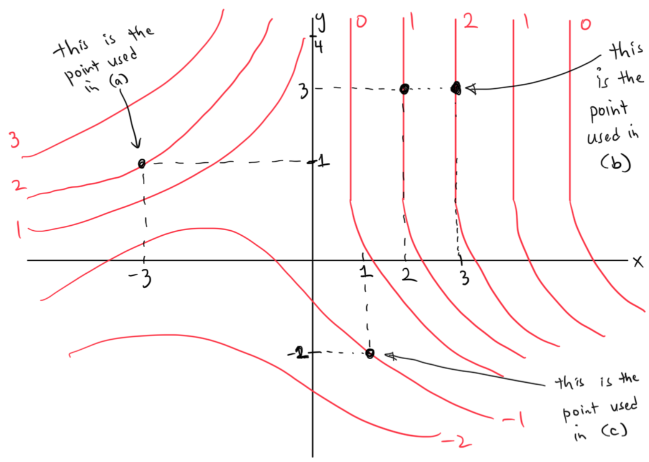
MATH 230-1: Written HW 5 Solutions
Northwestern University, Fall 2023

1. Below are some level curves of a continuous function $f(x, y)$.



Assume level curves which are not drawn occur at values of z strictly between those that are drawn. The portions of the level curves in the first quadrant that appear to be vertical are indeed meant to be vertical.

(a) When standing at the point $(-3, 1, 2)$, explain why the graph of f slopes downward when facing the direction of the vector $\mathbf{i}$ but slopes upward when facing the direction of the vector $\mathbf{j}$.

(b) Explain why $f(3, 3) \geq f(x, 3)$ for all x in some interval around 3. What can you say about the behavior of $f(3, y)$ for y in the interval $(1, 4)$?

(c) Find the value of the following limit, with justification.

$$\lim_{(x,y)\to(1,-2)} (xe^{f(x,y)} - 3)$$

Solution. (a) When standing at $(-3, 1, 2)$ and varying only the x-coordinate, the values of the function decrease from $z = 2$ at $(-3, 1)$ to values smaller than 2 and eventually 1 once we hit the level curve at $z = 1$. A bit to the left of $(-3, 1)$, the values of the function are larger than 2 since the level curves here that are not drawn occurs at values of z strictly between 3 and 2. The upshot is that the values of f decrease at $(-3, 1)$ if we vary the x-coordinate only, so the graph should slope downward at this point in the direction of $\mathbf{i}$.

When standing at $(-3, 1, 2)$ and varying the y-coordinate only, we instead have values of f that get larger from being smaller than 2 for $y < 1$, to $z = 2$ at $y = 1$, to z larger than 2 for $y > 1$, as we can tell based on the level curves and the values of z at which they occur. Hence f increases at $(-3, 1)$ when varying y, so the graph slopes upward when facing the direction of $\mathbf{j}$.

(b) The value of $f(x, y)$ at $(3, 3)$, which is $z = 2$, is larger than what it is at points nearby with the same y coordinate but slightly different x coordinate. Indeed, $(3, 3)$ sits on the level curve at $z = 2$ but moving to $(x, 3)$ for x slightly smaller or larger than 2 puts us on a level curve for a value of z smaller than 2. (We know it is smaller than 2 since it has to be strictly between $z = 2$ and $z = 1$.) Thus the single-variable function $f(x, 3)$ has a local maximum at $x = 3$, so $f(3, 3) \geq f(x, 3)$ for x near 2.

If we instead consider varying y in $f(3, y)$, then the value of f does no change since such points remain on the level curve at $z = 2$, at least for y between 1 and 4. Thus $f(3, y)$ is a constant function for y in $(1, 4)$.

(c) First note that $xe^{f(x,y)} - 3$ is a continuous expression in terms of x and y. Indeed, $f(x,y)$ is continuous and the exponential function is continuous, so the composition $e^{f(x,y)}$ of these two is continuous. Then x is continuous, and the product of continuous functions is continuous, so $xe^{f(x,y)}$ is continuous. Finally, the constant function -3 is continuous, and sums of continuous functions are continuous, so $xe^{f(x,y)} - 3$ is continuous too.

The limit of a continuous function as we approach a point is just the value of that function at the point being approached, so

$$\lim_{(x,y)\to(1,-2)} xe^{f(x,y)} - 3 = 2e^{f(1,-2)} - 3 = 2e^{-1} - 3,$$

where the value $f(1,-2) = -1$ comes from the fact that $(1,-2)$ is on the level curve at $z = -1$. $\square$

2. The goal of this problem is to find the value of the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$$

in three different ways using the sandwich theorem. It will be useful to know that the inequality $|a| \leq b$ is equivalent to the chain of inequalities $-b \leq a \leq b$, so if you want to show justify the latter inequalities, you can instead justify the former.

(a) Justify the fact that

$$-\frac{1}{2}|x^2-y^2| \leq \frac{xy(x^2-y^2)}{x^2+y^2} \leq \frac{1}{2}|x^2-y^2|$$

and then apply the sandwich theorem. Hint: Show that $|xy| \leq \frac{1}{2}(x^2+y^2)$ using the fact that $(|x|-|y|)^2 \geq 0$.

(b) Justify the fact that

$$-|xy| \leq \frac{xy(x^2-y^2)}{x^2+y^2} \leq |xy|$$

and then apply the sandwich theorem. Hint: Which of x^2+y^2 or $|x^2-y^2|$ is larger?

(c) Rewrite the given limit in terms of polar coordinates and then use the sandwich theorem.

