Engineering Analysis I, Fall 2015

Midterm 2

Name _____

Section number _____

Section number	Lecture time
20	8:00 a.m.
21	10:00 a.m.
22	11:00 a.m.
23	12:00 noon

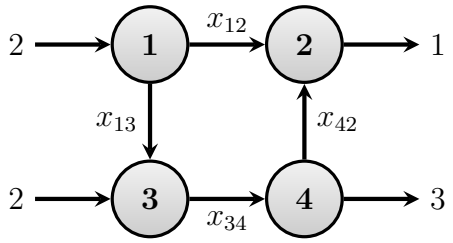
Answer the questions in the spaces provided on the question sheets. There are 7 problems worth 100 points total. This exam is closed-book and closed-notes. You will not need, and you are not allowed to use calculators, computers, phones, or other computing/communication devices.

Students should skip this page—it is only for graders.

Question	Points	Score
1	12	
2	10	
3	26	
4	15	
5	13	
6	12	
7	12	
Total:	100	

Problem 1

Consider the following network flow diagram and flow balance equations:



$$\boldsymbol{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{34} \\ x_{42} \end{bmatrix}$$

$$\begin{aligned} x_{12} + x_{13} &= 2 \\ x_{12} + x_{42} &= 1 \\ -x_{13} + x_{34} &= 2 \\ x_{34} - x_{42} &= 3 \end{aligned}$$

(a) [2 points] Write down the above flow balance equations in the form $A\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{x}$ as above.

(b) [3 points] Find the reduced row echelon form of the augmented matrix $[A \;\; \boldsymbol{b}]$.

(c) [3 points] Write the solution set for your system of equations in parametric vector form.

(d) [2 points] Draw the solution you get when you set all free parameters equal to zero.

(e) [2 points] Draw a solution which has $x_{12} = 0$.

Problem 2

Let

$$A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 1 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 2 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) [2 points] A^{-1}

(b) [2 points] A^{10}

(c) [2 points] AB

(d) [2 points] $((B^T B)C)A$

(e) [2 points] $\det(C)$

Problem 3

(a) i. [3 points] Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix}1\\3\end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix}-1\\2\end{bmatrix}$. What is $T(\begin{bmatrix}0\\1\end{bmatrix})$?

ii. [3 points] Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ that maps the vector $\mathbf{x} = \begin{bmatrix}0.5\\1.5\end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix}2\\3\\-1\end{bmatrix}$. What is $T(\begin{bmatrix}1\\3\end{bmatrix})$?

(b) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations.

i. [3 points] T reflects around the axis $x_2 = -x_1$.

ii. [3 points] T first rotates by 90 degrees clockwise, then projects onto the x_1 axis.

iii. [3 points] T first reflects around the x_2 axis, and then scales vertically by a factor of 4.

(c) [1½ points] Select all transformations from part (b) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i. ☐ ii. ☐ iii.

(d) [1½ points] Select all transformations from part (b) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i. ☐ ii. ☐ iii.

(e) Now consider T to be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 2x_2 + 4x_3 + x_4 \\ 2x_1 + ax_2 + 4x_3 \end{bmatrix}$ where a is a constant.

i. [2 points] Write the standard matrix A of this transformation.

ii. [2 points] For what values of a (if any) is T NOT **onto**?

ii. _____

iii. [2 points] For what values of a (if any) is T **one-to-one**?

iii. _____

iv. [2 points] For what values of a (if any) is T **invertible**?

iv. _____

Problem 4

(a) Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

Here a, b, c, and d represent constants. If this matrix is in *reduced* row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [1 point] What must be the value of a?

i. _____

ii. [1 point] What must be the value of b?

ii. _____

iii. [1 point] What must be the value of c?

iii. _____

iv. [1 point] What must be the value of d?

iv. _____

(b) [2 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

(c) Consider a system of linear equations of the form $A\boldsymbol{x} = b$, where the matrix A is 3-by-6 and the column vector $\boldsymbol{x}$ consists of the five unknown variables $x_1, x_2, \ldots, x_6$. Suppose the reduced row echelon form of the augmented matrix is given by

$$\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

i. [2 points] Which of the unknown variables are basic?

ii. [4 points] Write the solution set in parametric vector form.

iii. [3 points] Write the solution set to the corresponding homogeneous system of equations $A\boldsymbol{x} = \mathbf{0}$ as the span of a set of vectors.

Problem 5

Answer each question in the space provided.

(a) [2 points] Consider a matrix A with the form: $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & a & b \\ 3 & -1 & 4 \end{bmatrix}$. Give one choice of a and b for which this matrix is singular. *Note: you can answer by inspection.*

(b) [2 points] Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using A from part (a) with a and b chosen such that A is singular, what is the inverse of $B \cdot A$? If it does not exist, state that.

(c) [2 points] Using A from part (a), where a and b are chosen such that A is singular, what is the inverse of A^T? If it does not exist, state that.

(d) [3 points] Let $A = \begin{bmatrix} 0 & -4 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ be the standard matrix of a transformation. Find a matrix B such that whenever $A\boldsymbol{x} = \boldsymbol{y}$ for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then $B\boldsymbol{y} = \boldsymbol{x}$

(e) [1 point] Is the matrix B from part (d) invertible?

(f) [3 points] Let $D = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 7 & -3 \\ 2 & -4 & 1 \end{bmatrix}$. Find the inverse of D.

Problem 6 [12 points]

Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (1 point each)

(a) The reduced row echelon form of a matrix is unique.

(a) _____

(b) The solution set for a system of linear equations of the form $Ax = b$ cannot contain exactly three distinct vectors.

(b) _____

(c) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and suppose the reduced row echelon form of its standard matrix has a pivot in every column. Then T must be onto.

(c) _____

(d) Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$, and suppose the reduced row echelon form of its standard matrix has a pivot in every column. Then T must be onto.

(d) _____

(e) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ cannot be one-to-one.

(e) _____

(f) A matrix is invertible if and only if its reduced row echelon form is the identity matrix.

(f) _____

(g) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is invertible if and only if it is onto.

(g) _____

(h) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is invertible if and only if it is one-to-one.

(h) _____

(i) A square matrix is invertible if and only if its determinant is nonzero.

(i) _____

(j) If a matrix product AB exists, then B^TA^T exists and is equal to $(AB)^T$.

(j) _____

(k) If a matrix product AB exists and has all zero entries, then either A or B must have all zero entries.

(k) _____

(l) Suppose the matrix products AB and BA both exist. If $AB = BA$, then A and B must be square and of the same size.

(l) _____

Problem 7 [12 points]

Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (2 points each)

(a) If a matrix is in reduced row echelon form, then the entry in its upper left-hand corner must be 1.

(a) _____

(b) If a matrix A has a column of all zeros, then the system of linear equations $Ax = b$ must have infinitely many solutions.

(b) _____

(c) If a matrix A has a row of all zeros, then the system of linear equations $Ax = b$ must be inconsistent.

(c) _____

(d) Suppose the matrix product AB exists, and suppose the rows of A are not linearly independent. Then the rows of AB cannot be linearly independent.

(d) _____

(e) Suppose the matrix product AB exists, and suppose the columns of B are not linearly independent. Then the columns of AB cannot be linearly independent.

(e) _____

(f) If a matrix product AB exists and is invertible, then both A and B must be invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

(f) _____