# SOME ASPECTS OF NONCOMMUTATIVE GEOMETRY

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Abstract. We develop geometric notions such as regularity, coherence and flatness in the setting of prestable infinity categories.

We prove a corrected conjecture of Vladimir Sosnilo about discreteness of weight hearts in the presence of regularity. This is a consequence of a more general result which also implies that a regular bounded above $\mathbb{E}_2$-ring is coconnective.

Contents



In this paper, we explore some aspects of noncommutative geometry.

For us, the fundamental object in noncommutative geometry is a small idempotent-complete stable category equipped with a positive half of connective objects. We call the categories of such categories $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, and we think of these as categories of coherent sheaves over a noncommutative geometric object. Indeed geometric categories provide important examples, though they often are equipped with other kinds of structure, such as a symmetric monoidal structure (for examples coming from commutative geometry) or the structure of being linear over a commutative ring.

Alternatively, one can think of such a category as a globalization of the notion of an $\mathbb{E}_1$-algebra $R$. Indeed, $\mathbb{E}_1$-algebras embed fully faithfully into noncommutative geometry by considering their categories of perfect modules equipped with a unit object. The category

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of $\mathbb{E}_1$-algebras is enlarged by considering functors between their perfect module categories that don't necessarily preserve the unit, and in this enlargement, we only remember the $\mathbb{E}_1$-algebra up to Morita equivalence[1]. Morita theory then tells us that in noncommutative geometry, everything is built canonically from the perfect module categories of $\mathbb{E}_1$-algebras, and hence can be thought of as a globalization.

One of the goals of this paper is to develop and explore some notions in noncommutative geometry that robustly generalize what happens in the commutative setting. For example, we define $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ to be regular if the positive half of $\mathcal{C}$ is the positive half of a $t$-structure. By taking $\mathcal{C}$ to be a perfect module category of an $\mathbb{E}_1$-ring, this defines a notion of regularity $\mathbb{E}_1$-rings, which we show in Section 2 to coincide with notions of regular rings for discrete and connective rings that have previously been studied. We show that the class of regular rings is fairly well behaved, and provide many examples to show the range of the definition.

Another notion we study is that of coherence, which is a weaker finiteness condition than regularity one often encounters. We show that beyond being a well behaved notion (closed under natural operations), it also interacts well with regularity. For example, the if $R$ is a regular $\mathbb{E}_1$-ring, $R[x]$ may not be, but if $R[x]$ is coherent, then it is additionally regular (see Corollary 3.14).

Even without being regular, every $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ comes with a $t$-structure on $\mathrm{Ind}(\mathcal{C})$, with connective objects beinig Ind of its positive half. This allows one to talk about notions such as flatness of maps. We study notions related to flatness including flat localizations. We define a Zariski topology for any $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, and show that regularity is local in the Zariski topology and that our definition recovers the usual one for Noetherian discrete commutative rings.

We also explain the appearance of weight structures in noncommutative geometry. Namely, given $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, one naturally obtains a weight structure on $\mathrm{Ind}(\mathcal{C}^{op})$, where the positive objects are declared to generate the negative objects in the weight structure.

In Section 6, we study the relation between regularity, truncatedness, and commutativity. In particular, we prove the following result, which is a corrected form of a conjecture of Sosnilo [Sos21, Conjecture 3.3.6]:

**Theorem 0.1.** *Let $\mathcal{C}$ be a rigid monoidal stable category with compatible weight structure, and compatible adjacent $t$-structure such that the unit is bounded above in the $t$-structure. Then the weight heart is discrete.*

This is actually the consequence of a more general result which doesn't mention weight structures, and also has the following corollary:

**Theorem 0.2.** *Let $R$ be a regular bounded above $\mathbb{E}_2$-ring. Then $R$ is coconnective.*

## 1. Prestable categories

Recall from [Lur18, Appendix C] that a prestable category is one with finite colimits such that the functor $\Sigma$ is fully faithful, and for every map $f : x \to \Sigma y$ the fibre exists and is the desuspension of $\mathrm{cof}(f)$. These can be characterised as exactly the subcategories of stable categories closed under finite colimits and extensions [Lur18, C.1.2.2]. Moreover, there is a universal such stable category $\mathrm{SW}(\mathcal{C}_{\geq 0})$, the category obtained from $\mathcal{C}_{\geq 0}$ by inverting the endomorphism $\Sigma$. We will often denote prestable categories by $\mathcal{C}_{\geq 0}$ and use $\mathcal{C}$ to denote $\mathrm{SW}(\mathcal{C}_{\geq 0})$.

[1]Because we are considering the stable categories equipped with a positive half, really we remember the $\mathbb{E}_1$-algebra up to $t$-Morita equivalence, which includes classical Morita equivalences of discrete rings.

Our basic geometric object is $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, equipped with small idempotent complete prestable subcategory denoted $\mathcal{C}_{\geq 0}$ such that $\mathrm{SW}(\mathcal{C}_{\geq 0}) = \mathcal{C}$. Let $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ denote the category of such such objects with the right exact functors between them, i.e exact functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{C}_{\geq 0}$ to $\mathcal{D}_{\geq 0}$. Note that $\mathcal{C}_{\geq 0}$ completely determines $C$, so that $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is also the category of small idempotent complete prestable categories. Nevertheless, we will treat $\mathcal{C}_{\geq 0}$ as extra structure attached to $\mathcal{C}$, so for example we will write $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$.

If $\mathcal{C}_{\geq 0} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, then $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$. In the other direction, given any set of objects $\{X_\alpha\}$ in some $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ generating $\mathcal{C}$ as a thick subcategory, the subcategory of $\mathcal{C}$ generated by $\{X_\alpha\}$ under finite colimits, retracts, and extensions is in $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, and recovers $\mathcal{C}$ via SW.

Given $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, $\mathcal{C}_{\geq 0}$ is the connective objects of a $t$-structure on $\mathcal{C}$ iff $\mathcal{C}_{\geq 0}$ has finite limits [Lur18, C.1.2.9]. Nevertheless, $\mathrm{Ind}(\mathcal{C}_{\geq 0})$ is presentable, so has finite limits, and thus is the connective objects of a $t$-structure on $\mathrm{Ind}(\mathcal{C})$. In fact $\mathrm{Ind}(\mathcal{C}_{\geq 0})$ is a Grothendieck prestable category, meaning that $\Omega$ commutes with filtered colimits. A map $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$ induces a right t-exact functor $F^* : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$, whose right adjoint, $F_*$ is left t-exact.

**Lemma 1.1** ([Lur18, C.6.1.5])**.** *The functor sending $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ to $\mathrm{Ind}(\mathcal{C}_{\geq 0})$ defines an equivalence between $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ and the category of compactly generated Grothendieck prestable categories with colimit and compact preserving functors.*

**Lemma 1.2** ([Lur18, C.6.3.1])**.** *Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, and consider the $t$-structure $(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}_{\geq 0}))$. Then*

1. *An object is connective iff it can be built from $\mathcal{C}_{\geq 0}$ under filtered colimits*
2. *An object $A$ is coconnective iff $\mathrm{Map}(X, A) = 0$ for every $X \in \mathcal{C}_{\geq 0}$.*
3. *The $t$-structure is right complete and compatible with filtered colimits.*

The category $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is equipped with a symmetric monoidal structure refining the symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{perf}}$. Namely given $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$, $(\mathcal{C} \otimes \mathcal{D})_{\geq 0}$ is the subcategory of $\mathcal{C} \otimes \mathcal{D}$ generated under finite colimits, extensions, and retracts by $c \otimes d$ for $c \in \mathcal{C}_{\geq 0}, d \in \mathcal{D}_{\geq 0}$.

A key family of examples of $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ above come from rings.

**Definition 1.3.** Let $R$ be an $\mathbb{E}_1$-ring. We equip $\mathrm{Mod}(R)^\omega$ with $\mathrm{Mod}(R)^\omega_{\geq 0}$, the full subcategory generated under finite colimits and extensions by $R$. We define the *standard $t$-structure* to be the induced $t$-structure on $\mathrm{Mod}(R)$. In otherwords, $M \in \mathrm{Mod}(R)$ is connective iff it is generated under colimits and extensions by $R$, and coconnective iff $\mathrm{map}(R, M)$ is coconnective, i.e the underlying spectrum is coconnective.

Another family of examples come from abelian categories.

**Example 1.4.** Let $A$ be a small abelian category, and let $D^{\mathrm{perf}}(A)$ be its perfect derived category. $D^{\mathrm{perf}}(A)$ can be equipped with a $t$-structure such that the connective objects are generated by $A$, and such that $A = C^\heartsuit$. Thus $D^{\mathrm{perf}}(A) \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$.

## 2. Regularity

**Definition 2.1.** We say that $\mathcal{C} \in \mathrm{Cat}^{perf}_{\geq 0}$ is *regular* if $(\mathcal{C}, \mathcal{C}_{\geq 0})$ defines a $t$-structure on $\mathcal{C}$.

Regularity can be rephrased as asking that the $t$-structure on $\mathrm{Ind}(\mathcal{C})$ restrict to compact objects.

**Lemma 2.2.** *$\mathcal{C}$ is regular if and only if for every $c \in \mathcal{C}$ its homotopy groups $\pi_i^{\heartsuit} c \in \mathrm{Ind}(C)$ are compact.*

*Proof.* Let $x \in \mathcal{C}$, so that $\mathcal{C}$ is regular iff for all such $x$, $\tau_{\leq n}x$ is in $\mathcal{C}$ for each $n$. Now observe that $x$ is bounded below in the $t$-structure on $\mathrm{Ind}(\mathcal{C})$ because $\mathcal{C} = \mathrm{SW}(\mathcal{C}_{\geq 0})$, so for any $x \in \mathcal{C}$, a sufficiently large suspension of $x$ is in $\mathcal{C}_{\geq 0}$. So because $x$ is bounded below, the collection of objects $\tau_{\leq n}x$ are built from $\pi_i^{\heartsuit}x$ via finitely many extensions, and conversely $\pi_i^{\heartsuit}x$ is up to a shift the cofibre of $\tau_{\leq i}x \to \tau_{\leq i-1}x$ so the condition that these two collections of objects be compact is equivalent. □

## 2.1. Regularity for $\mathbb{E}_1$-rings.

There are many notions of regularity that already exist for rings, so we first go about explaining how they relate to ours. First, we show that for a discrete ring, $R$ is left regular coherent iff it is regular in our sense.

**Definition 2.3.** $R$ is *regular* if $\mathrm{Mod}(R)^{\omega}$ is regular, i.e the standard $t$-structure restricts to compact objects.

