MA 109: Calculus - I

Tutorial Solutions

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Contents

§1. Week 1

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Sheet 1.

2 (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Solution. We will utilise the fact that $n^{1/n} \geq 1$ for all $n \in \mathbb{N}$. (Why is this true?) We define $h_n := n^{1/n} - 1$. Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. For $n \geq 2$, we have

$$n = (1 + h_n)^n \geq 1 + \binom{n}{1} h_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

Cancelling out the n's, we get

$$h_n^2 < \frac{2}{n-1} \implies h_n < \sqrt{\frac{2}{n-1}}$$

Thus for $n \geq 2$, we have

$$0 \leq h_n < \sqrt{\frac{2}{n-1}}$$

Notice that the limit of the sequence on the right exists and is equal to 0. Thus, utilising Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$. Recalling how we defined h_n, we get $\lim_{n\to\infty} n^{1/n} = 1$. $\square$

3 (ii) Prove that the sequence $a_n := \left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is not convergent.

Solution. We will prove this result by contradiction. First, observe that the sequence $b_n := \frac{(-1)^n}{n}$ is convergent and its limit is 0. This is true because its absolute value behaves the same way as $\frac{1}{n}$ (try proving this with the ϵ-N definition to work out the details). We also know that the sequence $\{(-1)^n\}_{n\geq 1}$ is not convergent. (Why?) Now, let us assume that the given sequence (a_n) converges. We have

$$a_n := \left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\} = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

We also know that the the sum of two convergent sequences is convergent. Since a_n is assumed to be convergent and b_n is convergent, we have that $c_n := a_n + b_n = \frac{(-1)^n}{2}$ must also converge. However, the convergence of c_n implies that the sequence $(-1)^n$ also converges. Hence, we arrive at a contradiction and thus, the sequence (a_n) is not convergent.

$\square$

5 (iii) Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit.

$$a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \ \forall n \in \mathbb{N}$$

Solution. We first claim that $a_n < 6$ for all $n \in \mathbb{N}$. To prove this, we will use mathematical induction. The base case, $n = 1$ is immediate as $2 < 6$. Assume that the claim holds for some $n = k$. Now,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$$

By induction, the claim follows. Hence, a_n is bounded above.

Next, we claim that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. We have

$$a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2}$$

We just showed that $a_n < 6$ for all $n \in \mathbb{N}$. It thus follows that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Hence, (a_n) is a monotonically increasing sequence that is bounded above. Thus, it must converge. To find the limit of (a_n), we utilise the fact that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ (Sheet 1 : Problem 6). Let L denote the limit of (a_n). Taking the limit of the recursive definition (and using some limit properties), we have that

$$L = 3 + \frac{L}{2} \implies L = 6$$

Thus, the sequence (a_n) converges to 6. (Notice that this was the upper bound we chose for (a_n)) $\square$

7 If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0$$

Solution. We will use the $\epsilon - N$ definition to prove this result. Choose $\epsilon = \dfrac{|L|}{2}$. Since $L \neq 0$, we have $\epsilon > 0$. Now, as $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. From triangle inequality, we have

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \implies -\epsilon < |a_n| - |L| \quad \forall n \geq n_0$$

Substituting the value of ϵ, we get that

$$|a_n| > \frac{|L|}{2}$$

for all $n \geq n_0$, as desired. □

9 For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following statements:

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

Solution. This is a relatively short question. Both the statements are **false**. Verify that $a_n := 1$ and $b_n := (-1)^n$ acts as a counterexample for both the statements. □

11 Let $f, g\colon (a,b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a,b]$. Prove or disprove the following statements.

(i) $\lim_{x\to c} [f(x)g(x)] = 0$.

(ii) $\lim_{x\to c} [f(x)g(x)] = 0$ if g is bounded.

(iii) $\lim_{x\to c} [f(x)g(x)] = 0$ if $\lim_{x\to c} g(x)$ exists.

Solution. (i) This statement is **false**. As a counterexample, define $a = -1, b = 1$ and $c = 0$. Define $f, g\colon (-1,1) \to \mathbb{R}$ as

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$$

Clearly, $\lim_{x\to 0} f(x) = 0$. However, $\lim_{x\to 0} [f(x)g(x)]$ does not exist.

(ii) This statement is **true**. Since g is bounded, there exists $M > 0$ such that

$$|g(x)| \leq M$$

for all $x \in (a,b)$. Thus, we have

$$0 \leq |f(x)g(x)| \leq M|f(x)|$$

for all $x \in (a,b)$. Using Sandwich Theorem, we see that

$$\lim_{x\to c} |f(x)g(x)| = 0$$

which in turn implies that

$$\lim_{x\to c} [f(x)g(x)] = 0$$

(iii) This statement is **true**. Since $\lim_{x\to c} g(x)$ exists, we have $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 0$.

$\square$

§2. Week 2

2nd December, 2020

Sheet 1.

13 (ii) Discuss the continuity of the following function :

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. At all points other than $x = 0$, the given function is trivially continuous (since it is the product and composition of continuous functions). All that remains is to check the continuity of f at the point $x = 0$. Note that

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

for all $x \neq 0$. Thus, we have

$$0 \leq |f(x)| \leq |x|$$

Utilising Sandwich Theorem, we see that

$$\lim_{x \to 0} f(x) = 0$$

Since $f(0)$ is given to be 0, we see that $\lim_{x \to 0} f(x) = f(0)$, proving continuity of f at $x = 0$. Thus, f is continuous everywhere. $\square$

15 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows.

