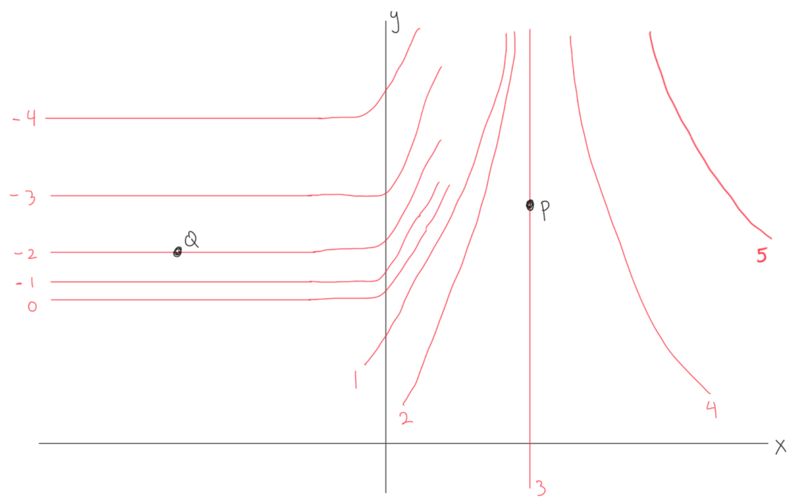
MATH 230-1: Written HW 6 Solutions
Northwestern University, Fall 2023

1. Below are some level curves of a function $f(x, y)$. Let us assume that the first and second-order partial derivatives of f exist and are continuous at all points.



Also assume the level curves that are not drawn occur at values of z strictly between those that are drawn, and that pieces that look vertical are meant to be vertical and pieces that look horizontal are meant to be horizontal.

(a) Determine whether the first-order partial derivatives

$$\frac{\partial f}{\partial x}(P),\ \frac{\partial f}{\partial y}(P),\ \frac{\partial f}{\partial x}(Q),\ \text{and}\ \frac{\partial f}{\partial y}(Q)$$

are positive, negative, or zero. Justify your answer.

(b) Determine whether the second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(P),\ \frac{\partial^2 f}{\partial y^2}(P),\ \frac{\partial^2 f}{\partial x^2}(Q),\ \text{and}\ \frac{\partial^2 f}{\partial y^2}(Q)$$

are positive, negative, or zero. Justify your answer. Hint: Second-order derivatives taken with respect to the same variable twice measure concavity, just as ordinary second derivatives do.

(c) Determine whether the mixed second-order derivatives

$$\frac{\partial^2 f}{\partial y \partial x}(P)\ \text{and}\ \frac{\partial^2 f}{\partial x \partial y}(Q)$$

are positive, negative, or zero. Justify your answer. Hint: View, for example,

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{as} \quad \frac{\partial g}{\partial y}\ \text{where}\ g = \frac{\partial f}{\partial x},$$

so that the sign of this mixed second-order partial derivative depends on whether $g = \frac{\partial f}{\partial x}$ is increasing or decreasing with respect to y. What happens to the "slopes in the x-direction" measured by $\frac{\partial f}{\partial x}$ as we move in the y-direction at P? Do they get smaller or larger?

Solution. (a) The values of $f(x,y)$ are increasing in the x-direction when at P, as we can tell by looking at the labels on the level curves. Thus $\frac{\partial f}{\partial x}(P) > 0$. In the y-direction however, $f(x,y)$ remains constant near P, so $\frac{\partial f}{\partial y}(P) = 0$.

At Q, $f(x,y)$ remains constant when moving in the x-direction, so $\frac{\partial f}{\partial x}(Q) = 0$. In the y-direction, the values of $f(x,y)$ are decreasing (i.e., getting more negative), so $\frac{\partial f}{\partial y}(Q) < 0$.

