HOMOTOPY THEORY

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1. Fibrations

Definition 1.1. *We say that a map $g : A \to B$ satisfies the* ***homotopy lifting property*** *with respect to a space X, if for any homotopy $H_t : X \times [0,1] \to B$ and for any lift $h_0 : X \to A$ of H_0, h_0 can be extended to a lift of the whole homotopy. In other words, the dashed arrow h_t in the diagram below can be filled in:*

$$\begin{tikzcd} X \arrow[r, "h_0"] \arrow[d, "i"] & A \arrow[d, "g"] \\ X \times [0,1] \arrow[r, "H_t"] \arrow[ur, dashed, "h_t"] & B \end{tikzcd}$$

Definition 1.2. *A (Hurewicz)* ***fibration*** *is a map $\pi : E \to B$ that satisfies the homotopy lifting property with respect to any space.*

We call E the total space and B the base space.

Fibrations, like fibre bundles, are closed under pullbacks.

Lemma 1.3. *The pullback of a fibration along any map is a fibration.*

Proof. Use the homotopy lifting property and the universal property of pullbacks to verify that the pullback also satisfies the homotopy lifting property. □

For example, a fibre bundle is an example of a fibration. Indeed, a fibration is a weaker version of a fibre bundle in that the fibres are no longer homeomorphic, but rather homotopy equivalent.

Proposition 1.4. *The fibres of a fibration $\pi : E \to B$ over a path-connected base are homotopy equivalent.*

Proof. First note that by Lemma 1.3, by pulling back paths between two points on the base, it suffices to prove this where the $B = [0,1]$, for the fibres of 0 and 1. Indeed, let $F_i = \pi^{-1}(i)$. Then let r_t be the lift of $R_t : F_0 \times [0,1] \to [0,1]$ the projection, where r_0 is the inclusion, and similarly let $s_t : F_0 \times [0,1] \to [0,1]$ be the lift of $S_t : F_1 \times [0,1] \to [0,1]$ the projection, where s_1 is the inclusion.

Then $s_0 : F_1 \to F_0$, and $r_1 : F_0 \to F_1$ are homotopy inverses. Indeed, $s_t \circ r_{1-t}$ is a homotopy from s_0 □