Practice Final Exam

Engineering Analysis 1

Name ____________________Solution____________________ **Section** ____________

Clearly circle or box your solutions.

You may leave answers as fractions, where appropriate.

1. (16 points total)

   (a) The questions below are independent of each other and use the following matrices and vectors: $B = \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$ $\quad \mathbf{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $\quad \mathbf{u} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

      i. (2 points) Calculate $\mathbf{y}^{\,T} B^2$.

         $\mathbf{y}^{\,T} B^2 = \begin{bmatrix} 8 & 16 \end{bmatrix}$

      ii. (2 points) Calculate $\mathbf{u} \cdot \mathbf{v}$.

         $\mathbf{u} \cdot \mathbf{v} = 7$

      iii. (2 points) Find the orthogonal projection, $\text{proj}_{\mathbf{v}}\mathbf{u}$, of $\mathbf{u}$ onto $\mathbf{v}$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = 7/38 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/19 \\ 21/38 \\ 35/38 \end{bmatrix}$$

(b) (3 points) Put A into reduced row echelon form, and circle the pivot positions.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$, 1 point for reducing correctly, 2 points for circling pivots

(c) (4 points) Let $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. What is $\det(P)$?

$\det(P) = 0$

(d) (3 points) Assume that you are given a vector `b` of unknown length with at least one element. Fill in the following loop to sum all of the elements of `b` and put the result in `bSum`:

```
bSum = 0;
```

for ii = 1:length(b)

bSum = bSum + b(ii);

end

```
disp(bSum);
```

2. (10 points, 1 point each) True or false:

   (a) If $\|\mathbf{v}\| = 0$, then $\mathbf{v} = 0$.
   True

   (b) There is no vector that is orthogonal to every other vector.
   False

   (c) If $T(\mathbf{x}) = A\mathbf{x}$ is a one-to-one and onto transformation, then A must be a square matrix.
   True

   (d) For any matrix A, $A^T A = I$.
   False

   (e) If $\mathbf{y}$ is in $\operatorname{Col} A$, then there is a vector in the domain of the transformation $T(\mathbf{x}) = A\mathbf{x}$ that is mapped to $\mathbf{y}$.
   True

   (f) If $T(\mathbf{x})$ is one-to-one, then the domain of the transformation is the same as the range.
   False

   (g) If $\dim \operatorname{Nul} A = 1$, then $A\mathbf{x} = \mathbf{y}$ has an infinite number of solutions for any $\mathbf{y}$.
   False

   (h) If A is $m \times n$ and B is $p \times q$, then AB is only defined if $n = p$.
   True

   (i) The line $x_2 = 3x_1 + 2$ is a subspace of R^2.
   False

   (j) For $\mathbf{y}$ and $A\mathbf{x}$ in the same subspace,
   $\|\mathbf{y} - A\mathbf{x}\|^2 = \|\mathbf{y}\|^2 + 2\mathbf{y} \cdot A\mathbf{x} - \|A\mathbf{x}\|^2$.
   False

3. (12 points) Write a MATLAB function called **extrema** that is passed a single matrix A and returns four arguments

   1) `mx`, the maximum value of A
   2) `mn`, the minimum value of A
   3) `mxNum`, the number of times `mx` appears in A
   4) `mnNum`, the number of times `mn` appears in A

   Do not use built-in MATLAB functions `max`, `min`, `sort`, or `find`.

   12 points, 2 for correct function header, 2 for correctly initializing variables (no partial credit), 2 for correctly constructing for loops, 3 for finding mx and mn, 3 for correctly counting mxNum and mnNum. No error checking or H1 comment needed.

```
function [mx, mn, mxNum, mnNum] = extrema(A)
mxNum = 0;
mnNum = 0;
mx = A(1,1);
mn = A(1,1);
for i = 1:length(A,1)
   for j=1:length(A,2)
       if A(i,j) > mx
           mx = A(i,j);
           mxNum=1;
       elseif A(i,j)==mx
           mxNum=mxNum+1;
        end

       if A(i,j) < mn
           mn = A(i,j);
           mnNum=1;
       elseif A(i,j) == mn
           mnNum=mnNum+1;
       end
    end
end
```

4. (12 points total) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ be a basis for the subspace W.

