Vectors are objects that move around space

Ishan Mukherjee

April 2023

Contents

1 The length of a vector

If the coordinate space is constructed out of two orthogonal unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, a vector $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Notice that $\boldsymbol{v}$ forms the hypotenuse of a right angle whose other two sides are $a\hat{\boldsymbol{\imath}}$ and $b\hat{\boldsymbol{\jmath}}$. So, we can use the Pythagorean theorem to find the **length**, also called the **size**, of the vector $\boldsymbol{v}$, denoted by $|\boldsymbol{v}|$:

$$\boxed{|\boldsymbol{v}| = \sqrt{a^2 + b^2}}$$

2 The dot product

2.1 Definition

The dot product is one way of several of "multiplying" two vectors. It is a scalar and is defined by the $\cdot$ (dot operator) in this way:

$$\boxed{\boldsymbol{r} \cdot \boldsymbol{s} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = r_1 s_2 + r_2 s_2 + \cdots + r_n s_n}$$

The dot product is also called

- the **inner product**,
- the **scalar product**, or
- the **projection product**.

2.2 Properties

2.2.1 Commutativity

$$\boxed{\boldsymbol{r} \cdot \boldsymbol{s} = \boldsymbol{s} \cdot \boldsymbol{r}}$$

Proof.

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{s} &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \\ &= r_1 s_1 + r_2 s_2 + \cdots + r_n s_n \\ &= s_1 r_1 + s_2 r_2 + \cdots + s_n r_n \\ &= \boldsymbol{s} \cdot \boldsymbol{r} \end{aligned}$$

$\square$

2.2.2 Distributivity over addition

$$\boxed{\boldsymbol{r} \cdot (\boldsymbol{s} + \boldsymbol{t}) = \boldsymbol{r} \cdot \boldsymbol{s} + \boldsymbol{r} \cdot \boldsymbol{t}}$$

Proof.

$$\begin{aligned}
\boldsymbol{r}\cdot(\boldsymbol{s}+\boldsymbol{t}) &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \cdot \left(\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \right) \\
&= r_1\left(s_1+t_1\right) + r_2\left(s_2+t_2\right) + \cdots + r_n\left(s_n+t_n\right) \\
&= r_1s_1 + r_1t_1 + r_2s_2 + r_2t_2 + \cdots + r_ns_n + r_nt_n \\
&= \left(r_1s_1 + r_2s_2 + \cdots + r_ns_n\right) + \left(r_1t_1 + r_2t_2 + \cdots + r_nt_n\right) \\
&= \boldsymbol{r}\cdot\boldsymbol{s} + \boldsymbol{r}\cdot\boldsymbol{t}
\end{aligned}$$

□

2.2.3 Associativity over scalar multiplication

$$\boxed{\boldsymbol{r}\left(k\boldsymbol{s}\right) = k\left(\boldsymbol{r}\cdot\boldsymbol{s}\right)}$$

Proof.

$$\begin{aligned}
\boldsymbol{r}\left(k\boldsymbol{s}\right) &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \cdot \begin{bmatrix} ks_1 \\ ks_2 \\ \vdots \\ ks_n \end{bmatrix} \\
&= r_1(ks_1) + r_2(ks_2) + \cdots + r_n(ks_n) \\
&= k\left(r_1s_1 + r_2s_2 + \cdots + r_ns_n\right) \\
&= k\left(\boldsymbol{r}\cdot\boldsymbol{s}\right)
\end{aligned}$$

□

2.3 The length of a vector

By "dotting" a vector with itself, we can find its length.

$$\boxed{\boldsymbol{r}\cdot\boldsymbol{r} = \left|\boldsymbol{r}\right|^2}$$

Proof. :

$$\begin{aligned}
\boldsymbol{r}\cdot\boldsymbol{r} &= r_1r_1 + r_2r_2 + \cdots + r_nr_n \\
&= r_1^2 + r_2^2 + \cdots + r_n^2 \\
&= \left(\sqrt{r_1^2 + r_2^2 + \cdots + r_n^2}\right)^2 \\
&= \left|\boldsymbol{r}\right|^2
\end{aligned}$$