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. Clearly, f is differentiable for all $x \neq 0$. Using the chain rule and product rule, we compute f' as

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

for $x \neq 0$. Now, all that remains to be checked is the differentiability of f at $x = 0$. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

From the previous question, this limit exists and is equal to 0. Thus, f is differentiable on all of $\mathbb{R}$ and its derivative is defined as

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly, f' is continuous at all $x \neq 0$. All that remains is to check continuity of f' at $x = 0$. It turns out that f' is in fact *not* continuous at $x = 0$. We will use the sequential criterion of continuity to prove this. Consider the sequence:

$$x_n := \frac{1}{2n\pi}, \quad n \in \mathbb{N}$$

Clearly, $x_n \to 0$ as $n \to \infty$. However,

$$f'(x_n) = \frac{2}{\cancel{2n\pi}} \cancel{\cdot \sin(2n\pi)} - \cos(2n\pi) = -1$$

We see that $\lim\limits_{n \to \infty} f(x_n)$ is -1, which is not equal to $f'(0)$. Hence, f' is not continuous at $x = 0$. This is an example of a differentiable function whose derivative is not continuous. $\square$

18 Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x) \cdot f(y) \quad \text{for all } x, y \in \mathbb{R}$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$.

Solution. We have that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. On substituting $x = y = 0$, we obtain

$$f(0) = f(0) \cdot f(0) \implies f(0) = 0 \text{ or } 1$$

First, we consider the case that $f(0) = 0$. We have

$$f(x) = f(x+0) = f(x) \cdot f(0) \implies f(x) = 0$$

for all x. Thus, $f \equiv 0$ is trivially differentiable and $f'(c) = 0 = f'(0) \cdot f(c)$ for all $c \in \mathbb{R}$.

Now consider that $f(0) = 1$. For all $c \in \mathbb{R}$, we have

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c)f(h) - f(c)f(0)}{h} = f(c) \cdot \left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \right)$$

If f is differentiable at 0, then the above limit exists. Thus, if f is differentiable at 0, then it is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$. $\square$

Optional Exercises.

7 Let $f\colon (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following statements are equivalent.

(i) f is differentiable at c.

(ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$ and a function $\epsilon_1\colon (-\delta,\delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$$

for all $h \in (-\delta,\delta)$.

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0}\left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|}\right) = 0$$

Solution. To show the equivalence of statements (i)-(iii), we must show that every statement implies every other statement, that is, a total of 6 implications. However, we can get away with just showing three implications. We will show that $(i) \to (ii)$, $(ii) \to (iii)$ and $(iii) \to (i)$. This is sufficient to conclude the equivalence of the three statements. (Why?)

$(i) \to (ii)$: Since we are given that f is differentiable at c, $f'(c)$ exists. We first pick $\delta := \min\{c-a, b-c\}$. Clearly $\delta > 0$ and $(c-\delta, c+\delta) \subset (a,b)$. Now, since f is differentiable at c, $f'(c)$ exists. Define $\alpha := f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Since $(c-\delta, c+\delta) \subset (a,b)$, $f(c+h)$ is well defined for all $h \in (-\delta,\delta)$. Now,

$$\lim_{h\to 0} \epsilon_1(h) = \underbrace{\left(\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}\right)}_{\alpha} - \alpha = 0$$

Further, some simple algebraic manipulation yields that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for $h \in (-\delta,\delta), h \neq 0$. Verify that this equation also holds for $h = 0$. It then follows that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for all $h \in (-\delta,\delta)$ and $\lim_{h\to 0} \epsilon_1(h) = 0$, as desired.

$(ii) \to (iii)$: By (ii), we have the existence of $\delta > 0, \alpha \in \mathbb{R}$ and the function ϵ_1. We have

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h \to 0} |\epsilon_1(h)| = 0$$

$(iii) \to (i)$: By (iii), we have the existence of some $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

Now,

$$\lim_{h \to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Thus, f is differentiable at c, as desired.

Since we have shown $(i) \to (ii)$, $(ii) \to (iii)$ and $(iii) \to (i)$, we get that the three statements are thus equivalent. □

10 Show that any continuous function $f\colon [0,1] \to [0,1]$ has a fixed point. x is said to be a fixed point of f if $f(x) = x$

Solution. Consider the function $g(x) = f(x) - x$. A fixed point of f is then a root of g. Note that g is continuous. Since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, we have

$$g(0) = f(0) \implies g(0) \geq 0$$

and

$$g(1) = f(1) - 1 \implies g(1) \leq 0$$

First consider the case that at least one of the two equalities hold. That is, either $g(0) = 0$ or $g(1) = 0$ or both. In either of the three cases, we have at least one fixed point (0 or 1 or both, respectively). Now, consider that $g(0) > 0$ and $g(1) < 0$. Since g is continuous, we can appeal to Intermediate Value Theorem. By IVT, there exists some $x_0 \in (0,1)$ such that $g(x_0) = 0$. This point x_0 is also a fixed point of f. Thus, we have shown that any continuous function mapping the unit interval to itself has a fixed point, as desired. □

Sheet 2.

