MATH 230-1: Written HW 2 Solutions
Northwestern University, Fall 2023

1. The goal of this problem is to find the point on the line with parametric equations

$$x = 2 - 3t,\ y = -4 + t,\ z = 1 + t$$

that is closest to the point $(-3, 2, 1)$ in multiple ways. The point is to illustrate how the same problem can be approached by different methods.

(a) Find the function $f(t)$, depending on t, that gives the distance between $(-3, 2, 1)$ and a point $(2 - 3t, -4 + t, 1 + t)$ on the given line. Then, using single-variable calculus, find the value of t that minimizes this distance, and hence find the desired closest point.

(b) Take any two specific points P and Q (your choice!) on this line. Set $R = (-3, 2, 1)$, and compute the vector projection of $\overrightarrow{PR}$ onto $\overrightarrow{PQ}$. Explain (draw a picture!) how to use the sum of $\overrightarrow{OP}$ and this projection to find the desired closest point, and hence find its coordinates.

(c) Use the method described in the book, using a cross product, to find the distance from $(-3, 2, 1)$ to the given line. Then, find the value of t so that the function $f(t)$ from part (a) gives the distance you found, and hence find the closest point.

Solution. (a) The distance from $(-3, 2, 1)$ to the point on the given line with coordinates $(2 - 3t, -4 + t, 1 + t)$ is

$$f(t) = \sqrt{(-3 - [2 - 3t])^2 + (2 - [-4 + t])^2 + (1 - [1 + t])^2}.$$

Minimizing this quantity is the same as minimizing its square, or equivalently the expression under the square root, so to save come computation and avoid differentiating a square root, we minimize

$$g(t) = (-3 - [2 - 3t])^2 + (2 - [-4 + t])^2 + (1 - [1 + t])^2$$

instead. (This was not required, and you can just as easily minimize the original $f(t)$ instead. Later on, however, it might be useful to think about minimizing squares in this way.) After simplifying, we get

$$g(t) = (-5 + 3t)^2 + (6 - t)^2 + t^2.$$

The minimum occurs at a point at which the derivative is zero, so we get

$$g'(t) = 2(-5 + 3t)(3) + 2(6 - t)(-1) + 2t = 0, \text{ or more simply } g'(t) = 22t - 42 = 0.$$

Hence the minimum occurs at $t = 42/22 = 21/11$, so the point with coordinates

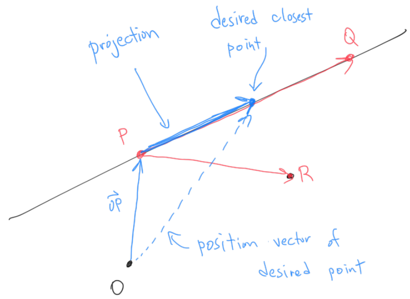
$$x = 2 - 3(21/11) = -41/11,\ y = -4 + 21/11 = -23/11,\ z = 1 + 21/11 = 32/11$$

is the point on the given line that is closest to $(-3, 2, 1)$. (It was not required here to justify the fact that this point does give a minimum and not some other type of critical point, but you should think about why this is!)

(b) Take $P = (2, -4, 1)$ and $Q = (-1, -3, 2)$, corresponding to $t = 0$ and $t = 1$ respectively, on the given line. Then $\overrightarrow{PR} = \langle -5, 6, 0 \rangle$ and $\overrightarrow{PQ} = \langle -3, 1, 1 \rangle$, so

$$\text{proj}_{\overrightarrow{PQ}} \overrightarrow{PR} = \left(\frac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\overrightarrow{PQ} \cdot \overrightarrow{PQ}} \right) \overrightarrow{PQ} = \frac{15 + 6 + 0}{9 + 1 + 1} \langle -3, 1, 1 \rangle = \langle -63/11, 21/11, 21/11 \rangle.$$

Based on the picture



we see that the sum of $\overrightarrow{OP}$ and this projection is the position vector of the desired closest point. Since

$$\overrightarrow{OP} + \operatorname{proj}_{\overrightarrow{PQ}} \overrightarrow{PR} = \langle 2, -4, 1\rangle + \langle -63/11, 21/11, 21/11\rangle = \langle -41/11, -23/11, 32/11\rangle\,,$$

$(-41/11, -23/11, 32/11)$ is the point on the given line closest to $(-3, 2, 1)$.

(c) Let us again take $P = (2, -4, 1)$ and $Q = (-1, -3, 2)$ as points on the given line, and $R = (-3, 2, 1)$. With the same $\overrightarrow{PR} = \langle -5, 6, 0\rangle$ and $\overrightarrow{PQ} = \langle -3, 1, 1\rangle$ as before, we get

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 6 & 0 \\ -3 & 1 & 1 \end{vmatrix} = \langle 6 - 0, -(-5 - 0), -5 - (-18)\rangle = \langle 6, 5, 13\rangle\,.$$

Hence the distance from $R = (-3, 2, 1)$ to the given line is

$$\frac{|\overrightarrow{PR} \times \overrightarrow{PQ}|}{|\overrightarrow{PQ}|} = \frac{\sqrt{36 + 25 + 169}}{\sqrt{9 + 1 + 1}} = \sqrt{\frac{230}{11}}.$$

To find the point on the line which has this specific distance to $(-3, 2, 1)$, we solve

$$f(t) = \sqrt{(-3 - [2 - 3t])^2 + (2 - [-4 + t])^2 + (1 - [1 + t])^2} = \sqrt{\frac{230}{11}}.$$

Simplifying a bit gives

$$(-5 + 3t)^2 + (6 - t)^2 + t^2 = \frac{230}{11}, \text{ or } 11t^2 - 42t + 61 = \frac{230}{11}.$$

Using the quadratic formula on $121t^2 - 462t + 441 = 0$ gives

$$t = \frac{462 \pm \sqrt{462^2 - 4(121)(441)}}{2(121)} = \frac{462}{242} = \frac{21}{11},$$

which in turn gives the point $(2 - 3(21/11), -4 + 21/11, 1 + 21/11) = (-41/11, -23/11, 32/11)$ as the one on the given line closest to $(-3, 2, 1)$.

