

# MA 109: Calculus - I

## Tutorial Solutions

Ishan Kapnadak

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# §1. Week 1

25th November, 2020

## Sheet 1.

2 (iv)  $\lim_{n \rightarrow \infty} (n)^{1/n}$ .

*Solution.* We will utilise the fact that  $n^{1/n} \geq 1$  for all  $n \in \mathbb{N}$ . (Why is this true?) We define  $h_n := n^{1/n} - 1$ . Then,  $h_n \geq 0$  for all  $n \in \mathbb{N}$ . For  $n \geq 2$ , we have

$$n = (1 + h_n)^n \geq 1 + \binom{n}{1} h_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

Cancelling out the  $n$ 's, we get

$$h_n^2 < \frac{2}{n-1} \implies h_n < \sqrt{\frac{2}{n-1}}$$

Thus for  $n \geq 2$ , we have

$$0 \leq h_n < \sqrt{\frac{2}{n-1}}$$

Notice that the limit of the sequence on the right exists and is equal to 0. Thus, utilising Sandwich Theorem, we get that  $\lim_{n \rightarrow \infty} h_n = 0$ . Recalling how we defined  $h_n$ , we get  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ . □

3 (ii) Prove that the sequence  $a_n := \left\{ (-1)^n \left( \frac{1}{2} - \frac{1}{n} \right) \right\}_{n \geq 1}$  is not convergent.

*Solution.* We will prove this result by contradiction. First, observe that the sequence  $b_n := \frac{(-1)^n}{n}$  is convergent and its limit is 0. This is true because its absolute value behaves the same way as  $\frac{1}{n}$  (try proving this with the  $\epsilon$ - $N$  definition to work out the details). We also know that the sequence  $\{(-1)^n\}_{n \geq 1}$  is not convergent. (Why?) Now, let us assume that the given sequence  $(a_n)$  converges. We have

$$a_n := \left\{ (-1)^n \left( \frac{1}{2} - \frac{1}{n} \right) \right\} = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

We also know that the sum of two convergent sequences is convergent. Since  $a_n$  is assumed to be convergent and  $b_n$  is convergent, we have that  $c_n := a_n + b_n = \frac{(-1)^n}{2}$  must also converge. However, the convergence of  $c_n$  implies that the sequence  $(-1)^n$  also converges. Hence, we arrive at a contradiction and thus, the sequence  $(a_n)$  is not convergent.

□

- 5 (iii) Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit.

$$a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \in \mathbb{N}$$

*Solution.* We first claim that  $a_n < 6$  for all  $n \in \mathbb{N}$ . To prove this, we will use mathematical induction. The base case,  $n = 1$  is immediate as  $2 < 6$ . Assume that the claim holds for some  $n = k$ . Now,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$$

By induction, the claim follows. Hence,  $a_n$  is bounded above.

Next, we claim that  $a_{n+1} > a_n$  for all  $n \in \mathbb{N}$ . We have

$$a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2}$$

We just showed that  $a_n < 6$  for all  $n \in \mathbb{N}$ . It thus follows that  $a_{n+1} > a_n$  for all  $n \in \mathbb{N}$ . Hence,  $(a_n)$  is a monotonically increasing sequence that is bounded above. Thus, it must converge. To find the limit of  $(a_n)$ , we utilise the fact that  $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$  (Sheet 1 : Problem 6). Let  $L$  denote the limit of  $(a_n)$ . Taking the limit of the recursive definition (and using some limit properties), we have that

$$L = 3 + \frac{L}{2} \implies L = 6$$

Thus, the sequence  $(a_n)$  converges to 6. (Notice that this was the upper bound we chose for  $(a_n)$ )  $\square$

7 If  $\lim_{n \rightarrow \infty} a_n = L \neq 0$ , show that there exists  $n_0 \in \mathbb{N}$  such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0$$