3 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Solution. Since $f(a)$ and $f(b)$ are of opposite signs and f is continuous, we know that there exists **at least** one $x_0 \in (a, b)$ such that $f(x_0) = 0$ (by IVP). Now, assume that there was some $y_0 (\neq x_0)$ in (a, b) such that $f(y_0) = 0$. We now have $f(x_0) = f(y_0)$. By Rolle's Theorem, there must exist some $c \in (x_0, y_0)$ such that $f'(c) = 0$. Since this c also lies in (a, b), we arrive at a contradiction. Hence, there is a unique x_0 in (a, b) such that $f(x_0) = 0$, as desired. □

5 Use the MVT to show that $|\sin(a) - \sin(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Solution. We will break this problem into two cases. First, consider $a = b$. The inequality is trivially satisfied in this case. Next, consider $a \neq b$. Define $f(x) = \sin(x)$. By MVT, there exists some c between a and b such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Since $f' = \cos$, we take modulus on both sides to obtain

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1$$

Rearranging, we get

$$|\sin a - \sin b| \leq |a - b|$$

for all $a, b \in \mathbb{R}$, as desired. □

§3. Week 3

9th December, 2020

Sheet 2.

8 In each case, find a function f that satisfies all the given conditions, or else show that no such function exists.

(ii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$.

Solution.

(ii) Possible. Verify that $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) := x + \dfrac{x^2}{2}$ is one such function.

(iii) Not possible. Assume that it was indeed possible to find such a function f. Then, we are given that f'' exists everywhere. Thus, f' is continuous and differentiable everywhere. As f'' is non-negative, f' must be increasing everywhere. Since $f'(0) = 1$, we have that $f'(c) \geq 1$ for all $c > 0$.

Let $x \in (0, \infty)$. By MVT, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

Since $c > 0$, we have $f'(c) \geq 1$ as shown above. Thus, $f(x) - f(0) \geq x$ for all $x > 0$. However, consider $x_0 := \max(101 - f(0), 1)$. Clearly, $x_0 > 0$ (as it is ≥ 1). Also, $f(x_0) > 100$, which contradicts the condition that $f(x) \leq 100$ for all $x > 0$. Hence, no such f can exist.

□

10 (i) Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local minima/maxima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

$$y = 2x^3 + 2x^2 - 2x - 1$$

Solution. We are given

$$f(x) = 2x^3 + 2x^2 - 2x - 1$$

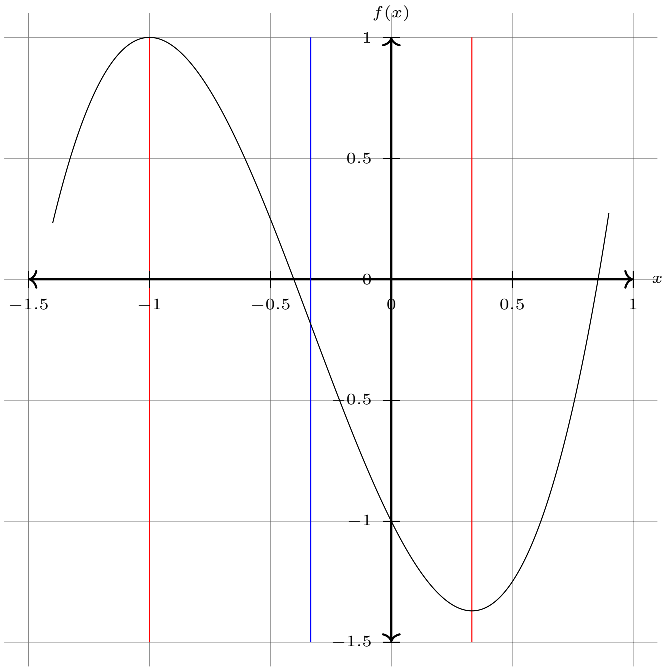
On differentiating, we get

$$f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$$

Thus, $f' > 0$ in $(-\infty, -1) \cup (\frac{1}{3}, \infty)$ and f is strictly increasing here. $f' < 0$ in $(-1, \frac{1}{3})$ and f is strictly decreasing here. Thus, f has a local maximum at -1 and a local minimum at $\frac{1}{3}$. Differentiating again, we see that

$$f''(x) = 12x + 4$$

Thus, f is convex in $(-\frac{1}{3}, \infty)$ and concave in $(-\infty, -\frac{1}{3})$, with a point of inflection at $-\frac{1}{3}$. A curve for f can be sketched as follows



□

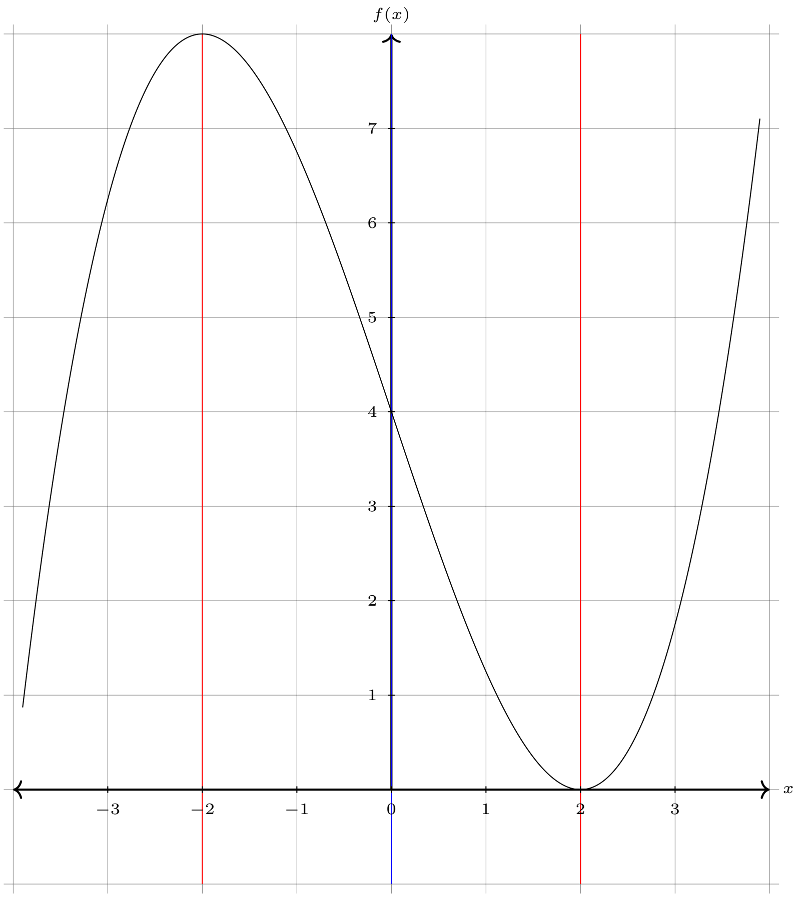
11 Sketch a continuous function having all the following properties :

