MATH 230-1: Discussion 4 Solutions
Northwestern University, Fall 2023

1. Consider the curve where the surfaces with equations $x + y^2 = 3$ and $z = \frac{2}{3}y^3 - 2$ intersect.
(a) Find parametric equations for this curve.
(b) Find parametric equations for the tangent line to this curve at the point $(-6, 3, 16)$.
(c) Find the arclength of the portion of this curve between $(-6, 3, 16)$ and $(-78, 9, 484)$.

Solution. (a) Set $y = t$ to be the parameter. In order for the desired curve to lie on the first surface, x must be $x = 3 - t^2$ since x and y together must satisfy $x + y^2 = 3$, and in order for the curve to lie on the second surface, z must be $z = \frac{2}{3}t^3 - 2$ since y and z must satisfy $z = \frac{2}{3}y^3 - 2$. Hence we take

$$x = 3 - t^2,\ y = t,\ z = \tfrac{2}{3}t^3 - 2,\ -\infty < t < \infty.$$

as parametric equations for this surface.

(b) In order to describe a tangent line we need a vector that gives its direction, which is a tangent vector. With

$$\mathbf{r}(t) = \left\langle 3 - t^2, t, \tfrac{2}{3}t^3 - 2 \right\rangle$$

as the vector-valued function describing the curve, we get

$$\mathbf{r}'(t) = \left\langle -2t, 1, 2t^2 \right\rangle$$

as the function describing tangent vectors. The point $(-6, 3, 16)$ is attained at $t = 3$, so

$$\mathbf{r}'(3) = \langle -6, 1, 18 \rangle$$

is the direction vector for the tangent line at $(-6, 3, 16)$. One choice of parametric equations of this line is thus

$$x = -6 - 6t,\ y = 3 + t,\ z = 16 + 18t,\ -\infty < t < \infty.$$

(c) We have

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 1 + 4t^4} = \sqrt{(2t^2 + 1)^2} = 2t^2 + 1.$$

Since $(-78, 8, 484)$ is the point occurring at $t = 8$ and $(-6, 3, 16)$ the point at $t = 3$, the desired arclength is

$$\int_3^8 |\mathbf{r}'(t)|\, dt = \int_3^8 (2t^2 + 1)\, dt = \left(\tfrac{2}{3}t^3 + t\right)\Big|_3^8 = \tfrac{1024}{3} + 8 - 18 - 3.$$

□

2. Find parametric equations for the Cartesian curve with polar equation $r = \theta$ for $0 \le \theta < 2\pi$, and verify that motion along this curve occurs at a speed which increases as you move along.

Solution. In polar coordinate we have $x = r\cos\theta, y = r\sin\theta$, and to describe this specific curve we need the value of r to be $r = \theta$, so

$$x = \theta\cos\theta,\ y = \theta\sin\theta,\ 0 \le \theta < 2\pi$$

is a valid choice of parametric equations. With $\mathbf{r}(\theta) = \langle \theta\cos\theta, \theta\sin\theta \rangle$ as the position vector of a point on the curve, the velocity vector is

$$\mathbf{r}'(\theta) = \langle \cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta \rangle.$$

The speed at a point is then

$$\begin{aligned}|\mathbf{r}'(\theta)| &= \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} \\ &= \sqrt{\cos^2\theta - 2\theta\cos\theta\sin\theta + \theta^2\sin^2\theta + \sin^2\theta + 2\theta\cos\theta\sin\theta + \theta^2\cos^2\theta} \\ &= \sqrt{(\cos^2\theta + \sin^2\theta) + \theta^2(\sin^2\theta + \cos^2\theta)} \\ &= \sqrt{1+\theta^2}.\end{aligned}$$

As θ increases, $|\mathbf{r}'(\theta)| = \sqrt{1+\theta^2}$ increases as well, so speed increases along the curve. □

3. Suppose a rocket moves through space with constant acceleration $\mathbf{a}(t) = 4\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$ and initial velocity $\mathbf{v}(\mathbf{0}) = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, where t is measured in seconds.

(a) If the rocket is initially at $(0,0,0)$, find the position vector $\mathbf{r}(t)$ of the rocket.

(b) Determine the point at which the rocket is at after traveling for 1000 seconds.

(c) What is the speed at which the rocket is traveling at the point in (b)?

Solution. (a) First we integrate acceleration to find velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = (4t + c_1)\,\mathbf{i} + (2t + c_2)\,\mathbf{j} + (-3t + c_3)\,\mathbf{k}.$$

Since we must have $\mathbf{v}(0) = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$, we need $c_1 = 1, c_2 = -2, c_3 = 1$. Thus the velocity vector is

$$\mathbf{v}(t) = (4t+1)\,\mathbf{i} + (2t-2)\,\mathbf{j} + (-3t+1)\,\mathbf{k}.$$

Next we integrate velocity to find position:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (2t^2 + t + d_1)\,\mathbf{i} + (t^2 - 2t + d_2)\,\mathbf{j} + (-\tfrac{3}{2}t^2 + t + d_3)\,\mathbf{k}.$$

Since the initial position $\mathbf{r}(0) = d_1\,\mathbf{i} + d_2\,\mathbf{j} + d_3\,\mathbf{k}$ should be $0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$, we get $d_1 = d_2 = d_3 = 0$. Thus the position is

$$\mathbf{r}(t) = (2t^2 + t)\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + (-\tfrac{3}{2}t^2 + t)\,\mathbf{k}$$

(b) At 1000 seconds, the position vector is

$$\mathbf{r}(1000) = [2(1000)^2 + 1000]\,\mathbf{i} + [(1000)^2 - 2000]\,\mathbf{j} + [-\tfrac{3}{2}(1000)^2 + 1000]\,\mathbf{k},$$

so the rocket is at the point $(2(1000)^2 + 1000, (1000)^2 - 2000, -\frac{3}{2}(1000)^2 + 1000)$.

(c) The speed of the rocket in general is given by

$$|\mathbf{r}'(t)| = |\mathbf{v}(t)| = \sqrt{(4t+1)^2 + (2t-2)^2 + (-3t+1)^2}.$$

Thus, at the point in (b) the speed is

$$|\mathbf{v}(1000)| = \sqrt{[4(1000)+1]^2 + [2(1000)-2]^2 + [-3(1000)+1]^2}$$

meters per second. □