MATH 230-1: Written HW 7 Solutions
Northwestern University, Fall 2023

1. Consider the surface defined by the equation

$$zx + x^2y - y^3 = 0.$$

(a) Find the tangent plane to this surface at $(1,1,0)$. (Note there are two ways in which this could be done, either by using a gradient or by using the function $f(x,y)$ described in part (b).)

(b) Solve for z in the given equation in order to express the surface near $(1,1,0)$ as the graph of a function $z = f(x,y)$, and use this to find the quadratic polynomial whose graph best approximates the given surface near $(1,1,0)$.

(c) Say you wanted to use the tangent plane in (a) to approximate the values of the function $f(x,y)$ in (b) for (x,y) near $(1,1)$ to within an accuracy of 0.001. Find a value of k such that the tangent plane approximation does give this desired accuracy on the rectangle $1 \leq x \leq k$, $1 \leq y \leq k$.

Solution. (a) First we use the gradient method. View the given surface as a level surface of the function

$$g(x,y,z) = zx + x^2y - y^3.$$

Then ∇g is orthogonal to the surface at any point, and so $\nabla g(1,1,0)$ gives a normal vector for the desired plane. We have $\nabla g = \left\langle z + 2xy, x^2 - 3y^2, x \right\rangle$, so $\nabla g(1,1,0) = \langle 2, -2, 1 \rangle$. Thus the tangent plane at $(1,1,0)$ has vector equation

$$\langle 2, -2, 1 \rangle \cdot \langle x - 1, y - 1, z \rangle = 0,$$

which becomes

$$2(x-1) - 2(y-1) + z = 0.$$

Alternatively, by solving for z we recognize that the given surface is the graph of the function

$$z = f(x,y) = -xy + \frac{y^3}{x},$$

at least for $x \neq 0$. For this function we have

$$f_x = -y - \frac{y^3}{x^2} \quad \text{and} \quad f_y = -x + \frac{3y^2}{x},$$

so the tangent plane at $(1,1,f(1,1))$ is

$$\begin{aligned} z &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= 0 - 2(x-1) + 2(y-1). \end{aligned}$$

Note that, after rearranging, this is the same equation as the one found using the gradient.

(b) The function $f(x,y)$ was found above: $f(x,y) = -xy + \frac{y^3}{x}$. We have

$$f_x = -y - \frac{y^3}{x^2} \quad \text{and} \quad f_y = -x + \frac{3y^2}{x},$$

so

$$f_{xx} = \frac{2y^3}{x^3} \qquad f_{xy} = -1 - \frac{3y^2}{x^2} \qquad f_{yy} = \frac{6y}{x}.$$

The best approximating quadratic polynomial near $(x, y) = (1, 1)$ is then

$$\begin{aligned} z &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &\qquad + \frac{1}{2}[f_{xx}(1,1)(x-1)^2 + 2f_{xy}(1,1)(x-1)(y-1) + f_{yy}(1,1)(y-1)^2] \\ &= -2(x-1) + 2(y-1) + \frac{1}{2}[2(x-1)^2 - 8(x-1)(y-1) + 6(y-1)^2]. \end{aligned}$$

(c) The error in the tangent plane (i.e., linear) approximation is no larger than

$$\frac{1}{2}M(|\Delta x| + |\Delta y|)^2$$

where M is a bound on the absolute values of all second derivatives over the rectangle $1 \leq x \leq k, 1 \leq y \leq k$, and Δx and Δy denote the changes in the x, y we are considering from the values $x = 1, y = 1$ giving the point at which we are taking the approximation. We use the second derivative computations from (b):

$$f_{xx} = \frac{2y^3}{x^3} \qquad f_{xy} = -1 - \frac{3y^2}{x^2} \qquad f_{yy} = \frac{6y}{x}.$$

For $1 \leq x \leq k$ and $1 \leq y \leq k$, the largest that $|f_{xx}|$ will be is $2k^3$ at $y = k$ and $x = 1$ (note that a fraction is maximized when the denominator is a small as possible); the largest that $|f_{yy}|$ will be is $1 + 3k^2$; and the largest that $|f_{yy}|$ will be is $6k$. For k just barely larger than 1 (we need only consider what happens near $(1, 1)$), the largest among $2k^3, 1 + 3k^2$ and $6k$ will be $6k$, so we take $M = 6k$ as our bound on the second derivatives.

