MA 109: Calculus - I

Tutorial Solutions

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Contents

§1. Week 1

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Sheet 1.

2 (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Solution. We will utilise the fact that $n^{1/n} \geq 1$ for all $n \in \mathbb{N}$. (Why is this true?) We define $h_n \coloneqq n^{1/n} - 1$. Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. For $n \geq 2$, we have

$$n = (1 + h_n)^n \geq 1 + \binom{n}{1} h_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

Cancelling out the n's, we get

$$h_n^2 < \frac{2}{n-1} \implies h_n < \sqrt{\frac{2}{n-1}}$$

Thus for $n \geq 2$, we have

$$0 \leq h_n < \sqrt{\frac{2}{n-1}}$$

Notice that the limit of the sequence on the right exists and is equal to 0. Thus, utilising Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$. Recalling how we defined h_n, we get $\lim_{n\to\infty} n^{1/n} = 1$. $\square$

3 (ii) Prove that the sequence $a_n := \left\{(-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n\geq 1}$ is not convergent.

Solution. We will prove this result by contradiction. First, observe that the sequence $b_n := \frac{(-1)^n}{n}$ is convergent and its limit is 0. This is true because its absolute value behaves the same way as $\frac{1}{n}$ (try proving this with the ϵ-N definition to work out the details). We also know that the sequence $\{(-1)^n\}_{n\geq 1}$ is not convergent. (Why?) Now, let us assume that the given sequence (a_n) converges. We have

$$a_n := \left\{(-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)\right\} = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

We also know that the the sum of two convergent sequences is convergent. Since a_n is assumed to be convergent and b_n is convergent, we have that $c_n := a_n + b_n = \frac{(-1)^n}{2}$ must also converge. However, the convergence of c_n implies that the sequence $(-1)^n$ also converges. Hence, we arrive at a contradiction and thus, the sequence (a_n) is not convergent.

$\square$

5 (iii) Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit.

$$a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \ \forall n \in \mathbb{N}$$

Solution. We first claim that $a_n < 6$ for all $n \in \mathbb{N}$. To prove this, we will use mathematical induction. The base case, $n = 1$ is immediate as $2 < 6$. Assume that the claim holds for some $n = k$. Now,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$$

By induction, the claim follows. Hence, a_n is bounded above.

Next, we claim that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. We have

$$a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2}$$

We just showed that $a_n < 6$ for all $n \in \mathbb{N}$. It thus follows that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Hence, (a_n) is a monotonically increasing sequence that is bounded above. Thus, it must converge. To find the limit of (a_n), we utilise the fact that $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n$ (Sheet 1 : Problem 6). Let L denote the limit of (a_n). Taking the limit of the recursive definition (and using some limit properties), we have that

$$L = 3 + \frac{L}{2} \implies L = 6$$

Thus, the sequence (a_n) converges to 6. (Notice that this was the upper bound we chose for (a_n)) □

7 If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0$$

Solution. We will use the $\epsilon - N$ definition to prove this result. Choose $\epsilon = \frac{|L|}{2}$. Since $L \neq 0$, we have $\epsilon > 0$. Now, as $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. From triangle inequality, we have

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \implies -\epsilon < |a_n| - |L| \quad \forall n \geq n_0$$

Substituting the value of ϵ, we get that

$$|a_n| > \frac{|L|}{2}$$

for all $n \geq n_0$, as desired. □

9 For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following statements:

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

Solution. This is a relatively short question. Both the statements are **false**. Verify that $a_n := 1$ and $b_n := (-1)^n$ acts as a counterexample for both the statements. □

11 Let $f, g\colon (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

(i) $\lim_{x\to c} [f(x)g(x)] = 0$.

(ii) $\lim_{x\to c} [f(x)g(x)] = 0$ if g is bounded.

(iii) $\lim_{x\to c} [f(x)g(x)] = 0$ if $\lim_{x\to c} g(x)$ exists.

