THE COHEN-LENSTRA MOMENTS OVER FUNCTION FIELDS VIA THE STABLE HOMOLOGY OF DIHEDRAL GROUP HURWITZ SPACES

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ABSTRACT. We compute the average number of surjections from class groups of quadratic function fields over $\mathbb{F}_q(t)$ onto finite odd order groups H, once q is sufficiently large. These yield the first known moments of these class groups, as predicted by the Cohen-Lenstra heuristics, apart from the case $H = \mathbb{Z}/3\mathbb{Z}$. The key input to this result is a topological one, where we compute the stable rational homology groups of certain Hurwitz spaces associated to generalized dihedral groups.

CONTENTS

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1. INTRODUCTION

The Cohen–Lenstra heuristics, introduced by Cohen and Lenstra in [CL84], predict the distribution of the odd part of class groups of quadratic fields, and have been one of the driving conjectures in arithmetic statistics over the last four decades. Let $\mathbb{F}_q$ be a finite field with odd characteristic. If K is a quadratic extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$, and $\mathscr{O}_K$ denotes the normalization of $\mathbb{Z}$ or $\mathbb{F}_q[t]$ in K, the Cohen-Lenstra heuristics predict the distribution of the odd part of $\mathrm{Cl}(\mathscr{O}_K)$.

One of the primary approaches to determining this distribution is to compute its moments, by which we mean the average number of surjections $|\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|$, for H a finite abelian group of odd order, as K ranges over imaginary quadratic fields or real quadratic fields. As further motivation for considering these moments, it was shown in [WW21, Theorem 1.3] that the Cohen-Lenstra distribution for imaginary quadratic fields or real quadratic fields is determined by its associated moments. Hence, if one were able to compute all conjectured moments, one would prove the Cohen-Lenstra heuristics.

Although the problem of computing the average size of the ℓ-torsion (corresponding to the $\mathbb{Z}/\ell\mathbb{Z}$ moment) in the class groups of quadratic fields may appear substantially more tractable than determining the entire distribution of these class groups, the only ℓ for which this has been carried out is $\ell = 3$. This was computed over $\mathbb{Q}$ by Davenport and Heilbronn [DH71] and the analog over function fields has been verified by Datskovsky and Wright in [DW88]. To the best of our knowledge, since 1988, no additional odd order moments of class groups of quadratic fields have been computed. We note that the original Cohen-Lenstra conjectures have been extended from odd order abelian groups to all finite abelian groups by Gerth, and the moments associated to abelian 2-group in this sense have been computed by Smith [Smi22, Theorem 1.9].

Over function fields of the form $\mathbb{F}_q(t)$, significant progress was made when a weaker version of these moments were computed where one takes a large q limit in [EVW16]. But even via this approach, no moment over a fixed $\mathbb{F}_q(t)$ has been computed, or even shown to exist.

In this paper, we compute the moment associated to *any* finite order abelian group associated to class groups of quadratic extensions of $\mathbb{F}_q(t)$ over suitably large finite fields $\mathbb{F}_q$ of suitable characteristic. Our main new input which lets us accomplish this is a computation of the stable rational homology of certain Hurwitz spaces.

1.1. **Main Results.** Let q be an odd prime power and let $\mathcal{MH}_{n,q}$ denote the set of function fields K of monic smooth hyperelliptic curves of the form $y^2 = f(x)$, where $f(x)$ is a monic squarefree degree n polynomial with coefficients in $\mathbb{F}_q$. (We consider monic smooth hyperelliptic curves to be the analog of quadratic number fields in the function field case.) Let $\mathscr{O}_K$ denote the normalization of $\mathbb{F}_q[t]$ in the quadratic extension K and let $\mathrm{Cl}(\mathscr{O}_K)$ denote its class group.

Theorem 1.1.1. *Suppose H is a finite abelian group of odd order. Let q be an odd prime power with $\gcd(|H|, q(q-1)) = 1$. There is an integer C, depending only on H, so that if $q > C$ and $i \in \{0,1\}$,*

$$\lim_{\substack{n\to\infty \\ n\equiv i \bmod 2}} \frac{\sum_{K\in\mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|}{\sum_{K\in\mathcal{MH}_{n,q}} 1} = \begin{cases} 1 & \text{if } i = 1 \\ \frac{1}{|H|} & \text{if } i = 0. \end{cases} \tag{1.1}$$

Theorem 1.1.1 is the special case of Theorem 1.1.5, stated below, where we take the number h, defined there, to be 1.

Remark 1.1.2. Even though for any given H, we compute the H-moment of the class group of quadratic fields over $\mathbb{F}_q(t)$ for sufficiently large q, we do not prove the Cohen-Lenstra heuristics. Although it is true that knowledge of all moments do determine the Cohen-Lenstra distribution, for a fixed value of q, we are only able to compute moments associated to sufficiently small groups H relative to q. We do so by computing the *stable* homology of related Hurwitz spaces. If one were also able to obtain a better understanding of the *unstable* homology of these Hurwitz spaces, one might be able to verify the entire predictions of the Cohen-Lenstra heuristics over $\mathbb{F}_q(t)$.

Remark 1.1.3. Since it was geometrically more natural, we took a limit in (1.1) of the H-moments of the class group as $K \in \mathcal{MH}_{n,q}$, where K has fixed log discriminant degree n. However, from this we can easily also deduce

$$\lim_{\substack{n\to\infty \\ n\equiv i \bmod 2}} \frac{\sum_{K\in\cup_{j\le n, j\equiv n \bmod 2}\mathcal{MH}_{j,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|}{\sum_{K\in\cup_{j\le n, j\equiv n \bmod 2}\mathcal{MH}_{j,q}} 1} = \begin{cases} 1 & \text{if } i = 1 \\ \frac{1}{|H|} & \text{if } i = 0 \end{cases} \tag{1.2}$$

by summing the terms in (1.1) over all integers up to n with the same parity as n.

Remark 1.1.4. One may wonder whether the analog of Theorem 1.1.1 would hold if one averaged over all smooth hyperelliptic curves instead of just monic ones. We explain in Remark 2.1.6 why the resulting averages will be the same in the case n is odd if one takes non-monic smooth hyperelliptic curves ramified over ∞ in place of monic ones ramified over ∞. In the case that n is even, the statistics of non-monic smooth hyperelliptic curves seem somewhat more subtle. We believe it would be quite interesting and doable to work this case out in detail.

We note that Theorem 1.1.1 assumes $\gcd(q-1, |H|) = 1$, so that there are no roots of unity in the base field of order dividing $|H|$. There has been a significant amount of work toward stating and proving the correct version of the Cohen-Lenstra heuristics in the presence of such roots of unity, see [Mal08, Mal10, AM15, Gar15, LST20, SW23]. To this end, we prove the following generalization which removes the condition that $\gcd(q-1, |H|) = 1$. For G a finite abelian group and $h \in \mathbb{Z}_{>0}$, we use $G[h]$ to denote the h-torsion subgroup of G.

Theorem 1.1.5. *Suppose H is a finite abelian group of odd order. Fix q an odd prime power with $\gcd(|H|, q) = 1$ and let $h := \gcd(|H|, q-1)$. There is an integer C depending only on H so that,* *if* $q > C$ *and* $i \in \{0,1\}$,

$$\lim_{\substack{n\to\infty \\ n \equiv i \bmod 2}} \frac{\sum_{K \in \mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|}{\sum_{K \in \mathcal{MH}_{n,q}} 1} = \begin{cases} \wedge^2 H[h] & \text{if } i = 1 \\ \frac{\wedge^2 H[h]}{|H|} & \text{if } i = 0. \end{cases} \tag{1.3}$$

Theorem 1.1.5 is an immediate consequence of Theorem 5.3.1, (with the constant C^2 there renamed with C here,) which proves a refined statement about the rate of convergence of the limit.

1.1.6. In order to prove Theorem 1.1.1, we adapt the general strategy outlined in [EVW12].[1] Namely, one constructs certain Hurwitz spaces parameterizing covers of $\mathbb{P}^1$ branched at n points over $\mathbb{A}^1$. Via the Grothendieck-Lefschetz trace formula, computing the above moments amounts to obtaining a precise count of the number of $\mathbb{F}_q$ points of these Hurwitz spaces. It was shown in [EVW16] that the homology groups of these spaces stabilize in a linear range. The authors in loc. cit. used this stability to prove a weaker version of our results, which included a large q limit. In order to compute the desired moments, it suffices to compute the stable value of these homology groups. Therefore, after the reductions mentioned above, the key new result of this paper is to compute the stable homology groups of certain Hurwitz spaces, which we discuss next.

1.2. **Stable homology of Hurwitz spaces.** There are a number of situations in algebraic geometry where one may be interested in determining the stable homology or cohomology groups of a sequence of spaces. One natural example is the sequence of spaces $\{\mathscr{M}_g\}_{g\geq 2}$. In [Har85], Harer proved that the ith cohomology of $\mathscr{M}_g$ stabilizes as g grows, see also [Wah08] and [Wah13]. Later, Madsen and Weiss [MW07] computed the stable values of these cohomology groups.

As a companion to the moduli spaces of curves, Hurwitz stacks also form prominent objects of study in algebraic geometry. There are a number of equivalent viewpoints on how to think about Hurwitz stacks. If one fixes a finite group G, a conjugacy class c, and an integer n, the associated Hurwitz stack parameterizes G covers of $\mathbb{A}^1$ branched at a degree n divisor with inertia at each point of the divisor lying in c.

It is natural to ask whether the ith homology of these Hurwitz stacks also stabilizes as n grows. Ellenberg, Venkatesh, and Westerland [EVW16] showed that this is indeed the case for very special types of (G, c). These include the following case, which is relevant for the Cohen-Lenstra heuristics: $G = H \rtimes \mathbb{Z}/2\mathbb{Z}$, for H and odd order abelian group and the generator of $\mathbb{Z}/2\mathbb{Z}$ acting on H via inversion, and c the conjugacy class of order 2 elements. However, the value of these stable homology groups remained open. In this paper, we will compute this stable value of the homology of Hurwitz spaces associated to $(H \rtimes \mathbb{Z}/2\mathbb{Z}, c)$, which are relevant to the Cohen-Lenstra heuristics.

We will consider a slight variant of the above notion of Hurwitz stack, which we refer to as the pointed Hurwitz space $\mathrm{Hur}_n^{G,c}$, which we define precisely in Definition 2.1.3 and Notation 2.1.8. This variant is a finite étale cover of the Hurwitz stack described above, and involves marking a point over ∞. We briefly mention four equivalent descriptions of this space. The equivalence between the descriptions given below in §1.2.1, §1.2.2, and

[1] Before publishing the preprint [EVW12], Ellenberg, Venkatesh, and Westerland sent their paper to Oscar Randal-Williams, who identified a serious error, which the authors were unable to address. See [ell] for further details. The main result of this paper can be viewed as a resolution of that error.

§1.2.4 is shown in [EVW16, §2.3], while the equivalence between these and the description in §1.2.3 is given in [RW20, Remark 6.4].

1.2.1. *In topology as covers of the disc.* One description of our pointed Hurwitz spaces, $\mathrm{Hur}_n^{G,c}$, is that they parameterize pointed branched covers of the disc, branched at n points, where the inertia type of each branch point lies in a conjugacy class $c \subset G$.

1.2.2. *In homotopy theory as an orbit space.* The pointed Hurwitz space $\mathrm{Hur}_n^{G,c}$ can also be described as the homotopy quotient $(c^n)_{hB_n}$, where B_n is the braid group on n strands.

1.2.3. *In homotopy theory as a free $\mathbb{E}_2$ algebra.* Conjugation by G gives a G-action on $\mathrm{Hur}_n^{G,c}$, and $\coprod_{n \geq 0} \mathrm{Hur}_n^{G,c}$ can also be described as the free $\mathbb{E}_2$-algebra generated by c in the category of G-crossed spaces, see [RW20, Remark 6.4].

1.2.4. *In algebraic geometry as a moduli space.* Algebro-geometrically, we can also think of complex valued points of $\mathrm{Hur}_n^{G,c}$ as parameterizing G-covers $X \to \mathbb{P}^1_{\mathbb{C}}$, branched at a degree n divisor in $\mathbb{A}^1_{\mathbb{C}}$ with inertia at these branch points lying in $c \subset G$ and with a marked point over $\infty \in \mathbb{P}^1_{\mathbb{C}}$.[2] See Definition 2.1.3 for a formal algebro-geometric definition of these pointed Hurwitz spaces. We call them pointed because they involve making a choice of marked point of the cover X over ∞.

Notation 1.2.5. Let Conf_n denote the configuration space of n unordered, distinct points in $\mathbb{A}^1_{\mathbb{C}}$. There is a natural map $\mathrm{Hur}_n^{G,c} \to \mathrm{Conf}_n$ which sends a branched cover $X \to \mathbb{P}^1$ to the intersection of its branch locus with $\mathbb{A}^1$.

We will use the following notation for our main homological stability theorem.

Notation 1.2.6. Let H be an odd order abelian group and let the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ act on H via inversion. Take $G := H \rtimes \mathbb{Z}/2\mathbb{Z}$ and $c \subset G$ to be the conjugacy class of order 2 elements, which is a single conjugacy class by Sylow's theorem.

Theorem 1.2.7. *With notation as in Notation 1.2.6, there are constants I and J depending only on G so that for any $i \geq 0$ and $n > iI + J$ and any connected component $Z \subset \mathrm{Hur}_{n,\mathbb{C}}^{G,c}$, the map $H_i(Z;\mathbb{Q}) \to H_i(\mathrm{Conf}_n;\mathbb{Q})$ is an isomorphism.*

We prove Theorem 1.2.7 in §4.2.3.

Remark 1.2.8. Theorem 1.2.7 is known to holds for certain special components Z by work of Bianchi and Miller [BM23, Corollary C′] and the text following it. Via personal communication, we learned this argument was also known to Ellenberg-Venkatesh-Westerland. In particular, this applies when G is the dihedral group of order $2n$, for n odd, to components Z whose boundary monodromy (see Definition 2.2.1) is a generator of H. However, it does not apply, for example, when G is the dihedral group of order $2n$ and the boundary monodromy of Z is trivial.

Remark 1.2.9. The subtlety in understanding homological stability for $\mathrm{Hur}_n^{G,c}$ (which includes points corresponding to disconnected covers) is that it does not stabilize with respect to the obvious operator $V \in H_0(\mathrm{Hur}^{G,c}_{\mathrm{ord}(c)\cdot|c|,\mathbb{C}};\mathbb{Q})$ corresponding to $\prod_{g \in c}[g]^{\mathrm{ord}(g)}$.

[2] When the monodromy at ∞ is nontrivial of order r, we need to replace $\mathbb{P}^1_{\mathbb{C}}$ with a root stack of order r at ∞ to make this technically correct.

Rather, [EVW16] show that the operator $\sum_{g \in c}[g]^{\mathrm{ord}(g)}$ stabilizes the homology, which is not an operation that exists at the level of spaces. However on the subspace $\mathrm{CHur}_n^{G,c} \subset \mathrm{Hur}_{n,\mathbb{C}}^{G,c}$ corresponding to G-covers that are connected, it follows from Theorem 1.2.7 that for the G, c in Theorem 1.2.7, V does stabilize the homology, and that the stable value on each component agrees with that of Conf_n. Oscar Randal-Williams mentions in his Bourbaki survey article that "homological stability of the spaces $\mathrm{CHur}_n^{G,c}$ with respect to the maps V should serve as a guiding problem for mathematicians working in this subject." [RW20, p. 24]

1.3. **The stable homology for S_3 and degree 3 covers.** Before proceeding further, we pause to highlight a seemingly elementary case of Theorem 1.2.7 which, surprisingly, was previously unknown. Consider the special case that $G = S_3$, and the conjugacy class $c \subset G$ consists of transpositions. The pointed Hurwitz space $\mathrm{CHur}_n^{G,c}$ associated to this parameterizes connected degree 3 simply branched covers of $\mathbb{P}^1$, branched over a degree n divisor in $\mathbb{A}^1$, see Notation 2.1.7 for a more formal definition. Simply branched triple covers have been studied extensively in algebraic geometry, and it is natural to ask about the homology of the space of such covers as the number of branch points grows.

Corollary 1.3.1. *There are constants I and J so that for any $i \geq 0$ and $n > Ii + J$*

$$h^i([\mathrm{CHur}_n^{S_3,\{(12),(13),(23)\}}/S_3]; \mathbb{Q}) = \begin{cases} 1 & \text{if } i \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.3.2. If one considers the space of all smooth trigonal curves, instead of just simply branched ones, the stable cohomology of such spaces has been computed in [Zhe24] using Vassiliev's method See [Tom05], [Gor05], and [Vas99] for references on Vassiliev's method. However, via personal communication, we learned that it is unclear whether a similar method can be used to compute the stable cohomology of the space of simply branched trigonal curves.

Remark 1.3.3. There is a cycle class map from the Chow ring of these Hurwitz spaces to the cohomology of these Hurwitz spaces. The same explicit parameterizations which allowed Davenport and Heilbronn [DH71] to compute the number of cubic fields allowed Patel and Vakil to compute the stable Chow rings of these Hurwitz spaces [PV15, Theorem C], see also [CL22]. However, such a parameterization appears to be of little use for computing the stable homology.

1.4. **Conjectures on stable homology of Hurwitz spaces.** Before proceeding to describe the new ideas in the proof of Theorem 1.2.7, we give a conjecture for what we think the stable homology of Hurwitz spaces should look like in general.

Notation 1.4.1. To set up notation, let G be a finite group and $c \subset G$ denote a union of conjugacy classes. That is, $c = c_1 \cup \cdots c_k$, where each $c_i \subset G$ is a conjugacy class. Let $\mathrm{CHur}_{n_1,\ldots,n_k}^{G,c_1,\ldots,c_k}$ denote the Hurwitz space parameterizing connected G covers of $\mathbb{A}^1_{\mathbb{C}}$ with a trivialization at ∞ and n_i branch points over $\mathbb{A}^1_{\mathbb{C}}$ with inertia in the conjugacy class of any element of c_i. We also let $\mathrm{Conf}_{n_1,\ldots,n_k}$ denote the multi-colored configuration space in $\mathbb{A}^1_{\mathbb{C}}$ parameterizing n_i points of color i so that all $n_1 + \cdots + n_k$ points in $\mathbb{A}^1_{\mathbb{C}}$ are distinct. That is, $\mathrm{Conf}_{n_1,\ldots,n_k}$ is the quotient of ordered configuration space of $n_1 + \cdots + n_k$ points by the action of $S_{n_1} \times \cdots \times S_{n_k}$, where S_{n_i} permutes points $n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_i$.

Remark 1.4.2. One might require a weaker notion that c is only conjugation invariant, in the sense that for any $x, y \in c$, we also have $x^{-1}yx \in c$. However, if c generates G, such a subset must actually be a union of conjugacy classes. In general, without this assumption, such a conjugation invariant subset can be viewed as a union of conjugacy classes in the subgroup it generates.

Remark 1.4.3. Because there will not be any *connected* G covers with inertia in c unless c generates G, using notation from Notation 1.4.1, $\mathrm{CHur}^{G,c_1,\ldots,c_k}_{n_1,\ldots,n_k}$ will be the empty set unless the conjugacy classes $c_1, \ldots, c_k$ jointly generate G.

Our main conjecture on the stable homology of Hurwitz spaces is the following, which predicts that the only stable homology is the "obvious" stable homology.

Conjecture 1.4.4. Fix an integer i. With notation as in Notation 1.4.1, suppose all $n_1, \ldots, n_k$ are sufficiently large (with how large they must be depending on i) and let $Z \subset \mathrm{CHur}^{G,c_1,\ldots,c_k}_{n_1,\ldots,n_k}$ denote a connected component. Then, the natural map sending a cover to its branch locus $Z \to \mathrm{Conf}_{n_1,\ldots,n_k}$ induces an isomorphism $H_i(Z;\mathbb{Q}) \to H_i(\mathrm{Conf}_{n_1,\ldots,n_k};\mathbb{Q})$.

The special case of the above conjecture where $G = S_d$, $k = 1$, and c_1 is the conjugacy class of transpositions is given in [EVW16, Conjecture 1.5]. The above conjecture was suggested to us by Melanie Wood, who in turn pointed out to us that a statement very closely related to this conjecture was formulated in [EVW12, Corollary 5.8].

Remark 1.4.5. Admittedly, the evidence for Conjecture 1.4.4 in the literature is somewhat limited. The case that G is abelian is fairly trivial, since in that case, the relevant Hurwitz space of G-covers is precisely identified with the corresponding configuration space: indeed, in the case G is abelian, all elements correspond to distinct conjugacy classes, so one can read off the total monodromy of the cover from the conjugacy classes of the inertia and the locations of the branch points, and therefore recover the cover. To our knowledge, the only nonabelian cases of this conjecture which are known are those in the present paper, namely those verified in Theorem 1.2.7. However, this conjecture is largely motivated by Malle's conjecture on counting G extensions of number fields. See [VE10] for a detailed description of this connection.

Remark 1.4.6. The authors are currently working on proving additional cases of Conjecture 1.4.4.

Remark 1.4.7. Note that we make Conjecture 1.4.4 for arbitrary $G, c_1, \ldots, c_k$ even though it is not known whether the homology of Hurwitz spaces stabilize at all, except in the case that $k = 1$ and c_1 satisfies a certain non-splitting condition as in [EVW16, Definition 3.1].

We also ask whether the analogous statement could possibly also hold for Chow groups of Hurwitz spaces.

Question 1.4.8. Fix an integer i. With notation as in Notation 1.4.1, suppose all $n_1, \ldots, n_k$ are sufficiently large (with how large they must be depending on i). Do the ith rational Chow groups of $\mathrm{CHur}^{G,c_1,\ldots,c_k}_{n_1,\ldots,n_k}$ stabilize?

Even more ambitiously, we ask if the Chow rings could stabilize to those of configuration space.

Question 1.4.9. In the setting of Question 1.4.8, let $Z \subset \mathrm{CHur}^{G,c_1,\ldots,c_k}_{n_1,\ldots,n_k}$ denote a connected component. Does the natural map $Z \to \mathrm{Conf}_{n_1,\ldots,n_k}$ sending a cover to its branch locus induce an isomorphism on ith rational Chow groups?

1.5. **Idea of the Proof.** We now describe the new ideas going into our proof of Theorem 1.1.1, and more generally Theorem 1.1.5, the computation of the moments predicted by the Cohen-Lenstra conjectures over function fields. A standard reduction also outlined in §1.1.6, which we carry out in §5, reduces us to proving Theorem 1.2.7.

We next explain the idea of the proof of Theorem 1.2.7, which computes the stable value of the homology for certain Hurwitz spaces. We fix (G, c) as in Theorem 1.2.7. We use $\mathrm{Hur}^{G,c}$ to denote the union over n of the Hurwitz spaces $\mathrm{Hur}^{G,c}_n$ and Conf to denote the union over n of Conf_n. We use $\mathrm{Conf}^{G,c}$ to denote $\mathrm{Conf} \times_{\pi_0 \mathrm{Conf}} \pi_0 \mathrm{Hur}^{G,c}$; in other words, it has the same set of components as $\mathrm{Hur}^{G,c}$, but we replace each component of $\mathrm{Hur}^{G,c}_n$ with Conf_n. There is a projection $v : \mathrm{Hur}^{G,c} \to \mathrm{Conf}^{G,c}$, which can be thought of as sending a cover to its branch locus, together with the data of which component the cover came from. If we use U to denote the stabilization map $\mathrm{Hur}^{G,c}$ (the map under which the homology of Hurwitz space was shown to stabilize in [EVW16, Theorem 6.1]) we can rephrase our goal as showing $v[U^{-1}] : H_*(\mathrm{Hur}^{G,c}; \mathbb{Q})[U^{-1}] \to H_*(\mathrm{Conf}^{G,c}; \mathbb{Q})[U^{-1}]$ is an equivalence.

