Northwestern University

MATH 230-1 Final Examination
Winter Quarter 2022
March 14, 2022

Last name: ________________ Email address: ________________

First name: ________________ NetID: ________________

Instructions

- Mark your section.

Section	**Time**	**Instructor**	
41	10:00	Bentsen	
51	11:00	Cuzzocreo	
61	12:00	Cuzzocreo	

- This examination consists of 16 pages, not including this cover page. Verify that your copy of this examination contains all 16 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 10 questions for a total of 100 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Clearly circle final answers.
- Use pages at the back for scratchwork if needed.
- Show all of your work and justify answers completely, unless otherwise indicated. Unsupported answers may not earn credit.

1. For each problem, circle either **True** or **False**. You do not need to justify your answer.

   (a) (2 points) The function $f(x, y) = x^4 - y^6$ has a saddle point at $(0, 0)$.

   True **False**

   (b) (2 points) If vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$ must be 0.

   True **False**

   (c) (2 points) Each trace of a hyperboloid in each direction is a hyperbola.

   True **False**

   (d) (2 points) Let $f(x, y)$ be a continuous function with continuous first partial derivatives on all of $\mathbb{R}^2$, and let (a, b) be a point. Let $\mathbf{u} = \dfrac{1}{|\vec{\nabla} f(a, b)|}\vec{\nabla} f(a, b)$. Then

$$D_{\mathbf{u}} f(a, b) = |\vec{\nabla} f(a, b)|.$$

   True **False**

For each problem, circle either **True** or **False**. You do not need to justify your answer.

(e) (2 points) If a function f attains a local maximum at a point P_0 in a closed, bounded region R, then f must attain an absolute maximum on R at P_0.

True **False**

(f) (2 points) The curve defined in polar coordinates by the equation $r = 2\sin\theta$ is a circle.

True **False**

(g) (2 points) If $\mathbf{r}(t)$ for $a \leq t \leq b$ parameterizes the curve $\mathcal{C}$ with respect to arc length, then the length of $\mathcal{C}$ is $b - a$.

True **False**

(h) (2 points) The two lines parameterized as $\mathbf{r}_1(t) = \mathbf{a}_1 + t\mathbf{v}_1$ and $\mathbf{r}_2(t) = \mathbf{a}_2 + t\mathbf{v}_2$ intersect if and only if there exists a value $t = t_0$ such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$.

True **False**

2. Let $\mathbf{u} = \langle 3, -1, 1\rangle, \mathbf{v} = \langle 0, 0, -2\rangle, \mathbf{w} = \langle a, b, 1\rangle$.

   (a) (3 points) Compute the vector projection $\text{proj}_{\mathbf{u}}\mathbf{v}$. Show all work and justify your answer completely.

   (b) (3 points) Find all real values of the constants a and b such that both of the following conditions hold:

   - $\mathbf{w}$ is orthogonal to $\mathbf{u}$
   - $||\mathbf{w} \times \mathbf{v}|| = 2$.

   Show all work and justify your answer completely.

3. Let $P = (1, 1, 2)$, and let ℓ be the line parametrized by $\mathbf{r}(t) = \langle 1 + t, 3 - 2t, 3t \rangle$.

   (a) (4 points) Find the distance between P and ℓ. Show all work and justify your answer completely.

   (b) (4 points) Give an equation for the plane containing P and ℓ. Show all work and justify your answer completely.

4. For $t > 0$ the smooth curve $\mathcal{C}$ is parametrized as

$$\mathbf{r}(t) = \left\langle \frac{t^2}{2},\ 2t,\ \frac{4}{3}t^{3/2} \right\rangle.$$

(a) (5 points) Find the equation of the tangent line to $\mathcal{C}$ at the point $\left(\frac{1}{2},\ 2,\ \frac{4}{3}\right)$. Show all work and justify your answer completely.

(b) (5 points) Find the arc length of $\mathcal{C}$ between $t = 1$ and $t = 2$. Show all work and justify your answer completely.

