# Northwestern University

MATH 230-1 Midterm 2
Fall Quarter 2022
November 15, 2022

Last name: SOLUTIONS  Email address: ______

First name: ______  NetID: ______

**Instructions**

- Mark your section.

| Section | Time | Instructor | |
|---|---|---|---|
| 31 | 9:00 | Lee | |
| 41 | 10:00 | Lee | |
| 51 | 11:00 | Cañez | |
| 61 | 12:00 | Schrader | |
| 71 | 1:00 | Tamarkin | |
| 81 | 2:00 | Tamarkin | |

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has four parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

(a) (5 points) If a ball is thrown from the ground with acceleration $\mathbf{a}(t) = -g\mathbf{j}$ and initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$, it will follow the path of the parabola $y = -\frac{1}{2}gx^2 + x$.

$$\vec{v}(t) = \int a(t)\,dt = c_1\vec{i} + (-gt + c_2)\vec{j}$$

$$\vec{v}(0) = \vec{i} + \vec{j} \Rightarrow c_1 = 1,\ c_2 = 1$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int (\vec{i} + (-gt+1)\vec{j})\,dt$$

$$= (t + d_1)\vec{i} + (-\tfrac{1}{2}gt^2 + t + d_2)\vec{j}$$

$$\vec{r}(0) = 0\vec{i} + 0\vec{j} \Rightarrow d_1 = 0 = d_2$$

$$\vec{r}(t) = \underbrace{t}_{x}\vec{i} + \underbrace{(-\tfrac{1}{2}gt^2 + t)}_{y} \qquad y = -\tfrac{1}{2}gx^2 + x \qquad \boxed{\text{true}}$$

(b) (5 points) The level curves of $f(x, y) = e^{x+y}$ are either empty or straight lines.

Set $z = k$ :

$k = e^{x+y}$ if $k \le 0$ no points satisfy this, so empty.

if $k > 0$ $k = e^{x+y} \Rightarrow \ln k = x + y$

which is a line

$\boxed{\text{true}}$

(c) (5 points) The limit of $\frac{x^2y}{x^3-3xy^2+y^3}$ as $(x,y)$ approaches $(0,0)$ exists.

Along $x=0$: $\lim_{(0,y)\to(0,0)} \frac{0}{0+0+y^3} = 0$

Along $y=x$: $\lim_{(x,x)\to(0,0)} \frac{x^3}{x^3-3x^3+x^3} = \lim_{x\to 0} \frac{x^3}{-x^3}$

$= -1$

$\boxed{\text{False}}$

(d) (5 points) There is a direction in which the directional derivative of $f(x,y) = x + xy - 2y$ at $(1,1)$ is $-10$.

$\nabla f = \langle 1+y, x-2 \rangle$

$\nabla f(1,1) = \langle 2, -1 \rangle$

$|\nabla f(1,1)| = \sqrt{5}$

min value of directional derivative

is $-|\nabla f(1,1)| = -\sqrt{5}$

but $-10 < -\sqrt{5}$. $\boxed{\text{False}}$

2. Consider the curve described by the vector function

$$\mathbf{r}(t) = \langle \sin t, 2t^{3/2}, \cos t \rangle, \ t > 0.$$

(a) (10 points) Find parametric equations for the tangent line to this curve at the point where it intersects the plane $y = \pi^{3/2}/4$.

intersects plane when $2t^{3/2} = \frac{\pi^{3/2}}{4} \Rightarrow t^{\frac{3}{2}} = \frac{\pi^{\frac{3}{2}}}{8}$

$\Rightarrow t^{1/2} = \frac{\pi^{1/2}}{2}$

$t = \pi/4$

$\vec{r}\,'(t) = \langle \cos t, 3t^{1/2}, -\sin t \rangle$

$\vec{r}\,'(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, 3(\frac{\pi}{4})^{1/2}, -\frac{\sqrt{2}}{2} \rangle$

$r(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, 2(\frac{\pi}{4})^{3/2}, \frac{\sqrt{2}}{2} \rangle$

tangent line is

$$\boxed{x = \frac{\sqrt{2}}{2} + t\frac{\sqrt{2}}{2}, \quad y = 2\left(\frac{\pi}{4}\right)^{3/2} + t \cdot 3\left(\frac{\pi}{4}\right)^{1/2}, \quad z = \frac{\sqrt{2}}{2} - t\frac{\sqrt{2}}{2}}$$

(b) (10 points) Find the arclength of the portion of the curve that begins at the point $(1, 2(\pi/2)^{3/2}, 0)$ and ends at the point $(0, 2\pi^{3/2}, -1)$. The expression for your final answer should not involve an integral but can be left unsimplified in terms of square roots.

