Integer Programming Formulations for Vehicle Routing Problems

Author | Ishank Juneja

May 25, 2018

1 Introduction

VRPs are problems concerning the distribution of goods from depots to final users (customers) Solving a VRP is equivalent to determining a set of routes that start and end at the same depot, such that operational constraints, customer requirements are satisfies and cost is minimized.
The road network is modelled as a Graph with edges (May be directed) as roads and vertices correspond to (potential) customer locations.

The parameters associated with a **customer/vertex** are

1. Vertex where customer is located
2. Demand associated with Vertex
3. Time windows during which a customer can be serviced
4. Overheads : Like loading and unloading times at depot or customer

The parameters associated with our **fleet of vehicles** is

1. Home depot/node of the vehicle
2. Capacity of vehicle (Number of Passengers). Capacity may be different for different goods
3. Subset of edges of graph that vehicle can traverse (Large Truck/Bus limitation)
4. Cost associated with utilizing a vehicle

Constraints on **Routes**
Its possible that there is a precedence associated with vertices (Pickup and Delivery problem/ VRP with Backhauls)
Objectives of VRPs are

1. Minimization of total transportation cost
2. Minimization of the number of vehicles
3. Balancing routes (to maintain traffic balance)
4. Minimization of penalties/tardiness

2 Classes of VRPs

2.1 Capacitated VRP : CVRP

Assumptions : Demands are known beforehand, vehicles are identical, there is a single central depot (home), there are capacity restrictions on vehicles.
The objective is to minimize total cost.

Notation

G = (V,A) is a complete Graph (Edge between every pair of distinct vertices), which is even stronger than the graph being strongly connected.
V = {0, ..., n} is the set of vertices, 0 is the 'home' node. A is the set of edges.
A cost $c_{ij} \geq 0$ is associated with every edge $(i, j) \in A, c_{ii} = \infty \quad \forall \quad i \in V$. If a direct edge (i,j) does not exist then the cost associated with it is ∞
If the graph is symmetric then $c_{ij} = c_{ji} \quad \forall \quad i, j \in V$
For any edge $e \in E$, $\alpha(e)$ and $\beta(e)$ denote the the endpoint vertices.
$\Delta^+(i)$ and $\Delta^-(i)$ denote the forward star (Nodes that are directly reachable)and backward star (Nodes from which i is directly reachable) of vertex i.
For any set $S \subseteq V$, $\delta(S)$ and E(S) denote sets of edges that have exactly one or both endpoints in S, respectively. $\delta(\{i\}) \cong \delta(i)$.
Each customer i (i = 1,...,n) is associated with a demand $d_i \geq 0$, $d_0 = 0$, the notation d(S) means $\sum_{i \in S} d_i$.
The number of identical vehicles is $K \geq K_{min}$, each has capacity C. We assume $d_i \leq C \quad \forall \quad i \in V$. K_{min} the minimum number of vehicles is obtained by solving an associated Bin packing problem (BPP).
For any $S \subseteq V$ {0}, r(S) is the min. number of vehicles required to serve customers in S (BPP solution associated with S). A lower bound on r(S) is

$$\lceil \frac{d(S)}{C} \rceil$$

Problem Objective

Find K simple (No vertex repeats) circuits (same start and end point paths) with minimum cost such that

- Each circuit visits the home/depot
- Each customer vertex is only visited by one circuit(vehicle)
- Sum of demands of vertices in a circuit is less than C. A variant of the problem may have separate capacities C_k, k = (1,...,K) for each vehicle.

2.2 Distance-Constrained VRP

The maximum capacity constraint is replaced with a max. distance/time constraint. A length $t_{ij}/t_e \geq 0$ is associated with each edge and the total route length should never exceed T
There can be service times s_i associated with the stop over at each node.
These service times can be absorbed in the t values as

$$t_{ij} = t'_{ij} + s_i/2 + s_j/2$$

If both capacity and distance have restrictions then the problem is called Distance-Constrained VRP (DCVRP).

2.3 VRP with Time Windows

Each customer i is associated with the additional parameter $[a_i, b_i]$, ie a time window.
The service to each customer must start within the time window. In case of early arrival, the vehicle must wait until a_i
VRPTW is modelled as an asymmetric problem (due to asymmetry introduced by time windows)

Problem Objective

Find a collection of K simple circuits with minimum cost such that

1. Each circuit visits the home/depot
2. Each customer vertex is only visited by one circuit(vehicle)
3. Sum of demands of vertices in a circuit is less than C.
4. The service for customer i starts within the window $[a_i, b_i]$ and the vehicle stops for s_i time instants. Not necessary that $a_i + s_i \leq b_i$

By taking $a_i = 0, b_i = \infty$ for each $i \in V\backslash\{0\}$ we get the CVRP.