5. Let

$$f(u,v) = u\cos(2v) - ve^{3u},$$
$$u = g(x,y,z) \quad \text{and} \quad v = h(x,y,z).$$

Suppose the values of g, h and their partial derivatives at the point $(x,y,z) = (0,0,0)$ are given in the chart below.

$g(0,0,0) = 1$	$h(0,0,0) = 0$
$g_x(0,0,0) = -1$	$h_x(0,0,0) = 2$
$g_y(0,0,0) = 6$	$h_y(0,0,0) = 0$
$g_z(0,0,0) = 1$	$h_z(0,0,0) = -1$

Let $F(x,y,z) = f(g(x,y,z), h(x,y,z))$. Answer the following, and show all work and justify your answer completely.

(a) (6 points) Find

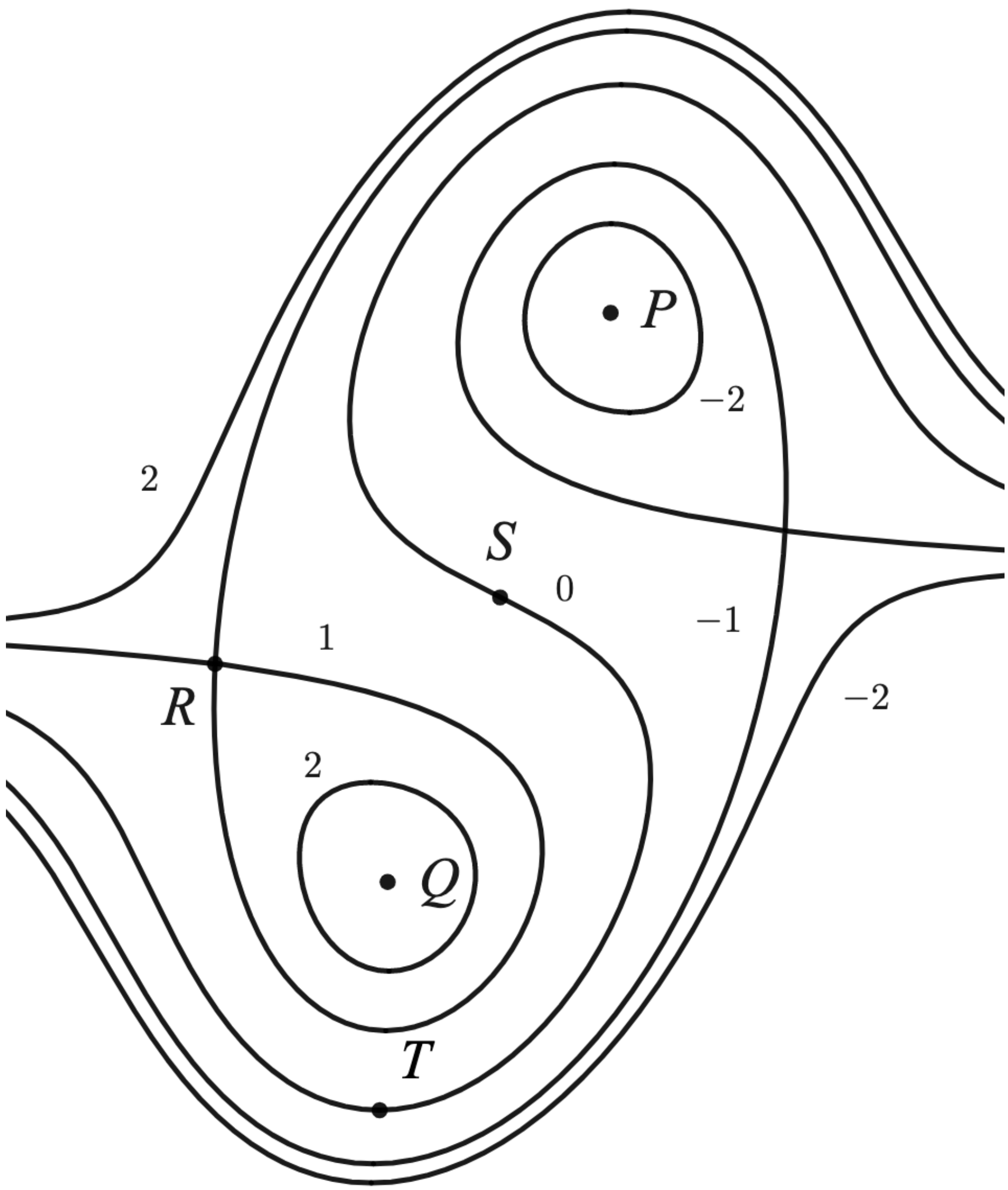
i. $\left.\frac{\partial F}{\partial x}\right|_{(x,y,z)=(0,0,0)}$

