MATH 230-1: Discussion 2 Solutions
Northwestern University, Fall 2023

1. Let $\mathbf{u} = \langle 3, -1, 2 \rangle$ and $\mathbf{v} = \langle -1, 4, 1 \rangle$. Find the vector of length 3 that points in the direction directly opposite that of the sum of the vector projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

Solution. We compute

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = (\tfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}})\mathbf{v} = (\tfrac{-5}{18}) \langle -1, 4, 1 \rangle = \left\langle \tfrac{5}{18}, -\tfrac{20}{18}, -\tfrac{5}{18} \right\rangle$$

and

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = (\tfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}})\mathbf{u} = (\tfrac{-5}{14}) \langle 3, -1, 2 \rangle = \left\langle -\tfrac{15}{14}, \tfrac{5}{14}, -\tfrac{10}{14} \right\rangle.$$

Thus

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} + \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left\langle \tfrac{5}{18} - \tfrac{15}{14}, -\tfrac{20}{18} + \tfrac{5}{14}, -\tfrac{5}{18} - \tfrac{10}{14} \right\rangle.$$

The vector pointing in the direction opposite this sum is its negative:

$$\left\langle -\tfrac{5}{18} + \tfrac{15}{14}, \tfrac{20}{18} - \tfrac{5}{14}, \tfrac{5}{18} + \tfrac{10}{14} \right\rangle.$$

To get a vector pointing in this opposite direction of length 3, we first divide by length to get a unit vector in the correct direction:

$$\frac{1}{\sqrt{(-\frac{5}{18} + \frac{15}{14})^2 + (\frac{20}{18} - \frac{5}{14})^2 + (\frac{5}{18} + \frac{10}{14})^2}} \left\langle -\tfrac{5}{18} + \tfrac{15}{14}, \tfrac{20}{18} - \tfrac{5}{14}, \tfrac{5}{18} + \tfrac{10}{14} \right\rangle,$$

and then scale by 3 to get the correct length:

$$\frac{3}{\sqrt{(-\frac{5}{18} + \frac{15}{14})^2 + (\frac{20}{18} - \frac{5}{14})^2 + (\frac{5}{18} + \frac{10}{14})^2}} \left\langle -\tfrac{5}{18} + \tfrac{15}{14}, \tfrac{20}{18} - \tfrac{5}{14}, \tfrac{5}{18} + \tfrac{10}{14} \right\rangle.$$

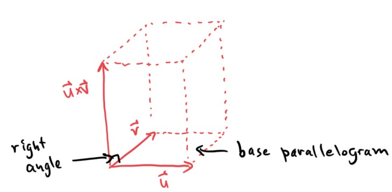
(No need to worry about simplifying this result! Indeed, leaving the answer in this form makes it simple to see how it was obtained in the first place.) □

2. A *parallelepiped* is a three-dimensional analogue of a parallelogram, which we can visualize as a "slanted" box. Find the volume of the parallelepiped with edges formed by the vectors $\mathbf{u} = \langle 5, 0, -2 \rangle$, $\mathbf{v} = \langle 1, 1, 1 \rangle$, and $\mathbf{u} \times \mathbf{v}$. The fact that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is important to determining the correct way of computing the volume. Hint: Think of the parallelogram with edges $\mathbf{u}$ and $\mathbf{v}$ as being the *base* of this parallelepiped.

Solution. First we compute

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix} = (0 - (-2))\,\mathbf{i} - (5 - (-2))\,\mathbf{j} + (5 - 0)\,\mathbf{k} = \langle 2, -7, 5 \rangle.$$

The parallelepiped in question looks like



. Since the $\mathbf{u} \times \mathbf{v}$ edge is orthogonal to the other two edges, the volume of the parallelepiped is equal to the area of the base parallelogram (with edges $\mathbf{u}$ and $\mathbf{v}$) times the height $|\mathbf{u} \times \mathbf{v}|$. The area of the base parallelogram is $|\mathbf{u} \times \mathbf{v}|$, so the volume is

$$|\mathbf{u} \times \mathbf{v}||\mathbf{u} \times \mathbf{v}| = |\mathbf{u} \times \mathbf{v}|^2 = 4 + 49 + 25 = 78.$$

□

3. Consider the line with parametric equations

$$x = 3 + t,\ y = -1 + 4t,\ z = 2 - t$$

and the line with parametric equations

$$x = 1 + 4t,\ y = 1 + 2t,\ z = -3 + 4t.$$

Determine whether or not these lines intersect. If they do intersect, find the point of intersection, and if they do not intersect, determine if they are *skew*, meaning that they are not parallel and do not intersect. (In fact, skew lines lie in different parallel planes.)

Solution. In order for these lines to intersect, the same point (x, y, z) must arise from each set of equations, albeit for possibly different values of the parameters. That is, we must have

$$\begin{aligned} 3 + t_1 &= 1 + 4t_2 \\ -1 + 4t_1 &= 1 + 2t_2 \\ 2 - t_1 &= -3 + 4t_2 \end{aligned}$$

for some values of t_1 and t_2. The first equation gives

$$t_1 = -2 + 4t_2,$$

and substituting into the second gives

$$-1 + 4(-2 + 4t_2) = 1 + 2t_2.$$

This simplifies to $-9 + 16t_2 = 1 + 2t_2$, or $14t_2 = 10$. Thus $t_2 = \frac{10}{14} = \frac{5}{7}$, and

$$t_1 = -2 + 4(\tfrac{5}{7}) = -2 + \tfrac{20}{7} = \tfrac{6}{7}.$$

These are the only values of t_1 and t_2 that satisfy the first pair of equation given at the start, but for these we have

$$2 - t_1 = 2 - \tfrac{6}{7} = \tfrac{8}{7} \text{ and } -3 + 4(\tfrac{5}{7}) = -3 + \tfrac{20}{7} = -\tfrac{1}{7}.$$

Hence $2 - t_1 \neq -3 + 4t_2$, so there are no values of t_1, t_2 which will satisfy all equations at once, meaning that these lines do not intersect.

The first line is parallel to the vector $\langle 1, 4, -1\rangle$ and the second line is parallel to $\langle 4, 2, 4\rangle$, which we find from the coefficient of the parameter t. Since these vectors are not multiples of one another, they are not parallel, so the given lines are not parallel. Hence the lines are skew. □