Northwestern University

MATH 230-1 Midterm Examination 2
Winter Quarter 2022
February 21, 2022

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

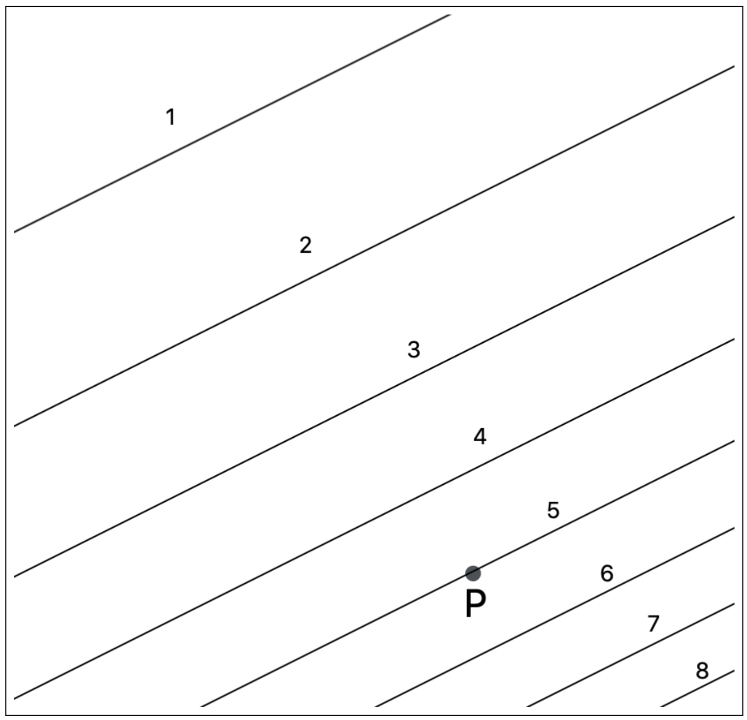
- Mark your section.

Section	**Time**	**Instructor**	
41	10:00	Bentsen	
51	11:00	Cuzzocreo	
61	12:00	Cuzzocreo	

- This examination consists of 12 pages, not including this cover page. Verify that your copy of this examination contains all 12 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 60 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Use pages at the back for scratchwork if needed.
- Show all of your work. Unsupported answers may not earn credit.

1. (6 points) The figure below depicts a contour map for a function $f(x, y)$, with each level curve labeled with the value of f on that curve. Determine the signs of all first and second partial derivatives of f at the point P by filling in the blanks below with either "<" or ">."

   You may assume that the signs of all first and second partial derivatives of f do not change at any point in the domain of f.



(a) $f_x(P)$ __ $>$ __ 0

(d) $f_{xy}(P)$ __ $<$ __ 0

(b) $f_y(P)$ __ $<$ __ 0

(e) $f_{yx}(P)$ __ $<$ __ 0

(c) $f_{xx}(P)$ __ $>$ __ 0

(f) $f_{yy}(P)$ __ $>$ __ 0

2. Let $\mathcal{C}$ be the curve parametrized for $t \geq 0$ by

$$\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t\rangle; \qquad t \geq 0$$

(a) (6 points) Find the arc length of $\mathcal{C}$ between the points $(0,1)$ and $(\pi,-1)$.

We first note that these points correspond to $t = 0$ and $t = \pi$ since

$$\begin{aligned}\mathbf{r}(0) &= \langle 0, 1\rangle \\ \mathbf{r}(\pi) &= \langle \pi, -1\rangle\end{aligned}$$

We then compute

$$\begin{aligned}\mathbf{r}'(t) &= \langle \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle \\ &= \langle t\sin t, t\cos t\rangle \\ |\mathbf{r}'(t)| &= |\langle t\sin t, t\cos t\rangle| \\ &= \sqrt{t^2\sin^2 t + t^2\cos^2 t} \\ &= \sqrt{t^2(\sin^2 t + \cos^2 t)} \\ &= |t| \\ &= t \quad \text{since } t \geq 0\end{aligned}$$

So we compute

$$\begin{aligned}\int_0^\pi |\mathbf{r}'(t)|\, dt &= \int_0^\pi t\, dt \\ &= \frac{t^2}{2}\Big]_0^\pi \\ &= \boxed{\frac{\pi^2}{2}}\end{aligned}$$

(b) (4 points) Find an arc length parametrization of $\mathcal{C}$ beginning at the point $\mathbf{r}(0)$.

We have

$$\begin{aligned}s(t) &= \int_0^t |\mathbf{r}'(\tau)|\, d\tau \\ &= \int_0^t \tau\, d\tau \\ &= \frac{\tau^2}{2}\Big]_0^t \\ &= \frac{t^2}{2}\end{aligned}$$

Inverting gives

$$t(s) = \sqrt{2s}.$$

Hence the arc length parameterization is

$$\boxed{\mathbf{r}(t(s)) = \langle \sin(\sqrt{2s}) - \sqrt{2s}\cos(\sqrt{2s}), \cos(\sqrt{2s}) + \sqrt{2s}\sin(\sqrt{2s})\rangle; \qquad s \geq 0}$$

3. Consider the function

$$f(x,y) = \frac{2x+4y}{x^2+y^2+1}$$

(a) (2 points) Find the domain of f.

