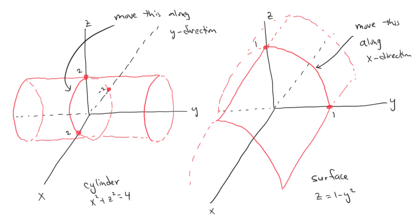
MATH 230-1: Discussion 1 Solutions
Northwestern University, Fall 2023

1. For the surface with equation $x^2 + z^2 = 4$ and for the surface with equation $z = 1 - y^2$, sketch or describe its intersection with the surfaces with each of the following equations:

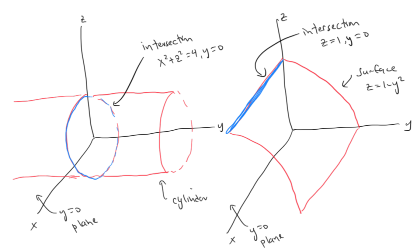
$$(a)\ y = 0 \quad (b)\ y = 1 \quad (c)\ x = 1 \quad (d)\ x = 2$$

Solution. First note that $x^2 + z^2 = 4$ describes a cylinder of radius 2 centered along the y-axis (which is obtained by taking the circle described by $x^2 + z^2 = 4$ in the xz-plane and sliding it along the y-axis) and $z = 1 - y^2$ describes the surface obtained by taking the downward-facing parabola $z = 1 - y^2$ in the yz-plane and sliding it along the x-axis:

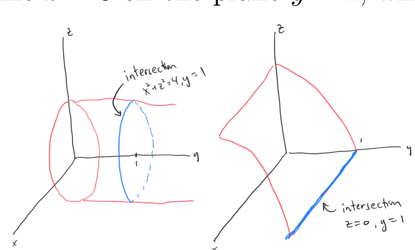


Also note that each of $y = 0, y = 1, x = 1$, and $x = 2$ describe planes.

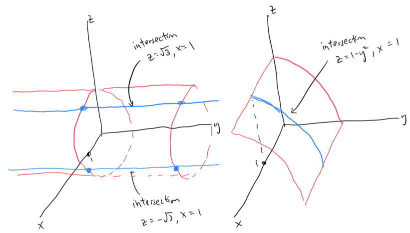
(a) The intersection of the cylinder $x^2 + z^2 = 4$ with the plane $y = 0$ is just the part of the cylinder that lies in the xz-plane (i.e., the plane where points have y-coordinate 0), so this is a circle of radius 2 centered at the origin of the xz-plane. The intersection of the surface $z = 1 - y^2$ with $y = 0$ is the object described by $z = 1 - 0^2 = 1$, so this is a line at $z = 1$ on the xz-plane:



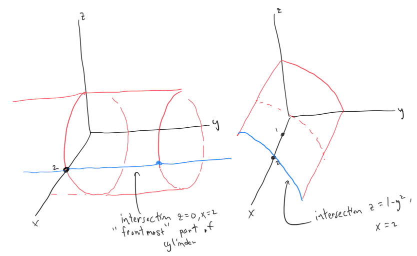
(b) The intersection of the cylinder with the plane $y = 1$ is the same circle as before, only shifted to lie on the plane $y = 1$. The intersection of the surface $z = 1 - y^2$ with $y = 1$ is described by $z = 1 - 1 = 0$, so this is the line $z = 0$ on the plane $y = 1$, which is parallel to the xz-plane:



(c) The intersection of the cylinder with the plane $x = 1$ is described by $1 + z^2 = 4$, or equivalently $z^2 = 3$, or $z = \pm\sqrt{3}$, which is a pair of lines. The intersection of the surface $z = 1 - y^2$ with $x = 1$ is just the downward-facing parabola $z = 1 - y^2$ drawn on the plane $x = 1$:



(d) The intersection of the cylinder and the plane $x = 2$ is described by $4 + z^2 = 4$, or $z = 0$, which is a line. The intersection of the surface $z = 1 - y^2$ with the plane $x = 2$ is the same parabola as before, only now lying at $x = 2$:



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2. The equation $x^2 - 2x + y^2 + 8y + z^2 - 10z + 38 = 0$ describes a sphere. Find the point on this sphere closest to the xy-plane, the point closest to the xz-plane, and the point closest to the plane with equation $x = -5$. Also, sketch or describe the intersection of this sphere with the yz-plane.

