Northwestern University

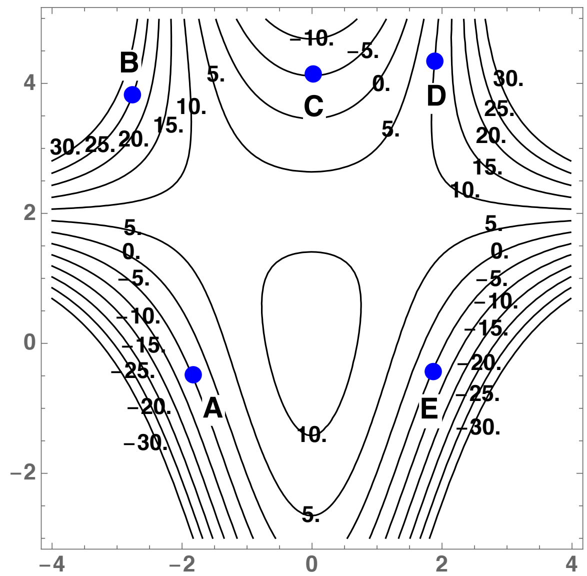
Math 230-1 Second Midterm Examination
Fall Quarter 2019
Tuesday 19 November

SOLUTIONS

1. (10 points) Below you find a contour diagram of $z = f(x, y)$ with five points labelled A-E.

   For each condition below, write the letter of the labelled point on the contour diagram that satisfies it.

   No justification required!



(i) $f_x(P) > 0$ and $f_{xx}(P) > 0$
D

(ii) $f_x(P) > 0$ and $f_{xx}(P) < 0$
A

(iii) $f_x(P) < 0$ and $f_{xx}(P) > 0$
B

(iv) $f_x(P) < 0$ and $f_{xx}(P) < 0$
E

(v) $f_x(P) = 0$
C

2. (15 points) Let $f(x,y) = x^2 - 2xy + y^2 - 3y$.

   (a) Find the directional derivative of f at $P = (1,2)$ in the direction of the vector $\langle 1, 1\rangle$.

   (b) Find the direction in which the directional derivative of f at $P = (1,2)$ is maximized. Your answer must be a *unit* vector. Justify!

   (c) Find a direction in which the directional derivative of f at $P = (1,2)$ is equal to 0. Your answer must be a *unit* vector. Justify!

Solution:
(a) First compute $\nabla f = \langle 2x - 2y, -2x + 2y - 3\rangle$ and $\nabla f(P) = \langle 2-4, -2+4-3\rangle = \langle -2, -1\rangle$.

Compute a unit vector in the direction of $\langle 1, 1\rangle$: $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$,

Then $D_{\mathbf{u}}f(P) = \nabla f(P)\cdot \mathbf{u} = \dfrac{1}{\sqrt{2}}\langle -2, -1\rangle \cdot \langle 1, 1\rangle = \dfrac{-3}{\sqrt{2}}$.

(b) $D_{\mathbf{u}}f(P)$ is maximized when $\mathbf{u}$ points in the same direction as $\nabla f(P) = \langle -2, -1\rangle$.

Thus we pick $\mathbf{u} = \dfrac{1}{\sqrt{5}}\langle -2, -1\rangle = -\dfrac{1}{\sqrt{5}}\langle 2, 1\rangle$.

(c) $D_{\mathbf{u}}(P) = 0$ when $\mathbf{u}$ is orthogonal to $\nabla f(P) = \langle -2, -1\rangle$: i.e., when $\langle -2, -1\rangle \cdot \mathbf{u} = 0$.

It is easy to see that $\langle 1, -2\rangle$ is orthogonal to $\langle -2, -1\rangle$. Thus pick $\mathbf{u} = \frac{1}{\sqrt{5}}\langle 1, -2\rangle$.

3. (10 points) Let $\mathcal{S}$ be the surface defined by the equation $x\ln y + y\ln z - x = 0$.
(a) Verify that $P = (1, 1, e)$ lies on $\mathcal{S}$.
(b) Find an equation of the tangent plane to $\mathcal{S}$ at P.

Solution:
(a) For $x = y = 1$ and $z = e$, we have

$$\begin{aligned} x\ln y + y\ln z - x &= 1\ln 1 + 1\ln e - 1 \\ &= 0 + 1 - 1 \qquad\qquad (\ln 1 = 0,\ \ln e = 1) \\ &= 0 \end{aligned}$$

(b) Let $g(x, y, z) = x\ln y + y\ln z - x$, so that $\mathcal{S}$ is the surface with equation $g(x, y, z) = 0$.

Compute $\nabla g = \langle \ln y - 1, \dfrac{x}{y} + \ln z, \dfrac{y}{z}\rangle$ and $\nabla g(1, 1, e) = \langle -1, 2, \frac{1}{e}\rangle$.

The tangent plane to $\mathcal{S}$ at $P = (1, 1, e)$ has normal vector $\nabla g(1, 1, e) = \langle -1, 2, \frac{1}{e}\rangle$.

The equation for the plane is thus

$$-(x - 1) + 2(y - 1) + \frac{1}{e}(z - e) = 0.$$

4. (10 points) A particle moves in $\mathbb{R}^2$ along the curve $\mathcal{C}$ with parametrization

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3},\ t^2 \right\rangle, \quad -\infty < t < \infty$$

Find the distance it travels as it moves along $\mathcal{C}$ from point $P_1 = (6, 9)$ to point $P_2 = (0, 0)$.
Consolation: things factor nicely in the integral you need to compute.

