METRIC GEOMETRY

ISHAN LEVY

Definition 0.1. *A* ***metric space*** *is a set X with a distance function or metric $d : X \times X \to \mathbb{R}_{\geq 0} \cup \infty$ such that $d(a, b) = 0 \iff a = b, d(a, b) + d(b, c) \geq d(a, c), d(a, b) = d(b, a)$.*

A standard example comes from normed vector spaces. If V is a normed vector space, we can define $d(x, y) = |x-y|$. Another example is the c**-discrete metric**, where $d(x, y) = c\delta_{xy}$.

If the first condition is not satisfied we call X a **pseudo-metric space** and by identifying distance 0 objects, we can force it to become a metric space. (pseudo)-metric spaces form a category by declaring the morphisms to be those functions $f : X \to Y$ for which $d_Y \circ f \times f = d_X$, and the notion of isomorphism is called **isometry**.

Given a function $f : X \to Y$ and a metric on Y, we can pullback the metric to a pseudo-metric on X via $d^*_f(x, y) = d(f(x), f(y))$, which is called the **induced metric**. This makes the category of (pseudo-)metric spaces fibred over Set.

Lemma 0.2. *Let X be a set and let d_i be a collection of metrics on X. $\liminf d_i$ is a pseudo-metric on X and $\limsup d_i$ is a metric on X.*

Proof. Symmetry is clear, and the triangle inequality clearly passes through. □

Given a metric space (X, d) and a function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, h(0) = 0$, there is another metric d^h_{in} called the h**-intrinsic metric**. The construction is as follows: let $p, q \in X$. We will define a collection of pseudo-metrics on X d^h_ϵ depending on $\epsilon > 0$ as follows: let $d^h_\epsilon(p, q)$ be the infimum over all collections $p = x_0, x_1, \ldots, x_n = q$ such that $d(x_i, x_{i-1}) < \epsilon$ of $\sum_1^n h \circ d(x_i, x_{i-1})$. The $d^h_\epsilon(p, q)$ are an increasing sequence of metrics, and by the lemma give the metric as the lim sup. When h is the identity function, we just write d_{in} and call it the intrinsic metric. For example, if X is a C^1 embedding of the circle into $\mathbb{R}^2$, d^{in}_f will give the arc length between any two points. A function between metric spaces $f : X \to Y$ is h**-length preserving** if the h-intrinsic metric associated to the induced metric is agrees with the usual one. A metric space is h**-path coherent** if the identity map is h-length preserving. h-path coherent metric spaces along with h-length preserving maps form a category that is a reflective subcategory of metric spaces. Once again we remove the h from the notation when it is the identity function. Any metric coming from a normed vector space is path coherent, as for p, q, we can linearly interpolate between them to see that $d_\epsilon = d$ for all ϵ.

If p is a point in a metric space X, let $B_\epsilon(p)$ be the open ball of radius ϵ around p, or the collection of points of X with distance less than ϵ from p. The $B_\epsilon(p)$ form a subbasis for a topology called the topology induced by the metric. By the triangle inequality, it follows that a set is open iff around any point in it there is an open ball containing it in the set.

Often metric spaces have holes, but they can be filled in. We say that a metric is **complete** if every Cauchy sequence converges. The full subcategory of complete metric spaces has a reflection called the completion. A subspace of a complete metric space is complete iff it is closed.

Given a set U, and a collection of metrics on an open cover, given a function h as in the definition of a h-coherent metric, there is a canonical way to glue them to form a metric on U. Namely, take $d(p,q)$ to be the infimum over all sequences $p = x_0, x_1, \ldots, x_n = q$ with each x_i, x_{i-1} belonging to one of the same U_i of $\sum_i h \circ d(x_i, x_{i-1})$. Note that d^h_ϵ used to define the h-intrinsic metric is exactly this construction where the cover is all balls of radius ϵ. Note that if U'_β is a refinement of U_α, then the metric from the cover on U'_β is at least the metric from U_α, and so another possible definition of the h-induced metric would be the limit over all covers of this glued metric. However, this notion is the same. Clearly this metric is less than or equal to the h-induced metric. For the opposite inequality, fix two points p, q, and note that for any finite collection of points going from p to q, the minimum distance between any two consecutive points is positive, so for sufficiently small ϵ, $d_\epsilon(p,q)$ is at most the amount coming from this path.

