now, modifying the resulting element by a coboundary of an element in

$\oplus_{\alpha=0}^{i+2-j} \mathrm{W}_{n-j-\alpha}^{G,c,i-j} \otimes k\{c^j\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}$ so that we may assume each such term has component in $\mathrm{Z}_{n-j-\alpha}^{G,c,i-j+1}$ equal to a multiple of $f^*(w_{n-j-\alpha})$, as desired. □

The next lemma further constrains the form of z by using the above constructed representatives to show that many projections of z vanish.

Lemma 4.0.12. *Using notation as in Lemma 4.0.7, suppose Lemma 4.0.7 holds for $t-1$. (This is vacuous if $t=2$.) Assume z has been modified by a coboundary to satisfy the conclusion of Lemma 4.0.11. Suppose $w \in \mathrm{W}_n^{G,c,i}$ is a basis vector in the form* (4.1), *and w does not lie in*

$$\mathrm{W}_{n-t-\alpha}^{G,c,i-t+1} \otimes k\{c^t\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \oplus \mathrm{W}_{n-t-\alpha-1}^{G,c,i-t+1} \otimes k\{c^t\} \otimes k\{c\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}. \tag{4.5}$$

Then, for $0 \le \alpha \le i+1-t$, $\mathrm{pr}_w(z)$ the projection of z onto any such w, the projection of $\delta\left(\mathrm{pr}_w(z)\right)$ to

$$\mathrm{W}_{n-t-1-\alpha}^{G,c,i-t+1} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \tag{4.6}$$

vanishes. Moreover, the projection of z onto (4.5) *lands in*

$$\mathrm{Z}_{n-t-\alpha}^{G,c,i-t+1} \otimes k\{c^t\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \oplus \mathrm{Z}_{n-t-\alpha-1}^{G,c,i-t+1} \otimes k\{c^t\} \otimes k\{c\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}. \tag{4.7}$$

Proof. First, the final statement that the projection onto (4.5) factors through (4.7) follows from Lemma 4.0.9. This importantly uses that α appearing in the sum in (4.3) runs up to $i+2-j$ while α appearing in (4.2) only runs up to $i+1-j$, so that Lemma 4.0.9 applies to both terms in in (4.5).

Using our inductive hypothesis that Lemma 4.0.7 holds for j with $2 \le j < t$, the projection of z onto all terms of the form

$$\mathrm{W}_{n-t-\alpha}^{G,c,i-t+1} \otimes k\{c^v\} \otimes k\{c^{t-v}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}} \tag{4.8}$$

with $0 \le \alpha \le i+1-t$, $v \ge 1$, and $t-v \ge 2$ vanishes. (This uses that the above condition holds when $\alpha \le i+1-(t-v)$, so it holds in particular when $\alpha \le i+2-t$.) The only other term whose coboundary can contribute to the projection onto (4.6) is

$$\mathrm{W}^{G,c,i-t}_{n-t-\alpha-1} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}. \tag{4.9}$$

However, the contribution from the coboundary of this term to (4.6) necessarily lies in

$$\mathrm{B}^{G,c,i-t+1}_{n-t-1-\alpha} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}. \tag{4.10}$$

On the other hand, using notation from Notation 4.0.10, each simple tensor in the expansion of the image of the projection of z to (4.7) along the $\mathrm{Z}^{G,c,i-t+1}_{n-t-\alpha-1}$ factor consists of a cocycle that is a multiple of $f^*(w_{n-j-\alpha})$.

Since such a cocycle is not a coboundary unless it is 0, the sum of the terms lying in the subspace (4.10) must vanish. In other words, the projection of z to (4.9) in fact lies in $\mathrm{Z}^{G,c,i-t}_{n-t-\alpha-1} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{c\} \cdots \otimes k\{c\}}_{\alpha \text{ times}}$. This establishes our claim. $\square$

The above lemmas are relatively straightforward reductions that allow us to significantly simplify the form of z. The next lemma is really the key step, which uses the two-sided $\mathcal{K}$-complex introduced in the previous section. Exactness of that complex gives us exactness in the next lemma, which we will then use to further simplify the form of z.

