MATH 230-1: Discussion 6 Problems
Northwestern University, Fall 2023

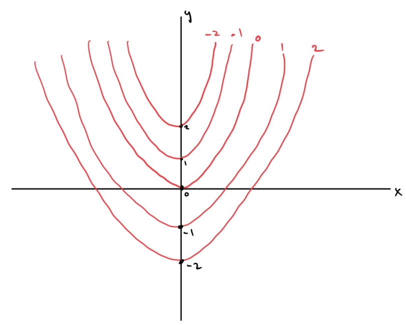
1. Let $f(x, y) = 2x^2 - y$.

(a) Draw the level curves of f at $z = -2, -1, 0, 1, 2$.

(b) Based on the level curves, explain why it makes sense that $f_x(0, 0)$ is zero, $f_y(0, 0)$ is negative, and $f_{yy}(0, 0)$ is zero.

(c) In which directions at $(3, -2)$ should we face so that the graph of f is as steep (either upward or downward) as possible?

Solution. (a) The level curves look like



(b) The value of f at $(0, 0)$ is 0. As we move away from $(0, 0)$ to the left or to the right (only horizontally) the value f increases as indicated by the value at which these level curves to the right and left occur at. Thus $f(0, 0) \leq f(x, 0)$ for x near $= 0$, so $(0, 0)$ is a local minimum of f in the x-direction, so the rate of change $f_x(0, 0)$ in this direction is zero. As we move vertically through $(0, 0)$ and consider points of the form $(0, y)$, we see that the value of f decreases as we do so, so the rate of change $f_y(0, 0)$ in this direction is negative.

The value of $f_y(0, 0)$ is negative, as is the value of $f_y(0, y)$ for y a bit above $y = 0$ and a bit below as well. However, these negative slopes remain the same based on the fact the level curves moving vertically through $(0, 0)$ do not get closer to one another: the space between the level curves at $z = 2$ and $z = 1$ is the same as that between $z = 1$ and $z = 0$, and between $z = 0$ and $z = -1$. This means that the value of $f_(0, y)$ remains constant as y varies, so $f_yy(0, 0)$, which is the derivative with respect to y of $f_y(0, y)$ at $y = 0$, is zero.

(c) The direction of steepest positive slope is given by the gradient of f at $(3, -2)$. Since $\nabla f = \langle f_x, f_y \rangle = \langle 4x, -1 \rangle$, we have $\nabla f(3, -2) = \langle 12, -1 \rangle$. Thus $\langle 12, -1 \rangle$ gives the direction of steepest positive (upward) slope, and $-\nabla f(3, -2) = \langle -12, 1 \rangle$ gives the direction of steepest negative (downward) slope. □

2. (a) The water temperature, in polar coordinates, at a point (r, θ) in a lake is given by

$$T(r, \theta) = r^3 \sin(\theta/2)$$

in degrees celsius. Suppose a swimmer moves along some path in the lake, and that at the point $(r, \theta) = (2, \frac{\pi}{2})$ experiences a change in water temperature with respect to time of 10 degrees celsius per second. If at this instant the value of θ is changing at a rate of $\frac{\pi}{2}$ radians per second, find the rate at which the value of r is changing with respect to time. Your answer can be left unsimplified.

(b) In which Cartesian direction is T increasing most rapidly at the Cartesian point with polar coordinates $(r, \theta) = (2, \frac{\pi}{2})$?

Solution. (a) With T depending on r and θ and r and θ depending on time t, the chain rule gievs

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial r}\frac{dr}{dt} + \frac{\partial T}{\partial \theta}\frac{d\theta}{dt} \\ &= 3r^2\sin(\theta/2)\frac{dr}{dt} + \frac{1}{2}r^3\cos(\theta/2)\frac{d\theta}{dt}.\end{aligned}$$

When $r = 2$ and $\theta = \frac{\pi}{2}$, we have $\frac{dT}{dt} = 10$ and $\frac{d\theta}{dt} = \frac{\pi}{2}$, so

$$10 = 3(2)^2\sin(\pi/4)\frac{dr}{dt} + \tfrac{1}{2}(2)^3\cos(\pi/4)(\tfrac{\pi}{2}) = 6\sqrt{2}\tfrac{dr}{dt} + \pi\sqrt{2}.$$

Thus at this point we have that the radius is changing at a rate of

$$\frac{dr}{dt} = \frac{10 - \pi\sqrt{2}}{6\sqrt{2}}$$

with respect to time.

