Engineering Analysis I, Fall 2023

Midterm 2

SOLUTIONS

Section number (from list below) ______________

Net ID ______________

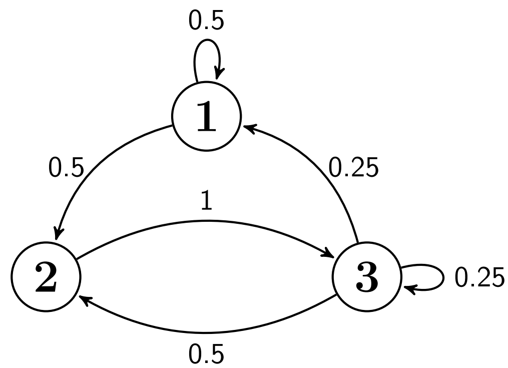
Section number	Discussion time	Instructor
22	8:00 a.m.	Prem Kumar
23	9:00 a.m.	Ilya Mikhelson
24	10:00 a.m.	Prem Kumar
25	11:00 a.m.	Michael Honig
27	12:00 p.m.	Michael Honig

This exam is closed-book and closed-notes. Calculators, computers, phones, or other computing/communication devices are not allowed.

Answer each question in the space provided. Write your final answer and nothing else in the box under each question. There are 7 questions for a total of 100 points. **Please check to make sure that your exam has all of its pages before you begin** (page numbers are indicated at the bottom of each page).

Problem 1

(a) [2 points] Consider the following Markov chain:



Write the transition matrix P.

$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

-1: transpose

(b) [3 points] Suppose you have the transition matrix $P = \begin{bmatrix} 0 & 1 \\ 0.8 & 0.2 \end{bmatrix}$. If the state probabilities at $k = 7$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find the state probabilities at $k = 8$.

$$\mathbf{s}(8) = P^T\mathbf{s}(7) = \begin{bmatrix} 0 & 0.8 \\ 1 & 0.2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

-2: multiply by P, not P’
-1: small math error
-1: probabilities don’t sum to 1

(c) [3 points] For the transition matrix given in part (b), find the steady-state state probabilities, $\boldsymbol{s}^\star$.

$$\boldsymbol{s}^\star = P^T\boldsymbol{s}^\star \Rightarrow (I - P^T)\boldsymbol{s}^\star = \mathbf{0}$$

$$\begin{bmatrix} 1 & -0.8 & 0 \\ -1 & 0.8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{s}^\star = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}$$

-1: probabilities don't sum to 1

-1: right equations, small error in row reduction

-2: used P, not P'

(d) [1 point] For the transition matrix given in part (b), does this Markov chain have an absorbing state? (Yes/No)

(d) No

(e) [1 point] Is the solution you found in part (c) unique? In other words, is this the only valid steady-state distribution? (Yes/No)

(e) Yes

(f) [1 point] For the Markov chain in part (a), if $\mathbf{s}(4) = \mathbf{s}(5)$, can we say that the steady state distribution is $\mathbf{s}(7)$? (Yes/No)

(f) Yes

Problem 2

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -2 & -2 \end{bmatrix}$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) [2 points] A^2

$$\boxed{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(b) [2 points] A^{-1}

$$\boxed{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

(c) [2 points] C^2

$\boxed{\text{Not defined}}$

(d) [2 points] BC

$$\boxed{\begin{bmatrix} -7 & 4 \\ 8 & 9 \end{bmatrix}}$$

(e) [2 points] AC

$\boxed{\text{Not defined}}$

Problem 3

(a) [6 points] Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Which of the following statements **can never** be correct? Put a check mark $\checkmark$ in the box next to **EACH** statement which must be **incorrect**.

$\checkmark$ $T(\begin{bmatrix}0\\0\end{bmatrix}) = \begin{bmatrix}0\\0\end{bmatrix}$,

$\checkmark$ $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $T(\begin{bmatrix}2\\2\end{bmatrix}) = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$\checkmark$ The Co-domain of T is $\mathbb{R}^2$.

$\checkmark$ $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $T(\begin{bmatrix}2\\2\end{bmatrix}) = \begin{bmatrix}0\\2\\2\end{bmatrix}$

☐ $T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\-1\\1\end{bmatrix}$,

$T(\begin{bmatrix}-1\\1\end{bmatrix}) = \begin{bmatrix}-2\\1\\3\end{bmatrix}$,

$T(\begin{bmatrix}5\\4\end{bmatrix}) = \begin{bmatrix}10\\-5\\-3\end{bmatrix}$

$\checkmark$ The range of T is $\mathbb{R}^2$.

