Northwestern University

MATH 230-1 Midterm 1
Fall Quarter 2022
October 18, 2022

Last name: SOLUTIONS Email address: ____________

First name: ____________ NetID: ____________

Instructions

- Mark your section.

Section	Time	Instructor	
31	9:00	Lee	
41	10:00	Lee	
51	11:00	Cañez	
61	12:00	Schrader	
71	1:00	Tamarkin	
81	2:00	Tamarkin	

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has four parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

   (a) (5 points) The vector projection of $\langle 1, 2\rangle$ onto $\langle 3, -2\rangle$ is shorter, meaning has a smaller magnitude, than the vector projection of $\langle 3, -2\rangle$ onto $\langle 1, 2\rangle$.

$$\text{proj}_{\langle 3,-2\rangle}\langle 1,2\rangle = \frac{\langle 1,2\rangle\cdot\langle 3,-2\rangle}{\langle 3,-2\rangle\cdot\langle 3,-2\rangle}\langle 3,-2\rangle = -\frac{1}{13}\langle 3,-2\rangle$$

$$\text{proj}_{\langle 1,2\rangle}\langle 3,-2\rangle = \frac{\langle 3,-2\rangle\cdot\langle 1,2\rangle}{\langle 1,2\rangle\cdot\langle 1,2\rangle}\langle 1,2\rangle = -\frac{1}{5}\langle 1,2\rangle$$

$$|\text{proj}_{\langle 3,-2\rangle}\langle 1,2\rangle| = \frac{1}{13}|\langle 3,-2\rangle| = \frac{\sqrt{13}}{13} = \frac{1}{\sqrt{13}}$$

$$|\text{proj}_{\langle 1,2\rangle}\langle 3,-2\rangle| = \frac{1}{5}|\langle 1,2\rangle| = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$\frac{1}{\sqrt{13}} < \frac{1}{\sqrt{5}} \checkmark$$

$$\boxed{\text{TRUE}}$$

   (b) (5 points) The line with parametric equations $x = 1+t, y = t, z = 3$ intersects the quadric surface with equation $z = -(x-1)^2 - y^2$.

$$3 = -(1+t-1)^2 - t^2$$

$$= -2t^2 \quad \text{is not satisfied}$$

by any value of t

$$\boxed{\text{FALSE}}$$

(c) (5 points) The plane with equation $x+y+z=1$ intersects the plane with equation $2x+2y+2z=1$.

The planes are parallel since their normal vectors $\langle 1,1,1\rangle$ and $\langle 2,2,2\rangle$ are parallel, but the planes are not the same plane since $(1,0,0)$ is on the first but not the second. So don't intersect. $\boxed{\text{FALSE}}$

(d) (5 points) The point $(0,0)$ is on the curve described by the polar equation $r = 1+\cos\theta$.

When $\theta = \pi$, $r = 1 + \cos \pi$

$= 1 - 1 = 0$

and $r=0$ gives the point $(0,0)$.

$\boxed{\text{TRUE}}$

2. Consider the triangle with vertices $(1,-1,-2)$, $(3,1,-4)$, and $(0,2,-1)$.

(a) (10 points) Verify that this is not a right triangle.

Set $P = (1,-1,-2)$, $Q = (3,1,-4)$, $R = (0,2,-1)$

Then $\overrightarrow{PQ} = \langle 2,2,-2 \rangle$, $\overrightarrow{PR} = \langle -1,3,1 \rangle$,

and $\overrightarrow{QR} = \langle -3,1,3 \rangle$.

No two sides are orthogonal since

$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 2 \neq 0$, $\overrightarrow{PQ} \cdot \overrightarrow{QR} = -10 \neq 0$,

and $\overrightarrow{PR} \cdot \overrightarrow{QR} = 12 \neq 0$.

(b) (10 points) Find the area of this triangle. Your answer should be expressed using square roots, but can be left unsimplified.