Considering $\mathrm{Hur}^{G,c}$ and $\mathrm{Conf}^{G,c}$ as associative monoids (or $\mathbb{E}_1$-algebras), our goal is equivalent to showing that the map $v[U^{-1}] : C_*(\mathrm{Hur}^{G,c}; \mathbb{Q})[U^{-1}] \to C_*(\mathrm{Conf}^{G,c}; \mathbb{Q})[U^{-1}]$ is an equivalence of associative ring spectra[3], where $C_*(X; \mathbb{Q})$ denotes the chains of a space X with $\mathbb{Q}$-coefficients, viewed as either a spectrum, or in the derived category of $\mathbb{Q}$-vector spaces.

We can view the center $Z(R)$ of $R := H_0(\mathrm{Hur}^{G,c}; \mathbb{Q})$ as a commutative ring acting on the source and target of the map v, and we study the map $v[U^{-1}]$ via decomposing it geometrically along open subsets covering $\mathrm{Spec}(Z(R))$. It then suffices to show that locally on each of these subsets the map $v[U^{-1}]$ becomes an equivalence. Algebraically, the way we implement restricting to an open subset of $\mathrm{Spec}(Z(R))$ is by using localizations in the setting of ring spectra (such as in [Lur17, Section 7.2.3]), and completions along a collection of elements, which we show in our case happens to also be a localization. Combining this strategy with a group theoretic argument shows that $v[U^{-1}]$ is an equivalence if v is an equivalence after inverting $\langle c' \rangle$, and then completing at $c - c'$, for every subset $c' \subset c$ closed under conjugation (i.e., $x, y \in c' \implies x^{-1}yx \in c'$).

When $c' = c$, this localized map was already known to be an equivalence. This is because inverting all elements of c gives the group completion of $\mathrm{Hur}^{G,c}$, whose homology can be understood via its classifying space $B\,\mathrm{Hur}^{G,c}$. The homology of the classifying space of is the rack homology of the rack c, which was computed in [EG03], see [RW20, Corollary 5.4].

When c' is a proper subset of c, the key notion helping us show the map becomes an equivalence is the notion of a homological epimorphism. A map $R \to S$ of $\mathbb{E}_1$-rings is called a homological epimorphism if the multiplication map $S \otimes_R S \to S$ is an equivalence in the ∞-category of spectra. A key result we use is a rigidity property of homological

[3] Since we are working over $\mathbb{Q}$, these associative ring spectra can be viewed as the dga computing the homology with its multiplication given via the Pontryagin product.

epimorphisms, which is that a homological epimorphism of connective associative ring spectra that is an isomorphism on π_0 is an equivalence. Applying the above results to the subgroup $\langle c' \rangle \subset G$ generated by c', we can reduce to showing that a certain 'restriction map' between the localized homology of Hurwitz spaces is a homological epimorphism. This fact is something that can be verified at the level of pointed spaces, by showing that a certain map between two sided bar constructions is a homotopy equivalence. We use an convenient topological model for these two sided bar constructions, the details of which are carried out in Appendix A. Using this topological model, we write down an explicit homotopy to prove this homotopy equivalence; see Figure 1 for a pictorial depiction.

Remark 1.5.1. It is also possible to prove Theorem 1.2.7 without the machinery of higher algebra. We originally came up with a more elementary argument but found that by translating things to higher algebra, the argument became significantly cleaner and more conceptual.

1.6. **Outline.** The sections of this paper are organized as follows. First, we set up our notation for Hurwitz spaces in §2. In §3, we introduce notation from higher algebra relevant to our paper, and prove some facts about homological epimorphisms, which will be crucial to our main results. In §4, we prove our main topological result, Theorem 1.2.7, computing the stable homology of certain Hurwitz spaces. We in fact compute the stable homology more generally of Hurwitz spaces associated to racks satisfying certain hypotheses. In §5, we prove our main result toward the Cohen-Lenstra heuristics, Theorem 1.1.5, by deducing it from our topological results about the stable homology of Hurwitz spaces. We conclude with Appendix A, which uses various scanning arguments to construct convenient topological models for certain bar constructions on Hurwitz spaces.

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2. NOTATION FOR HURWITZ SPACES

2.1. **The definition of Hurwitz spaces.** We begin by carefully defining the Hurwitz spaces we will work with. We start by defining the Hurwitz stack $[\mathrm{Hur}^{G,c}_{n,B}/G]$. This will be a quotient by an action of G of the pointed Hurwitz space $\mathrm{Hur}^{G,c}_{n,B}$, which we will define next, and this explains why there is a quotient by G in the notation for $[\mathrm{Hur}^{G,c}_{n,B}/G]$.

Definition 2.1.1. Let G be a finite group, $c \subset G$ be a union of conjugacy classes, and B be a scheme with $|G|$ invertible on B. Let $[\mathrm{Hur}^{G,c}_{n,B}/G]$ denote the *Hurwitz stack* whose T points $[\mathrm{Hur}^{G,c}_{n,B}/G](T)$ is the groupoid

$$\left(D, i : D \to \mathbb{P}^1_T, X, h : X \to \mathbb{P}^1_T\right)$$

satisfying the following conditions:

(1) D is a finite étale cover of T of degree n.
(2) i is a closed immersion $i : D \subset \mathbb{A}^1_T \subset \mathbb{P}^1_T$.
(3) X is a smooth proper relative curve over T, not necessarily having geometrically connected fibers.
(4) $h : X \to \mathbb{P}^1_T$ is a finite locally free Galois G-cover, (meaning that G acts simply transitively on the geometric generic fiber of h,) which is étale away from $\infty_T \cup i(D) \subset \mathbb{P}^1_T$.
(5) The inertia of $X \to \mathbb{P}^1_T$ over any geometric point of $i(D)$ lies in $c \subset G$.
(6) Two such covers are considered equivalent if they are related by the G-conjugation action.
(7) The morphisms between two points (D_i, i_i, X_i, h_i) for $i \in \{1, 2\}$ are given by (ϕ_D, ψ_X) where $\phi_D : D_1 \simeq D_2$ is an isomorphism so that $i_2 \circ \phi_D = i_2$ and $\psi_X : X_1 \simeq X_2$ is an isomorphism such $h_2 \circ \psi_X = h_1$ and $\psi_X = g^{-1}\psi_X g$ for every $g \in G$.

Remark 2.1.2. The definition of Hurwitz spaces we give here is a special case of the definition in [EL23, Definition 2.4.2]. See [EL23, Remark 2.4.3] for an explanation of why these Hurwitz stacks are indeed stacks. It essentially follows from [ACV03, §1.3.2 and Appendix B].

One important idea in this paper is to work with pointed Hurwitz spaces, which deal with the case of "real quadratic" or "ramified at ∞" quadratic fields. Variants of these were also used in [EL23], but prior work in the context of the Cohen-Lenstra heuristics appears to primarily focus on the case over covers unramified at ∞. These Hurwitz spaces parameterize covers of a stacky $\mathbb{P}^1$, with a root stack at ∞ of order 2. We now define these pointed Hurwitz spaces.

Definition 2.1.3. Fix an integer w and define $\mathscr{P}^w$ to be the root stack of order w along ∞ of $\mathbb{P}^1$. (See [Cad07, Definition 2.2.4], where this is notated as $\mathbb{P}^1_{(\infty,w)}$.) The fiber of this root stack over ∞ is the stack quotient $[(\operatorname{Spec}_B \mathscr{O}_B[x]/(x^r))\,/\mu_r]$ of the relative spectrum $\operatorname{Spec}_B \mathscr{O}_B[x]/(x^r)$ by μ_r. Let $\widetilde{\infty} : B \to \mathscr{P}^w$ denote the section over σ corresponding to map $B \to [(\operatorname{Spec}_B \mathscr{O}_B[x]/(x^r))\,/\mu_r]$ given by the trivial μ_r torsor over B, $\mu_r \to B$, and the μ_r equivariant map $\mu_r \to B \to \operatorname{Spec}_B \mathscr{O}_B[x]/(x^r)$. We use notation as in Definition 2.1.1,

Define the *w-pointed Hurwitz space*, $\left(\mathrm{Hur}^{G,c}_{n,B}\right)^w$, to be the algebraic space whose T points are the set parameterizing data of the form

$$\left(D, h' : X \to \mathscr{P}^w_T, t : T \to X \times_{h', \mathscr{P}^w_T, \widetilde{\infty}_T} T, i : D \to \mathbb{P}^1_T, X, h : X \to \mathbb{P}^1_T\right),$$

where D, i, X, and h satisfy the properties listed in Definition 2.1.1. We also assume the order of inertia of h along ∞ is w and define $\widetilde{\infty}_T$ to be the base change of the section $\widetilde{\infty}$ defined above to T. We additionally impose the condition that h' is a finite locally free G-cover, étale over $\widetilde{\infty}$, such that the composition of $h' : X \to \mathscr{P}^w_T$ with the coarse space map $\mathscr{P}^w_T \to \mathbb{P}^1_T$ is h, and $t : T \to X \times_{h', \mathscr{P}^w_T, \widetilde{\infty}_T} T$ is a section of h' over $\widetilde{\infty}$.

In general, we define the *pointed Hurwitz space* as $\mathrm{Hur}^{G,c}_{n,B} := \coprod_{w \geq 1} \left(\mathrm{Hur}^{G,c}_{n,B}\right)^w$.

Remark 2.1.4. We note that $\mathrm{Hur}^{G,c}_{n,B}$, defined as an algebraic space, is in fact a scheme. Indeed, $\mathrm{Hur}^{G,c}_{n,B}$ is a finite étale cover of the configuration space of n unordered points in $\mathbb{A}^1_B$, as can be verified in an analogous fashion to [LWZB24, Proposition 11.4].

Remark 2.1.5. In the definition of pointed Hurwitz space, Definition 2.1.3, we choose a particular section $\widetilde{\infty} : B \to \mathscr{P}^w$ over $\infty : B \to \mathscr{P}^1_B$. However, one can show that if we chose a different section $\widetilde{\infty}' : B \to \mathscr{P}^w$, the resulting pointed Hurwitz space would be isomorphic. The idea is that the sections $\widetilde{\infty}$ and $\widetilde{\infty}'$ factor respectively through maps $\alpha : B \to [B/\mu_w]$ and $\alpha' : B \to [B/\mu_w]$, and so it suffices to produce a map $\beta : [B/\mu_w] \to [B/\mu_w]$ so that $\alpha \circ \beta = \alpha'$, as this map will induce a map taking $\widetilde{\infty}$ to $\widetilde{\infty}'$, which then induces an isomorphism of these two Hurwitz spaces. The desired map β is simply given as the composition of the structure map $[B/\mu_w] \to B$ with α'.

Remark 2.1.6. Note that monic smooth hyperelliptic curves $y^2 = f(x)$ with f of odd degree correspond to the choice of marked section given in Definition 2.1.3 while non-monic smooth hyperelliptic curves $y^2 = f(x)$ with f of with the same odd degree correspond to a different choice of marked section, see Lemma 2.3.2. Hence, the two relevant Hurwitz spaces will be isomorphic by Remark 2.1.5. From this, one can obtain the fact that the distributions of class groups of monic smooth hyperelliptic curves of odd degree are isomorphic to the distribution of class groups of non-monic smooth hyperelliptic curves of odd degree. This was mentioned in Remark 1.1.4.

There is also a more elementary way to see this equivalence, as we learned from [BW17, Corollary 6.5]. This can be viewed as a reformulation of the argument in the previous paragraph. Namely, there is a bijection between monic smooth hyperelliptic curves and non-monic smooth hyperelliptic curves which additionally preserves their isomorphism type, and hence their class groups. For any $\alpha \in \mathbb{F}_q - (\mathbb{F}_q)^2$ every non-monic smooth hyperelliptic curve can be written as $y^2 = f(x)$, where $f(x)$ has leading coefficient α. Using this representation of such non-monic smooth hyperelliptic curves, the map is given by sending a quadratic extension corresponding to a monic smooth hyperelliptic curve K over $\mathbb{F}_q(t)$ to the quadratic extension corresponding to a non-monic smooth hyperelliptic curve $K \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(s)$ where $t \mapsto \alpha s$.

Notation 2.1.7. We use $\mathrm{CHur}^{G,c}_{n,B}$ to denote the open and closed subscheme of $\mathrm{Hur}^{G,c}_{n,B}$ parameterizing covers $X \to \mathbb{P}^1_T$ so that the geometric fibers of X over T are connected. We use $[\mathrm{CHur}^{G,c}_{n,B}/G]$ to denote the open and closed substack of $[\mathrm{Hur}^{G,c}_{n,B}/G]$ parameterizing covers $X \to \mathbb{P}^1_T$ so that the geometric fibers of X over T are connected.

Notation 2.1.8. We will often work over the complex numbers, and hence it will be convenient to assume $B = \operatorname{Spec} \mathbb{C}$. Therefore, we define $[\mathrm{Hur}^{G,c}_{n}/G] := [\mathrm{Hur}^{G,c}_{n,\operatorname{Spec}\mathbb{C}}/G]$, $\mathrm{Hur}^{G,c}_{n} := \mathrm{Hur}^{G,c}_{n,\operatorname{Spec}\mathbb{C}}$, $[\mathrm{CHur}^{G,c}_{n}/G] := [\mathrm{CHur}^{G,c}_{n,\operatorname{Spec}\mathbb{C}}/G]$, $\mathrm{CHur}^{G,c}_{n} := \mathrm{CHur}^{G,c}_{n,\operatorname{Spec}\mathbb{C}}$. When R is a ring, we often use use $\mathrm{Hur}^{G,c}_{n,R}$ in place of $\mathrm{Hur}^{G,c}_{n,\operatorname{Spec} R}$, and use analogous notation for other variants of this Hurwitz space.

2.2. **Notation for components of Hurwitz spaces.** We now introduce some notation for the Hurwitz spaces we will work with. First, we describe the notion of boundary monodromy of a component. If we think of the component as parameterizing certain covers $X \to \mathbb{P}^1$, the boundary monodromy corresponds to the inertia of the cover over ∞.

Definition 2.2.1. With notation as in Definition 2.1.3, there is a map from the set of components of $\mathrm{Hur}^{G,c}_{n,B}$ to G defined as follows. We can represent connected components of $\mathrm{Hur}^{G,c}_{n,B}$ as tuples $(g_1, \dots, g_n)$ modulo the action of the braid group B_n, and we associate to this the element $g_1 \cdots g_n \in G$. For such a component mapping to $g \in G$, we say the *boundary monodromy* of this component is g. We define $\mathrm{CHur}^{G,c,g}_{n,B}$ denote the open and closed subscheme of $\mathrm{CHur}^{G,c}_{n,B}$ consisting of those components whose boundary monodromy is g.

2.3. Covers split completely at infinity.

Definition 2.3.1. Let $X \to \mathbb{P}^1_{\mathbb{F}_q}$ be a Galois G cover with branch divisor of degree n and inertia over $\mathbb{A}^1_{\mathbb{F}_q}$ all lying in a union of conjugacy classes $c \subset G$. Hence, this cover corresponds to a point $\operatorname{Spec} \mathbb{F}_q \to [\mathrm{Hur}^{G,c}_{n,\mathbb{F}_q}/G]$. We say this cover is *completely split over* ∞ if it is in the image of the map $\mathrm{Hur}^{G,c}_{n,\mathbb{F}_q}(\mathbb{F}_q) \to [\mathrm{Hur}^{G,c}_{n,\mathbb{F}_q}/G](\mathbb{F}_q)$.

Next, we wish to verify that monic smooth hyperelliptic curves precisely correspond to smooth covers split over infinity. This is fairly straightforward in the case n is even. It is a bit more involved when n is odd, but we are able to verify this via an explicit blow-up computation. For the next lemma, recall that a monic smooth hyperelliptic curve over $\mathbb{F}_q$ is by definition a hyperelliptic curve defined affine locally by an equation of the form $y^2 = f(x)$ with $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for $a_i \in \mathbb{F}_q$. Explicitly, we view the hyperelliptic curve as a cover of $\mathbb{P}^1_{\mathbb{F}_q}$ with x as above the coordinate on $\mathbb{A}^1_{\mathbb{F}_q}$, viewed as a subscheme of $\mathbb{P}^1 = \mathbb{P}(\mathbb{F}_q[X,Y])$ with $x = X/Y$.

Lemma 2.3.2. *Take $G = \mathbb{Z}/2\mathbb{Z}$ and c to be the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ in Definition 2.3.1. The $\mathbb{Z}/2\mathbb{Z}$ smooth covers of $\mathbb{P}^1_{\mathbb{F}_q}$ split completely over the point $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$ precisely correspond to monic smooth hyperelliptic curves $y^2 = f(x)$ over $\mathbb{F}_q$.*

Proof. First, we describe the case that n is even. Any hyperelliptic curve in characteristic not 2 which is unramified over ∞ can be uniquely written in the form $f(x) = \alpha x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where either $\alpha = 1$ or $\alpha \in \mathbb{F}_q^\times$ is a fixed non-square. Such a curve is split completely over infinity if and only if its fiber over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_q)$ consists of two $\operatorname{Spec} \mathbb{F}_q$ points. The fiber over ∞ of this curve can be identified with the fiber over 0 of the curve $z^2 = \alpha + a_{n-1}u + \cdots + a_1 u^{n-1} + a_0 u^n$, where we have set $u = 1/x$ and then $z = y/x^{n/2}$. This is the same as $\operatorname{Spec} \mathbb{F}_q[y]/(y^2 - \alpha)$, which will be two copies of $\mathbb{F}_q$ when α is a square and a single copy of $\mathbb{F}_q^2$ when α is a non-square. Therefore, the curves where α is a square (which are all isomorphic to such curves with $\alpha = 1$) precisely correspond to those completely split over ∞.

We next consider the case that n is odd. As above, we can still write our hyperelliptic curve in the form $f(x) = \alpha x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with α either 1 or a fixed non-square. Changing coordinates to $u = 1/x$, we can write the curve in the form $z^2 = u(\alpha + a_{n-1}u + \cdots + a_1 u^{n-1} + a_0 u^n)$ where $u = 1/x$ and $z = y/x^{n+1/2}$. Let X denote the proper regular curve corresponding to the above equation. We can identify the stack quotient $[X/(\mathbb{Z}/2\mathbb{Z})]$ Zariski locally around the residual gerbe at ∞ with a Zariski open of the root stack $\mathscr{P}$, which is a root stack of $\mathbb{P}^1$ over the divisor ∞, so $\mathscr{P}$ has a single stacky point of order 2.

Let $\widetilde{\infty} : \operatorname{Spec} \mathbb{F}_q \to \mathscr{P}$ denote the map which factors through the residual gerbe $B\mu_2$ at infinity and corresponds to the map $\operatorname{Spec} \mathbb{F}_q \to B\mu_2$ associated to the trivial double cover $\operatorname{Spec} \mathbb{F}_q \coprod \operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathbb{F}_q$. We wish to show that the fiber product $X \times_{\mathscr{P},\infty} \operatorname{Spec} \mathbb{F}_q$ consists of two copies of $\operatorname{Spec} \mathbb{F}_q$. The fiber product above is identified with the fiber product $X \times_{\mathscr{P}} \mathbb{P}^1 \times_{\mathbb{P}^1,\infty} \operatorname{Spec} \mathbb{F}_q$, where the map $\mathbb{P}^1 \to \mathscr{P}$ is the double cover branched over 0. (In particular, on coarse spaces $\mathbb{P}^1 \to \mathscr{P}$ corresponds to the map given in local coordinates by $x \mapsto x^2$.) The above fiber product $X \times_{\mathscr{P}} \mathbb{P}^1$ is a normal curve in a neighborhood over ∞, and hence it is, locally around ∞, the normalization of $X \times_{\mathbb{P}^1_u} \mathbb{P}^1_t$, where again the map $\mathbb{P}^1_t \to \mathbb{P}^1_u$ is induced by the double cover $u \mapsto t^2$. In a neighborhood of ∞, we can express the projection map $X \times_{\mathbb{P}^1_u} \mathbb{P}^1_t \to X$ as

$$\begin{aligned} &\operatorname{Spec} k[u,z,t]/(t^2 - u, z^2 - u(\alpha + a_{n-1}u + \cdots + a_1 u^{n-1} + a_0 u^n) \\ &\to \operatorname{Spec} k[u,z]/(z^2 - u(\alpha + a_{n-1}u + \cdots + a_1 u^{n-1} + a_0 u^n). \end{aligned}$$

The source is then isomorphic to

$$\operatorname{Spec} k[t,z]/(z^2 - t^2(\alpha + a_{n-1}t^2 + \cdots + a_1 t^{2(n-1)} + a_0 t^{2n}).$$

This is not normal, but its blow up at the origin is normal in a neighborhood of the origin, and its blow up is isomorphic to

$$\operatorname{Spec} k[Z/U,t]/((Z/U)^2 - (\alpha + a_{n-1}t^2 + \cdots + a_1 t^{2(n-1)} + a_0 t^{2n}).$$

The fiber of this over the point $z = t = 0$ is then simply $k[Z/U]/((Z/U)^2 - \alpha)$, which is a copy of $\mathbb{F}_{q^2}$ if α is a non-square, while it is two copies of $\mathbb{F}_q$ if α is a square. This fiber product is precisely the fiber product $X \times_{\mathscr{P},\infty} \operatorname{Spec} \mathbb{F}_q$ we wished to compute, and so we are done. $\square$

3. Algebra setup

The goal of this section is to recall and set up some of the algebraic machinery that we use. After introducing some of our notation that we use from higher category theory in §3.1, we study the notion of a homological epimorphism in §3.2, which is key to our study of stable homology. Throughout the rest of the section, we collect a number of basic facts in higher algebra, primarily related to homological epimorphisms. In §3.3 we check localization and certain completions are homological epimorphisms. In §3.4 we record a rigidity property for homological epimorphisms, which shows in many cases that they are actually equivalences. Finally, in §3.5 we prove some well known lemmas about nilpotence of endomorphisms.

3.1. Algebra notations and recollections.

Algebra notations and recollections. Here, we recall some basic notions from higher algebra that we use here to study the rational homology of Hurwitz spaces. A convenient foundation for studying objects in homotopy theory is that of ∞-categories, as developed by Joyal and Lurie (see [Lur09], [Lur17] or [Lur18]).

3.1.1. *The infinity category of spaces.* We use Spc to denote the ∞-category of spaces, which we view as symmetric monoidal with respect to the product. We use Spc_* to denote the ∞-category of pointed spaces, which is symmetric monoidal with respect to the smash product $\wedge$. Recall that for (X, x) and (Y, y) two pointed spaces, $X \wedge Y$ is defined as the quotient of $X \times Y$ by collapsing the subspace $x \times Y \cup y \times X$. We have a symmetric monoidal functor $(-)_+ : \mathrm{Spc} \to \mathrm{Spc}_*$ given by adding a disjoint base point to a space.

There is a functor $L : \mathrm{Top} \to \mathrm{Spc}$ from the category of topological spaces to the ∞-category of spaces, obtained by universally inverting all morphisms that are weak homotopy equivalences.

3.1.2. $\mathbb{E}_1$ *algebras.* Given a symmetric monoidal ∞-category $\mathcal{C}$, we can form the symmetric monoidal ∞-categories $\mathrm{Alg}(\mathcal{C})$of associative (or $\mathbb{E}_1$) algebras in $\mathcal{C}$.

3.1.3. *Stable ∞ categories.* We recall that an ∞-category $\mathcal{C}$ is *stable* if it admits finite limits and finite colimits, a zero object, and pullback squares and pushout squares coincide. We use Sp to denote the symmetric monoidal stable ∞-category of spectra. Given a stable ∞-category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, we use $\mathrm{map}_{\mathcal{C}}(X, Y)$ to denote the spectrum of maps between X and Y.

