Northwestern University

MATH 230-1 Final Examination
Winter Quarter 2022
March 14, 2022

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- Mark your section.

Section	**Time**	**Instructor**	
41	10:00	Bentsen	
51	11:00	Cuzzocreo	
61	12:00	Cuzzocreo	

- This examination consists of 18 pages, not including this cover page. Verify that your copy of this examination contains all 18 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 10 questions for a total of 100 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Clearly circle final answers.
- Use pages at the back for scratchwork if needed.
- Show all of your work and justify answers completely, unless otherwise indicated. Unsupported answers may not earn credit.

1. For each problem, circle either **True** or **False**. You do not need to justify your answer.

(a) (2 points) The function $f(x, y) = x^4 - y^6$ has a saddle point at $(0, 0)$.

$\boxed{\textbf{True}}$ **False**

(b) (2 points) If nonzero vectors **a** and **b** are orthogonal, then the scalar projection of **a** onto **b** must be 0.

$\boxed{\textbf{True}}$ **False**

(c) (2 points) Each trace of a hyperboloid in each direction is a hyperbola.

True $\boxed{\textbf{False}}$

(d) (2 points) Let $f(x, y)$ be a continuous function with continuous first partial derivatives on all of $\mathbb{R}^2$, and let (a, b) be a point. Let $\mathbf{u} = \frac{1}{|\vec{\nabla} f(a,b)|}\vec{\nabla} f(a, b)$. Then

$$D_{\mathbf{u}} f(a, b) = |\vec{\nabla} f(a, b)|.$$

$\boxed{\textbf{True}}$ **False**

For each problem, circle either **True** or **False**. You do not need to justify your answer.

(e) (2 points) If a function f attains a local maximum at a point P_0 in a closed, bounded region R, then f must attain an absolute maximum on R at P_0.

True $\boxed{\textbf{False}}$

(f) (2 points) The curve defined in polar coordinates by the equation $r = 2\sin\theta$ is a circle.

$\boxed{\textbf{True}}$ **False**

(g) (2 points) If $\mathbf{r}(t)$ for $a \leq t \leq b$ parameterizes the curve $\mathcal{C}$ with respect to arc length, then the length of $\mathcal{C}$ is $b - a$.

$\boxed{\textbf{True}}$ **False**

(h) (2 points) The two lines parameterized as $\mathbf{r}_1(t) = \mathbf{a}_1 + t\mathbf{v}_1$ and $\mathbf{r}_2(t) = \mathbf{a}_2 + t\mathbf{v}_2$ intersect if and only if there exists a value $t = t_0$ such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0)$.

True $\boxed{\textbf{False}}$

2. Let $\mathbf{u} = \langle 3, -1, 1\rangle, \mathbf{v} = \langle 0, 0, -2\rangle, \mathbf{w} = \langle a, b, 1\rangle$.

(a) (3 points) Compute the vector projection $\text{proj}_{\mathbf{u}}\mathbf{v}$. Show all work and justify your answer completely.

$$\begin{aligned}\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||^2}\mathbf{u}\\ &= \frac{-2}{11}\langle 3, -1, 1\rangle\\ &= \boxed{\left\langle -\frac{6}{11}, \frac{2}{11}, -\frac{2}{11}\right\rangle}\end{aligned}$$

(b) (3 points) Find all real values of the constants a and b such that both of the following conditions hold:

- $\mathbf{w}$ is orthogonal to $\mathbf{u}$
- $||\mathbf{w}\times\mathbf{v}|| = 2$.

Show all work and justify your answer completely.

