Northwestern University

Math 230-1 Second Midterm Examination
Fall Quarter 2021
Tuesday 16 November

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- **Show and justify all of your work**. Unsupported answers may not earn credit.
- This examination consists of 6 questions for a total of 50 points.
- Read all problems carefully before answering.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.

1. (10 points) For the following limits, either compute them or explain why they do not exist.

(a) (3 points) $\lim_{(x,y)\to(0,0)} \frac{x+1}{x+y+1}$

$\lim_{(x,y)\to(0,0)} x+1 = 1$, $\lim_{(x,y)\to(0,0)} x+y+1 = 1 \neq 0$

$\to \lim_{(x,y)\to(0,0)} \frac{x+1}{x+y+1} = \frac{1}{1} = \boxed{1}$

(b) (3 points) $\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\sin(x+y)-1}{\sqrt{\sin(x+y)}-1}$

$= \lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{(\sqrt{\sin(x+y)}-1)(\sqrt{\sin(x+y)}+1)}{\sqrt{\sin(x+y)}-1} = \lim_{(x,y)\to(\frac{\pi}{2},0)} \sqrt{\sin(x+y)}+1$

$= 1+1 = \boxed{2}$

(c) (4 points) $\lim_{(x,y)\to(0,0)} \frac{y^4}{y^4+x^3}$

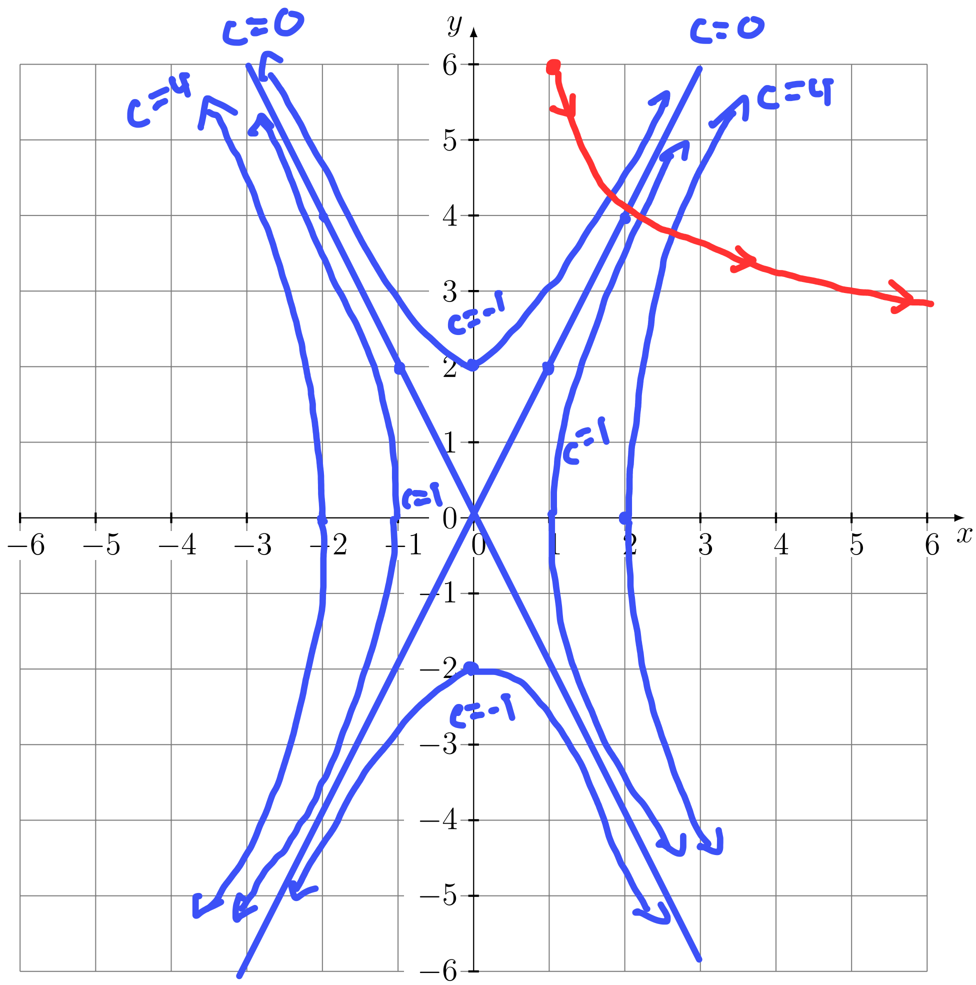
Along $y=0$: $\lim_{x\to 0} \frac{0}{x^3} = 0$