$$f(-2) = 8, f(0) = 4, f(2) = 0; f'(-2) = f'(2) = 0;$$

$$f'(x) > 0 \text{ for } |x| > 2,\ f'(x) < 0 \text{ for } |x| < 2;$$

$$f''(x) < 0 \text{ for } x < 0,\ f''(x) > 0 \text{ for } x > 0.$$

Solution. $f' > 0$ in $(-\infty, -2) \cup (2, \infty)$ and thus f is strictly increasing here. $f' < 0$ in $(-2, 2)$ and thus f is strictly decreasing here. Thus, f has a local maximum at -2 and a local minimum at -2. The function values at these points are 8 and 0 respectively. Also, f is convex in $(0, \infty)$ and concave in $(-\infty, 0)$ with an inflection point at 0. Putting all these together, we can sketch a curve for f as:



□

Sheet 3.

1 (ii) Write down the Taylor expansion of $\arctan(x)$ around the point 0. Also write a precise remainder term $R_n(x)$.

Solution. Let f denote the arctangent function. Let g denote its derivative

$$g(x) = f'(x) = \frac{1}{1+x^2}$$

For $|x| < 1$, we can expand the latter as a geometric series. Thus, we have

$$g(x) = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

for $|x| < 1$. Let us now evaluate the n^{th} derivative of f at $x = 0$. For $n \geq 1$, we have

$$f^{(n)} = g^{(n-1)}$$

where $f^{(r)}$ and $g^{(r)}$ denote the r^{th} derivatives of f and g respectively. To evaluate the derivatives of g, we will consider two cases. First, we will evaluate all odd derivatives (derivatives of the order $2n-1$). On differentiating g, r times, we will be left with a power series where the powers of x are of the form $(2k-r)$ for integer k. When r is odd, no exponent of x vanishes. As a result, all the terms of the power series vanish when we plug in $x = 0$. Thus, all odd derivatives of g vanish at 0. I leave it to you to compute the even order derivatives at $x = 0$. The derivatives of g at 0 are then given by

$$g^{(2n-1)}(0) = 0, \quad g^{(2n)}(0) = (-1)^n \cdot (2n)!$$

for $n \geq 1$. Now, we have

$$f^{2n}(0) = g^{(2n-1)}(0) = 0$$

and

$$f^{(2n-1)}(0) = g^{(2n-2)}(0) = (-1)^{n-1} \cdot (2n-2)!$$

for $n \geq 1$. We shall first compute the zeroth Taylor Polynomial. We have

$$T_0(x) = f(0) = 0$$

Let us now compute the n^{th} Taylor polynomial $T_n(x)$ of f at 0 for $n \geq 1$. Define $M := \lfloor\left(\frac{n+1}{2}\right)\rfloor$. For $n \geq 1$, we then have

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

where $f^{(0)} = f$. With a bit of manipulation, we can write

$$T_n(x) = \sum_{k=1}^{M} \frac{(-1)^{k-1} \cdot (2k-2)!}{(2k-1)!} \cdot x^{2k-1}$$

Thus, the n^{th} Taylor polynomial for arctan at 0 is given by

$$T_n(x) = \sum_{k=1}^{M} \frac{(-1)^k}{2k-1} x^{2k-1} \quad , \quad M = \left\lfloor \left(\frac{n+1}{2} \right) \right\rfloor$$

Writing it out in a neater way, we have

$$T_{2n-1}(x) = x - \frac{x^3}{3} + \ldots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$$

and

$$T_{2n}(x) = T_{2n-1}(x)$$

The remainder term is then just the difference of the arctangent function at x and its Taylor polynomial. More precisely, we have

$$R_n(x) = \arctan(x) - \sum_{k=0}^{M} \frac{(-1)^k}{2k-1} x^{2k-1}$$

with M defined as previously. Let us now calculate the remainder term $R_{2n-1}(x)$ more explicitly. We have

$$\arctan' = {\color{cyan} 1 - x^2 + x^4 + \ldots + (-1)^{n-1} x^{2n-2}} + (-1)^n x^{2n} \left[1 - x^2 + x^4 - \ldots \right]$$

$$\therefore \arctan' = {\color{cyan} 1 - x^2 + x^4 + (-1)^{n-1} x^{2n-2}} + (-1)^n \frac{x^{2n}}{1+x^2}$$

On integrating both sides from 0 to x, the cyan-coloured term just becomes $T_{2n-1}(x)$. (Verify!) Thus, we have

$$\arctan(x) = T_{2n-1}(x) + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} \, \mathrm{d}t$$

Thus,

$$R_{2n-1}(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} \, \mathrm{d}t$$

and

$$R_{2n}(x) = R_{2n-1}(x)$$

$\square$

2 Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.