*Solution.* We will use the  $\epsilon - N$  definition to prove this result. Choose  $\epsilon = \frac{|L|}{2}$ . Since  $L \neq 0$ , we have  $\epsilon > 0$ . Now, as  $a_n \rightarrow L$ , there exists  $n_0 \in \mathbb{N}$  such that  $|a_n - L| < \epsilon$  for all  $n \geq n_0$ . From triangle inequality, we have

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \implies -\epsilon < |a_n| - |L| \quad \forall n \geq n_0$$

Substituting the value of  $\epsilon$ , we get that

$$|a_n| > \frac{|L|}{2}$$

for all  $n \geq n_0$ , as desired. □

9 For given sequences  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$ , prove or disprove the following statements:

- (i)  $\{a_n b_n\}_{n \geq 1}$  is convergent if  $\{a_n\}_{n \geq 1}$  is convergent.
- (ii)  $\{a_n b_n\}_{n \geq 1}$  is convergent if  $\{a_n\}_{n \geq 1}$  is convergent and  $\{b_n\}_{n \geq 1}$  is bounded.

*Solution.* This is a relatively short question. Both the statements are **false**. Verify that  $a_n := 1$  and  $b_n := (-1)^n$  acts as a counterexample for both the statements. □

11 Let  $f, g: (a, b) \rightarrow \mathbb{R}$  be functions and suppose that  $\lim_{x \rightarrow c} f(x) = 0$  for  $c \in [a, b]$ . Prove or disprove the following statements.

- (i)  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$ .
- (ii)  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$  if  $g$  is bounded.
- (iii)  $\lim_{x \rightarrow c} [f(x)g(x)] = 0$  if  $\lim_{x \rightarrow c} g(x)$  exists.

*Solution.* (i) This statement is **false**. As a counterexample, define  $a = -1, b = 1$  and  $c = 0$ . Define  $f, g: (-1, 1) \rightarrow \mathbb{R}$  as

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$$

Clearly,  $\lim_{x \rightarrow 0} f(x) = 0$ . However,  $\lim_{x \rightarrow 0} [f(x)g(x)]$  does not exist.

(ii) This statement is **true**. Since  $g$  is bounded, there exists  $M > 0$  such that

$$|g(x)| \leq M$$

for all  $x \in (a, b)$ . Thus, we have

$$0 \leq |f(x)g(x)| \leq M|f(x)|$$

for all  $x \in (a, b)$ . Using Sandwich Theorem, we see that

$$\lim_{x \rightarrow c} |f(x)g(x)| = 0$$

which in turn implies that

$$\lim_{x \rightarrow c} [f(x)g(x)] = 0$$

(iii) This statement is **true**. Since  $\lim_{x \rightarrow c} g(x)$  exists, we have  $\lim_{x \rightarrow c} [f(x)g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = 0$ .

□

## §2. Week 2

2nd December, 2020

### Sheet 1.

13 (ii) Discuss the continuity of the following function :

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

*Solution.* At all points other than  $x = 0$ , the given function is trivially continuous (since it is the product and composition of continuous functions). All that remains is to check the continuity of  $f$  at the point  $x = 0$ . Note that

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

for all  $x \neq 0$ . Thus, we have

$$0 \leq |f(x)| \leq |x|$$

Utilising Sandwich Theorem, we see that

$$\lim_{x \rightarrow 0} f(x) = 0$$

Since  $f(0)$  is given to be 0, we see that  $\lim_{x \rightarrow 0} f(x) = f(0)$ , proving continuity of  $f$  at  $x = 0$ . Thus,  $f$  is continuous everywhere.  $\square$

15 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined as follows.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that  $f$  is differentiable on  $\mathbb{R}$ . Is  $f'$  a continuous function?

