MATH 230-1: Written Homework 4
Northwestern University, Fall 2023

1. Consider the curve which is the intersection of the surface with equation $y = e^x$ with the surface with equation $x^2 + z^2 = 4$.

(a) Find parametric equations for this curve.

(b) Find the points on this curve at which it intersects the surface with equation $\ln y = z$.

(c) Find parametric equations for the tangent line to this curve at one of the points (your choice!) found in part (b).

Solution. (a) Given that we need the x and z coordinates of points on this curve to satisfy $x^2+z^2 = 4$, we first take

$$x = 2\cos t \text{ and } z = 2\sin t \text{ for } 0 \leq t \leq 2\pi.$$

Then since points on the curve must also satisfy $y = e^x$, we take $y = e^{2\cos t}$, so altogether we get the parametric equations

$$x = 2\cos t,\ y = e^{2\cos t},\ z = 2\sin t \quad 0 \leq t \leq 2\pi.$$

Note that taking $x = t$ as the parametric equation for x does not quite work out. With this choice, we get $z^2 = 4 - t^2$, so that $z = \pm\sqrt{4-t^2}$. The problem is that we end up with *two* sets of parametric equations here, one where we take $z = \sqrt{4-t^2}$ and one where we take $z = -\sqrt{4-t^2}$, and each of these only describes a portion of the curve but not the entire curve all at once. The intent is to find a single set of parametric equations that gives the entire curve of intersection, so we must resort to using trigonometric functions.

(b) The curve in question intersects the surface with equation $\ln y = z$ when the coordinates we found above satisfy $\ln y = z$. This gives the requirement

$$\ln(e^{2\cos t}) = 2\sin t, \text{ or simply } \cos t = \sin t.$$

The values of the parameter t that satisfy this are $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$, so these give the desired intersections. For $t = \frac{\pi}{4}$ we get

$$(2\cos(\pi/4), e^{2\cos(\pi/4)}, 2\sin(\pi/4)) = (\sqrt{2}, e^{\sqrt{2}}, \sqrt{2})$$

as the point of intersection, and for $t = \frac{5\pi}{4}$ we get

$$(2\cos(5\pi/4), e^{2\cos(5\pi/4)}, 2\sin(5\pi/4)) = (-\sqrt{2}, e^{-\sqrt{2}}, -\sqrt{2})$$

as the point of intersection

(c) The problem only asked to find one tangent line, but we'll find both here. To find parametric equations for a line we need a point on the line (either one found in part (b)), and a direction vector for the line. The tangent line is parallel to the tangent vector at the given point, so we use this tangent vector as a direction vector. For $\mathbf{r}(t) = \langle 2\cos t, e^{2\cos t}, 2\sin t\rangle$, we have

$$\mathbf{r}'(t) = \left\langle -2\sin t, -2(\sin t)e^{2\cos t}, 2\cos t\right\rangle,$$

so the tangent vector at the point corresponding to $t = \frac{\pi}{4}$ is

$$\mathbf{r}'(\pi/4) = \left\langle -\sqrt{2}, -\sqrt{2}e^{\sqrt{2}}, \sqrt{2}\right\rangle$$

and the tangent vector at the point corresponding to $t = \frac{5\pi}{4}$ is

$$\mathbf{r}'(5\pi/4) = \left\langle \sqrt{2}, \sqrt{2}e^{-\sqrt{2}}, -\sqrt{2} \right\rangle.$$

Thus, the tangent line to the original curve at $(\sqrt{2}, e^{\sqrt{2}}, \sqrt{2})$ has parametric equations

$$x = \sqrt{2} - t\sqrt{2},\ y = e^{\sqrt{2}} - t\sqrt{2}e^{\sqrt{2}},\ z = \sqrt{2} + t\sqrt{2}$$

and the tangent line at $(-\sqrt{2}, e^{-\sqrt{2}}, -\sqrt{2})$ has parametric equations

$$x = -\sqrt{2} + t\sqrt{2},\ y = e^{-\sqrt{2}} + t\sqrt{2}e^{\sqrt{2}},\ z = -\sqrt{2} - t\sqrt{2}.$$

□

2. Suppose we launch a ball into the air from a height of 5 meters, at an angle of $\pi/4$ above the horizontal direction and at a speed of 100 meters per second. The only thing affecting the motion of the ball is the effect of gravity.

(a) Find the position vector $\mathbf{r}(t)$ of the ball at any time t.

(b) Determine the horizontal distance that the ball travels before it first hits the ground, and the maximal height attained along the way. Your answers should be actual numbers rounded to the nearest tenth, so use a calculator!

(c) Suppose now that a small rocket is attached to the ball, which gives it an additional (so gravity is still there) constant horizontal acceleration of 3 meters per second squared. The rest of the setup is the same as before. How far horizontally does the ball travel downfield now?