**Proposition 2.4.** *Let $R$ be a discrete ring. The standard $t$-structure on* $\mathrm{Mod}(R)$ *restricts to compact objects iff $R$ is left regular coherent.*

*Proof.* Suppose that the $t$-structure on $R$ restricts to compact objects. The heart of this $t$-structure is the finitely presented $R$-modules, which form an abelian category iff $R$ is coherent. Next we show that any finitely presented module $M$ has a finite projective resolution. Since $M$ is compact, it is $\leq i$ for some $i$ in the weight structure. Choose a projective $P_0 \to M$ surjecting onto $M$, and let $\Sigma M_1$ be the cofibre, which is also finitely presented because it is compact. Inductively we can produce $M_i$ with surjections of projectives $P_j \to M_j$, and define $\Sigma M_{j+1}$ to be the cofibre of this. $\Sigma^i M_i$ is then in the weight heart, so $M_i$ is flat by Lemma 5.6 and finitely presented, i.e projective. Thus $M$ has a finite projective resolution.

Conversely, suppose $R$ is left regular coherent, and let $M$ be a compact object. Since $R$ is coherent, any compact object has homotopy groups that are finitely presented. It suffices to show that $\tau_{\leq 0}M$ is compact in the standard $t$-structure. But this follows because we can choose compact projective modules surjecting onto the positive homotopy groups of $M$, and the cofibre will have homotopy groups the kernels of these surjections. After finitely many repetitions, the kernels will be themselves projective, so we can make the surjections isomorphisms, constructing $\tau_{\leq 0}M$ as a finite colimit of compact objects. □

Next we show that for a connective $\mathbb{E}_1$-ring, our notion is equivalent to Barwick and Lawson's [BL14] notion of almost regular (also studied in [Sos21]).

**Proposition 2.5.** *Suppose that $R$ is connective. Then $R$ is regular iff $\pi_0 R$ is a left regular coherent ring that is compact in* $\mathrm{Mod}(R)$*, and $\pi_i R$ is finitely presented over $\pi_0 R$.*

*Proof.* Suppose that the latter conditions are satisfied; we want to show that the homotopy groups of any compact object are compact. But since $\pi_n R$ are finitely presented over $\pi_0 R$ and $\pi_0 R$ is coherent, the homotopy groups of any compact object are always finitely presented over $\pi_0 R$. Thus they have a finite projective resolution, showing they are in the thick subcategory generated by $\pi_0 R$, which is assumed compact.

Conversely, assume that $R$ is regular. If $F : R \to \pi_0 R$ is the truncation functor, it induces $F^* : \mathrm{Mod}(R) \to \mathrm{Mod}(\pi_0 R)$ coming from the base change. The right adjoint $F_*$ of this is cocontinuous and t-exact. By assumption, this right adjoint preserves compact objects. We claim that the identity functor on $\mathrm{Mod}(\pi_0 R)$ is a retract of the functor $\pi_0 R \otimes_R (F_*(-))$.

Since both functors are cocontinuous, it suffices to show that $\pi_0 R \otimes_R (F_*(\pi_0 R))$ has $\pi_0 R$ as a retract. But this is clear since $\pi_0$ of the former is just $\pi_0 R$, so the retraction is given by applying $\tau_{\leq 0}$, and the section is given by choosing the element $1 \in \pi_0 R \otimes_R (F_*(\pi_0 R))$. It follows that $\tau_{\leq 0} M$ is compact for $M \in \mathrm{Mod}(\pi_0 R)^\omega$, since it is a retract of $\pi_0 R \otimes \tau_{\leq 0} F_* M$, so $\pi_0 R$ is regular.

By Proposition 2.4, we conclude that $\pi_0 R$ is left regular coherent. The homotopy groups of $R$ are compact, so must be finitely presented over $\pi_0 R$. $\square$

**Example 2.6.** Examples of connective regular rings include $\mathrm{BP}\langle n \rangle, \mathrm{tmf}, \mathrm{ko}, \mathrm{ku}$.

Note that the $t$-structure on the perfect module category of a regular ring is bounded iff the ring is bounded above.

**Example 2.7.** The property of an $\mathbb{E}_1$-ring being regular is not Morita invariant: consider $R = \mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p \oplus \Sigma^{-2}\mathbb{F}_p)$, which is Morita equivalent to $\mathbb{Z}_p$. Then $\pi_0^\heartsuit R$ in the standard $t$-structure of $R$ is $\Sigma^{-2}\mathbb{F}_p \oplus \mathbb{Q}_p$ as a $\mathbb{Z}_p$-module, which is not compact.

A similar example, where the ring itself has compact truncations is $R = \mathrm{End}_{k[x,y]}(k[x,y] \oplus \Sigma^{-2}k)$. This is Morita equivalent to $k[x,y]$ and is coconnected so it is in the heart of its standard $t$-structure. But $\pi_0^\heartsuit(k[y])$ under this $t$-structure is $k[y^{\pm 1}]$, which is not compact.

Nevertheless, regularity is invariant under $t$-Morita equivalences.

**Definition 2.8.** A *$t$-Morita equivalence* $\mathrm{Mod}(R) \to \mathrm{Mod}(R)'$ is an equivalence in $\mathrm{Cat}^{\mathrm{perf}_{\geq 0}}$.

Any two discrete rings that are classically Morita equivalent[2], are also $t$-Morita equivalent.

**Example 2.9.** By [BL21, Theorem 1.1], any ring $R$ with $\pi_0$ left regular coherent and $\mathrm{cof}(\pi_0 R \to R)$ tor amplitude in $[-\infty, -1]$ as a right $\pi_0 R$-module is regular. $\triangleleft$

All of the conditions in Example 2.9 fail in general for bounded above regular rings.

**Example 2.10.** Here is an example due to Sosnilo [Sos21, Construction 2.3.1] of a connective bounded above regular ring that is not coconnective. The category $\mathcal{C}$ of representations of the quiver $\cdot \to \cdot$ in $\mathrm{Mod}(k)^\omega$, is generated as a thick subcategory by the representation $k \xrightarrow{0} \Sigma^2 k$. Thus it is equivalent to perfect modules over $R = \mathrm{End}_{\mathcal{C}}(k \xrightarrow{0} \Sigma^2 k)^{op}$. $R$ is regular[3], and has nontrivial homotopy groups in only degrees 0 and 1. $\triangleleft$

The following still holds despite the above example:

**Lemma 2.11.** *Let $R$ be a regular bounded above $\mathbb{E}_1$-ring. Then $R$ is $t$-Morita equivalent to a coconnective ring.*

*Proof.* $R$ has finitely many nonzero homotopy groups in the standard $t$-structure since it is bounded above and connective. Let $R'$ be $\mathrm{End}_R(\oplus_i \pi_i^\heartsuit(R))^{op}$. This is a coconnective ring since it is the endomorphism ring of an object in the heart. $\oplus_i \pi_i^\heartsuit(R)$ clearly generates $\mathrm{Mod}(R)^\omega$ as a thick subcategory since $R$ is an extension of shifts of summands of it. It remains to check then that the standard $t$-structures agree. $R$ is connective in the $t$-structure generated by $\oplus_i \pi_i^\heartsuit(R)$ because it is an extension of nonnegative shifts of summands of it. Conversely, $\oplus_i \pi_i^\heartsuit(R)$ can be built from $R$ since by definition it is connective in the standard $t$-structure. Since these generate the connective objects, the $t$-structures agree. $\square$

[2] meaning their categories of discrete modules are equivalent

[3] this is easy to check since the quiver has finitely many indecomposable representations

In Section 6, we consider the question of whether a regular bounded above commutative ring is coconnective. For now, we return to showing that the conditions of Example 2.9 fail for general bounded above regular rings.

**Example 2.12.** There are coconnective regular rings where the tor dimension condition of [BL21, Theorem 1.1] fail, for example the ring in [BL21, Example 5.15].

More interestingly, there are coconnected regular rings with $\pi_0$ not even regular. For example, consider the degree 2 embedding $\mathbb{P}^1 \to \mathbb{P}^2$, and let $X$ be the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}^1 \arrow[d] \\ \mathbb{A}^3 - 0 \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

$X$ is the affine cone over the embedding of $\mathbb{P}^1$ with the cone point removed; in particular it is 0-affine, so its category of perfect coherent sheaves is generated by the unit. Since $X$ is regular, the endomorphism ring of the unit is regular. However $\pi_0$ of this ring is the global sections of the structure sheaf, which is not regular as the cone point is singular. ◁

## 2.2. **Proving regularity.**

Despite the above example, if a regular coconnective ring $R$ is augmented over its $\pi_0 R$, its $\pi_0 R$ must be regular. This is due to the following fact:

**Proposition 2.13.** *Regularity in* $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ *is a condition closed under retracts.*

*Proof.* Suppose that $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ with $\mathcal{D}$ regular, and $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$ realizes $\mathcal{C}$ as a retract of $\mathcal{D}$ via right $t$-exact functors.

If $c \in \mathcal{C}$ we need to show that $\tau_{\leq 0} c$ is compact. Since $\mathrm{Ind}(F), \mathrm{Ind}(G)$ are right $t$-exact, there are canonical natural transformations $F\tau_{\leq 0} \to \tau_{\leq 0} F$ and $G\tau_{\leq 0} \to \tau_{\leq 0} G$. Because the $t$-structure on $\mathrm{Ind}(\mathcal{D})$ restricts to compact objects, $\tau_{\leq 0} Fc$ is compact. Thus the composition

$$\tau_{\leq 0} c = GF\tau_{\leq 0} c \to G\tau_{\leq 0} Fc \to \tau_{\leq 0} GFc = \tau_{\leq 0} c$$

exhibits $\tau_{\leq 0} N$ as a retract of the compact object $G\tau_{\leq 0} Fc$. □

**Corollary 2.14.** *A retract of a regular ring is regular.*

**Remark 2.15.** In case the ring is discrete, commutative, and Noetherian, the corollary above is a result due to Costa [Cos77]. ◁

Regularity is closed under adding polynomial generators of nonzero degree.

We let $\mathcal{C}[x_n]$ denote $\mathcal{C} \otimes \mathrm{Mod}([x_n])^{\omega}$. When $\mathcal{C} = \mathrm{Mod}^{\omega}_R$, then this coinicides with $\mathrm{Mod}(R[x_n])^{\omega}$.

**Proposition 2.16.** *If* $\mathcal{C}$ *is regular, and* $n \neq 0$*, then* $\mathcal{C}[x_n]$ *is regular.*

*Proof.* If $n < 0$, then the functor $\otimes \mathbb{S}[x_n] : \mathcal{C} \to \mathcal{C}[x_n]$ satisfies the conditions of [BL21, Theorem 1.3]. Indeed, the composite of $\mathrm{Ind}(\otimes \mathbb{S}[x_n])$ with its right adjoint sends an object $c \in \mathcal{C}$ to $\bigoplus_0^{\infty} \Sigma^{-kn} c$, showing that hypothesis $(B')$ of Corollary 4.12 is satisfied.

