Northwestern University

Math 230-1 First Midterm Examination
Fall Quarter 2021
Tuesday 19 October

Last name: ________________ Email address: ________________

First name: ________________ NetID: ________________

Instructions

- This examination consists of 6 questions for a total of 60 points.
- Read all problems carefully before answering.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- **Show and justify all of your work**. Unsupported answers may not earn credit.
- **Terminology**: by "familiar named surface" we will mean a member of one of the following types of surfaces:

plane	cylinder	
ellipsoid	elliptic paraboloid	hyperbolic paraboloid
cone	hyperboloid of one sheet	hyperboloid of two sheets

1. Transform the following polar equations into Cartesian equations. Identify and sketch the resulting graph.

$$\boxed{r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta}$$

(a) (5 points) $r = 2\sin\theta$

$r = 2\sin\theta \rightarrow r^2 = 2r\sin\theta$

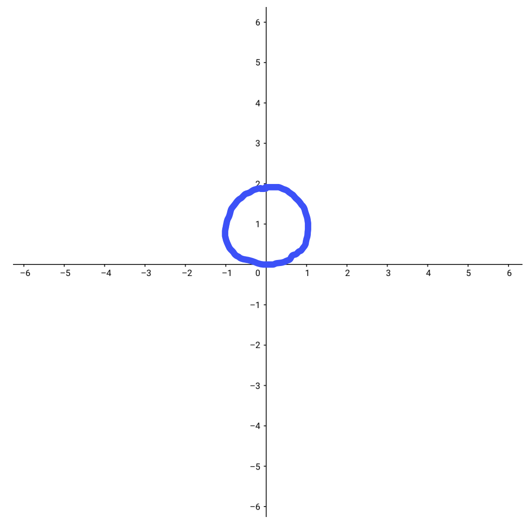
$\rightarrow x^2 + y^2 = 2y$

$\rightarrow x^2 + y^2 - 2y = 0$

$\rightarrow x^2 + (y-1)^2 - 1 = 0$

$\rightarrow x^2 + (y-1)^2 = 1.$

Circle of radius 1 centered at $(0,1)$.



(b) (5 points) $r = 4\tan\theta\sec\theta$

$r = 4\tan\theta\sec\theta \rightarrow r = 4\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$

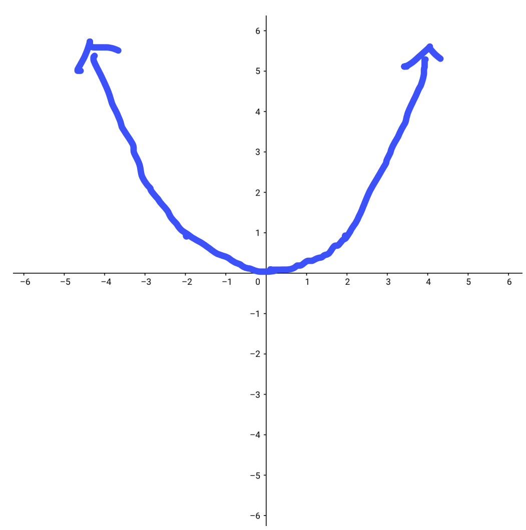
$\rightarrow r\cos^2\theta = 4\sin\theta$

$\rightarrow r^2\cos^2\theta = 4r\sin\theta$

$\rightarrow x^2 = 4y$

$\rightarrow y = \frac{1}{4}x^2.$

Parabola with vertex at $(0,0)$.



2. For each of the following, first identify the type of familiar named surface given by the equation and the point where it is centered. Then find an equation (in standard form) for the conic section given by the intersection of the given plane with the given surface. Finally, identify the type of conic section and the point at which it is centered.

(a) (5 points) The plane given by $y = -2$ and the surface given by

$$(x-1)^2 + \frac{(y+2)^2}{4} + \frac{z^2}{9} = 1$$

Ellipsoid centered at $(1,-2,0)$.

$y=-2 \to (x-1)^2 + \frac{z^2}{9} = 1$.

Ellipse in xz-plane centered at $(1,0)$.

(b) (5 points) The plane given by $x = 1$ and the surface given by

$$x^2 + 2x + y^2 - z^2 + 6z = 9$$

$x^2+2x+y^2-z^2+6z=9 \to (x+1)^2-1+y^2-(z^2-6z)=9$

$\to (x+1)^2+y^2-((z-3)^2-9)=9$

$\to (x+1)^2+y^2-(z-3)^2=1$.