3.1.4. $\mathbb{E}_1$ *rings.* We refer to an $\mathbb{E}_1$-algebra in spectra as an $\mathbb{E}_1$-ring. An ordinary associative ring R can be viewed as an $\mathbb{E}_1$-ring via the lax symmetric monoidal inclusion of abelian groups into spectra.

Given an $\mathbb{E}_1$-ring, we can form the stable ∞-categories $\mathrm{Mod}_{\mathrm{Sp}}(R)$ and $\mathrm{Mod}_{\mathrm{Sp}}(R^{\mathrm{op}})$ of left and right R-modules in spectra respectively. For example, when R is an ordinary associative ring, then $\mathrm{Mod}_{\mathrm{Sp}}(R)$ agrees with the derived ∞-category of R-modules, so that objects can be presented as (unbounded) chain complexes of R-modules. Given $M \in \mathrm{Mod}_{\mathrm{Sp}}(R)$, we use either $\pi_i M$ or $H_i M$ depending on the context to refer to π_i of the underlying spectrum of M. In the case R is an ordinary ring and M is presented as a chain complex of R-modules, this agrees with the i^{th} homology of the chain complex.

3.1.5. $\mathbb{E}_2$-*central elements.* Let R be an $\mathbb{E}_1$ ring. We call $r \in \pi_i R$ $\mathbb{E}_2$-*central* if there is an R-bimodule map $\Sigma^i R \to R$ sending 1 to r. In particular, if r is $\mathbb{E}_2$-central, it is also graded commutative in the homotopy ring $\pi_* R$ of R. Given an $\mathbb{E}_2$-central element r, we implicitly associate to it a choice of R-bimodule map refining r.

3.1.6. *Chains functor.* We have a symmetric monoidal functor $\tilde{C}_*(-; \mathbb{Q}) : \mathrm{Spc}_+ \to \mathrm{Sp}$, taking A to its reduced with coefficients in $\mathbb{Q}$, with the comultiplication coming from the diagonal map, which is dual to the cup product. Composing $\tilde{C}_*(-; \mathbb{Q})$ with the symmetric monoidal functor $(-)_+$, we get the symmetric monoidal functor $C_*(-; \mathbb{Q}) : \mathrm{Spc} \to \mathrm{Sp}$. For X a space, $H_i(X; \mathbb{Q}) = \pi_i C_*(X; \mathbb{Q})$.

3.2. **Homological epimorphisms.** Our goal is to study the algebra of chains on Hurwitz spaces, which forms an $\mathbb{E}_1$-ring. We will study this $\mathbb{E}_1$-ring by forming certain completions and localizations of it. An important property of these operations that makes them well behaved in this setting is that they are *homological epimorphisms*. To define this notion, we recall the following lemma:

Lemma 3.2.1 ([NK24, Lemma 2.9.8])**.** *For a map $R \to S$ of $\mathbb{E}_1$-rings with fiber I, the following conditions are equivalent:*

(1) The forgetful functor $\mathrm{Mod}_{\mathrm{Sp}}(S) \to \mathrm{Mod}_{\mathrm{Sp}}(R)$ is fully faithful.
(2) The multiplication map $S \otimes_R S \to S$ is an equivalence.
(3) $I \otimes_R S = 0$
(4) The multiplication map induces an equivalence $I \otimes_R I \cong I$.

Definition 3.2.2. A map $R \to S$ of $\mathbb{E}_1$-rings is called a *homological epimorphism* (or hom. epi) if any of the equivalent conditions of Lemma 3.2.1 are satisfied.

3.3. **Localization and completion.** Two sources of hom. epis for us will be certain localization and completion maps. We recall below that given an $\mathbb{E}_1$-ring R, some $i \in \mathbb{Z}$, and $r \in \pi_i R$, we can form the completion $M_r^\wedge$, and localization $M[r^{-1}]$, of an $M \in \mathrm{Mod}_{\mathrm{Sp}} R$, and that this forms an $\mathbb{E}_1$-ring when $M = R$.

$\mathrm{Mod}_{\mathrm{Sp}}(R[r^{-1}])$ is defined as the presentable localization (or Bousfield localization) of the ∞-category $\mathrm{Mod}_{\mathrm{Sp}} R$ away from $R/r = \mathrm{cof}(\Sigma^i R \xrightarrow{r} R)$. In other words, it is the full subcategory of $\mathrm{Mod}_{\mathrm{Sp}}(R)$ such that $\mathrm{map}(R/r, -)$ vanishes, and the left adjoint of the inclusion of this subcategory sends M to $M[r^{-1}]$.

Example 3.3.1. The map $R \to R[r^{-1}]$ is always a hom. epi [NK24, Example 2.9.14]. More generally for a set of elements S in the homotopy groups of R, we can form $M[S^{-1}]$, which can be described as the localization away from $R/r, r \in S$, and when S is a finite set, agrees with iteratively localizing away from each element of S in any order.

Remark 3.3.2. When $r \in \pi_i R$ satisfies the left Ore condition, as defined in [Lur17, Definition 7.2.3.1], the underlying spectrum of $M[r^{-1}]$ can be computed as the colimit of M along the left multiplication by r [Lur17, Proposition 7.2.3.20].

We will also want to show certain completions are homological epimorphisms. We next define completions, and later, in Lemma 3.3.5, we will show certain completions are homological epimorphisms by showing they are localizations.

Definition 3.3.3. For $r \in \pi_i R$, the *r-completion* of the ∞-category of R-modules is defined as the presentable localization of $\mathrm{Mod}_{\mathrm{Sp}} R$ away from $R[r^{-1}]$. In other words, $\mathrm{Mod}_{\mathrm{Sp}}(R)_r^\wedge$ is the full subcategory of R-modules such that $\mathrm{map}(R[r^{-1}], -)$ vanishes. The inclusion of this subcategory $\mathrm{Mod}_{\mathrm{Sp}}(R)_r^\wedge \subset \mathrm{Mod}_{\mathrm{Sp}} R$ has a left adjoint sending M to $M_r^\wedge$.

There are natural $\mathbb{E}_1$-ring maps $R \to R_r^\wedge$, $R \to R[r^{-1}]$ induced by taking the map induced on endomorphism rings of the R-module R via the completion and localization functor. Note also that $R[r^{-1}] = R[(r^n)^{-1}]$ and $R_r^\wedge = R_{r^n}^\wedge$ for each $n \geq 1$.

The following lemma is a standard manipulation of Bousfield localizations. See [Rav84, Lemma 1.34] for a closely related statement.

Lemma 3.3.4. *Let R be an $\mathbb{E}_1$-ring. Suppose that $f : M \to N$ is a map of R-modules and $r \in \pi_i R$ is an element. The following are equivalent:*

(1) f is an equivalence.
(2) $f[\frac{1}{r}]$ and $f_r^\wedge$ are equivalences.
(3) $f[\frac{1}{r}]$ and $R/r \otimes_R f$ are equivalences.

Proof. We have (1) $\implies$ (2) because localizations and completions preserve equivalences. Next, (2) $\iff$ (3) follows since M is an R module with r acting invertibly if and only if $(R/r) \otimes_R M = 0$.

Lastly, we show (3) $\implies$ (1). If (3) is true, then $\mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R/r, M) \to \mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R/r, N)$ is an equivalence, as is $\mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R[r^{-1}], M) \to \mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R[r^{-1}], N)$. Hence, R is an extension of $R[r^{-1}]$ by $\mathrm{fib}(R \to R[r^{-1}])$, which is generated under colimits by R/r, by definition of the Bousfield localization $(-)[r^{-1}]$. So, it follows that $M \cong \mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R, M) \cong \mathrm{map}_{\mathrm{Mod}_{\mathrm{Sp}}(R)}(R, N) \cong N$ is an equivalence. $\square$

The following lemma gives a situation under which completion happens to also be a localization.

Lemma 3.3.5. *Let R be an $\mathbb{E}_1$-ring, and $x \in \pi_0 R$. Suppose there is some integers n, m with $n > m$ and some $z \in \pi_0 R$ so that $zx^n = x^m$. Also assume $x^{m'}$ is central for some $m' > 0$.*

Then, $e := z^{nm'} x^{n(n-m)m'}$ is a central idempotent in $\pi_* R$ so that $R \cong R[e^{-1}] \times R[(1-e)^{-1}]$ as an $\mathbb{E}_1$-ring. We also have $R^\wedge_x \cong R[(1-e)^{-1}], R[e^{-1}] \cong R[x^{-1}]$, so that $R \to R^\wedge_x$ is in particular a localization. Moreover, $\pi_* R^\wedge_x$ is the quotient of $\pi_* R$ by the two sided ideal generated by x^m.

Proof. To see that $e = z^{nm'} x^{n(n-m)m'}$ is an idempotent, we have

$$(z^{nm'} x^{n(n-m)m'})^2 = z^{2nm'} x^{2n(n-m)m'} = z^{nm'} x^{n(n-m)m'},$$

where, the first equality uses that $x^{m'}$ is central and the second equality uses the equation $zx^n = x^m$ a total of nm'-times and the assumption that $n(n-m)m' \geq n$.

We next show that e is central. Indeed, for an arbitrary element $y \in \pi_* R$, we have

$$\begin{aligned} ey &= z^{nm'} x^{n(n-m)m'} y \\ &= z^{nm'} y x^{n(n-m)m'} \\ &= z^{nm'} y z^{nm'} x^{2n(n-m)m'} \\ &= z^{nm'} x^{2n(n-m)m'} y z^{nm'} \\ &= x^{n(n-m)m'} y z^n \\ &= ye. \end{aligned}$$

It follows that the localizations $R[e^{-1}], R[(1-e)^{-1}]$ are Ore localizations, so localizations can be computed as in Remark 3.3.2.

Since R is the pullback $R[e^{-1}] \times_{R[e^{-1},(1-e)^{-1}]} R[1-e^{-1}]$, and $R[e^{-1}, (1-e)^{-1}] = 0$, we have $R \cong R[e^{-1}] \times R[(1-e)^{-1}]$. Since e is sent to 1 in $R[x^{-1}]$, and x is invertible in $R[e^{-1}]$, we have $R[x^{-1}] = R[x^{-1}, e^{-1}] = R[e^{-1}]$. Since the completion is the localization away from $R[x^{-1}]$, it follows that the completion is the other summand $R^\wedge_x \cong R[(1-e)^{-1}]$. To see that $x^m = 0$ in the completion, we have

$$(1-e)x^m = x^m - z^{nm'} x^{n(n-m)m'+m} = z^{nm'} x^{n(n-m)m'+m} - z^{nm'} x^{n(n-m)m'+m} = 0.$$

Because x^m maps to 0 in the component $R[(1-e)^{-1}] \simeq R^\wedge_x$, the kernel of surjective map $\pi_* R \to \pi_* R^\wedge_x$ contains two sided ideal generated by x^m. From the above, this kernel is equal to $\pi_* R[x^{-1}]$. Since $\pi_* R[x^{-1}]$ is contained in the two sided ideal generated by x^m since any $y \in \pi_* R[x^{-1}]$ can be written as $yx^{-m}x^m$, we obtain that the kernel is equal to the two sided ideal generated by x^m. $\square$

Notation 3.3.6. Suppose that R is an $\mathbb{E}_1$-ring and $X := \{x_1, \ldots, x_{|X|}\}$ is a finite set of elements in $\pi_0 R$ each independently satisfying the conditions of Lemma 3.3.5. That is, for each i with $1 \le i \le |X|$, there are some n_i, m_i and m'_i with $n_i > m_i$ and $z \in \pi_0 R$ so that $zx^{n_i} = x_{m_i}$ and $x^{m'_i}$ is central. Then we use $R^{\wedge}_X$ to denote the iterated completion $(\cdots(R^{\wedge}_{x_1})\cdots)^{\wedge}_{x_{|X|}}$. By Lemma 3.3.5, this doesn't depend on the choice of ordering, since each completion is also a localization (see Example 3.3.1).

3.4. **Tools for homological epimorphisms.**

Lemma 3.4.1. *Suppose we have maps $R \to S \to S'$ such that $R \to S$ is a hom. epi. Then $R \to S'$ is a hom. epi iff $S \to S'$ is a hom. epi.*

Proof. We use the description in Lemma 3.2.1(1) of hom. epi. We have the composite of forgetful functors
$$\mathrm{Mod}(S') \to \mathrm{Mod}(S) \to \mathrm{Mod}(R),$$
where the second is fully faithful. It follows that the composite is fully faithful iff the first functor is. $\square$

The following proposition is the key to relate the notion of a hom. epi to homological stability. Heuristically, it says that hom. epis to connective rings are determined by their π_0. See [HS24, Theorem B] for a generalization of this.

Proposition 3.4.2 (Rigidity of homological epimorphisms)**.** *Consider a diagram of $\mathbb{E}_1$-rings*

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & S' \\ & S, \arrow[ur] & \end{tikzcd}$$

where $S \to S'$ is a map of connective $\mathbb{E}_1$-rings that is an isomorphism on π_0, and where $R \to S$ and $R \to S'$ are homological epimorphisms. Then the map $S \to S'$ is an equivalence.

Proof. Using Lemma 3.4.1, we learn that $S \to S'$ is a homological epimorphism. Letting I be the fiber of $S \to S'$, we have $I \otimes_S S' = 0$. I is the fiber of a map of connective $\mathbb{E}_1$-rings, so it is -1-connective. We will inductively show on $i \ge -1$ that $\pi_i I$ is 0. The inductive hypothesis implies that I is i-connective, so and since $S \to S'$ is an equivalence on π_0, we have $\pi_i I \cong \pi_i(I \otimes_S \pi_0 S) \cong \pi_i(I \otimes_S S') = 0$, where the first two equalities come from the fact that I is i-connective and the maps $S' \to \pi_0 S$ and $S \to \pi_0 S$ are 1-connective, so the map on tensor products is $i+1$-connective (see [Lev22, Lemma 3.1]). Thus $\pi_i I = 0$ for all I, so $I = 0$, and hence $S \to S'$ is an isomorphism. $\square$

The next lemma gives a criterion for being a hom. epi which is often easier to check in practice.

Lemma 3.4.3. *Let $f : R \to S$ be a map of connective $\mathbb{E}_1$-rings, and suppose that the map $\pi_0 S \otimes_R \pi_0 S \to \pi_0 S \otimes_S \pi_0 S$ is an equivalence. Then f is a hom. epi.*

Proof. Consider the subcategory C of pairs $(N, M) \in \mathrm{Mod}_{\mathrm{Sp}}(S^{\mathrm{op}}) \times \mathrm{Mod}_{\mathrm{Sp}}(S)$ such that $N \otimes_R M \cong N \otimes_S M$. Observe that C is closed under colimits and desuspensions in each variable.

Note that if N, M are connective, then the pair (N, M) is contained in C if $\tau_{\leq n}N, \tau_{\leq n}M$ is contained in C for all $n \in \mathbb{N}$. This is because $\lim_n \tau_{\leq n}N \otimes_R \tau_{\leq n}M \cong N \otimes_R M$, and similarly for S.

Thus to show that f is a hom. epi., i.e the pair (S, S) is in C, it suffices to show that $(\tau_{\leq n}S, \tau_{\leq n}S)$ is for each $n \in \mathbb{N}$. However, $\tau_{\leq n}S$ is bounded in the the t-structure on $\mathrm{Mod}_{\mathrm{Sp}}(S)$ and $\mathrm{Mod}_{\mathrm{Sp}}(S^{\mathrm{op}})$, and $\pi_0 S$ generates the hearts of these categories, so the result follows since we know by assumption that $(\pi_0 S, \pi_0 S) \in C$. □

The following lemma allows us to commute quotients by $\mathbb{E}_2$-central elements with homological epimorphisms.

Lemma 3.4.4. *If $f : R \to S$ is a hom. epi, and $x : R \to R$ is a bimodule map, then there is a natural isomorphism of R-modules* $\mathrm{cof}(x) \otimes_R (S \otimes_R M) \cong S \otimes_R \mathrm{cof}(x) \otimes_R M$

Proof. Consider the full subcategory of R-bimodules generated under colimits by R. If N is such a bimodule, then the functor $M \otimes_R (-)$ preserves the subcategory of R-modules that are S-modules, because it is true for the R-bimodule R itself, and the subcategory is closed under colimits. Thus there is a natural transformation from $S \otimes_R N \otimes_R M \to N \otimes_R (S \otimes_R M)$ defined when N is in this subcategory. To see that this map is an equivalence, we observe it is an equivalence for $N = R$, and the condition is closed under colimits. □

3.5. **Nilpotence of actions.** Here we record a few well known lemmas we use to understand the nilpotence of actions of $\mathbb{E}_2$-central elements:

Lemma 3.5.1. *Let $\mathcal{C}$ be a stable ∞-category, and consider a diagram as below:*

$$\begin{array}{ccccc} X & \xrightarrow{0} & X' & \longrightarrow & X'' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\ Y & \longrightarrow & Y' & \xrightarrow{0} & Y''. \end{array}$$

Then the composite of the associated maps on vertical cofibers $\mathrm{cof}(f) \to \mathrm{cof}(f') \to \mathrm{cof}(f'')$ *is nullhomotopic.*

In particular, if a natural endomorphism g of the identity functor acts by 0 on X and Y, then g^2 acts by 0 on $\mathrm{cof}(f)$ *for any map $X \to Y$.*

Proof. For the first claim, the composite $\mathrm{cof}(f) \to \mathrm{cof}(f') \to \Sigma X'$ is null since $X \to X'$ is null, so we obtain a factorization of $\mathrm{cof}(f) \to \mathrm{cof}(f')$ through Y'. Composing with the second horizontal map, since $Y' \to Y''$ is 0, we obtain a nullhomotopy of the desired map.

For the second claim, we apply the first claim when $X = X' = X'', Y = Y' = Y''$ and $f = f' = f'' = g$. □

For the next lemma, we say that a monoidal stable ∞-category is *exactly monoidal* if the tensor product is exact in each variable.

Lemma 3.5.2. *Let $\mathcal{C}$ be an exactly monoidal stable ∞-category, let $f : \mathbb{1} \to \mathbb{1}$ be a map in $\mathcal{C}$, where $\mathbb{1}$ is the monoidal unit. Then the map* $\mathrm{cof}(f) \otimes f^{\otimes 2}$ *is null.*

In particular, if R is an $\mathbb{E}_1$-algebra, $f : R \to \Sigma^i R$ a bimodule map, and $M \in \mathrm{Mod}_{\mathrm{Sp}}(R)$, then f^2 acts by 0 on $\mathrm{cof}\, f \otimes_R M$ as a map of R-modules.

Proof. By identifying the category of R-bimodules with the category of colimit preserving endofunctors of $\mathrm{Mod}_{\mathrm{Sp}}(R)$, which is a monoidal stable ∞-category, the second statement follows from the first.

We turn to proving the first statement. Let $\delta : \mathrm{cof}(f) \to \Sigma \mathbb{1}$ be the boundary map in the cofiber sequence. The map $\delta \circ \mathrm{cof}(f) \otimes f$ is null since it agrees with $\mathrm{cof}(f) \xrightarrow{\delta} \Sigma \mathbb{1} \xrightarrow{f} \Sigma \mathbb{1}$. It follows that $\mathrm{cof}(f) \otimes f$ factors through $\mathbb{1}$. But then since $\mathbb{1} \xrightarrow{f} \mathbb{1} \to \mathrm{cof}(f)$ agrees with $\mathbb{1} \to \mathrm{cof}(f) \xrightarrow{\otimes f} \mathrm{cof}(f)$, it follows that $f^{\otimes 2}$ vanishes on $\mathrm{cof}(f)$. □

4. Computing stable homology

The goal of this section is to compute the stable homology of the Hurwitz spaces of interest. First, in §4.1, we give some background on racks, which is the natural setting for our main stability result. Next, in §4.2 we state the main result, Theorem 4.2.2, and explain how this implies our main results on stable homology. Then, in §4.3 we prove that multiplication by certain components of Hurwitz spaces act centrally on homology. In §4.4 we reduce the proof of Theorem 4.2.2 to showing that certain map of pointed bar constructions is a homotopy equivalence. We prove this equivalence in §4.5, relying on a topological model for the relevant spaces provided by Appendix A.

4.1. **Background on racks.**

Although we will ultimately only be interested in working with Hurwitz spaces where the monodromy lives in a certain conjugacy class in a group, all of the arguments go through just as easily when working with racks. Hence, for this section, we will work with racks. One concise reference for the basics of racks is [EG03].

Definition 4.1.1. A *rack* is a set c with a binary operation $\triangleright : c \times c \to c$ such that the following two properties hold:

(1) For every $x \in c$, the map $\phi_x : c \to c, y \mapsto x \triangleright y$ is a bijection.
(2) For any $x, y, z \in c$, $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.

Example 4.1.2. The only type of rack we will need for our application to the Cohen-Lenstra heuristics and to prove Theorem 1.2.7 is a rack which is a conjugacy class c in a group G, in which case we take $x \triangleright y := x^{-1}yx$, where, on the right hand side, $x^{-1}yx$ is viewed as multiplication in the group.

We now introduce some further notation we will use for racks.

Notation 4.1.3. For c a rack and $g, g_1, \ldots, g_n \in c$, we will use the notation

$$(g_1 \cdots g_n) \triangleright g := (g_n \triangleright (\cdots (g_2 \triangleright (g_1 \triangleright g)) \cdots)).$$

Note the reversal of order in the iteration of the g_i, which is compatible with the convention from Example 4.1.2.

Notation 4.1.4. For c a rack, we use Hur^c_n to denote the *pointed Hurwitz space* of degree n for c, which is by definition the homotopy quotient $(c^n)_{hB_n}$ of c^n by the action of B_n, where the positive generator of $B_2 \simeq \mathbb{Z}$ acts via $(g, h) \mapsto (h, h \triangleright g)$. We use Hur^c to refer to the union of the Hurwitz spaces $\coprod_n \mathrm{Hur}^c_n$, and let Conf refer to the union of the configuration spaces $\coprod_n \mathrm{Conf}_n$, for Conf_n denoting the configuration space parameterizing unordered sets of n distinct points in $\mathbb{C}$.

Notation 4.1.5. We can identify the connected components of the pointed Hurwitz space Hur_n^c with orbits of the B_n action on c^n. Under this identification, for $x_1, \ldots, x_n \in c$ we use $[x_1] \cdots [x_n]$ to denote the connected component of $\mathrm{Hur}_{n,\mathbb{C}}^{G,c}$ corresponding to the B_n orbit of the tuple $(x_1, \ldots, x_n)$.

In order to state our main result on stable homology, we will need the notion of the structure group of a rack. A similar notion of structure group was defined in [EG03, Definition 2.1], but it will be convenient for us to use a slightly different convention with is compatible with our conventions from Notation 4.1.4 and Example 4.1.2.

Definition 4.1.6. Let c be a rack. The *structure group* G_c of c is the quotient of the free group generated by $\{x : x \in c\}$ by the relations $y \cdot x = x \cdot (x \triangleright y)$, where $\cdot$ denotes the product in the free group, for each $x, y \in c$.

Notation 4.1.7. If $f : c' \to c$ is a map of racks, then $G_{c'}$ acts on c on the right, via $a \cdot b = f(a) \triangleright b$, where $b \in c$ and $a \in c'$ is viewed as a generator of $G_{c'}$. We define $\pi_0 c$ to be the set of orbits of the c under this action of G_c.

The following definition is a version of the non-splitting condition of [EVW16, Definition 3.1] in the setting of racks.

Definition 4.1.8. We say that a rack c is *non-splitting* if every nonempty subrack $c' \subset c$ has $\pi_0 c' = *$.

Remark 4.1.9. Alternatively, one says a rack is connected if $\pi_0 c$ consists of a single orbit under the operation $\triangleright$. Then, a rack is non-splitting if and only if every nonempty subrack is connected.

If c comes from a conjugacy class in a group, this is implied by the non-splitting condition of [EVW16, Definition 3.1]. Note that G_c acts on c: if $y \in G_c, x \in c$, then we can write $y = g_1 \cdots g_n$ for $g_1, \ldots, g_n \in c$ and we define $y \cdot x := (g_1 \cdots g_n) \triangleright x$.