*Solution.* Clearly,  $f$  is differentiable for all  $x \neq 0$ . Using the chain rule and product rule, we compute  $f'$  as

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

for  $x \neq 0$ . Now, all that remains to be checked is the differentiability of  $f$  at  $x = 0$ . We have

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right)$$

From the previous question, this limit exists and is equal to 0. Thus,  $f$  is differentiable on all of  $\mathbb{R}$  and its derivative is defined as

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly,  $f'$  is continuous at all  $x \neq 0$ . All that remains is to check continuity of  $f'$  at  $x = 0$ . It turns out that  $f'$  is in fact *not* continuous at  $x = 0$ . We will use the sequential criterion of continuity to prove this. Consider the sequence:

$$x_n := \frac{1}{2n\pi}, \quad n \in \mathbb{N}$$

Clearly,  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ . However,

$$f'(x_n) = \frac{2}{2n\pi} \cdot \cancel{\sin(2n\pi)} - \cos(2n\pi) = -1$$

We see that  $\lim_{n \rightarrow \infty} f'(x_n)$  is  $-1$ , which is not equal to  $f'(0)$ . Hence,  $f'$  is not continuous at  $x = 0$ . This is an example of a differentiable function whose derivative is not continuous.  $\square$



18 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  satisfy

$$f(x+y) = f(x) \cdot f(y) \text{ for all } x, y \in \mathbb{R}$$

If  $f$  is differentiable at 0, then show that  $f$  is differentiable at every  $c \in \mathbb{R}$  and  $f'(c) = f'(0) \cdot f(c)$ .

*Solution.* We have that  $f(x+y) = f(x) \cdot f(y)$  for all  $x, y \in \mathbb{R}$ . On substituting  $x = y = 0$ , we obtain

$$f(0) = f(0) \cdot f(0) \implies f(0) = 0 \text{ or } 1$$

First, we consider the case that  $f(0) = 0$ . We have

$$f(x) = f(x+0) = f(x) \cdot f(0) \implies f(x) = 0$$

for all  $x$ . Thus,  $f \equiv 0$  is trivially differentiable and  $f'(c) = 0 = f'(0) \cdot f(c)$  for all  $c \in \mathbb{R}$ .

Now consider that  $f(0) = 1$ . For all  $c \in \mathbb{R}$ , we have

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0} \frac{f(c)f(h) - f(c)f(0)}{h} = f(c) \cdot \left( \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \right)$$

If  $f$  is differentiable at 0, then the above limit exists. Thus, if  $f$  is differentiable at 0, then it is differentiable at every  $c \in \mathbb{R}$  and  $f'(c) = f'(0) \cdot f(c)$ .  $\square$

### Optional Exercises.

7 Let  $f: (a, b) \rightarrow \mathbb{R}$  and  $c \in (a, b)$ . Show that the following statements are equivalent.

- (i)  $f$  is differentiable at  $c$ .
- (ii) There exists  $\delta > 0$ ,  $\alpha \in \mathbb{R}$  and a function  $\epsilon_1: (-\delta, \delta) \rightarrow \mathbb{R}$  such that  $\lim_{h \rightarrow 0} \epsilon_1(h) = 0$  and

$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h)$$

for all  $h \in (-\delta, \delta)$ .

- (iii) There exists  $\alpha \in \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \left( \frac{|f(c + h) - f(c) - \alpha h|}{|h|} \right) = 0$$

*Solution.* To show the equivalence of statements (i)-(iii), we must show that every statement implies every other statement, that is, a total of 6 implications. However, we can get away with just showing three implications. We will show that (i)  $\Rightarrow$  (ii), (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i). This is sufficient to conclude the equivalence of the three statements. (Why?)