The error in the linear approximation is then no larger than

$$\frac{1}{2}(6k)((k-1) + (k-1))^2 = 12k(k-1)^2,$$

where we use $\Delta x = k - 1$ and $\Delta y = k - 1$ as the largest changes in x and y in the rectangle $1 \leq x \leq k, 1 \leq y \leq k$. To achieve the desired accuracy, we want this error bound to be no larger than 0.001, so we need k such that

$$12k(k-1)^2 \leq 0.001.$$

A calculator then gives $k = 1.00908$ as a value of k larger than 1 for which this inequality is true, so we get this desired accuracy on the rectangle $1 \leq x \leq 1.00908, 1 \leq y \leq 1.00908$.

For a possible simplification, we can note that $6k$ is smaller than 7 for k just larger than 1, so we can also use $M = 7$ as a bound on the second derivatives. (The point is that we don't necessary need the most efficient bound, just one that works.) This gives the error bound as

$$\frac{1}{2}(7)((k-1) + (k-1))^2 = 14(k-1)^2,$$

in which we case we need k to satisfy

$$14(k-1)^2 \leq 0.001.$$

This gives $k = 1.00845$ as a valid option. The upshot is that there is no single "correct" answer to this problem, and what is important is justifying why your proposed answer is a valid one. □

2. Let $f(x, y) = x^2 + y^3 - 3y + 10$.

(a) Find and classify the critical points of $f(x, y)$.

(b) Find the absolute maximum and absolute minimum of $f(x, y)$ among points in the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$. Be sure to explain how you know $f(x, y)$ even has absolute extrema over this region in the first place.

(c) Find the absolute extrema of $f(x, y)$ among points in the disk $x^2 + (y-1)^2 \leq 1$. (Hint: Treat the problem of finding absolute extrema on the boundary circle as one of optimizing a function subject to a single constraint equation.)

Solution. (a) We have

$$f_x = 2x \quad \text{and} \quad f_y = 3y^2 - 3.$$

These are both zero when $x = 0$ and $y = \pm 1$, so the critical points of f are $(0, 1)$ and $(0, -1)$. The second derivatives of f are

$$f_{xx} = 2 \qquad f_{xy} = 0 = f_{yx} \qquad f_{yy} = 6y.$$

At $(0, 1)$ we have

$$f_{xx}f_{yy} - (f_{xy})^2 = 12 > 0,$$

so since $f_{xx}(0, 1) = 2 > 0$ we get that $(0, 1)$ is a local minimum of f. At $(0, -1)$ we have

$$f_{xx}f_{yy} - (f_{xy})^2 = -12 < 0,$$

so $(0, -1)$ is a saddle point of f.

(b) First, since $f(x, y)$ is continuous and the region in question is closed and bounded, the extreme value theorem guarantees that absolute extrema exist. Since none of the critical points found in (a) are in this triangle, the absolute maximum and minimum must occur on the boundary of the triangle. The value of $f(x, y)$ along the bottom edge $y = 0$ of the triangle is

$$f(x, 0) = x^2,$$

which has a (single-variable) critical point at $x = 0$. The value of $f(x, y)$ along the right edge $x = 1$ of the triangle is

$$f(1, y) = y^3 - 3y + 11,$$

which has a critical point when $3y^2 - 3 = 0$, so at $y = \pm 1$. But $y = -1$ does not give a point inside the region we are considering, so we need only consider $y = 1$ which gives the corner vertex $(1, 1)$.

The value of $f(x, y)$ along the diagonal edge $y = x$ of the triangle is

$$f(x, x) = x^3 + x^2 - 3x + 10.$$

This has a critical point when $3x^2 + 2x - 3 = 0$, so at

$$x = \frac{-2 \pm \sqrt{4 + 36}}{6} = \frac{-1 \pm \sqrt{10}}{3}.$$

Only the choice of positive square root gives a point inside the triangle, so there is a critical point at

$$\left(\tfrac{-1+\sqrt{10}}{3}, \tfrac{-1+\sqrt{10}}{3}\right)$$

along the diagonal edge of the triangle.

