Northwestern University

Math 230-1 First Midterm Examination
Fall Quarter 2019
Tuesday 22 October

Last name: SOLUTIONS

First name: SOLUTIONS

1. (5 points) Compute the angle θ (in radians) between $\mathbf{v} = \langle \sqrt{3}, 3, 2 \rangle$ and $\mathbf{w} = \langle -\sqrt{3}, -3, 2 \rangle$.

   Your answer cannot be expressed in terms of inverse trigonometric functions; i.e., the answer is a familiar angle.

Solution: We have

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{-8}{\sqrt{16}\sqrt{16}} = -\frac{1}{2}.$$

It follows that $\theta = \frac{2\pi}{3}$.

2. (5 points) Let $\mathcal{C}$ be the the conic in $\mathbb{R}^3$ defined by the following system of equations:

$$\frac{(x-1)^2}{9} + \frac{(z-2)^2}{25} = 1$$
$$y = 3$$

   (a) Describe $\mathcal{C}$ qualitatively: include what type of conic it is, what its center is, and how it is situated in $\mathbb{R}^3$.

   (b) Give a vector parametrization $\mathbf{r}(t)$ for $\mathcal{C}$. Include explicit bounds $a \le t \le b$ ensuring that the entire curve is parametrized. **No justification required**.

Solution: The curve $\mathcal{C}$ is an ellipse, centered at $(1, 3, 2)$, situated in the plane $y = 3$.

We have $\mathbf{r}(t) = \langle 1, 3, 2 \rangle + \langle 3\cos t, 3, 5\sin t \rangle = \langle 1 + 3\cos t, 3, 2 + 5\sin t \rangle$, $0 \le t \le 2\pi$.

3. (10 points) Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero vectors.

   (a) Give the dot product formula for $\operatorname{proj}_{\mathbf{w}} \mathbf{v}$. **No justification required**.

   (b) Now suppose $\mathbf{v}$ is parallel to $\mathbf{w}$. Show, using only the formula in (a), that $\operatorname{proj}_{\mathbf{w}} \mathbf{v} = \mathbf{v}$. You should begin by expressing with a vector equation what it means for $\mathbf{v}$ to be parallel to $\mathbf{w}$.

Solution:
(a) We have $\operatorname{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$.
(b) We assume $\mathbf{v} = c\mathbf{w}$ for some $c \neq 0$. Then

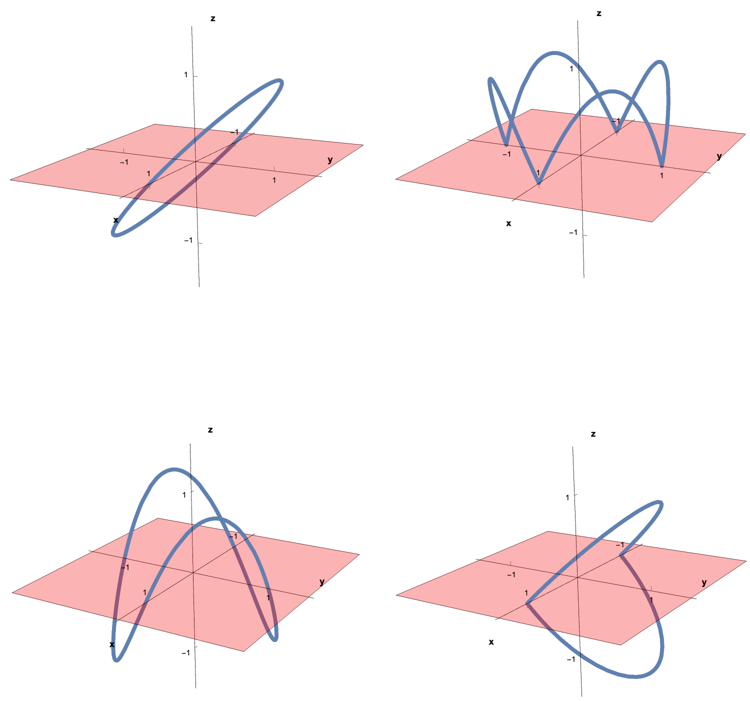
$$\begin{aligned}\operatorname{proj}_{\mathbf{w}} \mathbf{v} &= \frac{(c\mathbf{w}\cdot\mathbf{w})}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} \\ &= c\frac{\mathbf{w}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} \\ &= c\mathbf{w} \\ &= \mathbf{v},\end{aligned}$$

as claimed.

4. (15 points) Let $\mathcal{C}$ be the curve with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$.

   (a) Exactly one of the figures below is a graph of $\mathbf{r}(t)$ for $0 \le t \le 2\pi$. Identify which is correct via a process of elimination: that is, indicate each incorrect graph with an 'X' and briefly explain why it cannot be a graph of $\mathbf{r}(t)$; then indicate the correct graph with a checkmark.

   Note: I've included a shaded portion of the xy-plane in each figure to help you visualize the curve.



Solution: Th top left hits the xy-plane only twice, whereas $z = \sin(2t)$ is equal to 0 for $t = 0, \pi/2, \pi, 3\pi/2$.

The top right has only nonnegative z-coordinate values.

The bottom right has only nonnegative y-coordinates.

Thus the bottom left is the correct graph.

4. contd. Let $\mathcal{C}$ be the curve with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(2t)\rangle$.

(b) Give the parametric equations for the tangent line to $\mathcal{C}$ at $P = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1)$.