Suppose now that $n > 0$. It suffices to show that for every $M \in \mathcal{C}[x_n]$, $\tau_{\leq 0} M$ is compact. We first claim that $\tau_{\leq 0} M$ always belongs to the thick subcategory generated by $c \otimes \mathbb{S}[x_n]/x_n$ for $c \in \mathcal{C}$. To see this, for any object $M$ we can consider the object $M[x_n]^{-1} = M \otimes_{\mathbb{S}[x_n]} \mathbb{S}[x_n^{\pm 1}]$. For $c \in \mathcal{C}_{\geq 0}$, $\mathrm{map}(c \otimes \mathbb{S}[x_n], \tau_{\leq 0} M)$ is bounded above, so this vanishes after inverting $x_n$ for all $c$, and it follows that $\tau_{\leq 0} M[x_n^{\pm 1}] = 0$. The fibre of the map $\tau_{\leq 0} M \to (\tau_{\leq 0} M)[x_n^{-1}]$

is $\mathrm{colim}_i\, \tau_{\leq 0} M \otimes_{\mathbb{S}[x_n]} (\Sigma^i \mathbb{S}[x_n]/x_n^i)$, which is in Ind of the thick subcategory generated by $c \otimes (\mathbb{S}[x_n]/x_n)$, but it is also compact, so it is in that thick subcategory.

Thus it suffices to show that in the thick sucategory generated by $c \otimes_{\mathbb{S}[x_n]} \mathbb{S}[x_n]/x_n$, the $t$-structure restricts to compact objects. We claim this category is equivalent to

$$\mathcal{C}[\epsilon_{-n-1}] := \mathcal{C} \otimes \mathrm{Mod}(\mathbb{S}[\epsilon_{-n-1}])^{\omega}$$

, where $\mathbb{S}[\epsilon_{-n-1}])$ is the trivial square zero extension of $\mathbb{S}$ by $\mathbb{S}$ in degree $-n-1$. This is because of the fully faithful embedding $\mathrm{Mod}(\mathbb{S}[\epsilon_{-n-1}])^{\omega} \to \mathrm{Mod}(\mathbb{S}[x_n])^{\omega}$ given by sending the unit to $\mathbb{S}[x_n]/x_n$. Thus by applying [BL21, Theorem 1.3] to the map $\otimes \mathbb{S}[\epsilon_{-n-1}] : \mathcal{C} \to \mathcal{C}[\epsilon_{-n-1}]$, we find that the standard $t$-structure is bounded. □

Applying the above proposition for $\mathcal{C} = \mathrm{Mod}(R)^{\omega}$ with the standard $t$-structure gives

**Corollary 2.17.** *Let $R$ be a regular $\mathbb{E}_1$-ring. If $n \neq 0$, then $R[x_n]$ is regular.*

The case $n = 0$ is considered in more detail in Section 3.

**Example 2.18.** Regular rings are not closed under filtered colimits. For example, consider the rings $R_n = k[x, y, w_i, v_i, (v_i + w_i)^{-1}, 1 \leq i \leq n]/xw_i = yv_i$, where $k$ is a field. Each $R_n$ is regular, but their filtered colimit over $n$ is not even coherent: the ideal $(x, y)$ has infinitely many relations $xw_i = yv_i$.

Similarly, we note that regular rings are closed under finite products since the module category of a product is the sum of the module categories, but they are not closed under infinite products: for example $\Pi_0^{\infty} R_2$ is not even coherent, since the ideal generated by $x$ in each coordinate and $y$ in each coordinate is not finitely presented. ◁

## 3. Coherence

Often $\mathrm{Ind}(\mathcal{C})$ doesn't have enough compact objects to be regular, but it does satisfy a milder finiteness condition typified by the next example.

**Example 3.1.** Consider the category of modules for $R = k\langle\epsilon_0\rangle$. In this category $k = \ker\left(R \xrightarrow{\epsilon_0 \cdot -} R\right)$ isn't compact and therefore $\mathrm{Mod}(R)$ isn't regular. However, $k$ does have a relatively simple minimal free resolution

$$\to \cdots \to R \xrightarrow{\epsilon_0 \cdot -} R \xrightarrow{\epsilon_0 \cdot -} R \xrightarrow{\epsilon_0 \cdot -} R \to k.$$

Note that although this resolution has infinitely many terms, these terms are levelwise compact. ◁

In this subsection we recall the theory of coherent categories and almost compact objects from [Lur18, C.6] which is designed to extract the key properties of $\mathrm{Mod}_{k\langle\epsilon_0\rangle}$ and the object $k$.

**Definition 3.2.** An object $c \in \mathcal{C}_{\geq 0}$ is *almost compact* if its image in $\tau_{\leq n}\mathcal{C}_{\geq 0}$ is compact for every $n$. For a general object of $c \in \mathcal{C}$ we say that it is almost compact if it is bounded below and $\Sigma^k c$ is almost compact for some $k$ (chosen so that in $c \in \mathcal{C}_{\geq 0}$). ◁

The intuition behind this definition was that we are asking that $c$ have compact approximations which are correct through any given range (where our notion of range is fixed by the $t$-structure). We now make this precise.

**Lemma 3.3.** *$c \in \mathcal{C}$ is almost compact iff there exists a diagram*

$$c_0 \to c_1 \to \cdots \to c$$

*such that $\tau_{\leq n} c_n \to \tau_{\leq n} c$ is an equivalence. In particular, for any almost compact $c$, there exists a compact $c'$ such that $\pi_i c \simeq \pi_i c'$.*

*Proof.* The forward direction is easy, since the given conditions imply that $\tau_{\leq n} c$ is compact in $\tau_{\leq n} C_{\geq 0}$.

For the reverse direction we need to produce approximations. By induction it suffices to assume that $c$ is $\geq 0$ and to find a compact $c'$ and a map $c' \to c$ such that $\mathrm{cof}(c' \to c)$ is $\geq 1$. Pick a filtered colimit presentation $\varinjlim c_\alpha \simeq c$ of $c$ by compact objects which are each $\geq 0$. After applying $\tau_{\leq 0}$ to this filtered diagram the truncation map $c \to \tau_{\leq 0} c$ factors through some $c_\alpha$ since $c$ is almost compact. This means the map $c_\alpha \to c$ is surjective on $\pi_0$ as desired. ☐

In nature there are two major ways one encounters almost compact objects. The first is that a geometric realization of compact objects which are each $\geq 0$ may not be compact, but will always be almost compact [Lur18, C.6.4.4]. The second (really a special case of the first) is that if $X$ is compact and has a $G$-action (where $G$ is some finite group) then $X_{hG}$ is almost compact[4].

**Definition 3.4.** Let $\mathcal{C}^{\mathrm{ac}}$ denote the full subcategory of $\mathrm{Ind}(\mathcal{C})$ on the almost compact objects. We will say that $\mathcal{C}$ is *coherent* if the $t$-structure on $\mathrm{Ind}(\mathcal{C})$ restricted to $\mathcal{C}^{\mathrm{ac}}$. ◁

As with regularity (cf. ???), coherence is a finiteness property on homotopy groups of compact objects.

**Lemma 3.5** ([Lur18, C.6.5.6])**.** *$\mathcal{C}$ is coherent iff for every $c \in \mathcal{C}^{\mathrm{ac}}$ its homotopy groups $\pi_i c$ (which a priori live in $\mathrm{Ind}(\mathcal{C})$) are almost compact. In fact, by the previous lemma it suffices to consider only compact $c$.*

As with regularity, the property of being coherent is closed under restracts.

**Lemma 3.6.** *Coherence in $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is a condition closed under retracts.*

*Proof.* The same proof as used in Proposition 2.13 applies here. ☐

### 3.1. Coherence for $\mathbb{E}_1$-rings.

Mirroring our discussion of regularity for $\mathbb{E}_1$-algebras from **??**, the category of left modules for an $\mathbb{E}_1$-algebra $R$ is coherent exactly when the ring $R$ is coherent.

**Lemma 3.7** ([Lur18, C.6.5.3])**.** *A discrete ring $R$ is coherent (in the sense of Definition 3.4) iff it is left coherent (in the classical sense).*

**Example 3.8.** Suppose $k \to R$ is a discrete, commutative $k$-algebra where $k$ is a field. If $R$ is finitely presented, then $R$ is coherent. ◁

**Lemma 3.9** ([Lur18, C.6.5.3], [Lur17, 7.2.4.18])**.** *A connective $\mathbb{E}_1$-algebra $R$ is coherent (in the sense of Definition 3.4) iff $\pi_0 R$ is left coherent and $\pi_n R$ is finitely presented as a left $\pi_0 R$-module for each $n \geq 0$.*

[4]Note that this implies that many of the non-compact objects one encounters when thinking about power operations are almost compact.

3.2. **Stable coherence.**

In Proposition 2.16, it was shown that regularity is preserved under taking polynomial rings in nonzero degree. However, this is not in general true for $n = 0$.

**Example 3.10.** In [Gla89, Example 7.3.13], it is shown that an infinite product of the ring $\mathbb{Q}[[x, y]]$ is regular, but doesn't remain coherent after adjoining a polynomial variable.

Hence the following is a collection of finiteness conditions which are sligthly stronger than coherence, and which have applications to negative $K$-theory. See also the discussion in [AGH19, 3.4-3.5], which has some overlap.

We use $\mathbb{A}^n$ to denote either the ring $[x_1, \ldots, x_n]$ or its perfect module category.

**Definition 3.11.** A prestable category $\mathcal{C}$ is $\mathbb{A}^n$*-coherent* if $\mathcal{C} \otimes \mathbb{A}^n$ is coherent. If $\mathcal{C}$ is $\mathbb{A}^n$-coherent for all $n$, we will say $\mathcal{C}$ is *stably coherent.* Analogously, we will say that a prestable category $\mathcal{C}$ is $\mathbb{A}^n$*-regular* if $\mathcal{C} \otimes \mathbb{A}^n$ is regular and that it is *stably regular* if it is $\mathbb{A}^n$-regular for all $n$. ◁

**Remark 3.12.** From Lemma 3.6 we know that an $\mathbb{A}^n$-coherent category is also $\mathbb{A}^{n-1}$-coherent. It follows that $C$ is $\mathbb{A}^n$-coherent iff it is $\mathbb{A}^{n-1}$-coherent and $\mathcal{C}\otimes\mathbb{A}^{n-1}$ is $\mathbb{A}^1$-coherent. For this reason, the study of $\mathbb{A}^n$-coherence usually reduces to the the case $n = 1$. ◁

**Proposition 3.13.** *Suppose that $\mathcal{C}$ is $\mathbb{A}^1$-coherent. Then, the map $\mathcal{C} \to \mathcal{C}[t]$ has relative dimension at most 1 in the sense that every almost compact object in $e \in \mathrm{Ind}(\mathcal{C}[t])^\heartsuit$ is a retract of an object of the form*

$$\mathrm{cof}(d_1[t] \to d_2[t])$$

*where $d_1, d_2$ are almost compact objects in $\mathrm{Ind}(\mathcal{C})^\heartsuit$.*

*Proof.* Consider the adjunction $-[t] : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}[t]) : U(-)$ For any object $e \in \mathrm{Ind}(\mathcal{C}[t])$ there is a (functorial) 2-step resolution by induced objects given by

$$U(e)[t] \xrightarrow{U(t\cdot -)[t]-(t\cdot -)} U(e)[t] \to e.$$

In order to prove the proposition we will need to replace each copy of $U(e)[t]$ with an almost compact object.