Lemma 4.0.13. *Let $\alpha \geq 0, t \geq 2$ and $n \geq 0$ be integers with $\alpha + t < n$. Continuing to use notation as in Notation 4.0.4 and Notation 4.0.5, the restriction of the following three term sequence to the subcomplex spanned by tensors of elements* *which generate G is exact:*

$$
\begin{aligned}
(4.11)\qquad & \mathrm{H}^{G,c,i-t+1}_{n-t+1-\alpha} \otimes k\{c^{t-1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
& \oplus \mathrm{H}^{G,c,i-t+1}_{n-t-\alpha} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
& \oplus \mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
& \xrightarrow{\mu_\alpha} \mathrm{H}^{G,c,i-t+1}_{n-t-\alpha} \otimes k\{c^{t}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
& \oplus \mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha} \otimes k\{c^{t}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
& \xrightarrow{\nu_\alpha} \mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}.
\end{aligned}
$$

Proof. First, if $i-t+1 \notin \{0,1\}$, we are done, as this implies the above complex is 0, using that we are inductively assuming (via Notation 4.0.5 via Proposition 4.0.2) the $(i-t+1)$th cohomology is pulled back from configuration space, and this vanishes in degrees more than 1.

Hence, we now assume $i-t+1 \in \{0,1\}$. In this case, since $\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}$ is pulled back from configuration space, the module structure in the case $i-t+1 = 1$ is the same as the module structure in the case $i-t+1 = 0$, so we may assume $i-t+1 = 0$. Let $\widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \subset \mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}$ denote the codimension 1 subspaces spanned by all components other than the one with only monodromy g. There is a subcomplex of (4.11) given by replacing each $\mathrm{H}^{G,c,\bullet}_{\bullet}$ term with the corresponding codimension 1 subspace $\widetilde{\mathrm{H}^{G,c,\bullet}_{\bullet}}$.

Specifically, this subcomplex is given by

$$
\begin{aligned}
&\widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t+1-\alpha}} \otimes k\{c^{t-1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\quad \oplus \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha}} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\quad \oplus \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\xrightarrow{\mu_\alpha} \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha}} \otimes k\{c^{t}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\quad \oplus \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\xrightarrow{\nu_\alpha} \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}.
\end{aligned} \tag{4.12}
$$

This subcomplex is in fact exact using Lemma 3.3.1, as we next explain. The reason this is exact is that it can be identified, via the spectral sequence described in §3.2 as dual to part of the cohomology of the complex $\widetilde{\mathrm{H}^{G,c,i-t+1}_{\bullet}} \otimes_A k[g, g^{-1}]$. (This identification is quite similar to the description of $C_{\bullet,\bullet}$ in Lemma 3.4.4 and we omit further details.) The module $\widetilde{\mathrm{H}^{G,c,i-t+1}_{\bullet}}$ is then spanned by modules on which some $h \in c, h \neq g$ acts invertibly, and hence it follows from Lemma 3.3.1 that this subcomplex is exact. Now, let $\overline{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} := \mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha} / \widetilde{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}}$ denote the 1-dimensional quotient. Exactness of (4.12) implies that the cohomology of (4.11) is identified with the cohomology of the quotient complex, which is explicitly given by

$$
\begin{aligned}
&\overline{\mathrm{H}^{G,c,i-t+1}_{n-t+1-\alpha}} \otimes k\{c^{t-1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\oplus \overline{\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha}} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\oplus \overline{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t-1}\} \otimes k\{g\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\xrightarrow{\mu_\alpha} \overline{\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha}} \otimes k\{c^{t}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\oplus \overline{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t}\} \otimes k\{g\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}} \\
&\xrightarrow{\nu_\alpha} \overline{\mathrm{H}^{G,c,i-t+1}_{n-t-1-\alpha}} \otimes k\{c^{t+1}\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}.
\end{aligned}
\tag{4.13}
$$

To conclude, we wish to show the part of this complex spanned by tensors of elements generating G is exact. This part of the quotient complex identified as dual to the subcomplex $D^{-t-1}_{\bullet,\bullet}$ defined in Proposition 3.4.10. Hence, it is exact by Proposition 3.4.10. □

Combining the above lemmas, we now prove Lemma 4.0.7.

4.0.14. *Proof of Lemma 4.0.7.* Note that the second part holds by Lemma 4.0.6, so it remains to prove the first part, which we will do by induction on t. We assume Lemma 4.0.7 holds for $t-1$. (When $t=2$, this condition is vacuous.)

After modifying z by a coboundary, we may assume the projection of z to (4.7) lies in the kernel of ν_α, defined in (4.11), by Lemma 4.0.12.

We next claim we can modify z by an element in the image of μ_α so that the projection of z onto $\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha} \otimes k\{c^t\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}$ vanishes. First, when $\alpha = 0$, this follows from exactness of (4.11), established in Lemma 4.0.13. Applying this for each α with $0 \le \alpha \le i-t$, we can arrange that the projection of z onto $\mathrm{H}^{G,c,i-t+1}_{n-t-\alpha} \otimes k\{c^t\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}$ vanishes. Note we use here that the cocycles $f^*(w_n)$ are closed under the costabilization map.