$\boxed{\mathbb{R}^2}$

(b) (4 points) Compute $f_x(1,1)$ and $f_y(1,1)$.

$$f_x(x,y) = \frac{2(x^2+y^2+1) - 2x(2x+4y)}{(x^2+y^2+1)^2} \qquad f_x(1,1) = \frac{2(1+1+1) - 2(2+4)}{(1^2+1^2+1)^2} = \boxed{\frac{-2}{3}}$$

$$f_y(x,y) = \frac{4(x^2+y^2+1) - 2y(2x+4y)}{(x^2+y^2+1)^2} \qquad f_y(1,1) = \frac{4(1+1+1) - 2(2+4)}{(1^2+1^2+1)^2} = \boxed{0}$$

(c) Write and simplify the equation defining the level curve $f(x,y) = c$ for the following values of c.

i. (2 points) $c = -1$

$$\begin{aligned} f(x,y) &= -1 \\ \frac{2x+4y}{x^2+y^2+1} &= -1 \\ 2x+4y &= -x^2-y^2-1 \\ x^2+2x+y^2+4y &= -1 \\ x^2+2x+1+y^2+4y+4 &= 4 \\ \boxed{(x+1)^2+(y+2)^2 = 4} & \end{aligned}$$

ii. (2 points) $c = 0$

$$\begin{aligned} f(x,y) &= 0 \\ \frac{2x+4y}{x^2+y^2+1} &= 0 \\ \boxed{2x+4y = 0} & \end{aligned}$$

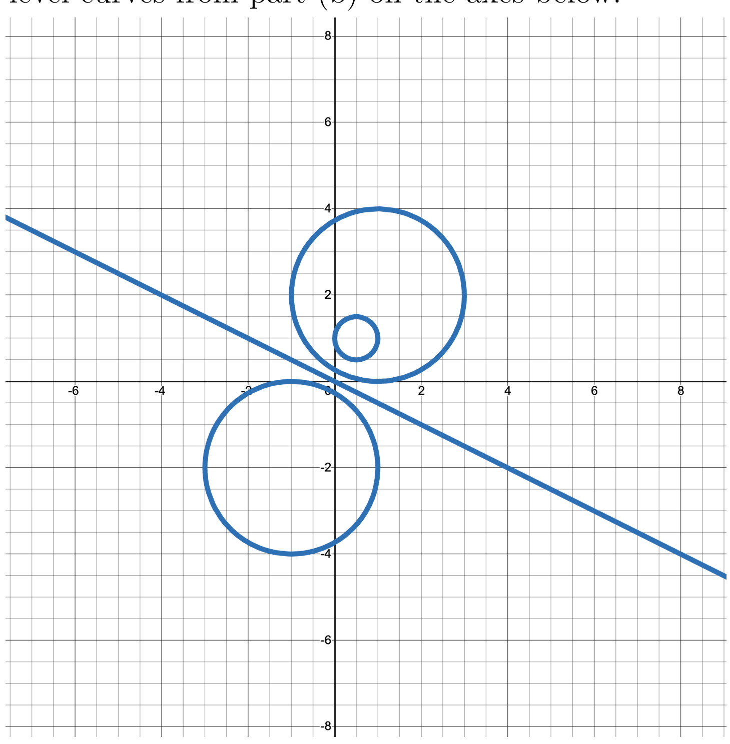
iii. (2 points) $c = 1$

$$\begin{aligned} f(x,y) &= 1 \\ \frac{2x+4y}{x^2+y^2+1} &= 1 \\ 2x+4y &= x^2+y^2+1 \\ x^2-2x+y^2-4y &= -1 \\ x^2-2x+1+y^2-4y+4 &= 4 \\ \boxed{(x-1)^2+(y-2)^2 = 4} \end{aligned}$$

iv. (2 points) $c = 2$

$$\begin{aligned} f(x,y) &= 2 \\ \frac{2x+4y}{x^2+y^2+1} &= 2 \\ 2x+4y &= 2x^2+2y^2+2 \\ x^2-x+y^2-2y &= -1 \\ x^2-x+\tfrac{1}{4}+y^2-2y+1 &= \tfrac{1}{4} \\ \boxed{\left(x-\tfrac{1}{2}\right)^2+(y-1)^2 = \tfrac{1}{4}} \end{aligned}$$

(d) (4 points) Sketch the level curves from part (b) on the axes below.