(i)  $\Rightarrow$  (ii) : Since we are given that  $f$  is differentiable at  $c$ ,  $f'(c)$  exists. We first pick  $\delta := \min \{c - a, b - c\}$ . Clearly  $\delta > 0$  and  $(c - \delta, c + \delta) \subset (a, b)$ . Now, since  $f$  is differentiable at  $c$ ,  $f'(c)$  exists. Define  $\alpha := f'(c)$  and

$$\epsilon_1(h) = \begin{cases} \frac{f(c + h) - f(c) - \alpha h}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Since  $(c - \delta, c + \delta) \subset (a, b)$ ,  $f(c + h)$  is well defined for all  $h \in (-\delta, \delta)$ . Now,

$$\lim_{h \rightarrow 0} \epsilon_1(h) = \underbrace{\left( \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} \right)}_{\alpha} - \alpha = 0$$

Further, some simple algebraic manipulation yields that  $f(c + h) = f(c) + \alpha h + h\epsilon_1(h)$  for  $h \in (-\delta, \delta)$ ,  $h \neq 0$ . Verify that this equation also holds for  $h = 0$ . It then follows that  $f(c + h) = f(c) + \alpha h + h\epsilon_1(h)$  for all  $h \in (-\delta, \delta)$  and  $\lim_{h \rightarrow 0} \epsilon_1(h) = 0$ , as desired.

(ii)  $\Rightarrow$  (iii) : By (ii), we have the existence of  $\delta > 0, \alpha \in \mathbb{R}$  and the function  $\epsilon_1$ . We have

$$\lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h \rightarrow 0} |\epsilon_1(h)| = 0$$

(iii)  $\Rightarrow$  (i) : By (iii), we have the existence of some  $\alpha \in \mathbb{R}$  such that

$$\lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

Now,

$$\lim_{h \rightarrow 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \implies \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Thus,  $f$  is differentiable at  $c$ , as desired.

Since we have shown (i)  $\Rightarrow$  (ii), (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (i), we get that the three statements are thus equivalent.  $\square$

- 10 Show that any continuous function  $f: [0, 1] \rightarrow [0, 1]$  has a fixed point.  *$x$  is said to be a fixed point of  $f$  if  $f(x) = x$*

*Solution.* Consider the function  $g(x) = f(x) - x$ . A fixed point of  $f$  is then a root of  $g$ . Note that  $g$  is continuous. Since  $0 \leq f(x) \leq 1$  for all  $x \in [0, 1]$ , we have

$$g(0) = f(0) \implies g(0) \geq 0$$

and

$$g(1) = f(1) - 1 \implies g(1) \leq 0$$

First consider the case that at least one of the two equalities hold. That is, either  $g(0) = 0$  or  $g(1) = 0$  or both. In either of the three cases, we have at least one fixed point (0 or 1 or both, respectively). Now, consider that  $g(0) > 0$  and  $g(1) < 0$ . Since  $g$  is continuous, we can appeal to Intermediate Value Theorem. By IVT, there exists some  $x_0 \in (0, 1)$  such that  $g(x_0) = 0$ . This point  $x_0$  is also a fixed point of  $f$ . Thus, we have shown that any continuous function mapping the unit interval to itself has a fixed point, as desired.  $\square$

## Sheet 2.

- 3 Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a)$  and  $f(b)$  are of different signs and  $f'(x) \neq 0$  for all  $x \in (a, b)$ , then show that there is a unique  $x_0 \in (a, b)$  such that  $f(x_0) = 0$ .

*Solution.* Since  $f(a)$  and  $f(b)$  are of opposite signs and  $f$  is continuous, we know that there exists **at least** one  $x_0 \in (a, b)$  such that  $f(x_0) = 0$  (by IVP). Now, assume that there was some  $y_0 (\neq x_0)$  in  $(a, b)$  such that  $f(y_0) = 0$ . We now have  $f(x_0) = f(y_0)$ . By Rolle's Theorem, there must exist some  $c \in (x_0, y_0)$  such that  $f'(c) = 0$ . Since this  $c$  also lies in  $(a, b)$ , we arrive at a contradiction. Hence, there is a unique  $x_0$  in  $(a, b)$  such that  $f(x_0) = 0$ , as desired.  $\square$

- 5 Use the MVT to show that  $|\sin(a) - \sin(b)| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ .

*Solution.* We will break this problem into two cases. First, consider  $a = b$ . The inequality is trivially satisfied in this case. Next, consider  $a \neq b$ . Define  $f(x) = \sin(x)$ . By MVT, there exists some  $c$  between  $a$  and  $b$  such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Since  $f' = \cos$ , we take modulus on both sides to obtain

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1$$

Rearranging, we get

$$|\sin a - \sin b| \leq |a - b|$$

for all  $a, b \in \mathbb{R}$ , as desired.  $\square$

### §3. Week 3

9th December, 2020

#### Sheet 2.

8 In each case, find a function  $f$  that satisfies all the given conditions, or else show that no such function exists.

(ii)  $f''(x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $f'(0) = 1$ ,  $f'(1) = 2$ .

(iii)  $f''(x) \geq 0$  for all  $x \in \mathbb{R}$ ,  $f'(0) = 1$ ,  $f(x) \leq 100$  for all  $x > 0$ .

*Solution.*

(ii) Possible. Verify that  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) := x + \frac{x^2}{2}$  is one such function.

(iii) Not possible. Assume that it was indeed possible to find such a function  $f$ . Then, we are given that  $f''$  exists everywhere. Thus,  $f'$  is continuous and differentiable everywhere. As  $f''$  is non-negative,  $f'$  must be increasing everywhere. Since  $f'(0) = 1$ , we have that  $f'(c) \geq 1$  for all  $c > 0$ .

Let  $x \in (0, \infty)$ . By MVT, there exists  $c \in (0, x)$  such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

Since  $c > 0$ , we have  $f'(c) \geq 1$  as shown above. Thus,  $f(x) - f(0) \geq x$  for all  $x > 0$ . However, consider  $x_0 := \max(101 - f(0), 1)$ . Clearly,  $x_0 > 0$  (as it is  $\geq 1$ ). Also,  $f(x_0) > 100$ , which contradicts the condition that  $f(x) \leq 100$  for all  $x > 0$ . Hence, no such  $f$  can exist.

□

- 10 (i) Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local minima/maxima, points of inflection and [asymptotes](#). How many times and approximately where does the curve cross the x-axis?

$$y = 2x^3 + 2x^2 - 2x - 1$$

*Solution.* We are given

$$f(x) = 2x^3 + 2x^2 - 2x - 1$$

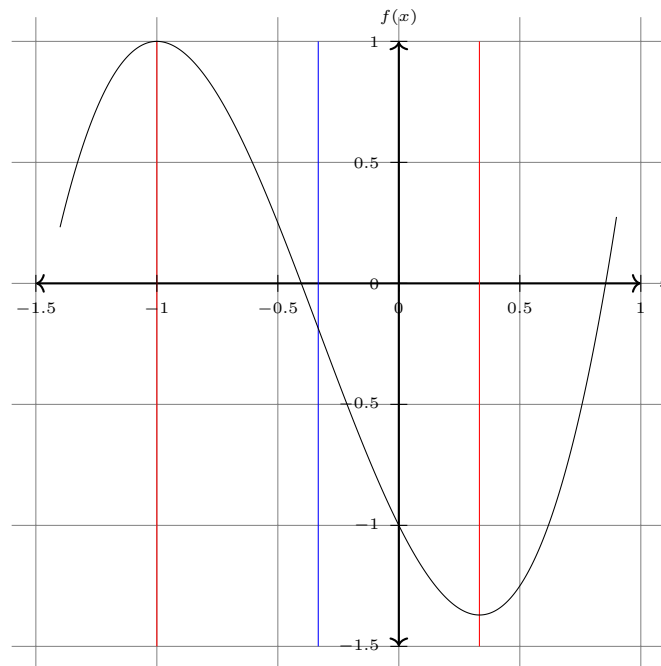
On differentiating, we get

$$f'(x) = 6x^2 + 4x - 2 = 2(x + 1)(3x - 1)$$

Thus,  $f' > 0$  in  $(-\infty, -1) \cup (\frac{1}{3}, \infty)$  and  $f$  is strictly increasing here.  $f' < 0$  in  $(-1, \frac{1}{3})$  and  $f$  is strictly decreasing here. Thus,  $f$  has a local maximum at  $-1$  and a local minimum at  $\frac{1}{3}$ . Differentiating again, we see that

