Engineering Analysis I, Fall 2018

Midterm 2

SOLUTIONS

Section number ______________

Section number	Discussion time	Instructor
30	9:00 a.m.	Randy Freeman
31	10:00 a.m.	Michael Honig
32	11:00 a.m.	Prem Kumar
33	12:00 noon.	Prem Kumar
35	11:00 a.m.	Michael Honig

This exam is closed-book and closed-notes. Calculators, computers, phones, or other computing/communication devices are not allowed.

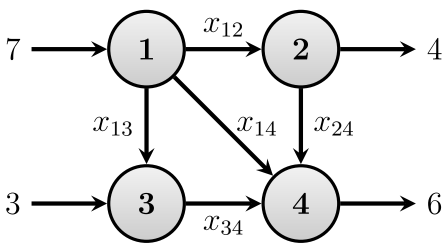
Students should skip this page—it is only for graders.

Question	Points	Score
1	12	
2	10	
3	23	
4	9	
5	10	
6	27	
7	9	
Total:	100	

Answer each question in the space provided. There are 7 questions for a total of 100 points.

Problem 1

Consider the following network flow diagram and flow balance equations:



$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} \qquad \begin{aligned} x_{12} + x_{13} + x_{14} &= 7 \\ x_{12} - x_{24} &= 4 \\ -x_{13} + x_{34} &= 3 \\ x_{14} + x_{24} + x_{34} &= 6 \end{aligned}$$

(a) [2 points] Write the flow balance equations in the form $A\mathbf{x} = \mathbf{b}$, with $\mathbf{x}$ as defined above.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \\ 6 \end{bmatrix}$$

(OK if rows are swapped or negative)

(b) [3 points] Find the reduced row echelon form of the augmented matrix $[A \;\; \mathbf{b}]$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) [3 points] Write the solution set for your system of equations in parametric vector form.

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(last 2 vectors can be multiples)
(accept x_{24} and x_{34} as parameters)

(d) [2 points] Find a parametric vector form solution in which the flows in the particular solution are all non-negative. There may be more than one right answer.

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

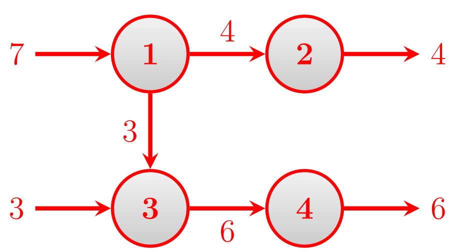
OR

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

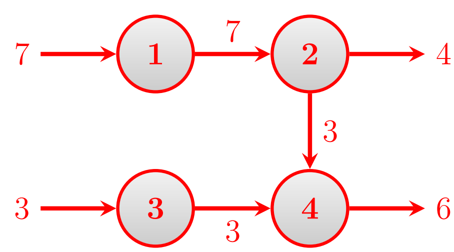
OR

$$\mathbf{x} = \begin{bmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 5 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(e) [2 points] Draw a solution to the network flow problem which has $x_{14} = 0$. Make sure no flows are negative! There may be more than one right answer.



OR



Problem 2

Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 1 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -2 & -2 \end{bmatrix}$$

Find each of the following quantities or write "not defined" if the operation is not defined.

(a) [2 points] A^2

$$\boxed{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

(b) [2 points] A^{-1}

$$\boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

(c) [2 points] B^2

Not defined

(d) [2 points] BC

Not defined

(e) [2 points] CB

$$\boxed{\begin{bmatrix} 7 & 0 & -5 & 7 \\ 0 & -7 & 4 & 7 \\ -6 & 4 & 2 & -10 \end{bmatrix}}$$

Problem 3

(a) i. [3 points] Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and the vector $\mathbf{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. What is the image of $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ under T?

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

ii. [3 points] Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ that maps the vector $\mathbf{x} = \begin{bmatrix} -1 \\ 1.5 \\ 2 \end{bmatrix}$ to $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Write a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$. Is this $\mathbf{x}$ unique?

$$\begin{bmatrix} -0.5 \\ 0.75 \\ 1 \end{bmatrix}, \text{ not unique}$$

(b) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T(\mathbf{x}) = A\mathbf{x}$. Write the standard matrix A for each of the following transformations.

i. [2 points] T reflects about the line $x_2 = -x_1$.

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

ii. [2 points] T first rotates by 90 degrees counterclockwise, then projects onto the x_2 axis.

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

iii. [2 points] T first scales horizontally by a factor of 2 then reflects about the x_2 axis.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) [1½ points] Select all transformations from part (b) that are **onto**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(d) [1½ points] Select all transformations from part (b) that are **one-to-one**. Put a check mark ✓ in the box next to **EACH** correct answer.

✓ i. ☐ ii. ✓ iii.

