MATH 230-1: Discussion 3 Solutions
Northwestern University, Fall 2023

1. Justify the fact that there does not exist a plane containing the line with parametric equations

$$x = 4 - 2t,\ y = -9 + 2t,\ z = t \quad -\infty < t < \infty$$

which is perpendicular to the line with parametric equations

$$x = -7 + 4t,\ y = 3 + 2t,\ z = 6 - t \quad -\infty < t < \infty.$$

Then, find a way to modify the equation for z in this second line to make it so that there *is* such a plane, and find the distance from $(1, 1, 1)$ to this new plane.

Solution. If a plane were to contain the line with the first set of parametric equations, then it must contain the direction vector of this line, which is $\langle -2, 2, 1\rangle$. This plane would be perpendicular to the direction vector $\langle 4, 2, -1\rangle$ of the second line, so $\langle -2, 2, 1\rangle$ would have to be perpendicular to $\langle 4, 2, -1\rangle$, which it is not since

$$\langle -2, 2, 1\rangle \cdot \langle 4, 2, -1\rangle = -8 + 4 - 1 = 5$$

is nonzero. Hence no such plane exists.

In order for such a plane to exist, the direction vector $\langle -2, 2, 1\rangle$ of the first line should be perpendicular to the direction vector $\langle 4, 2, k\rangle$ of the second line (with a to-be-determined equation for z as $z = 6 + kt$). This gives the requirement

$$\langle -2, 2, 1\rangle \cdot \langle 4, 2, k\rangle = -8 + 4 + k = 0,$$

so $k = 4$ and hence $z = 6 + 4t$ is one possible equation for z that works. The desired plane then has normal vector $\mathbf{n} = \langle 4, 2, 4\rangle$ and contains the point $P = (4, -9, 0)$ on the first line. The distance from $Q = (1, 1, 1)$ to this plane is the magnitude of the projection

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ} = \left(\tfrac{\overrightarrow{PQ}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\tfrac{-12+20+4}{16+4+16}\right)\langle 4, 2, 4\rangle = \tfrac{12}{36}\langle 4, 2, 4\rangle$$

where $\overrightarrow{PQ} = \langle -3, 10, 1\rangle$. The desired distance is thus

$$|\operatorname{proj}_{\mathbf{n}} \overrightarrow{PQ}| = |\tfrac{12}{36}\langle 4, 2, 4\rangle| = \tfrac{12}{36}\sqrt{36} = \tfrac{12}{\sqrt{36}}.$$

□

2. Consider the quadric surface with equation

$$-3x^2 + 2y^2 + 5z^2 = 1.$$

(a) Sketch the cross-sections of this surface at $x = 0, \pm 1, \pm 2, \pm 3$.

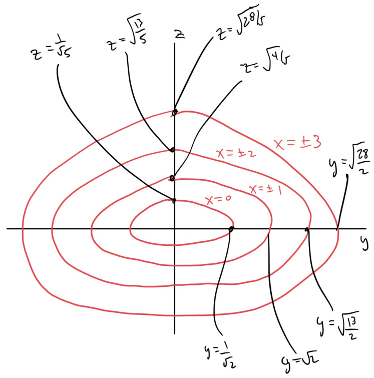
(b) There are two values of k such that the cross-sections at $z = k$ are pairs of straight lines. Determine these values of k.

(c) Identify the surface and find the points on it that are closest to the origin.

Solution. (a) The cross-section at $x = k$ has equation $-3k^2 + 2y^2 + 5z^2 = 1$, or

$$2y^2 + 5z^2 = 1 + 3k^2.$$

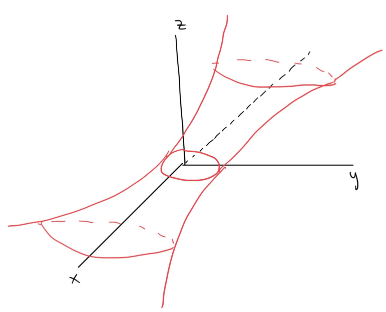
Since $1 + 3k^2$ is always positive, this always describes an ellipse in the yz-plane. At $x = 0$ this gives $2y^2 + 5z^2 = 1$, which is an ellipse with y-intercepts at $y = \pm 1\sqrt{2}$ when $z = 0$ and z-intercepts $z = \pm 1/\sqrt{5}$ when $y = 0$. At $x = \pm 1$ we have $2y^2 + 5z^2 = 4$, which has y-intercepts at $y = \pm\sqrt{2}$ and z-intercepts at $z = \pm\sqrt{4/5}$. At $x = \pm 2$ we get $2y^2 + 5z^2 = 13$, which has y-intercepts at $y = \pm\sqrt{13/2}$ and z-intercepts at $z = \pm\sqrt{13/5}$. And at $x = \pm 3$ we have $2y^2 + 5z^2 = 28$, with y-intercepts at $y = \pm\sqrt{28/2}$ and z-intercepts at $z = \pm\sqrt{28/5}$. These cross-sections thus look like