$$f''(x) = 12x + 4$$

Thus,  $f$  is convex in  $(-\frac{1}{3}, \infty)$  and concave in  $(-\infty, -\frac{1}{3})$ , with a point of inflection at  $-\frac{1}{3}$ . A curve for  $f$  can be sketched as follows



□

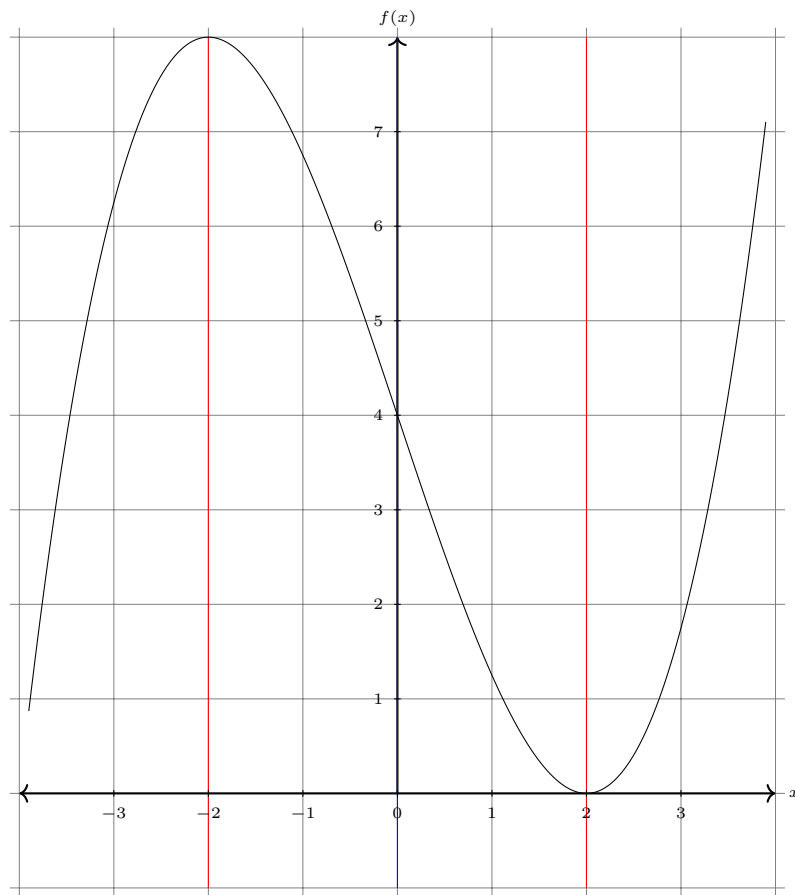
11 Sketch a continuous function having all the following properties :

$$f(-2) = 8, f(0) = 4, f(2) = 0; f'(-2) = f'(2) = 0;$$

$$f'(x) > 0 \text{ for } |x| > 2, f'(x) < 0 \text{ for } |x| < 2;$$

$$f''(x) < 0 \text{ for } x < 0, f''(x) > 0 \text{ for } x > 0.$$

*Solution.*  $f' > 0$  in  $(-\infty, -2) \cup (2, \infty)$  and thus  $f$  is strictly increasing here.  $f' < 0$  in  $(-2, 2)$  and thus  $f$  is strictly decreasing here. Thus,  $f$  has a local maximum at  $-2$  and a local minimum at  $2$ . The function values at these points are 8 and 0 respectively. Also,  $f$  is concave in  $(-\infty, 0)$  and convex in  $(0, \infty)$  with an inflection point at 0. Putting all these together, we can sketch a curve for  $f$  as:



□

### Sheet 3.

- 1 (ii) Write down the Taylor expansion of  $\arctan(x)$  around the point 0. Also write a precise remainder term  $R_n(x)$ .