View triangle as half of parallelogram,

So area $= \frac{1}{2}$ (area of parallelogram)

$= \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \boxed{\frac{1}{2}\sqrt{64+64}}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -2 \\ -1 & 3 & 1 \end{vmatrix} = \langle 8,0,8 \rangle$$

3. Consider the lines with parametric equations

$$\begin{cases} x = -3 + 2t \\ y = 2 + t \\ z = 1 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = 3 + t \\ y = 2 - t \\ z = -1 + 2t. \end{cases}$$

(a) (10 points) Verify that these lines intersect and find their point of intersection

$-3+2t_1 = 3+t_2 \rightarrow t_2 = -6+2t_1$

$2+t_1 = 2-t_2 \rightarrow 2+t_1 = 2-(-6+2t_1) = 8-2t_1$

$1-3t_1 = -1+2t_2$

So $3t_1 = 6 \Rightarrow \boxed{t_1 = 2}$

$\boxed{t_2 = -6+2t_1 = -6+2(2) = -2}$

Check z coordinate: $1-3(2) = -5$ agrees with $-1+2(-2)$

$\boxed{\text{So intersect at } (1,4,-5)}$

(b) (10 points) Find parametric equations for the line which is perpendicular to both of these lines and passes through their point of intersection. If you could not find the point of intersection in part (a), use (x_0, y_0, z_0) as the point of intersection.

perpendicular direction = cross product of direction vectors of two lines

$= \langle 2,1,-3\rangle \times \langle 1,-1,2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -1 & 2 \end{vmatrix}$

$= \langle -1,-7,-3\rangle$

$\vec{r}(t) = \langle 1,4,-5\rangle + t\langle -1,-7,-3\rangle$

gives $\boxed{x = 1-t,\ y = 4-7t,\ z = -5-3t}$

4. Consider the plane that contains the point $(1,-1,2)$ and is perpendicular to the line with parametric equations $x = 1-2t, y = 3+t, z = -3+2t$.

(a) (5 points) Find an equation for this plane.

normal vector = direction of perpendicular line

$= \langle -2, 1, 2 \rangle$

plane is

$$\boxed{-2(x-1) + (y+1) + 2(z-2) = 0}$$

(b) (10 points) Take R to be the point at which the line with parametric equations

$$x = t, \; y = t, z = t$$

intersects the plane found in part (a). Find the distance from R to the plane with equation $2x-3y+z = 1$. Your answer should be expressed using square roots, but can be left unsimplified.

Find R: $-2(t-1) + (t+1) + 2(t-2) = 0$

$\Rightarrow -2t + 2 + t + 1 + 2t - 4 = 0$

$\Rightarrow t = 1$ so $R = (1,1,1)$

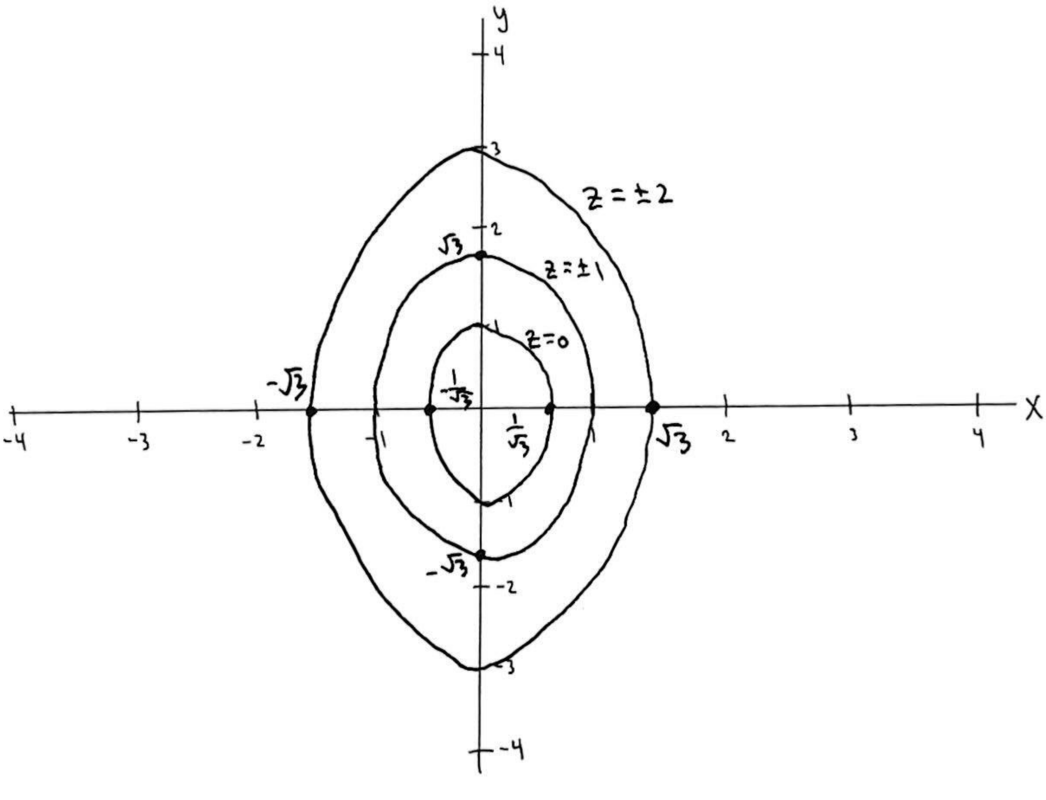
~~distance~~ Take $P = (0,0,1)$ on plane $2x-3y+z=1$ and $\vec{n} = \langle 2,-3,1 \rangle$. So $\overrightarrow{PR} = \langle 1,1,0 \rangle$.

