# Engineering Analysis I, Fall 2015

# Midterm 2

SOLUTIONS

Section number ______________

| Section number | Lecture time |
|---|---|
| 20 | 8:00 a.m. |
| 21 | 10:00 a.m. |
| 22 | 11:00 a.m. |
| 23 | 12:00 noon |

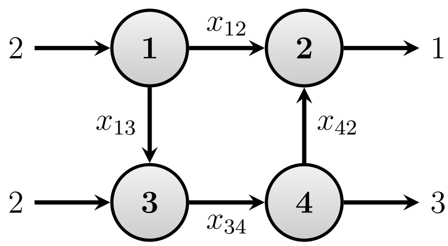
Answer the questions in the spaces provided on the question sheets. There are 7 problems worth 100 points total. This exam is closed-book and closed-notes. You will not need, and you are not allowed to use calculators, computers, phones, or other computing/communication devices.

Students should skip this page—it is only for graders.

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 10 | |
| 3 | 26 | |
| 4 | 15 | |
| 5 | 13 | |
| 6 | 12 | |
| 7 | 12 | |
| Total: | 100 | |

## Problem 1

Consider the following network flow diagram and flow balance equations:




$$\boldsymbol{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{34} \\ x_{42} \end{bmatrix}$$

$$\begin{aligned} x_{12} + x_{13} &= 2 \\ x_{12} + x_{42} &= 1 \\ -x_{13} + x_{34} &= 2 \\ x_{34} - x_{42} &= 3 \end{aligned}$$

(a) [2 points] Write down the above flow balance equations in the form $A\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{x}$ as above.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{34} \\ x_{42} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

(OK if rows are swapped or negative)

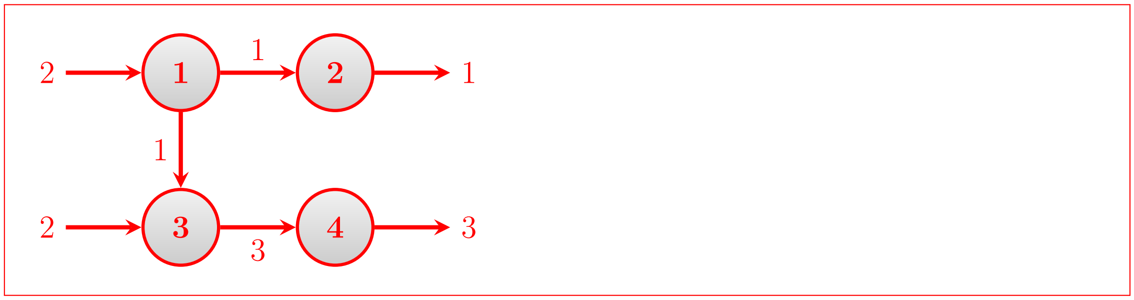
(b) [3 points] Find the reduced row echelon form of the augmented matrix $[A \;\; \boldsymbol{b}]$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

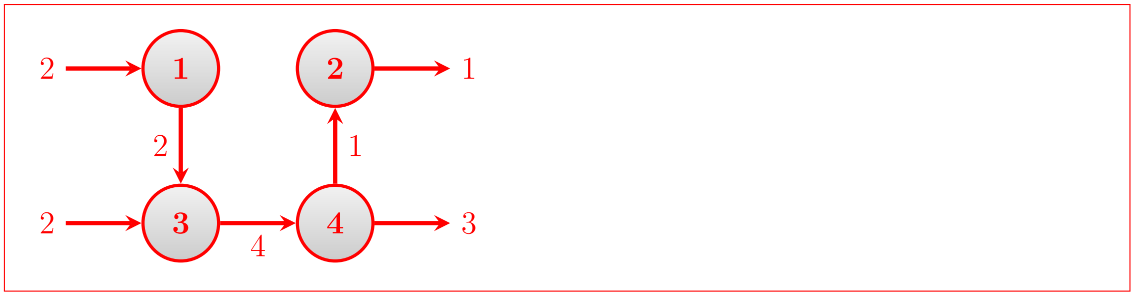
(c) [3 points] Write the solution set for your system of equations in parametric vector form.

$$\boldsymbol{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{34} \\ x_{42} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(d) [2 points] Draw the solution you get when you set all free parameters equal to zero.




(e) [2 points] Draw a solution which has $x_{12} = 0$.



**Problem 2**

Let

$$A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 1 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & 2 & 1 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) [2 points] $A^{-1}$

not defined

(b) [2 points] $A^{10}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) [2 points] $AB$

$$\begin{bmatrix} 6 & -4 & -2 & 10 \\ -6 & 4 & 2 & -10 \end{bmatrix}$$

(d) [2 points] $((B^T B)C)A$

not defined

(e) [2 points] $\det(C)$

2

**Problem 3**

(a) i. [3 points] Suppose $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. What is $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$?

$[-2 \quad -1]^T$

ii. [3 points] Suppose $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ that maps the vector $\mathbf{x} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. What is $T(\begin{bmatrix} 1 \\ 3 \end{bmatrix})$ ?

$[4 \;\; 6 \quad -2]^T$

(b) Suppose $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix $A$ for each of the following transformations.

i. [3 points] $T$ reflects around the axis $x_2 = -x_1$.