Solution. The Taylor series is just $(x-1)^3$. Let us see why. We wish to expand

$$f(x) = x^3 - 3x^2 + 3x - 1$$

about the point $a = 1$. We have

$$f(1) = 0$$

$$\begin{aligned} f^{(1)}(1) &= 0 \\ f^{(2)}(1) &= 0 \\ f^{(3)}(1) &= 6 \\ f^{(n)}(1) &= 0 \ \text{ for all } n \geq 4 \end{aligned}$$

Thus, we have

$$\begin{aligned} P_0(x) = P_1(x) &= P_2(x) = 0 \\ P_3(x) &= \frac{6}{3!}(x-1)^3 = (x-1)^3 \end{aligned}$$

and

$$P_n(x) = P_3(x) \ \text{ for all } n \geq 4$$

We also have

$$R_n(x) := f(x) - P_n(x) = 0 \ \text{ for all } n \geq 3$$

Thus, $R_n(x) \to 0$ for **all** x. Thus, the Taylor series of the function about the point 1 is simply given by $(x-1)^3$. □

4 Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. Prove that it converges as follows. Choose $N > 2|x|$. We see that for $n > N$,

$$\left|\frac{x^{n+1}}{(n+1)!}\right| < \frac{1}{2} \cdot \left|\frac{x^n}{n!}\right|$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$) is convergent.

Solution. Let the partial sums of the series be denoted as $S_m(x)$. That is,

$$S_m(x) := \sum_{k=0}^{m} \frac{x^k}{k!}$$

We wish to show that the difference $|S_m(x) - S_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Assume that $m > n > N$. We see that

$$|S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{m-n}} \right) \leq \left| \frac{x^n}{n!} \right| < \left| \frac{x^N}{N!} \right|$$

Now for any $\epsilon > 0$, we can pick N large enough such that

$$\left| \frac{x^N}{N!} \right| < \epsilon$$

This is possible because the sequence

$$a_n = \frac{|x|^n}{n!}$$

is convergent (it is eventually decreasing and bounded below) and its limit is 0. Thus, for all $m > n > N$, we have

$$|S_m(x) - S_n(x)| < \epsilon$$

Hence, the given series is Cauchy and thus convergent.

(Remark: During the tutorial session, I had showed that the term $|S_m(x) - S_n(x)|$ can be made arbitrarily small by picking n large enough. However, this is incorrect! We want to show that the term is smaller than ϵ for any n, m greater than N. So really we have to make N large enough and conclude. This is what I have now done.) $\square$

5 Using Taylor series, write down a series for the integral

$$\int \frac{e^x}{x} \, \mathrm{d}x$$

Solution. We will assume that a Taylor series can be integrated term by term and then proceed. Recall that the Taylor series for e^x is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We have

$$\begin{aligned}
\int \frac{e^x}{x} \, \mathrm{d}x &= \int \left(\frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) \mathrm{d}x \\
&= \int \frac{1}{x} \, \mathrm{d}x + \int \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) \mathrm{d}x
\end{aligned}$$

Since the latter term is a Taylor series, we can integrate it term by term to obtain

$$\int \frac{e^x}{x} \, \mathrm{d}x = \log x + \sum_{k=1}^{\infty} \left(\int \frac{x^{k-1}}{k!} \, \mathrm{d}x \right)$$

Thus, a series representation of the integral is given by

$$\int \frac{e^x}{x} \, \mathrm{d}x = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

□

§4. Week 4

16th December, 2020

Sheet 4.

2 (a) Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

Solution. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ denote a partition of $[a,b]$. Define $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Further, we define

$$m_i = \inf\{f(x)\colon x_{i-1} \leq x \leq x_i\}$$

Since $f(x) \geq 0$ for all $x \in [a,b]$, it follows that $m_i \geq 0$ for all i. The lower sum is now defined as

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Since $m_i \geq 0$ and $\Delta x_i > 0$ for all i, it follows that $L(P,f) \geq 0$ for any partition P. Thus, we also see that $L(f) \geq 0$ since $L(f)$ is the supremum of $L(P,f)$ over all partitions P. Since f is Riemann integrable, we have

$$\int_a^b f(x)\,\mathrm{d}x = L(f) \geq 0$$

as desired.

Now, let us further assume that f is continuous and that $\int_a^b f(x)\,\mathrm{d}x = 0$. If f is not identically zero, then there exists $c \in [a,b]$ such that $f(c) > 0$. Continuity of f implies that there exists a $\delta > 0$ such that, if $x \in [a,b]$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{f(c)}{2} \implies f(x) > \frac{f(c)}{2}$$

We may now assume $c \in (a,b)$ without any loss of generality [1] Further, pick $\delta > 0$ small enough so that $(c - \delta, c + \delta) \subset (a,b)$. Now, consider the partition

$$P = \left\{a, c - \frac{\delta}{2}, c + \frac{\delta}{2}, b\right\}$$

[1] If $c = a$ or $c = b$, then we can pick another point $\tilde{c}$ in (a,b) such that $f(\tilde{c}) \neq 0$.

