MA 109: Calculus - I

Tutorial Solutions

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Contents

§1. Week 1

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Sheet 1.

2 (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Solution. We will utilise the fact that $n^{1/n} \geq 1$ for all $n \in \mathbb{N}$. (Why is this true?) We define $h_n \coloneqq n^{1/n} - 1$. Then, $h_n \geq 0$ for all $n \in \mathbb{N}$. For $n \geq 2$, we have

$$n = (1 + h_n)^n \geq 1 + \binom{n}{1} h_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2$$

Cancelling out the n's, we get

$$h_n^2 < \frac{2}{n-1} \implies h_n < \sqrt{\frac{2}{n-1}}$$

Thus for $n \geq 2$, we have

$$0 \leq h_n < \sqrt{\frac{2}{n-1}}$$

Notice that the limit of the sequence on the right exists and is equal to 0. Thus, utilising Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$. Recalling how we defined h_n, we get $\lim_{n\to\infty} n^{1/n} = 1$. $\square$

3 (ii) Prove that the sequence $a_n := \left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is not convergent.

Solution. We will prove this result by contradiction. First, observe that the sequence $b_n := \frac{(-1)^n}{n}$ is convergent and its limit is 0. This is true because its absolute value behaves the same way as $\frac{1}{n}$ (try proving this with the ϵ-N definition to work out the details). We also know that the sequence $\{(-1)^n\}_{n\geq 1}$ is not convergent. (Why?) Now, let us assume that the given sequence (a_n) converges. We have

$$a_n := \left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\} = \frac{(-1)^n}{2} - \frac{(-1)^n}{n}$$

We also know that the the sum of two convergent sequences is convergent. Since a_n is assumed to be convergent and b_n is convergent, we have that $c_n := a_n + b_n = \frac{(-1)^n}{2}$ must also converge. However, the convergence of c_n implies that the sequence $(-1)^n$ also converges. Hence, we arrive at a contradiction and thus, the sequence (a_n) is not convergent.

$\square$

5 (iii) Prove that the following sequence is convergent by showing that it is monotone and bounded. Also find its limit.

$$a_1 = 2, a_{n+1} = 3 + \frac{a_n}{2} \; \forall n \in \mathbb{N}$$

Solution. We first claim that $a_n < 6$ for all $n \in \mathbb{N}$. To prove this, we will use mathematical induction. The base case, $n = 1$ is immediate as $2 < 6$. Assume that the claim holds for some $n = k$. Now,

$$a_{k+1} = 3 + \frac{a_k}{2} < 3 + \frac{6}{2} = 6$$

By induction, the claim follows. Hence, a_n is bounded above.

Next, we claim that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. We have

$$a_{n+1} - a_n = 3 - \frac{a_n}{2} = \frac{6 - a_n}{2}$$

We just showed that $a_n < 6$ for all $n \in \mathbb{N}$. It thus follows that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Hence, (a_n) is a monotonically increasing sequence that is bounded above. Thus, it must converge. To find the limit of (a_n), we utilise the fact that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ (Sheet 1 : Problem 6). Let L denote the limit of (a_n). Taking the limit of the recursive definition (and using some limit properties), we have that

$$L = 3 + \frac{L}{2} \implies L = 6$$

Thus, the sequence (a_n) converges to 6. (Notice that this was the upper bound we chose for (a_n)) $\square$

7 If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0$$

Solution. We will use the $\epsilon - N$ definition to prove this result. Choose $\epsilon = \dfrac{|L|}{2}$. Since $L \neq 0$, we have $\epsilon > 0$. Now, as $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$. From triangle inequality, we have

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \implies -\epsilon < |a_n| - |L| \quad \forall n \geq n_0$$

Substituting the value of ϵ, we get that

$$|a_n| > \frac{|L|}{2}$$

for all $n \geq n_0$, as desired. □

9 For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following statements:

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

Solution. This is a relatively short question. Both the statements are **false**. Verify that $a_n := 1$ and $b_n := (-1)^n$ acts as a counterexample for both the statements. □

11 Let $f, g\colon (a,b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a,b]$. Prove or disprove the following statements.

(i) $\lim_{x\to c} [f(x)g(x)] = 0$.

(ii) $\lim_{x\to c} [f(x)g(x)] = 0$ if g is bounded.

(iii) $\lim_{x\to c} [f(x)g(x)] = 0$ if $\lim_{x\to c} g(x)$ exists.

