TOPOLOGY

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1. Basics

A topological space is a set X with a collection of subsets called open sets, that contain the empty set, X and are closed under arbitrary unions and finite intersections. There is the dual notion of a closed set which is just the complement of an open set, and we say that an open set is a neighborhood of all of the subsets it contains. A continuous map between topological spaces is one for which preimages of open sets are open. In this way, topological spaces become a category, which is complete and cocomplete. This category is concrete, and the forgetful functor has a left and right adjoint, the **discrete** and **indiscrete** topologies. Moreover, the functor taking a space to its set of open sets is corepresented by the **Sierpinski space**.

Unfortunately, topological spaces can be quite pathological, so we like to look at regularity conditions that we can impose on our spaces. For example, we can look at how "separated" our space is, with the indiscrete topology being the least separated and the discrete topology being the most separated. For example, a space is **T1** if every point is closed. This is a weak condition, and topologies such as the one where the closed sets are either finite or everything (this is called the **cofinite** topology) is an example of a T1 space. A somewhat stronger condition is for a space X to be **T2** or **Hausdorff** meaning that every pair of distinct points have disjoint neighborhoods. This is the same as saying that the diagonal is closed inside $X \times X$.

We can strengthen the condition of Hausdorff to being **regular** or **T3**, meaning that every disjoint point and closed set can be separated by disjoint neighborhoods. Finally, an even stronger condition is being **normal** or **T4**, meaning that every pair of disjoint closed sets can be separated by disjoint neighborhoods.

A great example of a topology is $\mathbb{R}$. Its topology can be described in two ways. First, it can be derived from the ordering on $\mathbb{R}$, namely the set of open sets can be generated by $\{x|a < x < b\}$ for different a and b. Alternatively it can be derived from the metric on it given by $d(a, b) = |a - b|$. Metrizable spaces are normal, so $\mathbb{R}$ is as well.

An important notion is that of compactness. Namely, X is compact if every open cover has a finite subcover. This is a very abstract notion but is equivalent to being closed and bounded, if X is a subset of $\mathbb{R}^n$ (This is the Heine-Borel Theorem). Nice properties of compact spaces include being closed whenever they are subspaces of a Hausdorff space, and being sent to compact spaces via continuous maps.

Thus the unit interval $[0, 1]$ is a great example of a compact space. An nice characterization of normal spaces is that they are those in which closed sets can be separated by a continuous function to $[0, 1]$. This is the Urysohn Lemma.

Theorem 1.1 (Urysohn's Lemma)**.** *If X is normal and A, B are disjoint closed subsets, then there is a continuous function $X \to [0, 1]$ that sends A to 0 and B to 1.*

Proof. Let U_1 be open sets containing A whose closure $\overline{U}_1$ doesn't intersect B. Thus choose $U_{\frac{1}{2}}$ containing $\overline{U}_1$ whose closure doesn't intersect A. Then choose $U_{\frac{3}{4}}$ containing $\overline{U}_1$ whose closure doesn't contain the complement of $U_{\frac{1}{2}}$, and $U_{\frac{1}{4}}$ a neighborhood of $\overline{U}_{\frac{1}{2}}$ whose closure doesn't intersect A. Continuing this way, we can make our continuous function to $[0,1]$ by sending x to $\sup\{i|x \in U_i\}$. □

Theorem 1.2 (Tietze's Extension Theorem)**.** *If $A \subset X$ is a closed subset of a normal space, and $f : A \to B$ where $B = [-1,1], (-1,1)$ is a continuous function, then f can be extended to a continuous function $F : X \to B$.*