$(1, 2(\frac{\pi}{2})^{3/2}, 0)$ is at $t = \pi/2$

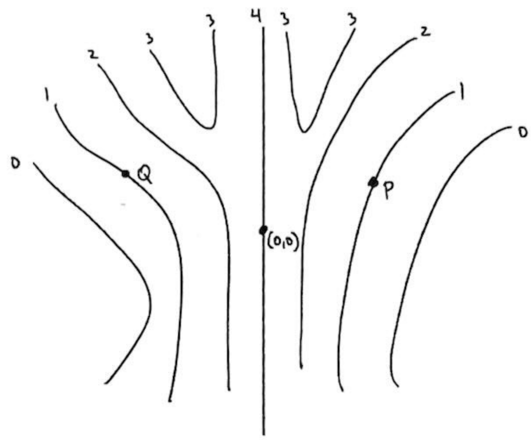
$(0, 2\pi^{3/2}, -1)$ is at $t = \pi$

$$\text{arclength} = \int_{\pi/2}^{\pi} |\vec{r}\,'(t)|\,dt = \int_{\pi/2}^{\pi} \sqrt{\cos^2 t + 9t + \sin^2 t}\,dt$$

$$= \int_{\pi/2}^{\pi} (1+9t)^{1/2}\,dt = \frac{2}{27}(1+9t)^{3/2} \Big|_{\pi/2}^{\pi}$$

$$\boxed{= \frac{2}{27}(1+9\pi)^{3/2} - \frac{2}{27}\left(1 + \frac{9\pi}{2}\right)^{3/2}}$$

3. (This problem has two parts and continues on the next page.) Suppose $z = f(x, y)$ is a continuous function with level curves




The labels on the level curves are the values of $z$ at which they occur. Assume that other level curves occur at values of $z$ strictly between those that are drawn. The $x$-axis (not drawn) passes through $(0, 0)$ horizontally, and the $y$-axis is the level curve drawn at $z = 4$.

(a) (10 points) Assuming that $f_x$ and $f_y$ exist at all points, determine whether each of the following is positive, negative, or zero.

$$f_x(Q),\ f_y(Q),\ f_x(0,0),\ f_y(0,0),\ f_x(P),\ f_y(P)$$

Justify your answer for $f_x(0,0)$ and $f_y(0,0)$ only.

$f_x(Q) > 0$ $f_y(Q) > 0$ $f_x(P) < 0$ $f_y(P) > 0$

$f_x(0,0) = 0$ since f is at a maximum at $(0,0)$ in the x-direction because $f(0,0) \geq f(x,0)$ for x close to 0

$f_y(0,0) = 0$ since f is constant in the y-direction at $(0,0)$

(b) (15 points) Determine, with appropriate justification, the value of the following limit.

$$\lim_{(x,y)\to(0,0)} \left( f(x,y) + \frac{2x^2+3xy}{\sqrt{x^2+y^2}} \right).$$

Since $f$ is continuous, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 4.$

For $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2+3xy}{\sqrt{x^2+y^2}}$, convert to ~~po~~ polar coordinates

Get $\lim\limits_{r\to 0} \frac{2r^2\cos^2\theta + 3r^2\cos\theta\sin\theta}{r}$

$= \lim\limits_{r\to 0} r\left[2\cos^2\theta + 3\cos\theta\sin\theta\right]$

Since $-5r \leq r\left[2\cos^2\theta + 3\cos\theta\sin\theta\right] \leq 5r$

and $-5r \to 0$ and $5r \to 0$ as $r \to 0$,

$\lim\limits_{r\to 0} r\left[2\cos^2\theta + 3\cos\theta\sin\theta\right] = 0$ by sandwich theorem.

