MATH 230-1: Written Homework 3
Northwestern University, Fall 2023

1. Consider the line L_1 with parametric equations

$$x = 4 - 6t,\ y = -3 - t,\ z = -2 + 3t \quad -\infty < t < \infty$$

and the line L_2 with parametric equations

$$x = t,\ y = 7 - 2t,\ z = 3 + 4t \quad -\infty < t < \infty.$$

(a) Justify the fact that these lines do not intersect, and are not parallel.

(b) The fact that L_1 and L_2 do not intersect and are not parallel implies they are *skew*, which means that they lie in different parallel planes. Find equations for these two planes. Hint: Two planes are parallel if they can be described using the same normal vector.

(c) Pick any point on L_1 and compute its distance to the plane containing L_2. Then pick any point on L_2 and compute its distance to the plane containing L_1. (You should get the same value, which is what we interpret as the distance between the skew lines, or between the corresponding parallel planes.)

Solution. (a) If we denote by t_1 the parameter at which a potential intersection would occur on L_1, and by t_2 the parameter at which the intersection would occur on L_2, the intersection then comes from satisfying the equations

$$\begin{aligned} 4 - 6t_1 &= t_2 \\ -3 - t_1 &= 7 - 2t_2 \\ -2 + 3t_1 &= 3 + 4t_2. \end{aligned}$$

Plugging the first into the second gives

$$-3 - t_1 = 7 - 2(4 - 6t_1) = -1 + 12t_1, \text{ so that } t_1 = -2/13.$$

Then the first equation gives $t_2 = 4 - 6(-2/13) = 64/13$. Thus $t_1 = -2/13$ and $t_2 = 64/13$ are the only values satisfying the first two equations above. However, for these values we have

$$-2 + 3t_1 = -2 + 3(-2/13) = -32/13 \text{ while } 3 + 4t_2 = 3 + 4(64/13) = 295/13,$$

so the third equation for the potential intersection is not satisfied. Thus there are no values of t_1 and t_2 that satisfy all three required equations, so the lines do not intersect.

The given lines are not parallel since their direction vectors are not parallel: L_1 is parallel to the vector $\langle -6, -1, 3\rangle$, and L_2 is parallel to $\langle 1, -2, 4\rangle$, and these vectors are not parallel since neither is a scalar multiple of the other.

(b) The common normal vector $\mathbf{n}$ we seek should be perpendicular to both L_1 and L_2, since it is perpendicular to the planes containing L_1 and L_2 respectively. Thus, we can take the cross product of the direction vectors of the given lines as a suitable normal vector. We get

$$\mathbf{n} = \langle -6, -1, 3\rangle \times \langle 1, -2, 4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -1 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \langle 2, 27, 13\rangle.$$

The plane containing L_1 contains the point $(4,-3,-2)$ on L_1, so with the normal vector $\mathbf{n}$ computed above we get

$$\langle 2,27,13\rangle \cdot (x-4, y+3, z+2) = 0, \text{ or } 2(x-4)+27(y+3)+13(z+2)=0$$

as an equation for the plane containing L_1. Th plane containing L_2 contains the point $(0,7,3)$ on L_2, so with normal vector $\mathbf{n}$ we get

$$\langle 2,27,13\rangle \cdot (x, y-7, z-3) = 0, \text{ or } 2x+27(y-7)+13(z-3)=0$$

as an equation for the plane containing L_2.

(c) Let us use $P = (-2,-4,1)$ as a point on L_1. (Any point will work, we just chose the one corresponding to $t=1$.) With $P_0 = (0,7,3)$ being a point on the plane containing L_2, the distance from P to this plane is the magnitude of the projection of $\overrightarrow{P_0P} = \langle -2,-11,-2\rangle$ onto the normal vector $\mathbf{n} = \langle 2,27,13\rangle$. We have

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \frac{-4-297-26}{4+729+169}\langle 2,27,13\rangle = -\frac{327}{902}\langle 2,27,13\rangle,$$

so the distance from $(-2,-4,1)$ to the plane containing L_2 is

$$|\operatorname{proj}_{\mathbf{n}} \overrightarrow{P_0P}| = \frac{327}{902}|\langle 2,27,13\rangle| = \frac{327}{902}\sqrt{902} = \frac{327}{\sqrt{902}}.$$

