MANIFOLDS

ISHAN LEVY

1. Introduction

A central object in topology is the **manifold**, which is a space that locally is homeomorphic to $\mathbb{R}^n$ for some n. We usually also require that a manifold be Hausdorff and second countable to avoid dealing with pathologies such as the long line or the line with two origins.

Often to make more methods available in studying manifolds as well as to make them less wild, we can require that they have additional structures. In this case, we call a manifold without any additional structure a topological manifold. An example of another type of manifold is a **smooth manifold**, which is a space that is locally diffeomorphic to $\mathbb{R}^n$. This means that the transition maps between Euclidean neighborhoods are diffeomorphisms rather than just isomorphisms. Thus smooth manifolds have natural notions of smoothness. The category of smooth manifolds is in many ways nicer than that of topological manifolds, because smooth maps are still flexible enough but cannot be too wild. For example there is a theorem of Sard which says that the critical values of a map between smooth manifolds is measure 0. Another advantage of using smooth manifolds is that one can put Riemannian metrics on them which allows one to use geometric tools to study manifolds.

Another category of manifolds is the **PL category**, where PL means "piecewise linear". Here the transition maps between Euclidean neighborhoods are piecewise linear isomorphisms. One can think of a PL-structure as a kind of triangulation, and once again PL-maps are much nicer than the generic continuous map.

Luckily, in dimensions ≤ 3, the topological, smooth, and PL categories coincide in that each topological manifold has a unique smooth and PL structure. It is generally true that every smooth manifold can be uniquely triangulated (i.e. given a PL structure) and PL and smooth are the same in dimension 4 but in general PL manifolds are not smoothable. Here we will mostly stick to smooth manifolds unless it is particularly convenient to think about triangulations.