Assuming that $e$ is an almost compact object living in the heart, we argue as follows: Using the hypothesis that $\mathcal{C}$ is coherent pick a presentation of $U(e)$ as a filtered colimit of almost compact objects $\{d_\alpha\}$ in the heart along injective maps. Let $f$ denote the map $U(t\cdot -)[t] - (t\cdot -)$ whose cofiber we want to take. Using the assumption that each $d_\alpha$ is almost compact, for each $\alpha$ there exists an $\alpha'$ such that $d_\alpha[t] \to U(e)[t] \xrightarrow{f} U(e)[t]$ restricts to $d_{\alpha'}$ (and the space of such lifts is contractable because of the injectivitiy condition above). The sequence of cofibers $e_\alpha \coloneqq \mathrm{cof}(d_\alpha[t] \to d_{\alpha'}[t])$ now provide a filtered diagram of almost compact objects in $\mathrm{Ind}(\mathcal{C}[t])$ presenting $e$. To conclude we will show that there exists an $\alpha$ for which $e$ splits off of $e_\alpha$. Since $e$ was by assumption almost compact and the $e_\alpha$ are uniformly bounded above, the indentity on $e$ factors through some finite stage of the filtered diagram of the $e_\alpha$, which provides the desired splitting.

□

**Corollary 3.14.** *If $\mathcal{C}$ is regular and $\mathbb{A}^n$-coherent, then it is $\mathbb{A}^n$-regular.*

*Proof.* By induction it suffices to prove the $n = 1$ case. By Lemma 2.2 it suffices to prove that the homotopy groups of compact objects of $\mathcal{C}[t]$ are compact. The coherence assumption guarantees that these homotopy groups are almost compact. Proposition 3.13 implies that each of these homotopy groups is a retract of an object that has a 2-step resolution by induced almost compact objects. However, since $\mathcal{C}$ is regular, almost compact bounded objects are compact, which implies that this object is compact. $\square$

Here we explain how our results can be extended to negative $K$-theory under stronger assumptions about the hearts of the $t$-structures.

**Lemma 3.15.** *If $\mathcal{C}$ is $\mathbb{A}^n$-coherent, then $\mathcal{C}[t^{\pm 1}]$ is $\mathbb{A}^{n-1}$-coherent. If $\mathcal{C}$ is $\mathbb{A}^n$-regular, then $\mathcal{C}[t^{\pm 1}]$ is $\mathbb{A}^{n-1}$-regular.*

*Proof.* We begin by observing that applying the $n = 1$ version of this lemma to $\mathcal{C} \otimes \mathbb{A}^{n-1}$ proves the full version.

The invert $t$ functor

$$\mathcal{C}[t] \to \mathcal{C}[t^{\pm 1}]$$

is a finite flat localization with kernel generated by objects of the form $\operatorname{cof}(c[t] \xrightarrow{t} c[t])$ with $c \in \mathcal{C}$. As a consequence of Corollary 4.21, $\mathcal{C}[t^{\pm 1}]$ is coherent (regular) if $\mathcal{C}[t]$ is coherent (regular).

$\square$

**Lemma 3.16.** *Suppose we are give a stable category $\mathcal{A}$ with a $t$-structure and a functor $F : \mathcal{A} \to \mathcal{B}$, such that $F$ is fully faithful on the heart of $\mathcal{A}$. Then the induced functor*

$$F : \mathcal{A} \otimes \mathbb{A}^n \to \mathcal{B} \otimes \mathbb{A}^n$$

*is fully faithful on the heart as well.*

*Proof.* Note that since this lemma is in the presentable world we don't need to worry about extending $t$-structures. In fact, the $t$-structure on $\mathcal{A} \otimes \mathbb{A}^n$ is nicely characterized by the property that the underlying object functor $\mathcal{A} \otimes \mathbb{A}^n \to \mathcal{A}$ (which is conservative) is $t$-exact.

By induction it suffices to treat the case where $n = 1$. Let $T$ denote the category obtained from the free arrow by identifying the source and target. Functors $T \to \mathcal{C}$ are equivalent to a choice of object and an endomorphism of that object. This provides us with an equivalence

$$\operatorname{Fun}(T, \operatorname{Ind}(\mathcal{C})) \simeq \operatorname{Ind}(\mathcal{C}[t]).$$

Restricting to hearts gives an equivalence

$$\operatorname{Fun}(T, \mathcal{A}^{\heartsuit}) \simeq (\mathcal{A} \otimes \mathbb{A}^1)^{\heartsuit}.$$

The functor we wish to prove is fully faithful is now just

$$\operatorname{Fun}(T, \mathcal{A}^{\heartsuit}) \to \operatorname{Fun}(T, \mathcal{B})$$

and this follows from the fact that if a functor $G$ is fully faithful, then the functor $\operatorname{Fun}(X, G)$ is also fully faithful. $\square$

The following lemma generalizes the discussion in [AGH19, Section 3.5].

**Lemma 3.17.** *If $\mathcal{C}$ is $\mathbb{A}^n$-regular and bounded, then $\tau_{\geq -n-1} K^{\mathrm{nc}}(\mathcal{C}) \simeq K(\mathcal{C})$.*

*Proof.* Since $\mathcal{C}$ is $\mathbb{A}^n$-regular we can use Lemma 3.15 several times to conclude that $\mathcal{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ admits a bounded $t$-structure. By [AGH19] this implies that its $K_{-1}$ vanishes.

Iterating the usual decomposition of the $K$-theory of $\mathcal{C}[t^{\pm}]$ we learn that $\Sigma^n K(\mathcal{C}[x_1, \ldots, x_n]^{\mathrm{nil}})$ splits off $K(\mathcal{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. The "trivial action" and "underlying" functors tell us that a copy of $K(\mathcal{C})$ splits off $\mathcal{C}[x_1, \ldots, x_n]^{\mathrm{nil}})$. Using the vanishing statement for $K_{-1}$ this implies that $K_{-1-n}(\mathcal{C}) = 0$. Since $\mathbb{A}^n$-regular implies $\mathbb{A}^{n-1}$-regular, we learn $K(\mathcal{C})$ vanishes in the range $[-1 - n, -1]$. $\square$

**Proposition 3.18.** *Let $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$ and let $\mathcal{C}$ be equipped with a bounded $t$-structure which is $\mathbb{A}^n$-coherent. If we assume that*

(A) *the image of $F$ contains a collection of generators of $\mathcal{D}$*
(B) *$F$ is fully faithful when restricted to $\mathcal{C}^\heartsuit$*

*then there is a corresponding $\mathbb{A}^n$-regular bounded $t$-structure on $\mathcal{D}$ for which $F$ is $t$-exact. Moreover, the induced maps on nonconnective $K$-theory*

$$\begin{tikzcd} K^{\mathrm{nc}}(\mathcal{C}^\heartsuit) \arrow[r] \arrow[d] & K^{\mathrm{nc}}(\mathcal{D}^\heartsuit) \arrow[d] \\ K^{\mathrm{nc}}(\mathcal{C}) \arrow[r] & K^{\mathrm{nc}}(\mathcal{D}) \end{tikzcd}$$

*are all equivalences after taking $(-n-1)$-connected covers because these covers are connective.*

*Proof.* Applying Corollary 3.14 we learn that $\mathcal{C}[x_1, \ldots, x_n]$ has a bounded $t$-structure. We now apply [BL21, Theorem 1.3] to the induced functor

$$\mathcal{C}[x_1, \ldots, x_n] \to \mathcal{D}[x_1, \ldots, x_n]$$

This functor is fully faithful on the heart by Lemma 3.16 and hits a collection of generators since the induced objects form a collection of generators. As a consequence $\mathcal{D}$ is $\mathbb{A}^n$-regular. The conclusions about $K$-theory now follow form Lemma 3.17. $\square$

Extending [BL21, Theorem 1.3] to negative K theory would be easy if the following question had a positive answer:

**Question 3.19.** Does the theorem of the heart and devissage always hold in negative K theory?

We know this to be true when the heart is Noetherian since the negative K theories vanish by [AGH19], but [Nee21] shows that this vanishing doesn't in general happen. The example doesn't seem to immediately show that the above question is false.

## 4. Flat maps

Classically, an $R$-module $M$ is flat when the functor $M \otimes_R -$ is $t$-exact. This generalizes to morphisms between prestable categories almost without change.

**Definition 4.1.** Let $F : \mathcal{C}_{\geq 0} \to \mathcal{D}_{\geq 0}$ be an exact functor of prestable categories. We will say that $F$ is *flat* if it is $t$-exact. ◁

In the situation where we have a map $f : A \to B$ of discrete rings this recovers the usual notion of a (right) flat map.

**Example 4.2.** The functor $B \otimes_A -$ is $t$-exact exactly when $B$ is flat as a right $A$-module. ◁

**Example 4.3.** If $R$ is a coconnective $\mathbb{E}_1$-algebra, then the connective cover map $\pi_0 R \to R$ is flat iff $R$ has tor amplitude $[-\infty, 0]$ as a right $\pi_0 R$-module. ◁

### 4.1. flat maps and regularity.

For the purposes of this paper the key feature of flat maps is the way they interact well with regularity. As an example, in Example 2.18 we saw that regularity need not be preserved under filtered colimits. However if we require the transition maps to be flat, then regularity is preserved.

**Lemma 4.4.** *Let $\mathcal{C}_i$ be a a filtered diagram of regular prestable categories and flat functors. The colimit $\mathcal{C}_\infty = \operatorname{colim} \mathcal{C}_i$ is regular and each of the functors $\mathcal{C}_i \to \mathcal{C}_\infty$ is flat.*

*Proof.* We begin by observing that filtered colimits in $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ are computed at the level of the underlying category. As a consequence every object of $\mathcal{C}_\infty$ is the image of an object from some $\mathcal{C}_i$.

To show that the maps $\mathcal{C}_i \to \mathcal{C}_\infty$ are flat we observe that for every $c_i \in (\mathcal{C}_i)_{<0}$ and $c_j \in (\mathcal{C}_j)_{\geq 0}$ we have

$$\mathrm{Map}_{\mathcal{C}_\infty}(c_j, c_i) \cong \operatorname*{colim}_k \mathrm{Map}_{\mathcal{C}_k}(c_j, c_i) \cong \operatorname*{colim}_k * \cong *.$$

In order to prove regularity we only need to check that for $c \in \mathcal{C}_i$ there is an object $\pi_0 c$ in $\mathcal{C}_\infty$. Since the map $\mathcal{C}_i \to \mathcal{C}_\infty$ is flat this follows the regularity assumption on $\mathcal{C}_i$. □

**Corollary 4.5.** *Discrete, left regular coherent rings are closed under flat filtered colimits.*

**Example 4.6.** As a consequence of Corollary 4.5 if we assume that $R[x_1, \ldots, x_n]$ is regular for each finite $n$ (where each $x_i$ is placed in degree zero), then $R[S]$ is regular for any set $S$.