Note if $\mathbf{w}$ is orthogonal to $\mathbf{u}$, then we have $3a - b + 1 = 0$, or

$$b = 3a + 1.$$

Then compute

$$\begin{aligned}\mathbf{w}\times\mathbf{v} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & 1\\ 0 & 0 & -2\end{vmatrix}\\ &= (-2b)\mathbf{i} - (-2a)\mathbf{j} + 0\mathbf{k}\\ &= \langle -2b, 2a, 0\rangle\end{aligned}$$

Then $||\mathbf{w}\times\mathbf{v}|| = 2$ implies $\sqrt{4a^2 + 4b^2} = 2$ or

$$a^2 + b^2 = 1$$

. Replacing b by $3a + 1$ gives $a^2 + 9a^2 + 6a + 1 = 1$, or

$$10a^2 + 6a = 0.$$

This has solutions $a = 0$ and $a = -\frac{3}{5}$. Hence we must have

$$\boxed{a = 0 \text{ and } b = 1 \qquad \text{or} \qquad a = -\frac{3}{5} \text{ and } b = -\frac{4}{5}}$$

3. Let $P = (1, 1, 2)$, and let ℓ be the line parametrized by $\mathbf{r}(t) = \langle 1 + t, 3 - 2t, 3t \rangle$.

(a) (4 points) Find the distance between P and ℓ. Show all work and justify your answer completely.

We first find a point on ℓ, $S = (1, 3, 0)$. We also find the direction vector of ℓ, $\mathbf{v} = \langle 1, -2, 3 \rangle$. We then recall that the distance between P and ℓ is

$$
\begin{aligned}
d &= \frac{||\overrightarrow{PS} \times \mathbf{v}||}{||\mathbf{v}||} \\
&= \frac{||\langle 0, 2, -2 \rangle \times \langle 1, -2, 3 \rangle||}{||\langle 1, -2, 3 \rangle||} \\
&= \frac{\left|\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -2 \\ 1 & -2 & 3 \end{vmatrix}\right|\right|}{||\langle 1, -2, 3 \rangle||} \\
&= \frac{||2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}||}{||\langle 1, -2, 3 \rangle||} \\
&= \boxed{\frac{\sqrt{12}}{\sqrt{14}}} \\
&= \boxed{\sqrt{\frac{6}{7}}}
\end{aligned}
$$

(b) (4 points) Give an equation for the plane containing P and ℓ. Show all work and justify your answer completely.

This can be given as the plane containing P with normal vector equal to the vector $\overrightarrow{PS} \times \mathbf{v}$ found above, with $S = (1, 3, 0)$ lying on ℓ and $\mathbf{v} = \langle 1, -2, 3 \rangle$ the direction vector of ℓ. We thus have

$$\mathbf{n} = \overrightarrow{PS} \times \mathbf{v} = \langle 0, 2, -2 \rangle \times \langle 1, -2, 3 \rangle = \langle 2, -2, -2 \rangle$$

The plane is then given by

$$\boxed{2(x - 1) - 2(y - 1) - 2(z - 2) = 0}$$

or

$$\boxed{x - y - z = -2}$$

4. For $t > 0$ the smooth curve $\mathcal{C}$ is parametrized as

$$\mathbf{r}(t) = \left\langle \frac{t^2}{2},\ 2t,\ \frac{4}{3}t^{3/2} \right\rangle.$$

(a) (5 points) Find the equation of the tangent line to $\mathcal{C}$ at the point $\left(\frac{1}{2},\ 2,\ \frac{4}{3}\right)$. Show all work and justify your answer completely.

The given point corresponds to $t = 1$. The tangent vector is then given as $\mathbf{r}'(1)$:

$$\begin{aligned} \mathbf{r}(t) &= \left\langle \frac{t^2}{2},\ 2t,\ \frac{4}{3}t^{3/2} \right\rangle \\ \mathbf{r}'(t) &= \langle t, 2, 2\sqrt{t} \rangle \\ \mathbf{r}'(1) &= \langle 1, 2, 2 \rangle \end{aligned}$$

The line is then given in parametric form as

$$\boxed{\begin{aligned} x &= \frac{1}{2} + t \\ y &= 2 + 2t \\ z &= \frac{4}{3} + 2t \end{aligned}}$$

(b) (5 points) Find the arc length of $\mathcal{C}$ between $t = 1$ and $t = 2$. Show all work and justify your answer completely.

