MATH 230-1: Discussion 7 Solutions
Northwestern University, Fall 2023

1. (a) Find an equation for the tangent plane to the surface

$$z - x^2y + xy^3 = 2$$

at the point $(2, 1, 4)$. It might help to find a function $z = f(x, y)$ whose graph is this surface.

(b) Use the tangent plane found in (a) to approximate the value of $(2.1)^2(0.8) - (2.1)(0.8)^3 + 2$.

(c) Find a bound on the error in approximation used in (b).

Solution. (a) Rewrite the given equation as $z = 2 + x^2y - xy^3$, so that the given surface is the graph of $z = f(x, y) = 2 + x^2y - xy^3$. We have

$$f_x = 2xy - y^3 \quad \text{and} \quad f_y = x^2 - 3xy^2,$$

so $f_x(2, 1) = 3$, $f_y(2, 1) = -2$. The tangent plane to the given surface at $(2, 1, 4)$ thus has equation

$$z = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 4 + 3(x - 2) - 2(y - 1).$$

(b) We want to approximate the value of $f(2.1, 0.8)$, so the linear approximation is

$$\begin{aligned} f(2.1, 0.8) &\approx 4 + 3(2.1 - 2) - 2(0.8 - 1) \\ &= 4 + 3(0.1) - 2(-0.2) \\ &= 4 + 0.3 + 0.4 \\ &= 4.7. \end{aligned}$$

Since $(2.1, 0.8)$ is fairly close to $(2, 1)$, we expect that $f(2.1, 0.8) \approx 4.7$ is a fairly OK approximation.

(c) We have

$$f_{xx} = 2y, \; f_{xy} = 2x - 3y^2, \; f_{yy} = -6xy.$$

For $2 \le x \le 2.1$ and $0.8 \le y \le 1$, we thus have that

$$|f_{xx}| = 2y \le 2, \; |f_{xy}| = |2x - 3y^2| \le 2x + 3y^2 \le 2(2.1) + 3 = 7.2, \; |f_{yy}| = 6xy \le 6(2.1) = 12.6.$$

Thus $M = 12.6$ is a bound on all second-order partial derivatives over the range $2 \le x \le 2.1$, $0.8 \le y \le 1$, so the error in the approximation $f(2.1, 0.8) \approx 4.7$ is no worse than

$$\tfrac{1}{2}M(|\Delta x| + |\Delta y|)^2 = \tfrac{1}{2}(12.6)(0.1 + 0.2)^2 = 6.3(0.3)^2 = 0.567.$$

(If you use a calculator, you get that $(2.1)^2(0.8) - (2.1)(0.8)^3 + 2 = 4.4528$, so the approximation 4.7 we got in (b) is indeed within 0.567 of the actual value!) □

2. (a) Find and classify the critical points of $f(x, y) = x^2 + y^2 + xy^2$.

(b) Find the absolute maximum and absolute minimum of $f(x, y) = x^2 + y^2 + xy^2$ among points (x, y) in the rectangle $-1 \le x \le 1, -1 \le y \le 1$. How do you know such absolute extrema exist before you find them?

Solution. (a) We have $\nabla f = \langle 2x + y^2, 2y + 2xy \rangle$, so critical points occur when

$$2x + y^2 = 0 \quad \text{and} \quad 2y + 2xy = 0.$$

The second equation gives $2y(1 + x) = 0$, so $y = 0$ or $x = -1$. If $y = 0$, the first equation becomes $2x = 0$, so $x = 0$ and hence $(0, 0)$ is a critical point. If $x = -1$, the first equation becomes $-2 + y^2 = 0$, so $y = \pm\sqrt{2}$ and hence $(-1, \sqrt{2})$ and $(-1, -\sqrt{2})$ are critical points.

We have $f_{xx} = 2, f_{xy} = 2y, f_{yy} = 2x$, so

$$\begin{aligned} \text{at } (0,0), &\quad f_{xx}f_{yy} - (f_{xy})^2 = 2(0) - 0 = 0 \\ \text{at } (-1,\sqrt{2}), &\quad f_{xx}f_{yy} - (f_{xy})^2 = 2(-1) - (2\sqrt{2})^2 < 0 \\ \text{at } (-1,-\sqrt{2}), &\quad f_{xx}f_{yy} - (f_{xy})^2 = 2(-1) - (-2\sqrt{2})^2 < 0. \end{aligned}$$

Thus $(-1, \sqrt{2})$ and $(-1, -\sqrt{2})$ are saddle points, and the second derivative test is inconclusive at $(0, 0)$. However, since

$$f(x, y) = x^2 + y^2(1 + x)$$

is positive for $(x, y) \neq (0, 0)$ close to the origin ($1 + x$ is positive even if x is negative but closer to 0), we see that $f(0, 0) = 0$ is a local minimum value. (On the final you can expect that there will only be such problems where the second derivative test *is* always applicable.)

