Northwestern University

MATH 230-1 Final Exam
Spring Quarter 2022
June 7, 2022

Last name: ____________________ Email address: ____________________

First name: ____________________ NetID: ____________________

Instructions

- Mark your section.

Section	**Time**	**Instructor**	
41	10:00am	Hille	
51/53	11:00am	Hille	
61	12:00pm	Getzler	

- This examination consists of 18 pages, not including this cover page. Verify that your copy of this examination contains all 18 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 9 questions for a total of 110 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Answers without sufficient justification may not earn full credit.

1. Mark each statement **True or False**. You do not need to justify your answers (no partial credit will be awarded).

   (a) (2 points) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three distinct vectors in $\mathbb{R}^3$. The vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is always orthogonal to $\mathbf{a}$.

   ☐ True ☐ False

   (b) (2 points) The lines $\mathbf{r}_1(t) = \langle t, t, t \rangle$ and $\mathbf{r}_2(s) = \langle s+1, s, s \rangle$ intersect.

   ☐ True ☐ False

   (c) (2 points) The intersection of two distinct planes in Euclidean space $\mathbb{R}^3$ is either empty or a line.

   ☐ True ☐ False

   (d) (2 points) The intersection of the cone $x^2 + y^2 - z^2 = 1$ with the plane $x = 2$ is an ellipse.

   ☐ True ☐ False

(e) (2 points) Let $f(x, y, z)$ be a function such that $f(1, 1, 1) = 0$ and $\nabla f(1, 1, 1) = \langle 2, 2, 0\rangle$. Then, $\mathbf{r}(t) = \langle 1 + t, 1 - t, 1 + t\rangle$ is a line tangent to the level surface $f(x, y, z) = 0$ at $(1, 1, 1)$

☐ True ☐ False

(f) (2 points) The tangent plane to the surface $x^2 - z^3 = 0$ in $\mathbb{R}^3$ at $(1, 0, 1)$ is parallel to the xy-plane.

☐ True ☐ False

(g) (2 points) There is a function $f(x, y)$ whose graph has the tangent plane $x + y = 0$ at the point $(0, 0, f(0, 0))$.

☐ True ☐ False

(h) (2 points) The Hessian of a function $f(x, y)$ is zero at a critical point that is not a local extremum.

☐ True ☐ False

2. (a) (5 points) Let $\mathbf{r}(t)$ be the parametric curve $\mathbf{r}(t) = \langle 2, t^2, t \rangle$ for $2 \leq t \leq 3$. Parametrize the *line segment* $\mathbf{R}(s)$ joining the endpoints of the curve.

(b) (5 points) Show that a plane contains the lines $\mathbf{r}(t) = \langle t, t, t \rangle$ and $\mathbf{R}(s) = \langle s, 2s - 1, 4s - 3 \rangle$, and find an equation for it.

3. (10 points) Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + 4y^2}, & \text{if } (x,y) \neq 0, \\ 0, & \text{if } (x,y) = 0. \end{cases}$$

Is $f(x,y)$ continuous at $(0,0)$? Justify your answer.

4. Let S be the surface given by the equation

$$xy + xz + yz = -1.$$

(a) (5 points) Find the *line normal* to S at $(1, -1, -1)$.

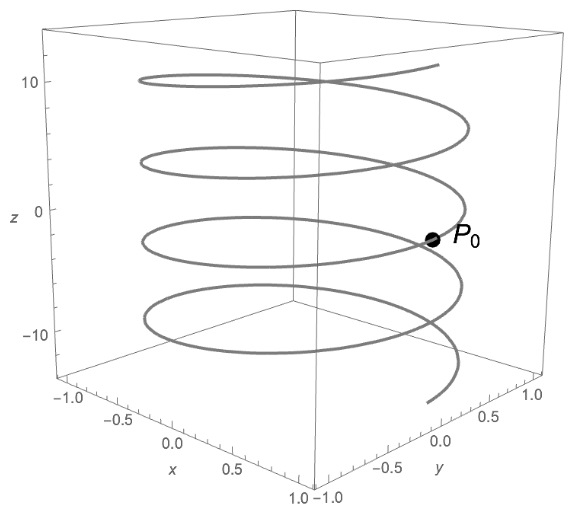
(b) (5 points) Find the tangent plane to S at $(1, -1, -1)$.

5. (10 points) Let C be the curve in $\mathbb{R}^3$ depicted below and parametrized by

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle, \quad t \in \mathbb{R}.$$

Find and sketch in the figure below the point on C at distance $5\pi/\sqrt{2}$ from the point $P_0 = (1, 0, 0)$ along the curve in the *negative direction*.