$\neq \Rightarrow$ DNE

Along $x=0$: $\lim_{y\to 0} \frac{y^4}{y^4} = 1$

2. (10 points) (a) (6 points) Let $f(x, y) = x^2 - \frac{y^2}{4}$. On the following sketch the level curves of $f(x, y) = k$ for $k = -1, 0, 1, 4$. Make sure to label the curves.

(b) (4 points) On the same plot below, give a rough sketch of the path a particle takes if it starts at $(1, 6)$ and always moves in the direction of the gradient $\nabla f(x, y)$. You do not need to explicitly calculate the gradient!



$x^2 - \frac{y^2}{4} = -1 \rightarrow \frac{y^2}{4} - \frac{x^2}{1} = 1$

$x^2 - \frac{y^2}{4} = 0 \rightarrow x^2 = \frac{y^2}{4} \rightarrow (2x)^2 = y^2 \rightarrow y = 2x$ or $-2x$

$x^2 - \frac{y^2}{4} = 1$

$x^2 - \frac{y^2}{4} = 4 \rightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1$

3. (10 points) The acceleration of a particle moving through space is given by

$$\mathbf{a}(t) = \langle -t^{-2}, -\pi^2 \sin(\pi t), 6t \rangle,\ t > 0.$$

At time $t = 1$, the particle passes through the origin with velocity vector $\langle 1, -\pi, 3 \rangle$. Find the vector-valued function $\mathbf{r}(t), t > 0$ which describes the particle's motion.

$$\vec{a}(t) = \langle -t^{-2}, -\pi^2 \sin(\pi t), 6t \rangle$$

$$\rightarrow \vec{v}(t) = \langle t^{-1}, \pi \cos(\pi t), 3t^2 \rangle + \vec{c}_1.$$

$$\langle 1, -\pi, 3 \rangle = \vec{v}(1) = \langle 1, -\pi, 3 \rangle + \vec{c}_1$$

$$\rightarrow \vec{c}_1 = \langle 0, 0, 0 \rangle.$$

$$\rightarrow \vec{r}(t) = \langle \ln t, \sin(\pi t), t^3 \rangle + \vec{c}_2.$$

$$\langle 0, 0, 0 \rangle = \vec{r}(1) = \langle 0, 0, 1 \rangle + \vec{c}_2$$

$$\rightarrow \vec{c}_2 = \langle 0, 0, -1 \rangle.$$

so,

$$\boxed{\vec{r}(t) = \langle \ln t, \sin(\pi t), t^3 - 1 \rangle}$$

4. (10 points) Suppose f is a continuous function of two variables with continuous partial derivatives. Several values of f, f_x, and f_y are given in the table below

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$
$(0,4)$	1	3	1
$(0,5)$	2	-1	2
$(1,2)$	3	1	-1
$(4,1)$	0	2	3

(a) (2 points) Find the directional derivative of f at $(4,1)$ in the direction of the vector $\langle 2,3\rangle$.

$$\vec{u} = \frac{\langle 2,3\rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$$D_{\vec{u}} f \Big|_{(4,1)} = \nabla f \Big|_{(4,1)} \cdot \vec{u} = \langle 2,3\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$$= \frac{4}{\sqrt{13}} + \frac{9}{\sqrt{13}} = \boxed{\sqrt{13}}$$

(b) (2 points) Find an equation of the line tangent to the level set $f(x,y) = 1$ at the point $(0,4)$.

Tangent Line:

$$f_x(0,4)(x-0) + f_y(0,4)(y-4) = 0$$

$$\rightarrow \quad 3x + (y-4) = 0$$

$$\rightarrow \quad \boxed{3x + y = 4}$$

(c) (3 points) Let $x(s,t) = t^2 \sin(\pi s)$ and $y(s,t) = st^2 + e^{st-2}$. Calculate $x_s(s,t), x_t(s,t), y_s(s,t)$, and $y_t(s,t)$.

