NAME:____

INSTRUCTOR:__________________________

Question 1 (of 8)

(i) (1 pt) Find $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 5 & 8 \\ 3 & 2 \end{bmatrix}$$

Grading: 1 pt, no partial credit

(ii) (1 pt) Find a 2×2 matrix A that maps (0,0)→(0,0), (1,0)→(4,0), (0,1)→(4,2) and (1,1)→(8,2).

Grading: 1 pt, no partial credit

$$\begin{bmatrix} 4 & 4 \\ 0 & 2 \end{bmatrix}$$

(iii) (1 pt) Find the $m \times n$ matrix A that yields the linear transformation

$$T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 2x_1 - 3x_2 + 2x_4 \\ 5x_3 \\ 6x_1 - 2x_2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -3 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 6 & -2 & 0 & 0 \end{bmatrix}$$

Grading: 1 pt if correct or mostly correct –up to three components wrong

POINTS:

(iv) (2 pts) For $A = \begin{bmatrix} -2 & 0 & 3 & 1 \\ 1 & 3 & 4 & 2 \\ 4 & 0 & -2 & 3 \end{bmatrix}$, determine if the associated transformation is one-to-one, onto, both or neither.

onto

Echelon form: $\begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 6 & 11 & 5 \\ 0 & 0 & 4 & 5 \end{bmatrix}$.

Grading: 2 pts: 1 pt for correct reduction, 1 pt for correct conclusion

(v) (2 pts) Consider the vectors v_1, v_2 and v_3 below. For what values of h are v_1, v_2 and v_3 linearly independent? (If you can find no such values for h, state so.)

$$v_1 = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5 \\ 6 \\ h \end{pmatrix}.$$

$h \sim= 12$

Grading: 2 pts: 1 pt for correct reduction (get *h-12* in the last component), 1 pt for correct conclusion

(vi) (2 pts) Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, find A^{-1}.

$$\begin{bmatrix} -2 & 3 & -3 \\ -1/2 & 1/2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Grading: 2 pts for correct solution (up to 2 components wrong).
Only 1 pt if final answer wrong but correct procedure.

(vii) (1 pt) Simplify the expression $A(A^{-1}B)(B^{-1}C)(C^{-1}I)$ assuming that all matrix row-column dimensions allow the indicated operations and the inverse matrices exist.

Grading: 1 pt, no partial credit

I

POINTS:

Question 2 (of 8)

Indicate whether each statement below is TRUE or FALSE.

(i) (1 pt) The product AB of two matrices A and B is always defined when the number of rows of A is equal to the number of columns of B.

False

(ii) (1 pt) The matrix product $A^T A$ is defined for all matrices.

True

(iii) (1 pt) If the transformation $T(x) = Ax$ for the matrix A is both one-to-one and onto, then A must be invertible.

True

(iv) (1 pt) If the matrix product AB yields the zero matrix, then either A or B must be the zero matrix.

False

(v) (1 pt) If the matrix product AB yields the identity matrix, then B must equal the transpose of A.

False

(vi) (1 pt) If the matrix product $AB = BA$, then both matrices must be square.

True

(vii) (1 pt) If the matrix product AB is invertible, then both A and B must be invertible.

False

(viii) (1 pt) The inverse of the transpose of a matrix A must equal the transpose of the inverse of A.

True

Grading: 1 pt each, no partial credit

POINTS:

Question 3 (of 8)

Suppose you are given the matrix A and its reduced echelon form as follows:

$$A = \begin{bmatrix} 2 & 1 & 4 & 2 & 6 \\ 1 & 0 & 1 & 3 & 4 \\ 3 & 1 & 5 & 2 & 7 \\ 1 & 1 & 3 & 7 & 10 \end{bmatrix}, \quad \texttt{rref(A)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) (1 pt) What is the dimension of the column space of A?

3

Grading: 1 pt, no partial credit

(ii) (1 pt) What is the dimension of the null space of A?

2

Grading: 1 pt, no partial credit

(iii) (2 pts) What is a basis for the column space of A?

$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 7 \end{bmatrix}$$

Grading: 2 pts if all vectors correct; accept obvious typos only.
1 pt if two vectors correct; accept obvious typos only.

(iv) (2 pts) What is a basis for the null space of A?

$$\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Grading: 2 pts if both vectors *entirely* correct;
1 pt if only one vector *entirely* correct or up to one component wrong in each vector.