By way of contrast, a polynomial ring $R$ over a field $k$ with a set $S$ of generators placed in degree 2 is regular exactly when $\pi_2 R$ is compact (i.e. when $S$ is finite). Note how this collection of examples fails to be closed under filtered colimits. ◁

### 4.2. faithfully flat maps.

The notion of a faithfully flat map also carries over to the prestable setting. In order to motivate our definition we begin by looking at several equivalent forms of the condition that a map be faithfully flat in the discrete setting.

**Lemma 4.7.** *In the situation where we are looking at a map of discrete rings $A \to B$ (with $A$ left regular coherent) and the functor $B \otimes_A -$ between their categories of modules the following are equivalent:*

1. *The map $A \to B$ is right faithfully flat.*
2. *The functor $B \otimes_A - : \mathrm{Mod}_A^\heartsuit \to \mathrm{Mod}_B$ is faithful.*
3. *The map $A \to B$ is injective and $\operatorname{coker}(A \to B)$ is flat as a right $A$-module.*

Both conditions (2) and (3) generalize to the prestable setting. Surprisingly, it is condition (3) that offers us the most flexibility.

**Definition 4.8.** Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of stable categories and let $\mathcal{A}_0 \subset \mathcal{B}$ be a (not necessarily stable) full subcategory. We will say that $F$ is *$n$-ff* on $\mathcal{A}_0$ if for every $x, y \in \mathcal{A}_0$, the spectrum

$$\operatorname{cof}\left(\operatorname{map}_{\mathcal{A}}(x, y) \to \operatorname{map}_{\mathcal{B}}(Fx, Fy)\right)$$

is $\leq -n$. ◁

**Remark 4.9.** We will almost exclusively use Definition 4.8 in the situation where $\mathcal{A} = \mathrm{Ind}(\mathcal{C})$ and $\mathcal{A}_0 = \mathrm{Ind}(\mathcal{C})^\heartsuit$. In this situation the condition that $F$ be *$n$-ff* on heart can be reformulated as saying that the functor $\operatorname{cof}(\mathrm{Id} \to GF)$ takes objects which are $\leq 0$ to objects which are $\leq -n$ (where here $G$ refers to the right adjoint of $F$). ◁

In the case where $n = 0$, the condition that $F$ be 0-*ff* coincides with the other natural condition.

**Lemma 4.10.** *Given a functor $F$ between prestable categories $\mathcal{C}$ and $\mathcal{D}$, the following conditions are equivalent:*

(1) *$F$ is 0-ff on the heart.*
(2) *$F$ is faithful on the heart.*

*Proof.* The condition that $F$ be faithful on $\mathcal{C}_0$ means that for every $x, y \in \mathcal{C}_0$ the map

$$\mathrm{map}(x, y) \to \mathrm{map}(Fx, Fy)$$

induces an isomorphism on $\pi_i$ for $i > 0$ and an inclusion on $\pi_0$. This is clearly equivalent to asking that the cofiber be $\leq 0$. $\square$

For other small values of $n$ this condition has similar interpretations.

**Lemma 4.11.** *In the situation of Definition 4.8,*

- *If we assume that $\mathcal{C}_0 \subset \mathcal{C}$ is closed under extensions, then $F$ is 1-ff on $\mathcal{C}_0$ iff $F$ is fully faithful on $\mathcal{C}_0$.*
- *Again assuming that $\mathcal{C}_0 \subset \mathcal{C}$ is closed under extensions, $F$ being 2-ff on $\mathcal{C}_0$ implies that the image of $\mathcal{C}_0$ in $\mathcal{D}$ is closed under extensions.*

*Proof.* For the first statement the forward implication is clear. For the reverse implication we start by noting that the previous point tells us that $F$ is 0-*ff* at $\mathcal{C}_0$. Using the condition that $F$ is fully faithful at $\mathcal{C}_0$ the only way $F$ could fail to be 1-*ff* is if there exists a non-trivial extension $a \to b \to c$ with $a, c \in \mathcal{C}_0$ that splits on applying $F$. However, the assumption that $\mathcal{C}_0$ is closed under extensions implies that such a $b$ also be in $\mathcal{C}_0$ and then fully faithfulness of $F$ at $\mathcal{C}_0$ would allow us to lift any splitting which existed after applying $F$.

The second statement is clear. $\square$

**Corollary 4.12.** *Condition (B) in* [BL21, Theorem 1.3] *is equivalent to asking that $F$ be 1-ff on $C^{\heartsuit}$, this in turn is equivalent to:*

(B′) *For every $c \in \mathcal{C}^{\heartsuit}$, the cofiber of the unit map $c \to GF(c)$ is $\leq -1$ in the t-structure on $\mathrm{Ind}(\mathcal{C})$.*

When we restrict to the case of a map of $\mathbb{E}_1$-algebras, the notion of *n-ff* naturally recovers the tor amplitude condition which appeared in [BL21, Theorem 1.1]

**Corollary 4.13.** *A map $A \to B$ of $\mathbb{E}_1$-algebras is n-ff iff the $\mathrm{cof}(A \to B) \otimes_A -$ sends $A$-modules which are $\leq 0$ to $A$-modules which are $\leq -n$.*

**Example 4.14.** For a coconnective ring $R$, the connected cover map $\pi_0 R \to R$ is *n-ff* iff $\tau_{<0} R$ has a tor-amplitude $[-\infty, -n]$ as a right $\pi_0 R$-module. In particular, condition (2) of [BL21, Theorem 1.1] asks that the connective cover map be 1-*ff* .

*Proof.* The first part follows from Lemma 4.11 and the rest is easy from the definition of 1-*ff* . $\square$

Although we will not delve too deeply into this can of worms, we point out that the *n-ff* maps are closed under base-change (when that makes sense).

**Example 4.15.** Maps which are *n-ff* can be produced via base change in the following sense: given a square

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ R' \arrow[r, "f'"] & S' \end{tikzcd}$$

such that the map $S \otimes_R R' \to S'$ is an equivalence, if $f$ is $n$-*ff* , then $f'$ is $n$-*ff* as well.

### 4.3. *ff* maps and regularity.

**4.4. regularity and flat localizations.** Regularity is a local property with respect to flat localizations. If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, given an accessible localization of $L : \mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$, we can define $\mathcal{D}_{\geq 0}$ to be the presentable prestable category generated by the image of $\mathcal{C}_{\geq 0}$.

A *compact localization* $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$ is a functor such that $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is an accessible localization[5]. If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, the localization functor obtains a natural lift to $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ by having $\mathcal{D}_{\geq 0}$ be the prestable category generated by the image of $\mathcal{C}_{\geq 0}$.

A *finite localization* is a compact localization such that the category of acyclic objects of $\mathrm{Ind}(\mathcal{C})$ is compactly generated.

**Example 4.16.** The functor $\mathrm{Mod}^{\omega}_{\mathbb{S}} \to \mathrm{Mod}^{\omega}_{L_n\mathbb{S}}$ is a compact localization that is a finite localization iff the telescope conjecture is true.

There is a simple characterization of when localizations are flat.

**Lemma 4.17.** *Let $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ be a compact localization killing the subcategory $\mathcal{A} \subset \mathrm{Ind}(\mathcal{C})$. Then $F$ is flat iff $\mathcal{A}$ is closed under the $t$-structure and $\mathcal{A}^{\heartsuit}$ is closed under subobjects.*

*Proof.* Suppose that $L_{\mathcal{A}}$ is flat. Then given $a \in \mathcal{A}$ and applying $L_{\mathcal{A}}$ to the cofibre sequence $\tau_{\geq 0}a \to a \to \tau_{<0}a$, we see $L_{\mathcal{A}}\tau_{<0}a = \Sigma^{-1}L_{\mathcal{A}}\tau_{\geq 0}a$. It follows that $L_{\mathcal{A}}\tau_{\geq 0}a$ is in the heart, but then $L_{\mathcal{A}}\tau_{\geq 1}a$ vanishes, so $\tau_{\geq 1}a \in A$, i.e $A$ is closed under the $t$-structure. To see $\mathcal{A}^{\heartsuit}$ is closed in $\mathcal{C}^{\heartsuit}$ under subobjects, if $f : a \to b$ is a subobject of $b \in \mathcal{A}$, then applying $L_{\mathcal{A}}$ to the cofibre sequence $\mathrm{fib}(f) \to a \to b$ shows that $L_{\mathcal{A}}\,\mathrm{fib}(f) = L_{\mathcal{A}}a$, so both must vanish.

Conversely suppose that $\mathcal{A}$ is closed under truncations and $\mathcal{A}^{\heartsuit}$ is closed under subobjects so that we would like to show that $L_{\mathcal{A}}$ is flat, i.e that it preserves coconnective objects. Since the generators of connective objects are in the image, the right adjoint $\mathrm{Ind}(L_{\mathcal{A}}\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ preserves and detects coconnective objects. Thus it suffices to show that the composite $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(L_{\mathcal{A}}\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ preserves coconnective objects. Given any coconnective object $y \in \mathrm{Ind}(\mathcal{C})$, we have a cofibre sequence $\mathcal{C}_{\mathcal{A}}y \to y \to L_{\mathcal{A}}y$ in $\mathrm{Ind}(\mathcal{C})$, where $\mathcal{C}_{\mathcal{A}}y \in \mathcal{A}$ is the terminal object in $\mathcal{A}$ mapping to $\mathcal{C}$. Since $\mathcal{A}$ is closed under truncations and we have a factorization $\mathcal{C}_{\mathcal{A}}y \to \tau_{\leq 0}\mathcal{C}_{\mathcal{A}}y \to y$, we must have $\mathcal{C}_{\mathcal{A}}y = \tau_{\leq 0}\mathcal{C}_{\mathcal{A}}y$, so that $L_{\mathcal{A}}y$ is 1-coconnective in the $t$-structure. The map $\pi_0^{\heartsuit}\mathcal{C}_{\mathcal{A}}y \to \pi_0^{\heartsuit}y$ must be a monomorphism because the kernel is in $\mathcal{A}^{\heartsuit}$. The result follows. $\square$

**Example 4.18.** An example of a thick subcategory not closed under truncations is the category of $x_{-1}$-torsion modules in $\mathrm{Mod}^{\omega}_{k[x_{-1}]}$.

Finite localizations are very close to surjective.