$x_s = \pi t^2 \cos(\pi s)$.

$y_s = t^2 + t e^{st-2}$.

$x_t = 2t \sin(\pi s)$.

$y_t = 2st + s e^{st-2}$.

(d) (3 points) Use the table provided and the definitions of $x(s,t)$ and $y(s,t)$ from part c) to calculate $\frac{\partial f}{\partial s}\big|_{(s,t)=(1,2)}$.

$x(1,2) = 0$, $y(1,2) = 5$, $f_x(0,5) = -1$, $f_y(0,5) = 2$

$x_s(1,2) = -5\pi$, $y_s(1,2) = 6$

$$\left.\frac{\partial f}{\partial s}\right|_{(s,t)=(1,2)} = \left.\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right|_{(1,2)} = (-1)(-5\pi) + (2)(6)$$

$$= \boxed{5\pi + 12}$$

5. (10 points) (a) (7 points) Find an equation for the plane tangent to the graph of $f(x,y) = xe^y - ye^x$ at the point $(-1,-1,0)$.

Let $F(x,y,z) = xe^y - ye^x - z$

$$\Rightarrow \nabla F\big|_{(-1,-1,0)} = \langle e^y - ye^x, xe^y - e^x, -1 \rangle\big|_{(-1,-1,0)}$$

$$= \langle \tfrac{1}{e} + \tfrac{1}{e}, -\tfrac{1}{e} - \tfrac{1}{e}, -1 \rangle$$

$$= \langle \tfrac{2}{e}, -\tfrac{2}{e}, -1 \rangle$$

$\Rightarrow$ Tangent Plane:

$$\boxed{\frac{2}{e}(x+1) - \frac{2}{e}(y+1) - z = 0.}$$

(b) (3 points) Compute all unit vectors $\mathbf{u}$ for which $D_{\mathbf{u}}f|_{(-1,-1)} = 5$, or explain why no such vector exists.

$\nabla f|_{(-1,-1)} = \langle \frac{2}{e}, -\frac{2}{e} \rangle$, and $|\langle \frac{2}{e}, -\frac{2}{e} \rangle| = \sqrt{\frac{8}{e^2}} = \frac{2\sqrt{2}}{e}$

Since $D_{\vec{u}} f \leq |\nabla f|$ always, and since

$\frac{2\sqrt{2}}{e} < 5$, then no such $\vec{u}$ exists.

6. (10 points) Consider the vector-valued function

$$\mathbf{r}(t) = \langle 3t, 2\sin(2t), 2\cos(2t)\rangle, \ -\infty < t < \infty.$$

(a) (5 points) Find the arc length parameter $s(t)$ for $\mathbf{r}(t)$, using base point $\langle 0, 0, 2\rangle$.

Base point $\langle 0,0,2\rangle$ corresponds to $t=0$

$$\rightarrow s(t) = \int_0^t |\vec{r}\,'(\tau)|\,d\tau = \int_0^t |\langle 3, 4\cos 2\tau, -4\sin 2\tau\rangle|\,d\tau$$

$$= \int_0^t (9 + 16\cos^2 2\tau + 16\sin^2 2\tau)^{\frac{1}{2}}\,d\tau$$

$$= \int_0^t \sqrt{25}\,d\tau = 5\tau\Big|_0^t = \boxed{5t}$$

(b) (5 points) Reparametrize $\mathbf{r}(t)$ in terms of arc length, and use this to determine the position of a particle which begins at $\langle 0, 0, 2\rangle$ and travels 10π units along the curve.

$$t(s) = \frac{s}{5}; \quad \vec{r}(t(s)) = \left\langle \frac{3s}{5}, 2\sin\left(\frac{2s}{5}\right), 2\cos\left(\frac{2s}{5}\right)\right\rangle$$

$$\rightarrow \boxed{\vec{r}(t(10\pi)) = \langle 6\pi, 0, 2\rangle}$$

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