So, in the end the absolute max and min must occur among the points found above along with all corner/vertex points, so we must test:

$$(0,0),\ (1,0),\ (1,1),\ (\tfrac{-1+\sqrt{10}}{3}, \tfrac{-1+\sqrt{10}}{3}).$$

Plugging into $f(x, y)$ gives

$$f(0,0) = 10,\ f(1,0) = 11,\ f(1,1) = 9,\ f((\tfrac{-1+\sqrt{10}}{3}, \tfrac{-1+\sqrt{10}}{3})) \approx 8.73,$$

where we used a calculator for the last one, so the absolute maximum value of f over this triangle is 11 and occurs at $(1, 0)$, while the absolute minimum value is about 8.73 and occurs at $((-1 + \sqrt(10))/3, (-1 + \sqrt(10))/3)$.

(c) The only critical point of f that falls within this disk is $(0, 1)$, so this is the only one we need to consider. To find the absolute extrema for the function along the boundary circle $x^2 + (y - 1)^2 = 1$, we use the method of Lagrange multipliers. With $f(x, y) = x^2 + y^3 - 3y + 10$ subject to the constraint $g(x, y) = x^2 + (y - 1)^2 = 1$, the max and min must be among the points satisfying

$$\nabla f = \lambda \nabla g, \text{ or } \left\langle 2x, 3y^2 - 3\right\rangle = \lambda \left\langle 2x, 2(y-1)\right\rangle$$

for some scalar λ. Comparing components and including the constraint gives the equations

$$\begin{aligned} 2x &= \lambda 2x \\ 3y^2 - 3 &= \lambda(2y - 2) \\ x^2 + (y-1)^2 &= 1. \end{aligned}$$

The first equation gives that either $x = 0$ or $\lambda = 1$. If $x = 0$ the constraint gives $(y - 1)^2 = 1$, so $y = 0, 2$ and thus $(0, 0)$ and $(0, 2)$ are candidates to consider. If $\lambda = 1$, then the second equation above becomes

$$3y^2 - 3 = 2y - 2, \text{ or equivalently } 3y^2 - 2y - 1 = 0.$$

This becomes $(3y + 1)(y - 1) = 0$, so $y = -\frac{1}{3}$ or $y = 1$. But $y = -\frac{1}{3}$ does not give a point in our disk, so we only consider $y = 1$. The constraint then gives $x^2 = 1$, so $x = pm1$.

Thus we are left consider the points $(0, 1)$ in the interior of the disk, and $(0, 0), (0, 2), (1, 1)$, and $(-1, 1)$ on the boundary circle. We have

$$f(0,1) = 8,\ f(0,0) = 10,\ f(0,2) = 12,\ f(1,1) = 10,\ f(-1,1) = 9,$$

so the absolute maximum value of f over this disk is 12 and occurs at $(0, 2)$, while the absolute minimum value is 8 and occurs at $(0, 1)$. □

3. A certain company produces three products. The cost for producing the first product is p_1 dollars per unit, the cost of producing the second product is p_2 dollars per unit, and the cost of producing the third is p_3 dollars per unit. Assume the value derived from producing x_1 units of the first product, x_2 units of the second, and x_3 units of the third is given by the *utility function*

$$U(x_1, x_2, x_3) = x_1 x_2^2 x_3^3.$$

The larger the value of the utility function, the more worthwhile producing that many products is to the company. You can take it for granted that given some budget constraint as in (a) and (b) below, there are values of x_1, x_2, x_3 that maximize utility subject to that constraint.

(a) Assume the company has a budget of $100,000$ dollars to spend on producing its products. If $p_1 = 1$, $p_2 = 3$, and $p_3 = 5$, find the values of x_1, x_2, x_3 needed to maximize utility.

(b) More generally, with a budget of B dollars and arbitrary prices p_1, p_2, p_3, determine how much of this budget the company should devote to producing the first product, to the second product, and to the third product in order to maximize utility.