Solution. (i) This statement is **false**. As a counterexample, define $a = -1, b = 1$ and $c = 0$. Define $f, g\colon (-1,1) \to \mathbb{R}$ as

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0 \end{cases}$$

Clearly, $\lim_{x\to 0} f(x) = 0$. However, $\lim_{x\to 0} [f(x)g(x)]$ does not exist.

(ii) This statement is **true**. Since g is bounded, there exists $M > 0$ such that

$$|g(x)| \leq M$$

for all $x \in (a,b)$. Thus, we have

$$0 \leq |f(x)g(x)| \leq M|f(x)|$$

for all $x \in (a,b)$. Using Sandwich Theorem, we see that

$$\lim_{x\to c} |f(x)g(x)| = 0$$

which in turn implies that

$$\lim_{x\to c} [f(x)g(x)] = 0$$

(iii) This statement is **true**. Since $\lim_{x\to c} g(x)$ exists, we have $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 0$.

□

§2. Week 2

2nd December, 2020

Sheet 1.

13 (ii) Discuss the continuity of the following function :

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. At all points other than $x = 0$, the given function is trivially continuous (since it is the product and composition of continuous functions). All that remains is to check the continuity of f at the point $x = 0$. Note that

$$|f(x)| = \left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

for all $x \neq 0$. Thus, we have

$$0 \leq |f(x)| \leq |x|$$

Utilising Sandwich Theorem, we see that

$$\lim_{x \to 0} f(x) = 0$$

Since $f(0)$ is given to be 0, we see that $\lim_{x \to 0} f(x) = f(0)$, proving continuity of f at $x = 0$. Thus, f is continuous everywhere. □

15 Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined as follows.

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

Solution. Clearly, f is differentiable for all $x \neq 0$. Using the chain rule and product rule, we compute f' as

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

for $x \neq 0$. Now, all that remains to be checked is the differentiability of f at $x = 0$. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \; h \sin\left(\frac{1}{h}\right)$$

From the previous question, this limit exists and is equal to 0. Thus, f is differentiable on all of $\mathbb{R}$ and its derivative is defined as

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x}\right) - \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly, f' is continuous at all $x \neq 0$. All that remains is to check continuity of f' at $x = 0$. It turns out that f' is in fact *not* continuous at $x = 0$. We will use the sequential criterion of continuity to prove this. Consider the sequence:

$$x_n := \frac{1}{2n\pi}, \quad n \in \mathbb{N}$$

Clearly, $x_n \to 0$ as $n \to \infty$. However,

$$f'(x_n) = \cancel{\frac{2}{2n\pi} \cdot \sin(2n\pi)} - \cos(2n\pi) = -1$$

We see that $\lim\limits_{n \to \infty} f(x_n)$ is -1, which is not equal to $f'(0)$. Hence, f' is not continuous at $x = 0$. This is an example of a differentiable function whose derivative is not continuous. $\square$

18 Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x) \cdot f(y) \text{ for all } x, y \in \mathbb{R}$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$.

Solution. We have that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$. On substituting $x = y = 0$, we obtain

$$f(0) = f(0) \cdot f(0) \implies f(0) = 0 \text{ or } 1$$

First, we consider the case that $f(0) = 0$. We have

$$f(x) = f(x+0) = f(x) \cdot f(0) \implies f(x) = 0$$

for all x. Thus, $f \equiv 0$ is trivially differentiable and $f'(c) = 0 = f'(0) \cdot f(c)$ for all $c \in \mathbb{R}$.

Now consider that $f(0) = 1$. For all $c \in \mathbb{R}$, we have

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{f(c)f(h) - f(c)f(0)}{h} = f(c) \cdot \left(\lim_{h\to 0} \frac{f(h) - f(0)}{h} \right)$$

If f is differentiable at 0, then the above limit exists. Thus, if f is differentiable at 0, then it is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0) \cdot f(c)$. $\square$

Optional Exercises.

7 Let $f\colon (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following statements are equivalent.

(i) f is differentiable at c.

(ii) There exists $\delta > 0$, $\alpha \in \mathbb{R}$ and a function $\epsilon_1\colon (-\delta,\delta) \to \mathbb{R}$ such that $\lim\limits_{h\to 0} \epsilon_1(h) = 0$ and

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$$

for all $h \in (-\delta,\delta)$.