2.4 VRP with Backhauls

The customer set $V\backslash\{0\}$ is partitioned into two subsets. Linehaul (Regular customers) : L = {1,...,n} and Backhaul (Like Suppliers) B = {n+1,...,n+m}
Just as before, $d_i \geq 0$ is associated with each customer, as either the demand or rhte quantity to be picked up.

Problem Objective

Find a collection of K simple circuits with minimum cost such that

1. Each circuit visits the home/depot
2. Each customer vertex is only visited by one circuit(vehicle)
3. Sum of demands of L and B vertices in a circuit do not separately exceed C.
4. In each circuit, all L vertices are visited before any B vertices.

Let K_L and K_B denote the min. number of vehicles required to service L and B only. Then $K \geq max\{K_L, K_B\}$
VRPB generalises to CVRP when B = ϕ

2.5 VRP with pickup and delivery - VRPPD

Two numbers d_i and p_i, the delivery and pickup demands, are associated with each vertex i.
O_i denotes the origin for the demand d_i, D_i denotes the vertex that is the destination for the pickup demand p_i

Problem Objective

Find a collection of K simple circuits with minimum cost such that

1. Each circuit visits the home/depot
2. Each customer vertex is only visited by one circuit(vehicle)
3. The instantaneous load of the vehicle travelling along a circuit should never exceed C
4. If for any i, O_i is different from the home/depot, then it must be served in the same circuit as i and before i.
5. For each i, if D_i is different from home/depot, then it must be serevd in the same circuit as i, after i.

VRPPD reduces to CVRP when $O_i = D_i = 0$ and $p_i = 0 \; \forall \quad i \in V$

3 Modelling as Programming Problems

The three approaches are

1. Vehicle Flow formulations
2. Commodity Flow Formulation
3. Set-Partitioning problem (SPP) using binary variables

3.1 Vehicle Flow Models

An IP formulation (VRP1) for Asymmetric CVRP is described.
Variable $x_{ij} = 1$ if (i,j)$\in A$ and $=$ 0 otherwise.

$$\text{Minimize} \quad \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{i \in V} x_{ij} = 1 \quad \forall j \in V \backslash \{0\}, \tag{2}$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall j \in V \backslash \{0\}, \tag{3}$$

2 and 3 imply that there is only one edge in and out of each customer.

$$\sum_{i \in V} x_{i0} = K, \tag{4}$$

$$\sum_{j \in V} x_{0j} = K, \tag{5}$$

4 and 5 are true since for the home vertex there are K in and out.

$$\sum_{i \notin S} \sum_{j \in S} x_{ij} \geq r(S) \quad \forall S \subseteq V \backslash \{0\}, S \neq \phi, \tag{6}$$

6 specifies that the cut (V \ S, S) is crossed by more edges than the min. number of vehicles required. Further each such cut is crossed in both directions (in and out) the same number of times. In fact 6 can be restated as

$$\sum_{i \notin S} \sum_{j \in S} x_{ij} \geq r(V \backslash S) \quad \forall S \subset V, 0 \in S. \tag{7}$$

Using 2, 3, 4 and 5 along with 6 we get

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - r(S) \quad \forall S \subseteq V \backslash \{0\}, S \neq \phi, \tag{8}$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j \in V \tag{9}$$

In the 6-8, the cardinality of constraints grow exponentially with n. It is practically impossible to solve directly the LP relaxation of of the problem in this form
A way out is to consider a subset of the constraints, and add **constraints only if needed**. Not sure what this means, related to sensitivity analysis in simplex algorithm and separation procedures.
The order of the number of constraints may be brought down (Polynomial cardinality) by introducing additional variables. This is similar to the treatment of sub tour elimination constraints of TSP.
Introducing $u_i, i \neq 0$ as the load of a vehicle after visiting node i. Some more constraints are

$$u_i - u_j + Cx_{ij} \leq C - d_j \quad \forall i, j \in V \backslash \{0\}, i \neq j, \text{ such that } d_i + d_j \leq C \tag{10}$$

$$d_i \leq u_i \leq C \quad \forall i \in V \backslash \{0\} \tag{11}$$

10 and 11 impose C and connectivity requirements for the CVRP.

3.2 Modifications of VRP-1

The ideas of VRP 1 may be modified to obtain the below formulations, described in [1].

1. VRP 2 : Symmetric counterpart of VRP 1
2. VRP 3 : Symmetric counterpart of VRP 1 but single customer routes are not allowed
3. VRP 4 : is a 3 index formulation that explicitly associates a vehicle index with an edge
4. VRP 5 : is te symmetric undirected version of VRP 4
5. VRP 6 : Solves the problem of VRP 1 but with the additional constraint that the graph is not complete, ie some edges are missing. Can be done by associating an M = INF cost with these missing edges.
   But VRP 6 considers the case of a sparse graph $|A| \ll n^2$.