Hint: Sketch the direction of C induced by the parametrization.



6. Let $f(x, y) = (1 + x)\sin(x + y)$ and $(x_0, y_0) = (0, 0)$.

   (a) (5 points) Find the Taylor polynomial of degree 1 (i.e. the linearization) $T_1(x, y)$ of $f(x, y)$ at (x_0, y_0), and give an approximation for the value of $f(1/200, 1/200)$.

   Note: At this point, the question does *not* ask for a bound of the approximation error!

(b) (5 points) Find an upper bound for the error $|f(x, y) - T_1(x, y)|$ in the square

$$R = [-1/100, 1/100] \times [-1/100, 1/100]$$

(that is, the set $\{-1/100 \leq x \leq 1/100 \text{ and } -1/100 \leq y \leq 1/100\}$).

7. Let $f(x, y) = (x^4 - x^2)\sin(y)$ be the function with domain D equal to the rectangle $D = [0, 2] \times [0, 2\pi]$ (that is, the set $\{0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\pi\}$).

   (a) (8 points) Find the critical points of $f(x, y)$ in the *interior of its domain.* List the value of $f(x, y)$ at each critical point.

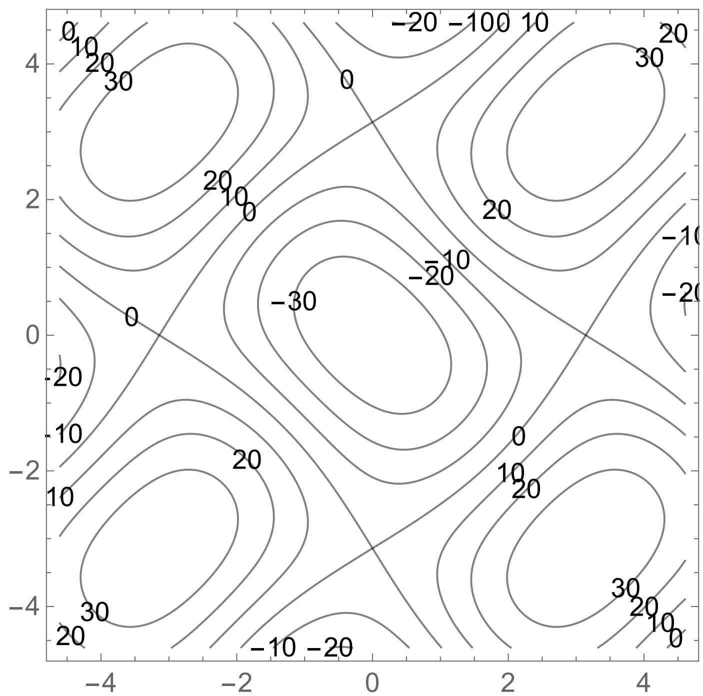
(b) (7 points) Find the global extrema (absolute maxima and minima) of $f(x, y)$ in D.

8. (10 points) Find all points on the ellipse $x^2 + 2y^2 = 4$ that are closest to the point $P_0 = (1/2, 0)$.

YOU CAN CONTINUE YOUR ANSWER FOR QUESTION 8 ON THE NEXT PAGE

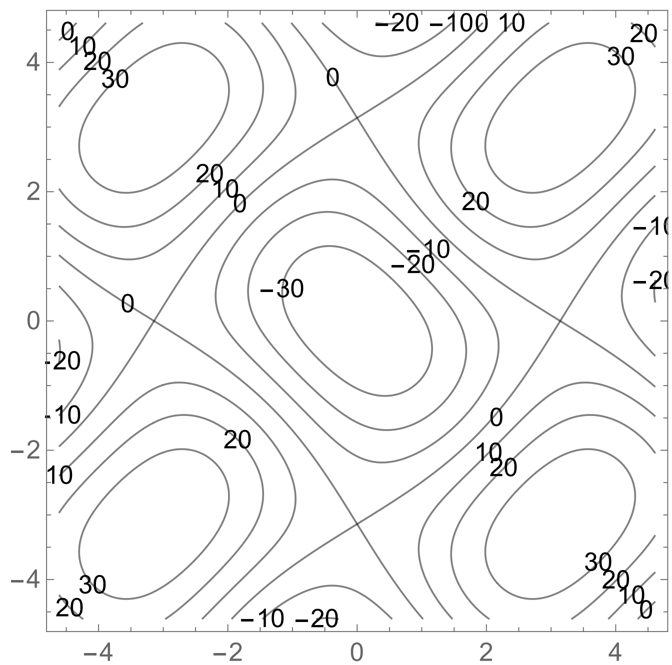
THIS PAGE IS FOR WORK ON QUESTION 8 ONLY

9. Below is the contour diagram of a function $f(x, y)$, for which all its first and second order partial derivatives exist and are continuous.

