EA1 Homework Program 8: Column Space, Null Space, and Coordinates
Due Friday, November 17, 2023, at 6:00am

In this homework, you will write a MATLAB function called `find_bases` to find basis vectors for the column space and null space of a given matrix. The function will also display the dimension of each of these subspaces, and the space they are a subspace of. Furthermore, given a vector, the function will determine if the vector is in the column space or null space, and if so, it will calculate the coordinates of that vector in terms of the basis you found earlier. The inputs to the function are a matrix `A`, and an optional vector `b`. The outputs are a matrix `cs`, a matrix `ns`, a vector `coords_cs`, and a vector `coords_ns`. The columns of `cs` form a basis for the column space of `A`, the columns of `ns` form a basis for the null space of `A`, `coords_cs` contains the coordinates of `b` if it is in the column space (or an empty matrix otherwise), and `coords_ns` contains the coordinates of `b` if it is in the null space (or an empty matrix otherwise). If the user does not provide a vector `b`, then the outputs `coords_cs` and `coords_ns` should be empty. Note: you are not allowed to use the built-in `null` or `orth` functions in MATLAB.

A sample set of inputs and outputs is below.

$$\mathtt{A} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \quad \mathtt{b} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ -5 \\ 1 \end{bmatrix}$$

The program will then display:

```
The column space is a 3 dimensional subspace of R^3
The null space is a 2 dimensional subspace of R^5

b is in R^5 and is in the null space of A, and its coordinates are:
1.000000
1.000000
```

The outputs of the program will be:

$$\mathtt{cs} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathtt{ns} = \begin{bmatrix} -2 & -1 \\ -3 & 2 \\ 1 & 0 \\ 0 & -5 \\ 0 & 1 \end{bmatrix}, \quad \mathtt{coords\_cs} = [\,], \quad \mathtt{coords\_ns} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Below is an outline of the program's functionality. You do not have to use the suggested methods, but your functionality should be consistent with what is specified.

1. Use an `arguments` block to check that the input matrix `A` is not empty. If vector `b` is not provided, set its default value to an empty matrix.

2. Use `rref` to find the pivot columns.

   (a) Use the pivot columns to find a basis for the column space.

3. Use `setdiff` to find the non-pivot columns. Read the documentation!
   (a) Use the non-pivot columns to find a basis for the null space. Think about what you would do by hand when finding this basis, and translate that to code.
4. Display the dimensions of the column space and null space, as well as the space they are in (i.e. $\mathbb{R}^n$).
5. Check if `b` was provided (i.e. that it is not the default empty matrix). If not, set `coords_cs` and `coords_ns` to empty and skip this step. Otherwise, check if `b` is in each of the subspaces. Always display the space(s) it is in. In addition:
   (a) If its dimensions indicate that it is not a candidate for either subspace, display that to the user.
   (b) If its dimensions indicate that it is a candidate for the column space:
      i. Check whether it is in the column space.
      ii. If not, tell that to the user.
      iii. If it is, display its coordinates in terms of the basis you found previously.
   (c) If its dimensions indicate that it is a candidate for the null space:
      i. Check whether it is in the null space.
      ii. If not, tell that to the user.
      iii. If it is, display its coordinates in terms of the basis you found previously.
   (d) Make sure that the `coords_cs` and `coords_ns` outputs are defined regardless of what case `b` falls into. Make them empty if `b` is not in that subspace. Also, note that a given vector can be in both the column space and the null space.

Test Cases

Make sure your program can run all of these. Copy and paste the displayed output of each, as well as the output variables, at the end of your file. Make sure to include all four output variables from the function.

1. `A = [1 2 -4 -3 0;-2 -3 5 8 8;2 2 -2 -9 -13]; b = [1;8;2];`
2. `A = [1 2 -4 -3 0;-2 -3 5 8 8;2 2 -2 -9 -13]; b = [1;8;2;3];`
3. `A = [1 2 -4 -3 0;-2 -3 5 8 8;2 2 -2 -9 -13]; b = [1;8;2;3;-1];`
4. `A = [1 2 -4 -3 0;-2 -3 5 8 8;2 2 -2 -9 -13]; b = [1;8;2;3;-2];`
5. `A = [1 0 2;0 1 3;0 0 0]; b = [1;2;3];`
6. `A = [2 -2;2 -2];b = [1;1];`
7. `A = zeros(3);b = zeros(3,1);`
8. `A = eye(3);b = zeros(3,1);`
9. `A = [1 0 0 1;0 1 0 2;0 0 1 3;0 0 -1/3 -1]; b = [1;2;3;-1];`
10. `A = [1 0 0 1;0 1 0 2;0 0 1 3;0 0 -1/3 -1];`

Sample Complete Output to Paste at End of Program (1)

In this case, `b` is of the proper dimensions to be in the null space.
(`A = [1 0 2 0 1;0 1 3 0 -2;0 0 0 1 5];b = [-3; -1; 1; -5; 1;];`)

```
The column space is a 3 dimensional subspace of R^3
The null space is a 2 dimensional subspace of R^5

b is in R^5 and is in the null space of A, and its coordinates are:
1.000000
1.000000

cs =

     1     0     0
     0     1     0
     0     0     1


ns =

    -2    -1
    -3     2
     1     0
     0    -5
     0     1


coords_cs =

     []


coords_ns =

    1.0000
    1.0000
```

Sample Complete Output to Paste at End of Program (2)

In this case, `b` is of the proper dimensions to be in both the column space and null space. (`A = [1 0 2;0 1 3;0 0 0];b = [1;2;3];`)

```
The column space is a 2 dimensional subspace of R^3
The null space is a 1 dimensional subspace of R^3

b is in R^3 but is not in the column space of A
b is in R^3 but is not in the null space of A

cs =

     1     0
     0     1
     0     0


ns =

    -2
    -3
     1


coords_cs =

     []


coords_ns =

     []
```

Sample Complete Output to Paste at End of Program (3)

In this case, `b` has dimensions that are invalid for both the column space and null space. (`A = [1 0 2;0 1 3;0 0 0];b = [1;2;3;4];`)

```
The column space is a 2 dimensional subspace of R^3
The null space is a 1 dimensional subspace of R^3

b is in R^4 and is neither in the column space nor the null space,
based on its dimensions

cs =

     1     0
     0     1
     0     0


ns =

    -2
    -3
     1


coords_cs =

     []


coords_ns =

     []
```

Sample Complete Output to Paste at End of Program (4)

In this case, `b` is in both the column space and null space.
(`A = [-2 4;-1 2];b = [8;4];`)

```
The column space is a 1 dimensional subspace of R^2
The null space is a 1 dimensional subspace of R^2

b is in R^2 and is in the column space of A, and its coordinates are:
-4.000000
b is in R^2 and is in the null space of A, and its coordinates are:
4.000000

cs =

    -2
    -1


ns =

     2
     1


coords_cs =

    -4


coords_ns =

     4
```

Sample Complete Output to Paste at End of Program (5)

In this case, `b` is not provided.
(`A = [-2 4;-1 2];`)

```
The column space is a 1 dimensional subspace of R^2
The null space is a 1 dimensional subspace of R^2

cs =

    -2
    -1


ns =

     2
     1


coords_cs =

     []


coords_ns =

     []
```