The arc length is given as the integral

$$\begin{aligned} L &= \int_1^2 |\mathbf{r}'(t)|\, dt = \int_1^2 |\langle t, 2, 2\sqrt{t} \rangle|\, dt \\ &= \int_1^2 \sqrt{t^2 + 4 + 4t}\, dt = \int_1^2 \sqrt{(t+2)^2}\, dt \\ &= \int_1^2 t + 2\, dt = \frac{t^2}{2} + 2t \Big]_1^2 \\ &= (2 + 4) - (\tfrac{1}{2} + 2) \\ &= \boxed{\frac{7}{2}} \end{aligned}$$

5. Let

$$f(u,v) = u\cos(2v) - ve^{3u},$$
$$u = g(x,y,z) \quad \text{and} \quad v = h(x,y,z).$$

Suppose the values of g, h and their partial derivatives at the point $(x,y,z) = (0,0,0)$ are given in the chart below.

$g(0,0,0) = 1$	$h(0,0,0) = 0$
$g_x(0,0,0) = -1$	$h_x(0,0,0) = 2$
$g_y(0,0,0) = 6$	$h_y(0,0,0) = 0$
$g_z(0,0,0) = 1$	$h_z(0,0,0) = -1$

Let $F(x,y,z) = f(g(x,y,z), h(x,y,z))$. Answer the following, and show all work and justify your answer completely.

(a) (6 points) Find

i. $\left.\frac{\partial F}{\partial x}\right|_{(x,y,z)=(0,0,0)}$

Note that when $x = y = z = 0$, $u = g(0,0,0) = 1$ and $v = h(0,0,0) = 0$.

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &= (\cos(2v) - 3ve^{3u})(-1) + (-2u\sin(2v) - e^{3u})(2)\\ &= (-1-0)(-1) + (0 - e^3)(2)\\ &= \boxed{1 - 2e^3}\end{aligned}$$

ii. $\left.\frac{\partial F}{\partial y}\right|_{(x,y,z)=(0,0,0)}$

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\\ &= (\cos(2v) - 3ve^{3u})(6) + (-2u\sin(2v) - e^{3u})(0)\\ &= (1-0)(6)\\ &= \boxed{6}\end{aligned}$$

iii. $\left.\frac{\partial F}{\partial z}\right|_{(x,y,z)=(0,0,0)}$

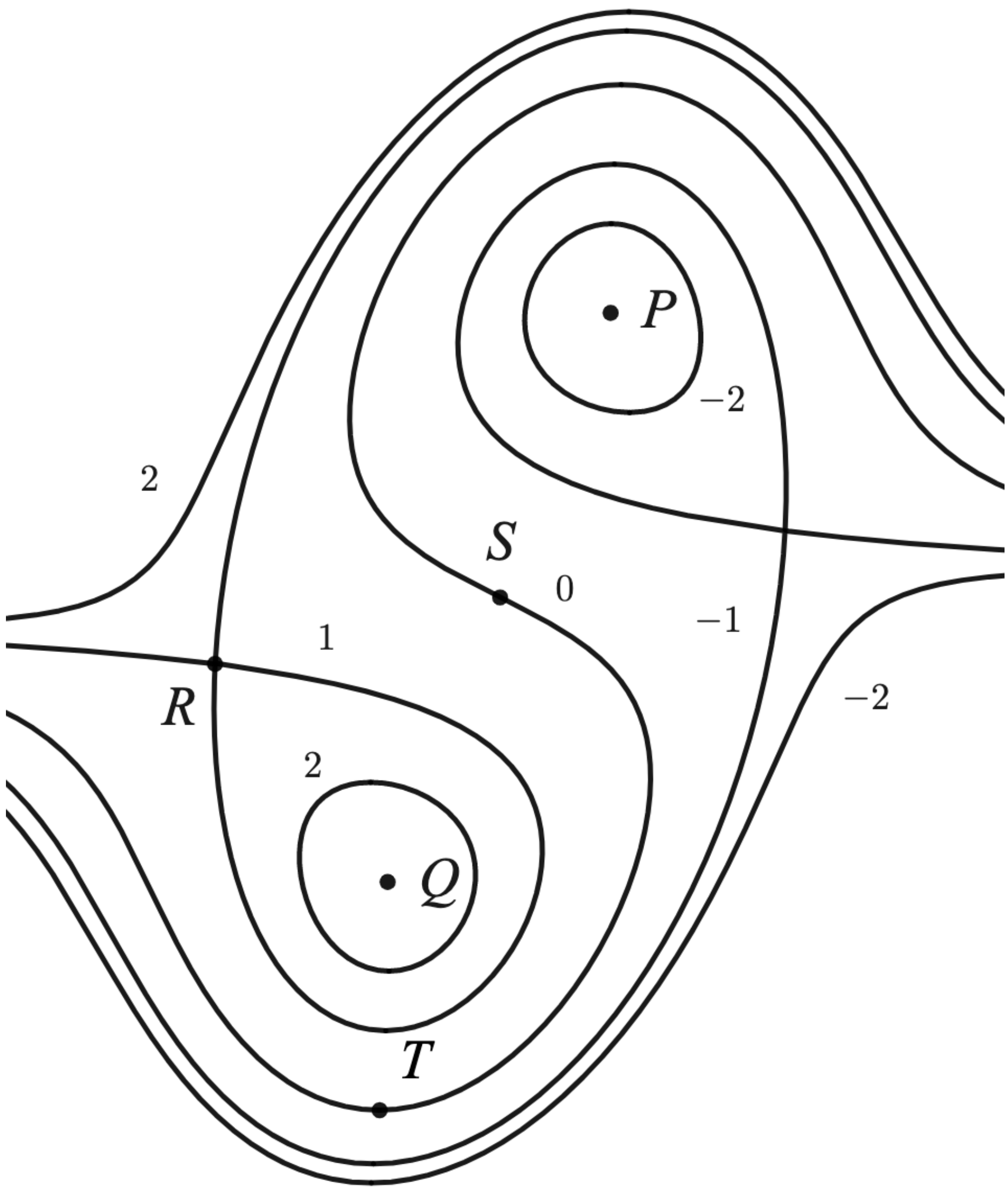
$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\\ &= (\cos(2v) - 3ve^{3u})(1) + (-2u\sin(2v) - e^{3u})(-1)\\ &= (1-0)(1) + (0 - e^3)(-1)\\ &= \boxed{1 + e^3}\end{aligned}$$

