Northwestern University

MATH 230-1 Final Exam
Fall Quarter 2022
December 7, 2022

Last name: SOLUTIONS Email address: ______

First name: ______ NetID: ______

Instructions

- Mark your section.

Section	Time	Instructor	
31	9:00	Lee	
41	10:00	Lee	
51	11:00	Cañez	
61	12:00	Schrader	
71	1:00	Tamarkin	
81	2:00	Tamarkin	

- This examination consists of 13 pages, not including this cover page. Verify that your copy of this examination contains all 13 pages. If your examination is missing any pages, then obtain a new copy of the examination immediately.
- This examination consists of 8 questions for a total of 100 points.
- You have two hours to complete this examination.
- Do not use books, notes, calculators, computers, tablets, or phones.
- Write legibly and only inside of the boxed region on each page.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. (This problem has five parts and continues on the next page.) Determine whether each of the following statements is true or false. Justify your answer.

(a) (4 points) The area of the parallelogram in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,1,1)$, $(1,-2,-1)$, and $(2,-1,0)$ is smaller than 3.

edges are $\langle 1,1,1\rangle$ and $\langle 1,-2,-1\rangle$

cross product $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \langle 1, 2, -3\rangle$

area $= |\langle 1,2,-3\rangle| = \sqrt{1+4+9} > \sqrt{9} = 3$ FALSE

(b) (4 points) There is a value of k such that the cross-section of the hyperbolic paraboloid $z = x^2 - y^2$ at $y = k$ is a hyperbola.

cross-section is $z = x^2 - k^2$,

which is a parabola.

FALSE

(c) (4 points) The limit $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$ exists.

$\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}$ is continuous at $(1,2)$

so limit exists TRUE

(d) (4 points) There is a point $(a, b, f(a, b))$ on the graph of $f(x, y) = xy^2$ at which the tangent plane is parallel to the plane $z = 9x + 6y$.

$f_x = y^2 \quad f_y = 2xy$

at $a = 1, b = 3$ get tangent plane

$z = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$

$= 9 + 9(x-1) + 6(y-3)$

which is parallel to $z = 9x + 6y$ $\boxed{\text{TRUE}}$

(e) (4 points) The function $f(x, y) = \cos(e^x + y)$ has an absolute maximum among points in the disk $(x-2)^2 + (y+1)^2 \leq 4$.

$f(x,y)$ is continuous and disk

is closed and bounded,

so absolute max exists by

extreme value theorem

$\boxed{\text{TRUE}}$

2. Consider the planes with equations

$$x + y + z = 1 \quad \text{and} \quad x - 2y - z = 2.$$

(a) (5 points) Find parametric equations for the line in which these planes intersect.

$z = 0 \rightsquigarrow x + y = 1 \to x = 1 - y \to x = 4/3$

$x - 2y = 2 \to 1 - y - 2y = 2 \Rightarrow y = -1/3$

so $(4/3, -1/3, 0)$ is on line of intersection

direction of line = cross product of normals = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \langle 1, 2, -3 \rangle$

So $\boxed{x = \frac{4}{3} + t,\ y = -\frac{1}{3} + 2t,\ z = -3t \quad -\infty < t < \infty}$

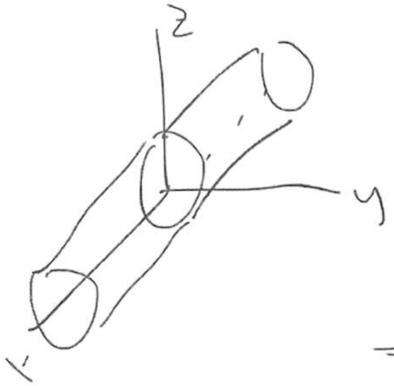
(b) (5 points) Determine if the line you found in part (a) intersects the plane $x + y + z = 0$.

line is on the plane $x + y + z = 1$, which parallel but not equal to $x + y + z = 0$. so line does not intersect $x + y + z = 0$

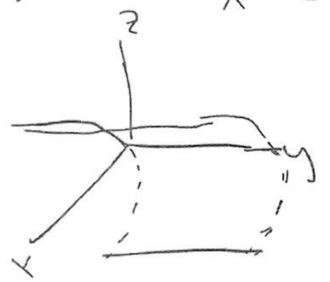
3. Consider the surfaces with equations $y^2 + z^2 = 1$ and $x = z^2$

(a) (5 points) Sketch or describe these surfaces. A valid description should indicate how the surface can be swept out by moving a specific curve in a specific plane in a specific direction.