**Lemma 4.19.** *If $L : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$ is a finite localization, given $c, c' \in \mathcal{C}$ and any map $f : F(c) \to F(c')$, there are maps $c \xrightarrow{g} x \xleftarrow{h} c'$ in $\mathcal{C}$ such that that $L(g) = f$ and $L(h)$ is an equivalence. There are also maps $c \xleftarrow{h'} y \xrightarrow{g'} c'$ with $L(g') = f$ and $L(h')$ an equivalence.*

[5]The word compact refers to the fact that the target of the localization of $\mathrm{Ind}(\mathcal{C})$ is compactly generated

*Proof.* Since the localization is finite, we can write $F(c')$ as a filtered colimit of objects under $c'$ with the same localization as $c'$. The map $c \to F(c) \to F(c')$ must factor through one of these, so we can take this factorization to be $g$. If we take $y$ to be the pullback of the cospan $c \xrightarrow{g} x \xleftarrow{h} c'$, this shows the second statement. □

**Lemma 4.20.** *If $L : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$ is a finite localization, then for every $d \in \mathcal{D}$, there is $c \in \mathcal{C}$ such that $L(c) = d \oplus \Sigma d$.*

*Proof.* We will show that $d \oplus \Sigma d$ is in the image of $\mathcal{C}$, by showing that the collection of $d$ satisfying this condition is thick subcategory of $\mathcal{D}$. The condition is clearly closed under shifts, and using Lemma 4.19 we find it is closed under cofibre sequences. To see it is closed under retracts, if $d \oplus e \oplus \Sigma d \oplus \Sigma e = L(z)$, use Lemma 4.19 to lift the idempotent projecting onto the latter three factors to a map $g : z \to z'$. Then $L(\operatorname{cof} g) = d \oplus \Sigma d$. □

The above lemma implies flat finite localizations are relative dimension 0.

**Corollary 4.21.** *Regularity in $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is closed under flat finite localizations.*

*Proof.* It suffices to show the homotopy groups of an object of a localization are compact, but by Lemma 4.20, they are retracts of compact objects so are compact. □

**Example 4.22.** $L_n BP\langle n\rangle, L_1 ko, L_2 tmf$ are flat finite localizations of $BP\langle n\rangle, ko, tmf$ respectively, so it follows that these are all regular.

**Remark 4.23.** As pointed out in [BL14], localization sequences such as those of Blumberg–Mandell [BM08] can be explained via regularity and an application of theorem of the heart. In our language, these are all examples of flat finite localizations away from bounded objects. Namely, given $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ regular and a flat finite localization $\mathcal{C} \to \mathcal{D}$ with kernel $\mathcal{B}$, if $\mathcal{B}$ is bounded, then we obtain a localization sequence

$$K(\mathcal{B}^{\heartsuit}) \to K(\mathcal{C}) \to K(\mathcal{D})$$

by applying theorem of the heart to $\mathcal{B}$.

Regularity glues along finite localizations. We explain how this works below for pullbacks.

**Lemma 4.24.** *Finite localizations and flat finite localizations are closed under pullbacks.*

*Proof.* Given a pullback square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "F"] \\ A' \arrow[r, "F'"] & D \end{tikzcd}$$

where $F$ is a flat finite localization, if $B$ is the category killed by $F$, the inclusion $B \to A$ lifts to $\mathcal{C}$ by declaring the $A'$ component to be 0. This is clearly fully faithful, and is killed in the projection to $A'$. Thus there is a map from $L_B\mathcal{C} \to A'$ where $L_B\mathcal{C}$ is the finite localization away from $B$. $\mathrm{Ind}(L_B\mathcal{C})$ embeds in $\mathrm{Ind}(\mathcal{C})$ as the triples $(a', a, f : F'a' \simeq Fa)$ such that $a \in A$ is local with respect to $B$. But this is exactly the pullback $\mathrm{Ind}(A') \times_{\mathrm{Ind}(D)} \mathrm{Ind}(D) = \mathrm{Ind}(A')$.

To see that flatness can also be carried along, we need to show that if $m = (a, a', f : Fa' \simeq Fa) \in \mathrm{Ind}(\mathcal{C})$ has $a$ not coconnective in $A'$, then there is a nonzero map from some object in $\mathcal{C}_{\geq 0}$. If $a$ is not coconnective, there exists some $x' \in A'_{\geq 0}$ with a nonzero map to $a$. $a'$ is a filtered colimit of compact objects, so $F'(x') \to F'(a)$ factors through $F(c)$ for some compact $c \in A$ By Lemma 4.19 and Lemma 4.20, since $F$ is a finite localization, we can find a map $f : x \to c \oplus \Sigma c$ such that $Ff$ is the map $F'(x' \oplus \Sigma x') \to F(c \oplus \Sigma c)$. Because $F$ is flat, we can replace $x$ with $\tau_{\geq 0} x$ in order to assume $x$ is connective. Then $x' \oplus \Sigma x'$ and $x$ glue together along the identification $F(x) = F'(x' \oplus \Sigma x')$ to form an object in $\mathcal{C}_{\geq 0}$ with a nonzero map to $m$. □

We say that a flat localization is generated by an collection of objects if the acyclic objects are generated by those objects under colimits, desuspensions, truncations, and subobjects.

**Definition 4.25.** The *Zariski site* on $(\mathrm{Cat}^{\mathrm{perf}}_{\geq 0})^{op}$ is the site with covers generated by jointly conservative maps $\mathcal{C} \to \mathcal{C}_i \in \mathrm{Cat}^{\mathrm{perf}}, 1 \leq i \leq n$ such that each is a compact flat localizations generated by a single compact object. The *pro-Zariski site* on $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is the site with almost the same covers, except we allow the compact flat localizations to be generated by an arbitrary collection of compact objects.

**Proposition 4.26.** *Let $A' \xrightarrow{F'} D \xleftarrow{F} A$ be a cospan of flat finite localizations in $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, and let $\mathcal{C}$ be the pullback $A' \times_D A$. Then the projections $\mathcal{C} \to A, A'$ are pro-Zariski cover of $\mathcal{C}$. If $A'$ and $A$ are regular, so is $\mathcal{C}$. Conversely, if $\mathcal{C}$ is regular, every cover by two generating pro-Zariski opens arises from such a pullback square.*

*Proof.* The fact that these are a pro-Zariski cover follows from Lemma 4.24. If $A, A'$ are regular, then $\mathcal{C}$ must be as well since truncations are computed componentwise due to the fact that the projections to $A, A'$ are flat.

Conversely, let $\mathcal{C} \to \mathcal{C}_1, \mathcal{C}_2$ be jointly surjective compact flat localizations generated by collections of compact objects. First, we observe that the localizations are actually finite localizations. To see this, we claim that if $c \subset c'$ is a subobject with $c' \in \mathcal{C}^{\heartsuit}$, then $c$ is a filtered colimit of compact subobjects. This is because $c$ is a filtered colimit of compact objects in the heart, but the images of these in $c'$ are also compact and have $c$ as their colimit.

Let $\mathcal{C}_{1,2}$ be the localization away from the union of the acyclic categories for $\mathcal{C}_1, \mathcal{C}_2$. Using Corollary 4.21, we see that the square

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d, "L_2"] \\ \mathcal{C}_2 \arrow[r, "L_1"] & \mathcal{C}_{1,2} \end{tikzcd}$$

consists of flat finite localizations of regular categories. It remains to identify $\mathcal{C}$ with the pullback $\mathcal{C}_1 \times_{\mathcal{C}_{1,2}} \mathcal{C}_2$. The fracture square for the localizations for $\mathcal{C}_1, \mathcal{C}_2$ show that map from $\mathcal{C}$ to the pullback is fully faithful. It then suffices to show conservativity of the right adjoint, i.e that given an object $(x, y, f : L_1 x \simeq L_2 y)$ of the pullback, the pullback of $x, y$ along $L_1 x$ in $\mathcal{C}$ is nonzero. But if it were 0, then we would have $L_1 x \simeq x \oplus y \simeq L_2 y$, which implies $x, y = 0$. □

In other words, regularity is a local property with respect to the pro-Zariski site.

We sketch the connection to the usual Zariski site on commutative rings. In the case of a Noetherian commutative discrete ring $R$, our Zariski site agrees with taking perfect modules on the usual Zariski covers, i.e it is generated by finite collections jointly conservative maps of the form $\mathrm{Mod}(R)^{\omega} \to \mathrm{Mod}^{\omega}_{R[f^{-1}]}$. Indeed, such a map is a compact flat localization, since it is flat and generated by $\mathrm{cof}(f)$. On the other hand, the theory of primary decomposition for a Noetherian ring shows that compact localization generated by any perfect $R$-module is the localization away from the support of its homotopy groups in the Zariski spectrum, which is a compact closed set. Thus any Zariski cover in our sense can be dominated by a cover using maps of the form above.

## 5. Weight structures

In this section, we explain the relationship between weight structures and noncommutative geometry. Weight structures on a stable category $\mathcal{C}$ axiomatize the notion of cell structures. The key definition is Definition 5.4, where we show that any $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ naturally induces a weight structure on $\mathrm{Ind}(\mathcal{C}^{op})$, which accounts for most interesting weight structures. The reason we can do this is because of the more general fact that in a presentable stable category, weight structures are easy to construct: we can always take some small collection of objects and declare them to generate the negative part of a weight structure. The lemma below is the analog of [Lur17, Proposition 1.4.4.11] for weight structures.

**Lemma 5.1.** *Let $\mathcal{C}$ be a presentable stable category, and let $\{X_a\}$ be a small collection of objects. Then $\mathcal{C}$ admits a weight structure where an object $Y$ is $\leq 0$ in the weight structure iff it is generated from $\{\Sigma^{-n}X_a\}, n \geq 0$ via filtered colimits and extensions, and is $> 0$ if* $\mathrm{map}(X_a, Y)$ *is connected.*

*Proof.* The semi-invariance and orthogonality axioms are obvious, so it remains to show that any object $Y$ fits into a cofibre sequence $Y' \to Y \to Y''$ where $Y' \leq 0$ and $Y'' > 0$.

Fix a regular cardinal $\kappa$ such that each object in the set $\{\Sigma^{-n}X_a\}$ is $\kappa$-compact. We will inductively construct a tower of maps $f_\alpha : Y \to Y_\alpha$ indexed on the ordinals such that $\mathrm{fib}(f_\alpha) \leq 0$ and for $\alpha$ sufficiently large $Y_\alpha > 0$.

On limit ordinals, the we let $Y_\alpha$ be the filtered colimit of its predecessors. On sucessor ordinals, we will make it so that for every map $Z \to Y_{\alpha-1}$ with $Z \in \{\Sigma^{-n}X_a\}$, the composite $Z \to Y_{\alpha-1} \to Y_\alpha$ is nulhomotopic. Furthermore, we will have the fibre of $Y_{\alpha-1} \to Y_\alpha$ be $\leq 0$. To do this, take all homotopy classes of maps from $\{\Sigma^{-n}X_a\}$ into $Y_{\alpha-1}$, and define $Y_\alpha$ to be the cofibre. Since the objects $\leq 0$ are closed under extensions and filtered colimits, it follows that the fibre of $Y \to Y_\alpha$ is $\leq 0$.