(e) Consider the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by $T(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 + 2x_3 \\ x_1 + 2x_2 + ax_3 \\ 2x_1 + 4x_2 + 4x_3 \\ x_2 \end{bmatrix}$

where a is a constant.

i. [2 points] Write the standard matrix A of this transformation.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & a \\ 2 & 4 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

ii. [2 points] For what values of a (if any) is T **onto**?

ii. none

iii. [2 points] For what values of a (if any) is T NOT **one-to-one**?

iii. $a = 2$

iv. [2 points] For what values of a (if any) is T **invertible**?

iv. none

Problem 4

(a) Consider the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$$

Here a, b, c, and d represent constants. If this matrix is in *reduced* row echelon form, and there are *three pivots*, then what must be the values of these four parameters? For each parameter, either write a specific number or write "any number" if the parameter can have any value.

i. [1½ points] What must be the value of a?

i. 0

ii. [1½ points] What must be the value of b?

ii. 1

iii. [1½ points] What must be the value of c?

iii. anything

iv. [1½ points] What must be the value of d?

iv. 0

(b) [3 points] What is the reduced row echelon form of the following matrix?

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 6 & 6 & 6 \end{bmatrix}$$

identity, I, or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 5

Answer each question in the space provided. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -5 & 3 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 2 & 6 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) [2 points] Is A invertible? If so, find its inverse.

$$\text{yes,} \quad A^{-1} = B = \begin{bmatrix} 1 & -5 & 3 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) [2 points] Is B invertible? If so, find its inverse.

$$\text{yes,} \quad B^{-1} = A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) [2 points] Is C invertible? If so, find its inverse.

no

(d) [2 points] Is the product AB invertible? If so, find its inverse.

yes, $AB = I$ so its inverse is also the 4×4 identity I

(e) [2 points] Is the product AC invertible? If so, find its inverse.

no

Problem 6 (27 points)
Answer TRUE or FALSE for each of the following statements. You do not have to explain your answer. (1.5 points each)

(a) In a network flows problem, you will always have more unknowns than equations.

(a) False

(b) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly independent set, then $\{\mathbf{x}, \mathbf{y}\}$ must be linearly independent.

(b) True

(c) Given the three row operations (exchange 2 rows, multiply a row by a non-zero scalar, and add a scalar multiple of a row to another row), any matrix can be put into echelon form without using the second of these operations.

(c) True

(d) Any two row equivalent matrices will have the same reduced echelon form.

(d) True

(e) If a matrix is in reduced row echelon form, then the entry in its bottom left-hand corner must be 0.

(e) False

(f) If a matrix A has a row of all zeros, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

(f) False

(g) There is exactly one way to write the parametric vector form solution for a system of equations having multiple solutions.

(g) False

(h) The span of a set of vectors can contain exactly one vector.

(h) True

(i) If a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ is one-to-one, then the span of the columns of its standard matrix must be $\mathbb{R}^4$.

(i) False

(j) If a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one, then the span of the columns of its standard matrix must be $\mathbb{R}^3$.

(j) True

(k) If a matrix product AB exists and is invertible, then both A and B must be invertible with $(AB)^{-1} = B^{-1}A^{-1}$.

(k) False

(l) If a matrix product AB exists and has all zero entries, then either A or B (or both) must have all zero entries.

(l) False

(m) If $A = B^T$, and the columns of A are linearly independent, then it must be the case that the linear transformation defined by the standard matrix B is onto.

(m) True

(n) A linear transformation defined by a matrix with more columns than rows cannot be onto.

(n) False

(o) A linear transformation defined by a matrix with more columns than rows must be onto.

(o) False

(p) A linear transformation defined by a matrix with more columns than rows cannot be one-to-one.

(p) True

(q) A linear transformation defined by a matrix with more columns than rows must be one-to-one.

(q) False

(r) If matrix A has more rows than columns, then $A\mathbf{x} = \mathbf{b}$ **cannot** have infinitely many solutions.

(r) False

Problem 7 (9 points)

Write a function called `in_span` that has 2 inputs and 1 output. The inputs are a matrix `A` and a column vector `b`, and the output is a logical scalar `y`. The columns of `A` form a set of vectors, i.e. `A` $= [\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n]$. The function should check if `b` is in the span of the columns of `A`, i.e. `b` $\in \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ If it is, then `y` should be `true`. Otherwise, it should be `false`. No error checking, help lines, etc. is necessary. Just provide the code. *Hint: use* `rref`.

```
function y = in_span(A,b)              (3 points)

[~,pivs] = rref([A b]);                (2 points)

if pivs(end) == size(A,2) + 1          (4 points for the whole block)
    y = false;
else
    y = true;
end

          or

y = pivs(end) ~= size(A,2) + 1;
```