Northwestern University

MATH 230-1 Midterm Examination 1
Winter Quarter 2022
January 31, 2022

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- Mark your section.

Section	**Time**	**Instructor**	
41	10:00	Bentsen	
51	11:00	Cuzzocreo	
61	12:00	Cuzzocreo	

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 60 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Use pages at the back for scratchwork if needed.
- Show all of your work. Unsupported answers may not earn credit.

1. Let $\mathbf{a} = \langle 1, 2, 2\rangle$, $\mathbf{b} = \langle 2, 3, 2\rangle$, and $\mathbf{c} = \langle 2, 1\rangle$. Determine whether the following expressions are well-defined or not. If they are well-defined, compute them, and circle your final answer. Answers which are vectors should be written as a single vector in component form. If the expression is not well-defined, write "UNDEFINED," and give a brief (one-sentence) explanation.

   (a) (2 points) $|\mathbf{c}|\mathbf{b} + \mathbf{a}$

$$\begin{aligned} |\mathbf{c}|\mathbf{b} + \mathbf{a} &= \sqrt{2^2 + 1^2}\langle 2, 3, 2\rangle + \langle 1, 2, 2\rangle \\ &= \sqrt{5}\langle 2, 3, 2\rangle + \langle 1, 2, 2\rangle \\ &= \boxed{\langle 2\sqrt{5} + 1, 3\sqrt{5} + 2, 2\sqrt{5} + 2\rangle} \end{aligned}$$

   (b) (2 points) $3\mathbf{a} + \mathbf{c}$

   $\boxed{\text{UNDEFINED}}$ since $3\mathbf{a}$ is a vector in $\mathbb{R}^3$ and $\mathbf{c}$ is a vector in $\mathbb{R}^2$

   (c) (2 points) $\mathbf{a} \times \mathbf{c}$

   $\boxed{\text{UNDEFINED}}$ since $3\mathbf{a}$ is a vector in $\mathbb{R}^3$ and $\mathbf{c}$ is a vector in $\mathbb{R}^2$

Let $\mathbf{a} = \langle 1, 2, 2\rangle$, $\mathbf{b} = \langle 2, 3, 2\rangle$, and $\mathbf{c} = \langle 2, 1\rangle$. Determine whether the following expressions are well-defined or not. If they are well-defined, compute them, and circle your final answer. Answers which are vectors should be written as a single vector in component form. If the expression is not well-defined, write "UNDEFINED," and give a brief (one-sentence) explanation.

(d) (2 points) $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$

$$\begin{aligned}\mathrm{proj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\\ &= \frac{2\cdot 1 + 3\cdot 2 + 2\cdot 2}{2^2+3^2+2^2}\mathbf{b}\\ &= \frac{12}{17}\langle 2, 3, 2\rangle\\ &= \boxed{\left\langle \frac{24}{17}, \frac{36}{17}, \frac{24}{17}\right\rangle}\end{aligned}$$

(e) (2 points) $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$(\mathbf{a}\cdot\mathbf{b})\mathbf{c} = 12\langle 2, 1\rangle = \boxed{\langle 24, 12\rangle}$$

(f) (2 points) $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{b}$

$\boxed{\text{UNDEFINED}}$ since $\mathbf{a}\cdot\mathbf{b}$ is a scalar and $\mathbf{b}$ is a vector.

2. Consider the points $P = (1, 2, -1)$, $Q = (0, 3, 1)$, $R = (4, -3, 0)$.

(a) (5 points) Find the angle of the triangle ΔPQR at the vertex P. You may express your answer in terms of an inverse trigonometric function. Show all work and justify your answer completely.

This is the angle between the vectors $\overrightarrow{PQ} = \langle -1, 1, 2\rangle$ and $\overrightarrow{PR} = \langle 3, -5, 1\rangle$. We have

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}\right)\\ &= \cos^{-1}\left(\frac{(-1)\cdot 3 + 1\cdot(-5) + 2\cdot 1}{\sqrt{(-1)^2+1^2+2^2}\sqrt{3^2+(-5)^2+1^2}}\right)\\ &= \boxed{\cos^{-1}\left(\frac{-6}{\sqrt{6}\sqrt{35}}\right)}\end{aligned}$$

(b) (5 points) Find a scalar equation for the plane containing P, Q, and R. Show all work and justify your answer completely.

