MEASURE THEORY

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1. Abstract Measures/Integration

An **algebra** is a set M of "measurable" subsets of a set X that are closed under finite unions, complements, and including the empty set. A σ**-algebra** is an algebra that is closed under countable unions as well. **measure** on a M is a function $\mu : M \to [0,\infty]$ that has the property that $\mu(\cup_0^\infty X_i) = \sum_0^\infty \mu(X_i)$ whenever the X_i are pairwise disjoint. A **measure space** is the data of a set, a σ-algebra on that set, and a measure on the σ-algebra. The measure space is said to be σ**-finite** if there is a countable cover of finite measure subspaces. A map $f : X \to Y$ of measure spaces is a map such that the preimage of a measurable set is measurable and such that the measure of the preimage of a subset is measure of the set.

Lemma 1.1. *Let μ be a measure. $\mu(\emptyset) = 0$. Say that $E_i \searrow E$ if it is a sequence of sets getting smaller and $\cap E_i = E$. Similarly say that $E_i \nearrow E$ if it is a sequence of sets getting larger and $\cup E_i = E$. If E_i are measurable and $E_i \nearrow E$, then $\mu(E_i) \to \mu(E)$ as $i \to \infty$. If E_1 is finite measure and $E_i \searrow E$ then $\mu(E_i) \to \mu(E)$ as $i \to \infty$.*

Proof. $\mu(\emptyset) + \mu(\emptyset) = \mu(\emptyset \cup \emptyset)$ gives the first part. If $E_i \nearrow E$, note that $\lim_{n\to\infty} \mu(E_n) = \mu(E_1) + \sum_1^\infty \mu(E_{i+1} - E_i) = \mu(E)$. Finally if $\mu(E) < \infty$ and $E_i \searrow E$, then $E_1 - E_i \nearrow E_1 - E$, so $\lim_{n\to\infty} \mu(E_i) = \mu(E_1) - \lim_{n\to\infty} \mu(E_1 - E_i) = \mu(E)$. $\square$

One way to construct measures is through an exterior or **outer measure**. This is a function $\mu^* : 2^X \to [0,\infty]$ that is a lattice homomorphism, meaning $\mu^*(\emptyset) = 0, A \subset B \implies \mu^*(A) \le \mu^*(B)$, and for a countably infinite family E_i, $\mu^*(\cup_1^\infty E_i) \le \sum_1^\infty \mu^*(E_i)$.

Given an outer measure on X, we say that a subset E is **Carathéodory measurable** or just measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for any A.

Theorem 1.2. *Given an outer measure, the set of measurable sets is a σ-algebra. Moreover, μ^* restricted to measureable sets is a measure.*

Proof. It is clearly closed under complements, and contains the empty set. To see it is closed under finite intersections, suppose E_1, E_2 measurable, and then $\mu^*(A\cap E_1\cap E_2)+\mu^*(A\cap(E_1\cap E_2)^c \le \mu^*(A\cap E_1)-\mu^*(A\cap E_1)+\mu^*(A\cap E_1\cap E_2^c)+\mu^*(A\cap E_1^c) = \mu^*(A\cap E_1)+\mu^*(A\cap E_1^c) = \mu^*(A)$. The opposite inequality is clear. To show that it is closed under countable unions $\cap E_i$, note that we only need to show that $\mu^*(A\cap\cap_1^\infty E_i)+\mu^*(A\cap(\cap_1^\infty E_i)^c) \le \mu^*(A)$. Now for infinitely many E_i, we will first observe by induction that $\mu^*(A\cap(\cup_1^n E_i)) = \sum_1^n \mu^*(A\cap(E_i - \cup_{j<i} E_j))$. Indeed the inductive step is solved by setting $E_i' = E_i$ for $i < n-1$ and $E_{n-1}' = E_{n-1} \cup E_n$, and using the inductive hypothesis and the fact that the sets are measurable. Then observe that $\mu(A) = \mu^*(A\cap(\cup_1^n E_i))+\mu^*(A\cap(\cup_1^n E_i)^c) \ge \sum_1^n \mu^*(A\cap(E_i - \cup_{j<i} E_j))+\mu^*(A\cap(\cup_1^\infty E_i)^c)$. Letting $n \to \infty$, we get $\mu^*(A) \ge \sum_1^\infty \mu^*(A \cap (E_i - \cup_{j<i} E_j)) + \mu^*(A \cap (\cup_1^\infty E_i)^c) \ge \mu^*(A \cap (\cup_1^\infty E_i)) + \mu^*(A\cap(\cup_1^\infty E_i)^c)$.and all the inequalities must be equalities. If the E_i are disjoint, these equalities imply that μ^* on the measurable sets is a measure. $\square$

It is easy to see that the measure constructed above is **complete**, meaning that if $F \subset E$, and E is measure 0, then F is measurable.

Given a topological space, a σ-algebra often of interest is the **Borel σ-algebra**, namely the smallest one containing the open sets. A measure on a σ-algebra containing this is called Borel. Given a metric space, an outer measure is called **metric** if it has the property that $d(A, B) > 0 \implies \mu^*(A \cup B) = \mu^*(A) \cup \mu^*(B)$.

Theorem 1.3. *A metric outer measure is Borel.*

Proof. We will show that a closed set A is measurable. Let A_n be the points of distance $\geq \frac{1}{n}$ from A, and since A is closed, $\cup A_i = A^c$. Let C WLOG have finite measure. Then $\mu(C) = \mu(C \cap (A \cup A_n)) \geq \mu(C \cap A) + \mu(C \cap A^c) + \sum_{n+1}^{\infty} \mu((A_{n+1} - A_n) \cap C)$ Let $D_n = \mu((A_{n+1} - A_n) \cap C)$, and note that by letting $n \to \infty$ it suffices to show $\sum_1^\infty D_n$ is bounded. To show this, note that if $|n - m| > 2$, the triangle inequality implies $d(D_n, D_m) > 0$. If the series diverges, then some subseries diverges where each consecutive term is spaced at least 2 apart, but this contradicts the fact that by the metric property of the measure, the partial sums of this subseries are bounded by $\mu(C)$. □

A Borel measure is **inner regular** if $\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}$ and **outer regular** if $\mu(A) = \inf\{\mu(U) : U \supset A \text{ is open}\}$. It is **regular** if both of these are true.

Theorem 1.4. *If a measure on the σ-algebra of Borel sets of a metric space is finite on balls of finite radius, then μ is outer regular and $\mu(A) = \sup\{\mu(K) : K \subset A \text{ is closed}\}$.*

Proof. Consider the subset of the Borel sets satisfying the theorem. It is not hard to see that this is a σ-algebra, so it only needs to be shown that it contains open sets. Indeed, we may assume we have some open set U in a ball of radius 1. Then consider K_i to be elements in U whose distance from U^c is $\geq \frac{1}{n}$. $K_i \nearrow U$ so by Lemma 1.1 we are done. □

There is a way to construct an outer measure from simpler information on the size of certain sets. A **premeasure** on an algebra A is a function $\mu_0 : A \to [0, \infty]$ such that if E_i are disjoint sets in A whose union are in A, then $\mu_0(\cup_1^\infty E_k) = \sum_1^\infty \mu_0(E_k)$.

Theorem 1.5. *A premeasure on A can be extended to an outer measure, such that A becomes measurable.*

Proof. Define $\mu^*(E) = \text{index}\{\sum_1^\infty \mu_0 E_j | E_j \text{ covers } E, E_j \in A\}$. First we will have to show that $\mu^*(E) = \mu_0(E)$ for $E \in A$. □