$y^2+z^2=1$: cylinder obtained by moving circle $y^2+z^2=1$ in yz-plane in x-direction.



$x=z^2$: obtained by taking parabola $x=z^2$ in xz-plan and moving in y-direction



(b) (5 points) Find a vector function $\mathbf{r}(t)$ describing the curve along which these surfaces intersect and then find the acceleration vector for a point that moves along this curve.

Set $y = \cos t$, $z = \sin t$, Then $x = \sin^2 t$

So $\vec{r}(t) = \langle \sin^2 t, \cos t, \sin t \rangle$ $0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle 2 \sin t \cos t, -\sin t, \cos t \rangle$

acceleration $= \vec{r}\,''(t) = \langle 2\cos^2 t - 2\sin^2 t, -\cos t, -\sin t \rangle$

4. Consider the function $f(x, y) = xe^{xy}$.

(a) (5 points) Find a vector giving a direction in which the rate of change of $f(x, y)$ at $(2, 1)$ is 0.

$$\nabla f = \langle e^{xy} + xye^{xy}, x^2 e^{xy} \rangle$$

$$\nabla f\,(2,1) = \langle 3e^2, 4e^2 \rangle$$

Need $D_{\vec{u}} f\,(2,1) = \nabla f\,(2,1) \cdot \vec{u} = 0$

So direction is orthogonal to $\langle 3e^2, 4e^2 \rangle$

$\langle -4, 3 \rangle$ is one example that works

(b) (5 points) Compute all second-order partial derivatives of $f(x, y)$.

$$f_x = (1 + xy)e^{xy} \qquad f_y = x^2 e^{xy}$$

$$f_{xx} = ye^{xy} + y(1+xy)e^{xy}$$

$$f_{xy} = xe^{xy} + x(1+xy)e^{xy}$$ same as

$$2x e^{xy} + x^2 y e^{xy} = f_{yx}$$

$$f_{yy} = x^3 e^{xy}$$

5. Consider the surfaces with equations $x^2 + y^2 - z^2 = 1$ and $xyz = 8$.

   (a) (5 points) Find normal vectors to each surface at an arbitrary point (x, y, z).

level surfaces of $f(x,y,z) = x^2+y^2-z^2$

and $g(x,y,z) = xyz$

normals are $\nabla f = \langle 2x, 2y, -2z \rangle$

$\nabla g = \langle yz, xz, xy \rangle$

   (b) (5 points) Justify the fact that there are no points at which these surfaces intersect at a right angle. (The angle between the surfaces at a point of intersection is just the angle between their normal vectors at that point.)

$\nabla f \cdot \nabla g = 2xyz + 2yxz - 2zxy$

$= 2xyz.$

this is zero only when $x=0, y=0,$ or $z=0$

but none gives a point on $xyz = 8$.

6. Consider the function $f(x, y) = e^{2x+3y}$.

(a) (5 points) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at $(0, 0, 1)$.

$$f_x = 2e^{2x+3y} \qquad f_y = 3e^{2x+3y}$$

tangent plane is

$$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$= 1 + 2x + 3y$$

(b) (5 points) Justify the fact that using the tangent plane found in part (a) to approximate the value of $e^{2(0.1)+3(0.2)}$ results in an error no larger than $\frac{9}{2}e(0.3)^2$. (Using the tangent plane to approximate the value is the same as using what the book calls the linearization to approximate the value.)

$$f_{xx} = 4e^{2x+3y}$$

$$f_{xy} = 6e^{2x+3y}$$

$$f_{yy} = 9e^{2x+3y}$$

So $M = 9e$ bounds 2nd derivatives in range

For $0 \le x \le 0.1$ and $0 \le y \le 0.2$, we get

$$|f_{xx}| \le 4e^{2(0.1)+3(0.2)}$$

$$= 4e^{0.8} \le 4e$$

$$|f_{xy}| \le 6e^{2(0.1)+3(0.2)} \le 4e$$

$$|f_{yy}| \le 9e^{2(0.1)+3(0.2)} \le 9e$$

$$\text{error} \le \frac{1}{2}M\left[|\Delta x| + |\Delta y|\right]^2 = \frac{9}{2}e(0.1+0.2)^2$$

7. Consider the function $f(x, y) = x^2 + y^2 + xy^2$.

(a) (10 points) Find all critical points of $f(x, y)$ and determine whether each is a local minimum, a local maximum, or a saddle point.