(c) The critical point $(0, 0)$ found in (a) lies within the given rectangle, but the other critical points $(-1, \pm\sqrt{2})$ do not, so we ignore these latter two. Along either the edge $y = -1$ or $y = 1$ we have

$$f(x, \pm 1) = x^2 + 1 + x, \text{ with derivative } 2x + 1.$$

This single-variable function thus has a critical point at $x = -1/2, y = \pm 1$. Along the edge $x = 1$ we have

$$f(1, y) = 1 + y^2 + y^2 = 1 + 2y^2, \text{ with derivative } 2y.$$

Thus we have a single-variable critical point at $y = 0, x = 1$ along this edge. Along the edge $x = -1$ we have

$$f(-1, y) = 1 + y^2 - y^2 = 1$$

and all points on $x = -1$ are single-variable critical points.

Thus have $(0, 0), (-1/2, \pm 1), (1, 0), (-1, y)$ and the endpoints $(-1, -1), (-1, 1), (1, -1), (1, 1)$ of the various boundary segments as candidates at which the absolute extrema occur. Plugging everything in gives

$$f(-1, y) = 1,\ f(0, 0) = 0,\ f(-1/2, \pm 1) = \tfrac{7}{4},\ f(1, 0) = 1,\ f(1, \pm 1) = 3,\ f(-1, \pm 1) = 1.$$

Thus the absolute maximum value of $f(x, y)$ within the given rectangle is 3 and occurs at $(1, \pm 1)$, and the absolute minimum value is 0 and occurs at 0. □

3. An aquarium in the shape of an open rectangular box without a top is to hold 81 cubic feet of water and is to be built using slate for the rectangular base and glass for the sides. Slate costs \$12 per square foot and glass costs \$2 per square foot. Find the dimensions of the aquarium which minimize the cost.

Solution. If x and y denote the width and length of the base of the aquarium and z the height of the sides, then the function giving the total cost is

$$C(x, y, z) = 12xy + 4xz + 4yz.$$

(The 4's come from the fact that we have two sides of area xz and two of area yz.) Thus, we want to optimize this function subject to the volume constraint $V = xyz = 81$. The method of Lagrange multipliers says that the points at which the optimal values occur satisfy

$$\nabla C = \lambda \nabla V \rightsquigarrow \langle 12y + 4z, 12x + 4z, 4x + 4y \rangle = \lambda \langle yz, xz, xy \rangle .$$

After we compare components on both sides and take the constraint into account, we see that the points we want satisfy

$$\begin{aligned} 12y + 4z &= \lambda yz \\ 12x + 4z &= \lambda xz \\ 4x + 4y &= \lambda xy \\ xyz &= 81. \end{aligned}$$

Multiplying the first equation through by x and the second through by y gives the same right-hand sides, so the resulting left-hand sides must be equal:

$$12yx + 4xz = 12xy + 4yz \rightsquigarrow 4xz = 4yz.$$

Since $z = 0$ does not satisfy the constraint, z must be nonzero and hence $4xz = 4yz$ gives $x = y$. Multiplying the second equation through by y and the third through by z implies that

$$12xy + 4yz = 4xz + 4yz \rightsquigarrow 12xy = 4xz.$$

We may assume x is nonzero since $x = 0$ does not satisfy the constraint, so $12xy = 4xz$ implies that $3y = z$. Hence with $x = y$ and $z = 3y$, the constraint becomes

$$yy(3y) = 81, \text{ so } y^3 = 27$$

and thus $y = 3$. Then we get $x = y = 3$ and $z = 3y = 9$, so the dimensions that optimize the cost subject to the given constraint are $x = 3, y = 3, z = 9$. These values minimize the cost since there can be no maximum: if we keep y fixed, take $x \to \infty$ and pick z so as to maintain the constraint $xyz = 81$, we see that the $12xy$ term in the cost can be made larger and larger, so no maximal cost subject to this constraint can exist. $\square$