We can find a normal vector to the plane by taking

$$\begin{aligned}\vec{n} &= \overrightarrow{PQ}\times\overrightarrow{PR}\\ &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 2\\ 3 & -5 & 1\end{vmatrix}\\ &= 11\vec{i} + 7\vec{j} + 2\vec{k}\end{aligned}$$

Using the coordinates of P, the equation of the plane can then be given as

$$\boxed{11(x-1) + 7(y-2) + 2(z+1) = 0}$$

or

$$\boxed{11x + 7y + 2z = 23}$$

3. Consider two planes given by

$$\mathcal{P}_1 : x - 2y + 6z = 1,$$
$$\mathcal{P}_2 : 2x - 3y - 4z = 9.$$

(a) (6 points) Compute the distance from the point $Q = (1, 0, 3)$ to the plane $\mathcal{P}_2$. Show all work and justify your answer completely.

First we locate a point P on $\mathcal{P}_2$. There are many possible choices, but $P = (0, -3, 0)$ is easy to find. We then compute $\overrightarrow{PQ} = \langle 1, 3, 3 \rangle$. Then, writing $\vec{n_2}$ for the normal vector $\langle 2, -3, -4 \rangle$ of $\mathcal{P}_2$, we compute

$$\begin{aligned}
\mathrm{dist}(\mathrm{Q}, \mathcal{P}_2) &= \left| \mathrm{proj}_{\tilde{\mathrm{n}_2}} \overrightarrow{PQ} \right| \\
&= \frac{\left| \overrightarrow{PQ} \cdot \vec{n_2} \right|}{|\vec{n_2}|} \\
&= \frac{|1 \cdot 2 + 3 \cdot (-3) + 3 \cdot (-4)|}{\sqrt{2^2 + (-3)^2 + (-4)^2}} \\
&= \boxed{\frac{19}{\sqrt{29}}}
\end{aligned}$$

(b) (8 points) Again, we consider two planes given by

$$\mathcal{P}_1 : x - 2y + 6z = 1,$$
$$\mathcal{P}_2 : 2x - 3y - 4z = 9.$$

Find parametric equations for the line of intersection of the planes $\mathcal{P}_1$ and $\mathcal{P}_2$. Show all work and justify your answer completely.

We must find a point on the line of intersection of the planes, as well as the direction vector of the line. We can find, e.g. the intersection of the line with the xy plane by setting $z = 0$ and solving the system

$$\begin{aligned} x - 2y &= 1 \\ 2x - 3y &= 9 \end{aligned}$$

which has the solution $x = 15, y = 7$. This gives the point $(15, 7, 0)$ on the line (there are many correct points one could use here).

We then find a direction vector by taking the cross product of the normal vectors of the planes:

$$\begin{aligned} \vec{v} &= \vec{n}_1 \times \vec{n}_2 \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 6 \\ 2 & -3 & -4 \end{vmatrix} \\ &= 26\vec{i} + 16\vec{j} + \vec{k} \end{aligned}$$

This gives the equation

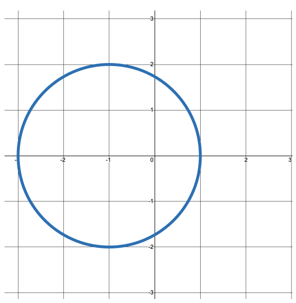
$$\boxed{\vec{r}(t) = \langle 15 + 26t, 7 + 16t, t \rangle}$$

(there are many other possible correct parameterizations)

4. Express the following polar equations in Cartesian (rectangular) coordinates. Then sketch the curve satisfying the equation inthe xy plane.