Proof. First assume $B = [-1,1]$. By the Urysohn Lemma, choose a function $F_1 : X \to [-\frac{1}{2}, \frac{1}{2}]$ that is $-\frac{1}{2}$ on $f^{-1}[-1, -\frac{1}{2}]$ and that is $\frac{1}{2}$ on $f^{-1}[\frac{1}{2}, 1]$. Then $f - F_1$ is a continuous function $A \to [-\frac{1}{2}, \frac{1}{2}]$. Repeating this process on $f - F_1$, we can get a sequence of functions $F_i : X \to [-\frac{1}{2^{i-1}}, \frac{1}{2^{i-1}}]$ such that $\sum_1^\infty F_i$ converges uniformly to a continuous function that restricts to f. Now if $B = (-1,1)$, using the Urysohn Lemma on the closed sets $f^{-1}(\{-1,1\}), A$, we can guarantee that the extension also lands inside B. □

As an application, we can prove that $\tilde{H}^p(\mathbb{R}^n - A) \cong \tilde{H}^p(\mathbb{R}^n - B)$ whenever A, B are closed homeomorphic subspaces.

Lemma 1.3. *If $\phi : A \to B$ is a homeomorphism where A and B are closed subspaces of $\mathbb{R}^n$, then viewing $\mathbb{R}^n$ inside of $\mathbb{R}^{2n}$ as the points with last n coordinates 0, ϕ extends to a homeomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

Proof. Let $\overline{\phi}$ be an extension to $\mathbb{R}^n$, $\overline{\phi^{-1}}$ be an extension of the inverse, and let $f(x,y) = (y + \overline{\phi}(x), x - \overline{\phi^{-1}}(y + \overline{\phi}(x)))$ with inverse $g(x,y) = (y + \overline{\phi^{-1}}(x), x - \overline{\phi}(y + \overline{\phi^{-1}}(x)))$. □

Lemma 1.4. *If $\mathbb{R}^n - A$ is nonempty, A closed, then for $k \geq 0$, $\tilde{H}^{p+k}(\mathbb{R}^{n+k} - A) = \tilde{H}^p(\mathbb{R}^n - A)$ for $p \geq 0$, and is 0 for $p < 0$.*

Proof. By induction it suffices to look at $\mathbb{R}^{n+1}$. Here for $p \geq 0$, use Mayer-Vietoris, and for $p = -1$ use the fact that $\mathbb{R}^{n+1} - A$ is connected. □

Theorem 1.5. *If A,B are homeomorphic closed proper subspaces of $\mathbb{R}^n$, then $\tilde{H}^p(\mathbb{R}^n - A) \cong \tilde{H}^p(\mathbb{R}^n - B)$.*

Proof. We have by the previous two lemmas $\tilde{H}^p(\mathbb{R}^n - A) \cong \tilde{H}^{n+p}(\mathbb{R}^{n+p} - A) \cong \tilde{H}^{n+p}(\mathbb{R}^{n+p} - B) \cong \tilde{H}^p(\mathbb{R}^n - B)$. □

A nice property of compact sets is Tychonoff's theorem, which says that an arbitrary product of compact sets is compact. This theorem is equivalent to Zorn's Lemma.

Lemma 1.6 (Alexander Subbase Theorem)**.** *If U_α is a subbase of X, and every cover of X with elements of U_α has a finite subcover, then X is compact.*

Proof. If X is not compact, choose a maximal counterexample, a cover B. Note that if $U \in B$ so is every open subset of U. In particular, B has a cover of basis elements, so it suffices to show for every basis element of B, there is a subbasis element of B containing it. By induction it suffices to show if $U \cap V \in B$, then U or V is in B. Indeed if not, then choose two finite subcovers of $B \cup \{U\}$ and $B \cup \{V\}$ and intersect them element-wise to obtain a contradiction. Thus B contains a subcover of subbase elements. □

Theorem 1.7 (Tychonoff)**.** *An arbitrary product of compact sets is compact.*

Proof. If $\Pi_i X_i$ is covered by subbase elements of the form $\pi_i^{-1}(U)$, there must be some j such that $\pi_j^{-1}(U_\alpha)$ is in the cover, where U_α cover X_j. Then by compactness of X_j and the Lemma 1.6 we are done. $\square$