ii. $\left.\frac{\partial F}{\partial y}\right|_{(x,y,z)=(0,0,0)}$

iii. $\left.\frac{\partial F}{\partial z}\right|_{(x,y,z)=(0,0,0)}$

(b) (4 points) Find the equation of the tangent plane to the level surface $F(x,y,z) = 1$ at $(x,y,z) = (0,0,0)$.

6. The function $f(x, y)$ has level curves shown below. The numbers represent the values of f on each level curve, and P, Q, R, S, and T are points in the xy-plane. Following the usual convention we take the x-axis to be horizontal and the y-axis to be vertical. Assume that f and all of its partial derivatives (including its second partial derivatives) are continuous everywhere.



The questions below refer to the contour map on the previous page. Circle the correct answer. There is only one correct answer for each problem. You do not need to justify your answer.

(a) (2 points) Which of the following could be the vector $\vec{\nabla} f(T)$?

(A) $\langle 1, 1\rangle$ (B) $\langle -1, 1\rangle$ (C) $\langle 0, 1\rangle$ (D) $\langle -1, 0\rangle$

(b) (2 points) Assume that the coordinates of R are $(-3, -2)$. Which of the following could be an equation of the tangent plane to the graph of f at R?

(A) $(x+3)+(y+2)=0$ (C) $z+(x+3)+(y+2)=0$

(B) $z-(x+3)-(y+2)=1$ (D) $z-1=0$

(c) (2 points) At which point could $f_x < 0$ be satisfied?

(A) T (B) Q (C) P (D) S.

(d) (2 points) At which point could f attain a local minimum?

(A) P (B) Q (C) R (D) S

(e) (2 points) For which of the following unit vectors $\mathbf{u}$ is $D_{\mathbf{u}}f(T) < 0$?

(A) $\mathbf{u} = \langle 1, 0\rangle$ (B) $\mathbf{u} = \langle -1, 0\rangle$ (C) $\mathbf{u} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$ (D) All of the above

7. Let $f(x, y) = x^3 - y^2$.

(a) (3 points) Find the linearization $L(x, y)$ for $f(x, y)$ at the point $(x, y) = (1, 2)$.

(b) (4 points) Find a reasonable upper bound for the error $|E(x, y)| = |f(x, y) - L(x, y)|$ in this approximation if (x, y) lies in the rectangle defined by $|x - 1| \leq 0.1$ and $|y - 2| \leq 0.2$. Show all work and justify your answer completely.

(c) (3 points) Find the quadratic approximation $Q(x, y)$ for $f(x, y)$ at the point $(x, y) = (1, 2)$.

8. (10 points) Let

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine whether f is a continuous function on $\mathbb{R}^2$ or not. In either case, give a one or two sentence explanation of your answer. Show all work and justify your answer completely.

9. (10 points) Find the absolute maximum and minimum values of the function

$$f(x, y, z) = x - 2y - z$$

in the region defined by $x^2 + 4y^2 + 2z^2 \leq 36$.

Clearly also indicate *at which points* these extrema are attained. Show all work and justify your answer completely.

10. (10 points) Find all of the critical points of the function

$$f(x, y) = x^2y^2 + x^2y + xy$$

and classify each one as corresponding to a local maximum, local minimum, or a saddle point of f. Show all work and justify your answer completely.

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YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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