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

ii. [3 points] $T$ first rotates by 90 degrees clockwise, then projects onto the $x_1$ axis.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

iii. [3 points] $T$ first reflects around the $x_2$ axis, and then scales vertically by a factor of 4.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

(c) [1½ points] Select all transformations from part (b) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(d) [1½ points] Select all transformations from part (b) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(e) Now consider $T$ to be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 2x_2 + 4x_3 + x_4 \\ 2x_1 + ax_2 + 4x_3 \end{bmatrix}$$ where $a$ is a constant.

i. [2 points] Write the standard matrix $A$ of this transformation.

$$\boxed{A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 2 & a & 4 & 0 \end{bmatrix}}$$

ii. [2 points] For what values of $a$ (if any) is $T$ NOT **onto**?

ii. $a = 2$

iii. [2 points] For what values of $a$ (if any) is $T$ **one-to-one**?

iii. none

iv. [2 points] For what values of $a$ (if any) is $T$ **invertible**?

iv. none

**Problem 4**

(a) Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

Here $a$, $b$, $c$, and $d$ represent constants. If this matrix is in *reduced* row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [1 point] What must be the value of $a$?

i. 0

ii. [1 point] What must be the value of $b$?

ii. 1

iii. [1 point] What must be the value of $c$?

iii. 0

iv. [1 point] What must be the value of $d$?

iv. 0

(b) [2 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

identity, $I$, or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Consider a system of linear equations of the form $A\boldsymbol{x} = b$, where the matrix $A$ is 3-by-6 and the column vector $\boldsymbol{x}$ consists of the five unknown variables $x_1, x_2, \ldots, x_6$. Suppose the reduced row echelon form of the augmented matrix is given by

$$\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

i. [2 points] Which of the unknown variables are basic?

$x_1$, $x_3$ and $x_5$

ii. [4 points] Write the solution set in parametric vector form.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 6 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

iii. [3 points] Write the solution set to the corresponding homogeneous system of equations $A\boldsymbol{x} = \mathbf{0}$ as the span of a set of vectors.

$$span\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

**Problem 5**

Answer each question in the space provided.

(a) [2 points] Consider a matrix $A$ with the form: $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & a & b \\ 3 & -1 & 4 \end{bmatrix}$. Give one choice of $a$ and $b$ for which this matrix is singular. *Note: you can answer by inspection.*

$a = -1$, $b = 4$

(b) [2 points] Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using $A$ from part (a) with $a$ and $b$ chosen such that $A$ is singular, what is the inverse of $B \cdot A$? If it does not exist, state that.

Multiplying a singular matrix by a non-singular matrix will not make the product non-singular. Therefore, the inverse is **undefined**.

(c) [2 points] Using $A$ from part (a), where $a$ and $b$ are chosen such that $A$ is singular, what is the inverse of $A^T$? If it does not exist, state that.

Taking the transpose of a singular matrix will not make it non-singular. Therefore, the inverse is **undefined**.

(d) [3 points] Let $A = \begin{bmatrix} 0 & -4 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ be the standard matrix of a transformation. Find a matrix $B$ such that whenever $A\boldsymbol{x} = \boldsymbol{y}$ for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, then $B\boldsymbol{y} = \boldsymbol{x}$

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

(e) [1 point] Is the matrix $B$ from part (d) invertible?

Yes.

(f) [3 points] Let $D = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 7 & -3 \\ 2 & -4 & 1 \end{bmatrix}$. Find the inverse of $D$.

$$D^{-1} = \begin{bmatrix} -5 & -1 & 2 \\ -4 & -1 & 1 \\ -6 & -2 & 1 \end{bmatrix}$$

**Problem 6** [12 points]
Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (1 point each)

(a) The reduced row echelon form of a matrix is unique.

(a) True

(b) The solution set for a system of linear equations of the form $Ax = b$ cannot contain exactly three distinct vectors.

(b) True

(c) Let $T$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and suppose the reduced row echelon form of its standard matrix has a pivot in every column. Then $T$ must be onto.

(c) False

(d) Let $T$ be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$, and suppose the reduced row echelon form of its standard matrix has a pivot in every column. Then $T$ must be onto.

(d) True

(e) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ cannot be one-to-one.

(e) True

(f) A matrix is invertible if and only if its reduced row echelon form is the identity matrix.

(f) True

(g) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is invertible if and only if it is onto.

(g) True

(h) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is invertible if and only if it is one-to-one.

(h) True

(i) A square matrix is invertible if and only if its determinant is nonzero.

(i) True

(j) If a matrix product $AB$ exists, then $B^T A^T$ exists and is equal to $(AB)^T$.

(j) True

(k) If a matrix product $AB$ exists and has all zero entries, then either $A$ or $B$ must have all zero entries.

(k) False

(l) Suppose the matrix products $AB$ and $BA$ both exist. If $AB = BA$, then $A$ and $B$ must be square and of the same size.

(l) True

**Problem 7** [12 points]
Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (2 points each)

(a) If a matrix is in reduced row echelon form, then the entry in its upper left-hand corner must be 1.

(a) False

(b) If a matrix $A$ has a column of all zeros, then the system of linear equations $Ax = b$ must have infinitely many solutions.

(b) False

(c) If a matrix $A$ has a row of all zeros, then the system of linear equations $Ax = b$ must be inconsistent.

(c) False

(d) Suppose the matrix product $AB$ exists, and suppose the rows of $A$ are not linearly independent. Then the rows of $AB$ cannot be linearly independent.

(d) True

(e) Suppose the matrix product $AB$ exists, and suppose the columns of $B$ are not linearly independent. Then the columns of $AB$ cannot be linearly independent.

(e) True

(f) If a matrix product $AB$ exists and is invertible, then both $A$ and $B$ must be invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

(f) False