Since we have

$$\inf_{x \in [c - \frac{\delta}{2}, c + \frac{\delta}{2}]} f(x) \geq \frac{f(c)}{2}$$

it follows that

$$L(f) \geq L(P, f) \geq \frac{f(c)\delta}{2} > 0$$

Further, if f is Riemann integrable, we have that its integral over $[a, b]$ is equal to $L(f)$, which is strictly positive - a contradiction! Hence, f must be identically zero. □

Alternate. (easier)

Solution. Consider the trivial partition $P_0 = a, b$ of $[a, b]$. Since $f(x) \geq 0$ for all $x \in [a, b]$, we have

$$\inf_{x \in [a,b]} f(x) \geq 0$$

We have

$$L(f, P_0) = \left[\inf_{x \in [a,b]} f(x) \right] \cdot (b - a) \geq 0$$

and

$$L(f) \geq L(f, P_0) \geq 0$$

Since f is Riemann integrable, its integral is $L(f)$, which is non-negative, as desired.

For the second part, define $F\colon [a, b] \to \mathbb{R}$ as

$$F(x) = \int_a^x f(t)\, \mathrm{d}t$$

Since f is continuous, we get that F is differentiable with $F' = f$, from the Fundamental Theorem of Calculus (Part 1). Since $f \geq 0$, we have $F' \geq 0$ and hence, F is increasing. This implies that for all $x \in [a, b]$, we have

$$F(a) \leq F(x) \leq F(b)$$

However, since $F(a) = 0 = F(b)$, we get that F is constant and hence,

$$f(x) = F'(x) = 0$$

for all $x \in [a, b]$, as desired. □

2 (b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\,\mathrm{d}x = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

Solution. As we saw in the previous question, no continuous function can satisfy these conditions. Thus, we must look for a discontinuous function. We define f on $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{when } x \neq \frac{1}{2} \\ 1 & \text{when } x = \frac{1}{2} \end{cases}$$

Since f has only finitely many discontinuities, it is Riemann integrable. Also, $f(x) \geq 0$ for all $x \in [0, 1]$. Further, it is easy to show that its Riemann integral over the interval is 0. Lastly, we have $f(\frac{1}{2}) = 1 \neq 0$. Thus, $f(x) \neq 0$ for some $x \in [0, 1]$. Hence, this f satisfies our desired conditions. □

3 Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an ~~approximate~~ appropriate Riemann sum of a suitable function over a suitable interval.

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$ (iv) $S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n}$

We shall use the following theorem for both the parts.

> **Theorem**
>
> Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, T_n) be a sequence of tagged partitions of $[a,b]$ such that $\|P_n\| \to 0$. Then,
>
> $$R(P_n, T_n, f) \to \int_a^b f(t)\,\mathrm{d}t$$

(ii) *Solution.* Consider $f\colon [0,1] \to \mathbb{R}$ defined as $f(x) := \arctan(x)$. Then, we have

$$f'(x) = \frac{1}{1+x^2}$$

Since f' is continuous on $[0,1]$, it is Riemann integrable on $[0,1]$. Let $P_n := \left\{x_i = \frac{i}{n} \colon 0 \le i \le n\right\}$ be a tagged partition of $[0,1]$ for $n \in \mathbb{N}$ and let $T_n := \left\{t_i = \frac{i}{n} \colon 1 \le i \le n\right\}$ denote the tags of the partition.

We have $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$ for all $1 \le i \le n$. The Riemann sum corresponding to this tagged partition is given by

$$\begin{aligned} R(P_n, T_n, f') = \sum_{i=1}^{n} f'(t_i)\Delta x_i &= \sum_{i=1}^{n} \frac{1}{1+t_i^2} \cdot \frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{n}{i^2+n^2} = S_n \end{aligned}$$

Thus, $R(P_n, T_n, f') = S_n$ for all $n \ge 1$. Moreover,

$$\|P_n\| = \max\{x_i - x_{i-1} \colon 1 \le i \le n\} = \frac{1}{n}$$

Clearly, we have

$$\lim_{n\to\infty} \|P_n\| = 0$$

and thus,

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x$$

From the Fundamental Theorem of Calculus (Part 2), we have that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0) = \boxed{\frac{\pi}{4}}$$

$\square$

(iv) *Solution.* Consider $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) := \frac{1}{\pi}\sin(\pi x)$$

We then have $f'(x) = \cos(\pi x)$. Since f' is continuous on $[0,1]$, it is Riemann integrable on $[0,1]$. Let $P_n := \left\{x_i = \frac{i}{n} \colon 0 \le i \le n\right\}$ be a tagged partition of $[0,1]$ for $n \in \mathbb{N}$ and let $T_n := \left\{t_i = \frac{i}{n} \colon 1 \le i \le n\right\}$ denote the tags of the partition.

We have $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$ for all $1 \le i \le n$. The Riemann sum corresponding to this tagged partition is given by

$$\begin{aligned} R(P_n, T_n, f') = \sum_{i=1}^{n} f'(t_i)\Delta x_i &= \sum_{i=1}^{n} \cos(\pi t_i) \cdot \frac{1}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n} = S_n \end{aligned}$$

Thus, $R(P_n, T_n, f') = S_n$ for all $n \ge 1$. Moreover,

$$\|P_n\| = \max\left\{x_i - x_{i-1} \colon 1 \le i \le n\right\} = \frac{1}{n}$$

Clearly, we have

$$\lim_{n\to\infty} \|P_n\| = 0$$

and thus,

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x$$

From the Fundamental Theorem of Calculus (Part 2), we have that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0) = \boxed{0}$$

$\square$

4(b) Compute $\dfrac{\mathrm{d}F}{\mathrm{d}x}$ if for $x \in \mathbb{R}$,

(i) $F(x) = \displaystyle\int_1^{2x} \cos\left(t^2\right)\,\mathrm{d}t$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)\,\mathrm{d}t$

Solution. Before solving these two subparts, I will first prove a short lemma.