*Solution.* Let  $f$  denote the arctangent function. Let  $g$  denote its derivative

$$g(x) = f'(x) = \frac{1}{1+x^2}$$

For  $|x| < 1$ , we can expand the latter as a geometric series. Thus, we have

$$g(x) = 1 - x^2 + x^4 - x^6 + \dots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

for  $|x| < 1$ . Let us now evaluate the  $n^{\text{th}}$  derivative of  $f$  at  $x = 0$ . For  $n \geq 1$ , we have

$$f^{(n)} = g^{(n-1)}$$

where  $f^{(r)}$  and  $g^{(r)}$  denote the  $r^{\text{th}}$  derivatives of  $f$  and  $g$  respectively. To evaluate the derivatives of  $g$ , we will consider two cases. First, we will evaluate all odd derivatives (derivatives of the order  $2n - 1$ ). On differentiating  $g$ ,  $r$  times, we will be left with a power series where the powers of  $x$  are of the form  $(2k - r)$  for integer  $k$ . When  $r$  is odd, no exponent of  $x$  vanishes. As a result, all the terms of the power series vanish when we plug in  $x = 0$ . Thus, all odd derivatives of  $g$  vanish at 0. I leave it to you to compute the even order derivatives at  $x = 0$ . The derivatives of  $g$  at 0 are then given by

$$g^{(2n-1)}(0) = 0, \quad g^{(2n)}(0) = (-1)^n \cdot (2n)!$$

for  $n \geq 1$ . Now, we have

$$f^{2n}(0) = g^{(2n-1)}(0) = 0$$

and

$$f^{(2n-1)}(0) = g^{(2n-2)}(0) = (-1)^{n-1} \cdot (2n - 2)!$$

for  $n \geq 1$ . Let us now compute the  $n^{\text{th}}$  Taylor polynomial  $T_n(x)$  of  $f$  at 0. Note that  $f(0)$  is 0. Define  $M := \lfloor \frac{n+1}{2} \rfloor$ . We then have

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$



where  $f^{(0)} = f$ . With a bit of manipulation, we can write

$$T_n(x) = \sum_{k=1}^M \frac{(-1)^{k-1} \cdot (2k-2)!}{(2k-1)!} \cdot x^{2k-1}$$

Thus, the  $n^{\text{th}}$  Taylor polynomial for  $\arctan$  at 0 is given by

$$T_n(x) = \sum_{k=1}^M \frac{(-1)^k}{2k-1} x^{2k-1} \quad , \quad M = \left\lfloor \left( \frac{n+1}{2} \right) \right\rfloor$$

Writing it out in a neater way, we have

$$T_{2n-1}(x) = x - \frac{x^3}{3} + \dots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$$

and

$$T_{2n}(x) = T_{2n-1}(x)$$

The remainder term is then just the difference of the arctangent function at  $x$  and its Taylor polynomial. More precisely, we have

$$R_n(x) = \arctan(x) - \sum_{k=0}^M \frac{(-1)^k}{2k-1} x^{2k-1}$$

with  $M$  defined as previously. Let us now calculate the remainder term  $R_{2n-1}(x)$  more explicitly. We have

$$\arctan' = 1 - x^2 + x^4 - \dots + (-1)^{n-1} x^{2n-2} + (-1)^n x^{2n} [1 - x^2 + x^4 - \dots]$$

$$\therefore \arctan' = 1 - x^2 + x^4 - \dots + (-1)^{n-1} x^{2n-2} + (-1)^n \frac{x^{2n}}{1+x^2}$$

On integrating both sides from 0 to  $x$ , the cyan-coloured term just becomes  $T_{2n-1}(x)$ . (Verify!) Thus, we have

$$\arctan(x) = T_{2n-1}(x) + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt$$

Thus,

$$R_{2n-1}(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} dt$$

and

$$R_{2n}(x) = R_{2n-1}(x)$$

□

- 2 Write down the Taylor series of the polynomial  $x^3 - 3x^2 + 3x - 1$  about the point 1.

