MATH 230-1: Discussion 5 Problems
Northwestern University, Fall 2023

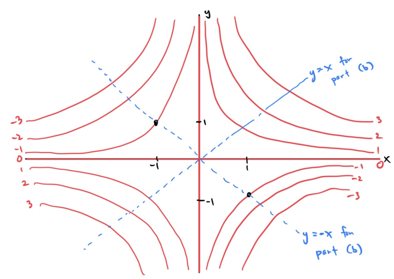
1. Consider the function $f(x, y) = xy$.

(a) Draw the level curves of f at $z = -3, -2, -1, 0, 1, 2, 3$.

(b) Based on the level curves, explain why $(0, 0)$ is sitting at a minimum of the single variable function $f(x, x)$, and at a maximum of the single variable function $f(x, -x)$. (Note, $f(x, x)$ gives the values of f along the line $y = x$, and $f(x, -x)$ gives the values of f along the line $y = -x$.)

(c) Based on the level curves, Find a point at which the graph of f slopes downward when facing in the direction of the vector $\mathbf{i}$ but upward when facing in the direction of the vector $\mathbf{j}$, and find a point at which the graph slopes upward in the direction of $\mathbf{i}$ but downward in the direction of $\mathbf{j}$.

Solution. (a) The equation $xy = k$ describes a hyperbola when $k \neq 0$ and a pair of lines when $k = 0$ (since $xy = 0$ gives $x = 0$ or $y = 0$), so the level curves look like



(b) At $(0, 0)$ we have $f(0, 0) = 0$. The values $f(x, x)$ are those of f along the line $y = x$, and from the level curves we see that moving along $y = x$ in the first quadrant causes the value of f to increase above 0, and moving along $y = x$ in the second quadrant also causes the value of f to increase. Thus the value of $f(x, x)$ is smallest at $x = 0$, so $(0, 0)$ gives the minimum value of f among points of the form (x, x).

The points $(x, -x)$ move along the line $y = -x$, and now as we move away from zero in either the second or fourth quadrants the value of f decreases into negative values. Thus the value of $f(x, -x)$ is largest at $x = 0$, so $(0, 0)$ gives the maximum value of f among points of the form $(x, -x)$.

(c) At the point $(-1, 1)$, which is sitting on the level curve at $z = -1$, the values of $f(x, 1)$ for x a bit less than -1 are smaller (more negative) than -1 while those of $f(x, -1)$ for x a bit larger than -1 are larger (less negative) than -1, so the values of $f(x, 1)$ increase through $x = -1$ and hence the graph slopes upward at $(-1, 1)$ in the direction of $\mathbf{i}$. But instead when looking at $f(-1, y)$ for y a bit less and a bit larger than $y = 1$, we see that the values of f get smaller (more negative) as y increases through 1, so the graph slopes downward in the direction of $\mathbf{j}$.

At $(1, -1)$ the opposite happens. The values of $f(x, -1)$ get smaller (more negative) as x increases through $x = 1$, and the values of $f(1, y)$ get larger (less negative) as y increases through -1, so the graph of f slopes downward at $(1, -1)$ in the direction of $\mathbf{i}$ and slopes upward in the direction of $\mathbf{j}$. $\square$

2. Set $f(x, y) = \frac{x^4 - y^4}{x^2 + y^2}$.

(a) Find $\lim_{(x,y)\to(0,0)} f(x, y)$ by algebraically simplifying the expression for $f(x, y)$.

(b) Find $\lim_{(x,y)\to(0,0)} f(x, y)$ by converting to polar coordinates.

(c) Find $\lim_{(x,y)\to(0,0)} \cos(f(x, y) + 4)$. Be sure to justify your answer by appropriately applying the notion of continuity.