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0$$

Solution. To show the equivalence of statements (i)-(iii), we must show that every statement implies every other statement, that is, a total of 6 implications. However, we can get away with just showing three implications. We will show that $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$. This is sufficient to conclude the equivalence of the three statements. (Why?)

$(i) \Rightarrow (ii)$: Since we are given that f is differentiable at c, $f'(c)$ exists. We first pick $\delta := \min\{c-a, b-c\}$. Clearly $\delta > 0$ and $(c-\delta, c+\delta) \subset (a,b)$. Now, since f is differentiable at c, $f'(c)$ exists. Define $\alpha := f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

Since $(c-\delta, c+\delta) \subset (a,b)$, $f(c+h)$ is well defined for all $h \in (-\delta,\delta)$. Now,

$$\lim_{h\to 0} \epsilon_1(h) = \underbrace{\left(\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \right)}_{\alpha} - \alpha = 0$$

Further, some simple algebraic manipulation yields that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for $h \in (-\delta,\delta), h \neq 0$. Verify that this equation also holds for $h = 0$. It then follows that $f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$ for all $h \in (-\delta,\delta)$ and $\lim\limits_{h\to 0} \epsilon_1(h) = 0$, as desired.

$(ii) \Rightarrow (iii)$: By (ii), we have the existence of $\delta > 0, \alpha \in \mathbb{R}$ and the function ϵ_1. We have

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \lim_{h \to 0} |\epsilon_1(h)| = 0$$

$(iii) \Rightarrow (i)$: By (iii), we have the existence of some $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0$$

Now,

$$\lim_{h \to 0} \left| \frac{f(c+h) - f(c)}{h} - \alpha \right| = 0 \implies \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Thus, f is differentiable at c, as desired.

Since we have shown $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$, we get that the three statements are thus equivalent. □

10 Show that any continuous function $f \colon [0,1] \to [0,1]$ has a fixed point. x is said to be a fixed point of f if $f(x) = x$

Solution. Consider the function $g(x) = f(x) - x$. A fixed point of f is then a root of g. Note that g is continuous. Since $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, we have

$$g(0) = f(0) \implies g(0) \geq 0$$

and

$$g(1) = f(1) - 1 \implies g(1) \leq 0$$

First consider the case that at least one of the two equalities hold. That is, either $g(0) = 0$ or $g(1) = 0$ or both. In either of the three cases, we have at least one fixed point (0 or 1 or both, respectively). Now, consider that $g(0) > 0$ and $g(1) < 0$. Since g is continuous, we can appeal to Intermediate Value Theorem. By IVT, there exists some $x_0 \in (0,1)$ such that $g(x_0) = 0$. This point x_0 is also a fixed point of f. Thus, we have shown that any continuous function mapping the unit interval to itself has a fixed point, as desired. □

Sheet 2.

3 Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a, b)$, then show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.

Solution. Since $f(a)$ and $f(b)$ are of opposite signs and f is continuous, we know that there exists **at least** one $x_0 \in (a, b)$ such that $f(x_0) = 0$ (by IVP). Now, assume that there was some $y_0 (\neq x_0)$ in (a, b) such that $f(y_0) = 0$. We now have $f(x_0) = f(y_0)$. By Rolle's Theorem, there must exist some $c \in (x_0, y_0)$ such that $f'(c) = 0$. Since this c also lies in (a, b), we arrive at a contradiction. Hence, there is a unique x_0 in (a, b) such that $f(x_0) = 0$, as desired. □

5 Use the MVT to show that $|\sin(a) - \sin(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Solution. We will break this problem into two cases. First, consider $a = b$. The inequality is trivially satisfied in this case. Next, consider $a \neq b$. Define $f(x) = \sin(x)$. By MVT, there exists some c between a and b such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Since $f' = \cos$, we take modulus on both sides to obtain

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1$$

Rearranging, we get

$$|\sin a - \sin b| \leq |a - b|$$

for all $a, b \in \mathbb{R}$, as desired. □

§3. Week 3

9th December, 2020

Sheet 2.

8 In each case, find a function f that satisfies all the given conditions, or else show that no such function exists.

(ii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$.

Solution.

(ii) Possible. Verify that $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) \coloneqq x + \dfrac{x^2}{2}$ is one such function.

(iii) Not possible. Assume that it was indeed possible to find such a function f. Then, we are given that f'' exists everywhere. Thus, f' is continuous and differentiable everywhere. As f'' is non-negative, f' must be increasing everywhere. Since $f'(0) = 1$, we have that $f'(c) \geq 1$ for all $c > 0$.