Solution. (a) First we note that since $0 \leq (|x|-|y|)^2$ for any x and y, we have

$$0 \leq x^2 - 2|x||y| + y^2, \text{ so } 2|x||y| \leq x^2+y^2, \text{ and thus } \frac{|xy|}{x^2+y^2} \leq \frac{1}{2}.$$

Hence

$$\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| = \frac{|xy|}{x^2+y^2}|x^2-y^2| \leq \frac{1}{2}|x^2-y^2|,$$

so

$$-\frac{1}{2}|x^2-y^2| \leq \frac{xy(x^2-y^2)}{x^2+y^2} \leq \frac{1}{2}|x^2-y^2|.$$

We have that the limit of both $-\frac{1}{2}|x^2-y^2|$ and $\frac{1}{2}|x^2-y^2|$ as (x,y) approaches $(0,0)$ is 0, so by the sandwich theorem we have

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0.$$

(b) First we note that

$$\frac{|x^2-y^2|}{x^2+y^2} \leq 1$$

for any x and y since the denominator of the fraction on the left is always larger than or equal to the numerator because adding two nonnegative numbers can never give something smaller than what you get by taking the absolute value of their difference. Thus

$$\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| = |xy|\frac{|x^2-y^2|}{x^2+y^2} \leq |xy|,$$

so

$$-|xy| \leq \frac{xy(x^2-y^2)}{x^2+y^2} \leq |xy|.$$

Since the limit of both $-|xy|$ and $|xy|$ as (x, y) approaches $(0, 0)$ is 0, the sandwich theorem gives

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0.$$

(c) Converting to polar coordinates gives

$$\frac{xy(x^2-y^2)}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2} = r^2\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta).$$

Since $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$ for all θ, we have

$$\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta) \leq 1 \cdot 1(1+1) = 2$$

and

$$-2 = -1 - 1 \leq \cos\theta\sin\theta(\cos^2\theta-\sin^2\theta).$$

Hence

$$-2r^2 \leq r^2\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta) \leq 2r^2.$$

Taking the limit as (x, y) approaches $(0, 0)$ is equivalent to taking the limit here as r approaches 0, and since the limit of both $-2r^2$ and $2r^2$ as r approaches 0 is 0, the sandwich theorem gives

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{r\to 0} r^2\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta) = 0.$$

□

3. The point of this problem is to demonstrate that checking the behavior of limits along lines alone is not enough to guarantee existence. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}.$$

(a) Find the limit when approaching $(0, 0)$ along the x-axis, the y-axis, and the line $y = x$.

(b) Find the limit when approaching $(0, 0)$ along *all* lines through $(0, 0)$.

(c) Find a curve passing through $(0, 0)$ such that the limit when approaching $(0, 0)$ along this curve is different than the value found in (b), and conclude that the given multivariable limit does not exist.

Solution. (a) The limit along the x-axis, where $y = 0$, is

$$\lim_{(x,0)\to(0,0)} \frac{0}{(x^2+0^4)^3} = \lim_{x\to 0} 0 = 0.$$

The limit along the y-axis, where $x = 0$, is

$$\lim_{(0,y)\to(0,0)} \frac{0}{(0^2+y^4)^3} = \lim_{y\to 0} 0 = 0.$$

Finally, the limit along $y = x$ is

$$\lim_{(x,x)\to(0,0)} \frac{x^4x^4}{(x^2+x^4)^3} = \lim_{x\to 0} \frac{x^8}{[x^2(1+x^2)]^3} = \lim_{x\to 0} \frac{x^8}{x^6(1+x^2)^3} = \lim_{x\to 0} \frac{x^2}{(1+x^2)^3} = \frac{0}{(1+0^2)^3} = 0.$$

(b) We give two approaches. First let us take us consider an arbitrary non-vertical line $y = mx$ through the origin. (The vertical line $y = 0$ was already considered in part (a).) The limit along this line is

$$\lim_{(x,mx)\to(0,0)} \frac{x^4(mx)^4}{(x^2+[m^4x^4])^3} = \lim_{x\to 0} \frac{mx^8}{[x^2(1+m^4x^2)^3} = \lim_{x\to 0} \frac{mx^2}{(1+m^4x^2)^3} = \frac{0}{(1+0)^3} = 0.$$

Alternatively, we can convert to polar coordinates:

$$\frac{x^4y^4}{(x^2+y^4)^3} = \frac{r^8\cos^4\theta\sin^4\theta}{(r^2\cos^2\theta+r^4\sin^4\theta)^3} = \frac{r^8\cos^4\theta\sin^4\theta}{r^6(\cos^2\theta+r^2\sin^4\theta)^3} = \frac{r^2\cos^4\theta\sin^4\theta}{(\cos^2\theta+r^2\sin^4\theta)^3}.$$

We can characterize any line through the origin as the set of points at which θ is some constant value, so we consider the limits along any of these lines. (Again, we ignore the y-axis given by $\theta = \frac{\pi}{2}$ as it was already considered in part (a).) For $\theta_0 \neq \frac{\pi}{2}$, approaching along $\theta = \theta_0$ gives

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{r\to 0,\theta=\theta_0} \frac{r^2\cos^4\theta\sin^4\theta}{(\cos^2\theta+r^2\sin^4\theta)^3} = \frac{0}{(\cos^2\theta_0+0)^3} = 0.$$

(c) Approaching along the parabola $x = y^2$ gives

$$\lim_{(y^2,y)\to(0,0)} \frac{y^8y^4}{(y^4+y^4)^3} = \lim_{y\to 0} \frac{y^8}{2^3y^8} = \frac{1}{8}.$$

Since approaching along $x = y^2$ gives a different value than approaching along any line through the origin, the limit in question does not exist. $\square$