$f_x = 2x + y^2 = 0 \qquad f_y = 2y + 2xy = 0$

$2y(1+x) = 0$

$y = 0$ or $x = -1$

$\underline{y=0}$ $2x = 0 \Rightarrow x = 0$

$\underline{x=-1}$ $-2+y^2 = 0 \Rightarrow y = \pm\sqrt{2}$

critical points are

$(0,0)$, $(-1, \sqrt{2})$, $(-1, -\sqrt{2})$

$f_{xx} = 2 \quad f_{xy} = 2y \quad f_{yy} = 2+2x$

@ $(0,0)$: $f_{xx} f_{yy} - (f_{xy})^2 = 2 \cdot 2 - 0 = 4$, $f_{xx} > 2 \rightarrow$ local min

@ $(-1, \pm\sqrt{2})$: $f_{xx} f_{yy} - (f_{xy})^2 = 2 \cdot 0 - (2(\pm\sqrt{2}))^2 < 0$ saddle points

(b) (10 points) Find the points at which $f(x, y)$ has an absolute maximum or an absolute minimum over the rectangle consisting of points (x, y) with $-3 \le x \le 2$ and $-1 \le y \le 1$.

Only critical point in rectangle is $(0,0)$

Along $y = -1$: $f(x,-1) = x^2 + 1 + x \longrightarrow$ derivative $= 2x+1$

critical point at $x = -1/2$

Along $x = -3$ $f(-3,y) = 9 - 2y^2 \longrightarrow$ derivative $= -4y$

critical at $y = 0$

Along $y = 1$: $f(x,1) = x^2 + 1 + x \longrightarrow$ critical $x = -1/2$

Along $x = 2$: $f(2,y) = 4 + 3y^2 \longrightarrow$ derivative $6y = 0$ critical

$y = 0$

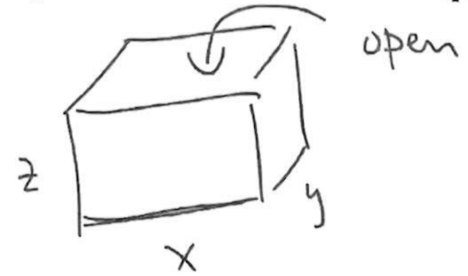
:

7b continued: Evaluate $f(0,0)=0$, $f(-\frac{1}{2},-1)=3/4=f(-\frac{1}{2},1)$

$f(-3,0)=9$, $f(2,0)=4$, $f(-3,-1)=7=f(-3,1)$, $f(2,-1)=3=f(2,1)$

max at $(-3,0)$, min at $(0,0)$

8. (10 points) Determine the dimensions of the open rectangular box of surface area 100 that maximizes volume among all open rectangular boxes of surface area 100. (To be clear, saying that the box is "open" means that there is no top, only four sides and a bottom.)



$$\text{area} = \underbrace{xy + 2yz + 2xz}_{g(x,y,z)} = 100$$

$$\text{Volume} = f(x,y,z) = xyz$$

$$\nabla f = \lambda \nabla g \rightarrow \langle yz, xz, xy \rangle = \lambda \langle y+2z, x+2z, 2y+2x \rangle$$

Get $yz = \lambda(y+2z) \longrightarrow xyz = \lambda(xy+2xz)$

$xz = \lambda(x+2z) \longrightarrow xyz = \lambda(xy+2yz)$

$xy = \lambda(2y+2x) \longrightarrow xyz = \lambda(2yz+2xz)$

$xy + 2yz + 2xz = 100$

None of x,y,z,λ is zero since volume $=0$ is not max

So get $2xz = 2yz \Rightarrow x = y$ $xy = 2yz \Rightarrow z = \frac{1}{2}x$

$\rightarrow x^2 + x^2 + x^2 = 100 \rightarrow x = \sqrt{\frac{100}{3}} = y$, $z = \frac{1}{2}\sqrt{\frac{100}{3}}$.