*Solution.* The Taylor series is just  $(x - 1)^3$ . Let us see why. We wish to expand

$$f(x) = x^3 - 3x^2 + 3x - 1$$

about the point  $a = 1$ . We have

$$f(1) = 0$$

$$f^{(1)}(1) = 0$$

$$f^{(2)}(1) = 0$$

$$f^{(3)}(1) = 6$$

$$f^{(n)}(1) = 0 \text{ for all } n \geq 4$$

Thus, we have

$$P_0(x) = P_1(x) = P_2(x) = 0$$

$$P_3(x) = \frac{6}{3!}(x - 1)^3 = (x - 1)^3$$

and

$$P_n(x) = P_3(x) \text{ for all } n \geq 4$$

We also have

$$R_n(x) := f(x) - P_n(x) = 0 \text{ for all } n \geq 3$$

Thus,  $R_n(x) \rightarrow 0$  for **all**  $x$ . Thus, the Taylor series of the function about the point 1 is simply given by  $(x - 1)^3$ .  $\square$

- 4 Consider the series  $\sum_{k=0}^{\infty} \frac{x^k}{k!}$  for a fixed  $x$ . Prove that it converges as follows. Choose  $N > 2|x|$ . We see that for  $n > N$ ,

$$\left| \frac{x^{n+1}}{(n+1)!} \right| < \frac{1}{2} \cdot \left| \frac{x^n}{n!} \right|$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of  $\mathbb{R}$ ) is convergent.

*Solution.* Let the partial sums of the series be denoted as  $S_m(x)$ . That is,

$$S_m(x) := \sum_{k=0}^m \frac{x^k}{k!}$$

We wish to show that the difference  $|S_m(x) - S_n(x)|$  can be made arbitrarily small whenever  $m$  and  $n$  are sufficiently large. Assume that  $m > n > N$ . We see that

$$|S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^m \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left( \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{m-n}} \right) \leq \left| \frac{x^n}{n!} \right| < \left| \frac{x^N}{N!} \right|$$

Now for any  $\epsilon > 0$ , we can pick  $N$  large enough such that

$$\left| \frac{x^N}{N!} \right| < \epsilon$$

This is possible because the sequence

$$a_n = \frac{|x|^n}{n!}$$

is convergent (it is eventually decreasing and bounded below) and its limit is 0. Thus, for all  $m > n > N$ , we have

$$|S_m(x) - S_n(x)| < \epsilon$$

Hence, the given series is Cauchy and thus convergent.

(Remark: During the tutorial session, I had showed that the term  $|S_m(x) - S_n(x)|$  can be made arbitrarily small by picking  $n$  large enough. However, this is incorrect! We want to show that the term is smaller than  $\epsilon$  for any  $n, m$  greater than  $N$ . So really we have to make  $N$  large enough and conclude. This is what I have now done.)  $\square$

5 Using Taylor series, write down a series for the integral

$$\int \frac{e^x}{x} dx$$

*Solution.* We will assume that a Taylor series can be integrated term by term and then proceed. Recall that the Taylor series for  $e^x$  is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We have

$$\begin{aligned} \int \frac{e^x}{x} dx &= \int \left( \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) dx \\ &= \int \frac{1}{x} dx + \int \left( \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) dx \end{aligned}$$

Since the latter term is a Taylor series, we can integrate it term by term to obtain

$$\int \frac{e^x}{x} dx = \log x + \sum_{k=1}^{\infty} \left( \int \frac{x^{k-1}}{k!} dx \right)$$

Thus, a series representation of the integral is given by

$$\int \frac{e^x}{x} dx = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

□