Definition 4.1.10. Let c be a rack and $c' \subset c$ be a subrack. Let $Z_{c'}(x) := \{y \in G_{c'} : y \cdot x = x\}$ denote the *centralizer* of x in $G_{c'}$, for $y \cdot x$ as defined above. Then we say c is *nullable for* c' if there is a map of sets $\xi : c - c' \to c'$ so that, for every $x \in c - c'$, $x \triangleright \xi(x) \notin c'$, and ξ sends centralizers in $G_{c'}$ to centralizers in $G_{c'}$ the sense that $Z_{c'}(x) \subset Z_{c'}(\xi(x))$ for every $x \in c - c'$. We say c is *nullable* if c is nullable for c' for every nonempty subrack $c' \subset c$.

The above definition may appear somewhat unusual, and it is simply the condition that the proof we give seems to require. The reason for the name is that we later will want to show a certain space is contractible. To show this, we need to construct a certain function, and that function is easily seen to exist when a rack c is *nullable* for every subrack $c' \subset c$.

Remark 4.1.11. If such a ξ as in Definition 4.1.10 exists, we may take $\xi : c - c' \to c'$ to moreover be a function satisfying $\xi(d \triangleright g) = d \triangleright \xi(g)$ for every $d \in c'$ and $g \in c - c'$. To see this simply choose some $g_{\mathcal{O}}$ in each c' orbit of $c - c'$ and replace ξ with the function given by $h \triangleright g_{\mathcal{O}} \mapsto h \triangleright \xi(g_{\mathcal{O}})$ for each $h \in c'$. This is well defined because of the assumption that $Z_{c'}(x) \subset Z_{c'}(\xi(x))$ for every $x \in c - c'$ from Definition 4.1.10.

Our main result computes the stable homology of Hurwitz spaces associated to non-splitting nullable racks c. We next give an example of such racks, which is relevant for the Cohen-Lenstra heuristics.

Example 4.1.12. Suppose H is an odd order abelian group and $G := H \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on H by inversion. Let $c \subset G$ denote the conjugacy class of order 2 elements. Viewing c as a rack via Example 4.1.2, we claim that for every nonempty subrack $c' \subset c$, c is nullable for c'. Indeed, every such subrack is conjugate to one of the form $c' = c \cap G'$, where $G' = H' \rtimes \mathbb{Z}/2\mathbb{Z}$ for $H' \subset H$ a subgroup. The centralizer of any g in G is $\{g, \mathrm{id}\}$ and hence if $g \in c - c'$ and $g' \in c'$ we have $\{\mathrm{id}\} = Z_{G'}(g) \subset Z_{G'}(g')$. Note that this implies that there is a containment $Z_{c'}(g) \subset Z_{c'}(g')$ since $Z_{c'}$ is the preimage of $Z_{G'}$ under the map $G_{c'} \to G'$. Finally, if we take any $g' \in c'$ and $g \notin c$, we can write $g = (h, 1), g' = (h', 1)$ via the semidirect product structure, with $h \in H', h \in H - H'$. Then, $g^{-1}g'g = (h,1)(h',1)(h,1) = (h' - 2h, 1) \notin c'$ since $h' - 2h \notin H'$. It follows that we can take ξ to be any function $c - c' \to c'$ to see that c is always nullable with respect to c'. Additionally, c is also a non-splitting rack, as follows from [EVW16, Lemma 3.2].

Question 4.1.13. Is there an example of a group G and a conjugacy class c so that c is non-splitting, but not nullable as defined in [EVW16, Definition 3.1]?

4.2. **The main stability result.** We now set up some notation we will use in this section and state our main result. The map $\mathrm{Hur}^c \to \mathrm{Conf}$ is a map in $\mathrm{Alg}(\mathrm{Spc})$. To do this, we introduce a suitable comparison map:

Notation 4.2.1. For a rack c, let Conf^c be the pullback $\mathrm{Conf} \times_{\pi_0 \mathrm{Conf}} \pi_0 \mathrm{Hur}^c$, so that there is a map $\mathrm{Hur}^c \to \mathrm{Conf}^c$ in $\mathrm{Alg}(\mathrm{Spc})$. Let $A := C_*(\mathrm{Hur}^c; \mathbb{Q})$ and $A' := C_*(\mathrm{Conf}^c; \mathbb{Q})$. Applying $C_*(-; \mathbb{Q})$ to the map $\mathrm{Hur}^c \to \mathrm{Conf}^c$, we obtain a map $v : A \to A'$ of $\mathbb{E}_1$-algebras in spectra, i.e in the ∞-category $\mathrm{Alg}(\mathrm{Sp})$.

The next result is our main theorem on stable homology. After stating it, we will explain why it implies Theorem 1.2.7 from the introduction. Recall our definition of nullable from Definition 4.1.10.

Theorem 4.2.2. *With notation as in Notation 4.2.1, suppose c is a finite non-splitting nullable rack. Let $U \in \pi_0 A$ be $\sum \alpha_i$, where each α_i comes from a class in $\pi_0 \mathrm{Hur}^c_n$ for $n > 0$ which is central. Then, the map $v[U^{-1}] : A[U^{-1}] \to A'[U^{-1}]$ is an equivalence.*

We prove this in §4.5.7. From this, we easily deduce Theorem 1.2.7 from the introduction, computing the stable homology of our Hurwitz spaces.

4.2.3. *Proof of Theorem 1.2.7 assuming Theorem 4.2.2.* First, note that there is an identification between the pointed Hurwitz space $\mathrm{Hur}^{G,c}_n$ and Hur^c_n as both can be described as the homotopy quotient $(c^n)_{hB_n}$, see §1.2.2. Let us fix a value of $i \geq 0$. By [EVW16, Theorem 6.1], there exist constants I, J, and D[4] so that for any $i \geq 0$ and $n > Ii + J$ and $U := \sum_{h \in c} \alpha_h^{D \operatorname{ord}(h)}$, $\pi_i A_n$ agrees with $\pi_i A[U^{-1}]_n$. Note that, (G, c) is non-splitting, in the sense of [EVW16, Definition 3.1] by [EVW16, Lemma 3.2], so the hypotheses of [EVW16, Theorem 6.1] are satisfied.

Because of the definition of Conf^c, it follows that $\pi_i A'_n$ also agrees with $\pi_i A'[U^{-1}]_n$ as soon as $\pi_0 A$ has stabilized. Thus it follows that for any $i \geq 0$ and $n > Ii + J$, the map $\pi_i A_n \to \pi_i A'_n$ is an isomorphism, since it agrees with the map induced by $v[U^{-1}]$ on π_i. But this is exactly saying that $H_i(\mathrm{Hur}^c_n; \mathbb{Q}) \cong H_i(\mathrm{Conf}^c_n; \mathbb{Q})$ for $n \gg 0$, which is a reformulation of the statement that for each component Z of Hur^c_n, we have $H_i(\mathrm{Hur}^c_n; \mathbb{Q}) \cong$

[4] In fact, one may take $D = 1$ by [EL23, Proposition A.3.1].

$H_i(\mathrm{Conf}_n;\mathbb{Q})$. Theorem 1.2.7 now follows from Theorem 4.2.2, using that for every subrack $c' \subset c$, c is nullable for c' by Example 4.1.12. $\square$

4.3. **Centrality.** In this subsection, we let c be an arbitrary rack. The goal of this subsection is to show that if $x \in \pi_0 \mathrm{Hur}^c$ is central, then x is actually central in a much stronger sense. Namely, it is $\mathbb{E}_2$-central in Hur^c, meaning that multiplication by x can be refined to a Hur^c-bimodule map.

Informally, in the case of Hurwitz spaces associated to groups, the reason for this stronger form of centrality is that there is a homotopy $[\alpha][\beta] \simeq [\beta][\beta^{-1}\alpha\beta]$ obtained by changing the direction from which one concatenates G-covers. Thus if β centralizes c, this identifies left and right multiplication by β. The above argument is not enough to make multiplication by β into a bimodule map, but it can be made as such. We now argue this below and further generalize from groups to racks:

Lemma 4.3.1. *Using notation as in Notation 3.3.6, we let c be a rack. Let $\gamma \in \pi_0 \mathrm{Hur}^c$ be central. If we view Hur^c as an $\mathbb{E}_1$-algebra in the ∞-category $\mathrm{Spc}_{/\,\mathrm{Conf}}$, then multiplication by γ in Hur^c can be refined to the structure of an Hur^c-bimodule map.*

Proof. We can consider the $\mathbb{E}_2$-monoidal ∞-category of local systems on Conf, $\mathrm{Spc}^{\mathrm{Conf}}$. This can be identified with the ∞-category of tuples $(X_i)_{i\in\mathbb{N}}$, where $X_i \in \mathrm{Spc}^{BB_i}$ is a space with an action of the braid group on i-letters. Indeed this follows because Conf is the free $\mathbb{E}_2$-algebras in spaces on a point, and is 1-truncated, and $\coprod_n BB_n$ is the free $\mathbb{E}_2$-monoidal category on an object [JS86, Theorem 4], and is a groupoid, so they agree.

We can view Hur^c as the colimit of the multiplicative local system given by the tuple $(c^i)_{i\in\mathbb{N}}$, where the action of the braid group is determined by the fact that the twist sends a pair $(x,y) \in c \times c$ to $(y, y \triangleright x)$, and the multiplication is given by concatenation. Note that since this is a local system of sets, we may view Hur^c as an object in the (ordinary) braided monoidal category $\mathrm{Fun}(\coprod_n BB_n, \mathrm{Set})$. Here the tensor product of $(X_i)_{i\in\mathbb{N}}, (Y_i)_{i\in\mathbb{N}}$ is given by $(\coprod_{i+j=n} \mathrm{Ind}_{B_i\times B_j}^{B_n}(X_i \times Y_j))_{n\in\mathbb{N}}$, and the braiding is determined by the formula given in [JS86, p7].

Let m be such that $\gamma \in \mathrm{Hur}^c_m$. Consider $m_!(*)$, where m is considered as the inclusion of the base point $* \to BB_m$. Explicitly, $m_!(*)$ is the local system which is B_m in degree m and empty otherwise. There is a left module map $(c^i)_{i\in\mathbb{N}} \otimes m_!(*) \to (c^i)_{i\in\mathbb{N}}$, given by right multiplying by the class γ on the identity element of B_m. Here we realize $(c^i)_{i\in\mathbb{N}} \otimes m_!(*)$ as a bimodule via the $\mathbb{E}_2$-monoidal structure, i.e using the twist map $(c^i)_{i\in\mathbb{N}} \otimes m_!(*) \to m_!(*) \otimes (c^i)_{i\in\mathbb{N}}$. The fact that this multiplication map is a bimodule map follows if the diagram

$$\begin{tikzcd}
(c^i)_{i\in\mathbb{N}} \otimes m_!(*) \arrow[rr, "\bullet_r"] \arrow[dr, "\mathrm{tw}"'] & & (c^i)_{i\in\mathbb{N}} \\
& m_!(*) \otimes (c^i)_{i\in\mathbb{N}} \arrow[ur, "\bullet_l"'] &
\end{tikzcd}$$

commutes, where $\bullet_l$ is left multiplication, $\bullet_r$ is right multiplication, and tw is the twist map. Unraveling the definitions, this holds because the twist sends a tuple $(x,\gamma), x \in c^i$ to (γ, x), which uses that γ is central in $\pi_0 \mathrm{Hur}^c$, so that $(\gamma\triangleright)$ acts trivially on c.

Thus we have constructed a bimodule map of multiplicative local systems on Conf induced by multiplication by γ. By applying the $\mathbb{E}_2$-monoidal functor colim : $\mathrm{Spc}^{\mathrm{Conf}} \to \mathrm{Spc}_{/\,\mathrm{Conf}}$, (where the map to Conf is obtained via the colimit of the terminal local system) we see that this bimodule map refines the multiplication by γ map. $\square$

Corollary 4.3.2. *Let $\gamma \in \pi_0 \mathrm{Hur}^c$ be central. Then the associated class $[\gamma] \in \pi_0 A$ is $\mathbb{E}_2$-central.*

Proof. This follows by applying the functor $C_*(-;\mathbb{Q})$ to the bimodule map produced in Lemma 4.3.1. $\square$

4.4. **Decomposition of the comparison map.** To check that the comparison map v is an equivalence, we geometrically decompose the algebra A based on central elements acting on it. We accomplish this decomposition in Lemma 4.4.4, which roughly says it suffices to check $v[U^{-1}]$ is an equivalence after suitable localizations and completions.

As a first step toward this, we note that certain completions of localization of A are well behaved in the sense of Lemma 3.3.5:

Notation 4.4.1. Let c be a finite rack. Given an element $h \in c$, let us use α_h to refer to the element $[h]$ in $\pi_0 \mathrm{Hur}^c_1$, as defined in Notation 4.1.5. For a subset $S \subset c$, we use α_S to denote the set $\{\alpha_s : s \in S\}$. We use $M[\alpha_S^{-1}]$ to denote the localization of some A-module M at the set $\{\alpha_s, s \in S\}$ and we similarly use $M^\wedge_{\alpha_S}$ to denote the completion of M at $\{\alpha_s : s \in S\}$ (when this doesn't depend on an order of S).

The following lemma is a generalization of [EVW16, Proposition 3.4] from groups to racks, whose proof requires no new ideas.

Lemma 4.4.2. *Let c be a finite rack with $\pi_0 c = *$, and let $g \in c$. For every N sufficiently large, any tuple of elements $(g_1, \ldots, g_N) \in c^N$ generating the rack c is equivalent under the braid group action to some tuple of the form $(g, g'_1, \ldots, g'_{N-1})$ where $g'_1, \ldots, g'_{N-1}$ generates c. Similarly, $(g_1, \ldots, g_N)$ is equivalent under the braid group action to some tuple of the form $(g''_1, \ldots, g''_{N-1}, g)$ where $g''_1, \ldots, g''_{N-1}$ generates c.*

Proof. We summarize the proof, leaving details to the reader, which can also be found in [EVW16, Proposition 3.4] in the case that c is a conjugacy class in a group. We will only prove the first statement regarding equivalence to $(g, g'_1, \ldots, g'_{N-1})$. The second statement regarding equivalence to $(g''_1, \ldots, g''_{N-1}, g)$ has an analogous proof.

Choose N large enough so that some $g' \in c$ appears $n+1$ times, where n is such that $\phi^n_{g'} = \mathrm{id}$. Using the action of the braid group, we can move all of these copies of g' to the left to see that $(g_1, \ldots, g_N)$ is equivalent under the braid group action to $(\underbrace{g', \ldots, g'}_{n+1}, h_1, \ldots, h_{N-n-1})$, where $g', h_1, \ldots, h_{N-n-1}$ still generate the rack. Because $\phi^n_{g'} = \mathrm{id}$, by using the element of the braid group that does a full twist of the last element around the rest, we see that $(\underbrace{g', \ldots, g'}_{n}, h) \in c^{n+1}$ is equivalent to $(\underbrace{h \triangleright g', \ldots, h \triangleright g'}_{n}, h) \in c^{n+1}$, where there are n copies of g'. Using this along with the fact that $g', h_1, \ldots, h_{N-n-1}$ generate the rack and $\pi_0 c = *$ shows that we can modify $(\underbrace{g', \ldots, g'}_{n+1}, h_1, \ldots, h_{N-n-1})$ by an element of the braid group to change the first n copies of g' into n copies of g, while keeping the fact that the rest of the elements generate the rack. $\square$

Lemma 4.4.3. *With notation as in Notation 4.4.1, let c be a finite non-splitting rack and let $c' \subset c$ be a subrack. In $A[\alpha_{c'}^{-1}]$, any element α_β with $\beta \in c - c'$ satisfies the conditions of Lemma 3.3.5. Thus for an A-module M, we can form $M[\alpha_{c'}^{-1}]^\wedge_{\alpha_{c-c'}}$ as in Notation 3.3.6.*

Proof. By Corollary 4.3.2, $\alpha_\beta^{\mathrm{ord}(\beta)}$ is $\mathbb{E}_2$-central where $\mathrm{ord}(\beta)$ is the order of ϕ_β as an automorphism of c. It remains to prove that $y\alpha_\beta^2 = \alpha_\beta$ for some $y \in \pi_* A[\alpha_{c'}^{-1}]$. To do this, choose $\gamma \in c'$. Note that because c is non-splitting, we have $\pi_0\langle c', \beta\rangle = *$. Applying Lemma 4.4.2 to the subrack $\langle c', \beta\rangle \subset c$ generated by c' and β, , for N sufficiently large, we can write $\alpha_{x_1} \cdots \alpha_{x_N}\alpha_\beta = \alpha_\gamma^N \alpha_\beta$, where $x_1, \ldots, x_N \in c$ together generate the same subrack as γ, β. Applying Lemma 4.4.2 again, we can write $\alpha_{y_1} \cdots \alpha_{y_{N-1}}\alpha_\beta = \alpha_{x_1} \cdots \alpha_{x_N}$. Taking $y := \alpha_\gamma^{-N}\alpha_{y_1} \cdots \alpha_{y_{N-1}}$, we therefore have $y\alpha_\beta^2 = \alpha_\beta$. This verifies the conditions of Lemma 3.3.5. ☐

The following reduces one to considering stable homology classes on which $\alpha_{c'}$, for a subrack $c' \subset c$, act by isomorphisms, and the complementary set act nilpotently.

Lemma 4.4.4. *Let c be a finite non-splitting nullable rack. Using notation from Notation 4.4.1 and Notation 4.2.1, and letting v, U be as in Theorem 4.2.2, the map $v[U^{-1}]$ is an equivalence if, for every subrack $c' \subset c$, $(v[\alpha_{c'}^{-1}])^\wedge_{\alpha_{c-c'}}$ is an equivalence, where the completion is taken as an A-module.*

Proof. By repeatedly applying Lemma 3.3.4, we learn that it is enough to show that for every subset $c' \subset c$,

$$(A/\alpha_{\beta_{|c-c'|}}^{\mathrm{ord}(\beta_{|c-c'|})}) \otimes_A \cdots \otimes_A (A/\alpha_{\beta_1}^{\mathrm{ord}(\beta_1)}) \otimes_A v[U^{-1}][\alpha_{c'}^{-1}] \tag{4.1}$$

is an equivalence, where $\coprod_{i=1}^{|c-c'|} \beta_i = c - c'$. Here, we have used Lemma 3.4.4 to commute the localization with the quotienting, and Corollary 4.3.2 to make sense of the iterated quotient. Namely, because $\alpha_{\beta_i}^{\mathrm{ord}(\beta_i)}$ is a bimodule map by Lemma 4.3.1, we can tensor with the $A - A$-bimodule $(A/\alpha_{\beta_1}^{\mathrm{ord}(\beta_1)})$ to make sense of the tensor product above as an A-module.

Note that on the source and target of this map, for each $\beta \in c - c'$, $\alpha_\beta^{\mathrm{ord}(\beta)n_\beta}$ acts by 0 for some n_β. This follows from applying Lemma 3.5.2 to see that after quotienting out by $\alpha_\beta^{\mathrm{ord}(\beta)}$, $\alpha_\beta^{2\,\mathrm{ord}(\beta)}$ acts by 0, and using Lemma 3.5.1 to see that after each further quotient, the smallest power of $\alpha_\beta^{\mathrm{ord}(\beta)}$ that acts by 0 gets multiplied by at most 2.

If c' is empty, then we claim that both the source and target of the map $(v)^\wedge_{\alpha_c}$ are 0. Some power of U acts by 0 on the source and target, since a large enough power of U, U^j, is in the two sided ideal of $\pi_* R$ generated by $\alpha_\beta^{\mathrm{ord}(\beta)n_\beta}$. This is because by the pigeonhole principle, one can see that any element of $\pi_0 \mathrm{Hur}_n^{G,c}$, written as a product of elements in c for n large enough contains $\max_\beta(\mathrm{ord}(\beta)n_\beta$ many copies of a single $\beta \in c$, and by pulling these to the left using the braid group action, we see the element is divisible by $\alpha_\beta^{\mathrm{ord}_\beta n_\beta}$.of U, U^j, is in the ideal generated by the smallest powers of $\alpha_\beta^{\mathrm{ord}(\beta)}$ that act trivially, and hence it acts trivially. Since U acts both as an isomorphism and nilpotently, the source and target of $(v)^{\wedge}_{\alpha_c}$ must both be 0.

We next claim that the source and target of (4.1) vanish unless $c' \subset c$ is a subrack. Indeed, observe that $\alpha_a \alpha_b = \alpha_b \alpha_{b \triangleright a}$. Hence, we learn that if α_a and α_b act invertibly, then so does $\alpha_{b \triangleright a}$. Thus the source and target of (4.1) vanish unless c' is closed under the operation $\triangleright$, i.e., c' is a subrack.

Suppose the map

$$(A/\alpha_{\beta_{|c-c'|}}^{\mathrm{ord}(\beta)}) \otimes_A \cdots \otimes_A (A/\alpha_{\beta_1}^{\mathrm{ord}(\beta)}) \otimes_A v[\alpha_{c'}^{-1}] \tag{4.2}$$

(which is similar to (4.1) but here we have not inverted U) is an equivalence for each subrack $c' \subset c$. Using Lemma 3.4.4 to commute the localization and quotienting, we find that if (4.2) is an equivalence for each subrack $c' \subset c$, (4.1) will also be an equivalence. Hence, it is enough to show (4.2) is an equivalence for each subrack $c' \subset c$. Finally, we note that (4.2) is an equivalence for a fixed $c' \subset c$ if and only if $(v[\alpha_{c'}^{-1}])^{\wedge}_{\alpha_{c-c'}}$ is an equivalence by using Lemma 4.4.3 and Lemma 3.3.4. □

4.5. **Showing the map is an equivalence.** In this section we show that the maps in Lemma 4.4.4 are an equivalence for each subrack $c' \subset c$. In the case that $c' = c$, the fact that the map in Lemma 4.4.4 is an equivalence comes from the fact that the homology of the group completion of Hur^c can be identified with the rack homology of the rack c, which is computed in [EG03]. We verify this in Proposition 4.5.1 For $c' \subset c$. In general, we show that completion at $\alpha_{c-c'}$ (see Notation 4.4.1) corresponds to projecting in pointed spaces to the components of $\mathrm{Hur}^{c'}$ at the level of homology, allowing us to again apply Proposition 4.5.1 for the rack c' in this case. Identifying the completion in terms of $\mathrm{Hur}^{c'}$ is at the core of our proof, and is proven using a homological epimorphism argument whose main input is a topological one identifying two bar constructions in pointed spaces as homotopy equivalent.

Proposition 4.5.1. *Use notation from Notation 4.2.1 and Notation 4.4.1, let c be a finite rack with $\pi_0 c = *$. The map $v[\alpha_c^{-1}]$ is an equivalence.*

Remark 4.5.2. Proposition 4.5.1 was essentially already known, and can be found in [RW20, Corollary 5.4]. Although this was stated for conjugacy classes of groups, the proof works equally well for racks. We now spell out the details of this argument for general racks.

Proof. By the group completion theorem, $v[\alpha_c^{-1}]$ can be identified with the map $\Omega B\,\mathrm{Hur}^c \to \Omega B\,\mathrm{Conf}^c$ on homology. It is then enough to show that the map $\Omega B\,\mathrm{Hur}^c \to \Omega B\,\mathrm{Conf}^c$ is a homotopy equivalence, so it is enough to show that $B\,\mathrm{Hur}^c \to B\,\mathrm{Conf}^c$ is a homotopy equivalence. Since $\pi_1 B\,\mathrm{Hur}^c \to \pi_1 B\,\mathrm{Conf}^c$ is an equivalence, as $\pi_0\,\mathrm{Hur}^c \to \pi_0\,\mathrm{Conf}^c$ is an equivalence by construction of Conf^c, it suffices to check $B\,\mathrm{Hur}^c \to B\,\mathrm{Conf}^c$ is an equivalence on rational homology.

