Northwestern University

Math 230-1 Final Examination
Fall Quarter 2019
Wednesday 11 December

Last name: ______________________ Email address: ______________________

First name: ______________________ NetID: ______________________

Instructions

- This examination consists of 8 questions for a total of 110 points.
- Read all problems carefully before answering.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- You will find an additional page after Exercise 6, as well as two additional pages at the end of the exam. If you make use of any of these, indicate on the original exercise page where to find the additional work.
- **Show and justify all of your work**. Unsupported answers may not earn credit.
- **Terminology**: by "familiar named surface" we will mean a member of one of the following types of surfaces:

plane	cylinder	
ellipsoid	elliptic paraboloid	hyperbolic paraboloid
cone	hyperboloid of one sheet	hyperboloid of two sheets

1. (15 points) Short answer.

   (a) Compute the angle θ between $\mathbf{v} = \langle\sqrt{6}, 2, \sqrt{6}\rangle$ and $\mathbf{w} = \langle\sqrt{2}, 2\sqrt{3}, \sqrt{2}\rangle$.
   Do not express your answer in terms of inverse trig functions.

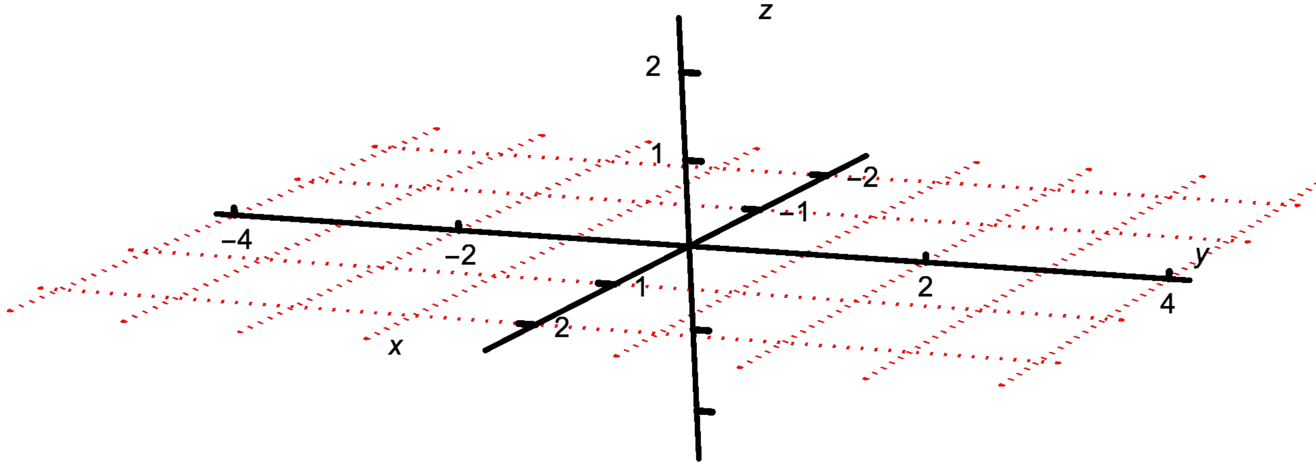
   (b) Let $f(x, y) = x^2y - y^2 + x$. Let $P = (1, 1)$, and let $\mathcal{C}$ be the level curve of f that P lies on. Find a vector $\mathbf{v}$ that points tangent to $\mathcal{C}$ at P. (The vector $\mathbf{v}$ does *not* need to be a unit vector.)

(c) Let $f(x, y) = x \sin(y)$. Compute the quadratic approximation $Q(x, y)$ of f at $(x_0, y_0) = (1, \pi/6)$.

2. (10 points) Let $w = f(x, y, z) = x^2 - \dfrac{y^2}{4} + z^2$.