4. Compute the following limits, or show that they do not exist.

(a) (6 points)

$$\lim_{(x,y)\to(1,0)} \frac{\cos(xy)-1}{\sqrt{\cos(xy)}-1}$$

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{\cos(xy)-1}{\sqrt{\cos(xy)}-1} &= \lim_{(x,y)\to(1,0)} \frac{\cos(xy)-1}{\sqrt{\cos(xy)}-1}\frac{\sqrt{\cos(xy)}+1}{\sqrt{\cos(xy)}+1} \\
&= \lim_{(x,y)\to(1,0)} \frac{(\cos(xy)-1)(\sqrt{\cos(xy)}+1)}{\cos(xy)-1} \\
&= \lim_{(x,y)\to(1,0)} \sqrt{\cos(xy)}+1 \\
&= \sqrt{\cos(0)}+1 \\
&= 1+1 \\
&= \boxed{2}
\end{aligned}$$

(b) (6 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^2+y^2}$$

METHOD 1:

Note

$$0 \le \frac{y^2}{x^2+y^2} \le 1$$

and so

$$0 \le \frac{x^4y^2}{x^2+y^2} \le x^4.$$

We then have

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} x^4 = 0$$

and so

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^2+y^2} = \boxed{0}$$

by the Squeeze Theorem.

METHOD 2:

Using polar coordinates,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^2+y^2} &= \lim_{r\to 0} \frac{r^4\cos^4\theta r^2\sin^2\theta}{r^2} \\
&= \lim_{r\to 0} r^4\cos^4\theta\sin^2\theta
\end{aligned}$$

Now,

$$0 \leq r^4 \cos^4\theta \sin^2\theta \leq r^4$$

and

$$\lim_{r\to 0} 0 = \lim_{r\to 0} r^4 = 0$$

and so

$$\lim_{r\to 0} r^4 \cos^4\theta \sin^2\theta = \boxed{0}$$

by the Squeeze Theorem.

(c) (6 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^2}{2x^4+y^2}$$

Parameterizing a line through the origin as $\mathbf{r}(t) = \langle at, bt \rangle$, we see that the limit of the given function along such a line is

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ x=at\\ y=bt}} \frac{x^4+y^2}{2x^4+y^2} &= \lim_{t\to 0} \frac{a^4t^4+b^2t^2}{2a^4t^4+b^2t^2} \\
&= \lim_{t\to 0} \frac{a^4t^2+b^2}{2a^4t^2+b^2} \\
&= \begin{cases} \dfrac{b^2}{b^2} = 1 & b \neq 0 \\ \dfrac{a^4}{2a^4} = \frac{1}{2} & b = 0 \end{cases}
\end{aligned}$$

$b = 0$ corresponds to the x axis, and thus we see that the limit is 1 along all lines other than the x axis, but the limit is $\frac{1}{2}$ along the x axis. Hence the limit does not exist.

5. (8 points) A tennis ball is launched inside of a wind tunnel. It is launched from the point $(0,0,0)$ with an initial speed of 50 m/s in the direction of the vector $\langle 0,4,3\rangle$ and its trajectory is affected by gravity and the wind such that its acceleration in m/s^2 is given by

$$\mathbf{a}(t) = \langle -3, -1, -10\rangle.$$

Determine the distance in meters between the launch point of the ball and the point at which it lands. (Give an exact numerical answer, but you do not need to simplify).

First we find the velocity function. Note that the initial velocity can be given as

$$\begin{aligned}\mathbf{v}(0) &= 50\frac{\langle 0,4,3\rangle}{|\langle 0,4,3\rangle|}\\ &= 50\frac{\langle 0,4,3\rangle}{\sqrt{4^2+3^2}}\\ &= 50\frac{\langle 0,4,3\rangle}{5}\\ &= \langle 0,40,30\rangle\end{aligned}$$

Then we integrate acceleration:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt\\ &= \int \langle -3,-1,-10\rangle\,dt\\ &= \langle -3t, -t, -10t\rangle + \mathbf{C}\end{aligned}$$

Plugging in $t=0$, we then have

$$\mathbf{C} = \mathbf{v}(0) = \langle 0,40,30\rangle$$

and so

$$\mathbf{v}(t) = \langle -3t, -t+40, -10t+30\rangle.$$

Next we integrate velocity to find position:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt\\ &= \int \langle -3t, -t+40, -10t+30\rangle\,dt\\ &= \left\langle -\tfrac{3}{2}t^2, -\tfrac{t^2}{2}+40t, -5t^2+30t\right\rangle + \mathbf{D}\end{aligned}$$

Plugging in $t=0$, we then have

$$\mathbf{D} = \mathbf{r}(0) = \langle 0,0,0\rangle$$

and so
$$\mathbf{r}(t) = \left\langle -\tfrac{3}{2}t^2, -\tfrac{t^2}{2} + 40t, -5t^2 + 30t \right\rangle.$$

We then solve for the time at which the ball hits the ground. Setting the third component $-5t^2+30t = 0$, we obtain $t = 6$. The position at time $t = 6$ is

$$\mathbf{r}(6) = \left\langle -\tfrac{3}{2}6^2, -\tfrac{6^2}{2} + 40(6), 0 \right\rangle = \langle -54, 222, 0\rangle.$$

Hence, the distance between the launch point and the landing is

$$|\mathbf{r}(6) - \mathbf{r}(0)| = |\langle -54 - 0, 222 - 0, 0 - 0\rangle| = \boxed{\sqrt{54^2 + 222^2}}$$

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