$\square$

2.4 The angle between two vectors

The angle θ between $\boldsymbol{r}$ and $\boldsymbol{s}$ has this relationship with the vectors' dot product:

$$\boxed{\boldsymbol{r} \cdot \boldsymbol{s} = |\boldsymbol{r}|\,|\boldsymbol{s}| \cos\theta}$$

Proof. Recall the **cosine rule**:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

We can represent the sides of a triangle using vectors. If two sides (originating from a common point) are $\boldsymbol{r}$ and $\boldsymbol{s}$, we can draw a vector diagram to confirm that the third side must be the difference between the two sides, $\boldsymbol{r} - \boldsymbol{s}$.

Then, we can write the relationship between the lengths of these sides using the cosine rule:

$$|\boldsymbol{r} - \boldsymbol{s}|^2 = |\boldsymbol{r}|^2 + |\boldsymbol{s}|^2 - 2\,|\boldsymbol{r}|\,|\boldsymbol{s}| \cos\theta \tag{1}$$

Recall the relationship between a vector's length and its dot product with itself, from section 2.3:

$$|\boldsymbol{r}|^2 = \boldsymbol{r} \cdot \boldsymbol{r}$$

So, we can rewrite the left hand side of equation 1 as:

$$\begin{aligned}
|\boldsymbol{r} - \boldsymbol{s}|^2 &= (\boldsymbol{r} - \boldsymbol{s}) \cdot (\boldsymbol{r} - \boldsymbol{s}) \\
&= \boldsymbol{r} \cdot \boldsymbol{r} + \boldsymbol{s} \cdot \boldsymbol{s} - 2\boldsymbol{r} \cdot \boldsymbol{s} \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{s}|^2 - 2\boldsymbol{r} \cdot \boldsymbol{s}
\end{aligned}$$

Substituting this into equation 1:

$$\begin{aligned}
\cancel{|\boldsymbol{r}|^2} + \cancel{|\boldsymbol{s}|^2} - 2\boldsymbol{r} \cdot \boldsymbol{s} &= \cancel{|\boldsymbol{r}|^2} + \cancel{|\boldsymbol{s}|^2} - 2\,|\boldsymbol{r}|\,|\boldsymbol{s}| \cos\theta \\
\cancel{-2}\boldsymbol{r} \cdot \boldsymbol{s} &= \cancel{-2}\,|\boldsymbol{r}|\,|\boldsymbol{s}| \cos\theta \\
\boldsymbol{r} \cdot \boldsymbol{s} &= |\boldsymbol{r}|\,|\boldsymbol{s}| \cos\theta
\end{aligned}$$

$\square$

2.5 Projection

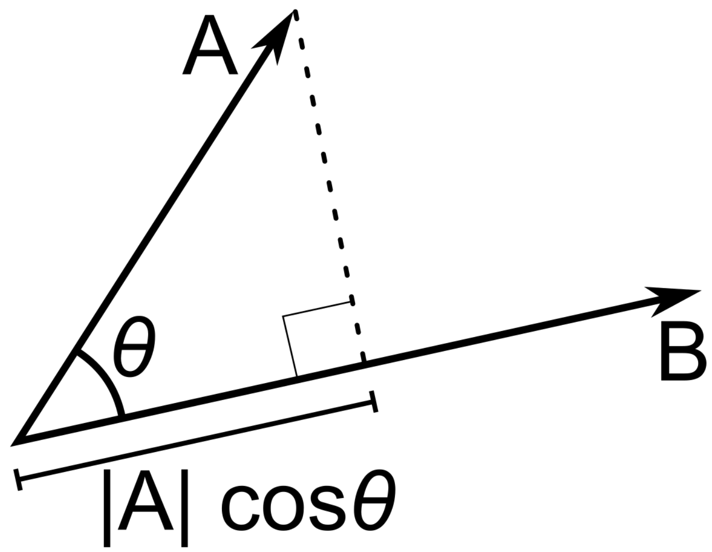

Figure 1: The **projection** of $\boldsymbol{A}$ onto $\boldsymbol{B}$, via Wikimedia

The **projection** of $\boldsymbol{A}$ onto $\boldsymbol{B}$ can be thought of as the shadow $\boldsymbol{A}$ casts on $\boldsymbol{B}$ when a light source orthogonal to $\boldsymbol{B}$ is shone on $\boldsymbol{A}$.