(b) (4 points) Find the equation of the tangent plane to the level surface $F(x, y, z) = 1$ at $(x, y, z) = (0, 0, 0)$.

From (a), the gradient of F at $(0, 0, 0)$ is $\langle -3, 6, 2\rangle$. The plane is then

$$\boxed{(1 - 2e^3)x + 6y + (1 + e^3) = 0}$$

6. The function $f(x, y)$ has level curves shown below. The numbers represent the values of f on each level curve, and P, Q, R, S, and T are points in the xy-plane. Following the usual convention we take the x-axis to be horizontal and the y-axis to be vertical. Assume that f and all of its partial derivatives (including its second partial derivatives) are continuous everywhere.



The questions below refer to the contour map on the previous page. Circle the correct answer. There is only one correct answer for each problem. You do not need to justify your answer.

(a) (2 points) Which of the following could be the vector $\overrightarrow{\nabla} f(T)$?

(A) $\langle 1, 1\rangle$ (B) $\langle -1, 1\rangle$ $\boxed{\text{(C) } \langle 0, 1\rangle}$ (D) $\langle -1, 0\rangle$

(b) (2 points) Assume that the coordinates of R are $(-3, -2)$. Which of the following could be an equation of the tangent plane to the graph of f at R?

(A) $(x + 3) + (y + 2) = 0$

(B) $z - (x + 3) - (y + 2) = 1$

(C) $z + (x + 3) + (y + 2) = 0$

$\boxed{\text{(D) } z - 1 = 0}$

(c) (2 points) At which point could $f_x < 0$ be satisfied?

(A) T (B) Q (C) P $\boxed{\text{(D) } S}$

(d) (2 points) At which point could f attain a local minimum?

$\boxed{\text{(A) } P}$ (B) Q (C) R (D) S

(e) (2 points) For which of the following unit vectors $\mathbf{u}$ is $D_{\mathbf{u}} f(T) < 0$?

(A) $\mathbf{u} = \langle 1, 0\rangle$ (B) $\mathbf{u} = \langle -1, 0\rangle$ $\boxed{\text{(C) } \mathbf{u} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle}$ (D) All of the above

7. Let $f(x, y) = x^3 - y^2$.

(a) (3 points) Find the linearization $L(x, y)$ for $f(x, y)$ at the point $(x, y) = (1, 2)$.