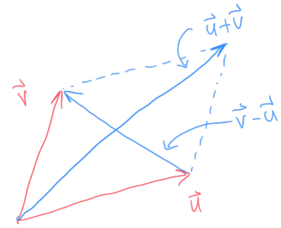
(Note that all three approaches gave the same answer in the end, which is a good sanity check! You should not expect numbers this messy on an exam—after all, calculators will not be allowed—but regardless of the numbers involved the point is to trust the process!) $\square$

2. The goal of this problem is to illustrate the use of the dot product in justifying some facts about some standard geometric shapes.

(a) A *rhombus* is a parallelogram whose sides all have the same length. Use vectors to justify the fact that the diagonals of a parallelogram intersect each other at right angles exactly when that parallelogram is actually a rhombus.

(b) Use vectors to justify the fact that the diagonals of a parallelogram are equal in length exactly when that parallelogram is actually a rectangle.

Solution. We use the following picture as a reference in both parts:



(a) To say that the diagonals intersect each other at right angles means that

$$(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{v}-\mathbf{u}) = 0.$$

But using the distributive properties of the dot product, we get that this equation is the same as

$$\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0.$$

Since $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, this equation simplifies further to

$$-\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 0, \text{ or } \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u}.$$

Now, $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ for any vector $\mathbf{a}$, so this final equation is the same as

$$|\mathbf{v}|^2 = |\mathbf{u}|^2, \text{ or } |\mathbf{v}| = |\mathbf{u}|$$

after taking square roots and using the fact that lengths are nonnegative. The upshot is that the equation $(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{v}-\mathbf{u}) = 0$ which says that the diagonals are orthogonal ends up being the same as the equation $|\mathbf{v}| = |\mathbf{u}|$ which says that the lengths of the sides are the same, so having orthogonal diagonals is equivalent to being a rhombus.

(b) To say that the diagonals are equal in length is to say that

$$|\mathbf{u}+\mathbf{v}| = |\mathbf{v}-\mathbf{u}|, \text{ or equivalently } |\mathbf{u}+\mathbf{v}|^2 = |\mathbf{v}-\mathbf{u}|^2$$

after squaring. This final equation is the same as

$$(\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = (\mathbf{v}-\mathbf{u}) \cdot (\mathbf{v}-\mathbf{u}).$$

Again using the distributive properties of the dot product, we see that this equation is the same as

$$\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u},$$

which simplifies further to $4\mathbf{u} \cdot \mathbf{v} = 0$ and hence to $\mathbf{u} \cdot \mathbf{v} = 0$. Hence the equation characterizing the condition that the diagonals are equal in length is equivalent to the equation $\mathbf{u} \cdot \mathbf{v} = 0$ which says that the sides are perpendicular, so having diagonals of equal length is equivalent to being a rectangle. □

3. Suppose we are given lines with parametric equations

$$x = 3 + t, \ y = 2 - t, \ z = -1 + 2t$$

and

$$x = -2 + 2t, \ y = 5 - t, \ z = 5 - 4t.$$

The goal of this problem is to find parametric equations for the line that is perpendicular to both of these lines and passes through their point of intersection.

(a) Find the point at which the lines above intersect.

(b) Find a vector that is perpendicular to both of these lines.

(c) Find parametric equations for the desired perpendicular line.

Solution. (a) The lines intersect at a point whose coordinates arise from both sets of parametric equations, albeit for possibly different values of the parameter. If we denote by t_1 the parameter value that gives the intersection point on the first line, and by t_2 the parameter value giving the intersection on the second line, we intersection occurs when

$$\begin{aligned} 3 + t_1 &= -2 + 2t_2 \\ 2 - t_1 &= 5 - t_2 \\ -1 + 2t_1 &= 5 - 4t_2. \end{aligned}$$

The first equation gives $t_1 = -5 + 2t_2$, and then the second gives

$$2 - (-5 + 2t_2) = 5 - t_2, \text{ or } 7 - 2t_2 = 5 - t_2.$$

This gives $t_2 = 2$, and then $t_1 = -5 + 2(2) = -1$. Hence $t_1 = -1, t_2 = 2$ gives the same x- and y-coordinates along the two lines, and we check that these also give the same z-coordinates:

$$-1 + 2(-1) = -3 \text{ agrees with } 5 - 4(2) = -3.$$

The lines thus intersect at $(2, 3, -3)$, occurring at $t_1 = -1$ on the first line and $t_2 = 2$ on the second.

(b) The first line has direction vector $\mathbf{v}_1 = \langle 1, -1, 2\rangle$, and the second has direction vector $\mathbf{v}_2 = \langle 2, -1, -4\rangle$, which we find by taking the coefficient of the parameter t. The cross product

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -1 & -4 \end{vmatrix} = \langle 4 - (-2), -(-4 - 4), -1 - (-2)\rangle = \langle 6, 8, 1\rangle$$

is then perpendicular to these direction vectors, and hence perpendicular to both lines.

(c) The line we want passes through $(2, 3, -3)$ and has direction vector $\langle 6, 8, 1\rangle$, so it has vector equation $\mathbf{r}(t) = \langle 2, 3, -3\rangle + t\,\langle 6, 8, 1\rangle = \langle 2 + 6t, 3 + 8t, -3 + t\rangle$, and hence parametric equations

$$x = 2 + 6t, \ y = 3 + 8t, \ z = -3 + t.$$

□