Finally, using Lemma 4.0.11, after modifying z by a coboundary, because its projection to $\mathrm{H}_{n-t-\alpha}^{G,c,i-t+1} \otimes k\{c^t\} \otimes \underbrace{k\{g\} \otimes \cdots k\{g\}}_{\alpha \text{ times}}$ vanishes, we may also assume its projection to (4.2) vanishes. $\square$

4.0.15. *Proof of Proposition 4.0.2.* By Proposition 4.0.8, we may assume that, after modifying z by a coboundary, $z = y_1 \otimes g$ for $y_1 \in \mathrm{W}_{n-1}^{G,c,i}$. In fact, we have, $y_i \in \mathrm{CW}_{n-1}^{G,c,i}$ using the definition of $\mathrm{CW}_{n-1}^{G,c,i}$ and the assumption that $y_1 \otimes g \in \mathrm{CW}_{n-1}^{G,c,i}$ using that $z \in H_i^{G,c,g}$. Then, applying Lemma 4.0.3 yields that $y_1 \otimes g$ is cohomologous to a cycle of the form $y_2 \otimes g \otimes g$, where, again, $y_2 \in \mathrm{CW}_{n-2}^{G,c,i}$. Applying Lemma 4.0.3 iteratively, we find there is y_i so that $y_i \otimes g = y_{i-1}$, and, when $i = n$, we find that z is cohomologous to a multiple of $\underbrace{g \otimes g \otimes \cdots \otimes g}_{n \text{ times}}$, and hence must be 0. $\square$

Example 4.0.16. In this example, we run through the argument for Proposition 4.0.2 in the case of $i = 1$, i.e., for the first (co)homology group. We fix an element $g \in c$ and assume we have an element $x \in H^1(\mathrm{CHur}_{n+1}^{G,c}, k)$ such that $x[h] = 0$, $x[g][h] = 0$, and $x[g]^2[h] = 0$ for all $h \neq g$. We can write x as an element of

$$\begin{aligned}
&\mathrm{W}_{n-2}^{G,c,0} \otimes k\{c^2\} \\
&\oplus \mathrm{W}_{n-3}^{G,c,0} \otimes k\{c^2\} \otimes k\{c\} \\
&\oplus \mathrm{W}_{n-4}^{G,c,1} \otimes k\{c^2\} \otimes k\{c\} \otimes k\{c\} \\
&\oplus \mathrm{W}_{n-3}^{G,c,1} \otimes k\{c\} \otimes k\{c\} \otimes k\{c\}.
\end{aligned}$$

We picture x on the left of Figure 2.

We next carry out Lemma 4.0.6 explicitly in this case. Write $x = x_1 + x_2 + x_3 + x_4$, where x_j is in the jth component above. The condition that $x[h] = 0$ implies that $x_i = y_i \otimes g$ for $2 \leq i \leq 4$. The further condition that $x[g][h] = 0$ implies $x_i = z_i \otimes g \otimes g$ for $3 \leq i \leq 4$. Finally, the condition that $x[g]^2[h] = 0$ implies $x_4 = w_4 \otimes g \otimes g \otimes g$. Hence, we have that x lies in

$$\begin{aligned}
&\mathrm{W}_{n-2}^{G,c,0} \otimes k\{c^2\} \\
&\oplus \mathrm{W}_{n-3}^{G,c,0} \otimes k\{c^2\} \otimes k\{g\} \\
&\oplus \mathrm{W}_{n-4}^{G,c,1} \otimes k\{c^2\} \otimes k\{g\} \otimes k\{g\} \\
&\oplus \mathrm{W}_{n-3}^{G,c,1} \otimes k\{g\} \otimes k\{g\} \otimes k\{g\}.
\end{aligned}$$