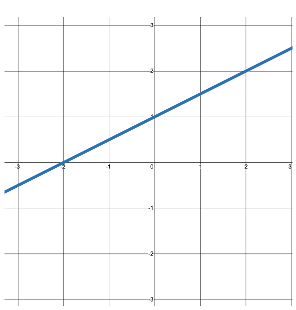
(a) (5 points) $r + 2\cos(\theta) = \dfrac{3}{r}$

$$\begin{aligned} r + 2\cos(\theta) &= \frac{3}{r} \\ r^2 + 2r\cos(\theta) &= 3 \\ x^2 + y^2 + 2x &= 3 \\ x^2 + 2x + 1 + y^2 &= 4 \\ \boxed{(x+1)^2 + y^2 = 4} \end{aligned}$$



(b) (5 points) $r = \dfrac{2}{2\sin\theta - \cos\theta}$

$$\begin{aligned} r &= \frac{2}{2\sin\theta - \cos\theta} \\ 2r\sin(\theta) - r\cos(\theta) &= 2 \\ \boxed{2y - x = 2} \\ \boxed{y = 1 + \tfrac{x}{2}} \end{aligned}$$



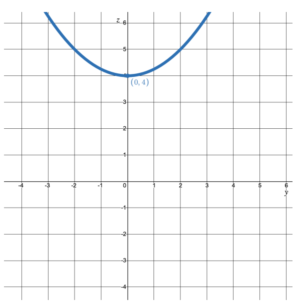
5. Consider the quadric surface defined by the equation

$$x^2 - \frac{1}{4}y^2 + z + 2x = 3$$

(a) Sketch the conic sections that arise as intersections (traces) of the surface with the following planes. Clearly label the center and any vertices of the conic section.

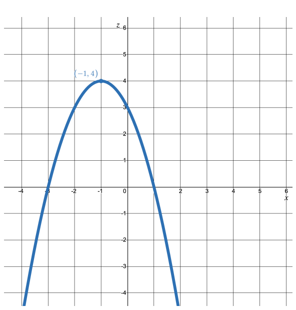
i. (3 points) $x = -1$

$$\begin{aligned} x^2 - \tfrac{1}{4}y^2 + z + 2x &= 3 \\ 1 - \tfrac{1}{4}y^2 + z - 2 &= 3 \\ z &= \tfrac{1}{4}y^2 + 4 \end{aligned}$$



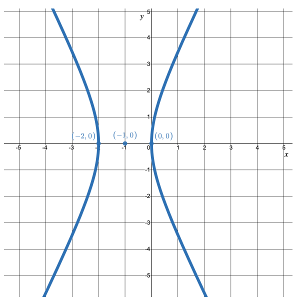
ii. (3 points) $y = 0$

$$\begin{aligned} x^2 - \tfrac{1}{4}y^2 + z + 2x &= 3 \\ x^2 + z + 2x &= 3 \\ x^2 + 2x + 1 + z &= 4 \\ z &= 4 - (x+1)^2 \end{aligned}$$



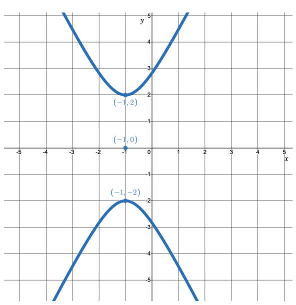
iii. (3 points) $z = 3$

$$\begin{aligned} x^2 - \tfrac{1}{4}y^2 + 3 + 2x &= 3 \\ x^2 - \tfrac{1}{4}y^2 + 2x &= 0 \\ x^2 + 2x + 1 - \tfrac{1}{4}y^2 &= 1 \\ (x+1)^2 - \tfrac{1}{4}y^2 &= 1 \end{aligned}$$



iv. (3 points) $z = 5$

$$\begin{aligned} x^2 - \tfrac{1}{4}y^2 + 5 + 2x &= 3 \\ x^2 - \tfrac{1}{4}y^2 + 2x &= -2 \\ x^2 + 2x + 1 - \tfrac{1}{4}y^2 &= -1 \\ (x+1)^2 - \tfrac{1}{4}y^2 &= -1 \end{aligned}$$



(b) (2 points) Give the name of the surface.

$\boxed{\text{Hyperbolic Paraboloid}}$

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