HOMOLOGICAL ALGEBRA

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1. Abelian Categories

Abelian categories are the setting for homological algebra. The notion is due to Grothendieck, and abelian categories appear in many places in mathematics.

A **preadditive category** is one that is enriched over Ab, the category of abelian groups. Note that the dual of a preadditive category is preadditive, so notions can be dualized.

Proposition 1.1. *In a preadditive category C, a map f is monic iff for any map g, $f \circ g = 0 \implies g = 0$.*

Proof. Take the original condition for monic and subtract. $\square$

Dually, a map f is epic iff for any map g, $g \circ f = 0 \implies g = 0$.

Theorem 1.2. *In a preadditive category C, finite (possibly empty) coproducts coincide with finite products.*

Proof. This follows from the fact that it is true in Ab. For a product, we have projections $A \times B \to A, B$. We can produce the inclusions $A, B \to A \times B$ as $1_B \times 0, 0 \times 1_A$. It is easy to check that these identify $A \times B$ with $A \coprod B$. Now let 0 be an initial object. $\mathrm{Hom}(0,0)$ is the trivial ring, so there must be a unique map to 0 as we can compose with the identity of 0, and the composite must be 0. $\square$

The finite product and coproduct in a preadditive category are also called the **biproduct**. The biproduct of $A_1 \dots A_n$ is also characterized as an object denoted $A_1 \oplus \cdots \oplus A_n$ such that there are projections π_j to each A_j with sections i_j such that $\sum_1^n i_j \circ \pi_j = 1_{A_1 \oplus \dots A_n}$ and $\pi_k \circ i_j = 0$ when $j \neq k$.

The **kernel** of a morphism f in a preadditive category is the equalizer with the 0 morphism. The **cokernel** is dual. By subtracting, having kernels is the same as having equalizers and likewise for cokernels.

Proposition 1.3. *A morphism f in a preadditive category is monic iff its kernel is 0.*

Proof. If its kernel is 0 and $f \circ g$ is 0, g factors through the kernel by definition, so is 0. Thus g is monic. Conversely if f is monic, 0 satisfies the universal property of the kernel. $\square$

A monomorphism f is **normal** if it is the kernel of another morphism. The dual notion for epimorphisms is also called normal.

An **abelian category** is a preadditive category that is finitely complete with every monomorphism and epimorphism is normal. It is also a self-dual concept.

Proposition 1.4 (First isomorphism theorem)**.** *A normal monic $f : A \to B$ is the kernel of its cokernel.*

Proof. f is the kernel of some morphism $g : B \to C$. the composite $g \circ f$ is 0, so g factors through the cokernel. Now the kernel of $B \to \text{coker}(f)$ has a natural map to A as the map $\text{coker}(f) \to C$ is 0. Its inverse exists since by definition the map $A \to B \to \text{coker}(f)$ is 0. ☐

Proposition 1.5. *In an abelian category, every morphism $f : A \to B$ factors as $A \to \text{im}(f) \to B$, an epic followed by a mono.* $\text{im}(A)$ *is called the **image**.*

Proof. Define $\text{im}(f)$ to be the cokernel of the map from the kernel, or alternatively the kernel of the map to the cokernel, which are the same ☐

Proposition 1.6. *The kernel of a map is* 0 *iff the map is mono, and the cokernel of a map is* 0 *iff the map is epi.*

We say that a sequence of morphisms $A_i, f_i : A_i \to A_{i+1}$ is **exact** if the $\text{coker}(f_i) = \ker(f_{i+1})$.