We first claim the map $B\,\mathrm{Conf}^c \to B\,\mathrm{Conf}$ is an equivalence on rational homology. It follows from the group completion theorem that

$$\Omega B\,\mathrm{Conf}^c \cong \Omega B\,\mathrm{Conf} \times_{\Omega B \pi_0 \mathrm{Conf}} \Omega B \pi_0\,\mathrm{Hur}^c .$$

The map $\Omega B \pi_0\,\mathrm{Hur}^c \to \Omega B \pi_0\,\mathrm{Conf}$ is surjective with kernel a group we call G. Here, G is finite because of the fact that $|\pi_0\,\mathrm{Conf}^c_n| = |\pi_0\,\mathrm{Hur}^c_n|$ stabilizes as n grows, which is a consequence of Lemma 4.4.2 and the fact that a sequence of surjective maps between finite sets eventually becomes isomorphisms. It follows that the fiber of the map $B\,\mathrm{Conf}^c \to B\,\mathrm{Conf}$ is BG. We then obtain the claim that $B\,\mathrm{Conf}^c \to B\,\mathrm{Conf}$ is an equivalence on rational homology using the Serre spectral sequence for this fiber sequence, since the rational homology of G is just $\mathbb{Q}$ in degree 0 with trivial action of $B\,\mathrm{Conf}$.

It remains then to show that the map $B\,\mathrm{Hur}^c \to B\,\mathrm{Conf}$ is an equivalence on rational homology. This follows as in [RW20, Corollary 5.4]: This map can be modelled as the map of rack spaces as in [FRS95], from the rack c to the trivial rack $*$, and the induced map on homology is the map on rack homology induced from this map of racks. It is evident that the map on rack homology is a split surjection (since the associated map of braided vectors spaces has a splitting sending 1 to $\frac{1}{|c|}\sum_{g\in c} g$), and on the other hand [EG03, Theorem 4.2] shows that the rack cohomology of both sides has rank 1 in each degree (using that $\pi_0 c = *$). Thus the map is an isomorphism. $\square$

Remark 4.5.3. In fact, the proof above (as well as this whole section) doesn't just work rationally, but also after inverting the order of G showing up in the proof and the order of the 'reduced structure group' of the rack c [EG03, Section 2]. In the case c comes from a conjugacy class in a group $\langle c\rangle$, all primes dividing $|G|$ appearing in the proof (which is potentially different from $\langle c\rangle$ and all primes dividing the reduced structure group of the rack are contained in the set of primes dividing $\langle c\rangle$, by [Woo21, Theorem 2.5]. So, we can compute stable homology after one inverts the order of $\langle c\rangle$.

We continue to use notation from Notation 4.1.4, Notation 4.2.1, and Notation 4.4.1.

Notation 4.5.4. For $c' \subset c$ a subrack, there is a map of $\mathbb{E}_1$-algebras $\bar{i}^c_{c'} : \mathrm{Conf}^{c'} \to \mathrm{Conf}^c$ that is an inclusion of components. Similarly, there is a map $\tilde{i}^c_{c'} : \mathrm{Hur}^{c'} \to \mathrm{Hur}^c$. We have a section of the induced map $\mathrm{Conf}^{c'}_+ \to \mathrm{Conf}^c_+$ of pointed $\mathbb{E}_1$-algebras $\bar{r}^{c'}_c : \mathrm{Conf}^c_+ \to \mathrm{Conf}^{c'}_+$ obtained by sending all components not in image of Conf^c to the base point. Similarly there is a map $\tilde{r}^{c'}_c : \mathrm{Hur}^c_+ \to \mathrm{Hur}^{c'}_+$ obtained the same way. Let $r^{c'}_c$ denote the associated $\mathbb{E}_1$-algebra map $A \to C_*(\mathrm{Hur}^{c'})$ obtained by taking reduced chains. In particular, this map of algebras sends all generators α_x, for $x \in c - c'$, to 0.

The following result is the key to proving the case when c' is a proper subrack of c.

Proposition 4.5.5. *Let c be a finite nonsplitting rack. With notation as in Notation 4.5.4, let c' be a proper subrack of c such that c is nullable with respect to c'. Then the map $r^{c'}_c$ induces an equivalence $A[\alpha_{c'}^{-1}]^\wedge_{\alpha_{c-c'}} \to C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{(c')^{-1}}]$.*

Proof. We first note that since $\alpha_{c-c'}$ acts by 0 on $C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}]$, completion doesn't change it, so the map $A[\alpha_{c'}^{-1}]^\wedge_{\alpha_{c-c'}} \to C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}]$ in the proposition statement is obtained by taking $r^{c'}_c$, localizing at $[\alpha_{c'}^{-1}]$ and completing at $\alpha_{c-c'}$.

We will prove the proposition by applying Proposition 3.4.2 (rigidity of hom. epis) to the triangle

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}] \\ & A[\alpha_{c'}^{-1}]^\wedge_{\alpha_{c-c'}}. \arrow[ur, "h"] & \end{tikzcd}$$

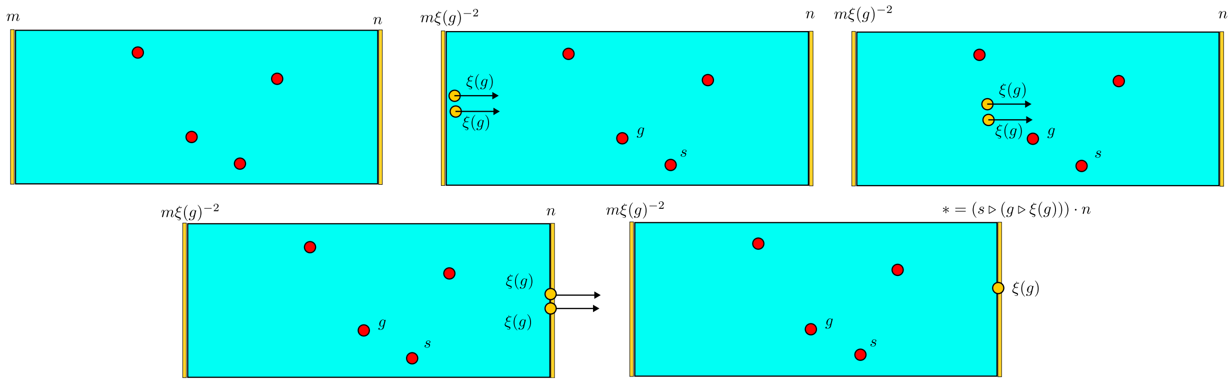

FIGURE 1. This is a pictorial description of the homotopy from Proposition 4.5.6. We take $s \in c'$ but $g \notin c'$. We pass the element $\xi(g)$ from the left to the right. When it meets the right boundary, it acts on the boundary by $s \triangleright (g \triangleright \xi(g)))$. By assumption $g \triangleright \xi(g) \notin c'$. Since $s \in c'$, $s \triangleright (g \triangleright \xi(g)))$ is also not in c'. Therefore, the right boundary becomes the base point, and this whole configuration is then identified with the base point.

The diagram is a triangle of $\mathbb{E}_1$-algebras since the completion on the source is obtained by inverting a central idempotent by Lemma 4.4.3.

Note that all of the objects in the triangle are connective. By Lemma 3.3.5, $\pi_0 A[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}}$ is the quotient of $\pi_0 A[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}}$ by the two sided ideal generated by $\alpha_{c-c'}$. This ideal is exactly the classes coming from components not in $\mathrm{Hur}^{c'}$. Thus the map h is an equivalence on π_0. Moreover, $A \to A[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}}$ is a hom. epi. by Lemma 4.4.3 and Example 3.3.1.

It remains to see that the map $A \to C_*(\mathrm{Hur}^{c'}; \mathbb{Q})[\alpha_{c'}^{-1}]$ is a hom. epi. Let $X = \pi_0 \mathrm{Hur}^{c'}[\alpha_{c'}^{-1}]$, so that $C_*(X;\mathbb{Q}) \cong \pi_0(C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}])$. By Lemma 3.4.3, it is enough to check that the map

$$C_*(X;\mathbb{Q}) \otimes_A C_*(X;\mathbb{Q}) \to C_*(X;\mathbb{Q}) \otimes_{C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}]} C_*(X;\mathbb{Q})$$

is an equivalence. Since $C_*(\mathrm{Hur}^{c'};\mathbb{Q}) \to C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{c'}^{-1}]$ is a hom. epi, we may replace the target with $C_*(X;\mathbb{Q}) \otimes_{C_*(\mathrm{Hur}^{c'};\mathbb{Q})} C_*(X;\mathbb{Q})$. Then this comparison is now induced from taking reduced chains of a map at the level of spaces. In what follows, the tensor product is the two sided bar construction of pointed spaces and we use notation from §3.1.1:

$$X_+ \otimes_{\mathrm{Hur}^c_+} X_+ \to X_+ \otimes_{\mathrm{Hur}^{c'}_+} X_+. \tag{4.3}$$

So, it is enough to verify the map (4.3) is an equivalence, and we do this in Proposition 4.5.6. □

The final hurdle before proving our main result is the following key result, which was needed for the proof of Proposition 4.5.5. For the next statement, the reader may wish to recall the notion of nullable from Definition 4.1.10.

Proposition 4.5.6. *Let c be a rack, $c' \subset c$ be a subrack, and X_+ be $\pi_0 \operatorname{Hur}^{c'}[\alpha_{c'}^{-1}]_+$, viewed as a $\operatorname{Hur}_+^{c'}$-bimodule. If c is nullable for c', the map*

$$X_+ \otimes_{\operatorname{Hur}_+^c} X_+ \to X_+ \otimes_{\operatorname{Hur}_+^{c'}} X_+$$

is an equivalence.

Proof. The map of interest $X_+ \otimes_{\operatorname{Hur}_+^c} X_+ \to X_+ \otimes_{\operatorname{Hur}_+^{c'}} X_+$ has a section induced from the inclusion of racks $c' \to c$. It thus suffices to show that this section induces a homotopy equivalence.

We now use the notation $\overline{Q}_\epsilon^*[M, \operatorname{Hur}_+^{c'}, N]$, as defined in Definition A.4.1 and Theorem A.4.9, which we describe below. By Theorem A.4.9, proving that this section is an equivalence is equivalent to showing that the inclusion

$$\iota_\epsilon : \overline{Q}_\epsilon^*[X_+, \operatorname{Hur}_+^{c'}, X_+] \to \overline{Q}_\epsilon^*[X_+, \operatorname{Hur}_+^{c}, X_+]$$

is an ind-weak equivalence (see Definition A.4.3) as ϵ approaches 0 where $0 < \epsilon < 1$.

We now recall the description of these spaces, which follows from Remark A.4.2. First, $Q_\epsilon[X_+, \operatorname{Hur}_+^c, X_+]$ is the space whose points consist of tuples $(m, n, x, l : x \to c)$ with the following properties: $m \in X_+, n \in X_+$, x is a finite subset x of $[0,1] \times [\epsilon, 1-\epsilon]$ such that if π_2 is the projection onto the second coordinate, then π_2 is injective on x, and the points in $\pi_2(x)$ are distance at least ϵ from each other, and $l : x \to c$ is a map of sets, which we colloquially refer to as a labeling function.

Viewing X_+ as having a left and right action of Hur^c, $\overline{Q}_\epsilon^*[X_+, \operatorname{Hur}_+^c, X_+]$ is the quotient of this space by the relations listed below:

(i) Suppose $x = \{p_1, \ldots, p_r, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_r) = 0$, so p_r lies on the left boundary, the data $(m, n, x, l : x \to c)$ is equivalent to $(m', n, x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to c)$, where $m' = ml(p_r)$, $l'(p_i) := l(p_r) \triangleright l(p_i)$ for $1 \le i < r$, $l'(p_i) := l(p_i)$ for $i > r$.

(ii) Suppose $x = \{p_1, \ldots, p_r, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_r) = 1$, so p_r lies on the right boundary, the data $(m, n, x, l : x \to c)$ is equivalent to $(m, n', x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to c)$, where $n' = ((l(p_{r+1}) \ldots l(p_k)) \triangleright l(p_r))n$, $l'(p_i) := l(p_i)$ for $i \neq r$.

(iii) All points where either n or m are the base point are identified to the base point.

The space $\overline{Q}_\epsilon^*[X_+, \operatorname{Hur}_+^{c'}, X_+]$ is defined similarly, except the the labelling function l must land in $c' \subset c$.

We first claim that for any $1 > \epsilon > 0$, ι_ϵ is a homeomorphism on all path components that do not contain the base point. Indeed, suppose we have some point of $Q_\epsilon[X_+, \operatorname{Hur}_+^c, X_+]$, given by $m \in X_+, n \in X_+, x \subset [0,1] \times [0,1]$ and $l : x \to c$.

We claim that such a point has image lying on the component of the base point in $\overline{Q}_\epsilon^*[X_+, \operatorname{Hur}_+^c, X_+]$ if and only if either m or n is the base point or the image of l is not in c'. Indeed, we can continuously move all of the points to the boundary, and using relations (i) and (ii) see that this data is equivalent to a point with empty configuration and values $m' \in X_+, n' \in X_+$, where one of m', n' is the base point exactly under this condition. Then relation (iii) shows that this point is the base point. Conversely, if m', n' aren't the base point and l has image in c', then it is easy to see that this property is preserved under the equivalence relation and on each component of $Q_\epsilon[X_+, \operatorname{Hur}_+^c, X_+]$.

On the source of each map ι_ϵ, for $\frac{1}{2} > \epsilon > 0$, the path component of the base point consists just of the base point. At this point, it may be useful to refer to the notion of two spaces being ind-weakly equivalent, defined in Definition A.4.3. To show that the ι_ϵ is an ind-weak equivalence, it suffices to show the following:

* $\star$ The component $S_\epsilon^{c,c'}$ of the base point of $\overline{Q}_\epsilon^*[X_+, \mathrm{Hur}_+^c, X_+]$ is ind-weakly equivalent to a point.

Fixing an $\epsilon > 0$, we will find a smaller ϵ' such that the inclusion $S_\epsilon^{c,c'} \to S_{\epsilon'}^{c,c'}$ is nullhomotopic.

Because c is nullable for c', by Remark 4.1.11, we can choose a function $\xi : c - c' \to c'$ with the following properties.

(a) $g \triangleright \xi(g) \notin c'$ for each $g \in c - c'$.
(b) $\xi(h \triangleright g) = h \triangleright (\xi(g))$ for each $g \in c - c', h \in c'$.

Let $\theta : T_\epsilon^{c,c'} \to S_\epsilon^{c,c'}$ be the map defined via the pullback

$$\begin{tikzcd}
T_\epsilon^{c,c'} \arrow[r] \arrow[d, "\theta"] & Q_\epsilon[X_+, \mathrm{Hur}_+^c, X_+] \arrow[d] \\
S_\epsilon^{c,c'} \arrow[r] & \overline{Q}_\epsilon^*[X_+, \mathrm{Hur}_+^c, X_+].
\end{tikzcd}$$

Fix a point $y \in T_\epsilon^{c,c'}$ given by the data

($\dagger$) $(m, n, x = \{p_1, \dots, p_k\}, l : x \to c)$ with $\pi_2(p_1) > \dots > \pi_2(p_k)$.

Let p_v be the lowest point of x with $l(p_v) \notin c'$. As explained above, such an v exists because this point lies in the component of the base point.

Let $\exp(c)$ be the *exponent* of the rack c, i.e the smallest positive number chosen so that the action of $(d\triangleright)^{\exp(c)}$ on c is trivial for all $d \in c$. Let $\epsilon' := \frac{\epsilon}{\exp(c)+1}$. We now construct a continuous homotopy $F : T_\epsilon^{c,c'} \times I \to S_{\epsilon'}^{c,c'}$ as follows: If either m, n is the base point, the point y is equivalent to the base point under relation (iii), and we choose the constant homotopy at the base point. So now assume this is not the case. Using relation (i) $\exp(c)$ times, we can change the label of m to $m\xi(l(p_v))^{-\exp(c)}$, and add $\exp(c)$ points $s_1, \dots, s_{\exp(c)}$ with s_j at $(0, \pi_2(p_v) + \frac{\epsilon j}{\exp(c)+1})$ for $1 \le j \le \exp(c)$, all labelled $\xi(l(p_v))$, to get an equivalent point of $S_\epsilon^{c,c'}$. We use the relations in order from top to bottom, i.e starting by adding in $s_{\exp(c)}$ and ending with adding in s_1. Here we use the fact that $(\xi(l(p_v))\triangleright)^{\exp(c)}$ acts trivially on c to see that the other labels don't change. Now, define the homotopy $F(y, t)$ by continuously changing the location of the points s_j to $(t, \pi_2(p_v) + \frac{\epsilon j}{\exp(c)+1})$ at time t. At $t = 0$, F is given by the composite $T_\epsilon^{c,c'} \xrightarrow{\theta} S_\epsilon^{c,c'} \to S_{\epsilon'}^{c,c'}$. Thus it will suffice to show:

(1) At $t = 1$, F is the constant map to the base point.
(2) F descends along θ to a continuous map $S_\epsilon^{c,c'} \times I \to S_{\epsilon'}^{c,c'}$.

Indeed, if these are shown, then F descends to a nullhomotopy of the inclusion $S_\epsilon^{c,c'} \to S_{\epsilon'}^{c,c'}$. Checking (1) is straightforward: for the point y, at $t = 1$, the coordinates of s_1 is

$(1, \pi_2(p_j) + \frac{\epsilon j}{\exp(c)+1}))$, and the points with second projection smaller than it in decreasing order are $p_v, \ldots, p_k$. Using relation (ii), this is equivalent to a point where s_1 is removed, and n changes to $n' = ((l(p_v) \ldots l(p_k)) \triangleright l(s_1))n$. But $l(p_v) \triangleright l(s_1) = g \triangleright \xi(l(p_v)) \notin c'$ by property (a) of ξ, and the action of $l(p_r)$ for $r > v$ preserves not being in c'. It follows that $((l(p_v) \ldots l(p_k)) \triangleright l(s_1)) \notin c'$, and hence n' is the base point. Thus, by relation (iii), this point is the base point of $S^{c,c'}_{\epsilon'}$, showing (1).

It remains to show (2), which posits that F is compatible with the equivalence relations (i), (ii), and (iii). By construction, F is compatible with the equivalence relation (iii), since if m or n is the base point, it is by definition the constant homotopy at the base point.

We next check F is compatible with the equivalence relation (ii). Indeed, the value of g remains unchanged under the identification from (ii), unless (ii) is applied to the point p_v. In this case, the homotopy is the constant homotopy at the base point, so is compatible with (ii).

It remains to check that F is compatible with the equivalence relation (i). We first deal with the case that the point p_r being removed in (i) has a label $l(p_r) \notin c'$. In this case the homotopy is the constant homotopy at the base point, because for any time t, the point p_r remains on the left hand side and one can use relations (i) and (iii) to identify it with the base point. Therefore, the nullhomotopy is compatible with the relation in this case.

It only remains to consider then the case that the point p_r being removed in (i) has $l(p_r) \in c'$. Note we cannot have $r = v$ because $l(p_v) \notin c'$.

We first work out the case $r > v$ at a point y presented as $(m, n, x, l : x \to c)$ as in (†) above. On the one hand, if we apply the nullhomotopy before applying (i), m changes to $m\xi(g)^{-\exp(c)}$, and we add in points $s_1, \ldots, s_{\exp(c)}$ with first coordinate t labelled by $\xi(l(p_v))$, and the rest of the labels remain unchanged. Applying relation (i) to the points throughout this nullhomotopy, m changes to $m\xi(g)^{-\exp(c)}l(p_r)$, the labels on p_j for $j < r$ change to $l(p_r) \triangleright l(p_j)$, and the labels on the s_j change to $l(p_r) \triangleright \xi(l(p_v))$.

On the other hand, if we first apply relation (i), m changes to $ml(p_r)$, the point p_r gets removed, and the points p_j for $j < r$ change to $l(p_r) \triangleright l(p_j)$. Note that p_v remains the lowest point with label not in c'. Applying the nullhomotopy F to this point, we add in the same points $s_1, \ldots, s_{\exp(c)}$, now with labels $\xi(l(p_r) \triangleright l(p_v))$, and change the element $ml(p_r)$ to $ml(p_r)\xi(l(p_r) \triangleright l(p_v))^{-\exp(c)}$.

Because of property (b) of the function ξ, $l(p_r) \triangleright \xi(l(p_v)) = \xi(l(p_r) \triangleright l(p_v))$, and because $\xi(g)^{-\exp(c)}$ is central, $m\xi(g)^{-\exp(c)}l(p_r) = ml(p_r)\xi(g)^{-\exp(c)}$. It follows that the nullhomotopy glues along relation (i) when $r > v$.

The case $r < v$ is similar, except that the labels on p_v and the s_j don't change, making it easier to check compatibility, because property (b) of ξ isn't used in this case. □

4.5.7. *Proof of Theorem 4.2.2.* By Lemma 4.4.4, it is enough to show that for each subrack $c' \subset c$, $v[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}}$ is an equivalence. When $c' = c$, the result is Proposition 4.5.1. When $c' \subset c$ is a proper subrack, we have a commutative square

$$\begin{tikzcd}
A[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}} \arrow[r] \arrow[d] & C_*(\mathrm{Hur}^{c'};\mathbb{Q})[\alpha_{(c')^{-1}}] \arrow[d] \\
A'[\alpha_{c'}^{-1}]^{\wedge}_{\alpha_{c-c'}} \arrow[r] & C_*(\mathrm{Conf}^{c'};\mathbb{Q})[\alpha_{(c')^{-1}}].
\end{tikzcd}$$

We wish to show the left vertical map is an equivalence, and so we will show the other three maps in the diagram are equivalences. The upper horizontal map is an equivalence by Proposition 4.5.5. The lower horizontal map is an equivalence because any homology class in $A'[\alpha_{c'}^{-1}]$ supported on a component that does not come from $\mathrm{Conf}^{c'}$ is divisible by $\alpha_{c-c'}$, and the completion is projection onto the summand where $\alpha_{c-c'}$ acts by 0 by Lemma 3.3.5. Finally, the right vertical map is an equivalence by Proposition 4.5.1, where c is replaced with c'. $\square$

5. Deducing the moments of the Cohen-Lenstra heuristics

In this section, we explain how we deduce Theorem 1.1.5, and hence Theorem 1.1.1, from our computation of the stable homology of Hurwitz spaces. The structure of this section is as follows. We first prove Lemma 5.1.2, which gives a criterion for computing the finite field point counts of a space in terms of its stable homology. In Proposition 5.2.1, we apply this lemma to Hurwitz spaces, and obtain a point count in terms of the number of geometrically irreducible components of that Hurwitz space. In Proposition 5.2.2, we count the components of the relevant Hurwitz spaces. Finally, in §5.3, we deduce Theorem 1.1.1 from the finite field point counts of these Hurwitz spaces.

5.1. **A preliminary lemma on point counting.** The next lemma is quite standard, and its proofs essentially appears in many places, such as [EVW16, Theorem 8.8] and also [EL23, Theorem 9.2.3]. The next lemma is a formal consequence of the Grothendieck-Lefschetz trace formula and Deligne's bounds on eigenvalues of Frobenius.

Remark 5.1.1. We prove the following lemma in the more general setting of stacks, but we will only apply it in the context of schemes.

In what follows, if x is a complex number, we use $\|x\|$ to denote its absolute value, while if X is a set, we use $|X|$ to denote its cardinality. For $\mathscr{X}$ a Deligne-Mumford stack, we follow the standard convention that $|\mathscr{X}(\mathbb{F}_q)|$ denotes $\sum_{x\in\mathscr{X}(\mathbb{F}_q)}\frac{1}{|\mathrm{Aut}(x)|}$, so points are weighted inversely proportionally to their automorphism groups. Of course, when $\mathscr{X}$ is a scheme, this agrees with the usual notion of $|\mathscr{X}(\mathbb{F}_q)|$.