Choose a $\kappa$-filtered ordinal $\alpha$. Then we claim $Y_\alpha$ is connnected in the weight structure. Indeed, for $Z \in \{\Sigma^{-n}X_a\}$, given a map $Z \to Y_\alpha$, by $\kappa$-compactness, it factors through $Y_\beta$ for some $\beta$ smaller than $\alpha$. But then it is nullhomotopic in $Y_{\beta+1}$, so the original map was nulhomotopic. □

**Example 5.2.** Let $A$ be a Grothendieck abelian category, let $D(A)$ be its derived category, and consider the weight structure on $D(A)$ with negative objects generated by $D(A)^\heartsuit$. The heart of this weight structure consists of injective objects in $D(A)^\heartsuit$, giving a proof that $A$ has enough injectives[6].

**Definition 5.3.** Given $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$, the induced weight structure on $\mathrm{Ind}(\mathcal{C}^{op})$ is the one constructed using Lemma 5.1 by declaring $(\mathcal{C}_{\geq 0})^{op}$ generate the negative objects.

The weight structures coming from Definition 5.3 account for essentially all weight structures on compactly generated stable categories that are useful.

**Definition 5.4.** Let $R$ be an $\mathbb{E}_1$-ring. The *standard weight structure* on $\mathrm{Mod}(R)$ is the weight structure induced from $(\mathrm{Mod}(R)^\omega)^{op} = \mathrm{Mod}^\omega_{R^{op}} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$. In otherwords, an object is $\leq 0$ if it is built out of $\Sigma^{-n}R, n \geq 0$ from extensions and filtered colimits, and an object is $> 0$ if the underlying spectrum is connected. ◁

[6] Of course, enough injectives are often used to construct $D(A)$

Recall that a $t$-structure and weight structure are *adjacent* iff their positive objects coincide. It follows that the standard weight structures and $t$-structures on the module category of a ring are adjacent iff the ring is connective. In this case, we have a characterization of the coconnective part and hearts of the weight structure, which is a consequence of Lurie's generalization of Lazard's theorem to connective rings.

**Lemma 5.5** ([Lur17, 7.2.2.15])**.** *For a connective ring $R$ with a connective module $N$, the following conditions are equivalent:*

(1) *$N$ is a filtered colimit of modules equivalent to $R^n$.*
(2) *$\pi_0 N$ is flat over $\pi_0 R$ and the map $\pi_n R \otimes_{\pi_0 R} \pi_0 N \to \pi_n R$ is an isomorphism.*
(3) *$N$ is flat, i.e $\otimes_R N : \mathrm{Mod}_{R^{op}} \to \mathrm{Sp}$ is $t$-exact.*

**Lemma 5.6.** *Suppose that $R$ is connective. Then the heart of the standard weight structure of $\mathrm{Mod}(R)$ consists of flat modules, and the objects $\leq 0$ are exactly those of tor amplitude in $[-\infty, 0]$.*

*Proof.* To see that any object $M$ that is $\leq 0$ in the weight structure is tor amplitude in $[-\infty, 0]$, observe that it is true for $\Sigma^{-n} R$, and the objects for which it is true are closed under filtered colimits and extensions.

Now let $M$ be an arbitrary object with tor amplitude in $[-\infty, 0]$. $M$ decomposes as $M' \to M \to M''$ with $M' \leq -1$ in the weight structure and $M''$ connective. $M''$ is an extension of $M$ and $\Sigma M'$, both of which are tor amplitude in $[-\infty, 0]$ so it is too. It follows from Lemma 5.5 that $M''$ is flat, so is a filtered colimit of copies of $R^n$. Thus we learn both that $M$ is $\leq 0$ in the weight structure, and that the heart of the weight structure is exactly the flat modules in the $t$-structure heart. □

We say that $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is *monoidal* if it is an associative algebra object. This is equivalent to the connective objects being closed under a monoidal structure on $\mathcal{C}$, and the unit being connective, which is exactly the condition that the $t$-structure on $\mathrm{Ind}(\mathcal{C})$ is compatible with the monoidal structure. A similar phenomenon happens for the weight structure generated by $\mathcal{C}$.

**Definition 5.7.** A weight structure on a monoidal stable category is compatible with the monoidal structure if the nonpositive objects in the weight structure are closed under the tensor product, and the unit is $\leq 0$.

Note that this is a weaker condition than the one in Sosnilo's definition [Sos21, Definition 3.2.1], which additionally requires that the nonnegative objects are closed under the monoidal structure. The discrepency between these definitions can be explained by the fact that if a compatible weight structure has a compatible adjacent $t$-structure, then the conditions of Sosnilo's definition are satisfied.

Our definition is more natural because of the following example:

**Example 5.8.** If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is monoidal, then the induced weight structure on $\mathrm{Ind}(\mathcal{C}^{op})$ is compatible with the monoidal structure in the sense that the negative weight objects are closed under the tensor product.

All interesting examples of compatible weight structures on compactly generated monoidal stable categories are explained by the above example. Note that if $\mathcal{C}$ is pivotal[7] monoidal, then the duality functor gives an equivalence $\mathcal{C} \simeq \mathcal{C}^{op}$, which allows us to interpret the

[7] A monoidal category is said to be pivotal if every object is dualizable (ie. it is rigid), and there is a chosen identification of the double dual with the identity functor.

weight structure on $\mathrm{Ind}(\mathcal{C}^{op})$ as one on $\mathrm{Ind}(\mathcal{C})$[8]. For example, the module category of an $\mathbb{E}_2$-ring $R$ is pivotal monoidal, and we get a weight structure on $\mathrm{Mod}(R)$ compatible with the monoidal structure that is exactly the standard weight structure.

Weight structures can be used to guarantee the existence of resolutions, and to bound the length of resolutions. Indeed, given an object $x \in \mathcal{C}$ such that $\mathcal{C}$ has a weight structure with $x$ bounded between $i$ and $j$ in the weight structure, we can form a filtration $w_{\leq i}x \dots w_{\leq j-2}x \to w_{\leq j-1}x \to x$ and the associated graded will show that $x$ is built out of $j-i$ objects in the weight heart. In the case of the standard and injective weight structures on the module category of a discrete ring, this recovers the fact that injective and flat resolutions of modules can terminate when the injective/tor dimension is reached.

[weight structures force finite resolutions to exist]

## 6. Regular truncated rings and Sosnilo's conjecture

Example 2.10, due to Sosnilo, gave a bounded above regular connective ring that is not coconnected. However, this example is noncommutative in an essential way, as the ring is an upper triangular matrix ring. Thus one may ask the following question, with the word 'commutative' to be broadly interpreted:

**Question 6.1.** Given a regular, bounded above, commutative ring $R$, is $R$ coconnective?

In this section, we answer this question. The following example shows that in general the kind of commutativity required must be strong.

**Example 6.2.** Consider the category of compact $\mathrm{End}_{k[x_2]}(k[x_2]/x_2^2)$-modules, Morita invariance tells us that $\mathrm{Hom}_{k[x_2]}(k[x_2]/x_2^2, -)$ induces an equivalence between the category of compact $k[x_2]$-modules on which $x_2$ acts nilpotently and left $\mathrm{End}_{k[x_2]}(k[x_2]/x_2^2)$-modules. Under this equivalence the generator of the latter category goes to $k[x_2]/x_2^2$.

Now, the category of compact $k[x_2]$-modules has a $t$-structure and when we restrict to objects on which $x_2$ acts nilpotently this becomes a bounded $t$-structure. The standard $t$-structure on $\mathrm{End}_{k[x_2]}(k[x_2]/x_2^2)$-modules restricts to this $t$-structure on compact objects. $\mathrm{End}_{k[x_2]}(k[x_2]/x_2^2)$ also has a commutative homotopy ring. $\triangleleft$

Nevertheless, if $R$ is in addition connective, a stronger form of commutativity may not be needed, as shown by the following result of Sosnilo.

**Theorem 6.3** ([Sos21, Theorem 2.2.1])**.** *Let $R$ be connective, bounded above, and regular, with $\pi_0 R$ Notherian and central in $\pi_* R$. Then $R$ is discrete.*

Sosnilo formulated the conjecture below, which is a many object version of Question 6.1 in the presense of a weight structure.

**Conjecture 6.4.** [Sos21, Conjecture 3.3.6] *If $\mathcal{C}$ is an idempotent complete symmetric monoidal stable category with compatible bounded weight structure and compatible adjacent $t$-structure, and such that the weight heart is bounded above, then the weight heart is discrete.*

This conjecture is false, as shown by the below example.

**Example 6.5.** Let $\mathcal{C}$ be the trivial square zero extension as a symmetric monoidal stable category of $\mathrm{Mod}_k^\omega$ by $\mathrm{Mod}(R)^\omega$, where $R$ is as in Example 2.10. In other words, the underlying stable category is $\mathrm{Mod}_k^\omega \times \mathrm{Mod}(R)^\omega$, and the tensor product is given by

[8] If $\mathcal{C}$ is only rigid monoidal, there are many choices of such an identification, given by odd iterations of either the left or right duality functor.

$(a, b) \otimes (c, d) = (a \otimes c, b \otimes c \oplus a \otimes d)$ using the fact that $R$ is a $k$-algebra. The sum of the standard $t$-structures gives one to $\mathcal{C}$, which is adjacent to the standard weight structures since $k$ and $R$ are connective. Moreover, it is clear that the connective objects are closed under symmetric monoidal product, but the weight heart is not discrete since $R$ is not discrete and is in the weight heart.

A property of the example above that doesn't occur in perfect module categories of commutative rings is that the object $R$ giving the counterexample is not dualizable. The perfect module category of an $\mathbb{E}_2$-ring is rigid, so if we add the word 'rigid' to Sosnilo's conjecture, it remains plausible despite the example:

**Conjecture 6.6.** *If $\mathcal{C}$ is an idempotent complete rigid symmetric monoidal stable category with compatible bounded weight structure admitting an adjacent $t$-structure, and such that the weight heart is bounded above, then the weight heart is discrete.*

We prove Conjecture 6.6 as a consequence of the more general result below that also completely answers Question 6.1. Given a monoidal category $\mathcal{C}$, let $(-)^{n*}$ be the operation of taking the iterated right dual $n$ times, when it exists. For $n = -1$ this is the left dual.

**Definition 6.7.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ be monoidal. $\mathcal{C}$ is (left/right) *t-rigid monoidal* if it is (left/right) rigid monoidal, and the (left/right) double dual functor preserves objects in $\mathcal{C}_{\geq 0}$.

Note that any pivotal monoidal category is t-rigid monoidal since the double dual functor is the identity.

We begin with the following general observations.

**Lemma 6.8.** *Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ be monoidal, and $x \geq 0$ a right dualizable object. Then for any $y \leq 0$, $y \otimes x^* \leq 0$.*

*If $x$ is in the heart, then $x^*$ is a retract of $x^* \otimes \tau_{\leq 0} \mathbb{1}$.*

*Proof.* The first claim follows from the fact that $\mathrm{Map}(z, y \otimes x^*) = \mathrm{Map}(z \otimes x, y) = 0$ when $z > 0$.