Let $x \in (0, \infty)$. By MVT, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

Since $c > 0$, we have $f'(c) \geq 1$ as shown above. Thus, $f(x) - f(0) \geq x$ for all $x > 0$. However, consider $x_0 \coloneqq \max(101 - f(0), 1)$. Clearly, $x_0 > 0$ (as it is ≥ 1). Also, $f(x_0) > 100$, which contradicts the condition that $f(x) \leq 100$ for all $x > 0$. Hence, no such f can exist.

$\square$

10 (i) Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local minima/maxima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

$$y = 2x^3 + 2x^2 - 2x - 1$$

Solution. We are given

$$f(x) = 2x^3 + 2x^2 - 2x - 1$$

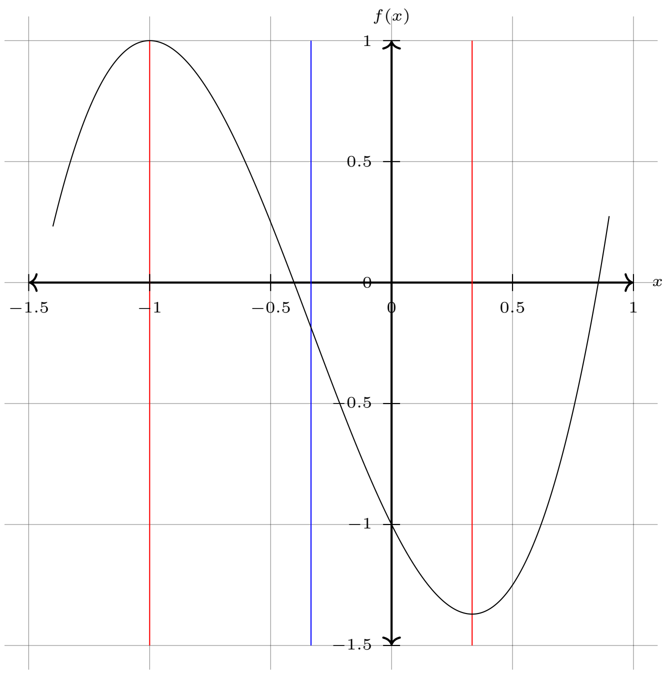
On differentiating, we get

$$f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$$

Thus, $f' > 0$ in $(-\infty, -1) \cup (\frac{1}{3}, \infty)$ and f is strictly increasing here. $f' < 0$ in $(-1, \frac{1}{3})$ and f is strictly decreasing here. Thus, f has a local maximum at -1 and a local minimum at $\frac{1}{3}$. Differentiating again, we see that

$$f''(x) = 12x + 4$$

Thus, f is convex in $(-\frac{1}{3}, \infty)$ and concave in $(-\infty, -\frac{1}{3})$, with a point of inflection at $-\frac{1}{3}$. A curve for f can be sketched as follows



□

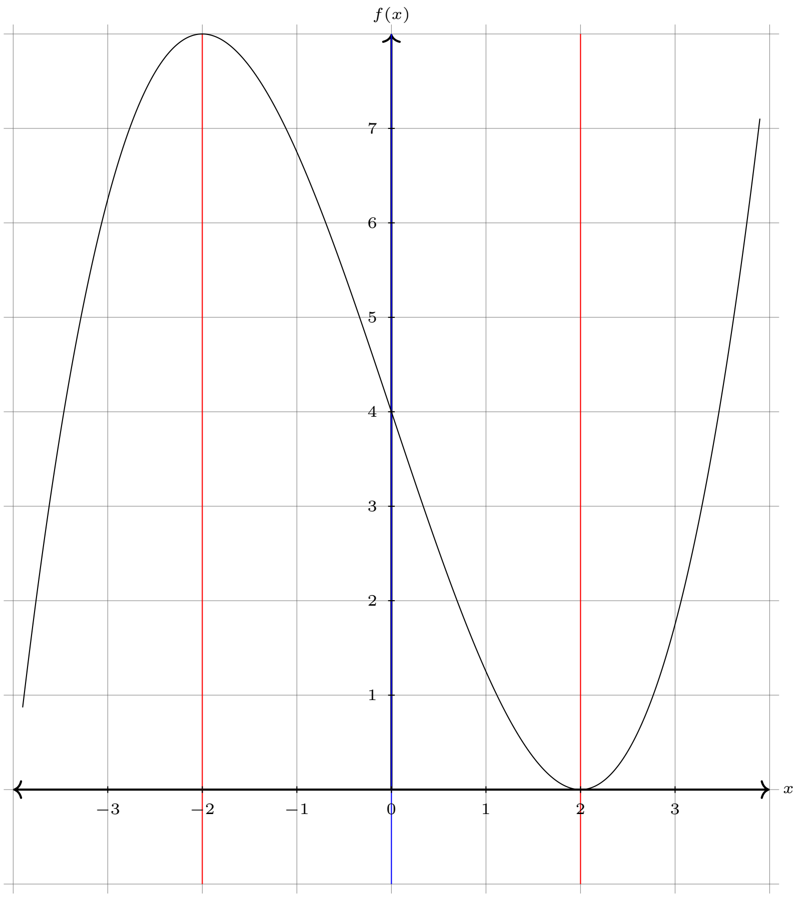
11 Sketch a continuous function having all the following properties :