> **Lemma**
>
> Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $v\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined as
>
> $$F(x) := \int_0^{v(x)} f(t)\,\mathrm{d}t$$
>
> Then,
>
> $$F'(x) = f\left(v(x)\right) \cdot v'(x)$$

Proof. First, we define $G\colon \mathbb{R} \to \mathbb{R}$ as

$$G(x) := \int_0^{x} f(t)\,\mathrm{d}t$$

Then, $G' = f$ by the Fundamental Theorem of Calculus (Part 1). Now,

$$F(x) = G\left(v(x)\right)$$

A simple application of chain rule yields

$$F'(x) = f\left(v(x)\right) \cdot v'(x)$$

as desired. □

(i) We have $v(x) = 2x$ and $f(t) = \cos(t^2)$. It thus follows from the above lemma that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \cos\left((2x)^2\right) \cdot (2x)' = \boxed{2\cos\left(4x^2\right)}$$

(ii) We have $v(x) = x^2$ and $f(t) = \cos(t)$. It thus follows from the above lemma that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \cos\left(x^2\right) \cdot \left(x^2\right)' = \boxed{2x\cos\left(x^2\right)}$$

□

6 Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}, \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \, \mathrm{d}t$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = g'(0) = 0$.

Solution. We will first make use of the identity $\sin(A-B) = \sin A \cos B - \cos A \sin B$. We have

$$\begin{aligned}
g(x) &= \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \, \mathrm{d}t \\
&= \frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t\right) \, \mathrm{d}t \\
&= \frac{1}{\lambda} \sin \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t - \frac{1}{\lambda} \cos \lambda t \int_0^x f(t) \sin \lambda t \, \mathrm{d}t
\end{aligned}$$

On applying the product rule and Fundamental Theorem of Calculus (Part 1), we get

$$\begin{aligned}
g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + \cancel{\frac{1}{\lambda} \sin \lambda x \cdot f(x)} \cdot \cos \lambda x \\
+ \sin \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t - \cancel{\frac{1}{\lambda} \sin \lambda x \cdot f(x)} \cdot \cos \lambda x
\end{aligned}$$

$$\therefore\ g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + \sin \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t$$

It is now easy to verify that both $g(0)$ and $g'(0)$ are indeed 0. We will differentiate g' in a similar manner to obtain

$$\begin{aligned}
g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + f(x) \cos^2 \lambda x \\
&\quad + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t + f(x) \sin^2 \lambda x \\
&= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t\right) \, \mathrm{d}t \right) \\
&= f(x) - \lambda^2 g(x)
\end{aligned}$$

It thus follows that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. $\square$

§5. Week 5

23rd December, 2020

Sheet 5

2 Describe the level curves and contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$.

(ii) $f(x, y) = x^2 + y^2$ (iii) $f(x, y) = xy$

(ii) *Solution.* $(x^2 + y^2 = c)$

For $c = -3, -2, -1$, level curves and contour lines are empty sets. For $c = 0$, the level curve is the point $(0, 0) \in \mathbb{R}^2$ and the contour line is the point $(0, 0, 0) \in \mathbb{R}^3$. For any $c \in \{1, 2, 3, 4\}$, the level curve is a circle in the xy plane, centered at the origin, with radius $\sqrt{c}$. More precisely, the level curve is the set $L = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = c\}$. The contour line is a cross-section in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$. That is, a circle in the plane $z = c$, centered at $(0, 0, c)$ and with radius $\sqrt{c}$. More precisely, the contour line is the set $L \times \{c\}$. □

(iii) *Solution.* $(xy = c)$

For $c = 0$, the level set is the union of the x and y axes in the xy-plane. Precisely, this is the set $L = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$. The contour line corresponding to $c = 0$ is the union of the x and y axes in the xyz-space. This is the set $L \times \{0\}$. For any non-zero c, the level curve is the rectangular hyperbola $xy = c$ and the contour line is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$. More precisely, the level curve is the set $L = \{(x, y) \in \mathbb{R}^2 \mid xy = c\}$ and the contour line is the set $L \times \{c\}$. For negative c, the level curve (and the contour line) has branches in the second and fourth quadrant while for positive c, the level curve (and the contour line) has branches in the first and third quadrants. □

4 Suppose $f, g \colon \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous.

(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$

(iii) $\max\{f(x), g(y)\}$ (iv) $\min\{f(x), g(y)\}$

Solution. We will use sequential criterion of continuity. Just to recall:

> **Theorem: Sequential Criterion**
>
> Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a function. Then, f is continuous at (x_0, y_0) if and only if for every sequence $((x_n, y_n))$ converging to (x_0, y_0), we have
>
> $$\lim_{n\to\infty} f(x_n, y_n) = f(x_0, y_0)$$

Let (x_0, y_0) be an arbitrary point of $\mathbb{R}^2$. Let (x_n, y_n) be an arbitrary sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x_0, y_0)$. We then have $x_n \to x_0$ and $y_n \to y_0$. Since f and g are continuous, it follows that $f(x_n) \to f(x_0)$ and $g(y_n) \to g(y_0)$. For (i) and (ii), note that we can now use algebra of limits to conclude that the given functions are indeed continuous.

