Basics of Linear Programming

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1 Overview

The standard formualtion is a set of a constraints and a variable that needs to be optimized subject to these constraints.
LPPs require the constraints and variable to be optimized should be linear functions of decision variables.

1.1 Std. Formulations of LPPs

- Production Planning Problem : Decision variables are Inventory and Current Production. Cost or Profit is to be optimized.
- Cutting Stock Problem : Decision variables are related to wastage associated with a pattern and the quantities of patterns are decision variables.
  Wastage is to be minimized while producing predifined qty. of each pattern. All patterns are from same material/fuel source.
  Number of pattrens determine number of decision variables.
  Example we want 9 inch, 8 inch, 7 inch and 6 inch wide strips from a 20 inch wide strip.
- Game theory problem : Compares decisions between competitors and their outcomes. Uses pay-off matrix.
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2 Modifying LPP form

Numerous formulations are possible for the same LPP. Guidelines are, in order:

1. Formulation with fewer constarints is superior.
2. A formulation which has fewer variables is superior to a formulation with more variables if the number of constarints are same.
3. Formualtion with inqualities is preferred over formulation which has equations.

3 Solutions to LPPs

1. Graphical method : Idea that only corner points of geometric regions can be an extreme point for the variable to be opyimized.

2. Algebriac Method : Closely related to the graphical method. Divides prospective solution set into Basic. Uses the idea of extra slack variables to convert inequality to equality (n variables, m equations hence ${}^{n}\mathcal{C}_{m}$).

3. Simplex Algorithm : Assume that constraints have non negative value on RHS. Steps in the Algorithm. Uses division of prospective solution set like method 2.

   (a) Reduce problem to std. form by introducing slack variables. std. form $\implies$ All constarints are equations and all variables are non-negative.

   (b) The slack variables will either be the undershoot or over shoot. for ex.

$$3x_1 + 2x_2 \leq 2 \qquad 3x_1 + 2x_2 + s_1 = 2, e_1 \geq 0$$

$$3x_1 + 2x_2 \geq 2 \qquad 3x_1 + 2x_2 - s_2 = 2, e_2 \geq 0$$

   (c) Assume n variables and m eq. st $n \geq m$. The set of Basic solutions can be obtained as follows

      i. Set n-m variables to 0 : Non Basic variables

      ii. Solve for m remaining variables : Basic variables

      iii. Solutions obtained are basic solutions

   A basic solution is feasible only if all var. are ≥ 0
   Geometrically speaking each feasible basic solution graphically corresponds to an extreme point. Example of solving: Source

4. Admissible solutions

Each basic solution of (LP=) for which all variables are nonnegative, is called an admissible basic solution. This admissible basic solution corresponds to an extreme point (corner solution).

5. Solution of a linear program (LP)

(LP)

Ex : $Max\ Z = 1000\, x_1 + 1200\, x_2$
$s.t. 10\, x_1 + 5x_2 \leq 200$
$2x_1 + 3x_2 \leq 60$
$x_1 \leq 34$
$x_2 \leq 14$
$x_1, x_2 \geq 0$

(LP)

Ex : $Max\ Z = 1000\, x_1 + 1200\, x_2$
$s.t. 10\, x_1 + 5x_2 + e_1 = 200$
$2x_1 + 3x_2 + e_2 = 60$
$x_1 + e_3 = 34$
$x_2 + e_4 = 14$
$x_1, x_2, e_1, e_2, e_3, e_4 \geq 0$

$(n - m) = 0$

$n = 6$ and $m = 4$

$(6 - 4) = 2$ variables $= 0$

Non-basic variables

if $x_1 = x_2 = 0$

then

Basic variables:

$e_1 = 200$
$e_2 = 60$
$e_3 = 34$
$e_4 = 14$

Step A: initial table

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic Var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

The initial table is written in the following way:

The bleu frame corresponds to the constraints of (LP=).

The green frame corresponds to z_j : the coefficients in $\times\ a_i$.

Example for the column of X_1 called (a_1) :

$$0 \times 10 + 0 \times 2 + 0 \times 1 + 0 \times 0 = 0$$

The pink frames correspond to the coefficients (C_j) of the variables in the objective function (Z).

The grey frame corresponds to the value of the basic variables.

The orange frame corresponds to the value of Z, i.e. the value of the objective function, calculated as follows :

$$0 \times 200 + 0 \times 60 + 0 \times 34 + 0 \times 14 = 0$$

Step B : selection of the entering variable (to the set of basic variables)

Maximum of the $C_j - z_j$ for maximum problems.

Minimum of the $C_j - z_j$ for the minimum problems.

In our example: x_2 has the greatest $C_j - z_j$; hence it enters in the set of basic variables.

Step C : selection of the leaving variable

In a problem of either min **OR** max, the leaving variable is the minimum of

$$\left.\frac{b_i}{a_{ik}}\right| a_{ik} > 0$$

In our example, we need to evaluate:

Entering variable

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic Var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

200/5 = 40

60/3 = 20

14/1 = 14 → is the minimum, hence e_4 is the variable that leaves the set of basic variables.

Step D : pivot

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

The blue cell is called the pivot. To go to the next table (and hence to carry out the first iteration), it is essential to use the pivot.

Pivoting goes like this:

One starts by dividing the line of the pivot by the pivot.

In our example, we divide by 1.

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1							
0	E_2							
0	E_3							
1200	X_2	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

We continue to construct the identity matrix for the basic variables. We write one the intersection of these variables and zero elsewhere.

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1		0	1	0	0		
0	E_2		0	0	1	0		
0	E_3		0	0	0	1		
1200	X_2	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

We need to calculate the values for the remaining cells from the previous table (or the initial table for the first iteration).

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1		0	1	0	0		
0	E_2		0	0	1	0		
0	E_3		0	0	0	1		
1200	X_2	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

Initial table:

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

In our example, the 10 in the red-framed cell is calculated with the following formula

$$10 - \frac{element\ on\ the\ line\ of\ the\ pivot * element\ in\ the\ column\ of\ the\ pivot}{pivot}$$

Hence, $10 - \frac{0*5}{1} = 10.$

Let us calculate the green-framed cell. We obtain -3 in the following way:

$$0 - \frac{3*1}{1} = -3$$

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	0	1	0	0	-5	
0	E_2	2	0	0	1	0	-3	
0	E_3	1	0	0	0	1	0	
1200	X_2	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

The remaining cells are calculated in the same way. When the table is full (such as the one below), one can continue to the second iteration, that will be carried out in the same way.

6. Stopping criterion

We stop when we reach the optimality criterion. The simplex algorithm stops when:

- $C_j - z_j \leq 0$ for a maximum problem
- $C_j - z_j \geq 0$ for a minimum problem

Example : Maximize Z

$$Z = 12x_1 + 16x_2$$

Subject to,

$$10x_1 + 20x_2 \leq 120$$

$$8x_1 + 8x_2 \leq 80$$

$$x_1, x_2 \geq 0$$

Introduce Slack variables

$$Z = 12x_1 + 16x_2 + 0S_1 + 0S_2$$

$$10x_1 + 20x_2 + S_1 + 0S_2 = 120$$

$$8x_1 + 8x_2 + 0S_1 + S_2 = 80$$