$$f(-2) = 8, f(0) = 4, f(2) = 0; f'(-2) = f'(2) = 0;$$

$$f'(x) > 0 \text{ for } |x| > 2,\ f'(x) < 0 \text{ for } |x| < 2;$$

$$f''(x) < 0 \text{ for } x < 0,\ f''(x) > 0 \text{ for } x > 0.$$

Solution. $f' > 0$ in $(-\infty, -2) \cup (2, \infty)$ and thus f is strictly increasing here. $f' < 0$ in $(-2, 2)$ and thus f is strictly decreasing here. Thus, f has a local maximum at -2 and a local minimum at -2. The function values at these points are 8 and 0 respectively. Also, f is convex in $(0, \infty)$ and concave in $(-\infty, 0)$ with an inflection point at 0. Putting all these together, we can sketch a curve for f as:



□

Sheet 3.

1 (ii) Write down the Taylor expansion of $\arctan(x)$ around the point 0. Also write a precise remainder term $R_n(x)$.

Solution. Let f denote the arctangent function. Let g denote its derivative

$$g(x) = f'(x) = \frac{1}{1+x^2}$$

For $|x| < 1$, we can expand the latter as a geometric series. Thus, we have

$$g(x) = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

for $|x| < 1$. Let us now evaluate the n^{th} derivative of f at $x = 0$. For $n \geq 1$, we have

$$f^{(n)} = g^{(n-1)}$$

where $f^{(r)}$ and $g^{(r)}$ denote the r^{th} derivatives of f and g respectively. To evaluate the derivatives of g, we will consider two cases. First, we will evaluate all odd derivatives (derivatives of the order $2n-1$). On differentiating g, r times, we will be left with a power series where the powers of x are of the form $(2k-r)$ for integer k. When r is odd, no exponent of x vanishes. As a result, all the terms of the power series vanish when we plug in $x = 0$. Thus, all odd derivatives of g vanish at 0. I leave it to you to compute the even order derivatives at $x = 0$. The derivatives of g at 0 are then given by

$$g^{(2n-1)}(0) = 0, \quad g^{(2n)}(0) = (-1)^n \cdot (2n)!$$

for $n \geq 1$. Now, we have

$$f^{2n}(0) = g^{(2n-1)}(0) = 0$$

and

$$f^{(2n-1)}(0) = g^{(2n-2)}(0) = (-1)^{n-1} \cdot (2n-2)!$$

for $n \geq 1$. We shall first compute the zeroth Taylor Polynomial. We have

$$T_0(x) = f(0) = 0$$

Let us now compute the n^{th} Taylor polynomial $T_n(x)$ of f at 0 for $n \geq 1$. Define $M := \left\lfloor \left(\frac{n+1}{2}\right) \right\rfloor$. For $n \geq 1$, we then have

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

where $f^{(0)} = f$. With a bit of manipulation, we can write

$$T_n(x) = \sum_{k=1}^{M} \frac{(-1)^{k-1} \cdot (2k-2)!}{(2k-1)!} \cdot x^{2k-1}$$

Thus, the n^{th} Taylor polynomial for arctan at 0 is given by

$$T_n(x) = \sum_{k=1}^{M} \frac{(-1)^k}{2k-1} x^{2k-1} \quad , \quad M = \left\lfloor \left(\frac{n+1}{2} \right) \right\rfloor$$