(b) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations, assuming that $\mathbf{x} = [x_1\ x_2]^T$.

i. [3 points] T reflects across the axis $x_2 = -x_1$.

$$\boxed{A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$$

ii. [3 points] T rotates by 90 degrees clockwise, then projects onto the x_1 axis.

$$\boxed{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

iii. [3 points] T first reflects across the x_2 axis, and then scales vertically by a factor of 4.

$$\boxed{A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}}$$

(c) [2 points] Select all transformations from part (b) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(d) [2 points] Select all transformations from part (c) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(e) Consider the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ given by $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 2x_2 + 4x_3 + x_4 \\ 2x_1 + ax_2 + 4x_3 \end{bmatrix}$ where a is a constant.

i. [2 points] Write the standard matrix A of this transformation.

$$\boxed{A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 2 & a & 4 & 0 \end{bmatrix}}$$

ii. [2 points] For what values of a (if any) is T **not onto**?

ii. $a = 2$

iii. [2 points] For what values of a (if any) is T **one-to-one**?

iii. none

iv. [2 points] For what values of a (if any) is T **invertible**?

iv. none

Problem 4

(a) Consider the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

Here a, b, c, and d represent constant parameters. If this matrix is in *reduced* row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [2 points] What must be the value of a?

i. 0

ii. [2 points] What must be the value of b?

ii. 1

iii. [2 points] What must be the value of c?

iii. any number

iv. [2 points] What must be the value of d?

iv. 0

(b) [2 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 4 \\ 3 & 3 & 0 \end{bmatrix}$$

identity, I, or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Consider a system of linear equations of the form $Ax = b$, where the matrix A is 4-by-5 and the column vector x consists of the five unknown variables $x_1, x_2, \ldots, x_5$, in that order. Suppose the reduced row echelon form of the *augmented* matrix is given by

$$\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i. [2 points] Which of the unknown variables are free?

x_2 and x_5

ii. [2 points] How many solutions are there?

infinitely many

iii. [1 point] Does the solution set pass through the origin in $\mathbb{R}^5$ (yes or no)?

no

Problem 5

Answer each of the following three questions in the space provided.

(a) [4 points] Consider a matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ a & b & c \\ 3 & -1 & 4 \end{bmatrix}$. In which cases *must* A be singular? Put a check mark ✓ in the box next to **EACH** correct answer.

✓ $a = b = c = 0$ ✓ $a = 2,\ b = 4,\ c = -2$

☐ $a = b = 0,\ c \neq 0$ ✓ $c = a - b$

(b) [4 points] Let $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. For the A in part (a), let $a = 0,\ b = 0,\ c = 1$. What is the inverse of $E^{-1} \cdot (A^{-1})^T$?

$$\boxed{\begin{gathered}\left(E^{-1} \cdot (A^{-1})^T\right)^{-1} = A^T \cdot E = \\ \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ -1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}\end{gathered}}$$

(c) [4 points] Let $D = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 7 & -3 \\ 2 & -4 & 1 \end{bmatrix}$. Find the inverse of D.

$$\boxed{D^{-1} = \begin{bmatrix} -5 & -1 & 2 \\ -4 & -1 & 1 \\ -6 & -2 & 1 \end{bmatrix}}$$

Problem 6 [18 points]
Answer TRUE or FALSE for each of the following statements (do NOT just write "T" or "F"). You do not have to explain your answer. (2 points each)

(a) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set and $\mathbf{v}$ is not in Span$\{\mathbf{x}, \mathbf{y}\}$ and is not a scalar multiple of $\mathbf{z}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}\}$ must be linearly independent.

(a) False

(b) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set and $\mathbf{v}$ is not in Span$\{\mathbf{x}, \mathbf{y}\, \mathbf{z}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$ must be linearly independent.

(b) False

(c) If a matrix is in reduced row echelon form, then the entry in its bottom right corner must be 1.

(c) False

(d) For Markov chains, the transition probability matrix is always invertible.

(d) False

(e) If a matrix A has a pivot in its last column, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(e) False

(f) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ cannot be one-to-one.

(f) True

(g) If the system $A\mathbf{x} = \mathbf{b}$ is consistent and A is $m \times n$, then the columns of A span $\mathbb{R}^m$.

(g) False

(h) If the system $A\mathbf{x} = \mathbf{b}$ has a unique solution and A is $m \times n$, then the columns of A span $\mathbb{R}^n$.

(h) False

(i) If the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent, then the linear transformation defined by $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$ must not be one-to-one.

(i) True

Problem 7 [7 points]

Write a MATLAB function called `advance_state` that advances the state of a Markov chain by a given number of time points. The function should have 3 inputs and 1 output. The inputs are a transition matrix `P`, a state vector `s_in`, and an integer `n`. The function should use the matrix `P` to advance the state `s_in` by `n` time points. The output, called `s_out`, should be the result of this process. Specifically, if the input `s_in` is $\mathtt{s}(k)$, then the output `s_out` should be $\mathtt{s}(k+n)$. You do **not** need to include any error checking (e.g. you can assume the dimensions of the inputs are proper), help lines, etc. - just provide the code.

```
function s_out = advance_state(P, s_in, n)

s_out = (P')^ n * s_in;

    OR

s_out = s_in;
for ii = 1:n
   s_out = P' * s_out;
end
```