Lemma 5.1.2. *Fix a prime power q and a sequence $\{Y_n\}_{n\in S}$ of nonempty smooth Deligne-Mumford stacks over $\operatorname{Spec}\mathbb{F}_q$ with Y_n of pure dimension n, where n traverses over an infinite sequence of increasing integers $S\subset\mathbb{Z}_{\geq 0}$. Suppose that the following cohomological conditions are satisfied:*

(1) Stability: Fix $i\geq 0$. There is some prime ℓ, possibly depending on n, and a fixed $\mathbb{Q}_\ell$ vector space V_i with the action of geometric Frobenius, Frob_q, such that V_i with its Frobenius action is independent of n. Assume there are constants I and J, independent of n, so that whenever $n > Ii+J$, there is a fixed vector space V_i with Frobenius action, so that $\mathrm{tr}(\mathrm{Frob}_q|V_i) = \mathrm{tr}(\mathrm{Frob}_q|H^i(Y_{n,\overline{\mathbb{F}}_q};\mathbb{Q}_\ell))$.

(2) Exp. Bound: There are constants C, C' independent of n so that for any i, $\dim H^i(Y_{n,\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\leq C'C^i$.

Then, if $q > C^2$,

$$\left\| \frac{|Y_n(\mathbb{F}_q)|}{q^n} - \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \right\| \leq \frac{2C'}{1 - \frac{C}{\sqrt{q}}} \left(\frac{C}{\sqrt{q}} \right)^{\frac{n-J}{I}}.$$

Proof. To ease notation, let $X_n := Y_n \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \overline{\mathbb{F}}_q$. Note that our stability assumption holds whenever $n > Ii + J$, or equivalently when $i < \frac{n-J}{I}$. Using our stability assumption (1) and the Grothendieck-Lefschetz trace formula [Beh93, Theorem 3.1.2], we have

$$\begin{aligned} \frac{|Y_n(\mathbb{F}_q)|}{q^n} &= \sum_{i=0}^{2n} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q^{-1} | H^i(X_n; \mathbb{Q}_\ell) \right) \\ &= \sum_{0 \leq i < \frac{n-J}{I}} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) + \sum_{i \geq \frac{n-J}{I}} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q^{-1} | H^i(X_n; \mathbb{Q}_\ell) \right) \end{aligned} \tag{5.1}$$

Subtracting $\sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i)$ from both sides of (5.1) and taking absolute values, we obtain

$$\begin{aligned} &\left\| \frac{|Y_n(\mathbb{F}_q)|}{q^n} - \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \right\| \\ &\leq \sum_{i \geq \frac{n-J}{I}} \left\| \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \right\| + \sum_{i \geq \frac{n-J}{I}} (-1)^i \left\| \operatorname{tr}\left(\mathrm{Frob}_q^{-1} | H^i(X_n; \mathbb{Q}_\ell) \right) \right\| \end{aligned} \tag{5.2}$$

We claim that $\sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i)$ converges absolutely as a function of q for $\sqrt{q} > C$. Indeed, using Sun's generalization of Deligne's bounds for algebraic stacks [Sun12, Theorem 1.4] and our exponential bound assumption (2) we obtain $\| \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \| \leq \frac{C' C^i}{q^{i/2}}$. It follows that

$$\sum_{i=0}^{\infty} \| (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \| \leq C' \sum_{i=0}^{\infty} \left(\frac{C}{\sqrt{q}} \right)^i = C' \frac{1}{1 - \frac{C}{\sqrt{q}}}.$$

We similarly find that

$$\sum_{i=\frac{n-J}{I}}^{\infty} \| (-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) \| \leq C' \sum_{i=\frac{n-J}{I}}^{\infty} \left(\frac{C}{\sqrt{q}} \right)^i = C' \left(\frac{C}{\sqrt{q}} \right)^{\frac{n-J}{I}} \frac{1}{1 - \frac{C}{\sqrt{q}}}. \tag{5.3}$$

This bounds the first term on the right hand side of (5.2). It remains to bound the second term. Again, using Sun's generalizations of Deligne's bounds and our exponential bound assumption (2), we similarly obtain $\| \operatorname{tr}(\mathrm{Frob}_q^{-1} | H^i(X_n; \mathbb{Q}_\ell)) \| \leq \frac{C' C^{2n-j}}{q^{i/2}}$. Hence, so long as $C < \sqrt{q}$, we can bound

$$\begin{aligned} \sum_{i \geq \frac{n-J}{I}} (-1)^i \left\| \operatorname{tr}\left(\mathrm{Frob}_q^{-1} | H^i(X_n; \mathbb{Q}_\ell) \right) \right\| &\leq C' \sum_{i \geq \frac{n-J}{I}} \left(\frac{C}{\sqrt{q}} \right)^i \\ &= C' \left(\frac{C}{\sqrt{q}} \right)^{\frac{n-J}{I}} \frac{1}{1 - \frac{C}{\sqrt{q}}}. \end{aligned}$$

□

5.2. **Applying the lemma to count points on Hurwitz spaces.** We next show that the conditions of Lemma 5.1.2 are satisfied for Hurwitz spaces using our computation of the stable homology. We use Lemma 5.1.2 to obtain a count of the finite field valued points of relevant Hurwitz spaces in terms of the number of their irreducible components which are also geometrically irreducible. It may be useful to recall the notation $\mathrm{CHur}_{n,B}^{G,c,g}$ for components of $\mathrm{CHur}_{n,B}^{G,c}$ whose boundary monodromy is g, as defined in Definition 2.2.1

Proposition 5.2.1. *Suppose G, H and c are as in Notation 1.2.6. We let q be an odd prime power satisfying* $\gcd(q, |H|) = 1$. *Fix $g \in G$. Assume that for any sufficiently large n, $\mathrm{CHur}_{n,\mathbb{F}_q}^{G,c,g}$ has r irreducible components which are also geometrically irreducible. Then there are constants C, I, J, all independent of n and q, but possibly depending on r, so that, long as $q > C^2$,*

$$\left\| \frac{\left|\mathrm{CHur}_{n,\mathbb{F}_q}^{G,c,g}(\mathbb{F}_q)\right|}{q^n} - r\left(1 - \frac{1}{q}\right) \right\| \leq \frac{2C}{1 - \frac{C}{\sqrt{q}}} \left(\frac{C}{\sqrt{q}}\right)^{\frac{n-J}{I}}.$$

Proof. This follows from Lemma 5.1.2, taking the constant C' there to be the same as C, once we verify its hypotheses. We will take $\ell > \max(n, |G|, q)$.

We first verify the stability assumption from Lemma 5.1.2(1). To do this, we start by constructing the vector spaces V_i, with the Frobenius action. For B a scheme, we use $\mathrm{Conf}_{n,B}$ denote the configuration space parameterizing n unordered points on $\mathbb{A}^1_B$. (Previously, we had used Conf_n as notation for this when $B = \operatorname{Spec} \mathbb{C}$.) Now, define $V_i := H^i(\mathrm{Conf}_{n,\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)^{\oplus r}$, as a vector space with Frobenius action, where r is the number of irreducible components of of $\mathrm{CHur}_{n,\mathbb{F}_q}^{G,c,g}$ which are also geometrically irreducible for sufficiently large n. Note that the dimension of V_i is independent of n and q once $n > 2$ as a vector space with Frobenius action, and the Frobenius action is given by and identity matrix of dimension r if $i = 0$, q times the identity matrix of dimension r when $i = 1$, and the 0 matrix if $i > 1$. In particular, we have $\sum_{i=0}^{\infty}(-1)^i \operatorname{tr}(\mathrm{Frob}_q^{-1} | V_i) = r(1 - 1/q)$.

To verify Lemma 5.1.2(1), we next need to show the trace of Frobenius on V_i agrees with the trace of Frobenius on $H^i(\mathrm{Hur}_{n,\overline{\mathbb{F}}_q}^{G,c}; \mathbb{Q}_\ell)$ for $n > iI + J$. There is a map

$$H^i(\mathrm{Conf}_{n,\mathbb{C}}; \mathbb{Q}_\ell)^{\oplus \pi_0(\mathrm{Hur}_{n,\mathbb{C}}^{G,c})} \to H^i(\mathrm{Hur}_{n,\mathbb{C}}^{G,c}; \mathbb{Q}_\ell), \tag{5.4}$$

which is an isomorphism for $n > iI + J$ by Theorem 1.2.7. It follows from [EVW16, Proposition 7.7] that there is an isomorphism

$$H^i(\mathrm{Hur}_{n,\overline{\mathbb{F}}_q}^{G,c}; \mathbb{Q}_\ell) \simeq H^i(\mathrm{Hur}_{n,\mathbb{C}}^{G,c}; \mathbb{Q}_\ell) \tag{5.5}$$

once we exhibit a suitable normal crossings compactification of configuration space. This is afforded by [FM94, Theorem 3]. (See also the proof of [EL23, Theorem 9.2.3] for further explanation.) Taking $B := \operatorname{Spec} \mathbb{Z}[1/|G|]$, there is an open and closed subscheme of $\mathrm{Hur}_{n,B}^{G,c}$ given by $\mathrm{CHur}_{n,B}^{G,c,g}$, since the properties of the covers being geometrically connected and their boundary monodromy being g is locally constant in families. Therefore, (5.5) restricts to an isomorphism

$$H^i(\mathrm{CHur}_{n,\overline{\mathbb{F}}_q}^{G,c,g}; \mathbb{Q}_\ell) \simeq H^i(\mathrm{CHur}_{n,\mathbb{C}}^{G,c,g}; \mathbb{Q}_\ell). \tag{5.6}$$

In particular, this implies $|\pi_0(\mathrm{CHur}^{G,c,g}_{n,\mathbb{C}})| = |\pi_0(\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q})|$. An analogous argument then shows

$$H^i(\mathrm{Conf}_{n,\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^{\oplus|\pi_0(\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q})|} \simeq H^i(\mathrm{Conf}_{n,\mathbb{C}};\mathbb{Q}_\ell)^{\oplus|\pi_0(\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q})|}. \tag{5.7}$$

Combining (5.4), (5.6), and (5.7), we obtain an isomorphism $U_i : H^i(\mathrm{Conf}_{n,\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)^{\oplus|\pi_0(\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q})|} \to H^i(\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$. We wish to show $\mathrm{tr}(\mathrm{Frob}_q\,|V_i) = \mathrm{tr}(\mathrm{Frob}_q\,|U_i)$. The subtle point here is that Frobenius may not act by a scalar on U_i because Frobenius acts nontrivially on the components of $\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q}$. The reason these two traces are equal is that V_i corresponds to the cohomology of those components fixed by the Frobenius action. Since Frobenius permutes the remaining components, the matrix restricted to the remaining components will have trace 0, as it all its diagonal terms are 0. This implies $\mathrm{tr}(\mathrm{Frob}_q\,|V_i) = \mathrm{tr}(\mathrm{Frob}_q\,|U_i)$, verifying the stability assumption Lemma 5.1.2(1).

Finally, we verify the exponential bound assumption, Lemma 5.1.2(2). This condition with $C = C'$ nearly follows from [EVW16, Proposition 7.8], using our assumption that $\ell > \max(n, |G|, q)$ as well as the comparison between étale and Betti cohomology. The slight difference we need is that [EVW16, Proposition 7.8] is proven for Hurwitz stacks, instead of pointed Hurwitz spaces. However, the same proof works for pointed Hurwitz spaces, since the analog of [EVW16, Corollary 6.2] without quotienting by the G conjugation action can be deduced in the same way from [EVW16, Theorem 6.1]. We note that the value of C is independent of q and ℓ satisfying our constraints because $\dim H^i(\mathrm{CHur}^{G,c,g}_{n,\mathbb{C}};\mathbb{Q}_\ell)$ is independent of q and ℓ. □

Proposition 5.2.1 lets us count the points of relevant Hurwitz spaces once we compute their number of geometrically irreducible components. We do so in the next proposition. It will be useful to recall the notion of being split completely over ∞ from Definition 2.3.1.

Proposition 5.2.2. *Suppose G, H, and c are as in Notation 1.2.6. Let q be an odd prime power with $\gcd(q, |H|) = 1$. Take $h := \gcd(|H|, q-1)$, and fix $g \in G$. For n sufficiently large, there are $\wedge^2 H[h]$ many irreducible components of $\mathrm{CHur}^{G,c,g}_{n,\mathbb{F}_q}$, which are also geometrically irreducible components, parameterizing H-covers of hyperelliptic curves which are split completely over ∞. There are constants constant C, I, J, independent of n and q, and depending only on H, so that for q satisfying $\gcd(|H|, q) = 1$, $q > C^2$, and any $g \in G$,*

$$\left\| \frac{\left|\mathrm{CHur}^{G,c,g}_{n,\mathbb{F}_q}(\mathbb{F}_q)\right|}{q^n} - |\wedge^2 H[h]| \left(1 - \frac{1}{q}\right) \right\| \leq \frac{2C}{1 - \frac{C}{\sqrt{q}}} \left(\frac{C}{\sqrt{q}}\right)^{\frac{n-J}{I}}. \tag{5.8}$$

Proof. Once we verify the claim about the number of irreducible components which are geometrically irreducible, (5.8) follows from Proposition 5.2.1. The remainder of the proof is essentially explained in [SW23, Theorem 3.1], and we now spell out the details. In the case that n is even, [LWZB24, Lemma 10.3] yields a bijection between components of $\mathrm{CHur}^{G,c,g}_{n,\overline{\mathbb{F}}_q}$ and components of $\mathrm{CHur}^{G,c,g}_{n,\mathbb{C}}$. Then, [LWZB24, Theorem 12.1(2)] and [LWZB24, Corollary 12.6] yield that the number of irreducible components of $\mathrm{CHur}^{G,c,g}_{n,\mathbb{C}}$ can be identified bijectively with a torsor for the set of elements of a certain finite group $H_2(G,c)$

which are equal to their qth power. There is an identification $H_2(G,c) \simeq (\wedge^2 H)^{\mathbb{Z}/2\mathbb{Z}}$ as is explained in the course of the proof of [SW23, Theorem 3.1]. Moreover, in our case, $\mathbb{Z}/2\mathbb{Z}$ acts on H by inversion, and hence it acts trivially on $\wedge^2 H$, and so $(\wedge^2 H)^{\mathbb{Z}/2\mathbb{Z}} \simeq \wedge^2 H$. In other words, the number of irreducible components which are geometrically irreducible is precisely the $q-1$ torsion subgroup of $\wedge^2 H$, $\wedge^2 H[q-1]$. Since $h := \gcd(|H|, q-1)$, this is also equal to $\wedge^2 H[h]$, and hence there are $\left|\wedge^2 H[h]\right|$ irreducible components which are geometrically irreducible components of $\mathrm{CHur}^{G,c,g}_{n,\mathbb{F}_q}$.

We conclude by explaining why the above argument still works when n is odd. The setting of [SW23] and [LWZB24] is slightly different from ours because they make the additional assumption that all G-covers are unramified over $\infty \in \mathbb{P}^1$. However, with our definition of Hurwitz spaces, Definition 2.1.3, the straightforward generalizations of all of the results cited in the above paragraph easily carry through. The main minor modification needed is that instead of working with marked covers $X \to \mathbb{P}^1_S$ as in [Woo21], one should instead work with marked covers $X \to \mathscr{P}_S$, where $\mathscr{P}_S$ is a root stack with an order 2 root over the ∞ section, $\infty : S \to \mathbb{P}^1_S$. Additionally, in [LWZB24, Theorem 12.4], one no longer requires that the inertia elements g_i need multiply to id, since the corresponding cover $X \to \mathbb{P}^1$ will be branched at ∞. With these minor modifications in place, the rest of the argument goes through for n odd and completes the proof.

At this point, the constant C may depend on the values of h and g, but since there are at most $|G|^2$ many choices for the pair (h, g), we can replace C with the constant from this finite set of values which maximizes the right hand side of (5.8). This concludes the proof. $\square$

5.3. **Proof of Theorem 1.1.5.** We conclude by proving our main result, Theorem 1.1.5, which determines the H-moment of the class group of quadratic extensions of $\mathbb{F}_q(t)$ for q sufficiently large relative to H. It will be useful to recall our notation for the class group of a function field extension of $\mathbb{F}_q(t)$ and for the set $\mathcal{MH}_{n,q}$ of monic smooth hyperelliptic curves over $\mathbb{F}_q$, defined prior to Theorem 1.1.1. The proof of Theorem 1.1.5 essentially follows immediately from Proposition 5.2.2 and a little bit of class field theory. In fact, we prove a slight refinement of Theorem 1.1.5 which moreover quantifies the error term in the asymptotic count of torsion in class groups of quadratic fields.

Theorem 5.3.1. *Suppose H is a finite abelian group of odd order. Fix q an odd prime power with $\gcd(|H|, q) = 1$ and let $h := \gcd(|H|, q-1)$. There is are integer constants I, J, and C depending only on H so that, if $q > C^2$,*

$$\left\| \frac{\sum_{K \in \mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|}{\sum_{K \in \mathcal{MH}_{n,q}} 1} - \alpha_n \left|\wedge^2 H[h]\right| \right\| \leq \left(\frac{1}{1-\frac{1}{q}}\right) \frac{2C}{1-\frac{C}{\sqrt{q}}} \left(\frac{C}{\sqrt{q}}\right)^{\frac{n-J}{I}}, \tag{5.9}$$

where $\alpha_n = 1$ if n is odd and $\alpha_n = \frac{1}{|H|}$ if n is even.

Proof. Note that the class group $\mathrm{Cl}(\mathscr{O}_K)$ is identified with the Galois group of the maximal abelian unramified extension of K, completely split completely over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$, as follows from class field theory [Hay92, 15.6]. (Also see the paragraph following [Hay92, 15.6] to show that what is called H_A there is in fact the maximal unramified abelian extension split completely over ∞.) Recall $G = H \rtimes (\mathbb{Z}/2\mathbb{Z})$.

We first deal with the case n is even. When n is even, by Lemma 2.3.2, every $K \in \mathcal{MH}_{n,q}$ corresponds bijectively to a smooth proper curve X split completely over ∞. For n even, we can then identify $\sum_{K \in \mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)| = \frac{\left|\mathrm{CHur}^{G,c,\mathrm{id}}_{n,\mathbb{F}_q}(\mathbb{F}_q)\right|}{\#H}$ by combining [LWZB24, Lemma 9.3 and Lemma 10.2]. Note that the denominator $\sum_{K \in \mathcal{MH}_{n,q}} 1 = q^n - q^{n-1}$ for $n > 1$, as it is identified with the $\mathbb{F}_q$ points of unordered configuration space of n points in $\mathbb{A}^1_{\mathbb{F}_q}$. The case that n is even now follows from Proposition 5.2.2 upon dividing both sides of (5.8) by $1 - \frac{1}{q}$.

It remains to deal with the case that n is odd. Recall notation for components with specified boundary monodromy from Definition 2.2.1. The case that n is odd is similar to the even case, except that we now claim $\sum_{K \in \mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)| = \frac{\left|\mathrm{CHur}^{G,c}_{n,\mathbb{F}_q}(\mathbb{F}_q)\right|}{\#H}$, with no condition imposed on the boundary monodromy. Indeed, any G cover corresponding to a point of $\mathrm{CHur}^{G,c}_{n,\mathbb{F}_q}$ has inertia type c at ∞, and so the the G extension corresponds to an H extension L/K followed by a quadratic extension $K/\mathbb{F}_q(t)$, where the quadratic extension is ramified over ∞ and the H extension is split completely over ∞. Here we are using that the quadratic extension $K/\mathbb{F}_q(t)$ here is split completely over ∞ in the stacky sense of Definition 2.3.1, and we are using Lemma 2.3.2 to identify $\mathcal{MH}_{n,q} \simeq \mathrm{Hur}^{\mathbb{Z}/2\mathbb{Z},\{1\}}_{n,\mathbb{F}_q}(\mathbb{F}_q)$. Hence, via the identification $\mathrm{Hur}^{\mathbb{Z}/2\mathbb{Z},\{1\}}_{n,\mathbb{F}_q}$ with the configuration space of n unordered points in $\mathbb{A}^1_{\mathbb{F}_q}$, we have $\sum_{K \in \mathcal{MH}_{n,q}} 1 = q^n - q^{n-1}$. Altogether, there are $|H|$ possible values for the boundary monodromy at ∞ when n is odd, which is why the H-moment in the case that n is odd is $|H|$ times the moment in the case that n is even. Thus, the result again follows by summing the result of Proposition 5.2.2 over those $|H|$ elements $g \in c = G - H$. $\square$

Appendix A. A topological model for the Hurwitz space bar construction

The goal of this appendix is to provide a good topological model for the two-sided bar construction $M \otimes_{\mathrm{Hur}^{G,c}_+} N$, where M is a pointed set with a right $\mathrm{Hur}^{G,c}$-action and N is a pointed set with a left $\mathrm{Hur}^{G,c}$ action. The end result that is used in the main body of the paper is Theorem A.4.9.

A.1. Background on Hurwitz spaces of braided sets.

A natural setting in which our constructions go through is that of braided sets, sometimes also called "sets equipped with an invertible solution of the Yang–Baxter equation," see [Dri90, LYZ00, Sol00]. We recall the definition:

Definition A.1.1. A *braided set* is a set X equipped with an automorphism $\sigma : X \times X \to X \times X$ satisfying the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, where σ_i is the automorphism of $X \times X \times X$ obtained by applying σ to factors i and $i+1$ and the identity on the other factor.

Construction A.1.2 (Hurwitz spaces for braided sets)**.** Let X be a braided set. Then there is a left action of the braid group on X^n, where the ith standard generator σ_i of the braid group acts by applying σ to the i and $i+1$ factors of X^n and acts as the identity on all other factors. The collection $(X^n)_{n \in \mathbb{Z}_{\geq 0}}$ can be viewed as an $\mathbb{E}_1$-algebra in the category of local system on $\mathrm{Conf} = \cup_{n \geq 0} BB_n$, where the multiplication is given by concatenation. We define Hur^X to be the $\mathbb{E}_1$-algebra in Spc given by the étale cover of Conf arising from this local system, and use Hur^X_n to denote the part lying over $\mathrm{Conf}_n = BB_n$. This construction is natural in maps of braided sets, and there is a natural $\mathbb{E}_1$-algebra map $\mathrm{Hur}^X \to \mathrm{Conf}$. Explicitly, the underlying space of Hur^X_n is the homotopy quotient $(X^n)_{hB_n}$.

Throughout the appendix, we use X to refer to a braided set.

Example A.1.3. The reader should keep in mind the following example: If G is a group and c is a union of conjugacy classes, then c forms a braided set, where the twist operation σ sends (x, y) to $(y, y^{-1}xy)$. Then the Hurwitz space $\cup_n \mathrm{Hur}^{G,c}_{n,\mathbb{C}}$ as defined in §1.2.2 agrees with Hur^X as defined in Construction A.1.2. More generally, if c is a rack, the twist operation on Hur^c is given by sending $(x, y) \mapsto (y, y \triangleright x)$.

The next remark will not be used in what follows, but it may help give a conceptual description of why braided sets is the level of generality we choose to work in.

Remark A.1.4. Consider the $\mathbb{E}_1$-algebra map $\mathbb{N} \to \mathrm{Conf}$ arising from the fact that $\mathbb{N}$ is the free $\mathbb{E}_1$-algebra in Spc on a point. Via Construction A.1.2, the category of braided sets identifies with the full subcategory of $\mathbb{E}_1$-algebras in local systems on Conf such that their restriction to $\mathbb{N}$ is a free algebra on a set in degree 1.

A.2. **A monoid model of Hurwitz spaces.** We begin by defining topological spaces that model Hur^X and Conf: First, we define a topological monoid, conf, which is a model for the $\mathbb{E}_1$-algebra structure on Conf.