(b) The function $\frac{\partial f}{\partial x}$ that gives the slope of the graph of f in the x-direction is positive at P, and also a bit to the left of P and a bit to the right of P. However, this slope is getting *less* positive as we move horizontally through P, as we can tell by how closely or far part the level curves are from one another; the further apart the level curves are, the less steep the graph is since it takes longer to move by a height (i.e., value of z) of 1. So, $\frac{\partial f}{\partial x}$ is actually decreasing as we move through P horizontally, so $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) < 0$. (In other words, the graph of f is concave down when facing the x-direction at P.)

Since $f(x,y)$ is constant in the y-direction at P, not only its first derivative but also its second derivative in the y-direction is 0, so $\frac{\partial^2 f}{\partial y^2}(P) = 0$. (In other words, the function $\frac{\partial f}{\partial y}$ that measures slope in the y-direction is zero at P and a bit above and below P as well, so it remains constant as y changes and hence $\frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = 0$.)

For the same reason, $\frac{\partial^2 f}{\partial x^2}(Q) = 0$, since $\frac{\partial f}{\partial x}$ is the constant zero function not only at Q but also a bit to the left and to the right of Q. The function $\frac{\partial f}{\partial y}$ is negative at Q, but it gets *less* negative as we move through Q in the y-direction since the decreasing slope of the graph in the y-direction is getting less steep, due to the level curves being spaced further apart. Thus $\frac{\partial^2 f}{\partial y^2}(Q) > 0$, or in other words $\frac{\partial f}{\partial y}$ is an increasing function at Q with respect to y. (Note that negative numbers which get less negative actually increase.)

(c) The function $\frac{\partial f}{\partial x}$ which measures slope in the x-direction is positive at P, and also a bit below P and a bit above P. But this positive x-direction slope gets more positive as we move vertically (level curves closer together), so $\frac{\partial f}{\partial x}$ actually increases with respect to y at P. Thus $\frac{\partial^2 f}{\partial y \partial x}(P) > 0$.

Finally, $\frac{\partial f}{\partial y}$ is a negative function at Q, and also a bit to the left of Q and to the right of Q. However, this negative y-direction slope is the same a bit to the left of Q as it is at Q, and the same a bit to right of Q as well since the level curves remain the same distance apart. Thus $\frac{\partial f}{\partial y}$ is a constant function near Q in the x-direction, so $\frac{\partial^2 f}{\partial x \partial y}(Q) = 0$. $\square$

2. Suppose $f(x,y)$ is a function written in terms of rectangular coordinates whose first-order partial derivatives exist and are continuous at all points.

(a) Express the first-order partial derivatives of f with respect to polar coordinates in terms of its partial derivatives with respect to rectangular coordinates. The expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ you get should involve only $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, r$, and θ.

(b) Express the first-order partial derivatives of f with respect to rectangular coordinates in terms of its partial derivatives with respect to polar coordinates. The expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ you get should involve only $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}, x$, and y.

(c) Suppose a particle moves along a curve given by some *polar* parametric equations

$$r = 9t^3 + t + \frac{4}{3},\ \theta = \frac{\pi}{2} - \pi t.$$

If at the Cartesian point $(x,y) = (\sqrt{3},1)$ we know that $f_x(\sqrt{3},1) = -2$ and $f_y(\sqrt{3},1) = 5$, find the rate at which f is changing with respect to t at $t = \frac{1}{3}$.

Solution. (a) The chain rule gives

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \text{ and } \frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta.$$

(b) The chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

Since $r = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$ and $\theta = \arctan(y/x)$, we have

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}, \ \frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}},$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}) = -\frac{y}{x^2+y^2}, \ \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}(\frac{1}{x}) = \frac{x}{x^2+y^2}.$$

Thus

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial f}{\partial \theta}\frac{y}{x^2+y^2} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\frac{x}{x^2+y^2}.$$

(c) With f depending on x and y, x and y depending on r and θ, and r and θ depending on t, the chain rule gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\frac{d\theta}{dt}.$$

At $t = 1/3$, we have $(r,\theta) = (2, \frac{\pi}{6})$, and then $(x,y) = (\sqrt{3}, 1)$, so these are the points at which the derivatives above should be evaluated.