Solution. (i) This statement is **false**. As a counterexample, define $a = -1, b = 1$ and $c = 0$. Define $f, g\colon (-1, 1) \to \mathbb{R}$ as

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$$

Clearly, $\lim_{x\to 0} f(x) = 0$. However, $\lim_{x\to 0} [f(x)g(x)]$ does not exist.

(ii) This statement is **true**. Since g is bounded, there exists $M > 0$ such that

$$|g(x)| \leq M$$

for all $x \in (a, b)$. Thus, we have

$$0 \leq |f(x)g(x)| \leq M|f(x)|$$

for all $x \in (a, b)$. Using Sandwich Theorem, we see that

$$\lim_{x\to c} |f(x)g(x)| = 0$$

which in turn implies that

$$\lim_{x\to c} [f(x)g(x)] = 0$$

(iii) This statement is **true**. Since $\lim_{x\to c} g(x)$ exists, we have $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 0$.

$\square$

§2. Week 2

2nd December, 2020

Sheet 1.

13 (ii) Discuss the continuity of the following function :

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. At all points other than $x = 0$, the given function is trivially continuous (since it is the product and composition of continuous functions). All that remains is to check the continuity of f at the point $x = 0$. Note that

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

for all $x \neq 0$. Thus, we have

$$0 \leq |f(x)| \leq |x|$$

Utilising Sandwich Theorem, we see that

$$\lim_{x \to 0} f(x) = 0$$

Since $f(0)$ is given to be 0, we see that $\lim\limits_{x \to 0} f(x) = f(0)$, proving continuity of f at $x = 0$. Thus, f is continuous everywhere. $\square$

15 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows.

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. Clearly, f is differentiable for all $x \neq 0$. Using the chain rule and product rule, we compute f' as

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

for $x \neq 0$. Now, all that remains to be checked is the differentiability of f at $x = 0$. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \ h \sin\left(\frac{1}{h}\right)$$

From the previous question, this limit exists and is equal to 0. Thus, f is differentiable on all of $\mathbb{R}$ and its derivative is defined as

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly, f' is continuous at all $x \neq 0$. All that remains is to check continuity of f' at $x = 0$. It turns out that f' is in fact *not* continuous at $x = 0$. We will use the sequential criterion of continuity to prove this. Consider the sequence:

$$x_n \coloneqq \frac{1}{2n\pi}, \quad n \in \mathbb{N}$$

Clearly, $x_n \to 0$ as $n \to \infty$. However,

$$f'(x_n) = \cancel{\frac{2}{2n\pi} \cdot \sin(2n\pi)} - \cos(2n\pi) = -1$$

We see that $\lim\limits_{n \to \infty} f(x_n)$ is -1, which is not equal to $f'(0)$. Hence, f' is not continuous at $x = 0$. This is an example of a differentiable function whose derivative is not continuous. $\square$

18 Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x) \cdot f(y) \quad \text{for all } x, y \in \mathbb{R}$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$.

Solution. We have that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. On substituting $x = y = 0$, we obtain

$$f(0) = f(0) \cdot f(0) \implies f(0) = 0 \text{ or } 1$$

First, we consider the case that $f(0) = 0$. We have

$$f(x) = f(x+0) = f(x) \cdot f(0) \implies f(x) = 0$$

for all x. Thus, $f \equiv 0$ is trivially differentiable and $f'(c) = 0 = f'(0) \cdot f(c)$ for all $c \in \mathbb{R}$.

Now consider that $f(0) = 1$. For all $c \in \mathbb{R}$, we have

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c)f(h) - f(c)f(0)}{h} = f(c) \cdot \left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \right)$$

If f is differentiable at 0, then the above limit exists. Thus, if f is differentiable at 0, then it is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$. $\square$

Optional Exercises.

7 Let $f\colon (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following statements are equivalent.

(i) f is differentiable at c.

(ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$ and a function $\epsilon_1\colon (-\delta,\delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$$

for all $h \in (-\delta,\delta)$.

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0$$

Solution. To show the equivalence of statements (i)-(iii), we must show that every statement implies every other statement, that is, a total of 6 implications. However, we can get away with just showing three implications. We will show that $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$. This is sufficient to conclude the equivalence of the three statements. (Why?)

$(i) \Rightarrow (ii)$: Since we are given that f is differentiable at c, $f'(c)$ exists. We first pick $\delta := \min\{c-a, b-c\}$. Clearly $\delta > 0$ and $(c-\delta, c+\delta) \subset (a,b)$. Now, since f is differentiable at c, $f'(c)$ exists. Define $\alpha := f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Since $(c-\delta, c+\delta) \subset (a,b)$, $f(c+h)$ is well defined for all $h \in (-\delta,\delta)$. Now,

$$\lim_{h\to 0} \epsilon_1(h) = \underbrace{\left(\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \right)}_{\alpha} - \alpha = 0$$

Further, some simple algebraic manipulation yields that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for $h \in (-\delta,\delta), h \neq 0$. Verify that this equation also holds for $h = 0$. It then follows that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for all $h \in (-\delta,\delta)$ and $\lim_{h\to 0} \epsilon_1(h) = 0$, as desired.

$(ii) \Rightarrow (iii)$: By (ii), we have the existence of $\delta > 0, \alpha \in \mathbb{R}$ and the function ϵ_1. We have

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h \to 0} |\epsilon_1(h)| = 0$$

$(iii) \Rightarrow (i)$: By (iii), we have the existence of some $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

Now,

$$\lim_{h \to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Thus, f is differentiable at c, as desired.

Since we have shown $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$, we get that the three statements are thus equivalent. $\square$

10 Show that any continuous function $f\colon [0,1] \to [0,1]$ has a fixed point. x is said to be a fixed point of f if $f(x) = x$

Solution. Consider the function $g(x) = f(x) - x$. A fixed point of f is then a root of g. Note that g is continuous. Since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, we have

$$g(0) = f(0) \implies g(0) \geq 0$$

and

$$g(1) = f(1) - 1 \implies g(1) \leq 0$$

First consider the case that at least one of the two equalities hold. That is, either $g(0) = 0$ or $g(1) = 0$ or both. In either of the three cases, we have at least one fixed point (0 or 1 or both, respectively). Now, consider that $g(0) > 0$ and $g(1) < 0$. Since g is continuous, we can appeal to Intermediate Value Theorem. By IVT, there exists some $x_0 \in (0,1)$ such that $g(x_0) = 0$. This point x_0 is also a fixed point of f. Thus, we have shown that any continuous function mapping the unit interval to itself has a fixed point, as desired. $\square$

Sheet 2.

3 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Solution. Since $f(a)$ and $f(b)$ are of opposite signs and f is continuous, we know that there exists **at least** one $x_0 \in (a, b)$ such that $f(x_0) = 0$ (by IVP). Now, assume that there was some $y_0 (\neq x_0)$ in (a, b) such that $f(y_0) = 0$. We now have $f(x_0) = f(y_0)$. By Rolle's Theorem, there must exist some $c \in (x_0, y_0)$ such that $f'(c) = 0$. Since this c also lies in (a, b), we arrive at a contradiction. Hence, there is a unique x_0 in (a, b) such that $f(x_0) = 0$, as desired. $\square$

5 Use the MVT to show that $|\sin(a) - \sin(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Solution. We will break this problem into two cases. First, consider $a = b$. The inequality is trivially satisfied in this case. Next, consider $a \neq b$. Define $f(x) = \sin(x)$. By MVT, there exists some c between a and b such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Since $f' = \cos$, we take modulus on both sides to obtain

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1$$

Rearranging, we get

$$|\sin a - \sin b| \leq |a - b|$$

for all $a, b \in \mathbb{R}$, as desired. $\square$