# Northwestern University

MATH 230-1 Midterm 1
Fall Quarter 2023
October 17, 2023

Last name: SOLUTIONS Email address: ____________

First name: ____________ NetID: ____________

**Instructions**

- Mark your section.

| Section | Time | Instructor | |
|---|---|---|---|
| 31 | 9:00 | Lee | |
| 41 | 10:00 | Lee | |
| 51 | 11:00 | Wunsch | |
| 61 | 12:00 | Cañez | |
| 71 | 1:00 | Coles | |
| 81 | 2:00 | Coles | |

- This examination consists of 11 pages, not including this cover page. Verify that your copy of this examination contains all 11 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 5 questions for a total of 100 points.
- You have one hour to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones. Use only material covered in this course (i.e., in the textbook or lecture) and not any formulas you may know from elsewhere that we did not cover.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has four parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

(a) (5 points) The line with parametric equations $x = -3, y = 1 - t, z = 2 + t$ completely lies on the plane with equation $x + y + z = 0$.

$$x + y + z = -3 + (1 - t) + (2 + t)$$
$$= -3 + 3 - t + t$$
$$= 0$$

$\boxed{\text{TRUE}}$

(b) (5 points) The vector projection of $\langle 4, -1\rangle$ onto $\langle -3, -2\rangle$ is orthogonal to the vector projection of $\langle 4, -1\rangle$ onto $\langle 3, -2\rangle$.

$\text{proj}_{\langle -3,-2\rangle}\langle 4,-1\rangle$ = nonzero multiple of $\langle -3,-2\rangle$

$\text{proj}_{\langle 3,-2\rangle}\langle 4,-1\rangle$ = nonzero multiple of $\langle 3,-2\rangle$

$\langle -3,-2\rangle \cdot \langle 3,-2\rangle \neq 0$ so not orthogonal

$\boxed{\text{FALSE}}$

(c) (5 points) The plane with equation $4x - 2y + 6z = 1$ is parallel to the plane with equation $-2x + y - 3z = 0$.

$4x - 2y + 6z = 1 \longrightarrow$ normal vector $\langle 4, -2, 6 \rangle$

$-2x + y - 3z = 0 \longrightarrow$ normal vector $\langle -2, 1, -3 \rangle$

1st normal $= -2$(2nd normal), so parallel

$\boxed{\text{TRUE}}$

(d) (5 points) The Cartesian point $(x, y) = (3, 5)$ is on the Cartesian curve consisting of points whose polar coordinates satisfy $r = 2\cos\theta$.

For $(x,y) = (3,5)$, $r = \sqrt{9+25} = \sqrt{34}$

is larger than 2, but $r = 2\cos\theta$

can only be between $-2 \leq r \leq 2$.

Alternatively, $r = 2\cos\theta$

is $x^2 + y^2 = 2x$, and $(3,5)$ does not satisfy this

$\boxed{\text{FALSE}}$

2. Consider the parallelogram with vertices $(1,-1,-2)$, $(3,1,-4)$, $(6,3,-3)$, and $(4,1,-1)$.

   (a) (10 points) Compute the vectors from $(1,-1,-2)$ to each of the other three vertices, and determine which of these two can be taken as sides of the parallelogram. Verify that this parallelogram is not a rectangle.

From $(1,-1,-2)$ to: $(3,1,-4)$ is $\langle 2,2,-2\rangle$

$(6,3,-3)$ is $\langle 5,4,-1\rangle$

$(4,1,-1)$ is $\langle 3,2,1\rangle$

$2^{nd}$ vector is sum of $1^{st}$ and $3^{rd}$, so $2^{nd}$ is diagonal and $\langle 2,2,-2\rangle$, $\langle 3,2,1\rangle$ are sides.

$\langle 2,2,-2\rangle \cdot \langle 3,2,1\rangle = 12 \neq 0$ so not rectangle (no right angle)

   (b) (10 points) Compute the area of this parallelogram. Your answer should be expressed using square roots but can otherwise be left unsimplified.

$$\langle 2,2,-2\rangle \times \langle 3,2,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= (2+4)\vec{i} - (2+6)\vec{j} + (4-6)\vec{k}$$

$$\text{area} = |\langle 6,-8,-2\rangle| = \sqrt{36+64+4}$$

3. The surface with equation

$$x^2 - 4x + y^2 + 6y + z^2 = 3$$

is a sphere.

(a) (5 points) Find parametric equations for the line passing through $(7, 2, 3)$ and the center of this sphere. To find the center you will need to complete the square in the equation of the sphere.

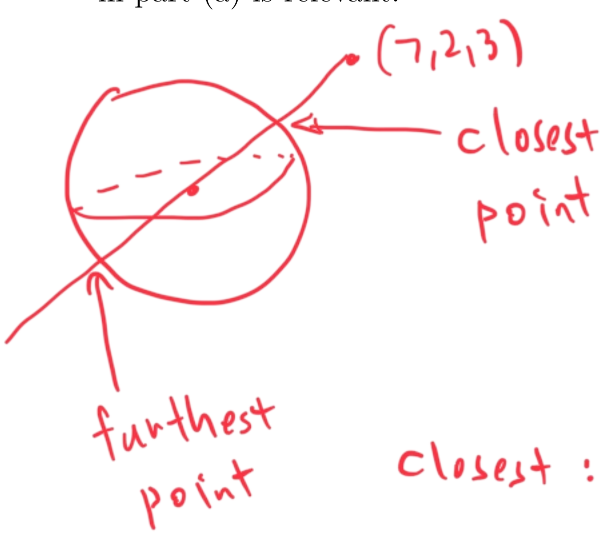
$$(x-2)^2 - 4 + (y+3)^2 - 9 + z^2 = 3$$

$$\rightarrow (x-2)^2 + (y+3)^2 + z^2 = 16$$

Center $= (2,-3,0)$, vector from $(2,-3,0)$ to $(7,2,3)$ is $\langle 5,5,3 \rangle$

line: $x = 2+5t$, $y = -3+5t$, $z = 3t$

(b) (10 points) Find the point on this sphere that is closest to $(7, 2, 3)$, and the point on this sphere that is furthest from $(7, 2, 3)$. Your answer should give values for the $x, y$, and $z$ coordinates of these points using square roots but can otherwise be left unsimplified. Hint: The line you found in part (a) is relevant.




Find intersections:

$$(x-2)^2 + (y+3)^2 + z^2 = 16$$

$$\rightarrow (5t)^2 + (5t)^2 + (3t)^2 = 16$$

$$\rightarrow 59t^2 = 16 \qquad t = \pm\sqrt{16/59}$$

closest: $\left(2+5\sqrt{16/59},\ -3+5\sqrt{16/59},\ 3\sqrt{16/59}\right)$

furthest: $\left(2-5\sqrt{16/59},\ -3-5\sqrt{16/59},\ -3\sqrt{16/59}\right)$

4. Consider the intersecting lines with parametric equations

$$\begin{cases} x = 4 + t \\ y = -3 - 2t \\ z = 1 - 3t \end{cases} \quad \text{and} \quad \begin{cases} x = -1 - 2t \\ y = 1 + t \\ z = 6 + t. \end{cases}$$

(a) (10 points) Find an equation of the plane that contains both of these lines.

point on plane : $(4,-3,1)$ from $1^{st}$ line

normal vector $= \langle 1,-2,-3\rangle \times \langle -2,1,1\rangle$ (product of direction vectors)

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ -2 & 1 & 1 \end{vmatrix} = \langle -2+3, -(1-6), 1-4\rangle$$

plane $(x-4)+5(y+3)-3(z-1)=0$

(b) (10 points) Find the distance from $(1, 1, 1)$ to the plane found in (a). Your answers should be expressed using square roots but can otherwise be left unsimplified.

point on plane : $(4,-3,1)$ vector to $(1,1,1)$ : $\langle -3,4,0\rangle$

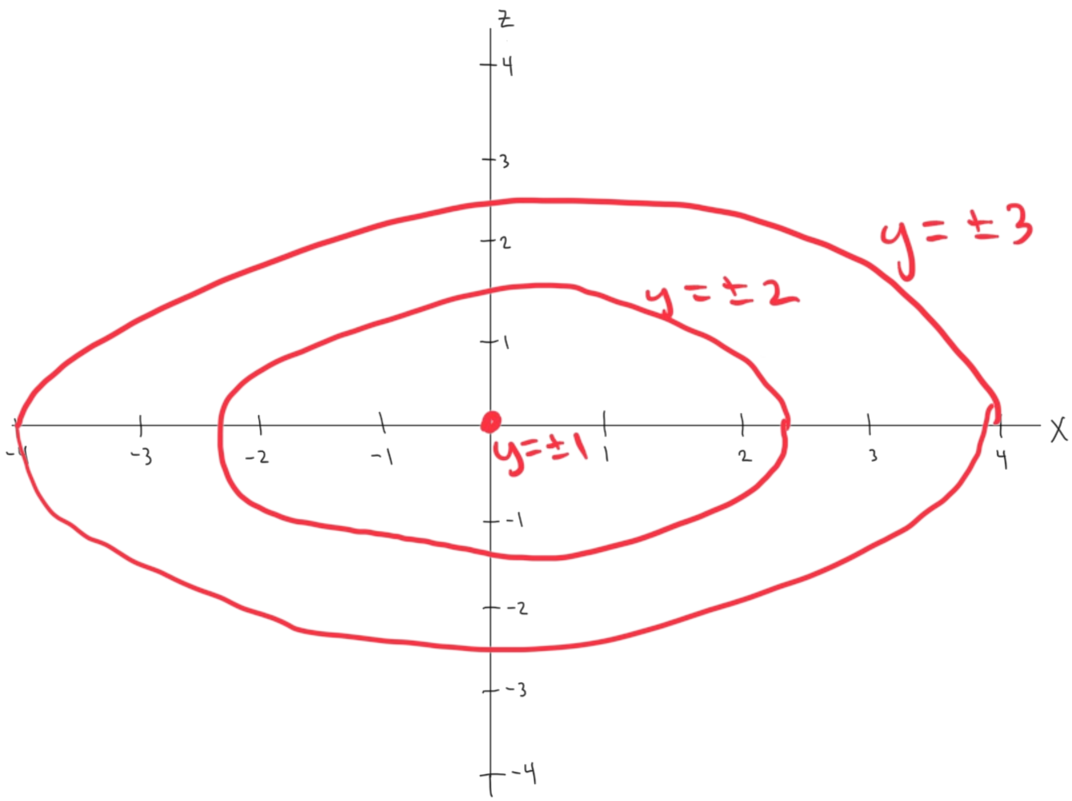
$$\text{proj}_{\langle 1,5,-3\rangle}\langle -3,4,0\rangle = \frac{17}{35}\langle 1,5,-3\rangle$$

distance to plane $= |\text{projection}| = \frac{17}{35}\sqrt{35} = \frac{17}{\sqrt{35}}$

5. (This problem has three parts and continues on the next two pages.) Consider the quadric surface with equation

$$-x^2 + 2y^2 - 3z^2 = 2.$$

(a) (10 points) Sketch the cross-sections of this surface at $y = \pm 1, \pm 2, \pm 3$ all on the given set of axes, clearly labeling which cross-section is which. Also, for each cross-section, find the points (if any) at which it intersects the $x$-axis and the points (if any) at which it intersects the $z$-axis.



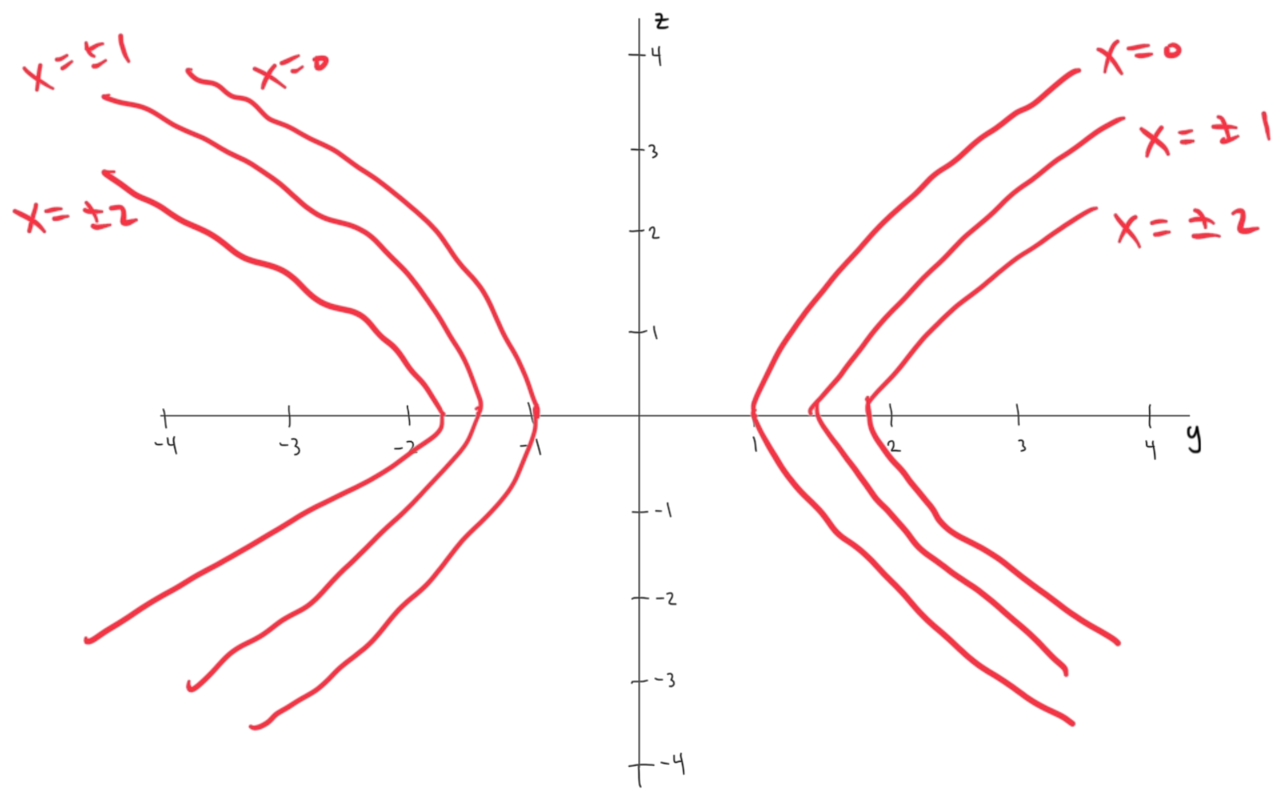

$x^2 + 3z^2 = 2y^2 - 2$

$y = \pm 1$ : $x^2 + 3z^2 = 0 \rightsquigarrow (x, z) = (0, 0)$ point
already its own intercepts

$y = \pm 2$ : $x^2 + 3z^2 = 6$ ellipse x-intercepts at $\pm\sqrt{6}$, z at $\pm\sqrt{2}$

$y = \pm 3$ : $x^2 + 3z^2 = 16$ ellipse, x-intercepts $\pm 4$, z at $\pm\sqrt{16/3}$

(b) (10 points) Sketch the cross-sections of this surface at $x = 0, \pm 1, \pm 2$ all on the given set of axes, clearly labeling which cross-section is which. Also, for each cross-section, find the points (if any) at which it intersects the $y$-axis and the points (if any) at which it intersects the $z$-axis.



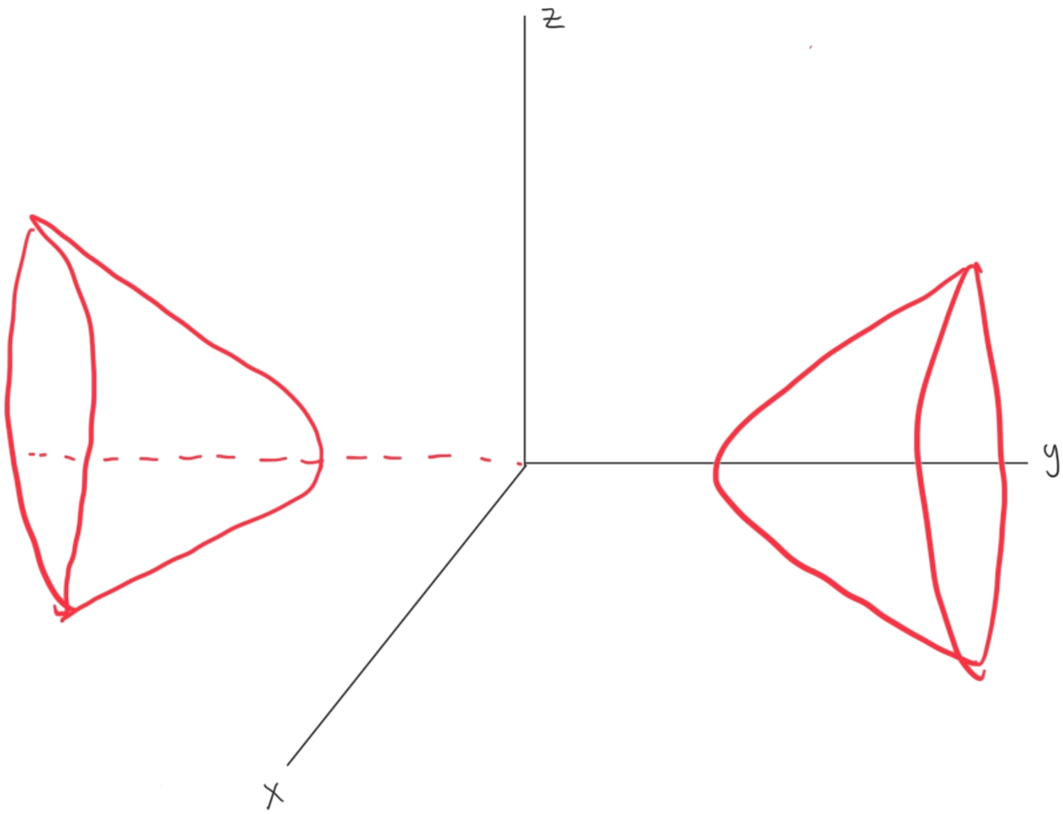

$2y^2 - 3z^2 = 2 + x^2$

$x=0$ : $2y^2 - 3z^2 = 2$ hyperbola y-intercepts $\pm 1$
no z-intercepts

$x=\pm 1$ : $2y^2 - 3z^2 = 3$ hyperbola y-intercepts $\pm\sqrt{3/2}$
no z

$x=\pm 2$ : $2y^2 - 3z^2 = 6$ hyperbola y-intercepts $\pm\sqrt{3}$
no z

(c) (5 points) Give a rough sketch of the surface which gives the basic shape on the given set of axes. There is no need to label its intercepts with axes.




hyperboloid of 2 sheets (name not required)

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