Solution. We put the given equation into the standard form of the equation for a sphere by completing the square. We have

$$x^2 - 2x = (x-1)^2 - 1, \; y^2 + 8y = (y+4)^2 - 16, \text{ and } z^2 - 10z = (z-5)^2 - 25,$$

so the original equation becomes

$$(x-1)^2 - 1 + (y+4)^2 - 16 + (z-5)^2 - 25 + 38 = 0,$$

which simplifies to

$$(x-1)^2 + (y+4)^2 + (z-5)^2 = 4.$$

This is thus a sphere of radius 2 centered at $(1, -4, 5)$.

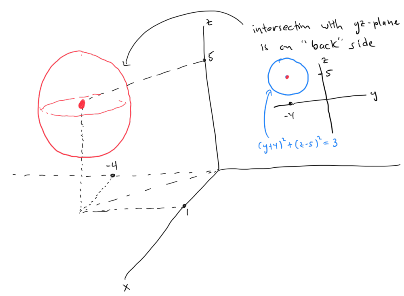
Since the center has z-coordinate 5 and radius 2, it lies completely above the xy-plane, so the point on the sphere closest to the xy-plane is the point with minimal z-coordinate, or in other words the "south" pole. This is the obtained by moving down from the center a distance equal to the radius, so $(1, -4, 3)$ is the point closest to the xy-plane.

The sphere lies completely to the left of the xz-plane (left means the direction where y is negative) since the center has y-coordinate -4 and the radius is only 2. The point on the sphere closest to the xz-plane is thus the "rightmost" point on the sphere, which is the one obtained by moving from the center to the right a distance equal to the radius, which gives $(1, -2, 5)$. The sphere lies completely in front of the plane $x = -5$, so the point on the sphere closest to this plane is the point on the sphere furthest "back", which is the point $(-1, -4, 5)$ obtained by moving from the center backwards a distance equal to the radius of 2.

The intersection of the sphere with the yz-plane consists of the points where $x = 0$, which means that the (y, z)-coordinates of these points must satisfy the equation

$$(0-1)^2 + (y+4)^2 + (z-5)^2 = 4, \text{ or } (y+4)^2 + (z-5)^2 = 3.$$

This intersection is thus a circle of radius $\sqrt{3}$ in the yz-plane centered at $(y, z) = (-4, 5)$:



$\square$

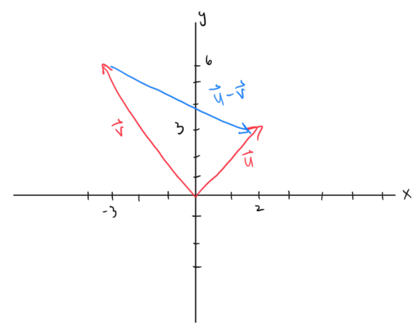
3. Set $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle -3, 6\rangle$, which we visualize as arrows drawn starting at the origin.

(a) Find the components of the vector of length 5 which points in the direction directly opposite that of the vector pointing from the endpoint of $\mathbf{v}$ to the endpoint of $\mathbf{u}$.

(b) Find a nonzero vector which is perpendicular to $2\mathbf{u} + \mathbf{v}$.

(c) If we write the vector $\langle -2, -1\rangle$ as $a\mathbf{u} + b\mathbf{v}$ for some numbers a and b, what will the signs of a and b be? Will b be smaller than or larger than 1? (Drawing a picture of $\mathbf{u}$, $\mathbf{v}$, and $\langle -2, -1\rangle$ may help!)