$|\boldsymbol{A}| \cos\theta$ is the **scalar projection** of $\boldsymbol{A}$ onto $\boldsymbol{B}$, or the *length* of the shadow. We know from section 2.4 that

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\theta$$

So,

$$\boxed{\text{scalar projection of } \boldsymbol{A} \text{ onto } \boldsymbol{B} = |\boldsymbol{A}| \cos\theta = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|}}$$

Suppose we also want to encode information about the *direction* of the projection into its representation. Then, we can multiply the scalar projection by the unit vector of $\boldsymbol{B}$ to get the **vector projection** of $\boldsymbol{A}$ onto $\boldsymbol{B}$.

So, the vector projection of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is literally a vector with the same origin as $\boldsymbol{A}$ that has the length $|\boldsymbol{A}| \cos\theta$.

$$
\begin{aligned}
\text{vector projection of } \boldsymbol{A} \text{ onto } \boldsymbol{B} &= \text{scalar projection} \times \hat{\boldsymbol{B}} \\
&= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|} \hat{\boldsymbol{B}} \\
&= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|} \frac{\boldsymbol{B}}{|\boldsymbol{B}|} \\
&= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|\,|\boldsymbol{B}|} \boldsymbol{B} \\
&= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|^2} \boldsymbol{B} \\
&= \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{B}} \boldsymbol{B}
\end{aligned}
$$

So,

$$
\boxed{\text{vector projection of } \boldsymbol{A} \text{ onto } \boldsymbol{B} = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{\boldsymbol{B} \cdot \boldsymbol{B}} \boldsymbol{B} = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{B}|^2} \boldsymbol{B}}
$$

The vector projection can be used to find the velocity of an object in the direction of an object moving with another velocity.

For example, a ship's velocity is $\boldsymbol{u}$ and a current's velocity is $\boldsymbol{v}$, both with respect to some coordinate axes. The velocity of the ship in the direction of the current is the vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$.

3 The reference frame

3.1 Basis

A **basis** is a set of n vectors that

i are **linearly independent**), and

ii **span the space**.

The space is then n-dimensional.

A **linear combination** is a vector obtained by adding two or more vectors (with different directions) which are multiplied by scalar values. The linear combination of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ with scalars $a_1, a_2, \ldots, a_n$ as coefficients is

$$
a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + \cdots + a_n\boldsymbol{v_n}
$$

A set of vectors has **linear independence** if no vector is a linear combination of another.

The **span** of set of vectors is the set of all linear combinations of the vectors. The set **spans the space** if every vector in a given space is a linear combination of the vectors in that set.

One nonzero vector $\boldsymbol{b_1}$ spans the 1D space. No other vector in the 1D space can be a valid second basis vector, since it could be expressed as a product of $\boldsymbol{b_1}$ with a scalar, i.e., as a linear combination of $\boldsymbol{b_1}$, violating the definition for the basis vector.

Two vectors $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ that are neither parallel nor antiparallel to each other span the 2D space. No other vector in the 2D space could be a valid third basis vector, since it could be expressed as a linear combination of $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$, violating the definition for the basis vector.

Similarly, three vectors could span the 3D space, four vectors span the 4D space, and so on.

Notice what basis vectors *do not* have to be: of length 1, or orthogonal to each other. However, calculations become much simpler if they are.

3.2 Changing basis

Consider the basis vectors

$$\hat{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{\boldsymbol{e_2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

used to describe

$$\boldsymbol{r_e} = 3\hat{\boldsymbol{e_1}} + 4\hat{\boldsymbol{e_2}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}_e$$

We have chosen the most commonly used set of basis vectors ($\hat{\boldsymbol{e_1}}$ and $\hat{\boldsymbol{e_2}}$ are usually written as $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ respectively), but we could have chosen some other set as well, like

$$\boldsymbol{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boldsymbol{b_2} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

How can we write $\boldsymbol{r}$ in this new set of basis vectors $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$, i.e, as $\boldsymbol{r_b}$? Stated another way,

$$\boldsymbol{r_b} = ?\boldsymbol{b_1} + ?\boldsymbol{b_2} = \begin{bmatrix} ? \\ ? \end{bmatrix}_b$$

First, notice that $\boldsymbol{r}$ is the sum of the vector projections of $\boldsymbol{r}$ onto $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$. (Draw a vector diagram to convince yourself of this if needed.)