distance $= \left| \text{proj}_{\vec{n}} \overrightarrow{PR} \right| = \left| \frac{-1}{14} \langle 2,-3,1 \rangle \right| = \boxed{\frac{1}{14}\sqrt{14}}$

5. (This problem has three parts and continues on the next two pages.) Consider the quadric surface with equation

$$3x^2 + y^2 - 2z^2 = 1.$$

(a) (10 points) Sketch the cross-sections of this surface at $z = 0, \pm 1, \pm 2$ on the axes below. Your drawing should clearly label the points at which these cross-sections intersect the axes. Draw all cross-sections on the same set of axes, and label which cross-section is which.



$\underline{z=0}$ $3x^2+y^2=1$ ellipse, x-intercepts $x=\pm 1/\sqrt{3}$

y-intercepts $y=\pm 1$

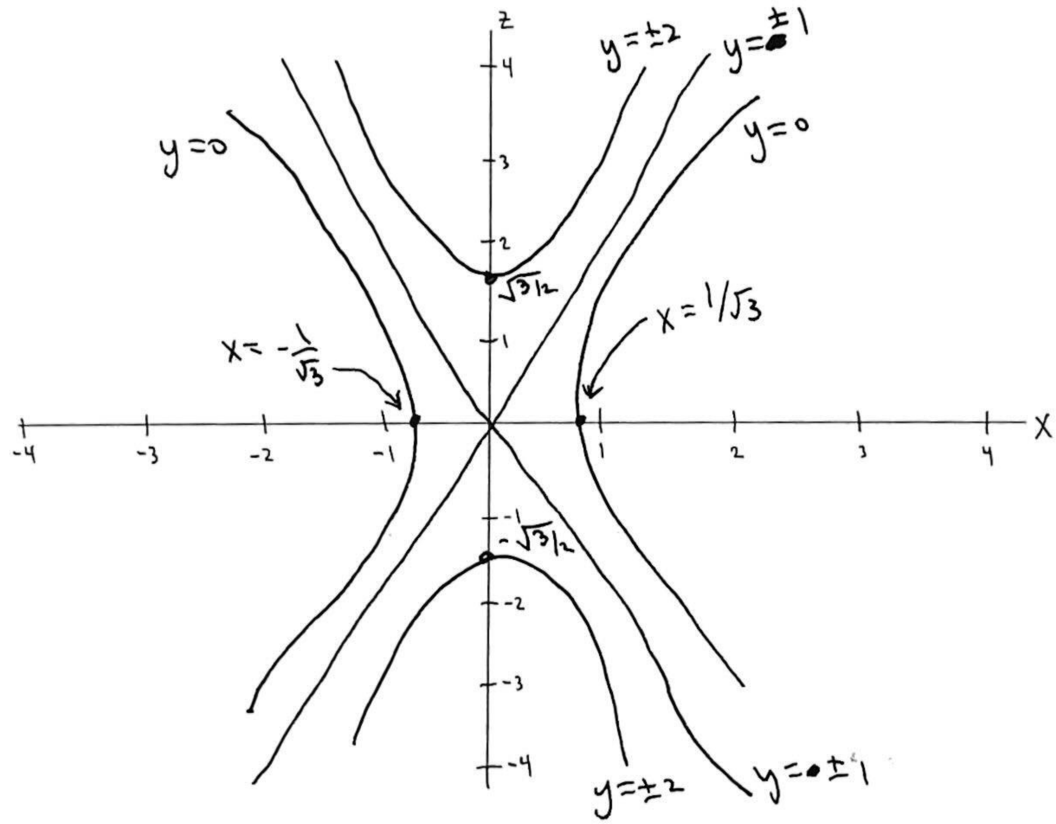
$\underline{z=\pm 1}$ $3x^2+y^2=3$ ellipse, x-intercepts $x=\pm 1$

