Abstract Algebra

Summer of Science

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Introduction

What is abstract algebra?

I could, of course, get away with saying that abstract algebra is the study of algebraic structures. But then, what are algebraic structures?

The way I see it, algebraic structures are basically sets along with one or more operations that follow some basic properties. Some examples include groups, rings, and fields. Abstract algebra is then the study of these objects: how they behave on their own, and how they interact with each other.

Motivation

Why did I choose abstract algebra?

Since my freshman year, I have found mathematics to be a subject that intrigues and allures me. When I had decided to undertake Summer of Science, I was pretty convinced that I wanted to do something in math.

Out of the plethora of math topics offered, abstract algebra was one that fascinated me the most. I was particularly keen on knowing how a simple set of mathematical objects gives rise to a field as rich and as beautiful as algebra. I was also interested in how these abstract objects connect to the real world. Taking up abstract algebra as my topic study seemed like the perfect way to uncover these mysteries.

Topics Covered

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I can broadly divide the vast list of topics I have covered, into three parts.

1. **Group Theory**: The study of groups, their properties, some examples, maps between groups, and classification of groups.
2. **Ring Theory**: The study of rings, special classes of rings (such as domains and fields), maps between rings.
3. **Field Theory/Galois Theory**: The study of fields, extensions of fields, and some special classes of these extensions.

The Plan for this Presentation

Although I could use this presentation to briefly summarise what I have covered, I prefer not to. Rather, I plan on talking about content that is *orthogonal* to what I have covered. In simpler words, I plan to let the rigorous math remain within my report, where it belongs, and use this opportunity to talk about some interesting applications of this subject.

Group Theory

A group is a set along with a binary operation on this set satisfying three axioms.

1. The operation is associative,
2. there is an identity element, and
3. every element has an inverse element.

A group is perhaps one of the simplest algebraic structures. Yet, it is surprisingly useful in capturing and “abstracting out” several real world phenomena.

Perhaps the most commonly discussed application of groups is its relation to the symmetries of shapes. Vaguely, one may understand a symmetry of an object to be a transformation that preserves the object, or maps the object onto itself. Under this definition, one may note the following.

1. Composition of symmetries is a symmetry.
2. Composition of symmetries is associative.
3. Not doing anything is a symmetry.
4. We can undo symmetries by using symmetries. (Or as Thanos would say: “I used the symmetries to destroy the symmetries”)

And voíla! We have a group! The beautiful thing about this exercise is that the axioms of a group abstract out very intuitive and natural properties of symmetries.

Groups also abstract out the phenomenon of rotation. Although this works for rotation in any number of dimensions, it is most easily visualised in two dimensions. As before, we note the following properties about rotations.

1. Composition of rotations is a rotation.
2. Composition of rotations is associative.
3. Not doing anything is a rotation.
4. We can undo rotations by using rotations.

Just as before, we have a group!

Group Theory - Conclusion

Although I have provided only two examples, one can already see how a simple mathematical object can abstract out natural properties of real world phenomena. The real fun begins when we work with these abstract objects and prove certain results about them. These results now also apply to the plethora of real world phenomena that can be described by groups! One may commonly encounter such approaches (often called a “group-theoretic approach”) being used to prove some commonly observed properties of such phenomena. Most notable fields of application include coding theory, chemistry, and molecular biology.

Thank You

That is all I have to say about the subject at this point. Do go through the report to see some of the more rigorous math involved.

Thank you!