Solution. (a) The acceleration vector of the ball is $\mathbf{a}(t) = -g\,\mathbf{j}$, where g is about $g \approx 9.8$ meters per second squared. We integrate once to find velocity:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\,\mathbf{j}\,dt = c_1\,\mathbf{i} + (-gt + c_2)\,\mathbf{j}$$

for some constants c_1 and c_2. The initial speed of the ball is $|\mathbf{v}(0)| = 100$ and the initial velocity $\mathbf{v}(0)$ makes an angle of $\pi/4$ with the horizontal direction, so the initial velocity vector is

$$\mathbf{v}(0) = 100\cos(\pi/4)\,\mathbf{i} + 100\sin(\pi/4)\,\mathbf{j} = 50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j}.$$

Comparing this to the velocity $\mathbf{v}(t)$ we computed above gives the requirement that

$$50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j} = c_1\,\mathbf{i} + (-g \cdot 0 + c_2)\,\mathbf{j},$$

so $c_1 = 50\sqrt{2}$ and $c_2 = 50\sqrt{2}$. Thus our velocity vector is

$$\mathbf{v}(t) = 50\sqrt{2}\,\mathbf{i} + (-gt + 50\sqrt{2})\,\mathbf{j}.$$

We integrate again to find position:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int [50\sqrt{2}\,\mathbf{i} + (-gt + 50\sqrt{2})\,\mathbf{j}]\,dt = (50\sqrt{2}t + d_1)\,\mathbf{i} + (-\frac{1}{2}gt^2 + 50\sqrt{2}t + d_2)\,\mathbf{j}$$

where d_1, d_2 are some constants. Since the initial position of the ball is $\mathbf{r}(0) = 5\,\mathbf{j}$ (5 meters in the vertical direction), we must have

$$5\,\mathbf{j} = (50\sqrt{2} \cdot 0 + d_1)\,\mathbf{i} + (-\frac{1}{2}g \cdot 0^2 + 50\sqrt{2} \cdot 0 + d_2)\,\mathbf{j},$$

so $d_1 = 0$ and $d_2 = 5$. Thus the position vector of the ball at time t is

$$\mathbf{r}(t) = 50\sqrt{2}t\,\mathbf{i} + (-\frac{1}{2}gt^2 + 50\sqrt{2}t + 5)\,\mathbf{j}.$$

(b) The ball hits the ground when the vertical component of $\mathbf{r}(t)$, which is the coefficient of $\mathbf{j}$, is zero. Thus this happens when

$$-\frac{1}{2}gt^2 + 50\sqrt{2}t + 5 = 0,$$

and the quadratic formula gives

$$t = \frac{-50\sqrt{2} \pm \sqrt{2 \cdot 50^2 + 4(g/2)5}}{-2(g/2)}.$$

Using $g = 9.8$, plugging into a calculator, and taking the positive value of t gives

$$t \approx 14.5 \text{ seconds}$$

as the time when the ball hits the ground. The horizontal distance of the ball at this time is the $\mathbf{i}$-component of $\mathbf{r}(14.5)$, so it is $50\sqrt{2}(14.5) \approx 1,025.3$ meters.

The maximal height the ball attains is obtained by finding the maximum of the $\mathbf{j}$-component $-\frac{1}{2}gt^2 + 50\sqrt{2}t + 5$ of $\mathbf{r}(t)$. Setting the derivative of this function equal to zero gives

$$-gt + 50\sqrt{2} = 0, \text{ so that } t = \frac{50\sqrt{2}}{g} \approx 7.2 \text{ seconds}$$

as the time at which the maximal height is attained. The maximal height is thus

$$-\frac{1}{2}g(7.2)^2 + 50\sqrt{2}(7.2) + 5 \approx 260.1 \text{ meters.}$$

(c) Now the acceleration of the ball is given by

$$\mathbf{a}(t) = 3\,\mathbf{i} - g\,\mathbf{j},$$

which we integrate to find velocity and position. Note that the $\mathbf{j}$-component of the answer will remain the same as it was in part (a) since only the $\mathbf{i}$-component of acceleration has changed. We get

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = (3t + c_1)\,\mathbf{i} + (-gt + c_2)\,\mathbf{j}.$$

With the same initial velocity $\mathbf{v}(0) = 50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j}$ as before with get $c_1 = 50\sqrt{2}$ and $c_2 = 50\sqrt{2}$, so our new velocity is

$$\mathbf{v}(t) = (3t + 50\sqrt{2})\,\mathbf{i} + (-gt + 50\sqrt{2})\,\mathbf{j}.$$

Then we get

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (\tfrac{3}{2}t^2 + 50\sqrt{2}t + d_1)\,\mathbf{i} + (-\tfrac{1}{2}gt^2 + 50\sqrt{2}t + d_2)\,\mathbf{j},$$

and $\mathbf{r}(0) = 5\,\mathbf{j}$ gives $d_1 = 0$ and $d_2 = 0$ as before, so our final position vector is

$$\mathbf{r}(t) = (\tfrac{3}{2}t^2 + 50\sqrt{2}t)\,\mathbf{i} + (-\tfrac{1}{2}gt^2 + 50\sqrt{2}t + 5)\,\mathbf{j}.$$