Writing it out in a neater way, we have

$$T_{2n-1}(x) = x - \frac{x^3}{3} + \ldots + \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$$

and

$$T_{2n}(x) = T_{2n-1}(x)$$

The remainder term is then just the difference of the arctangent function at x and its Taylor polynomial. More precisely, we have

$$R_n(x) = \arctan(x) - \sum_{k=0}^{M} \frac{(-1)^k}{2k-1} x^{2k-1}$$

with M defined as previously. Let us now calculate the remainder term $R_{2n-1}(x)$ more explicitly. We have

$$\arctan' = 1 - x^2 + x^4 + \ldots + (-1)^{n-1} x^{2n-2} + (-1)^n x^{2n} \left[1 - x^2 + x^4 - \ldots \right]$$

$$\therefore \arctan' = 1 - x^2 + x^4 + (-1)^{n-1} x^{2n-2} + (-1)^n \frac{x^{2n}}{1+x^2}$$

On integrating both sides from 0 to x, the cyan-coloured term just becomes $T_{2n-1}(x)$. (Verify!) Thus, we have

$$\arctan(x) = T_{2n-1}(x) + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} \, \mathrm{d}t$$

Thus,

$$R_{2n-1}(x) = (-1)^n \int_0^x \frac{t^{2n}}{1+t^2} \, \mathrm{d}t$$

and

$$R_{2n}(x) = R_{2n-1}(x)$$

$\square$

2 Write down the Taylor series of the polynomial $x^3 - 3x^2 + 3x - 1$ about the point 1.

Solution. The Taylor series is just $(x-1)^3$. Let us see why. We wish to expand

$$f(x) = x^3 - 3x^2 + 3x - 1$$

about the point $a = 1$. We have

$$f(1) = 0$$

$$\begin{aligned} f^{(1)}(1) &= 0 \\ f^{(2)}(1) &= 0 \\ f^{(3)}(1) &= 6 \\ f^{(n)}(1) &= 0 \ \text{ for all } n \geq 4 \end{aligned}$$

Thus, we have

$$\begin{aligned} P_0(x) = P_1(x) &= P_2(x) = 0 \\ P_3(x) &= \frac{6}{3!}(x-1)^3 = (x-1)^3 \end{aligned}$$

and

$$P_n(x) = P_3(x) \ \text{ for all } n \geq 4$$

We also have

$$R_n(x) := f(x) - P_n(x) = 0 \ \text{ for all } n \geq 3$$

Thus, $R_n(x) \to 0$ for **all** x. Thus, the Taylor series of the function about the point 1 is simply given by $(x-1)^3$. $\square$

4 Consider the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for a fixed x. Prove that it converges as follows. Choose $N > 2|x|$. We see that for $n > N$,

$$\left| \frac{x^{n+1}}{(n+1)!} \right| < \frac{1}{2} \cdot \left| \frac{x^n}{n!} \right|$$

It should now be relatively easy to show that the given series is Cauchy, and hence (by the completeness of $\mathbb{R}$) is convergent.

Solution. Let the partial sums of the series be denoted as $S_m(x)$. That is,

$$S_m(x) := \sum_{k=0}^{m} \frac{x^k}{k!}$$

We wish to show that the difference $|S_m(x) - S_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Assume that $m > n > N$. We see that

$$|S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{m-n}} \right) \leq \left| \frac{x^n}{n!} \right| < \left| \frac{x^N}{N!} \right|$$

Now for any $\epsilon > 0$, we can pick N large enough such that

$$\left| \frac{x^N}{N!} \right| < \epsilon$$

This is possible because the sequence

$$a_n = \frac{|x|^n}{n!}$$

is convergent (it is eventually decreasing and bounded below) and its limit is 0. Thus, for all $m > n > N$, we have

$$|S_m(x) - S_n(x)| < \epsilon$$

Hence, the given series is Cauchy and thus convergent.

