A First Course in Optimization

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Abstract

Lecture Notes for the course EE 659: A First Course in Optimization taught in Spring 2022 by Prof. Vivek Borkar. Additional references include *A first course in optimization* by Rangarajan K. Sundaram, *Optimization by vector space methods* by David Luenberger, and *Nonlinear programming* by Dimitri P. Bertsekas

Contents

§1. Lecture 1

Definition 1.1: Open Ball

The *open ball* of radius ϵ centered around $\mathbf{x}_0 \in \mathbb{R}^d$ is defined as

$$B_\epsilon(\mathbf{x}_0) := \left\{\mathbf{x} \in \mathbb{R}^d \colon \|\mathbf{x} - \mathbf{x}_0\| < \epsilon\right\}.$$

Definition 1.2: Closed Ball

The *closed ball* of radius ϵ centered around $\mathbf{x}_0 \in \mathbb{R}^d$ is defined as

$$\overline{B}_\epsilon(\mathbf{x}_0) := \left\{\mathbf{x} \in \mathbb{R}^d \colon \|\mathbf{x} - \mathbf{x}_0\| \le \epsilon\right\}.$$

Definition 1.3: Open and Closed Sets

A set $A \subset \mathbb{R}^d$ is said to be *open* if for all $\mathbf{x} \in A$, there exists an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset A$. A set A is said to be *closed* if A^c is open.

We have the following properties for open and closed sets.

1. Let $\mathcal{I}$ be an arbitrary index set. If A_α is open for each $\alpha \in \mathcal{I}$, then
$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha$$
is open. In other words, open sets are closed under arbitrary unions.
2. Let $\mathcal{I}$ be a finite index set. If A_α is open for each $\alpha \in \mathcal{I}$, then
$$\bigcap_{\alpha \in \mathcal{I}} A_\alpha$$
is open. In other words, open sets are closed under finte intersections.
3. Let $\mathcal{I}$ be an arbitrary index set. If A_α is closed for each $\alpha \in \mathcal{I}$, then
$$\bigcap_{\alpha \in \mathcal{I}} A_\alpha$$
is closed. In other words, closed sets are closed under arbitrary intersections.
4. Let $\mathcal{I}$ be a finite index set. If A_α is closed for each $\alpha \in \mathcal{I}$, then
$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha$$
is closed. In other words, closed sets are closed under finite unions.

Definition 1.4: Convergence of a sequence

Let $\langle \mathbf{x}_n \rangle$ be a sequence in $\mathbb{R}^d$. Then, $\langle \mathbf{x}_n \rangle$ converges to $\mathbf{x}^*$ (written $\mathbf{x}_n \to \mathbf{x}^*$) if for all $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\mathbf{x}_n \in B_\epsilon(\mathbf{x}^*) \quad \forall n > n_0.$$

Equivalently, $\|\mathbf{x}_n - \mathbf{x}^*\| \to 0$.

Definition 1.5: Closure

Let $A \subset \mathbb{R}^d$. The *closure* of A (denoted $\overline{A}$) is the smallest closed set containing A. Equivalently, $\overline{A}$ is the intersection of all closed sets containing A.

Definition 1.6: Interior

Let $A \subset \mathbb{R}^d$. The *interior* of A (denoted A^{o}) is the largest open set contained in A. Equivalently, A^{o} is the union of all open sets contained in A.

Note that by definition, we have $A^{\mathrm{o}} \subset A \subset \overline{A}$.

Definition 1.7: Boundary

Let $A \subset \mathbb{R}^d$. The *boundary* of A is defined as

$$\partial A := \overline{A} \setminus A^{\mathrm{o}}.$$

Note that for a closed set, $\overline{A} = A$, and for an open set, $A^{\mathrm{o}} = A$.