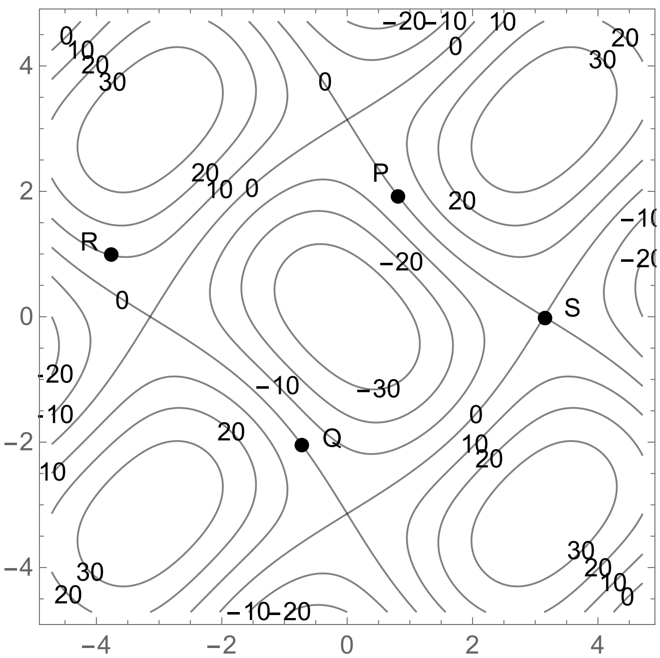


(a) (6 points) The contour diagram suggests a number of local maxima, local minima and saddle points. Identify on the diagram one local maximum, one local minimum and one saddle point. Justify your answer briefly.

(b) (4 points) All the critical points of the function $f(x, y)$ are nondegenerate (i.e. the Hessian of $f(x, y)$ is non-zero at any critical point). Use this information to identify two points A and B on the contour for which $f_{xx}(A) > 0$ and $f_{xx}(B) < 0$. Justify your answer.

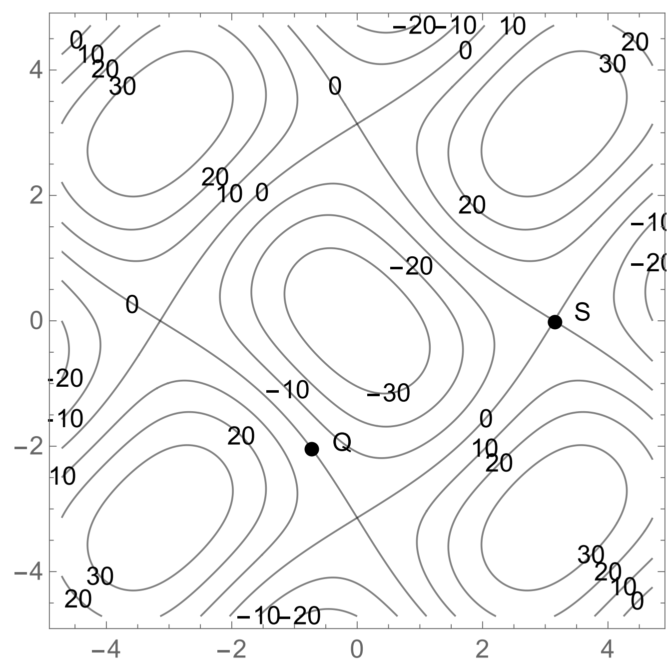


(c) (4 points) Four points P, Q, R and S are indicated in the contour diagram below. Let $\mathbf{v} = \langle 2, 1\rangle$. Order the directional derivatives $(D_{\mathbf{v}}f)(P)$, $(D_{\mathbf{v}}f)(Q)$, $(D_{\mathbf{v}}f)(R)$ and $(D_{\mathbf{v}}f)(S)$ in increasing order. Justify your answer.



(d) (5 points) Let Q and S be the same points as indicated previously. Let $\mathbf{u} = \overrightarrow{SQ}$ be the displacement vector from S in the direction of Q. Using the Mean Value Theorem, show that there is a point T on the line segment between S and Q for which $(D_{\mathbf{u}}(D_{\mathbf{u}}f))(T) > 0$.

Hint: Consider the restriction of $f(x, y)$ to the line passing through the point S in the direction of $\mathbf{u}$.



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