(Remark: During the tutorial session, I had showed that the term $|S_m(x) - S_n(x)|$ can be made arbitrarily small by picking n large enough. However, this is incorrect! We want to show that the term is smaller than ϵ for any n, m greater than N. So really we have to make N large enough and conclude. This is what I have now done.) $\square$

5 Using Taylor series, write down a series for the integral

$$\int \frac{e^x}{x} \, \mathrm{d}x$$

Solution. We will assume that a Taylor series can be integrated term by term and then proceed. Recall that the Taylor series for e^x is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We have

$$\begin{aligned} \int \frac{e^x}{x} \, \mathrm{d}x &= \int \left(\frac{1}{x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) \mathrm{d}x \\ &= \int \frac{1}{x} \, \mathrm{d}x + \int \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right) \mathrm{d}x \end{aligned}$$

Since the latter term is a Taylor series, we can integrate it term by term to obtain

$$\int \frac{e^x}{x} \, \mathrm{d}x = \log x + \sum_{k=1}^{\infty} \left(\int \frac{x^{k-1}}{k!} \, \mathrm{d}x \right)$$

Thus, a series representation of the integral is given by

$$\int \frac{e^x}{x} \, \mathrm{d}x = \log x + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}$$

$\square$

§4. Week 4

16th December, 2020

Sheet 4.

2 (a) Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)\,\mathrm{d}x = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

Solution. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ denote a partition of $[a,b]$. Define $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Further, we define

$$m_i = \inf\{f(x) \colon x_{i-1} \leq x \leq x_i\}$$

Since $f(x) \geq 0$ for all $x \in [a,b]$, it follows that $m_i \geq 0$ for all i. The lower sum is now defined as

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Since $m_i \geq 0$ and $\Delta x_i > 0$ for all i, it follows that $L(P,f) \geq 0$ for any partition P. Thus, we also see that $L(f) \geq 0$ since $L(f)$ is the supremum of $L(P,f)$ over all partitions P. Since f is Riemann integrable, we have

$$\int_a^b f(x)\,\mathrm{d}x = L(f) \geq 0$$

as desired.

Now, let us further assume that f is continuous and that $\int_a^b f(x)\,\mathrm{d}x = 0$. If f is not identically zero, then there exists $c \in [a,b]$ such that $f(c) > 0$. Continuity of f implies that there exists a $\delta > 0$ such that, if $x \in [a,b]$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{f(c)}{2} \implies f(x) > \frac{f(c)}{2}$$

We may now assume $c \in (a,b)$ without any loss of generality [1] Further, pick $\delta > 0$ small enough so that $(c-\delta, c+\delta) \subset (a,b)$. Now, consider the partition

$$P = \left\{a, c - \frac{\delta}{2}, c + \frac{\delta}{2}, b\right\}$$

[1] If $c = a$ or $c = b$, then we can pick another point $\tilde{c}$ in (a,b) such that $f(\tilde{c}) \neq 0$.

Since we have

$$\inf_{x \in [c-\frac{\delta}{2}, c+\frac{\delta}{2}]} f(x) \geq \frac{f(c)}{2}$$

it follows that

$$L(f) \geq L(P, f) \geq \frac{f(c)\delta}{2} > 0$$

Further, if f is Riemann integrable, we have that its integral over $[a, b]$ is equal to $L(f)$, which is strictly positive - a contradiction! Hence, f must be identically zero. □

Alternate. (easier)

Solution. Consider the trivial partition $P_0 = a, b$ of $[a, b]$. Since $f(x) \geq 0$ for all $x \in [a, b]$, we have

$$\inf_{x \in [a,b]} f(x) \geq 0$$

We have

$$L(f, P_0) = \left[\inf_{x \in [a,b]} f(x)\right] \cdot (b - a) \geq 0$$

and

$$L(f) \geq L(f, P_0) \geq 0$$

Since f is Riemann integrable, its integral is $L(f)$, which is non-negative, as desired.

For the second part, define $F\colon [a, b] \to \mathbb{R}$ as

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

Since f is continuous, we get that F is differentiable with $F' = f$, from the Fundamental Theorem of Calculus (Part 1). Since $f \geq 0$, we have $F' \geq 0$ and hence, F is increasing. This implies that for all $x \in [a, b]$, we have

$$F(a) \leq F(x) \leq F(b)$$

However, since $F(a) = 0 = F(b)$, we get that F is constant and hence,

$$f(x) = F'(x) = 0$$

for all $x \in [a, b]$, as desired. □

2 (b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\, \mathrm{d}x = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

Solution. As we saw in the previous question, no continuous function can satisfy these conditions. Thus, we must look for a discontinuous function. We define f on $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{when } x \neq \frac{1}{2} \\ 1 & \text{when } x = \frac{1}{2} \end{cases}$$

Since f has only finitely many discontinuities, it is Riemann integrable. Also, $f(x) \geq 0$ for all $x \in [0, 1]$. Further, it is easy to show that its Riemann integral over the interval is 0. Lastly, we have $f(\frac{1}{2}) = 1 \neq 0$. Thus, $f(x) \neq 0$ for some $x \in [0, 1]$. Hence, this f satisfies our desired conditions. $\square$

3 Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an ~~approximate~~ appropriate Riemann sum of a suitable function over a suitable interval.