Solution: First find the inputs t corresponding to P_1 and P_2.

$\mathbf{r}(t) = \langle 6, 9\rangle \Rightarrow t^2 = 9$ & $t - \frac{t^3}{3} = 6 \Rightarrow t = -3$.

$\mathbf{r}(t) = \langle 0, 0\rangle \Rightarrow t^2 = 0 \Rightarrow t = 0$.

The distance travelled by the particle is given by the arc length between P_1 and P_2:

$$\int_{-3}^{0} |\mathbf{r}'(t)|\ dt.$$

Now compute

$$\mathbf{r}'(t) = \langle 1 - t^2, 2t\rangle$$

$$\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{(1-t^2)^2 + (2t)^2} \\
&= \sqrt{1 - 2t^2 + t^4 + 4t^2} \\
&= \sqrt{1 + 2t^2 + t^4} \\
&= \sqrt{(1+t^2)^2} \\
&= \left|1 + t^2\right| = 1 + t^2 && \text{(since } 1 + t^2 \geq 0\text{)}
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\int_{-3}^{0} |\mathbf{r}'(t)|\ dt &= \int_{-3}^{0} 1 + t^2\ dt \\
&= (t + \frac{1}{3}t^3)\Big|_{-3}^{0} \\
&= -(-3 - 9) \\
&= \boxed{12}.
\end{aligned}$$

5. (10 points) Decide whether the function f defined below is continuous at $P = (0,0)$. Justify your answer, and indicate how you are using the definition of continuity at a point.

$$f(x,y) = \begin{cases} \frac{x^2}{2x^2+3y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution: The function f is continuous at $P = (0,0)$ if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and is equal to $f(0,0) = (0,0)$.

Thus me must investigate $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2+3y^2}$.
Consider the limit along an arbitrary line passing through $(0,0)$: i.e., along the path with parametric equations

$$\begin{aligned} x &= at \\ y &= bt \end{aligned}$$

Along such a path we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{t\to 0} f(at,bt) \\ &= \lim_{t\to 0} \frac{a^2t^2}{2a^2t^2+3b^2t^2} \\ &= \frac{a^2}{2a^2+3b^2} \end{aligned}$$

Clearly this value depends on a and b. Explicitly, if we pick the line corresponding to $a = 0$ and $b = 1$, we get a limit of 0; whereas if we pick the line corresponding to $a = 1$ and $b = 0$, we get a limit of 1/2.

We conclude that the limit does not exist, since we have found two paths yielding different limits.

Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, f is not continuous at $(0,0)$.

6. (15 points) Toxic green slime mold proliferates in dark and humid conditions. More precisely, we can model the concentration M of mold (in spores per m^2) with the function

$$M = f(L, h) = 100e^{-L^2}h^2,$$

where L is average ambient light (in lux) and h is percent humidity (in decimal form).

Suppose light and humidity vary in your room as

$$\begin{aligned} L &= \sin(\pi x) - \cos(\pi y) + 2 \\ h &= e^{-x^2-y^2}, \end{aligned}$$

where x is your distance (in meters) east of the center of the room, and y is your distance (in meters) north of the center of the room.

(a) **Use the chain rule** to compute $\frac{\partial M}{\partial x}$ at $(x, y) = (0, 0)$.

Solution: By the chain rule we have

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial M}{\partial L}\frac{\partial L}{\partial x} + \frac{\partial M}{\partial h}\frac{\partial h}{\partial x} \\ &= -200Le^{-L^2}h^2(\pi\cos(\pi x)) + 200e^{-L^2}h(-2xe^{-x^2-y^2}) \\ &= -200\pi e^{-L^2}Lh^2(\cos(\pi x)) - 400e^{-L^2}hxe^{-x^2-y^2}) \end{aligned}$$

To evaluate this expression at $(x, y) = (0, 0)$ note that for this position in our room the corresponding values of L and h are

$$\begin{aligned} L &= 1 \\ h &= 1 \end{aligned}$$

Thus

$$\begin{aligned} \left.\frac{\partial M}{\partial x}\right|_{(x,y)=(0,0)} &= (-200\pi e^{-L^2}h^2(\cos(\pi x)) - 400e^{-L^2}hxe^{-x^2-y^2})\big|_{(x,y)=(0,0)} \\ &= -200\pi e^{-1} - 0 \\ &= -\frac{200\pi}{e} \end{aligned}$$

6.contd. Recall, we have

$M = f(L, h) = 100e^{-L^2}h^2$
$L = \sin(\pi x) - \cos(\pi y) + 2$
$h = e^{-x^2-y^2}$

(b) Interpret your computation of $\frac{\partial M}{\partial x}\Big|_{(x,y)=(0,0)}$ in (a) as a statement about mold in your room. Your answer should be a full sentence, should be comprehensible to someone with no calculus background, and should include all numeric details of your result in (a), along with units.

(If you were not able to do part (a), just set $\frac{\partial M}{\partial x}\Big|_{(x,y)=(0,0)} = c$ for an undetermined constant c and express your answer in terms of c.)

Solution: Starting at the center of our room, as we move to the east, the concentration of mold decreases at a rate of $\dfrac{200\pi}{e}$ (spores/m^2) per meter.

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