We have

$$\frac{dr}{dt} = 27t^2 + 1, \text{ so } \frac{dr}{dt}(1/3) = 4,$$
$$\frac{d\theta}{dt} = -\pi, \text{ so } \frac{d\theta}{dt}(1/3) = -\pi$$
$$\frac{\partial x}{\partial r} = \cos\theta, \text{ so } \frac{\partial x}{\partial r}(2,\pi/6) = \sqrt{3}/2$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \text{ so } \frac{\partial x}{\partial \theta}(2,\pi/6) = -1$$
$$\frac{\partial y}{\partial r} = \sin\theta, \text{ so } \frac{\partial y}{\partial r}(2,\pi/6) = 1/2,$$
$$\frac{\partial y}{\partial \theta} = r\cos\theta, \text{ so } \frac{\partial y}{\partial \theta}(2,\pi/6) = \sqrt{3}.$$

Thus altogether we get that

$$\frac{df}{dt}(1/3) = -2(\sqrt{3}/2)(4) - 2(-1)(-\pi) + 5(1/2)(4) + 5(\sqrt{3})(-\pi) = -4\sqrt{3} - 2\pi + 10 - 5\pi\sqrt{3}.$$

$\square$

3. Suppose $f(x, y)$ is a function with continuous first-order partial derivatives such that $f_x(1, 2) = 5$ and $f_y(1, 2) = -2$.

(a) Is there a unit direction vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(1, 2) = 6$? Is there a unit direction vector $\mathbf{v}$ such that $D_{\mathbf{v}}f(1, 2) = -6$? Explain.

(b) In order for the directional derivative $D_{\mathbf{u}}f(1, 2)$ to equal $\frac{\sqrt{29}}{2}$, what are the possible values of the angle between $\mathbf{u}$ and $\langle 5, -2\rangle$?

(c) If $f(1, 2) = -6$, find the Cartesian equation of the tangent line to the level curve $f(x, y) = -6$ at the point $(1, 2)$.

Solution. (a) Since $\nabla f(1, 2) = \langle 5, -2\rangle$, the largest value the directional derivative of f at $(1, 2)$ can be in any direction is

$$|\nabla f(1, 2)| = |\,\langle 5, -2\rangle\,| = \sqrt{29}$$

and the smallest the value the directional derivative can be in any direction is

$$-|\nabla f(1, 2)| = -|\,\langle 5, -2\rangle\,| = -\sqrt{29}.$$

Since $6 > \sqrt{29}$, there is no direction in which the directional derivative at $(1, 2)$ is 6, and since $-6 < -\sqrt{29}$, there is no direction in which it is -6 either.

(b) In order to have a directional derivative value of $\sqrt{29}/2$, the direction vector $\mathbf{u}$ must satisfy

$$\frac{\sqrt{29}}{2} = D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = |\nabla f(1, 2)| \cos\theta.$$

Since $|\nabla f(1, 2)| = \sqrt{29}$, this happens only when $\cos\theta = \frac{1}{2}$, so $\theta = \pm\pi/3$. Thus, $\nabla f(1, 2) = \langle 5, -2\rangle$ would have to be rotated by $\theta = \pi/3$ or $\theta = -\pi/3$ (or equivalently $\theta = 5\pi/3$ for the latter) in order to give the desired directions.

(c) The key point is that $\nabla f(1, 2) = \langle 5, -2\rangle$ is orthogonal to the desired tangent line. For a Cartesian equation, we use that (x, y) is on the desired tangent line when $\nabla f(1, 2)$ is orthogonal to $\langle x - 1, y - 2\rangle$. This gives the requirement that

$$\langle 5, -2\rangle \cdot \langle x - 1, y - 2\rangle = 0, \text{ so } 5(x - 1) - 2(y - 2) = 0$$

is the Cartesian equation of the tangent line. $\square$