(c) Show that the velocity vector of a particle moving along $\mathcal{C}$ according to $\mathbf{r}(t)$ never points in the vertical direction: i.e., is never parallel to the z-axis.

Solution:
(b) First observe that $t = \pi/4$ is input corresponding to point P.
Next compute $\mathbf{r}'(t) = \langle -\sin t, \cos t, 2\cos(2t)\rangle$.
At $t = \pi/4$ we have $\mathbf{r}'(\pi/4) = \langle -\sqrt{2}/2, \sqrt{2}/2, 0\rangle$.
This is the direction vector for our tangent line. Taking the given P as our point on the line, we derive the parametric equations

$$\begin{aligned} x &= \sqrt{2}/2 - t\sqrt{2}/2 \\ y &= \sqrt{2}/2 + t\sqrt{2}/2 \\ z &= 1 \end{aligned}$$

(c) We saw above that $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -\sin t, \cos t, 2\cos(2t)\rangle$. For this to be vertical we need a t satisfying

$$-\sin t = 0 \tag{1}$$
$$\cos t = 0 \tag{2}$$

Equation (1) implies $t = \pi n$ for some n. But $\cos(\pi n) = (-1)^n \neq 0$. Thus there is no such t, and the velocity vector is never vertical.

5. (15 points) Let M be the plane through the points $P = (0,0,0)$, $Q = (1,-1,0)$, and $R = (1,0,1)$. Let N be the plane containing the point $S = (1,0,-2)$ with normal vector $\mathbf{n} = \langle 2,1,1\rangle$.
   (a) Find an equation for M.
   (b) Determine whether the planes M and N intersect. If they do intersect, find the parametric equations for their line of intersection.

Solution: (a) Using point $P = (0,0,0)$ and normal vector

$$\mathbf{n}' = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,-1,0\rangle \times \langle 1,0,1\rangle = \langle -1,-1,1\rangle,$$

we obtain the equation $-x - y + z = 0$, or $x + y - z = 0$ for M.

(b) Since the normal vectors for two planes are not parallel, the planes intersect.

To find the line of intersection L we observe that its direction vector, which lies in both planes, is orthogonal to $\mathbf{n}$ and $\mathbf{n}'$, and thus may be chosen as $\mathbf{v} = \mathbf{n} \times \mathbf{n}' = \langle 2,-3,-1\rangle$.

The equation of plane N is $2(x-1) + y + (z+2) = 0$, or $2x - y + z = 0$. To find a point on the intersection of the two planes we need to provide a solution to the system

$$\begin{aligned} x + y - z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

We see by inspection that $P = (0,0,0)$ itself is a solution. Using $P = (0,0,0)$ as our point on L, and $\mathbf{v} = \langle 2,-3,-1\rangle$ we obtain the parametric equations

$$\begin{aligned} x &= 2t \\ y &= -3t \\ z &= -t \end{aligned}$$

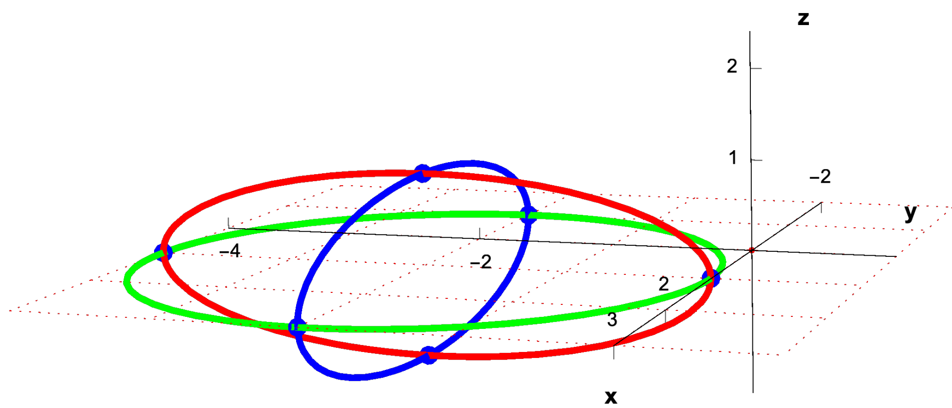
6. (10 points) Let $\mathcal{S}$ be the surface with equation $x^2 + y^2 + 4z^2 - 2x + 4y + 1 = 0$.

(a) Identify $\mathcal{S}$ as one of our familiar named surfaces. You should first do some algebra to bring the equation into a more standard form.

Justify your answer. You may reference your work in (b) if you like.

Solution: Completing squares yields the equation $(x-1)^2 + (y+2)^2 + 4z^2 = 4$, or $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} + z^2 = 1$. We recognize this as a standard form for an ellipsoid, shifted by $\langle 1, -2, 0\rangle$. Alternatively, as we see below, all three cross section types are ellipses.

(b) Find equations for the $(x = 1)$-, $(y = -2)-$ and $(z = 0)$-cross sections, and sketch these in the coordinate system below. Each cross section sketch must include at least 4 plotted points.



$x = 1$: $\frac{(y+2)^2}{4} + z^2 = 1$, an ellipse centered at $(1, -2, 0)$, parallel to the yz-plane. (In red)

$y = -2$: $\frac{(x-1)^2}{4} + z^2 = 1$, an ellipse centered at $(1, -2, 0)$, parallel to the xz-plane. (In blue)

$z = 0$: $(x - 1)^2 + (y + 2)^2 = 4$, a circle of radius 2 centered at $(1, -2, 0)$, in the xy-plane. (In green)

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