MA 109: Calculus - I

Tutorial Solutions

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Contents

§1. Week 1

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Sheet 1.

2 (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Solution. We will utilise the fact that $n^{1/n} \geq 1$ for all $n \in \mathbb{N}$. (Why is this true?) We define $h_n \coloneqq n^{1/n} - 1$. Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. For $n \geq 2$, we have

$$n = (1 + h_n)^n \geq 1 + \binom{n}{1} h_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

Cancelling out the n's, we get

$$h_n^2 < \frac{2}{n-1} \implies h_n < \sqrt{\frac{2}{n-1}}$$

Thus for $n \geq 2$, we have

$$0 \leq h_n < \sqrt{\frac{2}{n-1}}$$

Notice that the limit of the sequence on the right exists and is equal to 0. Thus, utilising Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$. Recalling how we defined h_n, we get $\lim_{n\to\infty} n^{1/n} = 1$. $\square$

3 (ii) Prove that the sequence $a_n := \left\{ (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\}_{n \geq 1}$ is not convergent.

Solution. We will prove this result by contradiction. First, observe that the sequence $b_n := \frac{(-1)^n}{n}$ is convergent and its limit is 0. This is true because its absolute value behaves the same way as $\frac{1}{n}$ (try proving this with the $\epsilon - N$ definition to work out the details). We also know that the sequence $\{(-1)^n\}_{n \geq 1}$ is not convergent. (Why?) Now, let us assume that the given sequence (a_n) converges. We have

$$a_n := \left\{ (-1)^n \left(\frac{1}{2} - \frac{1}{n} \right) \right\} = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

We also know that the the sum of two convergent sequences is convergent. Since a_n is assumed to be convergent and b_n is convergent, we have that $c_n := a_n + b_n = \frac{(-1)^n}{2}$ must also converge. However, the convergence of c_n implies that the sequence $(-1)^n$ also converges. Hence, we arrive at a contradiction and thus, the sequence (a_n) is not convergent.

$\square$

5 (iii) Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit.

$$a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \; \forall n \in \mathbb{N}$$

Solution. We first claim that $a_n < 6$ for all $n \in \mathbb{N}$. To prove this, we will use mathematical induction. The base case, $n = 1$ is immediate as $2 < 6$. Assume that the claim holds for some $n = k$. Now,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$$

By induction, the claim follows. Hence, a_n is bounded above.

Next, we claim that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. We have

$$a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2}$$

We just showed that $a_n < 6$ for all $n \in \mathbb{N}$. It thus follows that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Hence, (a_n) is a monotonically increasing sequence that is bounded above. Thus, it must converge. To find the limit of (a_n), we utilise the fact that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ (Sheet 1 : Problem 6). Let L denote the limit of (a_n). Taking the limit of the recursive definition (and using some limit properties), we have that

$$L = 3 + \frac{L}{2} \implies L = 6$$

Thus, the sequence (a_n) converges to 6. (Notice that this was the upper bound we chose for (a_n)) □

7 If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0$$

Solution. We will use the $\epsilon - N$ definition to prove this result. Choose $\epsilon = \frac{|L|}{2}$. Since $L \neq 0$, we have $\epsilon > 0$. Now, as $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. From triangle inequality, we have

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \implies -\epsilon < |a_n| - |L| \quad \forall n \geq n_0$$

Substituting the value of ϵ, we get that

$$|a_n| > \frac{|L|}{2}$$

for all $n \geq n_0$, as desired. □

9 For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following statements:

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

Solution. This is a relatively short question. Both the statements are **false**. Verify that $a_n := 1$ and $b_n := (-1)^n$ acts as a counterexample for both the statements. □

11 Let $f, g\colon (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

(i) $\lim_{x\to c} [f(x)g(x)] = 0$.

(ii) $\lim_{x\to c} [f(x)g(x)] = 0$ if g is bounded.

(iii) $\lim_{x\to c} [f(x)g(x)] = 0$ if $\lim_{x\to c} g(x)$ exists.

Solution. (i) This statement is **false**. As a counterexample, define $a = -1, b = 1$ and $c = 0$. Define $f, g\colon (-1, 1) \to \mathbb{R}$ as

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$$

Clearly, $\lim_{x\to 0} f(x) = 0$. However, $\lim_{x\to 0} [f(x)g(x)]$ does not exist.

(ii) This statement is **true**. Since g is bounded, there exists $M > 0$ such that

$$|g(x)| \leq M$$

for all $x \in (a, b)$. Thus, we have

$$0 \leq |f(x)g(x)| \leq M|f(x)|$$

for all $x \in (a, b)$. Using Sandwich Theorem, we see that

$$\lim_{x\to c} |f(x)g(x)| = 0$$

which in turn implies that

$$\lim_{x\to c} [f(x)g(x)] = 0$$

(iii) This statement is **true**. Since $\lim_{x\to c} g(x)$ exists, we have $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 0$.

$\square$