For the second claim, note that the coevaluation map $\mathbb{1} \to x \otimes x^*$ factors through $\tau_{\leq 0} \mathbb{1}$ by the first claim. Since the coevaluation map splits after left tensoring with $x^*$ by the triangle identity, so does $\mathbb{1} \to \tau_{\leq 0} \mathbb{1}$, giving the result. □

**Remark 6.9.** Whenever $x$ is in the heart, it is a retract of $x \otimes \tau_{\leq 0} \mathbb{1}$. This is because the map $x \otimes \tau_{>0} \mathbb{1} \to x$ is null, so the corresponding cofibre sequnce splits.

**Theorem 6.10.** *Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ be regular and t-rigid monoidal, and suppose $\mathbb{1}$ is bounded above in the t-structure. Then $(-)^*$ sends $\mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\leq 0}$. In particular, $\mathbb{1}$ is in the heart.*

*Proof.* Because $\mathcal{C}$ is t-rigid monoidal, taking double duals in either direction gives a self equivalence of $\mathcal{C}_{\geq 0}$. Since the double dual of the unit is itsefl, It follows that $\tau_{\leq 0} \mathbb{1}$ is its own double dual, so that the right and left duals of $\tau_{\leq 0} \mathbb{1}$ agree.

Now observe that if $\mathbb{1} \leq n$ in the $t$-structure and $x \geq 0$, then $x^* \leq n$, which follows from the equality $0 = \mathrm{Map}(y \otimes x, \mathbb{1}) = \mathrm{Map}(y, x^*)$, for $y \in \mathcal{C}_{>n}$.

It follows from Lemma 6.8 that $(\tau_{\leq 0} \mathbb{1})^* \otimes \tau_{\leq 0} \mathbb{1} = (\tau_{\leq 0} \mathbb{1})^{-1*} \otimes \tau_{\leq 0} \mathbb{1} \leq 0$, and since $(\tau_{\leq 0} \mathbb{1})^*$ is a retract of this, it is also $\leq 0$.

Choose $n \geq 0$ maximal such that $\pi_n^{\heartsuit} \mathbb{1} \neq 0$. We have a cofibre sequence $(\tau_{\leq 0} \mathbb{1})^* \to \mathbb{1} \to (\tau_{\geq 1} \mathbb{1})^*$, where $(\tau_{\geq 1} \mathbb{1})^* \leq n-1$ and $(\tau_{\leq 0} \mathbb{1})^* \leq 0$, so it follows from the long exact sequence on homotopy groups that we must have $n = 0$. □

**Remark 6.11.** The proof of Theorem 6.10 uses less than the full extent of its hypotheses. For example it works given a t-rigid monoidal structure just on the homotopy 1-category, and it is also not necessary to assume that $\mathcal{C}$ is small.

Moreover, in order to conclude that $\mathbb{1}$ is in the heart, it is only used that $\tau_{\leq 0}\,\mathbb{1}$ is dualizable and equivalent to its double dual. When this is the case, any dualizable connective object $c$ will have $c^*$ coconnective, which explains why the unit in Example 6.5 is discrete, and $R$ is not. ◁

Since the module category of an $\mathbb{E}_2$-ring is pivotal monoidal, and in particular t-rigid monnoidal, the following result follows from Theorem 6.10, answering Question 6.1.

**Corollary 6.12.** *Let $R$ be a regular bounded above $\mathbb{E}_2$-ring. Then $R$ is coconnective.*

The following corollary of Theorem 6.10 is a generalization of Conjecture 6.6.

**Theorem 6.13.** *Let $\mathcal{C}$ be a rigid monoidal stable category with compatible weight structure, and compatible adjacent t-structure such that the unit is bounded above in the t-structure. Then the weight heart is discrete.*

*Proof.* We claim that the right and left dual functors swap positive and negative objects in the weight structure. Indeed, if $X, Y$ are nonnegative in the weight structure, then $\mathrm{map}(X^*, Y) = \mathrm{map}(\mathbb{1}, X \otimes Y)$ is connective, so $X^*$ is nonpositive in the weight structure. Conversely, if $X, Z$ are nonpositive in the weight structure, then $\mathrm{map}(Z \otimes X, \mathbb{1}) = \mathrm{map}(Z, X^*)$ is connective so $X^*$ is nonnegative in the weight structure.

It follows that $\mathcal{C}$ is t-rigid monoidal, so that by applying Theorem 6.10, the dual of a connective object is $\leq 0$. But the weight heart is connective and self dual, so is in the $t$-structure heart, and hence discrete. □

## 7. Transfer of regularity

In this section, we interpret the results of [BL21] in the language developed here as a result about transferring regulariity along a map $\mathcal{C} \to \mathcal{D}$. One can consider this section as an elaboration of the contents of [BL21, Remark 3.7].

**Theorem 7.1** (Transfer of regularity)**.** *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a map in $\mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ is bounded regular. If we assume that*

(A) *$F$ is 0-affine*
(B) *$F$ is unipotent*

*then $F$ has a lift to $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ so that $\mathcal{D}$ is bounded regular, and $F$ is flat of relative dimension* 0.

*Proof.* (A) is by definition equivalent to (A) of [BL21, Theorem 1.3]. and it follows from Corollary 4.12 that (B) above is equivalent to (B) of [BL21, Theorem 1.3]. Thus we can apply that theorem along with [BL21, Proposition 2.4], to conclude. □

We now explain how the $K$-theory results can be recovered from Theorem 7.1. Since $F$ is flat, and $\mathcal{C}, \mathcal{D}$ are bounded regular, we can apply Barwick's theorem of the heart to obtain the commutative square

$$\begin{array}{ccc} K(\mathcal{C}^{\heartsuit}) & \longrightarrow & K(\mathcal{D}^{\heartsuit}) \\ \downarrow & & \downarrow \\ K(\mathcal{C}) & \longrightarrow & K(\mathcal{D}) \end{array}$$

on $K$-theory where the vertical maps are equivalences. Because $F$ is unipotent and relative dimension 0, we can apply devissage to conclude that the horizontal maps in the square are also equivalences.

Next, we see that under a stronger hypothesis, one actually obtains an equivalence on the level of hearts.

**Proposition 7.2.** *Given $F : \mathcal{C} \to \mathcal{D}$ a map in $\mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ regular, if we assume that*

(A) *$F$ is 0-affine and*
(C′) *for every $c \in \mathcal{C}^{\heartsuit}$, the cofiber of the unit map $c \to GF(c)$ is $\leq -2$ in the $t$-structure on $\mathrm{Ind}(\mathcal{C})$*

*and give $\mathcal{D}$ the $t$-structure from* [BL21, Theorem 1.3] *then the induced functor $\mathcal{C}^{\heartsuit} \to \mathcal{D}^{\heartsuit}$ is an equivalence.*

*Proof.* We begin by recalling from Lemma 4.11 that (C′) is equivalent to the condition (C) given below.

(C) *The restriction of $F$ to $\mathcal{C}^{\heartsuit}$ is fully faithful with image closed under extensions.*

Since (C) is stronger than (B), it follows that $\mathcal{D}$ has a $t$-structure such that $F$ is $t$-exact. To see that the hearts agree we observe that $F$ is fully faithful on the heart and every object of $\mathcal{D}^{\heartsuit}$ is built from (iterated) extensions of objects from $\mathcal{C}^{\heartsuit}$. Since the image of $\mathcal{C}^{\heartsuit}$ is closed under extensions by (C) we may conclude. □

As before, we give a more transparent condition when $F$ is the induction functor along the connective cover map for a coconnective ring.

**Corollary 7.3.** *Let $R$ denote a coconnective $\mathbb{E}_1$-algebra such that*

(1) *$\pi_0 R$ is left regular coherent,*
(2) *$\tau_{<0} R$ has tor amplitude in $[-\infty, -2]$ as a right $\pi_0 R$-module.*

*Then, if we equip $\mathrm{Mod}^{\omega}_{\pi_0 R}$ with its natural $t$-structure, $\mathrm{Mod}^{\omega}_{R}$ has a bounded $t$-structure such that the natural map $\pi_0 R \to R$ induces an equivalence on hearts.*

*Proof.* This is essentially the same as the proof of [BL21, Proposition 3.9] but using Proposition 7.2. □

Unlike [BL21, Theorem 1.3], the converse of Proposition 7.2 need not be true.

**Example 7.4.** Conside the diagram $k \xrightarrow{f} k[x_{-2}] \xrightarrow{g} k$ where $k$ is a field and the polynomial generator $x_{-2}$ is in degree $-2$. Applying Corollary 7.3 to $f$ we learn that $\mathrm{Mod}_k[x_{-2}]^{\omega}$ is regular with heart finite $k$ vector spaces.

Now we consider the augmentation map $g$. This map induces an equivalence of hearts, but does not satisfy (C′) since

$$\mathrm{cof}\left(\mathrm{map}_{k[x_{-2}]}(k[x_{-2}], k[x_{-2}]) \to \mathrm{map}_k(k,k)\right) \simeq \tau_{\leq 0}\Sigma k[x_{-2}]$$

which has non-trivial $\pi_{-1}$.

**Remark 7.5.** The above example can be explained by the fact that the condition really needed in the proof of Proposition 7.2 is that $\mathrm{map}(a,b) \to \mathrm{map}(Fa, Fb)$ is an equivalence on $-1$-connective covers for $a, b \in \mathcal{C}^{\heartsuit}$, which is slightly weaker than 2-ff and satisfied in the above example.

## References

[AGH19] Benjamin Antieau, David Gepner, and Jeremiah Heller. K-theoretic obstructions to bounded t-structures. *Inventiones mathematicae*, 216(1):241–300, 2019.

[BL14] Clark Barwick and Tyler Lawson. Regularity of structured ring spectra and localization in k-theory, 2014.

[BL21] Robert Burklund and Ishan Levy. On the k theory of regular coconnective rings, 2021.

[BM08] Andrew J. Blumberg and Michael A. Mandell. The localization sequence for the algebraic k-theory of topological k-theory. *Acta Mathematica*, 200(2):155–179, 2008.

[Cos77] Douglas L Costa. Retracts of polynomial rings. *Journal of Algebra*, 44(2):492–502, 1977.

[Gla89] Sarah Glaz. *Commutative coherent rings*, volume 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[Lur17] Jacob Lurie. Higher Algebra. 2017. available online.

[Lur18] Jacob Lurie. Spectral algebraic geometry. 2018. available online.

[Nee21] Amnon Neeman. A counterexample to vanishing conjectures for negative k-theory. *Inventiones mathematicae*, 225(2):427–452, Jan 2021.

[Sos21] Vladimir Sosnilo. Regularity of spectral stacks and discreteness of weight-hearts, 2021.

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