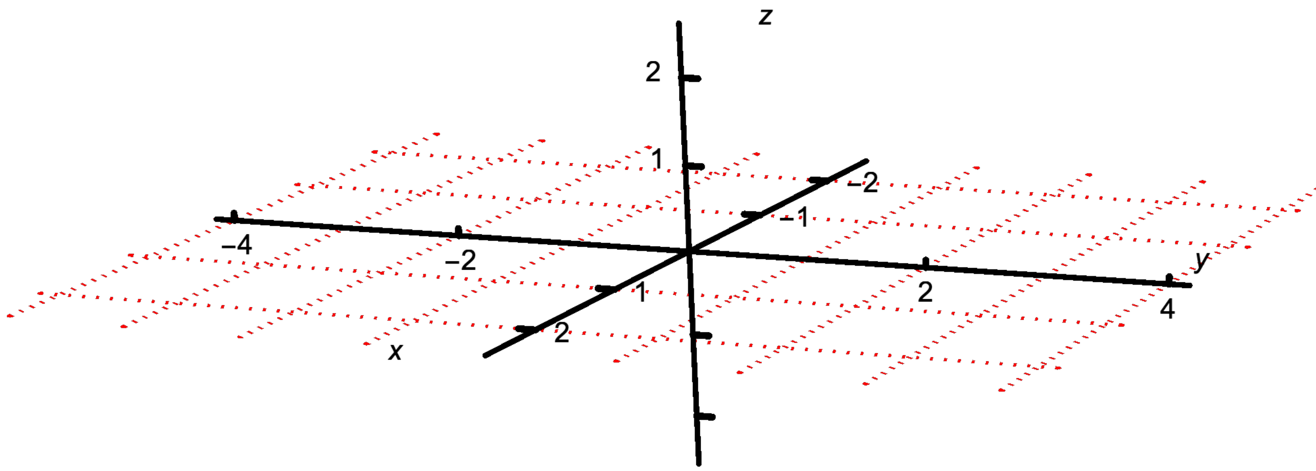
For parts (a) and (b) below, the following estimates may be useful: $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$.

(a) Sketch the $(w = -1)$-level surface of f and identify this as one of our familiar named surfaces. Your sketch must include the $(y = \pm 2, \pm 4)$-cross sections of the surface.



Exercise 2 contd. $w = f(x, y, z) = x^2 - \dfrac{y^2}{4} + z^2$
$\sqrt{2} \approx 1.4,\ \sqrt{3} \approx 1.7,\ \sqrt{5} \approx 2.2$

(b) Sketch the $(w = 1)$-level surface of f and identify this as one of our familiar named surfaces. Your sketch must include the $(y = 0, \pm 2, \pm 4)$-cross sections of the surface.



3. (15 points) Let L_1 be the line through the point $Q_1 = (2, 2, 1)$ in the direction of $\mathbf{v}_1 = \langle 1, -1, 1 \rangle$. Let L_2 be the line through the point $Q_2 = (1, 0, 2)$ in the direction of $\mathbf{v}_2 = \langle 1, 1, 1 \rangle$.

   (a) Show that L_1 and L_2 are skew.

   (b) Find equations for two parallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ containing L_1 and L_2, respectively. **Hint**: find a vector $\mathbf{n}$ that is normal to both lines.

Exercise 3 contd. $L_1\colon Q_1 = (2, 2, 1), \mathbf{v}_1 = \langle 1, -1, 1\rangle$ $L_2\colon Q_2 = (1, 0, 2), \mathbf{v}_2 = \langle 1, 1, 1\rangle$

(c) Compute the distance between L_1 and L_2.
You may assume that this is the same thing as the distance between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.
Tip: to see how to compute the distance, you may want to draw a general picture of two parallel planes, each with a labelled point on it. (Don't pick the points to be right above one another in your picture.)

