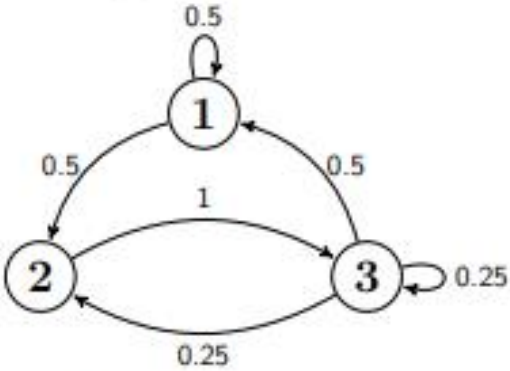
Answer each question in the space provided. There are 7 questions for a total of 100 points.

Problem 1

(a) [2 points] Consider the following Markov chain:



Write the transition matrix P

(b) [3 points] Suppose you have the transition matrix $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$. If the state probabilities at $k = 8$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find the state probabilities at $k = 9$.

(c) [3 points] For the transition matrix given in part (b), find the steady-state state probabilities, $\boldsymbol{\pi}$.

(d) [1 point] For the transition matrix given in part (b), does this Markov chain have an absorbing state?

(e) [1 point] Is the solution you found in part (c) unique? In other words, is this the only valid steady-state distribution?

(f) [1 point] In order to find the state distribution at $k = 100$, do you have to know **all** of the distributions for $k = 0, \ldots, 99$? If not, which one(s) would you need to know?

Problem 2 10 points

Let

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) A^{-1}

(b) A^2

(c) $(B^T A)C$

(d) $(B^T B)A$

(e) $C^T B^T$

Problem 3

(a) i. [2 points] Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $T(\mathbf{z}) = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$, what is a possible value for $\mathbf{z}$?

ii. [3 points] Suppose T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ to $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If $\mathbf{z} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, what is $T(\mathbf{z})$?

(b) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations.

i. [3 points] T first reflects about the line $x_2 = -x_1$, and then rotates 90 degrees clockwise about the origin.

ii. [3 points] T rotates by 90 degrees counterclockwise about the origin 5 times, and then projects onto the horizontal axis.

(c) [2 points] Select all transformations from part (b) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i. ☐ ii.

(d) [2 points] Select all transformations from part (b) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i. ☐ ii.

(e) Consider the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ given by $T(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ 2x_1 - x_2 + 2x_4 \\ x_1 + ax_2 + x_3 \end{bmatrix}$ where a is a constant.

i. [2 points] Write the standard matrix A of this transformation.

ii. [2 points] For what values of a (if any) is T **one-to-one**?

ii. __________

iii. [2 points] For what values of a (if any) is T **onto**?

iii. __________

iv. [2 points] For what values of a (if any) is T **invertible**?

iv. __________

Problem 4

(a) [8 points] This problem has 4 parts. Each question below may have more than one choice as the correct answer. You must list all correct choices to receive full credit. Consider the following four linear systems given by the **augmented matrices**:

$$E = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 1 & 3 & 5 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 3 & 2 & 5 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

i. Which of the linear systems, if any, are in row echelon form?

ii. Which of the linear systems, if any, are in reduced row echelon form?

iii. Which of the linear systems, if any, are consistent?

iv. Which of the linear systems, if any, has a unique solutions?

(b) Consider an **augmented matrix** $[A \;\; \mathbf{b}]$ that has reduced row echelon form

$$U = \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Denote the variables of the linear system by $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$.

i. [2 points] Which variables of the linear system are basic and which are free?

ii. [3 points] Write the solution to the linear system described by matrix U in parametric vector form.

(c) [3 points] Let $u = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $w = \begin{bmatrix} 2 \\ 6 \\ h \end{bmatrix}$. For what value(s) of h is the vector w in the span of $\{u, v\}$?

Problem 5 [12 points]

Answer each of the following three questions in the space provided.

(a) Consider matrices A and B below:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If matrix A is singular and B is nonsingular, which of the following matrices *must* be singular? Put a check mark ✓ in the box next to **EACH** correct answer. **Note**: The symbol $\circ$ denotes element-wise product (operation `.*` in MATLAB).

☐ $A \cdot B$ ☐ $A \circ B$

☐ $A + B$ ☐ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$

(b) Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and note that $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using B from part (a), what is the inverse of $E \cdot B^{-1}$?

(c) Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$. Is C^T invertible? If so, provide the **third column** of $(C^T)^{-1}$.

Problem 6 (18 points)

Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (1.5 points each)

(a) If a matrix is in row echelon form, then the entry in its upper left-hand corner must be non-zero.

(a) ____________

(b) If $A\mathbf{x} = \mathbf{0}$ has a free variable, where A is $m \times n$, then there must be an infinite number of solutions to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^m$.

(b) ____________

(c) If A and B can both be row reduced to the same sized identity matrix, then A must be equal to B.

(c) ____________

(d) If $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set, then the set of vectors $\{\mathbf{z}, \mathbf{w}, \mathbf{x}\}$ is also linearly independent.

(d) ____________

(e) If A is an $m \times n$ matrix whose columns are linearly independent, then it must be the case that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

(e) ____________

(f) If a vector $\mathbf{b}$ is in the span of the columns of matrix A, then A must have a pivot in every column.

(f) ____________

(g) Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and suppose the reduced row echelon form of its standard matrix has n pivots. Then T must be onto.

(g) ____________

(h) A linear transformation defined by a matrix with more columns than rows cannot be one-to-one.

(h) ____________

(i) If the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent, then the linear transformation defined by $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$ is not one-to-one.

(i) ____________

(j) A matrix is invertible if and only if its reduced row echelon form is the identity matrix.

(j) ____________

(k) If a matrix product $AB = I$, the identity matrix, then $A = B^{-1}$.

(k) ____________

(l) Given $n \times n$ matrices A and B, if A and B are invertible then A^2B^2 must also be invertible.

(l) ____________

Problem 7 (10 points)

Write a function called `is_cons` that has 2 inputs and 1 output. The inputs are a matrix `A` and a vector `b`, and the output is a logical scalar `cons`. The function should check if the system defined by `Ax = b` is consistent. If it is, then `cons` should be `true`. Otherwise, it should be `false`. No error checking or help lines are necessary. Just provide the code. *Hint: use* `rref`, *and note that* `rref` *has two outputs. Do not use loops.*