Notation A.2.1. Define the topological space conf as the space whose points are given by pairs (t, x) where, $t \in \mathbb{R}_{\geq 0}$, and x is a (possibly empty) configuration of finitely many distinct unordered points $(0, t) \times (0, 1)$, not intersecting $[0, \frac{1}{2}) \times [0, 1]$ or $(t - \frac{1}{2}, t] \times [0, 1]$. The multiplication is given by concatenation of configurations and adding the elements t. We use conf to denote this topological monoid, and conf_n to denote the component with configuration of n points. There is a map of topological monoids $t : \mathrm{conf} \to \mathbb{R}_{\geq 0}$ given by $(t, x) \mapsto t$. There is also a subset $\mathrm{ord} \subset \mathrm{conf}$ consisting of configurations such that the second coordinate of each point in x is $\frac{1}{2}$. For each n, the intersection $\mathrm{ord} \cap \mathrm{conf}_n$ is contractible. We will use the inclusion $\mathrm{ord}_n := \mathrm{ord} \cap \mathrm{conf}_n \subset \mathrm{conf}_n$ to view each component of conf as being equipped with a 'base point', i.e a fixed contractible subspace. We also use confbig to denote the variant of this topological monoid where we remove the condition that configurations not intersect $[0, \frac{1}{2}) \times [0, 1]$ and $(t - \frac{1}{2}, t] \times [0, 1]$, and use bigord and bigord_n to denote the corresponding subspaces. There is an inclusion of topological monoids $\mathrm{conf} \to \mathrm{confbig}$ which is clearly a homotopy equivalence.

Notation A.2.2. We identify $\pi_1(\mathrm{conf}_n, \mathrm{ord}_n)$ with the braid group such that the elementary twist σ_i (twisting strands i and $i + 1$) gets sent to the homotopy class of paths from ord_n that swaps the ith and $i + 1$th element of the configuration via rotating them clockwise around each other by π.

Convention A.2.3. By convention, the composition in the fundamental group/groupoid of a space is such that given two paths γ, γ', we have $[\gamma][\gamma']$ is the class of the path which is first γ' and then γ (assuming they are composable). This convention is the one that is compatible with the orientation of the interval in that we view γ' as a morphism from its starting point to its end point.

We next define a topological monoid hur^X which models Hur^X as an $\mathbb{E}_1$-algebra.

Notation A.2.4. Since Hur^X, viewed as an $\mathbb{E}_1$-algebra, is obtained as a multiplicative finite cover of Conf, we can construct a model hur^X for Hur^X as the corresponding multiplicative finite cover of conf in topological spaces. Namely, we first use the base points from the monoid map $\mathbb{N} \to \mathrm{conf}$ constructed in Notation A.2.1 to identify the fundamental group of conf with B_n, using the convention of Notation A.2.2. Then, hur^X_n, as a cover of conf_n, can be constructed as the quotient of $X^n \times \widetilde{\mathrm{conf}}_n$ by the braid group B_n, where $\widetilde{\mathrm{conf}}_n$ is the universal cover of conf_n; More explicitly, a point of hur^X_n is given by a B_n-equivalence class of data (x, t, γ, α_i), where $(x, t) \in \mathrm{conf}_n$, γ is a homotopy class of paths from this point to ord, and α_i, for $1 \le i \le n$, is an n tuple of points in X, where $n = |x|$. Here the braid group action is such that jth standard generator of B_n, σ_j, sends (x, t, γ, α_i) to $(x, t, \sigma_j\gamma, \sigma_j(\alpha_i))$ using Convention A.2.3 and Notation A.2.2 to get a left action of B_n on such γ, and using the left action of the braid group on X^n from Construction A.1.2.

To make $\mathrm{hur}^X := \cup_{n\ge 0} \mathrm{hur}^X_n$ into a topological monoid, we use the multiplication of conf, concatenate paths, and concatenate tuples of points in X. There is a map of topological monoids $\mathrm{hur}^X \to \mathrm{conf}$. Composing the time coordinate $t : \mathrm{conf} \to \mathbb{R}_{\ge 0}$ with this map, we obtain a composite map, (which we also call t,) $t : \mathrm{hur}^X \to \mathbb{R}_{\ge 0}$. This is a map of topological monoids. We also define a variant hurbig^X of hur^X by doing the same construction with confbig instead of conf. We note hurbig^X is homotopy equivalent to hur^X.

The key construction we use for Hurwitz spaces for braided sets is the following cutting construction. The idea for this construction is that if we start with a configuration in $[0, t'] \times [0, 1]$ which never contains points in the vertical line $t \times [0, 1]$, we can cut it into two configurations by cutting along $t \times [0, 1]$.

Construction A.2.5 (Cutting)**.** Let $t \in [0, \infty)$, and consider the open set $U(t)$ of hurbig^X consisting of $(x, t', \gamma, \alpha_i)$ such that $t' > t$ and no points of the configuration x lie on $t \times [0, 1]$. Then there is a continuous map $U(t) \to \mathrm{hurbig}^X \times \mathrm{hurbig}^X$ which we call 'cutting at t', given as follows: We first cut the configuration on $[0, t']$ into two configurations x', x'' on $[0, t]$ and $[0, t' - t] \cong [t, t']$. Choose a homotopy class of paths ϕ from x to ord with a representative that doesn't cross $t \times [0, 1]$. Using the braid group action, the pair $(x, t', \gamma, \alpha_i)$ is equivalent to (x, t', ϕ, β_i) for some β_i obtained by acting on the tuple (x_i) via the element of the braid group given by the class of $\phi\gamma^{-1}$. We can now similarly cut ϕ into homotopy classes of paths ϕ', ϕ'' from the configurations x', x'' to the respective base points, and cut the sequence β_i into α'_j, α''_k by unconcatenating it into two sequences corresponding to the number of points in the configuration in $[0, t] \times [0, 1]$ and $[t, t'] \times [0, 1]$ respectively.

Then the cutting map sends $(x, t', \gamma, \alpha_i)$ to $(x', t, \phi', \alpha'_j)$ and $(x'', t' - t, \phi'', \alpha''_k)$. Since the concatenation map $X^a \times X^b \to X^{a+b}$ is $B_a \times B_b$-equivariant, this doesn't depend on the choice of ϕ.

A.3. **A scanning map.** Next, we produce well behaved topological models for the source and target of Proposition 4.5.6. The following definition will be the key ingredient in constructing the models:

Notation A.3.1. Let M be a set with a right action of Hur^X and N be a set with a left action of Hur^X. Consider the space $B[M, \mathrm{Hur}^X, N]$ consisting of points which are of the form

$$(a, b, y) \tag{A.1}$$

where $a \in M, b \in N$ and $y = (x, 1, \gamma, x_i) \in \mathrm{hurbig}^X$, following notation from Notation A.2.4. The topology on $B[M, \mathrm{Hur}^X, N]$ has basis given as follows. Consider the following data:

(a) Numbers $d_1, d_2 \in (0, 1)$.
(b) A finite collection of pairwise disjoint open balls $U_1, \ldots, U_n$ in contained in the interior of $[d_1, d_2] \times [0, 1]$.
(c) A homotopy class of paths ϕ from the center of the balls U_i, viewed as an element of $\mathrm{confbig}_n$, to the base point.
(d) Elements $\alpha_1, \ldots, \alpha_n \in X$.
(e) Elements $a' \in M, b' \in N$.

We next define subsets $\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')$ which form a basis of the topology on $B[M, \mathrm{Hur}^X, N]$. A point of the form (A.1) lies in $\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')$ if the following conditions hold.

(1) None of the points in x lie in $[d_1, d_2] \times [0, 1] - \cup_1^n U_i$, and there is a unique point from x in each U_i.
(2) Cutting the element of hurbig^X twice to restrict it to the interval $[d_1, d_2]$ yields a point $y' \in \mathrm{hurbig}^X$ (see Construction A.2.5). Then, using the homotopy class of ϕ, the corresponding tuple of elements in X associated to y' is $\alpha_1, \ldots, \alpha_n$.
(3) Define $y_1 \in \mathrm{hurbig}^X$ to be the element of hur^X obtained by cutting and restricting to the interval $[0, d_1]$, and let y_2 be the element similarly obtained by using the interval $[d_2, 1]$. We then require that $ay_1 = a'$ and $y_2 b = b'$.

Remark A.3.2. The topology of Notation A.3.1 is such that points of the configuration x are allowed to collide with the boundary $\{0, 1\} \times [0, 1]$, at which point they disappear and multiply the elements m, n using the left action of Hur^X on M and the right action on N. Condition (3) guarantees that when a collection of points converges to the boundary, it multiplies m or n to the correct value.

We next construct a homotopy equivalence showing that the topological space $B[M, \mathrm{Hur}^X, N]$ models the two-sided bar construction $M \otimes_{\mathrm{hur}^X} N$, which we define next. Our proof is an adaptation of the proof of [BDPW23, Theorem 5.2.3]. Indeed, the case $M = N = X = *$ corresponds to the result there.

Notation A.3.3. Let H be a topological monoid and let M be a right module for H and let N be a left module for H. View Δ^n as the collection of tuples $(y_0, y_1, \ldots, y_n)$ with $\sum_{i=0}^n y_i = 1$. We will describe elements of $M \times N \times \Delta^n \times H^n$ via the notation

$$(a, b, (y_0, \ldots, y_n), (x_1, \ldots, x_n)). \tag{A.2}$$

We define the *two-sided bar construction*, $M \otimes_H N$, to be the quotient of $\coprod_{n \geq 0} M \times N \times \Delta^n \times H^n$, by the following equivalence relations:

(1) $(a, b, (0, y_1, \ldots, y_n), (x_1, x_2, \ldots, x_n)) \sim (ax_1, b, (y_1, \ldots, y_n), (x_2, \ldots, x_n))$,
(2) $(a, b, (y_0, \ldots, y_{n-1}, 0), (x_1, \ldots, x_{n-1}, x_n)) \sim (a, x_n b, (y_0, \ldots, y_{n-1}), (x_1, \ldots, x_{n-1}))$,

(3) $(a, b, (y_0, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n), (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n))$
$\sim (a, b, (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n), (x_1, \ldots, x_{i-1}, x_i x_{i+1}, \ldots, x_n))$,
(4) $(a, b, (y_0, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n), (x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n))$
$\sim (a, b, (y_0, \ldots, y_{i-1} + y_i, y_{i+1}, \ldots, y_n), (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$, where e denotes the identity of the monoid H.

Lemma A.3.4. *Let M be a set with a right action of hur^X and let N be a set with a left action of hur^X. Then there is a weak homotopy equivalence $\sigma : M \otimes_{\mathrm{hur}^X} N \to B[M, \mathrm{Hur}^X, N]$, natural in X, M, N.*

Proof. Recall that $M \otimes_{\mathrm{hur}^X} N$ denotes the two-sided bar construction, as defined in Notation A.3.3. In particular, since hur^X models Hur^X as an $\mathbb{E}_1$-algebra, $M \otimes_{\mathrm{hur}^X} N$ models the two-sided bar construction $M \otimes_{\mathrm{Hur}^X} N$.

We produce a map $\sigma : M \otimes_{\mathrm{hur}^X} N \to B[M, \mathrm{Hur}^X, N]$, as follows: Given data as in (A.2) for $H = \mathrm{hur}^X$, and using notation from Notation A.2.4, let $t_i := t(x_i) \in \mathbb{R}_{>0}$ denote the time associated to $x_i \in \mathrm{hur}^X$. We multiply $x_1, \ldots, x_n$ to give an element $x \in \mathrm{hur}^X$ with $t := t(x) = \sum_{i=1}^n t_i$. First, extend the interval on which x is defined from $[0, t] \times [0, 1]$ to $[-\frac{1}{2}, t + \frac{1}{2}] \times [0, 1]$. We then define $t' := \sum_{i=1}^n y_i \cdot (\sum_{j=1}^i t_j)$. At this point, it will be useful to recall the cutting construction from Construction A.2.5. Choose $\epsilon > 0$ small enough so that there are no points of the configuration x in $(t' - \frac{1}{2}, t' - \frac{1}{2} + \epsilon] \times [0, 1]$ and $[t' + \frac{1}{2} - \epsilon, t' + \frac{1}{2}) \times [0, 1]$. We can then cut x into three pieces, $w, x'', z \in \mathrm{hurbig}^X$ coming from the intervals $[-\frac{1}{2}, t' - \frac{1}{2} + \epsilon], [t' - \frac{1}{2} + \epsilon, t' + \frac{1}{2} - \epsilon], [t' + \frac{1}{2} - \epsilon, t + \frac{1}{2}]$. Extend x'' to be of length 1 by extending the length of the interval on both ends by ϵ, and let x' denote the resulting element of hurbig^X. Note that x' doesn't depend on ϵ, and neither does the class of w, z in $\pi_0 \mathrm{Hur}^X$.

The then define the map σ to sends data as in (A.2) to the point (aw, zb, x'), in $B[M, \mathrm{Hur}^X, N]$, with notation as in (A.1). To see this is well defined, one must check that this glues along the identifications (1)-(4) in the two sided bar construction given in Notation A.3.3. We omit this verification.

To verify that σ is continuous, it suffices to do check continuity on $M \times N \times \Delta^n \times (\mathrm{hur}^X)^n$. We now carry out this straightforward verification. We choose a basic open set of the form $\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')$. We wish to verify its preimage under σ is also open. If $(a, b, (y_0, \ldots, y_n), (x_1, \ldots, x_n))$ is a point in the preimage, then in a small neighborhood, (1) and (2) of Notation A.3.1 are clearly still satisfied. The path components of w and z are also unchanged in a small neighborhood, so condition (3) is satisfied as well.

Having shown σ is continuous, we next claim that σ is an surjective on path components. Every point in $B[M, \mathrm{Hur}^X, N]$ has a path to a point where the configuration on $[0, 1] \times [0, 1]$ is empty, by linearly pushing the configuration towards $0 \times [0, 1]$ on the boundary, and multiplying once points collide with the boundary. A point with an empty configuration is determined by $(a, b) \in M \times N$. This point is the image of the point $(a, b, (1), ()) \in M \otimes_{\mathrm{hur}^X} N$, showing the claimed surjectivity on path components. We view these points as above with empty configuration as giving a copy of $M \times N \subset M \otimes_{\mathrm{hur}^X} N$.

Recall we are aiming to show σ is a homotopy equivalence. To do this, we claim it suffices to show the following two statements.

(i) For every $v \geq 1$, given a map of pairs $f : (D^v, S^{v-1}) \to (B[M, \mathrm{hur}^X, N], M \times N)$, there is a lift, up to homotopy, to a map of pairs $g : (D^v, S^{v-1}) \to (M \otimes_{\mathrm{hur}^X} N, M \times N)$ such that $\sigma \circ g$ is homotopic to f.
(ii) For every $v \geq 1$ and a map of pairs $g : (S^v, *) \to (M \otimes_{\mathrm{hur}^X} N, M \times N)$ such that $\sigma \circ g$ is nullhomotopic, then g is nullhomotopic.

We now explain why (i) and (ii) above imply σ is a homotopy equivalence. First, (i) in the case $v = 1$, shows in particular that the map σ on π_0 is injective, since it shows that the equivalence relation describing π_0 as a quotient of $M \times N$ is the same for the source and target. Since we showed above that σ is a surjection on π_0, it will follow that σ is an isomorphism on π_0. Then (i) also shows that on each component, the map on π_v is surjective for each $v \geq 1$, and (ii) shows π_v is injective.

We turn to proving (i), and assume that we have a map f as given in the statement of (i). We set up notation to define the map g described in (i). Using compactness, we can find a finite open cover $D^v = \cup_{\lambda \in \Lambda} U_\lambda$ such that there is an $r > 0$ and for each $\lambda \in \Lambda$, there is a $t_\lambda \in (0,1)$ such that for all $u \in U_\lambda$, the configuration associated to $f(u)$ has no points in $(t_\lambda - r, t_\lambda + r) \times [0,1]$. Choose a partition of unity w_λ subordinate to the cover U_λ. Let u be a point in $U_{\lambda_0} \cap \cdots \cap U_{\lambda_n}$, and not in U_λ for $\lambda \in \Lambda - \{\lambda_1, \ldots, \lambda_n\}$. Assume that $t_{\lambda_0} < t_{\lambda_1} \cdots < t_{\lambda_n}$ (which can always be achieved by decreasing r and making all t_λ distinct). Then we consider the n-tuple of elements $x_1, \ldots, x_n \in \mathrm{hur}^X$, where x_i is obtained from the element of hur associated to $f(u)$ by cutting along $t_{\lambda_{i-1}} \times [0,1]$ and $t_{\lambda_i} \times [0,1]$, taking the configuration with points lying in $[t_{\lambda_{i-1}}, t_{\lambda_i}] \times [0,1]$, and rescaling the intervals by $\frac{1}{2r}$ so that the time function t applied to this new configuration is $(t_{\lambda_i} - t_{\lambda_{i-1}})\frac{1}{2r}$. Define u' and $u'' \in \mathrm{hurbig}^X$ to be the elements obtained respectively by cutting to restrict to the intervals $[0, t_{\lambda_0}]$ and $[t_{\lambda_n}, 1]$ respectively. We define the map $g : D^v \to M \otimes_{\mathrm{hur}^X} N$ by sending u to the tuple

$$(au', u''b, (w_{\lambda_0(u)}, \ldots, w_{\lambda_n(u)}), (x_1, \ldots, x_n)).$$

We will show there is a homotopy between $\sigma \circ g$ and f, which will verify (i). Choose the open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of D^v so that image of each component the boundary S^{v-1} is contained in exactly one U_{λ_i}. (When $v > 1$, the boundary is connected, so this is the same as saying the boundary S^{v-1} is contained in exactly one U_{λ_i}.) Then, g is a map of pairs because each component of the boundary is sent to a single point, as $M \times N$ is a discrete set.

We now check continuity of g at a point $u \in D^v$. Let j, k be such that $w_{\lambda_i}(u) = 0$ for $i < j$ and $i > k$ but not for $i = j$ or $i = k$. Any neighborhood of $g(u)$, contains an open set W consisting of data of the form $(a', b', (y_0, \ldots, y_n), (z_1, \ldots, z_n))$ where $|y_i - w_{\lambda_i}(u)| < \epsilon$ for some $\epsilon > 0$ small enough such that this guarantees that $y_j, y_k \neq 0$, and such that

($\star$) $a' z_1 \cdots z_{j-1} = au' x_1 \cdots x_{j-1}$ and $z_{k+1} \cdots z_n b' = x_{k+1} \cdots x_n u'' b$.

In order to show g is continuous at u, it is then enough that for any such open set W, $g^{-1}(W)$ contains an open neighborhood of u in D^v.

Choose a basic open set $\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')$ containing $f(u)$ and a small open neighborhood V containing u with $V \subset f^{-1}(\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')) \cap U_{\lambda_0} \cap \cdots \cap U_{\lambda_n}$. We may shrink $\mathfrak{B}(d_1, d_2, U_i, \phi, \alpha_i, a', b')$, to assume $d_1 \leq t_{\lambda_0}, d_2 \geq t_{\lambda_n}$, where, here, shrinking this neighborhood may involve increasing the number of U_λ so that there is one such open containing each point of the configuration x, appearing in the data for $f(u)$, in $(d_1, d_2) \times [0,1]$. By further shrinking V, we may assume the functions w_λ for $\lambda \in \Lambda - \{\lambda_0, \dots, \lambda_n\}$ vanish on V. Therefore, using identifications $(1) - (3)$ of the bar construction, for the purposes of proving continuity, we may assume our open cover consists solely of $U_{\lambda_0}, \dots, U_{\lambda_n}$, i.e, we can ignore the other U_λ. At this point, condition (3) of Notation A.3.1 enables us to further shrink V so that $(\star)$ holds for all $g(v)$ with $v \in V$. We have therefore found an open set in D^v containing u and contained in $g^{-1}(W)$ for W an open set containing $g(u)$. This proves that g is continuous.

For a point $z \in D^{v+1}$, $\sigma(g(z))$ is obtained from $f(z)$ by zooming in on the configuration on the interval around $\sum_{i=0}^n w_{\lambda_i} t_{\lambda_i}$ of radius r, rescaling it to be in the interval $[0,1]$, and then multiplying the element of hurbig obtained by cutting to the left of this interval with a and multiplying the element of hurbig obtained by cutting to the right with b. There is an evident continuous homotopy from f to $\sigma \circ g$, given by linearly rescaling around this interval, and letting points act on the boundary as they reach it. This gives us a homotopy between f and $\sigma \circ g$, which is a homotopy of pairs because of our assumption about g applied to the boundary sphere (each component of which is sent to a single point). Thus we have verified (i).

To prove (ii), given a map of pairs $g : (S^v, *) \to (M \otimes_{\mathrm{hur}^X} N, M \times N)$ and a nullhomotopy of $\sigma \circ g$, the same construction used above for (i) lifts the nullhomotopy of $\sigma \circ g$ to a pointed map $h : D^{v+1} \to M \otimes_{\mathrm{hur}^X} N$. It remains to show that $h|_{S^v}$ is homotopic to g as a pointed map. Here we follow the last paragraph of the proof of [BDPW23, Theorem 5.2.3]. Namely, given $u \in S^v$, suppose u is in $U_{\lambda_0} \cap \cdots \cap U_{\lambda_n}$ and not in U_λ for $\lambda \in \Lambda - \{\lambda_0, \dots, \lambda_n\}$. Associated to $g(u)$ is a sequence of κ elements in hur^X, for some integer κ. We can first linearly stretch these κ elements each by a factor of $\frac{1}{2r}$, and then use the identifications of (3) of the 2-sided bar construction Notation A.3.3 to make n cuts in the places specified by t_{λ_i} in $\sigma \circ g$. Let $\tilde{g}$ be the function of u obtained this way, so that $\tilde{g}$ is homotopic to g. Then, $\tilde{g}$ agrees with $h|_{S^v}$ except for the weights defining a point in Δ^n. Linearly changing these weights then gives a continuous homotopy between $\tilde{g}$ and $h|_{S^v}$. Thus g is homotopic to $h|_{S^v}$. □

A.4. **A simple topological model.** We will carry out a scanning argument to obtain a description of the homotopy type of $B[M, \mathrm{Hur}^X, N]$ in terms of explicit quotient spaces, which we define next. This identification is verified in the unpointed case in Lemma A.4.7, and in the pointed case in Theorem A.4.9. In the next definition, we have two somewhat complicated relations (i) and (ii). We must impose these relations so that the proof of Proposition A.4.6 goes through, the key idea being depicted in Figure 3.

Definition A.4.1. Let M be a set with a right action of Hur^X and let N be a set with a left action of Hur^X. Let $0 < \epsilon < 1$. We define $Q_\epsilon[M, \mathrm{Hur}^X, N]$ to be the set consisting of triples $(m, n, x, l : x \to X)$ with $m \in M, n \in N$, $x = \{p_1, \dots, p_k\}$ a finite subset of $[0,1] \times [\epsilon, 1-\epsilon]$ (with first projection $\pi_1 : [0,1] \times [0,1] \to [0,1]$ and second projection $\pi_2 : [0,1] \times [0,1] \to [0,1]$) such that for each $i \neq j$, the distance between $\pi_2(p_i)$ and $\pi_2(p_j)$ is at least ϵ, and $l : x \to X$ is a map of sets.

We define $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ to be the quotient space of $Q_\epsilon[M, \mathrm{Hur}^X, N]$ under the equivalence relation generated by the following two types of equivalences.

(*i*) Suppose $x = \{p_1, \ldots, p_v, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_v) = 0$, so p_r lies on the left boundary, the data $(m, n, x, l : x \to X)$ is equivalent to $(m', n, x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to X)$, defined as follows: write $(y_1, \ldots, y_r) := \sigma_1\sigma_2 \ldots \sigma_{r-1}(l(p_1), \ldots, l(p_r))$. Then $m' = my_1$, and $l'(p_i) := y_i$ for $1 \leq i < r$, $l'(p_i) := l(p_i)$ for $i > r$.