(b) The Cartesian point with polar coordinates $(r, \theta) = (2, \frac{\pi}{2})$ is $(x, y) = (2\cos\frac{\pi}{2}, 2\sin\frac{\pi}{2}) = (0, 2)$. The direction in which T increases most rapidly at this point is $\nabla T(0, 2)$, so we must compute T_x and T_y. Since T depends on r and θ and r and θ each depend on x and y, the chain rule gives (using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$)

$$\begin{aligned}\frac{\partial T}{\partial x} &= \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= 3r^2\sin(\theta/2)\frac{2x}{\sqrt{x^2+y^2}} + \frac{1}{2}r^3\cos(\theta/2)\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} \\ &= 3r^2\sin(\theta/2)\frac{2x}{\sqrt{x^2+y^2}} + \frac{1}{2}r^3\cos(\theta/2)\frac{-y}{x^2+y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial T}{\partial y} &= \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= 3r^2\sin(\theta/2)\frac{2y}{\sqrt{x^2+y^2}} + \frac{1}{2}r^3\cos(\theta/2)\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} \\ &= 3r^2\sin(\theta/2)\frac{2y}{\sqrt{x^2+y^2}} + \frac{1}{2}r^3\cos(\theta/2)\frac{x}{x^2+y^2}.\end{aligned}$$

(We use the fact that $\frac{d}{du}\arctan u = \frac{1}{1+u^2}$.) Evaluating at $(r, \theta) = (2, \frac{\pi}{2})$ and $(x, y) = (0, 2)$ we get

$$\frac{\partial T}{\partial x} = 6\sqrt{2}\left(\frac{0}{\sqrt{0+4}}\right) + 2\sqrt{2}\left(-\frac{2}{0+4}\right) = -\sqrt{2}$$

and

$$\frac{\partial T}{\partial y} = 6\sqrt{2}\left(\frac{4}{0+4}\right) + 2\sqrt{2}\left(\frac{0}{0+4}\right) = 6\sqrt{2},$$

so $\langle -\sqrt{2}, 6\sqrt{2}\rangle$ gives the Cartesian direction in which the value of T increases most rapidly as the given point. □

3. Let $f(x,y,z) = xy^2z^3 + xy - 3ye^{yz}$.

(a) Compute $\nabla f(x,y,z)$.

(b) Give two directions in which the rate of change of f at the point $(1,2,0)$ is zero.

(c) Find an equation for the tangent plane to the surface

$$xy^2z^3 + xy - 3ye^{yz} = -4$$

at the point $(1,2,0)$.

Solution. (a) We have

$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle y^2z^3 + y, 2xyz^3 + x - 3e^{yz} - 3yze^{yz}, 3xy^2z^2 - 3y^2e^{yz} \right\rangle.$$

(b) At the point $(1,2,0)$ we get

$$\nabla f(1,2,0) = \langle 2, 1-3-6, -12 \rangle = \langle 2, -8, -12 \rangle.$$

Directions in which the rate of change at $(1,2,0)$ is zero are given by unit vectors $\mathbf{u}$ for which $D_{\mathbf{u}} f(1,2,0) = \nabla f(1,2,0) \cdot \mathbf{u} = 0$, so we need only find two nonzero vectors that are orthogonal to $\nabla f(1,2,0) = \langle 2,-8,-12 \rangle$. One such direction is $\langle 8,2,0 \rangle$, and another is $\langle 0,12,-8 \rangle$.

(c) Note first that $f(1,2,0) = -4$, so $(1,2,0)$ does lie on the given surface. We can view this surface as the level surface of f at -4, so the gradient vector $\nabla f(1,2,0)$ is orthogonal to this surface at $(1,2,0)$. Hence we take $\nabla f(1,2,0) = \langle 2,-8,-12 \rangle$ as a normal vector for the desired tangent plane. The tangent plane is thus given by

$$2(x-1) - 8(y-2) - 12z = 0.$$

□