Using the formula for vector projection from section 2.5, we can write

$$\begin{aligned}
\text{vector projection of } \boldsymbol{r} \text{ onto } \boldsymbol{b_1} &= \frac{\boldsymbol{r_e} \cdot \boldsymbol{b_1}}{|\boldsymbol{b_1}|^2}\boldsymbol{b_1} \\
&= \frac{\begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix}}{\left(\sqrt{2^2+1^2}\right)^2}\boldsymbol{b_1} \\
&= 2\boldsymbol{b_1} \\
\text{vector projection of } \boldsymbol{r} \text{ onto } \boldsymbol{b_2} &= \frac{\boldsymbol{r_e} \cdot \boldsymbol{b_2}}{|\boldsymbol{b_2}|^2}\boldsymbol{b_2} \\
&= \frac{\begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}-2\\4\end{bmatrix}}{\left(\sqrt{(-2)^2+4^2}\right)^2}\boldsymbol{b_2} \\
&= \frac{1}{2}\boldsymbol{b_2}
\end{aligned}$$

By adding these vector projections, we can rewrite $\boldsymbol{r}$ as

$$\boldsymbol{r_b} = 2\boldsymbol{b_1} + \frac{1}{2}\boldsymbol{b_2} = \begin{bmatrix}2\\\frac{1}{2}\end{bmatrix}_b$$

We can also confirm that $\boldsymbol{r_b} = \boldsymbol{r_e}$:

$$\begin{aligned}
\boldsymbol{r_b} &= \begin{bmatrix}2\\\frac{1}{2}\end{bmatrix}_b \\
&= 2\boldsymbol{b_1} + \frac{1}{2}\boldsymbol{b_2} \\
&= 2\begin{bmatrix}2\\1\end{bmatrix}_e + \frac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix}_e \\
&= \begin{bmatrix}4\\2\end{bmatrix}_e + \begin{bmatrix}-1\\2\end{bmatrix}_e \\
&= \begin{bmatrix}4-1\\2+2\end{bmatrix}_e \\
&= \begin{bmatrix}3\\4\end{bmatrix}_e \\
&= \boldsymbol{r_e}
\end{aligned}$$

In general, given an old set of basis vectors $\boldsymbol{e}$, a vector $\boldsymbol{r}$ can be written in a new set of basis vectors $\boldsymbol{e'}$ as

$$\boxed{\boldsymbol{r_{e'}} = \begin{bmatrix} \dfrac{\boldsymbol{r_e} \cdot \boldsymbol{e'_1}}{\left|\boldsymbol{e'_1}\right|^2} \\ \dfrac{\boldsymbol{r_e} \cdot \boldsymbol{e'_2}}{\left|\boldsymbol{e'_2}\right|^2} \end{bmatrix}_{e'}}$$

Note:

1. The method described above for changing basis works *only* if the new basis vectors are orthogonal to each other. (You can check for yourself that $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are orthogonal to each other; their dot product is zero.)

   Why this restriction of orthogonality? Because the method relies on the assumption that the vector projections of $\boldsymbol{r}$ onto the new basis vectors add up to $\boldsymbol{r}$. This assumption does not hold if the new basis vectors are not orthogonal to each other. To see for yourself, draw a new pair of basis vectors for $\boldsymbol{r}$ that are not orthogonal to each other. Then, take the vector projections of $\boldsymbol{r}$ onto these new basis vectors, and add them. Their sum will not be $\boldsymbol{r}$.

2. An interesting implication of the fact that it is possible to change basis is that a vector has an existence in some deep mathematical sense, independently of the coordinate system used to describe it. No matter the basis, a vector will always represent the same distance along a fixed direction from the origin.