MATH 230-1: Written HW 1 Solutions
Northwestern University, Fall 2023

1. The collection of points (x, y, z) in $\mathbb{R}^3$ whose distance to $(1, 2, 3)$ is the same as their distance to $(-2, 1, 0)$ forms a plane with an equation of the form $ax + by + cz = d$, where a, b, c, and d are some constants. (It is not important at this stage to know why this equation describes a plane—we will see why in Week 3.)

(a) Find an equation for this plane. Hint: Setup an equation saying that the distance from (x, y, z) to $(1, 2, 3)$ is equal to the distance from (x, y, z) to $(-2, 1, 0)$, and simplify algebraically until your equation looks like $ax + by + cz = d$.

(b) Sketch the intersections of the plane you found with the xy-plane, with the yz-plane, and with the xz-plane.

(c) The portion of the plane in question that lies in the first octant (i.e., the region where $x \geq 0, y \geq 0$, and $z \geq 0$) has a triangular shape with a triangle as its boundary. Find the perimeter of this triangle.

Solution. (a) The distance from (x, y, z) to $(1, 2, 3)$ is given by

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$

and the distance from (x, y, z) to $(-2, 1, 0)$ is given by

$$\sqrt{(x+2)^2 + (y-1)^2 + z^2}.$$

We are considering points for which these two are equal, so our desired condition is

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x+2)^2 + (y-1)^2 + z^2}.$$

To put this into the desired form, we simplify by first squaring both sides:

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = (x+2)^2 + (y-1)^2 + z^2.$$

Expanding everyone out gives

$$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = x^2 + 4x + 4 + y^2 - 2y + 1 + z^2,$$

which further simplifies to

$$-2x - 4y - 6z + 14 = 4x - 2y + 5.$$

A bit more simplification to $6x + 2y + 6z = 9$ gives our desired equation. (The equation you found does not have to look exactly like this, but it should be possible to obtain it from this one by multiplying through by a nonzero number.)

(b) The intersection of the plane above with the xy-plane is found by setting $z = 0$ since $z = 0$ is the equation of the xy-plane. This intersection thus has equation

$$6x + 2y = 9,$$

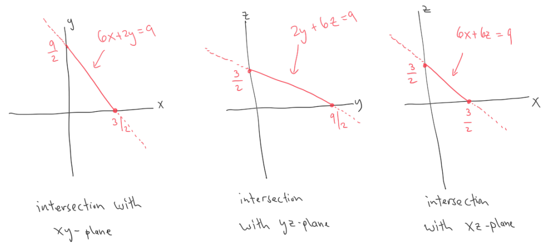
which is the equation of a line in the xy-plane. The yz-plane has equation $x = 0$, so the intersection of the plane in (a) with the yz-plane is found by setting $x = 0$, which gives

$$2y + 6z = 9,$$

which is a line in the yz-plane. Finally, the intersection with the xz-plane occurs when $y = 0$, so the intersection is the line

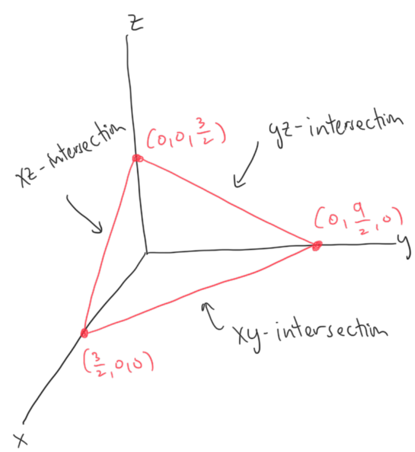
$$6x + 6z = 9$$

in the xz-plane. These intersections thus look like



(If you put all three intersections in the same drawing, as we do in part (c) below, that's fine as long as you're clear in your labeling about which intersection is which. Also, you did not have to necessarily find the equations for these intersections as we did above, but you should be clear about what the intersection actually is; for example, labeling the points where these intersections meet the axes is also a valid way to describe these specific lines.)

(c) The triangle in question has vertices at $(\frac{3}{2}, 0, 0)$, $(0, \frac{9}{2}, 0)$, and $(0, 0, \frac{3}{2})$:



The perimeter is found by adding the distances between these points. These distances are

$$\sqrt{(\tfrac{3}{2})^2 + (\tfrac{9}{2})^2 + 0^2} = \sqrt{\tfrac{90}{4}}, \ \sqrt{0^2 + (\tfrac{9}{2})^2 + (\tfrac{3}{2})^2} = \sqrt{\tfrac{90}{4}}, \ \text{ and } \sqrt{(\tfrac{3}{2})^2 + 0^2 + (\tfrac{3}{2})^2} = \sqrt{\tfrac{18}{4}},$$

so the perimeter is $\frac{\sqrt{90}}{2} + \frac{\sqrt{90}}{2} + \frac{\sqrt{18}}{2}$. (No need to simplify.) $\square$