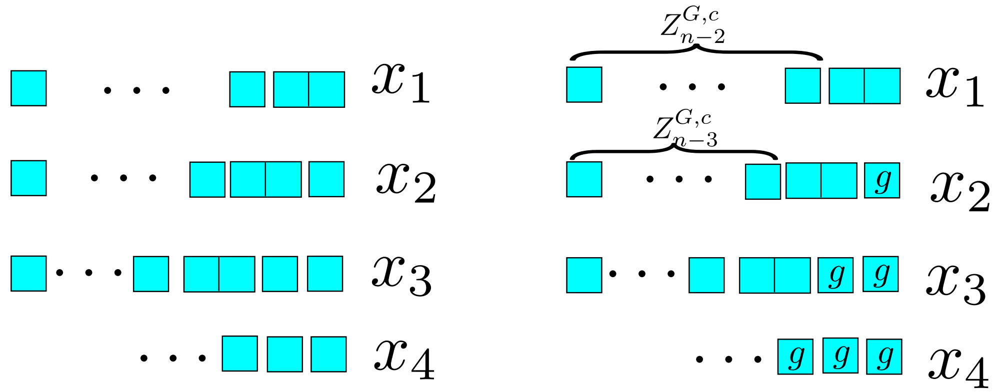

FIGURE 2. The left is a picture of the Fuks-Neuwirth cell structure of an element of the first stable cohomology. The right hand side pictures some additional constraints we may impose on the cell structure, as explained in Example 4.0.16.

We now carry out Lemma 4.0.9 explicitly in this case. Note that we can write x_1 as a sum of tensor products $\sum_j u_j \otimes v_j$, where $u_j \in \mathrm{W}_{n-2}^{G,c,0}$ and $v_j \in k\{c^2\}$. Using that $\delta(x) = 0$, we can arrange that each v_j satisfies $\delta(v_j) = 0$ because $0 = \delta(x) = \delta(x_1) + \delta(x_2) + \delta(x_3) + \delta(x_4)$ but $\delta(x_2)$, $\delta(x_3)$, and $\delta(x_3)$ all have no terms ending in $k\{c^2\}$, using that the shuffle of g with itself is 0. This shows that x_1 lies in $\mathrm{Z}_{n-2}^{G,c,0} \otimes k\{c^2\}$. Similarly, $x_2 \in \mathrm{Z}_{n-3}^{G,c,1} \otimes k\{c^2\} \otimes k\{g\}$. So, altogether, we obtain that x lies in

$$\begin{aligned}
&\mathrm{Z}_{n-2}^{G,c,0} \otimes k\{c^2\}\\
&\oplus \mathrm{Z}_{n-3}^{G,c,0} \otimes k\{c^2\} \otimes k\{g\}\\
&\oplus \mathrm{W}_{n-4}^{G,c,1} \otimes k\{c^2\} \otimes k\{g\} \otimes k\{g\}\\
&\oplus \mathrm{W}_{n-3}^{G,c,1} \otimes k\{g\} \otimes k\{g\} \otimes k\{g\}.
\end{aligned}$$

We picture the present form on x on the right of Figure 2.

Now, using exactness of the two-sided K-complex, in the form of Proposition 3.4.10, we claim that there is some element

$$s = s_1 + s_2 + s_3 \in \mathrm{H}_{n-1}^{G,c,0} \otimes k\{c\} \oplus \mathrm{H}_{n-2}^{G,c,0} \otimes k\{c\} \otimes k\{g\} \oplus \mathrm{H}_{n-3}^{G,c,0} \otimes k\{c\} \otimes k\{g\} \otimes k\{g\}$$

so that the projection of $\delta(s)$ to $\mathrm{W}_{n-2}^{G,c,0} \otimes k\{c^2\}$ agrees with x_1. This uses that $\mathrm{H}_{n-3}^{G,c,0} = \mathrm{Z}_{n-3}^{G,c,0}$, as there are no coboundaries in the 0th cohomology. We note that to work with the two sided K-complex, we really need to ignore terms in the subcomplex spanned by $g \otimes \cdots \otimes g$, but this is not a problem because

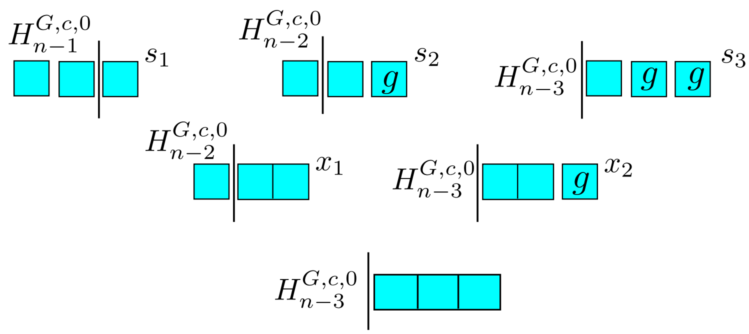

FIGURE 3. This is a picture of part of the two-sided K complex used to compute the stable first cohomology, as relevant to Example 4.0.16.

our element starts in the cohomology of the connected Hurwitz space. We picture s in the top row of Figure 3.

