Engineering Analysis 1 - Midterm 2 – Practice Problems

Problem 1

a)
Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

i) A^2

-5 10
-15 10

ii) B^2

not defined

iii) AB

4 -1 6
-2 3 12

iv) $\mathbf{x}^T\mathbf{x}$

10

v) $\mathbf{x}\mathbf{x}^T$

9 -3
-3 1

b)
Let T be a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations

i) T reflects through the x_1-axis

$$\boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}}$$

ii) T reflects through the line $x_2 = x_1$

$$\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}$$

iii) T expands in the x_1 direction by factor 3

$$\boxed{\begin{matrix} 3 & 0 \\ 0 & 1 \end{matrix}}$$

iv) T projects onto the x_1-axis

$$\boxed{\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}}$$

v) T first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2$+4$\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = x_1$

$$\boxed{\begin{matrix} 0 & 1 \\ 1 & 4 \end{matrix}}$$

Problem 2

a) Let A be an $m \times n$ matrix. Which of the following conditions imply that the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution for each choice of $\mathbf{b}$ in $\mathbf{R}^m$?
(Write all of the correct conditions in the box.)

A. The columns of A span $\mathbf{R}^m$.
B. A has more columns than rows.
C. A has a pivot in every row.

A, C

b) Let A be an $m \times n$ matrix. If the number of rows is greater than the number of columns then which of the following are always true?
(Write all of the correct conditions in the box.)

A. The columns of A cannot span $\mathbf{R}^m$.
B. If $A\mathbf{x} = \mathbf{b}$ has a solution it will be unique.
C. A has a pivot in every column.

A

c) Consider the vectors:

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, w = \begin{bmatrix} 6 \\ -4 \\ h \end{bmatrix}$$

Find all values of h for which $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

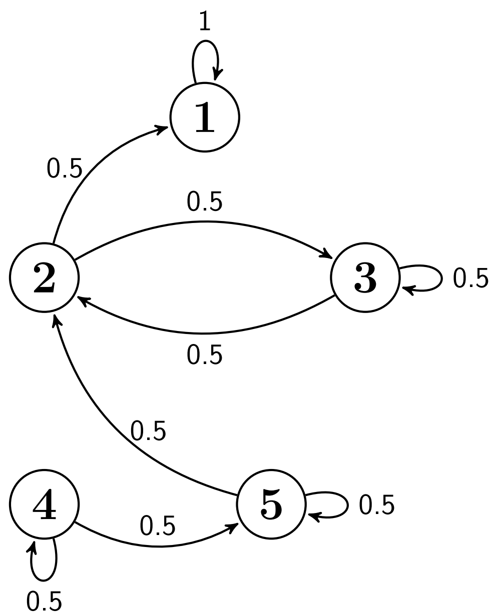
h = -6

d) For the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in part (c), give a geometric description of Span($\mathbf{v}_1$, $\mathbf{v}_2$) (i.e. is it a single point, a line, a plane, or all of $\mathbf{R}^3$).

a plane (or a plane in $\mathbf{R}^3$)

Problem 3

Consider the following Markov chain:



(a) Write the transition matrix P

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

(b) If the state probabilities at $k = 3$ are $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, find the state probabilities at $k = 5$.

$$\mathbf{s}(5) = P^T \mathbf{s}(4) = P^T (P^T \mathbf{s}(3)) = P^T P^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \\ 0 \\ 0 \end{bmatrix}$$

(c) Find the steady-state state probabilities, $\boldsymbol{\pi}$.

$$\boldsymbol{\pi} = P^T \boldsymbol{\pi} \Rightarrow (I - P^T)\boldsymbol{\pi} = \mathbf{0}$$

$$\sum_{i=1}^{n} \pi_i = 1$$

$$\begin{bmatrix} 0 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{\pi} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) Does this Markov chain have an absorbing state?

Yes

(e) Is the solution you found in part (c) unique? In other words, is this the only valid steady-state distribution?

Yes

Problem 4

Write MATLAB statements that perform the following operations. Assume that a matrix `A` has been defined previously and that the instruction `[m n]= size(A)` has been executed.

a) Given a pivot element of `A`, which may not be 1, use row operations to zero out the entries below the pivot. Let the row and column indices of the pivot be `ii` and `jj`, respectively.

```
for row = (ii +1):m
   A(row,:) = A(ii,jj) * A(row,:) - A(row,jj) * A(ii,:);
end
```

b) Given two row numbers, `nrow1` and `nrow2`, write MATLAB statements that will exchange rows `nrow1` and `nrow2` of `A`

```
temp = A(nrow1, :);
A(nrow1, :) = A(nrow2, :);
A(nrow2, :) = temp;

OR

A([nrow1 nrow2],:) = A([nrow2 nrow1],:);
```

Problem 5

This question has 5 parts. *A part may have more than one correct choice. You need to write down all the correct choices to receive full credit.* Each part of this question refers to the four linear systems given by the following **augmented** matrices.

$$(A.)\begin{bmatrix}1&2&5&3\\0&1&4&0\\0&0&1&2\end{bmatrix}\qquad(B.)\begin{bmatrix}1&0&5&0\\0&1&4&0\\0&0&0&1\end{bmatrix}\qquad(C.)\begin{bmatrix}1&2&5&3\\0&0&4&0\\0&0&0&0\end{bmatrix}\qquad(D.)\begin{bmatrix}1&2&5&3\\0&0&1&2\\0&0&1&0\end{bmatrix}$$

a) Which of the linear systems, if any, are in echelon form?

A, B, C

b) Which of the linear systems, if any, are in *reduced* echelon form?

B

c) Which of the linear systems are *not* consistent?

B, D

d) Which of the linear systems have a unique solution?

A

e) Suppose the linear system for each case is written in matrix form as $A\mathbf{x} = \mathbf{b}$. For which of the systems is $\mathbf{b}$ in the span of the columns of A?

A, C

Problem 6

Answer true or false for each of the following. You do not have to explain your answer.

a) If {**x**, **y**} is linearly independent and **z** is in Span{**x**, **y**}, then {**x**, **y**, **z**} is linearly independent.

_____False______

b) If T is a linear transformation, then $T(\mathbf{x})$ must be a vector that has the same number of elements as **x**.

_____False______

c) If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

______True_______

d) A linear transformation T is one-to-one if and only if the columns of T's standard matrix A are linearly independent.

______True_______

e) The second column of AB equals A times the second column of B.

______True_______

f) The transpose of AB, $(AB)^T$, always equals the transpose of A times the transpose of B, A^TB^T.

_____False______

g) The inverse of AB, $(AB)^{-1}$, always equals the inverse of A times the inverse of B, $A^{-1}B^{-1}$.

_____False______

h) If A is invertible, then the inverse of A^{-1}, $(A^{-1})^{-1}$, is A itself.

______True_______

i) If square matrices A and B satisfy $BA = I$, then $BA = AB$.

______True_______

j) If the square matrix A is $n \times n$ and invertible, then the linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

______True_______

k) If matrix A has m rows and 1 column, and matrix B has 1 row and m columns, then AB is an $m \times m$ matrix.

______True_______

l) If matrix A has more columns than rows, then $A\mathbf{x} = \mathbf{b}$ cannot have exactly one solution.

______True_______

m) If matrix A has more rows than columns, then $A\mathbf{x} = \mathbf{b}$ cannot have infinitely many solutions.

_____False______