(v) (2 pts) Suppose $y = A(:,3) + A(:,4)$. What are the coordinates of y relative to the basis you gave in part (iii)?

Grading: 2 pts if entirely correct; 1 pt if one component wrong.
If solution is wrong due to a wrong solution in part (iii), get 1 point if the procedure is correct.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(vi) (1 pt) Do the columns of A span $\mathbf{R}^3$?

Grading: 1 pt, no partial credit

no

POINTS:

Question 4 (of 8)

(i) (1 pt) Suppose A is a $p \times q$ matrix and the columns of A are linearly independent. What is the dimension of the null space of A?

zero

(ii) (1 pt) Suppose A is a $p \times q$ matrix, which is already in row reduced echelon form. Suppose that every row has at least one non-zero entry. What is the dimension of the column space of A?

p

(iii) (1 pt) If A is a $p \times q$ matrix where $p>q$, what is the maximum number of vectors in a basis of the column space of A?

q

(iv) (1 pt) If the columns of A form a basis for the column space of A, what are the possible number of solutions for $Ax=b$? (Give all the possibilities.)

0 or 1

(v) (1 pt) If A is a $p \times q$ matrix and $p<q$, then the rank of A cannot be greater than p. (true/false)

true

(vi) (1 pt) If the coordinates of some vector x relative to a basis $\boldsymbol{B}$ are $c_1 = 1$, $c_2 = 0$ and $c_3 = 3$, and the coordinates of a different vector y relative to the same basis $\boldsymbol{B}$ are $d_1 = -1$, $d_2 = 2$ and $d_3 = 1$, what are the coordinates of $2x+y$ relative to $\boldsymbol{B}$?

$$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Grading: 1 pt each, no partial credit

POINTS:

Question 5 (of 8)

(i) (2 pts) Given the vector $x = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \end{bmatrix}$ and the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}$, find the orthogonal projection of x onto Col (A).

Grading: 2 pts, if entirely correct.
1 pt if final solution wrong, but correct procedure.

$$\begin{bmatrix} 5 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

(ii) (2 pts) Suppose a vector y and its orthogonal projection onto a subspace W are given by

$$y = \begin{bmatrix} 2 \\ -3 \\ 3 \\ 1 \end{bmatrix}, \quad proj_W\, y = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

What is the distance from y to W?

6

Grading: 2 pts, if entirely correct.
1 pt if final solution wrong, but correct procedure.

(iii) (2 pts) Apply Gram-Schmidt to transform the set $\{x_1, x_2\}$ into an orthogonal set of vectors, where

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Grading: 2 pts, if entirely correct.
1 pt if final solution wrong, but correct procedure (Gram-Schmidt).

(iv) (2 pts) What is the least squares solution to $Ax = b$? $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -3 & 3 \\ 1 & -1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$

$$\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

Grading: 2 pts, if entirely correct.
1 pt if final solution wrong, but correct procedure (either normal equations, or orthogonal projection, or QR).

Question 6 (of 8)

Parts (i) to (viii) (1 pt each) Let B be an $p \times q$ matrix with **linearly independent columns** and **let x be a vector in the column space of B**. The following segments of code can be used to calculate `coor`—the coordinates of x relative to B— if B has certain properties. You can assume that x, b, m (# rows of B) and n (# columns of B) are all previously defined.

For each of the segments of code, what is the necessary property of B for the code segment to correctly calculate `coor`? Each segment of code has exactly one correct answer and each answer is used at least once.

a) All B
b) B is a square matrix
c) The columns of B are orthonormal.
d) None of the Above. More information about B is needed.

```
EXAMPLE:
coor = B\x
```
Answer: a

(*You do not need to give an explanation – just the final answer*). The answer is (a) —If x is in the column space of B, then B*$coor = x$ is consistent for any matrix B. The left division symbol will return the solution to B*$coor = x$ if the system is consistent, and this system is consistent for any B.