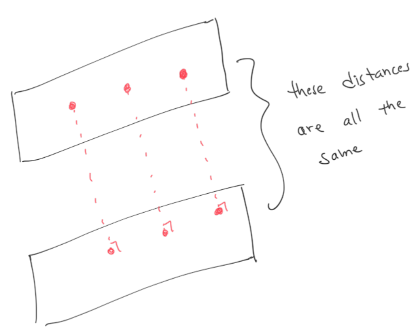
Now we use $Q = (1,5,7)$ as a point on L_2, corresponding to $t=1$. With $Q_0 = (4,-3,-2)$ as a point on the plane containing L_1, the distance form Q to this plane is the magnitude of the projection of $\overrightarrow{Q_0Q} = \langle -3,8,9\rangle$ onto $\mathbf{n} = \langle 2,27,13\rangle$. We have

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{Q_0Q} = \left(\frac{\overrightarrow{Q_0Q}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \frac{-6+216+117}{4+729+169}\langle 2,27,13\rangle = \frac{327}{902}\langle 2,27,13\rangle,$$

so the desired distance is

$$|\operatorname{proj}_{\mathbf{n}} \overrightarrow{Q_0Q}| = \frac{327}{902}|\langle 2,27,13\rangle| = \frac{327}{902}\sqrt{902} = \frac{327}{\sqrt{902}},$$

which agrees with the first distance. This makes sense geometrically, since what we are actually computing here is the distance between the two parallel planes:



Clearly, no matter which points we pick on the two planes, so that one occurs in the direction perpendicular to the other, the distances between these will always be the same since the planes are parallel. $\square$

2. Consider the quadric surface with equation

$$-2x^2 + 3y^2 - z^2 = 1.$$

(a) Sketch the cross-sections of this surface at $y = \pm 1/\sqrt{3}, \pm 1, \pm 2$ in a 2-dimensional xz-plane picture. For which values of k is there an empty cross-section at $y = k$? (In other words, for which k does the given surface not intersect the plane at $y = k$?)

(b) Sketch the given surface in 3-dimensional space, and in the same picture sketch the intersections of this surface with the planes at $z = 0, \pm 1, \pm 2$.

(c) Determine the point(s) on this surface that are closest to the origin.

Solution. (a) The cross-section at $y = \pm 1/\sqrt{3}$ has equation

$$-2x^2 + 1 - z^2 = 1, \text{ or } 2x^2 + z^2 = 0.$$

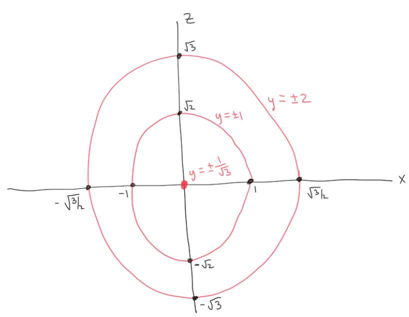
The only point satisfying this is $x = 0, z = 0$, so these cross-sections consist of single points. The cross section at $y = \pm 1$ has equation

$$-2x^2 + 3 - z^2 = 1, \text{ or } 2x^2 + z^2 = 2,$$

which describes an ellipse with x-intercepts at $x = \pm 1$ and z-intercepts at $z = \pm\sqrt{2}$. The cross sections at $y = \pm 2$ have equations

$$-2x^2 + 4 - z^2 = 1, \text{ or } 2x^2 + z^2 = 3,$$

which is an ellipse with x-intercepts $x = \pm\sqrt{3/2}$ and z-intercepts $z = \pm\sqrt{3}$. Hence we get



Now, the cross-section at $y = k$ in general has equation

$$-2x^2 + 3k^2 - z^2 = 1, \text{ or } 2x^2 + z^2 = 3k^2 - 1.$$

If the right side $3k^2 - 1$ is negative, there are no points satisfying this equation, so these cross-sections are empty. Since $3k^2 - 1 < 0$ when $k^2 < 1/3$, we get empty cross-sections for

$$-\frac{1}{\sqrt{3}} < k < \frac{1}{\sqrt{3}}.$$

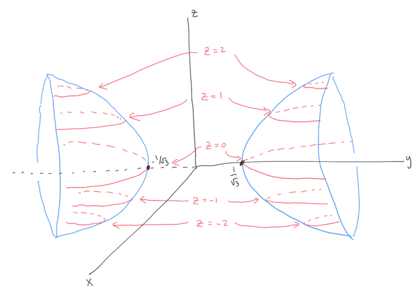
(b) The surface is a hyperboloid of two sheets centered along the y-axis. The cross-section at $z = 0$ has equation

$$-2x^2 + 3y^2 = 1,$$

which is a hyperbola opening in the y-direction. The cross-sections at $z = \pm 1$ and $z = \pm 2$ have equations $-2x^2 + 3y^2 - 1 = 1$ and $-2x^2 + 3y^2 - 4 = 1$ respectively, or equivalently

$$-2x^2 + 3y^2 = 2 \quad \text{and} \quad -2x^2 + 3y^2 = 5$$

respectively. These are also hyperbolas opening in the y-direction, so altogether the surface with these cross-sections look like