2. The goal of this problem is to visualize the region in $\mathbb{R}^3$ consisting of points (x, y, z) satisfying all of the inequalities

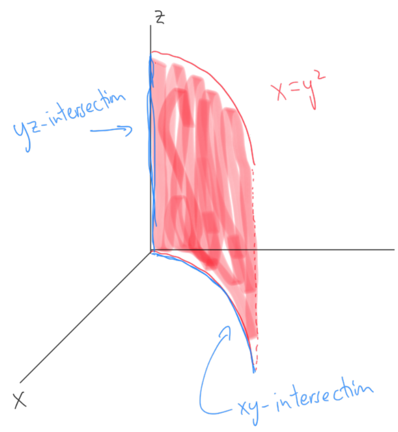
$$0 \le y \le 1, \ 0 \le x \le y^2, \text{ and } 0 \le z \le 1 - y.$$

(a) Sketch surface with equation $x = y^2$, labeling clearly its intersection with the xy-plane and with the yz-plane. Then, in a separate drawing, sketch the surface with equation $z = 1 - y$, labeling clearly its intersection with the both the yz-plane and the xy-plane. Only sketch the portions of these surfaces that lie in the first octant.

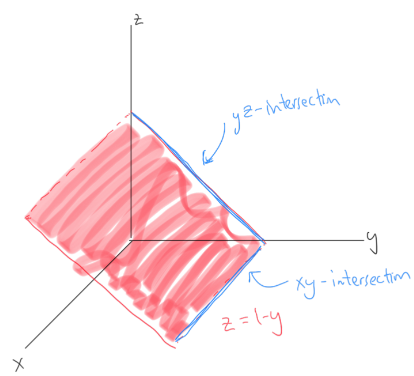
(b) The points $(0, 0, 1)$ and $(1, 1, 0)$ both lie on the surfaces in (a) since their coordinates satisfy the equation of these two surfaces. Thus, these two points lie on the intersections of these two surfaces. Use this to give a rough sketch of this intersection, which is some kind of curve. You DO NOT have to try to describe this intersection algebraically in terms of equations, only give a rough sketch. Hint: The curve in question should lie above the intersection of $x = y^2$ and the xy-plane drawn in (a), and in front of the intersection of $z = 1 - y$ and the yz-plane.

(c) Sketch the region described by the inequalities at the start of the problem as best you can.

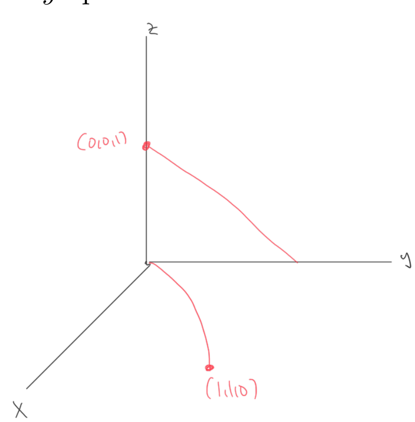
Solution. (a) The surface given by $x = y^2$ is obtained by taking the curve $x = y^2$ in the xy-plane and moving it along the z-direction to sweep out a surface. (This is because the given equation places no restriction on z, so taking any point on the curve $x = y^2$ in the xy-plane and modifying its z-coordinate will produce a point that is still on this surface.) This looks like



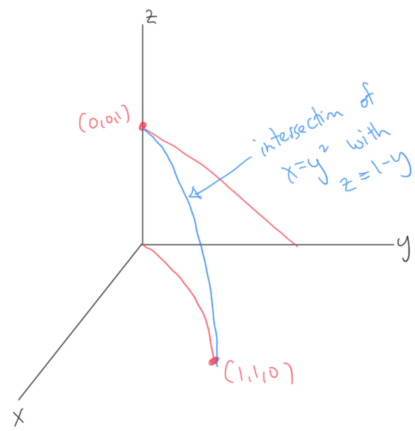
Similarly, the surface with equation $z = 1-y$ is obtained by taking the line with equation $z = 1-y$ in the yz-plane and moving it in the x-direction (since there is no restriction on x), which gives



(b) Let us start with the following picture, which gives the intersections of $x = y^2$ with the xy-plane and of $z = 1 - y$ with the yz-plane:

