Engineering Analysis I, Fall 2022

Midterm 2

SOLUTIONS

Section number (from list below) ______________

Net ID ______________

Section number	Discussion time	Instructor
23	9:00 a.m.	Prem Kumar
24	10:00 a.m.	Michael Honig
25	11:00 a.m.	Prem Kumar
26	8:00 a.m.	Randy Freeman
27	12:00 p.m.	Michael Honig

This exam is closed-book and closed-notes. Calculators, computers, phones, or other computing/communication devices are not allowed.

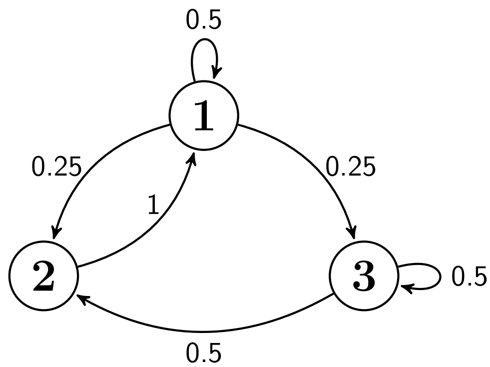
Students should skip this page—it is only for graders.

Question	Points	Score
1	11	
2	10	
3	23	
4	15	
5	12	
6	18	
7	11	
Total:	100	

Answer each question in the space provided. There are 7 questions for a total of 100 points.

Problem 1

(a) [2 points] Consider the following Markov chain:



Write the transition matrix P

$$\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

-1: transpose

(b) [3 points] Suppose you have the transition matrix $P = \begin{bmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \end{bmatrix}$. If the state probabilities at $k = 6$ are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the state probabilities at $k = 7$.

$$\mathbf{s}(7) = P^T \mathbf{s}(6) = \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

-2: multiply by P, not P'
-1: small math error
-1: probabilities don't sum to 1

(c) [3 points] For the transition matrix given in part (b), find the steady-state state probabilities, $\boldsymbol{s}^\star$.

$$\boldsymbol{s}^\star = P^T \boldsymbol{s}^\star \Rightarrow (I - P^T)\boldsymbol{s}^\star = \mathbf{0}$$

$$\begin{bmatrix} 0.25 & -0.75 & 0 \\ -0.25 & 0.75 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.75 \\ 0 & 1 & 0.25 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{s}^\star = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

-1: probabilities don't sum to 1

-1: right equations, small error in row reduction

-2: used P, not P'

(d) [1 point] For the transition matrix given in part (b), does this Markov chain have an absorbing state? (Yes/No)

(d) No

(e) [1 point] Is the solution you found in part (c) unique? In other words, is this the only valid steady-state distribution? (Yes/No)

(e) Yes

(f) [1 point] For the Markov chain in part (a), if $\mathbf{s}(10) = \mathbf{s}(11)$, can we say that the steady state distribution is $\mathbf{s}(11)$? (Yes/No)

(f) Yes

Problem 2 10 points

Let

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) A^2

$$\boxed{\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}}$$

(b) $(\frac{1}{\sqrt{6}}A)^{23}$

$$\boxed{\frac{1}{\sqrt{6}}A}$$

(c) $A(BC^T)$

$$\boxed{\begin{bmatrix} -4 & 6 \\ 3 & -9 \end{bmatrix}}$$

(d) $(C(B^TA^T))^T$

$\boxed{\text{same as (c)}}$

(e) $(C^TC)B$

$\boxed{\text{not defined}}$

Problem 3

(a) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to $T(\mathbf{y}) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

i. [1 point] Is the domain all of $\mathbb{R}^2$? (Yes/No)

i. Yes

ii. [1 point] Is the codomain all of $\mathbb{R}^2$? (Yes/No)

ii. Yes

iii. [1 point] Is the range all of $\mathbb{R}^2$? (Yes/No)

iii. No

(b) [2 points] Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ to $T(\mathbf{y}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\mathbf{z} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$, what is $T(\mathbf{z})$?

$$\boxed{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

(c) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations.

i. [3 points] T first reflects about the x_2 axis, then rotates 90 degrees counterclockwise about the origin, and then projects onto the x_2 axis.

$$\boxed{A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}$$

ii. [3 points] T rotates by 180 degrees clockwise about the origin, and then reflects about the x_1 axis.

$$\boxed{A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

(d) [2 points] Select all transformations from part (c) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i.   ✓ ii.

(e) [2 points] Select all transformations from part (c) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

☐ i.   ✓ ii.

