CALCULATIONS IN DIFFERENTIAL GEOMETRY

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1. Introduction

Here is what to remember when doing calculations in differential geometry.

A covariant tensor is one that is a contravariant functor and vice versa (i.e. a tangent vector is contravariant and a cotangent vector is covariant). Coefficients for tangent vectors get upper indices and likewise cotangent vectors get lower indices (remember that the Riemannian metric has lower indices g_{ij}). Similarly, the coordinate tangent vectors themselves get lower indices (∂_i), while the coordinate cotangent vectors get upper indices (dx^j), as do the coordinate functions. Using Einstein notation, whenever two indices appear both on top and on bottom, they are summed over, and cancel out. Moreover partial derivatives as well as a basis for tangent.

A tensor can be thought of as a generalized matrix, and by using notation very carefully, one can easily calculate with it in this way. For example, a Riemannian metric is a matrix that takes in tangent vectors both on the left and right, so can be written as

$$dx^i g_{ij} dx^j$$

or alternatively

$$g_{ij} dx^i dx^j$$

when the side doesn't matter. The side just indicates that when thinking of it as a matrix, it takes in the j tangent vectors on the left hand side, and i tangent vectors on the right hand side.

As another example, The "signed" way to write the Christoffel symbols of a connection Γ_{ij}^k would be

$$e_k \Gamma_{ij}^k e^j dx^i$$

where e^j are the coordinates of the vector bundle, e_k are the coordinates of the dual bundle.

This convention makes it more clear what matrix multiplication means. Namely, when multiplying tensors via evaluation maps, the covariant and contravariant tensors should match up, and the upper and lower indices should match up.

For example, the condition for an affine connection to be a metric connection can be written in three equivalent ways:

$$\partial_i \langle \partial_j, \partial_k \rangle = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_i} \partial_k \rangle$$

$$\partial_i g = \Gamma_i^\top g + g \Gamma_i$$

$$\partial_i g_{jk} = \Gamma^l_{ij} g_{lk} + g_{jl} \Gamma^l_{ik}$$

The second equation is in matrix form. You can see that Γ_i needs to be transposed in the second equation when the indices go from low to high in the third. g only takes in tangent vectors, and Γ_i takes in tangent vectors from the right and cotangent vectors from the left, so indeed, must be transposed as above.

2. Formulas

Here $x \wedge y = x \otimes y - y \otimes x$.

$$d\omega(X, Y) = d(\omega(X))Y - d(\omega(Y))X - \omega[X, Y]$$