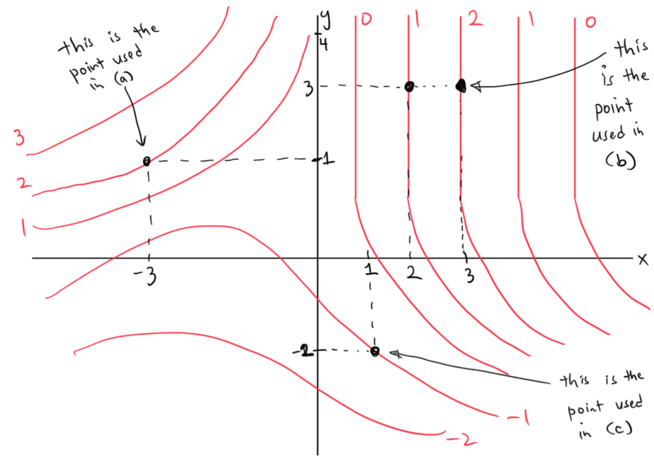
MATH 230-1: Written Homework 5
Northwestern University, Fall 2023

1. Below are some level curves of a continuous function $f(x, y)$.



Assume level curves which are not drawn occur at values of z strictly between those that are drawn. The portions of the level curves in the first quadrant that appear to be vertical are indeed meant to be vertical.

(a) When standing at the point $(-3, 1, 2)$, explain why the graph of f slopes downward when facing the direction of the vector $\mathbf{i}$ but slopes upward when facing the direction of the vector $\mathbf{j}$.

(b) Explain why $f(3, 3) \geq f(x, 3)$ for all x in some interval around 3. What can you say about the behavior of $f(3, y)$ for y in the interval $(1, 4)$?

(c) Find the value of the following limit, with justification.

$$\lim_{(x,y)\to(1,-2)} (xe^{f(x,y)} - 3)$$

2. The goal of this problem is to find the value of the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

in three different ways using the sandwich theorem. It will be useful to know that the inequality $|a| \leq b$ is equivalent to the chain of inequalities $-b \leq a \leq b$, so if you want to show justify the latter inequalities, you can instead justify the former.

(a) Justify the fact that

$$-\frac{1}{2}|x^2 - y^2| \leq \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq \frac{1}{2}|x^2 - y^2|$$

and then apply the sandwich theorem. Hint: Show that $|xy| \leq \frac{1}{2}(x^2 + y^2)$ using the fact that $(|x| - |y|)^2 \geq 0$.

(b) Justify the fact that

$$-|xy| \leq \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq |xy|$$

and then apply the sandwich theorem. Hint: Which of $x^2 + y^2$ or $|x^2 - y^2|$ is larger?

(c) Rewrite the given limit in terms of polar coordinates and then use the sandwich theorem.

3. The point of this problem is to demonstrate that checking the behavior of limits along lines alone is not enough to guarantee existence. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}.$$

(a) Find the limit when approaching $(0,0)$ along the x-axis, the y-axis, and the line $y = x$.

(b) Find the limit when approaching $(0,0)$ along *all* lines through $(0,0)$.

(c) Find a curve passing through $(0,0)$ such that the limit when approaching $(0,0)$ along this curve is different than the value found in (b), and conclude that the given multivariable limit does not exist.