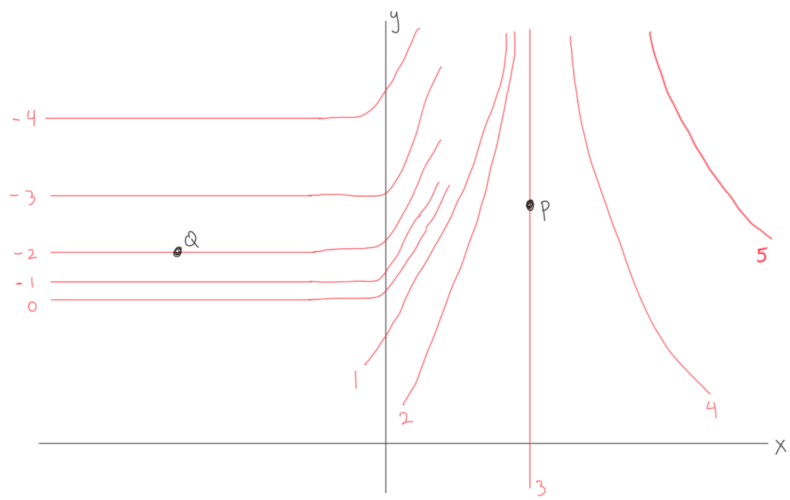
MATH 230-1: Written Homework 6
Northwestern University, Fall 2023

1. Below are some level curves of a function $f(x,y)$. Let us assume that the first and second-order partial derivatives of f exist and are continuous at all points.



Also assume the level curves that are not drawn occur at values of z strictly between those that are drawn, and that pieces that look vertical are meant to be vertical and pieces that look horizontal are meant to be horizontal.

(a) Determine whether the first-order partial derivatives

$$\frac{\partial f}{\partial x}(P),\ \frac{\partial f}{\partial y}(P),\ \frac{\partial f}{\partial x}(Q),\ \text{and}\ \frac{\partial f}{\partial y}(Q)$$

are positive, negative, or zero. Justify your answer.

(b) Determine whether the second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(P),\ \frac{\partial^2 f}{\partial y^2}(P),\ \frac{\partial^2 f}{\partial x^2}(Q),\ \text{and}\ \frac{\partial^2 f}{\partial y^2}(Q)$$

are positive, negative, or zero. Justify your answer. Hint: Second-order derivatives taken with respect to the same variable twice measure concavity, just as ordinary second derivatives do.

(c) Determine whether the mixed second-order derivatives

$$\frac{\partial^2 f}{\partial y \partial x}(P)\ \text{and}\ \frac{\partial^2 f}{\partial x \partial y}(Q)$$

are positive, negative, or zero. Justify your answer. Hint: View, for example,

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{as} \quad \frac{\partial g}{\partial y}\ \text{where}\ g = \frac{\partial f}{\partial x},$$

so that the sign of this mixed second-order partial derivative depends on whether $g = \frac{\partial f}{\partial x}$ is increasing or decreasing with respect to y. What happens to the "slopes in the x-direction" measured by $\frac{\partial f}{\partial x}$ as we move in the y-direction at P? Do they get smaller or larger?

2. Suppose $f(x,y)$ is a function written in terms of rectangular coordinates whose first-order partial derivatives exist and are continuous at all points.

(a) Express the first-order partial derivatives of f with respect to polar coordinates in terms of its partial derivatives with respect to rectangular coordinates. The expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ you get should involve only $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, r, and θ.

(b) Express the first-order partial derivatives of f with respect to rectangular coordinates in terms of its partial derivatives with respect to polar coordinates. The expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ you get should involve only $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$, x, and y.

(c) Suppose a particle moves along a curve given by some *polar* parametric equations

$$r = 9t^3 + t + \frac{4}{3}, \quad \theta = \frac{\pi}{2} - \pi t.$$

If at the Cartesian point $(x,y) = (\sqrt{3}, 1)$ we know that $f_x(\sqrt{3}, 1) = -2$ and $f_y(\sqrt{3}, 1) = 5$, find the rate at which f is changing with respect to t at $t = \frac{1}{3}$.

3. Suppose $f(x,y)$ is a function with continuous first-order partial derivatives such that $f_x(1,2) = 5$ and $f_y(1,2) = -2$.

(a) Is there a unit direction vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(1,2) = 6$? Is there a unit direction vector $\mathbf{v}$ such that $D_{\mathbf{v}}f(1,2) = -6$? Explain.

(b) In order for the directional derivative $D_{\mathbf{u}}f(1,2)$ to equal $\frac{\sqrt{29}}{2}$, what are the possible values of the angle between $\mathbf{u}$ and $\langle 5, -2 \rangle$?

(c) If $f(1,2) = -6$, find the Cartesian equation of the tangent line to the level curve $f(x,y) = -6$ at the point $(1,2)$.