One-sheet hyperbola centered at $(-1,0,3)$.

$x=1 \to y^2-(z-3)^2=-3 \to \frac{(z-3)^2}{3} - \frac{y^2}{3} = 1$

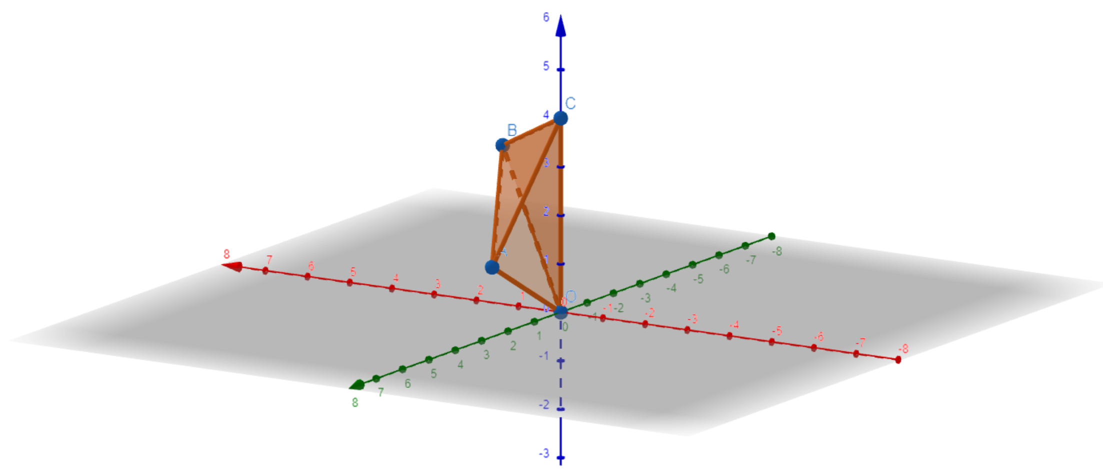
Hyperbola in yz-plane centered at $(0,3)$.

3.

(a) (2 points) What is the geometric interpretation of $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$? No justification is required.

This is the volume of the parallelopiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

(b) (8 points) Find the volume of the pyramid with vertices (0,0,0), (1,1,1), (2,-1,3), and (0,0,4).



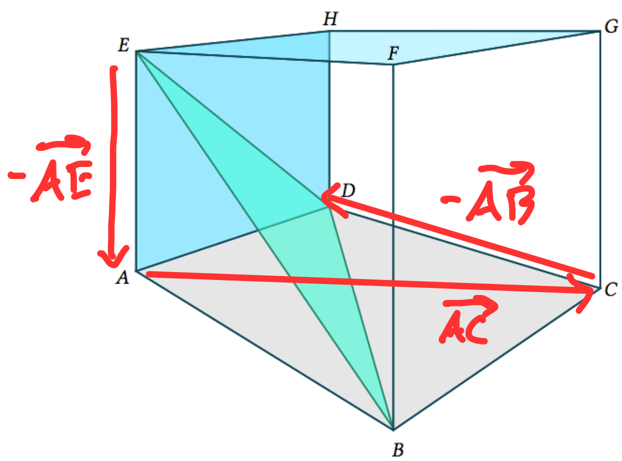
The pyramid has volume

$V = \frac{1}{6}$ (volume of parallelopiped determined by $\underbrace{\langle 1,1,1\rangle}_{\vec{u}}, \underbrace{\langle 2,-1,3\rangle}_{\vec{v}}, \underbrace{\langle 0,0,4\rangle}_{\vec{w}}$).

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \langle 4, -1, -3\rangle.$$

$$6V = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |\langle 4,-1,-3\rangle \cdot \langle 0,0,4\rangle| = |-12| = 12$$

$$\Rightarrow \boxed{V = 2}$$



4. The above figure shows a cube whose sides have length 1. Choose A to be the origin, and let the positive x, y and z axes lie along the sides AB, AD, and AE respectively.

   (a) (3 points) Write $\vec{ED}$ in terms of $\vec{AB}$, $\vec{AC}$, and $\vec{AE}$.

$$\vec{ED} = -\vec{AE} + \vec{AC} - \vec{AB}$$

   (b) (4 points) Find an equation for the plane M containing the points B, D, and E (the intersection of M with the cube is shown in the diagram).