(a) (2 points) Show that the two basis vectors are linearly independent.

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ only for $c_1, c_2 = 0$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors in $\mathcal{B}$ *or*

By observation $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other *or*

Row reducing the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$ results in a pivot in each column.

(b) (1 point) What is the dimension of W?

$\dim W = 2$

(c) (3 points) Let $\mathbf{y} = \begin{bmatrix} 12 \\ 10 \\ 14 \end{bmatrix}$. What is $[\mathbf{y}]_{\mathcal{B}}$?

$$[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$$

(d) (1 point) Show that the two basis vectors are orthogonal.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors in $\mathcal{B}$

(e) (3 points) What is the best approximation to $\mathbf{u} = \begin{bmatrix} 6 \\ 10 \\ 8 \end{bmatrix}$ in subspace W?

$$\hat{u} = \begin{bmatrix} 8 \\ 9 \\ 7 \end{bmatrix}$$

(f) (2 points) What is the distance from $\mathbf{u}$ to the nearest point in W?

$\sqrt{6}$

5. (10 points total) Suppose you are given the matrix A and its reduced echelon form as follows:

$$A = \begin{bmatrix} 1 & -3 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 15 \\ 0 & 0 & 2 & -3 & 24 \\ 2 & -6 & 0 & -5 & -2 \\ -4 & 12 & -12 & 0 & 28 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & -16 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (3 points) What is the dimension of the null space of A?

2

(b) (4 points) What is a basis for the null space of A?

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 0 \\ -3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

(c) (3 points) Denote the columns of A as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$. Let $\mathcal{B}$ be the basis $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$. If the coordinates of $\mathbf{z}$ with respect to basis $\mathcal{B}$ are $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, calculate $\mathbf{z}$.

$$\mathbf{z} = \begin{bmatrix} -2 \\ -2 \\ -3 \\ -3 \\ -4 \end{bmatrix}$$

6. (8 points) Let $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

(a) (4 points) Find all eigenvalues of A.

$\lambda = -1, -2$

(b) (4 points) For each eigenvalue found in part (a), find a corresponding eigenvector.

For $\lambda = -1$, $\mathbf{v_1} = \begin{bmatrix} c \\ -c \end{bmatrix}$ for any scalar c.

For $\lambda = -2$, $\mathbf{v_2} = \begin{bmatrix} d \\ -2d \end{bmatrix}$ for any scalar d.

7. (12 points) Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

(a) (10 points) Find the least squares solution of the system $Ax = b$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b) (2 points) Is the solution from part (a) unique? Circle Yes (Y) or No (N):

Yes

8. (20 points) In this problem you will write two MATLAB functions to check various properties of an $n \times n$ square matrix A.

   (a) Write a function `isInvertible(A)` that returns 1 if the matrix is invertible and 0 if the matrix is not invertible.

   (b) Write a function `isOrthonormal(A)` that returns 1 if the columns of A are orthonormal and 0 if they are not.

   Moreover,

   - Do not perform any error checking.
   - In your function you may use any of the MATLAB functions listed below if you are comfortable using them. (We will not explain what they do for you.)

`eye()`	`length()`	`min()`	`rank()`	`zeros()`
`inv()`	`max()`	`rref()`	`size()`	`det()`

```
function y = isInvertible(A)
   n=length(A)
   if rank(A)==n
      y=1;
   else
      y=0;
   end
end
```

```
function y = isOrthonormal(A)

[n,m]=size(A);  y=1;

% check that all columns have norm equal 1

for i= 1:m or 1:n (since matrix is square)

      if A(:,i)' * A(:,i) ~=1

          y=0;

          break

      end

end

% check orthogonality

for k=1:n-1

       for j= (k+1):n

              if A(:,k)'  * A(:,j)  ~=0

                    y= 0;

                    break

              end

        end

end
```