Solution. (a) We use this picture as a guide:



The vector pointing from the endpoint of $\mathbf{v}$ to the endpoint of $\mathbf{u}$ is

$$\mathbf{u} - \mathbf{v} = \langle 2, 3\rangle - \langle -3, 6\rangle = \langle 5, -3\rangle.$$

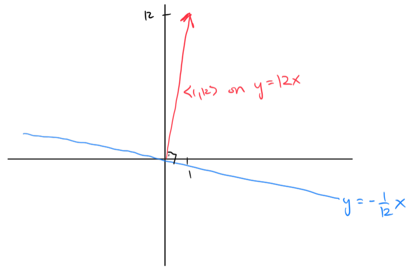
The vector point opposite this direction is $-(\mathbf{u} - \mathbf{v}) = \langle -5, 3\rangle$, and we can get a vector of the desired length by modifying the magnitude of this vector. Since $|\langle -5, 3\rangle| = \sqrt{25+9} = \sqrt{34}$, we first divide by $\sqrt{34}$ to get a unit vector in the desired direction:

$$\frac{\langle -5, 3\rangle}{|\langle -5, 3\rangle|} = \frac{1}{\sqrt{34}}\langle -5, 3\rangle,$$

and then we multiply by 5 to make the length equal 5:

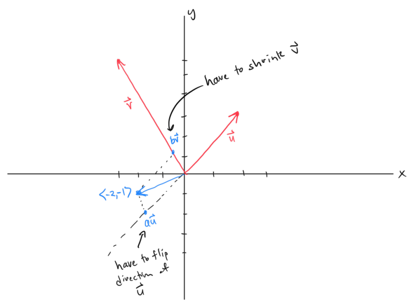
$$\tfrac{5}{\sqrt{34}}\langle -5, 3\rangle = \left\langle -\tfrac{25}{\sqrt{34}}, \tfrac{15}{\sqrt{34}}\right\rangle.$$

(b) We have $2\mathbf{u} + \mathbf{v} = 2\langle 2, 3\rangle + \langle -3, 6\rangle = \langle 4, 6\rangle + \langle -3, 6\rangle = \langle 1, 12\rangle$, which looks like



This vector points along a line of slope $\frac{12}{1} = 12$, so to get the perpendicular direction we need a slope of $-\frac{1}{12}$. The vector $\langle -12, 1 \rangle$, for example, points along this perpendicular line since its components satisfy $y = -\frac{1}{12}x$, so $\langle -12, 1 \rangle$ is one possible answer. (Any nonzero scalar multiple of $\langle -12, 1 \rangle$ will also work.)

(c) Consider the following picture:



Using the parallelogram interpretation of vector addition, we see that in order to obtain $\langle -2, -1 \rangle$ as a sum of $a\mathbf{u}$ and $b\mathbf{v}$, we would have to flip the direction of $\mathbf{u}$ and shrink the length of $\mathbf{v}$. Thus, a would have to be negative in order to make $a\mathbf{u}$ point in the direction opposite to that of $\mathbf{u}$, and b would have to be positive but smaller than 1 in order for $b\mathbf{v}$ to be shorter than $\mathbf{v}$ but still point in the same direction as $\mathbf{v}$. Thus $a < 0$ and $0 < b < 1$.

We can also determine this algebraically. In order to have

$$\langle -2, -1 \rangle = a \langle 2, 3 \rangle + b \langle -3, 6 \rangle = \langle 2a - 3b, 3a + 6b \rangle$$

be true, we would need a and b to satisfy

$$-2 = 2a - 3b \text{ and } -1 = 3a + 6b.$$

The first equation gives $a = -1 + \frac{3}{2}b$, and plugging into the second gives

$$-1 = 3(-1 + \tfrac{3}{2}b) + 6b = -3 + \tfrac{21}{2}b.$$

thus $b = (\frac{2}{21})2 = \frac{4}{21}$, which is positive and smaller than 1, and then $a = -1 + \frac{3}{2}(\frac{4}{21}) = -1 + \frac{12}{42}$, which is negative. (Although this can be answered algebraically by solving for a and b explicitly, this is too much work! The geometric solution based on the geometric interpretation of vector addition is more enlightening.) $\square$