We now conclude the argument that $x = 0$. With notation as above, $x - \delta(s)$ lies in $\mathrm{W}^{G,c,1}_{n-1} \otimes k\{g\}$. So, after replacing x by its sum with a coboundary, we can arrange that $x \in \mathrm{W}^{G,c,1}_{n-1} \otimes k\{g\}$. Applying Lemma 4.0.3 shows we may further assume $x \in \mathrm{W}^{G,c,1}_{n-2} \otimes k\{g\} \otimes k\{g\}$. Continuing to apply Lemma 4.0.3, we may assume x lies in the span of $g \otimes \cdots \otimes g$. Since we are assuming $x \in H^1(\mathrm{CHur}^{G,c}_{n+1}, k)$, we must have $x = 0$.

5. COMPUTING THE STABLE HOMOLOGY

In this section, we complete the proof of our main result by computing the stable homology of Hurwitz spaces associated dihedral groups. As a preparatory lemma, we show that components with boundary monodromy in $G - c - \mathrm{id}$ cannot have interesting stable homology. This argument essentially appears in [BM23], and via personal communication, we learned it was also known to Ellenberg, Venkatesh, and Westerland.

Lemma 5.0.1. *For G and c as in Notation 2.0.1 and $\zeta \in \mathbb{Z}/\ell\mathbb{Z} \subset G$ a generator,* $\ker(H^{G,c,\zeta}_i \to H^{\mathrm{id},\mathrm{id}}_{i,0}) = 0$.

Proof. This is a consequence of [BM23, Corollary C′] and the proof of [RW20, Corollary 5.4]. The first identifies $H^{G,c,\zeta}_i$ with the homology of one component of the group completion of $\coprod_{n \geq 0} \mathrm{Hur}^{G,c}_n$. The second shows that the map from the homology of each component of the group completion of $\coprod_{n \geq 0} \mathrm{Hur}^{G,c}_n$ to $H^{\mathrm{id},\mathrm{id}}_{i,0}$ is an isomorphism. □

Lemma 5.0.2. *For G and c as in Notation 2.0.1, for any $g \in c$ and any $h \in c$ with $h \neq g$, $[h]$ acts by 0 on $\ker(H_i^{G,c,g} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$. Moreover, $[g]^j[h]$ also acts by 0 on $\ker(H_i^{G,c,g} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$ for any $j \geq 0$.*

Proof. For $h \neq g$, with $h \in c, g \in c$, we have that $\zeta := gh$ is a generator of $\mathbb{Z}/\ell\mathbb{Z} \subset G$. Therefore, $[h]$ maps $\ker(H_i^{G,c,g} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$ onto $\ker(H_i^{G,c,\zeta} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$, and the latter vanishes by Lemma 5.0.1.

To show $[g]^j[h]$ acts by 0, simply observe $[g]^j[h] = [h][h^{-1}gh]^j$. So this acts by 0 because $[h]$ acts by 0. □

Finally, we are prepared to deduce our main result.

Theorem 5.0.3. *With notation as in Definition 2.0.6 $H_i^{G,c,g} \to H_{i,\beta}^{\mathrm{id},\mathrm{id}}$ is an isomorphism.*

Proof. One can see using transfer maps that the map $H_i^{G,c,g} \to H_{i,\beta}^{\mathrm{id},\mathrm{id}}$ is a split surjection, so it suffices to show that the kernel of this map is 0. First, we assume $g \in c$. Consider the stabilization map $\sum_{h\in c}[h]^2$ acting on $\ker(H_i^{G,c,g} \to H_{i,1}^{\mathrm{id},\mathrm{id}})$ for $g \in c$. Each $[h]^2$ for $h \neq g$ acts by 0 on this kernel by Lemma 5.0.2, while $[g]^2$ acts by 0 using Proposition 4.0.2. Note here that we may apply Proposition 4.0.2 because the assumptions of Notation 4.0.5 are satisfied by Lemma 5.0.2. Overall, this implies $\sum_{h\in c}[h]^2$ acts as 0 on $\ker(H_i^{G,c,g} \to H_{i,1}^{\mathrm{id},\mathrm{id}})$. But $\sum_{h\in c}[h]^2$ acts as the identity on this space by Notation 2.0.3, implying $\ker(H_i^{G,c,g} \to H_{i,1}^{\mathrm{id},\mathrm{id}}) = 0$.

The case that $g \notin c$ follows from the case $g \in c$ because each $[h]$ for $h \in c$ acts as 0 on $\ker(H_i^{G,c,g} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$, since it sends $\ker(H_i^{G,c,g} \to H_{i,0}^{\mathrm{id},\mathrm{id}})$ to $\ker(H_i^{G,c,gh} \to H_{i,1}^{\mathrm{id},\mathrm{id}})$, which we have just shown to be 0, and $\sum_{h\in c}[h]^2$ is the identity. □

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