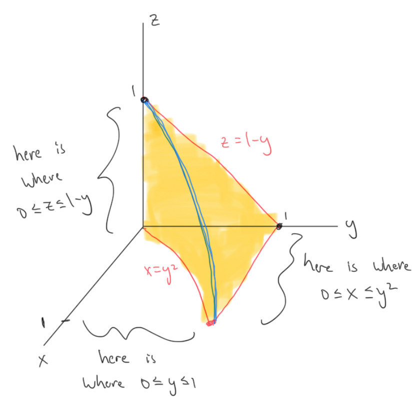


The curve in which $x = y$ and $1 - y$ intersect should be a path connecting $(0, 0, 1)$ to $(1, 1, 0)$ in this picture, so it should look like



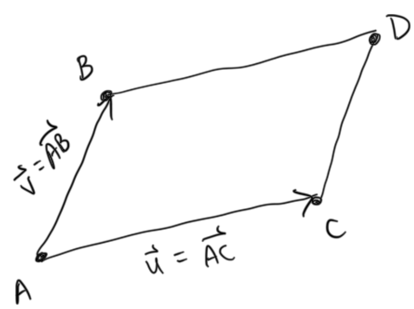
To be clear, we drew this curve here to lie "above" $x = y^2$ in the xy-plane since it lies on the surface with equation $x = y^2$, and we drew this curve to lie in "front" of $z = 1 - y$ in the yz-plane since it lies on the surface with equation $z = 1 - y$.

(c) The region in question looks like



Indeed, we first restrict to y coordinates between 0 and 1 due to the constraint $0 \leq y \leq 1$, and then we restrict the base of this region in the xy-plane using $0 \leq x \leq y^2$, which describes the portion between the y-axis and the curve $x = y^2$ in the xy-plane. Finally, $0 \leq z \leq 1 - y$ describes z-coordinates that are no smaller than $z = 0$ on the xy-plane and no larger than the ones on the plane $z = 1 - y$. In other words, the region we want lies below this plane. (Your picture did not have to be accurate as this, but it should demonstrate roughly the correct region.) $\square$

3. The goal of this problem is to use vector arithmetic to justify the fact that diagonals of parallelograms bisect each other, meaning cut each other in half. Consider the parallelogram



The diagonal of this parallelogram coming out of the point A is the sum $\mathbf{u} + \mathbf{v}$.

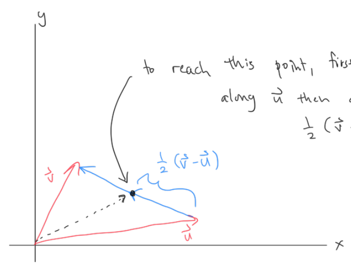
(a) What is the vector that begins at the point A and goes halfway up the diagonal $\mathbf{u}+\mathbf{v}$?

(b) Find, in terms of $\mathbf{u}$ and $\mathbf{v}$, the vector that describes the *other* diagonal of this parallelogram, namely the diagonal that goes between B and C. Then, find the vector that begins at the point A and ends halfway along this other diagonal.

(c) How do the results of (a) and (b) justify the fact about bisecting diagonals?

Solution. (a) The vector that goes from the origin to halfway along $\mathbf{u}+\mathbf{v}$ is $\frac{1}{2}(\mathbf{u}+\mathbf{v}) = \frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}$, since multiplying by $\frac{1}{2}$ has the effect of making $\frac{1}{2}(\mathbf{u}+\mathbf{v})$ has as long as $\mathbf{u}+\mathbf{v}$.

(b) The remaining diagonal of this parallelogram is given by the vector $\mathbf{v}-\mathbf{u}$ since it goes from the endpoint of $\mathbf{u}$ to the endpoint of $\mathbf{v}$. (We could also have used $\mathbf{u}-\mathbf{v}$ instead and modified the rest of this solution accordingly.) We want to describe the process of reaching the midpoint of this diagonal using vectors. To do so we first must move along $\mathbf{u}$, and then halfway along $\mathbf{v}-\mathbf{u}$:



Thus the sum $\mathbf{u}+\frac{1}{2}(\mathbf{v}-\mathbf{u})$ is the vector drawn to start at the origin and end at the midway point of the remaining diagonal.

(c) Note the vector found in (b) can be simplified as

$$\mathbf{u}+\tfrac{1}{2}(\mathbf{v}-\mathbf{u}) = \mathbf{u}+\tfrac{1}{2}\mathbf{v}-\tfrac{1}{2}\mathbf{u} = \tfrac{1}{2}\mathbf{u}+\tfrac{1}{2}\mathbf{v}.$$

The fact that this is the same vector as that found in (a) says that this common point is indeed where the two diagonals intersect; in other words, the diagonals intersect *somewhere*, and the work above verifies that the endpoint of $\frac{1}{2}\mathbf{u}+\frac{1}{2}\mathbf{v}$ lies on both diagonals, so it must the point at which the intersect. By construction, this point occurs halfway along both diagonals, so at this intersection both diagonals are bisected as claimed. $\square$