y-intercepts $y=\pm\sqrt{3}$

$\underline{z=\pm 2}$ $3x^2+y^2=9$ ellipse, x-intercepts $x=\pm\sqrt{3}$

y-intercepts $y=\pm 3$

(b) (10 points) Sketch the cross-sections of this surface at $y = 0, \pm 1, \pm 2$ on the axes below. Your drawing should clearly label the points at which these cross-sections intersect the axes. Draw all cross-sections on the same set of axes, and label which cross-section is which.



$\underline{y=0}$ $3x^2 - 2z^2 = 1$ hyperbola x-intercepts

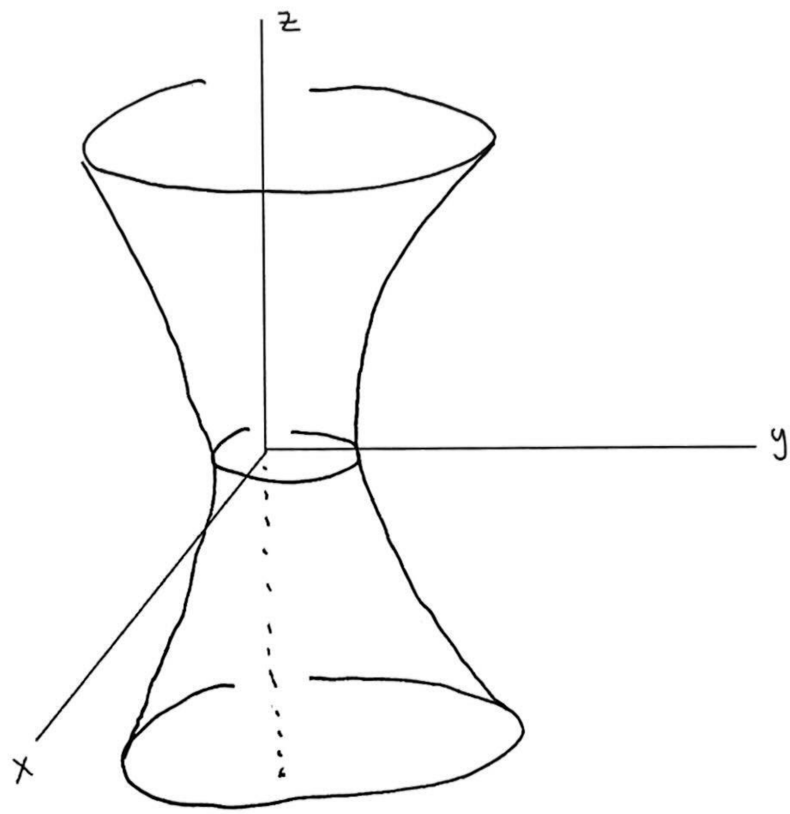
$x = \pm 1/\sqrt{3}$

$\underline{y=\pm 1}$ $3x^2 - 2z^2 = 0$ lines $z = \pm\sqrt{\frac{3}{2}}\, x$

$\underline{y=\pm 2}$ $3x^2 - 2z^2 = -3$ hyperbola

$-3x^2 + 2z^2 = 3$ z-intercepts $z = \pm\sqrt{\frac{3}{2}}$

(c) (5 points) Identify the surface by name and the axis along which it is centered, or if you forgot the name sketch the surface. (A sketch is only required if you do not know the name.)



hyperboloid of one sheet

centered along z-axis

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