(f) Consider the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by $T(\mathbf{x}) = \begin{bmatrix} x_1 - 2x_2 \\ 2x_1 - 4x_2 \\ ax_2 - x_1 \end{bmatrix}$ where a is a constant.

i. [2 points] Write the standard matrix A of this transformation.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & a \end{bmatrix}$$

Only 1 point for $\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ a & -1 \end{bmatrix}$

ii. [2 points] For what values of a (if any) is T **one-to-one**?

ii. $a \neq 2$ (or $a \neq 1/2$)

iii. [2 points] For what values of a (if any) is T **onto**?

iii. none

iv. [2 points] For what values of a (if any) is T **invertible**?

iv. none

Problem 4

(a) Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here a, b, c, and d represent constant parameters. If this matrix is in *reduced* row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [2 points] What must be the value of a?

i. 1

ii. [2 points] What must be the value of b?

ii. any number

iii. [2 points] What must be the value of c?

iii. 0

iv. [2 points] What must be the value of d?

iv. 1

(b) [2 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 2 & 1 & 0 & 4 & 5 & 6 \\ 0 & 7 & 0 & 9 & 8 & 7 \\ 0 & 0 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Consider a system of linear equations of the form $Ax = b$, where the matrix A is 3-by-5 and the column vector x consists of the five unknown variables $x_1, x_2, \ldots, x_5$. Suppose the reduced row echelon form of the augmented matrix is given by

$$\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i. [3 points] Which of the unknown variables are free?

x_1, x_3 and x_5
1 point for each correct variable, -1 for each incorrect variable

ii. [2 points] Does the solution set pass through the origin in $\mathbb{R}^5$ (yes or no)?

ii. No

Problem 5 [12 points]
Answer each of the following three questions in the space provided.

(a) Consider matrices A and B below:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

If matrix A is singular and B is nonsingular, which of the following matrices *must* be singular? Put a check mark ✓ in the box next to **EACH** correct answer.

✓ $B^{-1} \cdot A$ ✓ $A^2 \cdot B^T$

☐ $(A + B)B$ ☐ $\begin{bmatrix} 1 & 1 & 1 \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$

(b) Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Using B from part (a), what is the inverse of $B^{-1} \cdot (E^{-1})^T$?

$$\left(B^{-1} \cdot (E^{-1})^T\right)^{-1} = E^T \cdot B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} b_{11} + b_{21} & b_{12} + b_{22} & b_{13} + b_{23} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

(c) Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$. Is C^T invertible? If so, compute $(C^T)^{-1}$.

$$(C^T)^{-1} \to \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & -1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & -0.5 & -1 & 0.5 \end{array}\right]$$

Problem 6 (18 points)
Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (1.5 points each)

(a) If $\{\mathbf{x}, \mathbf{y}\}$ is a linearly independent set and $\mathbf{z}$ is **not** in Span$\{\mathbf{x}, \mathbf{y}\}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

(a) True

(b) If a matrix is in reduced row echelon form, then the entry in its upper left-hand corner must be 1.

(b) False

(c) The solution set for a system of linear equations of the form $A\mathbf{x} = \mathbf{b}$ can contain exactly three distinct vectors.

(c) False

(d) If a matrix A has a row of all zeros, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(d) False

(e) If a vector $\mathbf{b}$ is in the span of the columns of matrix A, then $A\mathbf{x} = \mathbf{b}$ must be consistent.

(e) True

(f) If a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is onto, it must also be invertible.

(f) True

(g) If a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is one-to-one, then the the span of the columns of its standard matrix must be the same as its codomain.

(g) True

(h) If a matrix product AB exists and is invertible, then both A and B must be invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

(h) False

(i) If a matrix product AB exists and has all zero entries, then either A or B (or both) must have all zero entries.

(i) False

(j) A linear transformation defined by a matrix with more columns than rows cannot be one-to-one.

(j) True

(k) If the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent, then the linear transformation defined by $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$ is one-to-one.

(k) False

(l) If A and B can both be row reduced to the identity matrix, then $A = B$.

(l) False

Problem 7 (11 points)

Write a function called `is_invertible` that checks if the linear transformation $T(x) = Ax$ is invertible. The function has one input, the standard matrix `A`, and one output, a logical scalar `inv`. The matrix `A` can have arbitrary size. If the transformation is invertible, `inv` should be `true`. Otherwise, it should be `false`. *For this problem, you may use any built-in MATLAB function, but you* **must** *use the output(s) of* `rref` *(or equivalently,* `reduce` *from Homework 5) to decide whether* `inv` *is true or false. Do not use loops.* No error checking, help lines, or comments are necessary. Assume the input `A` is not empty.

```
function inv = is_invertible(A)         (3 points)

[m,n] = size(A);
[~,pivs] = rref(A);         (2 points)

if (m == n) && (length(pivs) == m)
    inv = true;
else
    inv = false;
end

    or

inv = (m == n) && (length(pivs) == m);

(3 points for checking if A is square,
3 points for checking if pivot is in every row or column)
```