First compute

$$\begin{aligned} f(x,y) &= x^3 - y^2 & f(1,2) &= -3 \\ f_x(x,y) &= 3x^2 & f_x(1,2) &= 3 \\ f_y(x,y) &= -2y & f_y(1,2) &= -4 \end{aligned}$$

We then have

$$L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = \boxed{-3 + 3(x-1) - 4(y-2)}$$

(b) (4 points) Find a reasonable upper bound for the error $|E(x, y)| = |f(x, y) - L(x, y)|$ in this approximation if (x, y) lies in the rectangle defined by $|x - 1| \leq 0.1$ and $|y - 2| \leq 0.2$. Show all work and justify your answer completely.

We need the second partial derivatives of f:

$$\begin{aligned} f_{xx}(x,y) &= 6x & f(1,2) &= 6 \\ f_{xy}(x,y) &= 0 & f_x(1,2) &= 0 \\ f_{yy}(x,y) &= -2 & f_y(1,2) &= -2 \end{aligned}$$

We can then take $M = 6(1.1) = 6.6$ to be the upper bound on the absolute values of all second aprtial derivatives on the rectangle. We then have

$$|E(x,y) \leq \frac{M}{2}(|x-1| + |y-2|)^2 = 3.3(0.3)^2 = \boxed{3.3(0.09) < 0.33}$$

(c) (3 points) Find the quadratic approximation $Q(x, y)$ for $f(x, y)$ at the point $(x, y) = (1, 2)$.

Given the second partials computed above, we have

We then have

$$\begin{aligned} Q(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &\quad + \frac{1}{2} f_{xx}(1,2)(x-1)^2 + f_{xy}(1,2)(x-1)(y-2) + \frac{1}{2} f_{yy}(1,2)(y-2)^2 \\ &= \boxed{-3 + 3(x-1) - 4(y-2) + 3(x-1)^2 - (y-2)^2} \end{aligned}$$

8. (10 points) Let

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine whether f is a continuous function on $\mathbb{R}^2$ or not. In either case, give a one or two sentence explanation of your answer. Show all work and justify your answer completely.

We will show that f is not continuous on $\mathbb{R}^2$ by showing that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

Let $\mathbf{r}(t) = \langle at^3, bt \rangle$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} \\ &= \lim_{t\to 0} \frac{at^3(bt)^3}{(at^3)^2+(bt)^6} \\ &= \lim_{t\to 0} \frac{ab^3t^6}{(a^2+b^6)t^6} \\ &= \frac{ab^3}{a^2+b^6} \end{aligned}$$

This limit is clearly a non-constant function of a and b, and hence the limit is not independent of the path along which we approach the origin.

Thus the limit does not exist, and the function is cannot be continuous.

9. (10 points) Find the absolute maximum and minimum values of the function

$$f(x, y, z) = x - 2y - z$$

in the region defined by $x^2 + 4y^2 + 2z^2 \le 36$.

Clearly also indicate *at which points* these extrema are attained. Show all work and justify your answer completely.

We first look for critical points of f in the interior of the region. But we notice that

$$\nabla f(x, y, z) = \langle 1, -2, -1 \rangle$$

which is never 0, and hence f has no critical points at all.

We then use Lagrange Multipliers to find the extrema of f on the boundary of the region, given by the constraint equation $g(x, y, z) = x^2 + 4y^2 + 2z^2 = 36$.

$$\nabla g(x, y, z) = \langle 2x, 8y, 4z \rangle$$

and so the Lagrange Multipliers method gives us the system:

$$\begin{aligned} 1 &= 2\lambda x && (1) \\ -2 &= 8\lambda y && (2) \\ -1 &= 4\lambda z && (3) \\ x^2 + 4y^2 + 2z^2 &= 36 && (4) \end{aligned}$$