4. (10 points) The paraboloid $\mathcal{S}_1 \colon x^2 + 4y^2 + z^2 = 9$ and the cubic surface $\mathcal{S}_2 \colon x^3 + yz^2 = 5$ intersect in a curve $\mathcal{C}$. As is easily verified, the point $P = (1, 1, 2)$ lies on $\mathcal{C}$.
(a) For each $i = 1, 2$ find a normal vector $\mathbf{n}_i$ to $\mathcal{S}_i$ at P.
(b) Use (a) to find a vector parametrization of the tangent line to $\mathcal{C}$ at P. (Do *not* try and parametrize $\mathcal{C}$!)

5. (10 points) For each $f(x,y)$ below compute $\lim_{(x,y)\to(0,0)} f(x,y)$, or else show that this limit does not exist.
Hint 1: one of the limits exists, the other does not.
Hint 2: you don't need to use polar coordinates for either.

(a) $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

Exercise 5 cont.

Compute $\lim_{(x,y)\to(0,0)} f(x,y)$, or else show that this limit does not exist.

(b) $f(x,y) = \dfrac{x^2 y}{x^2 + y^4}$

6. (15 points) A ball is swung round on a string until the string snaps. Before the string snaps the ball's position vector is given by

$$\mathbf{r}_{\text{before}}(t) = \langle \sqrt{2}\sin t, \sqrt{2}\cos t, 3 + \sqrt{2}\sin t - \sqrt{2}\cos t \rangle$$

Here the x-, y-, and z-coordinates are meters measured from some point O on the ground and t is in seconds.

At time $t = \pi/4$ seconds, the string snaps. From this time on the only force acting on the ball is gravity, causing a downward acceleration of $g = 10$ m/s^2 until the ball hits the ground.

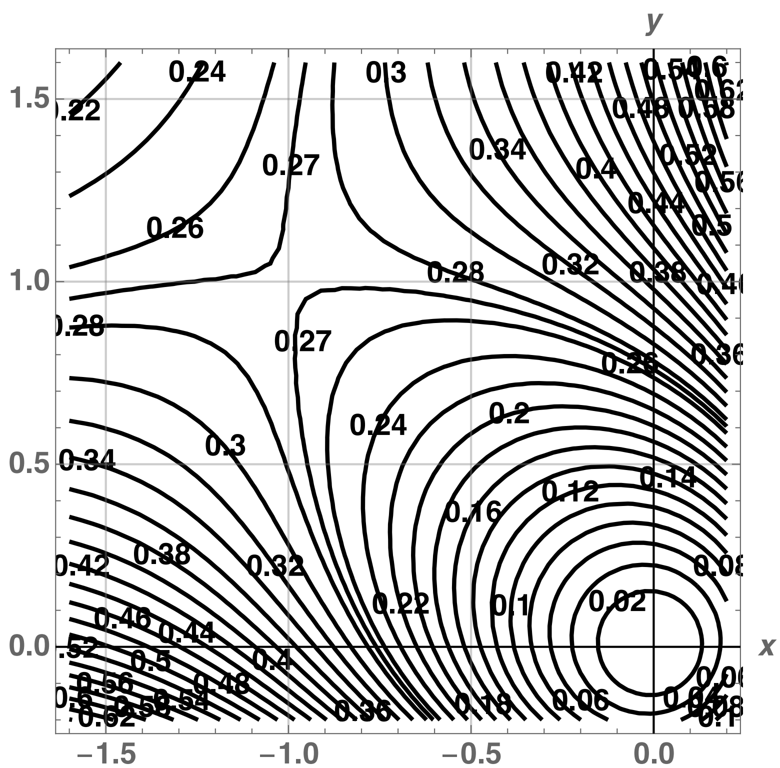
(a) Compute the position and velocity of the ball at the moment when the string snaps.

(b) Compute the xy-coordinates of the ball when it hits the ground.

Additional space for work on Exercise 6, if needed.

7. (20 points) Let $f(x,y) = (x^2 + y^2)e^{x-y}$.

(a) The contour diagram of f below clearly indicates a number of critical points.
(i) Mark these on the diagram, and give their xy-coordinates below the diagram.
(ii) Use the contour diagram to classify each critical point you found in (i) as a local min, local max, or saddle point.