Thus $\boxed{\lim\limits_{(x,y)\to(0,0)} \left[f(x,y) + \frac{2x^2+3xy}{\sqrt{x^2+y^2}}\right] = 4+0 = 4}$

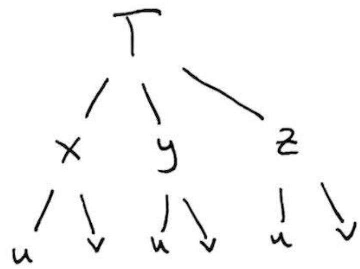
4. (10 points) Suppose the temperature at a point in $\mathbb{R}^3$ is given by a function $T(x,y,z)$ satisfying

$$\frac{\partial T}{\partial x}(0,0,0) = 1 = \frac{\partial T}{\partial y}(0,0,0),\ \frac{\partial T}{\partial x}(-1,0,0) = 2 = \frac{\partial T}{\partial y}(-1,0,0),\ \frac{\partial T}{\partial x}(-1,0,1) = 3 = \frac{\partial T}{\partial y}(-1,0,1)$$

and $\frac{\partial T}{\partial z}(x,y,z) = 4$ at all points. (Note that some of these values are irrelevant in this problem.) If $x, y, z$ are given in terms of two other variables $u$ and $v$ according to

$$x = \sin(u)\cos(v),\ y = \sin(u)\sin(v),\ z = \cos(u),$$

find $\frac{\partial T}{\partial u}$ and $\frac{\partial T}{\partial v}$ at $(u,v) = (\frac{\pi}{2}, \pi)$.




$$\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\partial T}{\partial x}\cos u \cos v + \frac{\partial T}{\partial y}\cos u \sin v + \frac{\partial T}{\partial z}(-\sin u)$$

At $(u,v) = (\frac{\pi}{2}, \pi)$, $(x,y,z) = (-1,0,0)$ so evaluate at this point

$$\boxed{\frac{\partial T}{\partial u} = 2(0)(-1) + 2(0)(0) + 4(-1) = -4}$$

$$\frac{\partial T}{\partial v} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial v}$$

$$= \frac{\partial T}{\partial x}\sin u(-\sin v) + \frac{\partial T}{\partial y}\sin u \cos v + \frac{\partial T}{\partial z}(0)$$

$$\boxed{\frac{\partial T}{\partial v} = 2(1)(0) + 2(1)(-1) + 0 = -2}$$

5. (This problem has three parts and continues on the next page.) Let $f(x, y) = \frac{3x}{2x+y}$.

(a) (10 points) Find the rate at which $f$ is changing at $(1, 1)$ in the direction of $2\mathbf{i} + 3\mathbf{j}$.

$$\nabla f = \left\langle \frac{(2x+y)3 - 3x \cdot 2}{(2x+y)^2}, -\frac{3x \cdot 1}{(2x+y)^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle, \quad \vec{u} = \frac{\langle 2, 3 \rangle}{\sqrt{13}}$$

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u}$$

$$= \left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle = \boxed{\frac{1}{3\sqrt{13}}}$$

(b) (10 points) Find the direction in which the rate of change of $f$ at $(1, 1)$ is as large as possible and the rate of change in this direction.

this is the direction of the gradient, so $\nabla f(1,1) = \boxed{\left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle}$

the rate of change in this direction is

$$|\nabla f(1,1)| = \left| \left\langle \frac{1}{3}, -\frac{1}{3} \right\rangle \right| = \boxed{\sqrt{2/9}}$$

(c) (5 points) Find parametric equations for the line passing through $(1, 1)$ which is perpendicular to the curve $\frac{3x}{2x+y} = 1$.

direction vector of perpendicular line

is $\nabla f(1,1) = \langle \frac{1}{3}, -\frac{1}{3} \rangle$ since

$\frac{3x}{2x+y} = 1$ is a level curve of

$f(x,y) = \frac{3x}{2x+y}$, namely the one

containing $(1,1)$.

line has vector equation

$\vec{r}(t) = \langle 1,1 \rangle + t \langle \frac{1}{3}, -\frac{1}{3} \rangle$

so
$$\boxed{\begin{aligned} x &= 1 + \tfrac{1}{3}t \\ y &= 1 - \tfrac{1}{3}t \end{aligned}}$$

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