6. m TSP : Replacing a single depot with K vertices (one for each vehicle). Implemented by defining $c_{ij}^{'}$ the cost as

$$c_{ij}^{'} := \begin{cases} c_{ij} & \text{for i,j} \in V\backslash\{0\} \\ c_{i0} & \text{for } i \in V\backslash\{0\}, j \in W \\ c_{0j} & \text{for } i \in W, j \in V\backslash\{0\} \\ \lambda & \text{for } i, j \in W \end{cases} \tag{12}$$

Where W is the set of k home depots and λ is a given weight If we pick $\lambda = M$ then using all the depots becomes necessary.

7. Cases where vehicles may be left unused are covered by replacing $=$ K constraints with $\leq K$ type constraints.
This may become crucial if there is some fixed cost associated with their use ex. Driver wages.

8. Non Homogeneous fleet : Easy to account for in 3 index model of VRP 4. A different C_k for each vehicle.

9. Prevention of single customer routes : In a CVRP, a customer j can be served alone in a route iff the remaining K-1 vehicles can satisfy the rest of the demand

$$r(V\backslash\{j\}) \leq K - 1$$

r(.) may be replaced by its trivial lower bound.

3.3 Commodity Flow Models

Formulation : VRP7

$G^{'} = (V^{'}, A^{'})$ which is derived from G but has n+1 as another depot like 0. This is done so that routes are paths from 0 to n+1 instead of circuits.
There are two non negative flow variables associated with every edge y_{ji} and y_{ij}. If a vehicle runs from i to j then y_{ij} is the vehicle load and $y_{ji} = C - y_{ij}$ is the vehicle capacity. The equation

$$y_{ij} + y_{ji} = C$$

holds for every edge (i, j) $\in A^{'}$.
The constraints are given below:

$$(1.62) \qquad \text{(VRP7)} \quad \min \sum_{(i,j)\in A'} c_{ij}x_{ij}$$

subject to

$$(1.63) \qquad \sum_{j\in V'} (y_{ji} - y_{ij}) = 2d_i \qquad \forall\, i \in V' \setminus \{0, n+1\},$$

$$(1.64) \qquad \sum_{j\in V'\setminus\{0,n+1\}} y_{0j} = d(V \setminus \{0, n+1\}),$$

$$(1.65) \qquad \sum_{j\in V'\setminus\{0,n+1\}} y_{j0} = KC - d(V \setminus \{0, n+1\}),$$

$$(1.66) \qquad \sum_{j\in V'\setminus\{0,n+1\}} y_{n+1\,j} = KC,$$

$$(1.67) \qquad y_{ij} + y_{ji} = Cx_{ij} \qquad \forall\, (i, j) \in A',$$

$$(1.68) \qquad \sum_{j\in V'} (x_{ij} + x_{ji}) = 2 \qquad \forall\, i \in V' \setminus \{0, n+1\},$$

$$(1.69) \qquad y_{ij} \geq 0 \qquad \forall\, (i, j) \in A',$$

$$(1.70) \qquad x_{ij} \in \{0, 1\} \qquad \forall\, (i, j) \in A'.$$

Figure 1: Constraint and formulation for VRP7. From [1]

3.4 Set-Partitioning Models

Formulation : VRP8

$\mathcal{H} = \{H_1, H_2, ..., H_q\}$ is the collection of all the feasible circuits associated with the original (G, A) formulation. Each circuit H_j has cost c_j. $a_{ij} \in \{0, 1\}$, $a_{i,j} = 1$ if vertex i is visited by route H_j, and =0 otherwise.
The binary variable x_j, $\quad j = 1, ..., q$ equals 1 iff H_j is part of the optimum solution.

$$(1.71) \qquad \text{(VRP8)} \quad \min \sum_{j=1}^{q} c_j x_j$$

subject to

$$(1.72) \qquad \sum_{j=1}^{q} a_{ij}x_j = 1 \qquad \forall\, i \in V \setminus \{0\},$$

$$(1.73) \qquad \sum_{j=1}^{q} x_j = K,$$

$$(1.74) \qquad x_j \in \{0, 1\} \qquad \forall\, j = 1, \ldots, q.$$

Figure 2: Constraint and formulation for VRP8. From [1]

If the cost matrix satisfies the triangle inequality then we can relax 1.71 and get

$$\sum_{j=1}^{q} a_{ij}x_j \geq 1 \tag{VRP8$'$}$$

Solved using *column generation*

4 References

[1] The Vehicle Routing Problem, Edited by Paolo Toth and Daniele Vigo

[2] NPTEL Lecture series : Advanced Operations Research - Prof. G Srinivasan, IIT Madras