Exercise 7 contd. $f(x, y) = (x^2 + y^2)e^{x-y}$.

(b) Now verify your answer in (a) by *algebraically* finding all critical points of f.
Your work must justify that you have indeed found *all* critical points of f.
Hint: factoring will be useful when solving the relevant system of equations.

Exercise 7 contd. $f(x, y) = (x^2 + y^2)e^{x-y}$.

(c) Now use the **second derivatives test** to classify each of your critical points as a local min, a local max, or neither. We will compute second derivatives for you!

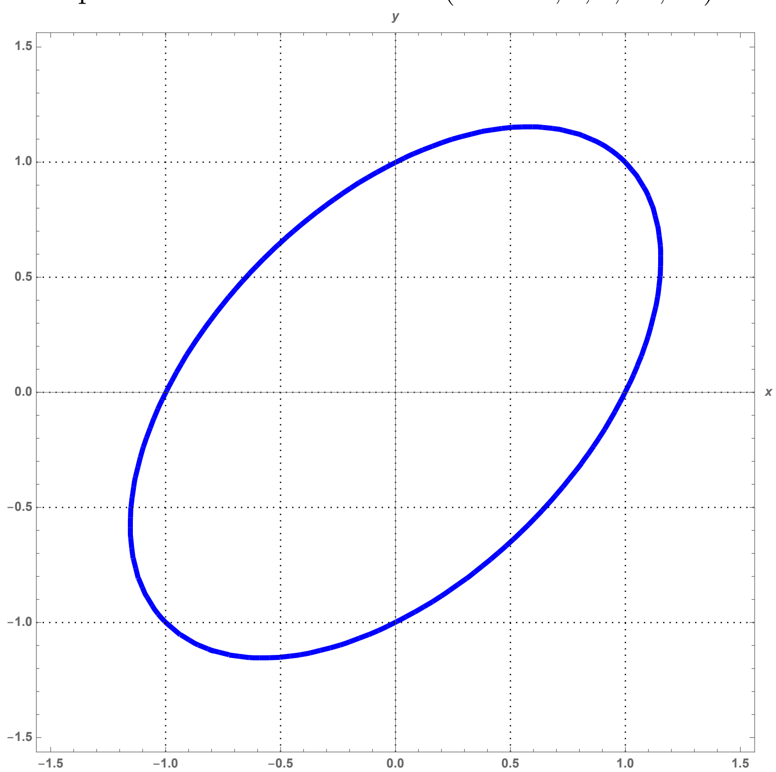
$f_{xx} = (2 + 4x + x^2 + y^2)e^{x-y}$ $\quad$ $f_{yy} = (2 - 4y + x^2 + y^2)e^{x-y}$ $\quad$ $f_{xy} = (2y - 2x - x^2 - y^2)e^{x-y}$

Note: even if you couldn't do part (b), you can use your critical points from part (a).

(d) Decide whether f has an *absolute* minimum value and/or an *absolute* maximum value on its entire domain. Justify your answer.

8. (15 points) Let $z = f(x, y) = 6x - 10y + 5$.

   (a) Below you find the ellipse with defining equation $x^2 - xy + y^2 = 1$. Draw a contour diagram of f on top of this that includes the ($z = -5, 0, 5, 10, 15$)-contours of f. Label your contours!



   (b) Your diagram will now suggest some *approximate* points on the ellipse where f obtains extreme values. Mark and name these points on the diagram, and for each determine whether it approximates an absolute minimum of f on the ellipse, an absolute maximum, or neither.

Exercise 8 contd. $f(x, y) = 6x - 10y + 5$. Ellipse: $x^2 - xy + y^2 = 1$.

(c) Now use the method of Lagrange multipliers to find the *exact* points on the ellipse where f obtains extreme values.

Furthermore, for each point you find, decide whether it represents an absolute minimum of f on the ellipse, an absolute maximum, or neither, and compute the value of f there.

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DO NOT WRITE ON THIS PAGE.