Engineering Analysis I, Fall 2017

Midterm 2

SOLUTIONS

Section number ______________

Section number	Discussion time	Instructor
30	9:00 a.m.	Ilya Mikhelson
31	10:00 a.m.	Ilya Mikhelson
32	10:00 a.m.	Iman Hassani Nia
33	11:00 a.m.	Iman Hassani Nia
34	12:00 noon	Randy Berry

This exam is closed-book and closed-notes. Calculators, computers, phones, or other computing/communication devices are not allowed.

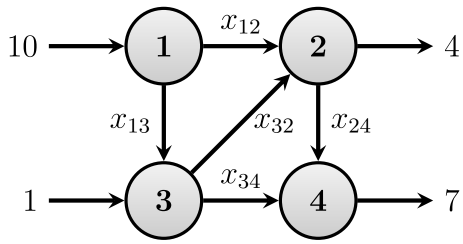
Students should skip this page—it is only for graders.

Question	Points	Score
1	12	
2	10	
3	13	
4	14	
5	10	
6	32	
7	9	
Total:	100	

Answer each question in the space provided. There are 7 questions for a total of 100 points.

Problem 1

Consider the following network flow diagram and flow balance equations:



$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} \qquad \begin{aligned} x_{12} + x_{13} &= 10 \\ x_{12} + x_{32} - x_{24} &= 4 \\ -x_{13} + x_{32} + x_{34} &= 1 \\ x_{24} + x_{34} &= 7 \end{aligned}$$

(a) [2 points] Write the flow balance equations in the form $A\mathbf{x} = \mathbf{b}$, with $\mathbf{x}$ as defined above.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 1 \\ 7 \end{bmatrix}$$

(OK if rows are swapped or negative)

(b) [3 points] Find the reduced row echelon form of the augmented matrix $[A \;\; \mathbf{b}]$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 11 \\ 0 & 1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) [3 points] Write the solution set for your system of equations in parametric vector form.

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 0 \\ 7 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

($\mathbf{x}_h$ (last 2 vectors) can be multiples)
(accept x_{32} and x_{34} as parameters.)

(d) [2 points] Find a parametric vector form solution in which the flows in the particular solution are all non-negative. There may be more than one right answer.

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 1 \\ 7 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

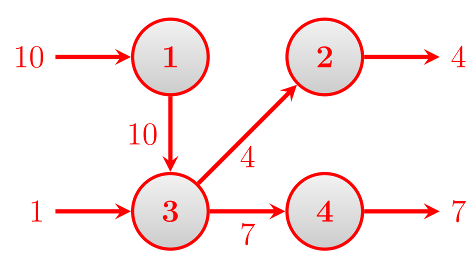
OR

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 2 \\ 7 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

OR

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{32} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 3 \\ 7 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(e) [2 points] Draw a solution to the network flow problem with $x_{12} = 0$ and $x_{24} = 0$. Make sure no flows are negative!



Problem 2

Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 1 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 3 & -4 \\ 0 & 1 & 4 & 2 \\ 2 & -2 & 6 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) [2 points] A^{100}

$$\boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

(b) [2 points] C^{-1}

$\boxed{\text{not defined}}$

(c) [2 points] $(C^T)^{-1}$

$\boxed{\text{not defined}}$

(d) [2 points] $(BB^T)C$

$\boxed{\text{not defined}}$

(e) [2 points] AB

$$\boxed{\begin{bmatrix} 3 & -2 & -1 & 5 \\ -3 & 2 & 1 & -5 \end{bmatrix}}$$

Problem 3

(a) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Write the standard matrix for each of the following transformations.

i. [2 points] T scales vectors vertically by a factor of 3.

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

ii. [2 points] T first reflects across the line at -45 degrees through the origin and then rotates clockwise by 90 degrees about the origin.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

iii. [2 points] T first projects onto the line at 45 degrees through the origin and then rotates clockwise by 90 degrees about the origin.

$$\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}$$

(b) [2 points] Select all transformations from part (a) that are **onto**. Put a check mark (✓) in the box next to **EACH** correct answer.

✓ i. ✓ ii. ☐ iii.

(c) [2 points] Select all transformations from part (a) that are **one-to-one**. Put a check mark (✓) in the box next to **EACH** correct answer.

✓ i. ✓ ii. ☐ iii.