(*ii*) Suppose $x = \{p_1, \ldots, p_r, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_r) = 1$, so p_r lies on the right boundary, the data $(m, n, x, l : x \to X)$ is equivalent to $(m, n', x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to X)$, defined as follows: write $(y_r, \ldots, y_k) := \sigma_{k-r-1} \cdots \sigma_1(l(p_r), \ldots, l(p_k))$. Then $n' = y_k n$, and $l'(p_i) := l(p_i)$ for $i < r$, $l'(p_i) = y_{i-1}$ for $i > r$.

Remark A.4.2. In the case that the braided set X comes from a rack c, the equivalences in Definition A.4.1 can be described more simply as follows.

(*i*) Suppose $x = \{p_1, \ldots, p_r, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_r) = 0$, so p_r lies on the left boundary, the data $(m, n, x, l : x \to X)$ is equivalent to $(m', n, x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to X)$, where $m' = ml(p_r)$, $l'(p_i) := l(p_r) \triangleright l(p_i)$ for $1 \leq i < r$, $l'(p_i) := l(p_i)$ for $i > r$.

(*ii*) Suppose $x = \{p_1, \ldots, p_r, \ldots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$. If $\pi_1(p_r) = 1$, so p_r lies on the right boundary, the data $(m, n, x, l : x \to X)$ is equivalent to $(m, n', x' := \{p_1, \ldots, p_{r-1}, p_{r+1}, \ldots, p_k\}, l' : x' \to X)$, where $n' = ((l(p_{r+1}) \ldots l(p_k)) \triangleright l(p_r))n$, $l'(p_i) := l(p_i)$ for $i \neq r$.

For technical reasons, we will want to work with inductive systems of topological spaces $(X_\epsilon)_{\epsilon > 0}$, which we view as presenting the colimit of their homotopy types. For example, we will consider $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ as an ind-system as ϵ approaches 0.

Definition A.4.3. Let $L : \mathrm{Top} \to \mathrm{Spc}$ and $L : \mathrm{Top}_* \to \mathrm{Spc}_*$ denote the functors of ∞-categories sending a (pointed) topological space to its weak homotopy type. Given a (pointed) map $f : X \to X'$ in $\mathrm{Ind}(\mathrm{Top})$ or $\mathrm{Ind}(\mathrm{Top}_*)$, we say that it is an *ind-weak equivalence* if $\mathrm{colim}\, Lf$ is an equivalence.

The following is an ind-variant of a well known lemma about weak homotopy equivalences. In what follows, we use the smash product $\wedge$, the definition of which is in §3.1.1.

Lemma A.4.4. *Let P be a filtered poset, and let $X'_\bullet \hookrightarrow X_\bullet, \bullet \in P$ be an injection of P-indexed diagrams of path connected pointed topological spaces (not necessarily with the subspace topology) such that for every pointed inclusion of compact spaces $K' \hookrightarrow K$, and commutative diagram*

$$\begin{tikzcd} K' \arrow[r, hook, "i"] \arrow[d, "\psi"] & K \arrow[d] \\ X'_\alpha \arrow[r, hook] & X_\alpha \end{tikzcd} \tag{A.3}$$

there is a $\beta > \alpha$ and a pointed homotopy $H : K \wedge [0,1]_+ \to X_\beta$ with the following two properties.

(1) The restriction of H along $i \wedge \mathrm{id}$ induces a pointed homotopy $K' \wedge [0,1]_+ \to X'_\beta$ whose restriction to the point $0 \in [0,1]_+$ is the composite $K' \xrightarrow{\psi} X'_\alpha \to X'_\beta$ where ψ is the map from (A.3).

(2) The restriction of H to the point $1 \in [0,1]_+$ defines a map $K \to X_\beta$, which we assume factors continuously through X'_β.

Then $X'_\bullet \to X_\bullet$ is an ind-weak homotopy equivalence.

Proof. Since $\pi_i(-) = \pi_{i-1}(\Omega -)$ and homotopy sets and based loop spaces commute with filtered colimits in Spc, it suffices by induction on i to show that $\Omega X'_\bullet \hookrightarrow \Omega X_\bullet$ satisfies the same hypotheses (1) and (2) as in the statement that $X'_\bullet \to X_\bullet$ satisfies (except that $\Omega X'_\bullet$ and $\Omega X_\bullet$ need not be connected), and moreover that $\Omega X'_\bullet \hookrightarrow \Omega X_\bullet$ is an ind-equivalence on π_0.

To see that the map $\Omega X'_\bullet \hookrightarrow \Omega X_\bullet$ satisfies (1) and (2), note that giving a pointed map $K \to \Omega X_\alpha$ is the same as giving a map $K \wedge S^1 \to X_\alpha$, and similarly for K'. Here, we implicitly choose a fixed point in S^1 and view S^1 as a pointed space to make sense of the wedge product. Applying the hypotheses to $K \wedge S^1$ in place of K, we get an pointed map $K \wedge S^1 \wedge [0,1]_+ \to X_\beta$, which is the same as a map $K \wedge [0,1]_+ \to \Omega X_\beta$ satisfying the desired properties (1) and (2).

We finish the proof by showing that show that $\operatorname{colim}_{\alpha \in P} \pi_0 \Omega X'_\alpha \to \operatorname{colim}_{\alpha \in P} \pi_0 \Omega X_\alpha$ is a bijection. We first check it is surjective. Given a point $* \in \Omega X_\alpha$, taking K to be the compact space $*_+$ and $K' \subset K$ the base point, the hypotheses applied to the pointed map $*_+ \to \Omega X_\alpha$ shows surjectivity. For injectivity, if $[0,1]_+ \to \Omega X_\alpha$ is a path between two points with the end points factoring through $\Omega X'_\alpha$, then applying the hypothesis to this map, we get a map $([0,1] \times [0,1])_+ \to \Omega X_\beta$, and the part of the boundary $\{0,1\} \times [0,1] \cup [0,1] \times \{1\}$ yields a path in $\Omega X'_\beta$ between the two points. □

A useful construction we will use to study the space $B[M, \mathrm{hur}^X, N]$ is a flow that pushes configurations outwards, which we define next.

Construction A.4.5. Choose a smooth function $f : [0,1] \to \mathbb{R}$ such that $f([0, \frac{1}{2})) < 0$ and $f(1-x) = -f(x)$. In particular, $f(1/2) = 0$. Consider the vector field $f(x)\frac{\partial}{\partial x}$ on $[0,1] \times [0,1]$, where x is the first coordinate. We can then form a continuous flow $\phi_t : B[M, \mathrm{Hur}^X, N] \times [0, \infty) \to B[M, \mathrm{Hur}^X, N]$ by moving points in a configuration along the flow, and having them act on the elements (m,n) in $M \times N$ when they reach the boundary. We can define such a flow (which we give the same name) on $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ via the same procedure.

The following proposition is key in relating $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ to $B[M, \mathrm{Hur}^X, N]$. This proposition characterizes the image of the natural map from the first to the second.

Proposition A.4.6. *Let M be a set with a right action of Hur^X and N be a set with a left action of Hur^X. Let $\pi_2 : [0,1] \times [0,1] \to [0,1]$ be the second projection. There is a natural continuous injection $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N] \to B[M, \mathrm{Hur}^X, N]$ whose image consists of those points (a, b, y) where $y := (x := \{p_1, \ldots, p_k\}, 1, \gamma, x_i)$ as in Notation A.3.1 with the property that $\epsilon \le \pi_i(p_j) \le 1 - \epsilon$ for $1 \le j \le k$ and any two distinct elements in $\{\pi_2(p_1), \cdots, \pi_2(p_k)\}$ differ by at least ϵ.*

Moreover, a continuous map $K \to B[M, \mathrm{Hur}^X, N]$ lifts to $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ if its image in the image of finitely many connected components of $Q_\epsilon[M, \mathrm{Hur}^X, N]$.

Proof. To shorten notation, in this proof only, we will use Q to denote $Q_\epsilon[M, \mathrm{Hur}^X, N]$ and $\overline{Q}$ to denote $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$. We will produce a continuous map

$$Q \to B[M, \mathrm{Hur}^X, N]$$

which induces a map $\overline{Q} \to B[M, \mathrm{Hur}^X, N]$ satisfying the claimed properties.

We define the interior Q° to be the dense subset of Q on which the configurations are contained in $(0,1) \times [\epsilon, 1-\epsilon]$. In other words, a point of Q° is given by $m \in M, n \in N$, such a configuration x and a function $l : x \to X$. We define a continuous function $g : Q^\circ \to B[M, \mathrm{Hur}^X, N]$ as we describe next.

We first continuously choose a homotopy class of paths γ from the configuration to bigord as follows: given a configuration, because the projection π_2 to the second coordinate is injective, we can linearly move all points so that their first coordinate is $\frac{1}{2}$. Then, we can rotate the configuration in $[0,1] \times [0,1]$ counterclockwise by $\frac{\pi}{2}$ so that the second coordinate of all the points is $\frac{1}{2}$, so that the path ends in bigord. This ends our description of γ.

Let $x := \{p_1, \dots, p_k\}$ with $\pi_2(p_1) > \cdots > \pi_2(p_k)$ and define $x_i := l(p_i)$. The map g then sends the above point $(m, n, x, l : x \to X) \in Q$ to the point $(m, n, y) \in B[M, \mathrm{Hur}^X, N]$, where $y \in \mathrm{hurbig}^X$ is given by $(x, 1, \gamma, x_i)$. We omit the straightforward verification that this map bijects Q° onto the image stated in the proposition statement.

We claim the above map $Q^\circ \to B[M, \mathrm{Hur}^X, N]$ continuously extends to a map $Q \to B[M, \mathrm{Hur}^X, N]$. To see this, we can produce a continuous inward flow φ_{-t} on Q by pushing points inwards along the vector field $-f(x)\frac{\partial}{\partial x}$ as in Construction A.4.5. For positive t, this flow lands inside Q°, so we can continuously extend the function g to a map $\tilde{g} : Q \to B[M, \mathrm{Hur}^X, N]$ via the formula $\phi_t \circ g \circ \varphi_{-t}$ for any $t > 0$, with ϕ_t as in Construction A.4.5. It is easy to see that the underlying configuration associated to the map $\tilde{g}$ is similar to the configuration of the original point, with the only difference being that all of the points on the boundary of $[0,1] \times [0,1]$ are removed.

Since the image of g was closed, and Q° is dense in Q, every element of Q is sent to the same point as some element of Q°. To show that the equivalence relation induced by $\tilde{g}$ is exactly determined by (i) and (ii) from Definition A.4.1, it is then enough to show that (i) and (ii) are satisfied, because one can repeatedly use (i) and (ii) to show that every point is equivalent to a point in the image of Q°, and the map g is a bijection, so this accounts for the entire equivalence relation.

To show relations (i) and (ii), we can reduce to the case where there is exactly one point on the boundary, since if we prove the relations in this case, by continuity, the general case will follow from continuity by perturbing the other points on the boundary and having them approach the boundary.

We now establish relation (ii) from Definition A.4.1 in this case, as (i) follows from a similar argument. Suppose that we have a point $w \in Q$ where y is the only point in the configuration associated to w on the boundary of $[0,1] \times [0,1]$. Moreover, assume y lies on $1 \times [0,1]$. We need to compute the limit of $\phi_s \circ g \circ \varphi_{-t}(w)$ as s approaches t from below. For the configuration associated to point $\varphi_{-t}(w)$, we can choose some $\delta < 1$ so that the point y' to which y flows in in $\varphi_{-t}(w)$ has $\pi_1(y') > \delta$ (where $\pi_1(y')$ denotes the first coordinate of y') and is the only point of the configuration with first coordinate $\geq \delta$. Let $[\gamma]$ be the homotopy class of paths from the configuration in $\varphi_{-t}(w)$ to bigord that is used

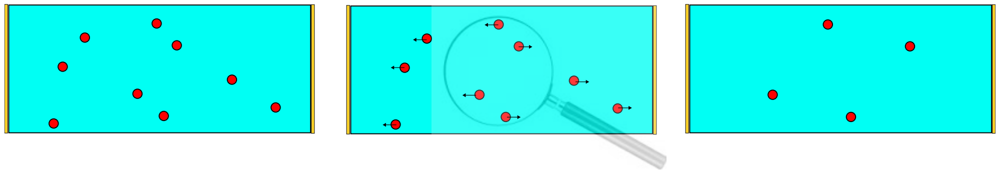

FIGURE 2. This is a picture of the scanning argument of Lemma A.4.7. In the diagrams, there vertical distance at least ε between any two red points, and all red points have second coordinate lie in $[\epsilon, 1-\epsilon]$.

in defining the map g. We can choose a different homotopy class of paths by first cutting up $[0,1]\times[0,1]$ into $[0,\delta]\times[0,1]$ and $[\delta,1]\times[0,1]$, doing the construction as in the map g on each piece[5], and concatenating to get a new homotopy class of paths $[\gamma']$. One checks that the class $[\gamma'][\gamma]^{-1}$ in the braid group is given by $\sigma_{k-1}\cdots\sigma_i$, where there are k points of the configuration, and where y has the ith highest second coordinate of all elements in the configuration. We have illustrated checking this in Figure 3. Thus $g\circ\varphi_{-t}(w)$ is equivalent to the point where the path $[\gamma']$ is used instead of $[\gamma]$, and the element of X^k is changed according to the action of $\sigma_{k-1}\cdots\sigma_i$. Letting s approach t, we see that this is exactly the image of the point as described in relation (ii).

We turn to proving the last statement of the proposition. Let $\overline{Q}'$ denote a subspace of $\overline{Q}$ that is the image of finitely many components of Q so that the image of K factors through $\overline{Q}'$. Each component of Q is compact, and so $\overline{Q}'$ is compact, since it is a quotient of finitely many such components. Since $B[M,\mathrm{Hur}^X,N]$ is Hausdorff, and a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the inclusion of $\overline{Q}'\to\overline{Q}$ is a homeomorphism onto its image. Thus, the unique lift of the map from K to $\overline{Q}'$ (and hence to $\overline{Q}$) is continuous. ☐

The following scanning argument is a ind-version of a quite standard scanning argument (see for example [RW20, Lemma 5.2] or [BDPW23, Proposition 5.2.7]), and is illustrated in Figure 2.

Lemma A.4.7. *The inclusions $\overline{Q}_\epsilon[M,\mathrm{Hur}^X,N]\to B[M,\mathrm{Hur}^X,N]$ of Proposition A.4.6 indexed over the poset of real numbers $1>\epsilon>0$ are an ind-weak equivalence.*

Proof. It is easy to see that this map is always an equivalence on path components for each $\frac{1}{2}\geq\epsilon>0$, and that every element is in the component of a point given by $(m,n)\in M\times N$ and an empty configuration of points. It suffices to prove the statement for each connected path component, so we now fix a path component Z, restrict to $\epsilon\leq\frac{1}{2}$, and fix some (m,n) on this component with empty configuration as the base point for all ϵ. We will now apply Lemma A.4.4 to the path component Z of this base point. Consider a diagram

$$\begin{tikzcd} K' \arrow[r, hook] \arrow[d] & K \arrow[d, "f"] \\ \overline{Q}_\epsilon[M,\mathrm{Hur}^X,N] \arrow[r, hook] & B[M,\mathrm{Hur}^X,N] \end{tikzcd}$$

[5]Really, the rotation used in the construction should be rescaled since $[0,\delta]\times[0,1]$ and $[\delta,1]\times[0,1]$ are not squares.

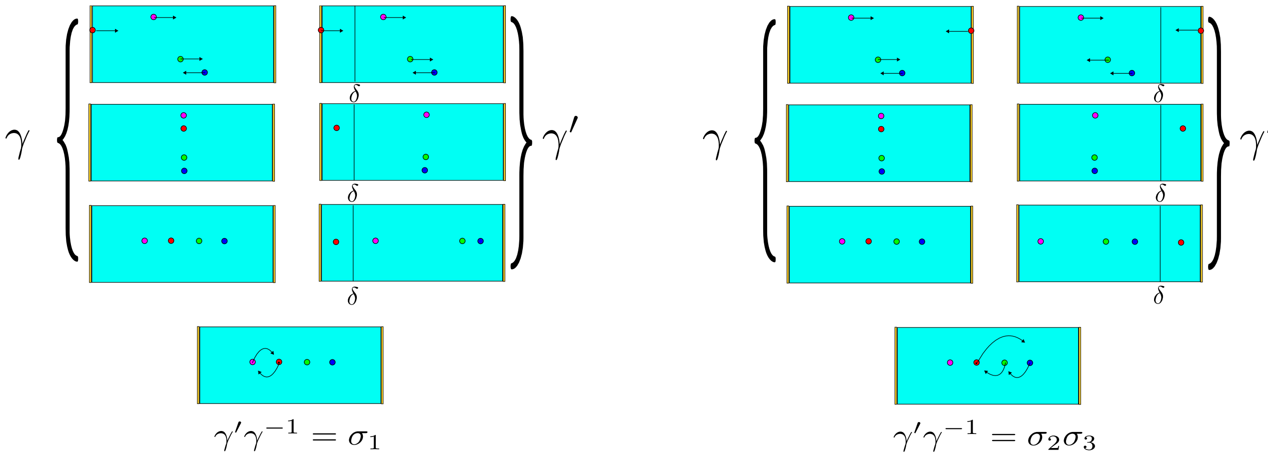

FIGURE 3. This is a picture depicting the compatibility of the map $\tilde{g}$ in the proof of Proposition A.4.6 with relations (i) and (ii) from Definition A.4.1. The first two columns depict the paths γ and γ' the points take during the map $\tilde{g}$ under relation (i), and the composite $\gamma'\gamma^{-1}$ is depicted underneath. The last two columns depict the paths γ and γ' which the points take under relation (ii), and the composite $\gamma'\gamma^{-1}$ is again depicted underneath. In the diagrams, there are 4 points, p_1 which is purple, p_2 which is red, p_3 which is green, and p_4 which is blue. So $r = 2$ and $k = 4$ in the notation of Definition A.4.1. The point p_2 is on the boundary. When p_2 is on the left boundary, the path is $\sigma_1 \cdots \sigma_{r-1} = \sigma_1$ since $r = 2$. When p_2 is on the right boundary, the path is $\sigma_r \cdots \sigma_{k-1} = \sigma_2\sigma_3$.

with K, K' compact. Choose a $\delta, \epsilon > 0$ such that for all points in the image of $f : K \to B[M, \mathrm{Hur}^X, N]$, the part of the configuration in $[\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [0, 1]$ projects injectively under the second projection and lands in $[\epsilon, 1 - \epsilon]$, and each two points in the image of this projection have distance at least ϵ from each other. Since the flow in Construction A.4.5 sends $\frac{1}{2} \pm \delta$ to the boundary in a finite amount of time, we can use the flow to produce a continuous homotopy starting from f and ending with a map landing in the image of $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$. First, we verify condition (2) from Lemma A.4.4. Indeed, since K is compact, at time $t = 1$ of the homotopy, the map satisfies the criterion of the last sentence of Proposition A.4.6, so factors through $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$. Finally, we verify condition (1) of Lemma A.4.4 by using the flow on $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$. Indeed, the homotopy also restricts on K' to a homotopy of pointed maps to $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$. This verifies the conditions of Lemma A.4.4, as desired. $\square$

We turn to proving a pointed version of Lemma A.4.7. As a first step, we prove a lemma which shows that certain pushouts of our topological models are homotopy pushouts.

Lemma A.4.8. *Let M be a right Hur^X module and N be a left Hur^X module. Suppose that $X \subset M \times N$ is a subset closed under the right and left actions of Hur^X. Let $A \subset \overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ denote the subset of points in the image of points in $Q_\epsilon[M, \mathrm{Hur}^X, N]$ whose projection to $M \times N$ lies in X. Then $A \subset \overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ has the homotopy extension property.*

Proof. It is easy to see that for each $1 > \epsilon > 0$, $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ admits the structure of a CW-complex of dimension $< \frac{2}{\epsilon}$. This is because each component of $Q_\epsilon[M, \mathrm{Hur}^X, N]$ is a union of disks, and the gluing relations to form $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ are along subcomplexes. It is finite dimensional since ϵ puts a bound on the number of points in the configurations that are allowed. For a suitably chosen CW structure on $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$, the inclusion $A \subset \overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ is that of a subcomplex, showing the lemma. $\square$

We are finally prepared to prove a pointed version of Lemma A.4.7, which is the main result of this section:

Theorem A.4.9. *Let M be a pointed set with a right Hur^X action and N be a pointed set with a left Hur^X action. There is a natural identification of $M \otimes_{\mathrm{Hur}^X_+} N$ with the ind-weak homotopy type of the quotient of the pointed spaces*

$$\overline{Q}^*_\epsilon[M, \mathrm{Hur}^X_+, N] := \overline{Q}_\epsilon[M, \mathrm{Hur}^X_+, N]/S_{M,X,N},$$

as ϵ approaches 0 (for $0 < \epsilon < 1$) where $S_{M,X,N} \subset \overline{Q}_\epsilon[M, \mathrm{Hur}^X_+, N]$ is the subspace consisting of all points whose projection to either M or N is the base point.

Proof. The pointed bar construction $M \otimes_{\mathrm{Hur}^X_+} N$ is the total cofiber of the square

$$\begin{array}{ccc} * \otimes_{\mathrm{Hur}^X} * & \longrightarrow & M \otimes_{\mathrm{Hur}^X} * \\ \downarrow & & \downarrow \\ * \otimes_{\mathrm{Hur}^X} N & \longrightarrow & M \otimes_{\mathrm{Hur}^X} N. \end{array} \tag{A.4}$$

In other words, $M \otimes_{\mathrm{Hur}^X_+} N$ is the cofiber of the map from the pushout of the diagram

$$M \otimes_{\mathrm{Hur}^X} * \leftarrow * \otimes_{\mathrm{Hur}^X} * \to * \otimes_{\mathrm{Hur}^X} N$$

to $M \otimes_{\mathrm{Hur}^X} N$.

By Lemma A.3.4 and Lemma A.4.7, $M' \otimes_{\mathrm{Hur}^X} N'$ can be modeled as the colimit over $\epsilon > 0$ of the homotopy types of $Q_\epsilon[M', \mathrm{Hur}^X, N']$ for any unpointed right module M' and unpointed left module N'. It then follows by applying Lemma A.4.8, that the maps

$$\overline{Q}_\epsilon[*, \mathrm{Hur}^X, *] \to \overline{Q}_\epsilon[*, \mathrm{Hur}^X, N]$$
$$\overline{Q}_\epsilon[*, \mathrm{Hur}^X, *] \to \overline{Q}_\epsilon[M, \mathrm{Hur}^X, *]$$
$$\overline{Q}_\epsilon[*, \mathrm{Hur}^X, N] \cup_{\overline{Q}_\epsilon[*, \mathrm{Hur}^X, *]} \overline{Q}_\epsilon[M, \mathrm{Hur}^X, *] \to \overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$$

are cofibrations, i.e., they both have the homotopy extension property. Hence, the total cofiber of the square

$$\begin{array}{ccc} \overline{Q}_\epsilon[*, \mathrm{Hur}^X, *] & \longrightarrow & \overline{Q}_\epsilon[M, \mathrm{Hur}^X, *] \\ \downarrow & & \downarrow \\ \overline{Q}_\epsilon[*, \mathrm{Hur}^X, N] & \longrightarrow & \overline{Q}_\epsilon[M, \mathrm{Hur}^X, N] \end{array} \tag{A.5}$$

can be modelled as the quotient of $\overline{Q}_\epsilon[M, \mathrm{Hur}^X, N]$ by $S_{M,X,N}$. The total cofiber of (A.4) is the colimit over ϵ of the total cofibers of the squares (A.5) for $\varepsilon > 0$, since the total cofiber is a colimit, so commutes with the filtered colimit over ε. Thus we obtain the desired result. $\square$

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