(b) The cross-section at $z = k$ has equation $-3x^2 + 2y^2 + 5k^2 = 1$, or $3x^2 + 2y^2 = 1 - 5k^2$. This describes a pair of straight lines when $1 - 5k^2$ is zero, so when $k = \pm 1/\sqrt{5}$. Indeed, at these k our cross-section equation is

$$-3x^2 + 2y^2 = 0, \text{ or } 2y^2 = 3x^2, \text{ or } y = \pm\sqrt{\tfrac{3}{2}}x.$$

(c) Since the cross-sections at any value of $x = k$ are ellipses, this is a hyperboloid of one sheet centered along the x-axis:



From the picture, the points closest to the origin occur on the thinnest part of the one-sheeted hyperboloid at $x = 0$, which we saw in (a) is an ellipse with equation $2y^2 + 5z^2 = 1$. The points on this ellipse closest to the origin are either its intercepts with the y-axis or with the z-axis. These intercepts were $y = \pm 1\sqrt{2}$ and $z = \pm 1/\sqrt{5}$ as found in (a), and since $\frac{1}{\sqrt{5}} < \frac{1}{\sqrt{2}}$, the points closest to the origin are the z-intercepts. Hence $(0, 0, \frac{1}{\sqrt{5}})$ and $(0, 0, -\frac{1}{\sqrt{5}})$ are the points on this surface that are closest to the origin. $\square$

3. Consider the quadric surface with equation

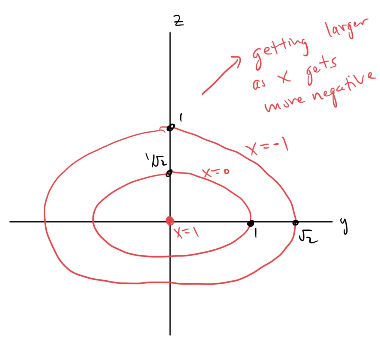
$$x = 1 - y^2 - 2z^2.$$

Sketch enough cross-sections that allow you to determine what this surface looks like, and then sketch the surface and identify it by name.

Solution. We consider cross sections at $x = k$, which have equations

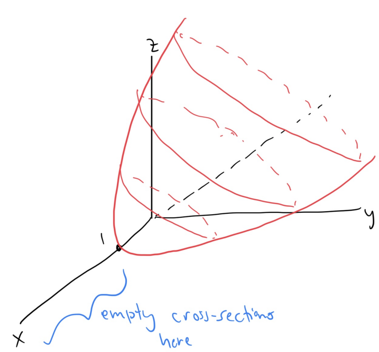
$$k = 1 - y^2 - 2z^2, \text{ or } y^2 + 2z^2 = 1 - k.$$

These are ellipses when $1 - k > 0$, a single point when $1 - k = 0$, and are empty (i.e., contain no points) when $1 - k < 0$. (The fact that we get ellipses—at least for $1 - k > 0$—instead of hyperbolas or parabolas as cross-sections is the reason why we chose to consider cross-sections at a fixed value of x as opposed to y or z.) So, at $x = 1$ we get $y^2 + 2z^2 = 0$, which describes the origin $(y, z) = (0, 0)$ of the yz-plane, at $x = 0$ we get $y^2 + 2z^2 = 1$ with y-intercepts $y = \pm 1$ and z-intercepts $z = \pm 1/\sqrt{2}$, and at $x = -1$ we get $y^2 + 2z^2 = 2$, with intercepts at $y = \pm\sqrt{2}$ and $z = \pm 1$:



For $x = 2$, or indeed any $x > 1$, we get an empty cross-section.

The surface is thus a paraboloid opening-up in the direction of decreasing $x \leq 1$:



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