Multiplying the first equation by $4y$, the second by x, and subtracting gives $4y + 2x = 0$ or

$$x = -2y$$

Multiplying the second equation by z, the third by $2y$, and subtracting gives $-2z + 2y = 0$ or

$$z = y$$

Plugging $x = -2y$ and $z = y$ into the constraint equation gives

$$\begin{aligned} x^2 + 4y^2 + 2z^2 &= 36 \\ (-2y)^2 + 4y^2 + 2y^2 &= 36 \\ 10y^2 &= 36 \\ y &= \pm\frac{6}{\sqrt{10}} \end{aligned}$$

We then obtain $x = \mp\frac{12}{\sqrt{10}}$ and $z = \pm\frac{6}{\sqrt{10}}$. The extrema then occur at the points

$$\left(\frac{12}{\sqrt{10}}, -\frac{6}{\sqrt{10}}, -\frac{6}{\sqrt{10}}\right) \qquad \text{and} \qquad \left(-\frac{12}{\sqrt{10}}, \frac{6}{\sqrt{10}}, \frac{6}{\sqrt{10}}\right)$$

We then compute

$$f\left(\frac{12}{\sqrt{10}}, -\frac{6}{\sqrt{10}}, -\frac{6}{\sqrt{10}}\right) = \left(\frac{12}{\sqrt{10}}\right) - 2\left(-\frac{6}{\sqrt{10}}\right) - \left(-\frac{6}{\sqrt{10}}\right) = \frac{30}{\sqrt{10}} = 3\sqrt{10}$$
$$f\left(-\frac{12}{\sqrt{10}}, \frac{6}{\sqrt{10}}, \frac{6}{\sqrt{10}}\right) = \left(-\frac{12}{\sqrt{10}}\right) - 2\left(\frac{6}{\sqrt{10}}\right) - \left(\frac{6}{\sqrt{10}}\right) = -\frac{30}{\sqrt{10}} = -3\sqrt{10}$$

Thus we conclude that on the set $x^2 + 4y^2 + 2z^2 \leq 36$, the maximum of f is $3\sqrt{10}$ at the point $\left(\frac{12}{\sqrt{10}}, -\frac{6}{\sqrt{10}}, -\frac{6}{\sqrt{10}}\right)$, and the minimum of f is $-3\sqrt{10}$ at the point $\left(-\frac{12}{\sqrt{10}}, \frac{6}{\sqrt{10}}, \frac{6}{\sqrt{10}}\right)$

10. (10 points) Find all of the critical points of the function

$$f(x, y) = x^2y^2 + x^2y + xy$$

and classify each one as corresponding to a local maximum, local minimum, or a saddle point of f. Show all work and justify your answer completely.

First we compute $f_x = 2xy^2 + 2xy + y$ and $f_y = 2x^2y + x^2 + x$. We then solve the system

$$\begin{aligned} 2xy^2 + 2xy + y &= 0 \\ 2x^2y + x^2 + x &= 0 \end{aligned}$$

Factoring gives

$$\begin{aligned} y(2xy + 2x + 1) &= 0 \\ x(2xy + x + 1) &= 0 \end{aligned}$$

For the first equation to be satisfied, either $y = 0$ or $2xy + 2x + 1 = 0$.

- If $y = 0$, then the second equation becomes $x(x + 1) = 0$, and $x = 0$ or $x = -1$.
- If $2xy + 2x + 1 = 0$, then the second equation becomes $x(2xy + x + 1) = x(2xy + 2x + 1 - x) = x(-x) = 0$ and thus $x = 0$. Plugging $x = 0$ back into $2xy + 2x + 1 = 0$, we get $1 = 0$, which is inconsistent.

Thus the only critical points of f are $(0, 0)$ and $(-1, 0)$.

Now we classify, using the second derivative test. First compute

$$\begin{aligned} f_{xx} &= 2y^2 + 2y \\ f_{xy} &= 4xy + 2x + 1 \\ f_{yy} &= 2x^2 \end{aligned}$$

At $(0, 0)$:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot 0 - 1^2 = -1 < 0$$

and hence by the second derivative test

$\boxed{f \text{ has a } \textit{saddle point} \text{ at } (0, 0).}$

At $(-1,0)$:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot (2) - 1^2 = -1 < 0$$

and hence by the second derivative test

$\boxed{f \text{ has a } \textit{saddle point} \text{ at } (-1,0).}$

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