Solution. (a) The cost of producing x_1 units of the first product is x_1, the cost of producing x_2 units of the second product is $3x_2$, and the cost of producing x_3 units of the third is $5x_3$, so our budget constraint is

$$x_1 + 3x_2 + 5x_3 = 100,000.$$

Thus we maximize $U(x_1, x_2, x_3) = x_1 x_2^2 x_3^3$ subject to $g(x_1, x_2, x_3) = x_1 + 3x_2 + 5x_3 = 100,000$. The equation obtained from the method of Lagrange multipliers is

$$\nabla U = \lambda \nabla g, \text{ or } \left\langle x_2^2 x_3^3, 2x_1 x_2 x_3^3, 3x_1 x_2^2 x_3^2 \right\rangle = \lambda \left\langle 1, 3, 5 \right\rangle.$$

Comparing components and including the constraint gives

$$\begin{aligned} x_2 x_3^3 &= \lambda \\ 2x_1 x_2 x_3^3 &= 3\lambda \\ 3x_1 x_2^2 x_3^2 &= 5\lambda \\ x_1 + 3x_2 + 5x_3 &= 100,000. \end{aligned}$$

Multiplying the first equation through by x_1, the second by x_2, and the third by x_3 gives

$$x_1 x_2^2 x_3^3 = \lambda x_1, \ 2x_1 x_2^2 x_3^3 = 3\lambda x_2, \ 3x_1 x_2^2 x_3^3 = 5\lambda x_3.$$

By comparing the left-hand sides here, we get that

$$\lambda 2x_1 = 3\lambda x_2 \quad \text{and} \quad \lambda 3x_1 = 5\lambda x_3.$$

If $\lambda = 0$, then from $x_2 x_3^3 = \lambda$ we that x_2 or x_3 would be zero, but then the utility function would have value 0, which is not the maximum (as we'll see) we are looking for. So we may as well assume $\lambda \neq 0$, and we can divide the equations above by λ to get

$$2x_1 = 3x_2 \quad \text{and} \quad 3x_1 = 5x_3.$$

The constraint then becomes

$$x_1 + 2x_1 + 3x_1 = 100,000, \text{ so } x_1 = \frac{100,000}{6}.$$

Then $x_2 = \frac{2}{3}x_1 = \frac{100,000}{9}$ and $x_3 = \frac{3}{5}x_1 = \frac{100,000}{10}$, so maximize utility with a budget of $100,000$ dollars we should produce $\frac{100,000}{6}$ units of product 1, $\frac{100,000}{9}$ units of product 2, and $\frac{100,000}{10}$ units of product 3.

(b) Now we take the constraint to be $p_1 x_1 + p_2 x_2 + p_3 x_3 = B$. The same process as above gives the gradient equation

$$\left\langle x_2^2 x_3^3, 2x_1 x_2 x_3^3, 3x_1 x_2^2 x_3^2 \right\rangle = \lambda p_1, p_2, p_3,$$

from which we get the equations

$$x_2 x_3^3 = \lambda p_1$$

$$\begin{aligned} 2x_1x_2x_3^3 &= \lambda p_2 \\ 3x_1x_2^2x_3^2 &= \lambda p_3 \\ p_1x_1 + p_2x_2 + p_3x_3 &= B. \end{aligned}$$

Multiplying the first equation by x_1, second by x_2, and third by x_3 then results in

$$2\lambda p_1x_1 = \lambda p_2x_2 \text{ and } 3\lambda p_1x_1 = \lambda p_3x_3$$

As before, we may assume that $\lambda \neq 0$ since otherwise the utility would be zero, so

$$2p_1x_1 = p_2x_2 \text{ and } 3p_1x_1 = p_3x_3.$$

The constraint then gives

$$6p_1x_1 = B, \text{ so } p_1x_1 = \frac{B}{6}.$$

This in turn gives $p_2x_2 = \frac{B}{3}$ and $p_3x_3 = \frac{B}{2}$. Thus, in order to maximize utility, we should devote half the budget towards producing product 3, one third towards product 2, and one sixth towards product 1.

(Note that the prices p_1, p_2, p_3 are irrelevant to this part, the budget allocations are the same regardless of price! Also, it makes sense intuitively that most of the budget should go towards product 3, since due to the third power in the utility function, the value of x_3 has the greatest impact on utility. The math makes practical sense!) □