(i)
```
coor = zeros(n, 1);
for index = 1:n
    coor(index) = dot (x, x)/dot(B(:, index), x);
end;
```
Answer: d

(ii)
```
A = rref[B x];
coor = A(:, n+1);
```
Answer: b

(iii)
```
coor = inv(B' * B) * B' * x;
```
Answer: a

(iv)
```
coor = (eye(m,n) - B)\x;
```
Answer: d

Grading: 1 pt each, no partial credit

(v)
```
coor = zeros(n, 1);
for index = 1:n
    coor(index) = dot (B(:, index), x);
end;
```
Answer: c

(vi)
```
coor = inv(B)* x;
```
Answer: b

(vii)
```
coor = B' * x;
```
Answer: c

(viii)
```
A = rref[B x];
coor = A(1:n, n+1);
```
Answer: a

(ix) (2 pts) The following segment of code is used to identify the pivot columns of matrix *A* and store their position in a vector **pivot**. However, this segment contains one incorrect line. Find the incorrect line of code in this segment and replace it with the code necessary to correctly find the pivots in matrix *A*.

```
B = rref(A);

[m n] = size (A);

row = 1;
col = 1;

1.    while (row <=m & col <= n)
2.         if (B(row, col))
3.              pivot(row) = col;
4.              col = col + 1;
5.              row = row + 1;
6.         else
7.              row = row+ 1;
8.         end;
9.    end;
```

line: 7 **new code:** `col = col + 1;`

Grading: 2 pts, no partial credit

POINTS:

Question 7 (of 8)

(i) (3 pts) Write a function called `projectit` whose input is a matrix A and whose output is a matrix Q. The matrix Q is obtained by normalizing each column of A, i.e., each column of Q is of norm 1. You can assume that A contains no zero columns.

```
function [Q] = projectit(A)

[m n] = size(A);
for jj = 1:n
     len = sqrt(dot(A(:,jj),A(:,jj)));
     Q(:,jj) = A(:,jj)/len;
end


Also accepted: len = sqrt(A(:,jj)'*A(:,jj));
Also accepted: len = sqrt(sum(A(:,jj).*A(:,jj)));
NOT ACCEPTED: len = sqrt(length(A(:,jj)));
```

Grading: 3 points total. –1 pt (up to 3) for each mistake, like wrong function definition, or wrong loop, or wrong computation of length, etc.

(ii) (3 pts) Write a function called `yonu` whose input is two vectors, y and u. The output is a vector w containing the projection of y onto the line containing u. You can assume that u is a non-zero vector.

```
function [w] = yonu(y,u)

w = (dot(y,u)/dot(u,u))*u;

Also accepted: w = ((y'*u)/(u'*u))*u;
Also accepted: w = (sum(y.*u)/sum(u.*u))*u;
```

Grading: 3 points total. –1 pt (up to 3) for each mistake, like wrong function definition, or put the vectors backwards (project u onto y), or compute the quotient of dot products but do not multiply by u at the end.

POINTS:

Question 8 (of 8)

Grading for (i), (ii) and (iii): 2 pts each. –1 pt (up to 2) for each mistake.

(i) (2 pts) The purpose of the function `make_triang` given below is to set all elements of B below the diagonal to zero. What is the missing code?

```
function  U= make_triang(B)
U=B;
[m n] = size(B);
for INSERT_CODE_1
     for INSERT_CODE_2
          U(irow,icol)= 0;
     end
end
```

```
INSERT_CODE_1:

irow=2:m

INSERT_CODE_2:

icol=1:(irow-1)
```

(ii) (2 points) The following Matlab script attempts to compute the least squares solution of $Ax = b$, for some predefined matrix A and vector b. Correct every line that is incorrect or write "OK" if the line is correct.

```
[m q]= size(A);
if rank(A) = q

  B= A*A;

  b= A'*b;

  xhat= B/b;

  disp(xhat);
else
  disp('dependent columns');
end
```

```
(1)   B = A'*A;

(2)   OK

(3)   xhat = B\b;
```

(iii) (2 pts) The following Matlab script applies Gram-Schmidt to the n columns of a matrix X, which has been defined earlier. What is the missing code?

```
V(:,1)= X(:,1);
for iv = 2:n
  V(:,iv)= X(:,iv);
  for k = INSERT_CODE_1
    num = INSERT_CODE_2*V(:,k);
    den = V(:,k)'*V(:,k);
    proj = (num/den)*INSERT_CODE_3;
    V(:,iv) = V(:,iv)-proj;
  end
end
```

```
INSERT_CODE_1:

1:(iv-1)

INSERT_CODE_2:

X(:,iv)'

INSERT_CODE_3:

V(:,k)
```

POINTS: