Northwestern University

Math 230-1 Final Examination
Fall Quarter 2019
Wednesday 11 December

Last name: SOLUTIONS   Email address: ____________

First name: SOLUTIONS   NetID: ____________

Instructions

- This examination consists of 8 questions for a total of 110 points.
- Read all problems carefully before answering.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- You will find an additional page after Exercise 6, as well as two additional pages at the end of the exam. If you make use of any of these, indicate on the original exercise page where to find the additional work.
- **Show and justify all of your work**. Unsupported answers may not earn credit.
- **Terminology**: by "familiar named surface" we will mean a member of one of the following types of surfaces:

plane	cylinder	
ellipsoid	elliptic paraboloid	hyperbolic paraboloid
cone	hyperboloid of one sheet	hyperboloid of two sheets

1. (15 points) Short answer.

   (a) Compute the angle θ between $\mathbf{v} = \langle\sqrt{6}, 2, \sqrt{6}\rangle$ and $\mathbf{w} = \langle\sqrt{2}, 2\sqrt{3}, \sqrt{2}\rangle$.
   Do not express your answer in terms of inverse trig functions.

   Solution: We have

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{2\sqrt{12}+4\sqrt{3}}{4\cdot 4}$$
$$= \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2}.$$

   It follows that $\theta = \pi/6$.

   (b) Let $f(x, y) = x^2y - y^2 + x$. Let $P = (1, 1)$, and let $\mathcal{C}$ be the level curve of f that P lies on. Find a vector $\mathbf{v}$ that points tangent to $\mathcal{C}$ at P. (The vector $\mathbf{v}$ does *not* need to be a unit vector.)

   Solution: First compute $\nabla f(x, y) = \langle 2xy + 1, x^2 - 2y\rangle$.
   At P we have $\nabla f(P) = \langle 3, -1\rangle$. This vector is orthogonal to $\mathcal{C}$ at P. Thus a tangent vector must be orthogonal to $\langle 3, -1\rangle$. The vector $\mathbf{v} = \langle 1, 3\rangle$ will do.

(c) Let $f(x, y) = x\sin(y)$. Compute the quadratic approximation $Q(x, y)$ of f at $(x_0, y_0) = (1, \pi/6)$.

Solution: First assemble the necessary ingredients:

$$
\begin{aligned}
& & f(1, \pi/6) &= \frac{1}{2} \\
f_x &= \sin y & f_x(1, \pi/6) &= \frac{1}{2} \\
f_y &= x\cos y & f_y(1, \pi/6) &= \frac{\sqrt{3}}{2} \\
f_{xx} &= 0 & f_{xx}(1.\pi/6) &= 0 \\
f_{xy} &= \cos y & f_{xy}(1, \pi/6) &= \frac{\sqrt{3}}{2} \\
f_{yy} &= -x\sin y & f_{yy}(1, \pi/6) &= -\frac{1}{2}
\end{aligned}
$$

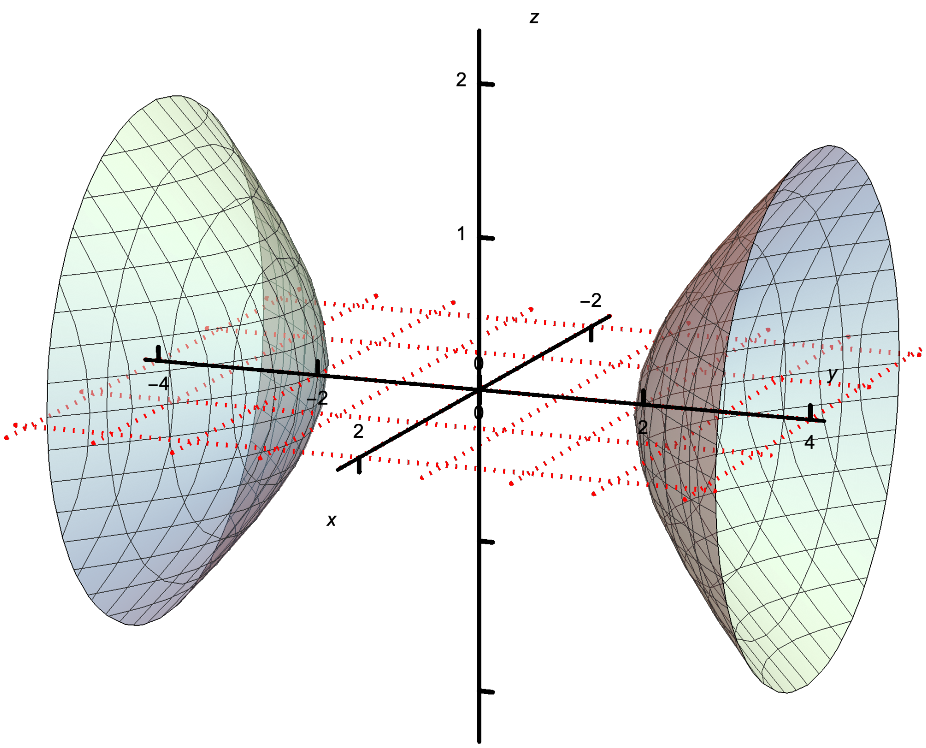
Let $P = (1, \pi/6)$. Then

$$
\begin{aligned}
Q(x, y) &= f(P) + f_x(P)(x-1) + f_y(P)(y - \pi/6) \\
&\quad + \frac{1}{2}f_{xx}(P)(x-1)^2 + f_{xy}(P)(x-1)(y-\pi/6) + \frac{1}{2}f_{yy}(P)(y-\pi/6)^2 \\
&= \frac{1}{2} + \frac{1}{2}(x-1) + \frac{\sqrt{3}}{2}(y-\pi/6) + \frac{\sqrt{3}}{2}(x-1)(y-\pi/6) - \frac{1}{4}(y-\pi/6)^2
\end{aligned}
$$

2. (10 points) Let $w = f(x, y, z) = x^2 - \dfrac{y^2}{4} + z^2$.

For parts (a) and (b) below, the following estimates may be useful: $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$.

(a) Sketch the $(w = -1)$-level surface of f and identify this as one of our familiar named surfaces. Your sketch must include the $(y = \pm 2, \pm 4)$-cross sections of the surface.



The surface has equation $x^2 - \frac{y^2}{4} + z^2 = -1$.
Cross sections:

$y = \pm 2$: $x^2 + z^2 = 0 \Longrightarrow (x, y, z) = (0, \pm 2, 0)$.

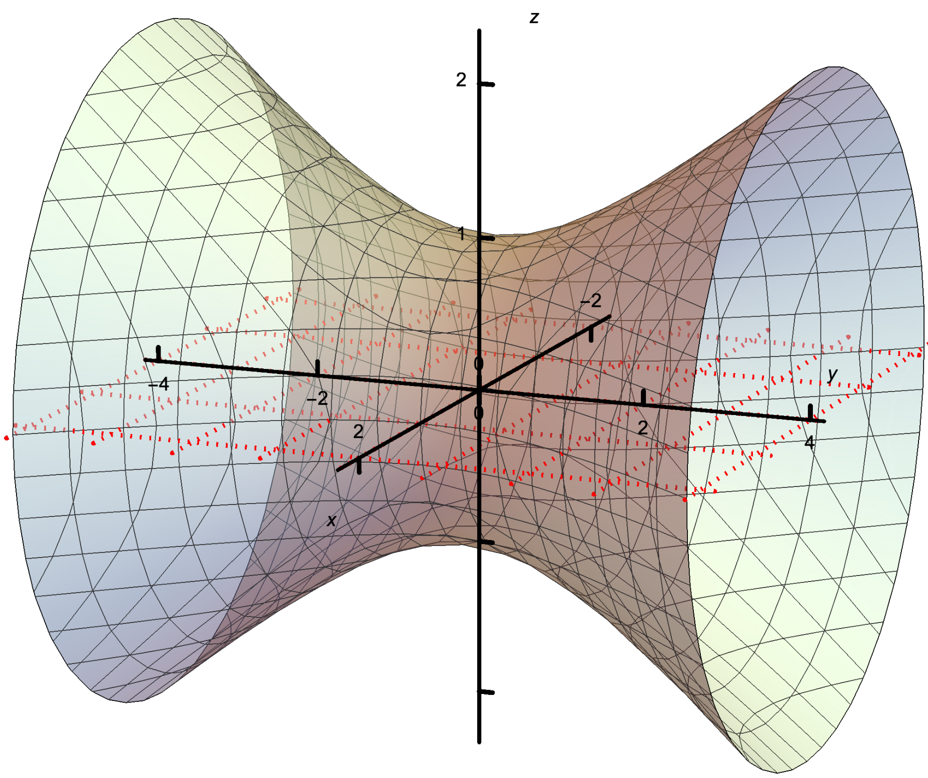
$y = \pm 4$: $x^2 + z^2 = 3 \Longrightarrow$ cross section is circle of radius $\sqrt{3}$.

In general we see cross sections are circles along y-axis, except between -2 and 2 where cross sections are empty.

The surface is thus seen to be a hyperboloid of two sheets.

Exercise 2 contd. $w = f(x, y, z) = x^2 - \dfrac{y^2}{4} + z^2$
$\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$

(b) Sketch the $(w = 1)$-level surface of f and identify this as one of our familiar named surfaces. Your sketch must include the $(y = 0, \pm 2, \pm 4)$-cross sections of the surface.



The surface has equation $x^2 - \frac{y^2}{4} + z^2 = 1$.
General $y = \pm k$ cross sections:

$y = \pm k$: $x^2 + z^2 = 1 + k^2/4 \Longrightarrow$ cross section is circle of radius $\sqrt{1 + k^2/4}$.

Thus $y = 0$ is circle of radius 1, $y = \pm 2$ cross section is circle of radius $\sqrt{2}$, $y = \pm 4$ cross section is circle of radius $\sqrt{5}$.

The surface is thus a hyperboloid of one sheet.

3. (15 points) Let L_1 be the line through the point $Q_1 = (2, 2, 1)$ in the direction of $\mathbf{v}_1 = \langle 1, -1, 1\rangle$. Let L_2 be the line through the point $Q_2 = (1, 0, 2)$ in the direction of $\mathbf{v}_2 = \langle 1, 1, 1\rangle$.

(a) Show that L_1 and L_2 are skew.

Solution: The lines have parametrizations $\mathbf{r}_1(t) = \langle 2+t, 2-t, 1+t\rangle$ and $\mathbf{r}_2(s) = \langle 1+s, s, 2+s\rangle$, respectively.

For the two lines to intersect there must be a t and s satisfying the system

$$2 + t = 1 + s \tag{1}$$
$$2 - t = s \tag{2}$$
$$1 + t = 2 + s \tag{3}$$

Adding equations (1)&(2) tells us that $4 = 1 + 2s$, or $\boxed{s = 3/2}$. It then follows from (1) that $\boxed{t = 1/2}$. But then equation (3) would imply that $1 + 1/2 = 2 + 3/2$: a contradiction!

Thus there are no such t and s satisfying the system, and we see that the lines do not intersect. Since the lines are clearly not parallel (their direction vectors are not scalar multiples of one another), it follows that the lines are skew.

(b) Find equations for two parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ containing L_1 and L_2, respectively.
Hint: find a vector $\mathbf{n}$ that is normal to both lines.

Solution: In general if line L passes through point Q and has direction vector $\mathbf{v}$, then L is contained in any plane $\mathcal{P}$ passing through Q having a normal vector $\mathbf{n}$ orthogonal to $\mathbf{v}$.
With this in mind let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -2, 0, 2\rangle$.

Since $\mathbf{n}$ is orthogonal to $\mathbf{v}_1$, the plane $\mathcal{P}_1\colon -2(x-2) + 0(y-2) + 2(z-1) = 0$ contains L_1.

Since $\mathbf{n}$ is orthogonal to $\mathbf{v}_2$, the plane $\mathcal{P}_2\colon -2(x-1) + 0(y) + 2(z-2) = 0$ contains L_2.

The two planes share a normal vector, and so are either parallel or equal. The latter is not the case, since Q_1 does not satisfy the defining equation of $\mathcal{P}_2$. Thus the two planes are parallel.

Exercise 3 contd. $L_1\colon Q_1 = (2,2,1), \mathbf{v}_1 = \langle 1,-1,1\rangle$ $L_2\colon Q_2 = (1,0,2), \mathbf{v}_2 = \langle 1,1,1\rangle$

(c) Compute the distance between L_1 and L_2.
You may assume that this is the same thing as the distance between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.
Tip: to see how to compute the distance, you may want to draw a general picture of two parallel planes, each with a labelled point on it. (Don't pick the points to be right above one another in your picture.)

Solution: To compute the distance $d(\mathcal{P}_1, \mathcal{P}_2)$ between $\mathcal{P}_1$ and $\mathcal{P}_2$, we can simply compute the distance from a single point on $\mathcal{P}_1$ to $\mathcal{P}_2$.

We will compute the distance $d(Q_1, \mathcal{P}_2)$ between Q_1 and $\mathcal{P}_2$. We do so using the "distance from point to plane" formula. The formula requires a choice of point on $\mathcal{P}_2$. We of course pick Q_2 and compute:

$$\begin{aligned} d(\mathcal{P}_1, \mathcal{P}_2) &= d(Q_1, \mathcal{P}_2) \\ &= \frac{\left|\overrightarrow{Q_2Q_1} \cdot \mathbf{n}\right|}{|\mathbf{n}|} \\ &= \frac{|\langle 1,2,-1\rangle \cdot \langle -2,0,2\rangle|}{\sqrt{8}} \\ &= \frac{4}{2\sqrt{2}} \\ &= \sqrt{2}. \end{aligned}$$

4. (10 points) The paraboloid $\mathcal{S}_1\colon x^2 + 4y^2 + z^2 = 9$ and the cubic surface $\mathcal{S}_2\colon x^3 + yz^2 = 5$ intersect in a curve $\mathcal{C}$. As is easily verified, the point $P = (1, 1, 2)$ lies on $\mathcal{C}$.
(a) For each $i = 1, 2$ find a normal vector $\mathbf{n}_i$ to $\mathcal{S}_i$ at P.
(b) Use (a) to find a vector parametrization of the tangent line to $\mathcal{C}$ at P. (Do *not* try and parametrize $\mathcal{C}$!)

Solution:
(a) Let $g_1(x, y, z) = x^2 + 4y^2 + z^2$ and $g_2(x, y, z) = x^3 + yz^2$.

Then $\nabla g_1(x, y, z) = \langle 2x, 8y, 2z\rangle$ and $\nabla g_2(x, y, z) = \langle 3x^2, z^2, 2yz\rangle$, and we can compute the normal vectors $\mathbf{n}_i$ at P as
$\mathbf{n}_1 = \nabla g_1(P) = \langle 2, 8, 4\rangle$
$\mathbf{n}_2 = \nabla g_2(P) = \langle 3, 4, 4\rangle$.

Solution: (b) Let $\mathbf{v}$ be a tangent vector to $\mathcal{C}$ at P. Since $\mathcal{C}$ is the intersection of both surfaces, $\mathbf{v}$ must lie in the tangent plane to $\mathcal{S}_1$ at P *and* the tangent plane to $\mathcal{S}_2$ at P. It follows that $\mathbf{v}$ must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus that $\mathbf{v}$ must be a scalar multiple of

$$\mathbf{n}_1 \times \mathbf{n}_2 = \langle 16, -4, -16\rangle.$$

We take $\mathbf{v} = \frac{1}{4}\langle 16, -4, -16\rangle = \langle 4, -1, -4\rangle$. The corresponding parametrization of the tangent line to $\mathcal{C}$ at P is

$$\mathbf{r}(t) = \langle 1 + 4t, 1 - t, 2 - 4t\rangle.$$

5. (10 points) For each $f(x, y)$ below compute $\lim_{(x,y)\to(0,0)} f(x, y)$, or else show that this limit does not exist.
Hint 1: one of the limits exists, the other does not.
Hint 2: you don't need to use polar coordinates for either.

(a) $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$

Solution: The limit does not exist.

Along the path L with parametric equations

$$x = t$$
$$y = t$$

the limit is computed as

$$\lim_{t\to 0} \frac{t^3}{t^2 + t^4}$$
$$= \lim_{t\to 0} \frac{t}{1 + t^2} = 0.$$

Along the path $\mathcal{C}$ with parametric equations

$$x = t^2$$
$$y = t$$

the limit is computed as

$$\lim_{t\to 0} \frac{t^4}{t^4 + t^4}$$
$$= \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since the two paths give us two different limits, we see that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Exercise 5 cont.
Compute $\lim_{(x,y)\to(0,0)} f(x,y)$, or else show that this limit does not exist.

(b) $f(x,y) = \dfrac{x^2 y}{x^2 + y^4}$

Solution: We have

$$\begin{aligned} \left|\frac{x^2 y}{x^2+y^4}\right| &= \frac{x^2\,|y|}{x^2+y^4} \\ &\le \frac{x^2\,|y|}{x^2} && \text{(since } x^2+y^4 \ge x^2\text{)} \\ &= |y| \end{aligned}$$

Thus $|f(x,y)| \le |y|$, which implies $-y \le f(x,y) \le y$.

Since both $-y$ and y limit to 0 as $(x,y) \to (0,0)$, it follows by the squeeze theorem that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

6. (15 points) A ball is swung round on a string until the string snaps. Before the string snaps the ball's position vector is given by

$$\mathbf{r}_{\text{before}}(t) = \langle \sqrt{2}\sin t, \sqrt{2}\cos t, 3 + \sqrt{2}\sin t - \sqrt{2}\cos t \rangle$$

Here the x-, y-, and z-coordinates are meters measured from some point O on the ground and t is in seconds.

At time $t = \pi/4$ seconds, the string snaps. From this time on the only force acting on the ball is gravity, causing a downward acceleration of $g = 10$ m/s^2 until the ball hits the ground.

(a) Compute the position and velocity of the ball at the moment when the string snaps.

Solution: First we compute $\mathbf{r}'_{\text{before}}(t) = \langle \sqrt{2}\cos t, -\sqrt{2}\sin t, \sqrt{2}(\cos t + \sin t) \rangle$.

The string snaps after $t = \pi/4$ seconds, at which point position and velocity are given by

$$\mathbf{r}_{\text{before}}(\pi/4)) = \langle 1, 1, 3 \rangle$$
$$\mathbf{r}'_{\text{before}}(\pi/4)) = \langle 1, -1, 2 \rangle$$

(b) Compute the xy-coordinates of the ball when it hits the ground.

Solution: Define $\mathbf{r}(t)$ to be the position of the ball t seconds *after* the string snaps.

Initial conditions: we have

$$\mathbf{r}(0) = \mathbf{r}_{\text{before}}(\pi/4)) = \langle 1, 1, 3 \rangle$$
$$\mathbf{r}'(0) = \mathbf{r}'_{\text{before}}(\pi/4)) = \langle 1, -1, 2 \rangle$$

After the ball snaps the only acceleration is due to gravity: $\mathbf{r}''(t) = \langle 0, 0, -10 \rangle$.

It follows that $\mathbf{r}'(t) = \langle c_1, c_2, c_3 - 10t \rangle$. Since $\mathbf{r}'(0) = \langle 1, -1, 2 \rangle$, we see easily that $c_1 = 1, c_2 = -1$ and $c_3 = 2$. Thus $\mathbf{r}'(t) = \langle 1, -1, 2 - 10t \rangle$.

Next, we see that $\mathbf{r}(t) = \langle t + d_1, -t + d_2, 2t - 5t^2 + d_3 \rangle$. Since $\mathbf{r}(0) = \langle 1, 1, 3 \rangle$ we see easily that $d_1 = 1, d_2 = 1, d_3 = 3$, and hence that $\mathbf{r}(t) = \langle t + 1, -t + 1, 3 + 2t - 5t^2 \rangle$.
To find the time when the ball hits the ground we solve $3 + 2t - 5t^2 = 0$ for t with the help of the quadratic formula:

$$t = \frac{-2 \pm \sqrt{4 + 60}}{-10} = 1 \text{ or } -3/5.$$

The solution that is meaningful to our model is of course $t = 1$. At this time we have

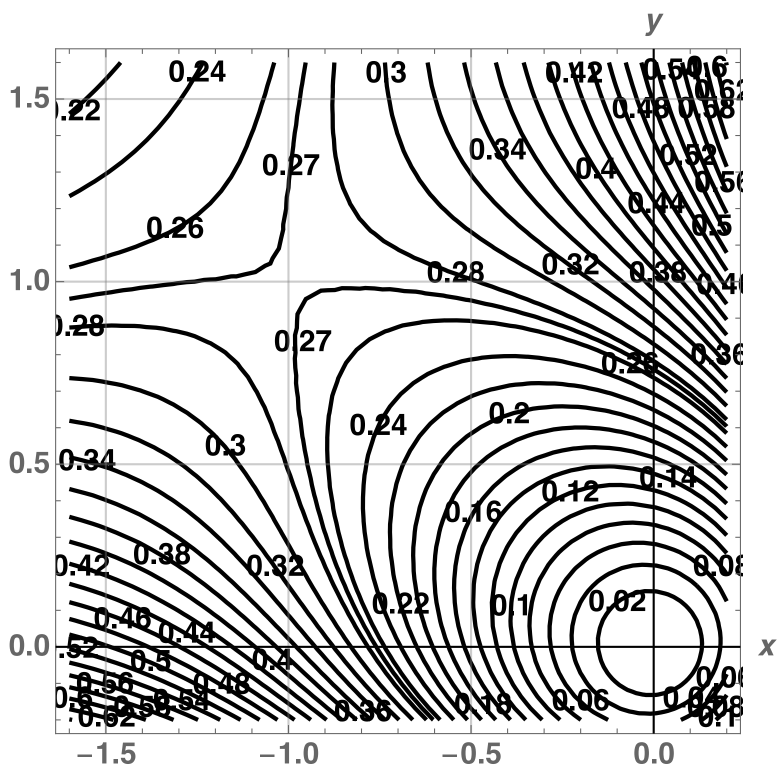
$$\mathbf{r}(1) = \langle 1 + 1, -1 + 1, 0 \rangle = \langle 2, 0, 0 \rangle.$$

We conclude that when the ball lands it's xy-coordinates are $x = 2$ and $y = 0$.

Additional space for work on Exercise 6, if needed.

7. (20 points) Let $f(x,y) = (x^2+y^2)e^{x-y}$.

(a) The contour diagram of f below clearly indicates a number of critical points.
(i) Mark these on the diagram, and give their xy-coordinates below the diagram.
(ii) Use the contour diagram to classify each critical point you found in (i) as a local min, local max, or saddle point.



Solution: We clearly have a saddle point at $P = (-1, 1)$ and a local min at $Q = (0, 0)$.

Exercise 7 contd. $f(x, y) = (x^2 + y^2)e^{x-y}$.

(b) Now verify your answer in (a) by *algebraically* finding all critical points of f. Your work must justify that you have indeed found *all* critical points of f.
Hint: factoring will be useful when solving the relevant system of equations.

Solution: We must solve the system

$$f_x = (2x + x^2 + y^2)e^{x-y} = 0 \quad (1)$$
$$f_y = (2y - x^2 - y^2)e^{x-y} = 0 \quad (2)$$

Since e^{x-y} is never equal to 0, we conclude

$$f_x = (2x + x^2 + y^2) = 0 \quad (3)$$
$$f_y = (2y - x^2 - y^2) = 0 \quad (4)$$

Adding equations (3) and (4) yields $2x + 2y = 0$: i.e.,

$$\boxed{y = -x}. \quad (5)$$

Substituting this back into (3) yields $2x + 2x^2 = 0$, which implies $x = 0$ or $x = -1$. Since we must have $y = -x$, we conclude the only critical points are $P = (-1, 1)$ and $Q = (0, 0)$, just as we saw in (a).

Exercise 7 contd. $f(x, y) = (x^2 + y^2)e^{x-y}$.

(c) Now use the **second derivatives test** to classify each of your critical points as a local min, a local max, or neither. We will compute second derivatives for you!
$f_{xx} = (2 + 4x + x^2 + y^2)e^{x-y}$ $\qquad$ $f_{yy} = (2 - 4y + x^2 + y^2)e^{x-y}$ $\qquad$ $f_{xy} = (2y - 2x - x^2 - y^2)e^{x-y}$
Note: even if you couldn't do part (b), you can use your critical points from part (a).

Solution: We run our second derivatives test on our two critical points.
Let $H(x, y) = f_{xx}f_{yy} - f_{xy}^2$.

Take $P = (-1, 1)$. We have

$$H(-1, 1) = f_{xx}(-1, 1)f_{yy}(-1, 1) - (f_{xy}(-1, 1))^2 = 0 \cdot f_{yy}(-1, 1) - 4e^{-4} = -4e^{-4} < 0.$$

We conclude P is a saddle point.

Take $Q = (0, 0)$. We have

$$\begin{aligned} H(0, 0) &= f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2 = 4 > 0 \\ f_{xx}(0, 0) &= 2 > 0 \end{aligned}$$

We conclude Q is a local min.

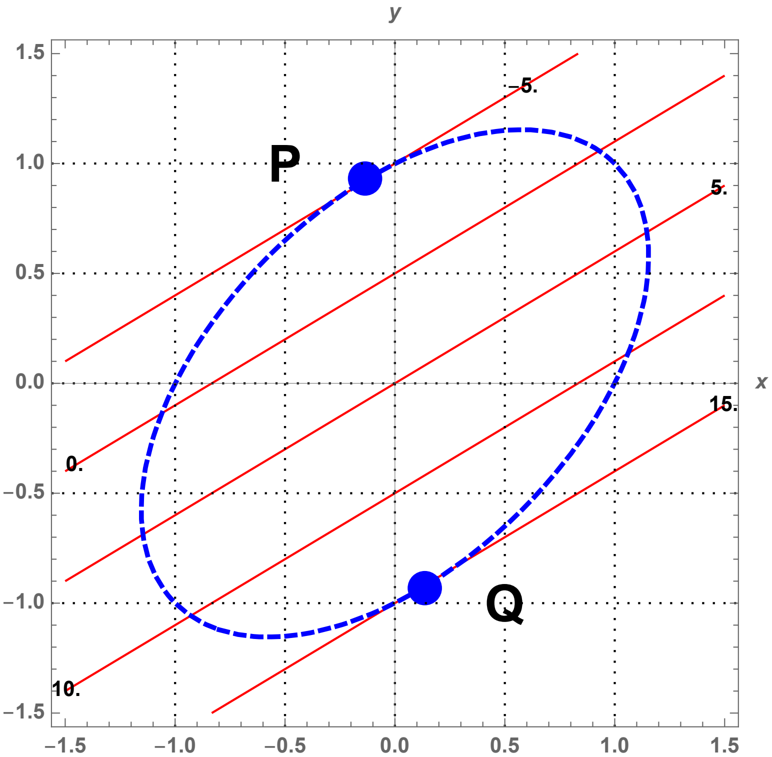
(d) Decide whether f has an *absolute* minimum value and/or an *absolute* maximum value on its entire domain. Justify your answer.

Solution: Observe that $f(x, y) = (x^2 + y^2)e^{x-y} \geq 0$ for all (x, y). Since $f(0, 0) = 0$, we conclude that 0 is an absolute minimum value of f.

Consider $f(t, 0) = t^2e^t$. As $t \to \infty$, $f(t, 0) \to \infty$. We conclude that f does not have an absolute maximum value.

8. (15 points) Let $z = f(x, y) = 6x - 10y + 5$.

(a) Below you find the ellipse with defining equation $x^2 - xy + y^2 = 1$. Draw a contour diagram of f on top of this that includes the $(z = -5, 0, 5, 10, 15)$-contours of f. Label your contours!



In general, the $z = c$-contour has equation $6x - 10y + 5 = c$, or $\frac{3}{5}x - \frac{c-5}{10} = y$. The contours are thus parallel lines of slope $3/5$ and y-intercept $(c - 5)/10$.

(b) Your diagram will now suggest some *approximate* points on the ellipse where f obtains extreme values. Mark and name these points on the diagram, and for each determine whether it approximates an absolute minimum of f on the ellipse, an absolute maximum, or neither.

Solution: We look for points where the level curves of f intersect the constraint curve (ellipse) tangentially: I've labelled these points P and Q above. (Note: in general we should also look for points on the ellipse where $f_x = f_y = 0$, but this is never the case for the given $f(x, y)$.) The point since $f(P) \approx -5$ and $f(Q) \approx 15$, we see that P is an approximate minimum of f on the ellipse, and Q is an approximate maximum.

Exercise 8 contd. $f(x,y) = 6x - 10y + 5$. Ellipse: $x^2 - xy + y^2 = 1$.

(c) Now use the method of Lagrange multipliers to find the *exact* points on the ellipse where f obtains extreme values.

Furthermore, for each point you find, decide whether it represents an absolute minimum of f on the ellipse, an absolute maximum, or neither, and compute the value of f there.

Solution: We use Lagrange multipliers to optimize $f(x,y)$ subject to the constraint $g(x,y) = x^2 - xy + y^2 = 1$.

(Note: since the ellipse is a bounded curve, there is guaranteed to be an absolute maximum and minimum of f on the ellipse.)

The relevant system of equations is

$$\begin{aligned} x^2 - xy + y^2 &= 1 && \text{(constraint)} && (1)\\ 6 &= \lambda(2x - y) && (f_x = \lambda g_x) && (2)\\ -10 &= \lambda(2y - x) && (f_y = \lambda g_y) && (3) \end{aligned}$$

(2)&(3) $\Rightarrow 6/(2x-y) = -10/(2y-x) \Rightarrow -20x + 10y = 12y - 6x \Rightarrow \boxed{y = -7x}$ (4).

(Note that equations (2)& (3) imply $2x - y \neq 0$ and $2y - x \neq 0$, so we may divide with impunity.)

(1)& (4)$\Rightarrow x^2 + 7x^2 + 49x^2 = 1 \Rightarrow 57x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{57}}$.

Since $y = -7x$ we see that the candidate points are $P = \left(-\frac{1}{\sqrt{57}}, \frac{7}{\sqrt{57}}\right)$ and $Q = \left(\frac{1}{\sqrt{57}}, -\frac{7}{\sqrt{57}}\right)$

Next we compute $f(P) = -6/\sqrt{57} - 70/\sqrt{57} + 5 = 5 - 67/\sqrt{57}$ and $f(Q) = 5 + 67/\sqrt{57}$.

Since P and Q are the only candidate points obtained via Lagrange multipliers, and since $f(P) < f(Q)$ we conclude that f obtains a minimum value of $5 - 67/\sqrt{57}$ on the ellipse at P, and f obtains a maximum value of $5 + 67/\sqrt{57}$ on the ellipse at Q.