(d) [3 points] Given a linear transformation $T : \mathbb{R}^7 \to \mathbb{R}^2$, suppose that $T(2\mathbf{x})=\begin{bmatrix} 10 \\ 2 \end{bmatrix}$ and $T(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ for two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^7$. What is $T(\mathbf{y})$?

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Problem 4

(a) Consider the following matrix:

$$\begin{bmatrix} a & 0 & c & 0 \\ 0 & 1 & 1 & 0 \\ 0 & b & 0 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here a, b, c, and d represent constant parameters. If this matrix is in **reduced** row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [2 points] What must be the value of a?

i. 1

ii. [2 points] What must be the value of b?

ii. 0

iii. [2 points] What must be the value of c?

iii. any number

iv. [2 points] What must be the value of d?

iv. 1

(b) [2 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 5 & 0 & 1 & 3 & 4 & 8 \\ 0 & 0 & 3 & 4 & 4 & 7 \\ 0 & 0 & 0 & 5 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Consider a system of linear equations of the form $Ax = b$, where the matrix A is 4-by-5 and the column vector x consists of the five unknown variables $x_1, x_2, \ldots, x_5$. Suppose the reduced row echelon form of the augmented matrix is given by

$$\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i. [2 points] Which of the unknown variables are free?

x_2 and x_5

ii. [1 point] How many solutions are there?

infinitely many

iii. [1 point] Does the solution set pass through the origin in $\mathbb{R}^5$ (yes or no)?

no

Problem 5

Answer each question in the space provided. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -5 & 3 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) [2 points] Is A invertible? If so, find its inverse.

$$\text{yes,} \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) [2 points] Is B invertible? If so, find its inverse.

no

(c) [2 points] Is C invertible? If so, find its inverse.

$$\text{yes,} \quad C = A^T \text{ so } C^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(d) [2 points] Is the product AB invertible? If so, find its inverse.

no (because B is not invertible)

(e) [2 points] Is the product AC invertible? If so, find its inverse.

$$\text{yes, } (AC)^{-1} = C^{-1}A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Problem 6 32 points

Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (2 points each)

(a) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set, then $\{\mathbf{x}, \mathbf{y}\}$ must be linearly independent.

(a) False

(b) If a matrix is in reduced row echelon form, then the entry in its bottom left-hand corner must be 0.

(b) False

(c) The solution set for a system of linear equations of the form $A\mathbf{x} = \mathbf{b}$ can contain exactly two distinct vectors.

(c) False

(d) If a matrix A has a row of all zeros, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(d) False

(e) No two matrices can have the same reduced row echelon form.

(e) False

(f) If a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ is one-to-one, then the span of the columns of its standard matrix must be the same as its codomain.

(f) True

(g) If a matrix product AB exists and is invertible, then both A and B must be invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

(g) False

(h) If a matrix product AB exists and has all zero entries, then either A or B (or both) must have all zero entries.

(h) False

(i) If $A = B^T$, and the columns of B are linearly independent, then it must be the case that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

(i) True

(j) If the span of the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ in $\mathbb{R}^m$ does not include all vectors in $\mathbb{R}^m$, then the linear transformation defined by the standard matrix $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$ is not onto.

(j) True

(k) A linear transformation defined by a matrix with more columns than rows cannot be onto.

(k) False

(l) A linear transformation defined by a matrix with more columns than rows must be onto.

(l) False

(m) A linear transformation defined by a matrix with more columns than rows cannot be one-to-one.

(m) True

(n) A linear transformation defined by a matrix with more columns than rows must be one-to-one.

(n) False

(o) If the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, then the linear transformation defined by A must be both onto and one-to-one.

(o) True

(p) If matrix A has more rows than columns, then $A\mathbf{x} = \mathbf{b}$ **cannot** have infinitely many solutions.

(p) False

Problem 7 9 points

Write a function called `check_cons` that has 2 inputs and 1 output. The inputs are a matrix `A` and a vector `b`, and the output is a logical scalar `y`. The function should check if the system defined by `Ax = b` is consistent. If it is, then `y` should be `true`. Otherwise, it should be `false`. No error checking, help lines, etc. is necessary. Just provide the code. *Hint: use* `rref`.

```
function y = check_cons(A,b)          (3 points)

[~,pivs] = rref([A b]);               (2 points)

if pivs(end) == size(A,2) + 1         (4 points for the whole block)
    y = false;
else
    y = true;
end

          or

y = pivs(end) ~= size(A,2) + 1;
```