Proposition 1.8

A set $A \subset \mathbb{R}^d$ is closed if and only if

$$\mathbf{x}_n \to \mathbf{x}^*, \mathbf{x}_n \in A \forall n \implies \mathbf{x}^* \in A.$$

Proof. Let $A \subset \mathbb{R}^d$ be closed, $\mathbf{x}_n \in A$ for all n, and $\mathbf{x}_n \to \mathbf{x}^*$. Assume to the contrary that $\mathbf{x}^* \notin A$. Then, $\mathbf{x}^* \in A^{\mathsf{c}}$, which is open by assumption. Thus, $\exists \epsilon > 0$ such that $B_\epsilon(\mathbf{x}^*) \subset A^{\mathsf{c}}$. This implies that $\mathbf{x}_n \notin B_\epsilon(\mathbf{x}^*)$ for all n, and thus $\mathbf{x}_n \not\to \mathbf{x}^*$, a contradiction. To prove the converse, assume that A is not closed. Thus, there exists a $\tilde{\mathbf{x}} \in \partial A$ such that $\tilde{\mathbf{x}} \notin A$. Then, for all $\epsilon > 0$, $B_\epsilon(\tilde{\mathbf{x}}) \cap A \neq \varnothing$. Let $\epsilon_n \downarrow 0$ and let $\mathbf{x}_n \in B_{\epsilon_n}(\tilde{\mathbf{x}}) \cap A$. Then, $\mathbf{x}_n \to \tilde{\mathbf{x}} \notin A$, a contradiction. □

Definition 1.9: Limit Point

Let $\langle \mathbf{x}_n \rangle$ be a sequence in $\mathbb{R}^d$. $\tilde{\mathbf{x}}$ is a *limit point* of $\langle \mathbf{x}_n \rangle$ if there exists a subsequence $\langle \mathbf{x}_{n_k} \rangle$ such that $\mathbf{x}_{n_k} \to \tilde{\mathbf{x}}$.

Proposition 1.10

$\langle \mathbf{x}_n \rangle$ converges if and only if $\langle \mathbf{x}_n \rangle$ has a unique limit point.

Definition 1.11: Supremum and Infimum

Let $A \subset \mathbb{R}$ be bounded. Then,

$$\begin{aligned} \sup A &:= \text{ smallest } x \in \mathbb{R} \cup \{+\infty\} \text{ such that } y \in A \implies y \leq x, \\ \inf A &:= \text{ largest } x \in \mathbb{R} \cup \{-\infty\} \text{ such that } y \in A \implies y \geq x. \end{aligned}$$

Definition 1.12: Cauchy Sequence

A sequence $\langle \mathbf{x}_n \rangle$ is said to be *Cauchy* if $\lim\limits_{m,n\uparrow\infty} \|\mathbf{x}_m - \mathbf{x}_n\| = 0$.

Proposition 1.13

Cauchy sequences are bounded.

Proof. Let $\langle \mathbf{x}_n \rangle$ be a Cauchy sequene and let $\epsilon > 0$. Pick N large enough such that

$$n, m > N \implies \|\mathbf{x}_m - \mathbf{x}_n\| < \epsilon.$$

We then have

$$\begin{aligned} &\mathbf{x}_n \in B_\epsilon(\mathbf{x}_m) \quad \forall n > N \\ &\implies \langle \mathbf{x}_n : n > N \rangle \text{ is bounded} \\ &\implies \langle \mathbf{x}_n \rangle \text{ is bounded.} \end{aligned}$$

□

Proposition 1.14

Cauchy sequences have at most one limit point.

Proof. Suppose $\langle \mathbf{x}_n \rangle$ is a Cauchy sequence having two limit points, $\tilde{\mathbf{x}}$ and $\overline{\mathbf{x}}$. Then, there exist subsequences $\mathbf{x}_{\tilde{n}_k} \to \tilde{\mathbf{x}}$ and $\mathbf{x}_{\overline{n}_l} \to \overline{\mathbf{x}}$. We then have

$$\lim_{\tilde{n}_k, \overline{n}_l \uparrow \infty} \|\mathbf{x}_{\tilde{n}_k} - \mathbf{x}_{\overline{n}_l}\| = 0 \implies \tilde{\mathbf{x}} = \overline{\mathbf{x}}. \qquad \square$$

Definition 1.15: Complete Space

A metric space is *complete* if every Cauchy sequence converges.

Theorem 1.16: Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.