The balls hits the ground at $t \approx 14.5$ seconds as before since the $\mathbf{j}$-component of position is the same as before, but now the $\mathbf{i}$-component of position at this time is

$$\tfrac{3}{2}(14.5)^2 + 50\sqrt{2}(14.5) \approx 1340.7 \text{ meters.}$$

$\square$

3. Consider the curve in the xy-plane with polar equation $r = \theta^2$ for $0 \le \theta < 2\pi$.

(a) Find parametric equations for this curve, using t as the parameter.

(b) Find the arclength parameter function $s(t)$ for this curve, measured starting at the point with Cartesian coordinates $(\frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}, \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2})$

(c) Find the point on this curve that is at a distance of 1 away from the starting point in (b) as measured along the curve in the direction of increasing values of the parameter t. Give the answer in terms of numbers rounded to the nearest tenth, so again use a calculator.

Solution. (a) We know that $x = r\cos\theta$ and $y = r\sin\theta$ in polar coordinates, and points where the r coordinate is $r = \theta^2$ have x and y coordinates given by

$$x = \theta^2 \cos\theta \text{ and } y = \theta^2 \sin\theta.$$

If we relabel θ as t we get

$$x = t^2 \cos t, \;\; y = t^2 \sin t$$

as the desired parametric equations. (We relabeled θ as t only to match the usual notation for the parameter we've seen in arclenghth problems. Keeping it at θ makes no material difference.)

(b) The point with Cartesian coordinates $(\frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}, \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2})$ is the point where $t = \frac{\pi}{4}$ in the parametric equations above. Thus, the distance from this point to some other point at parameter value $t > \frac{\pi}{4}$ is given by

$$s(t) = \int_{\pi/4}^{t} |\mathbf{r}'(u)|\, du$$

where $\mathbf{r}(t) = t^2 \cos t\, \mathbf{i} + t^2 \sin t\, \mathbf{j}$ is the position vector of the curve. (One subtle point is that this formula only applies for $t > \frac{\pi}{4}$, so points occurring *after* $(\frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}, \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2})$ in the direction of increasing t. For $t < \frac{\pi}{4}$ the integral above is negative, so the bounds must be switched to $\int_t^{\pi/4}$ to get the correct distance. No worries if you did not note this in your solution.) We compute

$$\mathbf{r}'(t) = \left\langle 2t\cos t - t^2 \sin t, 2t \sin t + t^2 \cos t \right\rangle,$$

and

$$\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{(2t\cos t - t^2 \sin t)^2 + (2t\sin t + t^2 \cos t)^2} \\
&= \sqrt{4t^2 \cos^2 t - 2t^3 \sin t \cos t + t^4 \sin^2 t + 4t^2 \sin^2 t + 2t^3 \sin t \cos t + t^4 \cos^2 t} \\
&= \sqrt{4t^2(\cos^2 t + \sin^2 t) + t^4(\sin^2 t + \cos^2 t)} \\
&= \sqrt{4t^2 + t^4} \\
&= \sqrt{t^2(4 + t^2)}.
\end{aligned}$$

Using $|\mathbf{r}'(t)| = t\sqrt{4+t^2}$, or $|\mathbf{r}'(u)| = u\sqrt{4+u^2}$ to match our variable of integration, we get

$$s(t) = \int_{\pi/4}^{t} |\mathbf{r}'(u)|\, du = \int_{\pi/4}^{t} u(4+u^2)^{1/2}\, du = \frac{1}{3}(4+u^2)^{3/2}\Big|_{\pi/4}^{t} = \frac{1}{2}(4+t^2)^{3/2} - \frac{1}{2}(4 + \tfrac{\pi^2}{16})^{3/2}.$$

If this was your answer, you'll get full credit, even though as explained before this is technically only valid for $t > \pi/4$.

(c) The point beyond $(\frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}, \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2})$ in the direction of increasing t on the curve that is at a distance of 1 away from this point is the point at which $s(t) = 1$. This equation looks like

$$\frac{1}{2}(4+t^2)^{3/2} - \frac{1}{2}(4+\tfrac{\pi^2}{16})^{3/2} = 1,$$

and solving gives

$$t = \pm\sqrt{(2+(4+\tfrac{\pi^2}{16})^{3/2})^{2/3} - 4} \approx \pm 1.1.$$

Of these, $t \approx 1.1$ is the one that gives a point beyond $(\frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2}, \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2})$ since -1.1 is not larger than $t = \pi/4$ where the starting point occurs, and this gives

$$(1.1^2 \cos(1.1), 1.1^2 \sin(1.1)) \approx (0.5, 1.0)$$

as the desired point. $\square$