Suppose X is a metric space with $d's$ image $V \subset \mathbb{R}_{\geq 0} \cup \infty$. Let f satisfy $f(\alpha x) \geq \alpha f(x)$ for $0 < \alpha < 1$, and $f(x) = 0 \iff x = 0$, and f' decreasing. Then $f \circ d$ is a metric on X. For example, for the standard metric on the unit interval, we can let f be the function $x \mapsto x^\alpha, 0 < \alpha < 1$. Also

A function $f : X \to Y$ between metric spaces is a **Lipschitz** map with dilation constant $\mathrm{dil}(f) = c$ if $d(f(x), f(y)) \leq c d(x,y)$. If the opposite inequality also holds for some constant, it is **bi-Lipschitz**. Complete metric spaces also are fully reflective in the Lipschitz category with the same reflection functor, and the unit is a bi-Lipschitz with constants 1. The notion of isomorphism in the Lipschitz category is called Lipschitz equivalence. This category is also fibred over Set. Note that any two finite valued metrics on a finite set are Lipschitz equivalent, and more generally any two finite valued metrics on a compact set that are topologically equivalent are Lipschitz equivalent.

Theorem 0.3 (Baire category theorem)**.** *A complete metric space is not the union of countably many nowhere dense subspaces.*

Proof. Let X_i be countably many nowhere dense subspaces of X. Since X_1 is nowhere dense, there is a small neighborhood $B_{\epsilon_1}(p_1)$ around a point p_1 that doesn't intersect X_1. Now inductively we can get smaller neighborhoods $B_{\epsilon_i}(p_i) \subset B_{\epsilon_{i-1}}(p_{i-1})$ that don't intersect X_i such that $\epsilon_i \to 0$. Then the p_i converge to a point not in any X_i. $\square$

The diameter of a metric space is $\sup_{x,y} d(x,y)$. If it is finite, the metric space is bounded. The distance between two subsets of a metric space is the infimum of the distance between any two points in the subsets. An ϵ-net is a set of points such that every other point is at most ϵ distance away from it. A metric space is totally bounded if for any $\epsilon > 0$ there is a finite ϵ-net.

Proposition 0.4. *A metric space is compact iff it is totally bounded and complete.*

Proof. The balls $B_\epsilon(p)$ have a finite subcover, and the centers give a finite ϵ-net. If a Cauchy sequence didn't converge it would have to have an infinite set of points. The closure would then be infinite, discrete, and compact. Conversely, if the space is totally bounded and complete, and x_i is an infinite set of points, for each ϵ there are infinitely many x_i that are ϵ away from the same point. Throw away the rest, and continue with smaller ϵ to obtain a subsequence that is Cauchy. $\square$

Lemma 0.5 (Lebesgue number lemma)**.** *Every open cover of a compact metric space has a refinement consisting of balls of size δ for some $\delta > 0$.*

Proof. Consider the function taking a point to the supremal ϵ such that B_ϵ is inside some open set in the cover. This is continuous, and achieves a non-zero minimum δ by compactness. $\square$

Lemma 0.6. *If $f : X \to Y \subset X$ is an isometry, then $Y = X$.*

Proof. If $Y \neq X$, some small ϵ ball of some point is not in the image. By continually applying f to this ball, we get infinitely many disjoint ϵ balls on our space, contradicting compactness. $\square$

Theorem 0.7. *If a function $f : X \to X$ satisfies $d \circ f \times f \geq d$ or $d \circ f \times f \leq d$ and is surjective, and X is compact, then f is an isometry. In particular any isometry of X with a subspace of X must actually be an isometry with all of X.*

Proof. In the first case, consider the maximum of $d \circ f \times f - d$, attained by $d(p, q)$

$\square$

Notion of a unit tangent cone purely in terms of the metric that works for riemannian manifolds: look at germs of distance-minimizing paths coming out of the point, put a natural pseudo-metric on it, and the quotient is the unit tangent cone.