(c) Based on the picture, the points on this two-sheeted hyperboloid closest to $(0,0,0)$ are the points where it intersects the y-axis, which are $(0, \pm 1/\sqrt{3}, 0)$. ☐

3. Consider the quadric surface with equation

$$y = 2(x-4)^2 - z^2.$$

(a) Sketch the cross-sections of this surface at $y = 0, \pm 1, \pm 2$ in a 2-dimensional xz-plane picture.
(b) What is the standard name we give to this type of surface?
(c) Determine the point(s) at which the line L_1 from Problem 1 intersects this surface.

Solution. (a) Let us first consider the simpler surface $y = 2x^2 - z^2$. The surface above is just a translated version of this one. For this untranslated surface, the cross section at $y = 0$ has equation

$$0 = 2x^2 - z^2, \text{ or } z = \pm\sqrt{2}x,$$

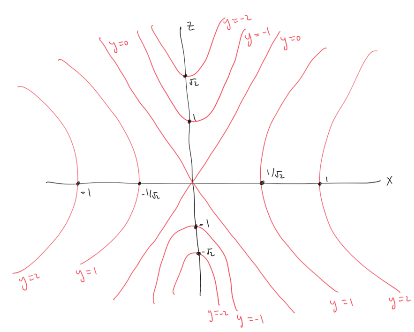
which describes a pair of lines passing through the origin. The cross-sections at $y = 1$ or $y = 2$ have respective equations

$$1 = 2x^2 - z^2 \text{ and } 2 = 2x^2 - z^2.$$

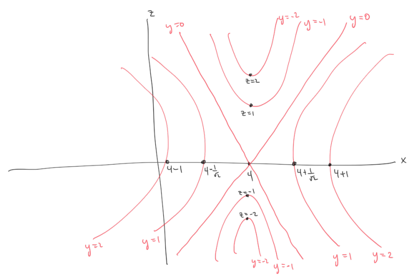
Both of these are hyperbolas opening in the x-direction, with the first having x-intercepts at $x = \pm 1/\sqrt{2}$ and the second $x = \pm 1$. The cross-sections at $y = -1$ and $y = -2$ have equations

$$-1 = 2x^2 - z^2 \text{ and } -2 = 2x^2 - z^2, \text{ or } 1 = -2x^2 + z^2 \text{ and } 2 = -2x^2 + z^2$$

respectively. These are hyperbolas opening in the z-direction, with the first having z-intercepts at $z = \pm 1$ and the second with z-intercepts $z = \pm\sqrt{2}$. Altogether, these cross sections look like



Now, for our original surface, we translate these pictures above by adding 4 to x due to the $(x-4)^2$ term. (In other words, $y = 2(x-4)^2 - z^2$ is "centered" at $x = 4, y = 0, y = 0$.) The cross-sections are thus



(b) This is a hyperbolic paraboloid, which is the formal name for a "saddle'.

(c) The line L_1 intersects this surface when the point with coordinates

$$x = 4 - 6t,\ y = -3 - t,\ z = -2 + 3t$$

satisfies $y = 2(x-4)^2 - z^2$. After substituting, we get

$$-3 - t = 2(-6t)^2 - (-2+3t)^2 = 63t^2 + 12t - 4,$$

which simplifies to $63t^2 + 13t - 1 = 0$. From the quadratic formula, we get

$$t = \frac{-13 \pm \sqrt{169 - 4(63)(-1)}}{2(63)} = \frac{-13 \pm \sqrt{421}}{126}.$$

Thus there are two points of intersection, one with coordinates

$$x = 4 - 6\left(\frac{-13+\sqrt{421}}{126}\right),\ y = -3 - \frac{-13+\sqrt{421}}{126},\ \text{and } z = -2 + 3\left(\frac{-13+\sqrt{421}}{126}\right),$$

and the other with coordinates

$$x = 4 - 6\left(\frac{-13-\sqrt{421}}{126}\right),\ y = -3 - \frac{-13-\sqrt{421}}{126},\ \text{and } z = -2 + 3\left(\frac{-13-\sqrt{421}}{126}\right).$$

$\square$