For (iii) and (iv), note the following:

$$\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \frac{|f(x)-g(y)|}{2}$$

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \frac{|f(x)-g(y)|}{2}$$

Again, consider $((x_n, y_n))$ to be an arbitrary sequence converging to (x_0, y_0). We then have $x_n \to x_0$ and $y_n \to y_0$. From the continuity of f, g it follows that $f(x_n) \to f(x_0)$ and $g(y_n) \to g(y)$. Thus, we have

$$f(x_n) + g(y_n) \to f(x_0) + g(y_0)$$

Since the modulus function is continuous, we also have

$$|f(x_n) + g(y_n)| \to |f(x_0) + g(y_0)|$$

It then follows that

$$\frac{f(x_n)+g(y_n)}{2} + \frac{|f(x_n)-g(y_n)|}{2} \longrightarrow \frac{f(x_0)+g(y_0)}{2} + \frac{|f(x_0)-g(y_0)|}{2}$$

which can be rewritten as

$$\max\{f(x_n), g(y_n)\} \to \max\{f(x_0), g(y_0)\}$$

concluding the proof for (iii). Similarly, the proof for (iv) follows.

Since the point (x_0, y_0) was arbitrary, it follows that the given functions are continuous on all of $\mathbb{R}^2$. $\square$

6 (ii) Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value for $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken to be zero.

$$\frac{\sin^2(x+y)}{|x|+|y|}$$

Solution. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given. That is,

$$f(x,y) = \begin{cases} \dfrac{\sin^2(x+y)}{|x|+|y|} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $h \neq 0$, we have

$$\frac{f(h,0)-f(0,0)}{h} = \left(\frac{\sin^2 h}{h|h|}\right)$$

It is easy to show that the above limit (as h goes to 0) does not exist (Take strictly positive and strictly negative sequences converging to zero). Hence, we see that $\dfrac{\partial f}{\partial x_1}(0,0)$ does not exist. Similar arguments show that the second partial does not exist either. □

8 Let f be defined as

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & x \neq 0, y \neq 0 \\ x\sin(1/x) & x \neq 0, y = 0 \\ y\sin(1/y) & x = 0, y \neq 0 \\ 0 & x = 0, y = 0 \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

Solution. Let us first show that the given function is continuous at $(0,0)$. Let (x_n, y_n) be a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. This gives us that $x_n \to 0$ and $y_n \to 0$. Now, note that

$$0 \leq |f(x_n, y_n)| \leq |x_n| + |y_n|$$

for all $(x_n, y_n) \in \mathbb{R}^2$. Since $(x_n, y_n) \to (0,0)$, we get that $f(x_n, y_n) \to 0 = f(0,0)$. Thus, the function is continuous at $(0,0)$.

Let us now show that neither partial derivatives of f at $(0,0)$ exist. For $h \neq 0$, we have

$$\frac{f(h,0) - f(0,0)}{h} = \sin\frac{1}{h}$$

The limit of the above expression as $h \to 0$, does not exist. Similar arguments show that the second partial derivative does not exist either. $\square$

10 Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Show that f is continuous at $(0,0)$, $\nabla_{\mathbf{u}} f(0,0)$ exists for every unit vector $\mathbf{u}$ and yet, f is not differentiable at $(0,0)$.

Solution. First, we will show that f is indeed continuous at $(0,0)$. We will now use the ϵ-δ condition (it's easier to work with in this case). Note that we have

$$|f(x,y) - f(0,0)| = \begin{cases} \sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Thus, in general, we have

$$|f(x,y) - f(0,0)| \leq \sqrt{x^2+y^2}$$

Now, given any $\epsilon > 0$, setting $\delta := \epsilon$ works.

Let $\mathbf{u} = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$. If $u_2 \neq 0$, then for $t \neq 0$, we have

$$\begin{aligned} \frac{f(u_1 t, u_2 t) - f(0,0)}{t} &= \frac{f(u_1 t, 0) - 0}{t} \\ &= \frac{0-0}{t} = 0 \end{aligned}$$

For $u_2 \neq 0$ and $t \neq 0$, we have

$$\begin{aligned} \frac{f(u_1 t, u_2 t) - f(0,0)}{t} &= \frac{1}{t}\frac{u_2 t}{|u_2 t|}\sqrt{(u_1^2+u_2^2)\,t} \\ &= \frac{1}{t}\frac{u_2 t}{|u_2 t|}|t| \\ &= \frac{u_2}{|u_2|} \end{aligned}$$

Thus, all directional derivatives exist and are given by

$$\nabla_{\mathbf{u}} f(0,0) = \begin{cases} 0 & u_2 = 0 \\ \dfrac{u_2}{|u_2|} & u_2 \neq 0 \end{cases}$$

Setting $\mathbf{u} = (1,0)$ and $(0,1)$ recovers the partial derivatives. We will now check if f is differentiable.

If f is differentiable at $(0,0)$ then its total derivative must be

$$Df(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(0,0) & \frac{\partial f}{\partial x_2}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

We will now check if this is indeed the total derivative. We must check whether

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - 0h - 1k|}{\sqrt{h^2+k^2}} = 0$$

For $k \neq 0$, we have that

$$\frac{|f(h,k) - f(0,0) - 0h - 1k|}{\sqrt{h^2+k^2}} = \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right|$$

Along the curve $h = k$ with $k \neq 0$, the above expression becomes

$$\left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| = \left| \frac{k}{|k|} - \frac{k}{\sqrt{2k^2}} \right| = \left(1 - \frac{1}{\sqrt{2}}\right)$$

Clearly the limit of this expression is not zero (which is what we wanted it to be). Thus, f is not differentiable at $(0,0)$.

<u>Remark:</u> The above does not show that the limit is $\left(1 - \frac{1}{\sqrt{2}}\right)$. In fact, the limit does not exist at all. However, this is sufficient to show that the limit is not zero (which is all we required). $\square$