Solution. (a) Since $x^4 - y^4 = (x^2+y^2)(x^2-y^2)$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0-0 = 0.$$

(b) In polar coordinates, we have

$$\frac{x^4-y^4}{x^2+y^2} = \frac{r^4\cos^4\theta - r^4\sin^4\theta}{r^2} = r^2(\cos^4\theta - \sin^4\theta).$$

Since $-1 \le \cos^4\theta \le 1$ and $-1 \le \sin^4\theta \le 1$ for all θ, we get

$$-2r^2 = r^2(-1-1) \le r^2(\cos^4\theta - \sin^4\theta) \le r^2(1+1) = 2r^2.$$

Because $-2r^2$ and $2r^2$ both approach 0 as $r \to 0$, the sandwich theorem gives

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r^2(\cos^4\theta - \sin^4\theta) = 0.$$

(c) Since the cosine function is continuous, we have

$$\lim_{(x,y)\to(0,0)} \cos(f(x,y)+4) = \cos(\lim_{(x,y)\to(0,0)} [f(x,y)+4]) = \cos(0+4) = \cos 4.$$

□

3. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-3xy-4y^2}{x^2+y^2}.$$

(a) Show that this limit does not exist by finding three lines passing through $(0,0)$ along which the limit gives three different values.

(b) Determine the value of the limit when approaching $(0,0)$ only along the curve $y = x^2$.

(c) Show that this limit does not exist by converting to polar coordinates.

Solution. (a) Approaching $(0,0)$ along the x-axis where $y=0$ gives

$$\lim_{(x,0)\to(0,0)} \frac{2x^2-0-0}{x^2+0} = \lim_{x\to 0}\frac{2x^2}{x^2} = \lim_{x\to 0} 2 = 2.$$

Approaching along the y-axis where $x = 0$ gives

$$\lim_{(0,y)\to(0,0)} \frac{0-0-4y^2}{0+y^2} = \lim_{y\to 0} -\frac{-4y^2}{y^2} = \lim_{y\to 0} -4 = -4.$$

The fact that the limits along these two curves are different is already enough to know that the given multivariable limit does not exist, but the setup ask for a third line as well. Approaching along $y = x$ gives

$$\lim_{(x,x)\to(0,0)} \frac{2x^2-3x^2-4x^2}{x^2+x^2} = \lim_{x\to 0}\frac{-5x^2}{2x^2} = \lim_{x\to 0} -\frac{5}{2} = -\frac{5}{2}.$$

(b) When approaching the origin along $y = x^2$, we have

$$\lim_{(x,x^2)\to(0,0)} \frac{2x^2-3x^3-4x^4}{x^2+x^4} = \lim_{x\to 0}\frac{x^2(2-3x-4x^2)}{x^2(1+x^2)} = \lim_{x\to 0}\frac{2-3x-4x^2}{1+x^2} = \frac{2-0-0}{1+0} = 2.$$

(c) In polar coordinates, we have

$$\frac{2x^2 - 3xy - 4y^2}{x^2 + y^2} = \frac{2r^2\cos^2\theta - 3r^2\cos\theta\sin\theta - 4r^2\sin^2\theta}{r^2} = 2\cos^2\theta - 3\cos\theta\sin\theta - 4\sin^2\theta.$$

We want to approach the origin, which only requires $r \to 0$ regardless of what θ is. But if we approach the origin along the line where $\theta = 0$ we get

$$\lim_{(r,\theta)\to(0,0)} [2\cos^2\theta - 3\cos\theta\sin\theta - 4\sin^2\theta] = 2 - 0 - 0 = 2,$$

whereas if we approach along the line where $\theta = \frac{\pi}{4}$ we get

$$\lim_{(r,\theta)\to(0,\frac{\pi}{4})} [2\cos^2\theta - 3\cos\theta\sin\theta - 4\sin^2\theta] = 2(\tfrac{1}{\sqrt{2}})^2 - 3(\tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) - 4(\tfrac{1}{\sqrt{2}})^2 = 1 - \tfrac{3}{2} - 2 = -\tfrac{5}{2}.$$

Since approaching the origin along these two directions (note that $\theta = 0$ corresponds to the x-axis and $\theta = \frac{\pi}{4}$ corresponds to the line $y = x$) gives different candidate values for the limit, the multivariable limit does not exist. $\square$