$\vec{n} = \vec{BD} \times \vec{BE}$, B a point on M

$\rightarrow$ M given by all points P such that

$$\boxed{\vec{n} \cdot \vec{BP} = 0}$$

(c) (3 points) Find an equation for the distance d from G to the plane M from part b). Your answer should involve dot product and magnitude of vectors.

$$d = \left| \operatorname{proj}_{\vec{n}} \overrightarrow{BG} \right|$$

$$= \left| \frac{\overrightarrow{BG} \cdot \vec{n}}{|\vec{n}|^2} \right|$$

$$\boxed{= \left| \frac{\overrightarrow{BG} \cdot \vec{n}}{|\vec{n}|} \right|}$$

5. Let $\mathbf{a} = \langle 2, 1 \rangle$, $\mathbf{b} = \langle -1, 2, -1 \rangle$ and $\mathbf{c} = \langle 1, 0, 1 \rangle$. Which of the following expressions are not well defined? Evaluate expressions which are well defined.

(a) (2 points) $3\mathbf{a} + \mathbf{b}$

Undefined

(b) (2 points) $\mathbf{b} + \mathbf{c}$

$$\vec{b} + \vec{c} = \langle -1, 2, -1 \rangle + \langle 1, 0, 1 \rangle = \boxed{\langle 0, 2, 0 \rangle}$$

(c) (2 points) $\mathbf{b} \cdot \mathbf{c}$

$$\vec{b} \cdot \vec{c} = \langle -1, 2, -1 \rangle \cdot \langle 1, 0, 1 \rangle = -1 + 0 - 1 = \boxed{-2}$$

(d) (2 points) $\mathbf{a} \times \mathbf{c}$

Undefined

(e) (2 points) $\text{proj}_{\mathbf{c}}\mathbf{b}$

$$\text{proj}_{\vec{c}}\vec{b} = \left(\frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2} \right) \vec{c} = \left(\frac{-2}{2} \right) \vec{c} = -\vec{c} = \boxed{\langle -1, 0, -1 \rangle}$$

6. We consider the planes M_1, M_2, and M_3 given by the following equations:

$$M_1 : x + y + z = -1,\ M_2 : x + 3y - 2z = 2,\ M_3 : -2x + 3y - 5z = 5.$$

(a) (7 points) Find a parametrization $\mathbf{r}_{1,2}(t)$ for the line of intersection between the planes M_1 and M_2.

M_1 has normal vector $\vec{n_1} = \langle 1,1,1 \rangle$.

M_2 has normal vector $\vec{n_2} = \langle 1,3,-2 \rangle$.

Let $\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \langle -5, 3, 2 \rangle$.

Find point on intersection of M_1, M_2:

$$\begin{array}{r} x + y + z = -1 \\ -\,(x + 3y - 2z = 2) \\ \hline -2y + 3z = -3 \end{array} \rightarrow \text{let } z = -1,\ y = 0$$

$\rightarrow$ (by $x+y+z=-1$) $x = 0$.

So $P = (0,0,-1)$ lies on line of intersection.

Parametrize

$$\boxed{\vec{r}_{1,2}(t) = \langle 0,0,-1 \rangle + t\langle -5,3,2 \rangle}$$

(b) (3 points) A parametrization for the line of intersection between the planes M_1 and M_3 is given by $\mathbf{r}_{1,3}(s) = \langle 8, -3, -6\rangle + s\langle -8, 3, 5\rangle$. Find the point of intersection between this line and the line from part a). Notice from the equations above (no work necessary) that this is the point of intersection between all three planes M_1, M_2, and M_3.

We need $\vec{r}_{1,2}(t) = \vec{r}_{1,3}(s)$

$\rightarrow \langle 0, 0, -1\rangle + t\langle -5, 3, 2\rangle = \langle 8, -3, -6\rangle + s\langle -8, 3, 5\rangle$

$\rightarrow \langle -5t, 3t, 2t-1\rangle = \langle 8-8s, 3s-3, 5s-6\rangle$

$\rightarrow -5t = 8-8s, \; 3t = 3s-3, \; 2t-1 = 5s-6.$

Since $3t = 3s-3$, then $t = s-1$.

So, $2(s-1)-1 = 5s-6 \rightarrow 2s-2-1 = 5s-6$

$\rightarrow s = 1$

$\rightarrow t = 0.$

So, point of intersection when $t=0$:

$\vec{r}_{1,2}(0) = \langle 0, 0, -1\rangle$

$\rightarrow$ $\boxed{\text{point of intersection } (0,0,-1)}$

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