(ii) $S_n = \sum\limits_{i=1}^{n} \dfrac{n}{i^2+n^2}$ (iv) $S_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} \cos\dfrac{i\pi}{n}$

We shall use the following theorem for both the parts.

> **Theorem**
>
> Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Suppose that (P_n, T_n) be a sequence of tagged partitions of $[a,b]$ such that $\|P_n\| \to 0$. Then,
>
> $$R(P_n, T_n, f) \to \int_a^b f(t)\,\mathrm{d}t$$

(ii) *Solution.* Consider $f\colon [0,1] \to \mathbb{R}$ defined as $f(x) \coloneqq \arctan(x)$. Then, we have

$$f'(x) = \frac{1}{1+x^2}$$

Since f' is continuous on $[0,1]$, it is Riemann integrable on $[0,1]$. Let $P_n \coloneqq \left\{x_i = \frac{i}{n} \colon 0 \le i \le n\right\}$ be a tagged partition of $[0,1]$ for $n \in \mathbb{N}$ and let $T_n \coloneqq \left\{t_i = \frac{i}{n} \colon 1 \le i \le n\right\}$ denote the tags of the partition.

We have $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$ for all $1 \le i \le n$. The Riemann sum corresponding to this tagged partition is given by

$$\begin{aligned} R(P_n, T_n, f') = \sum_{i=1}^{n} f'(t_i)\Delta x_i &= \sum_{i=1}^{n} \frac{1}{1+t_i^2}\cdot\frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{n}{i^2+n^2} = S_n \end{aligned}$$

Thus, $R(P_n, T_n, f') = S_n$ for all $n \ge 1$. Moreover,

$$\|P_n\| = \max\{x_i - x_{i-1} \colon 1 \le i \le n\} = \frac{1}{n}$$

Clearly, we have

$$\lim_{n\to\infty} \|P_n\| = 0$$

and thus,

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x$$

From the Fundamental Theorem of Calculus (Part 2), we have that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0) = \boxed{\frac{\pi}{4}}$$

$\square$

(iv) *Solution.* Consider $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) \coloneqq \frac{1}{\pi}\sin(\pi x)$$

We then have $f'(x) = \cos(\pi x)$. Since f' is continuous on $[0,1]$, it is Riemann integrable on $[0,1]$. Let $P_n \coloneqq \left\{x_i = \frac{i}{n} \colon 0 \le i \le n\right\}$ be a tagged partition of $[0,1]$ for $n \in \mathbb{N}$ and let $T_n \coloneqq \left\{t_i = \frac{i}{n} \colon 1 \le i \le n\right\}$ denote the tags of the partition.

We have $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$ for all $1 \le i \le n$. The Riemann sum corresponding to this tagged partition is given by

$$\begin{aligned} R(P_n, T_n, f') = \sum_{i=1}^{n} f'(t_i)\Delta x_i &= \sum_{i=1}^{n} \cos(\pi t_i)\cdot\frac{1}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n} \cos\frac{i\pi}{n} = S_n \end{aligned}$$

Thus, $R(P_n, T_n, f') = S_n$ for all $n \ge 1$. Moreover,

$$\|P_n\| = \max\{x_i - x_{i-1} \colon 1 \le i \le n\} = \frac{1}{n}$$

Clearly, we have

$$\lim_{n\to\infty} \|P_n\| = 0$$

and thus,

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x$$

From the Fundamental Theorem of Calculus (Part 2), we have that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0) = \boxed{0}$$

$\square$

4(b) Compute $\dfrac{\mathrm{d}F}{\mathrm{d}x}$ if for $x \in \mathbb{R}$,

(i) $F(x) = \displaystyle\int_1^{2x} \cos\left(t^2\right)\, \mathrm{d}t$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)\, \mathrm{d}t$

Solution. Before solving these two subparts, I will first prove a short lemma.

> **Lemma**
>
> Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $v\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined as
>
> $$F(x) := \int_0^{v(x)} f(t)\, \mathrm{d}t$$
>
> Then,
>
> $$F'(x) = f\left(v(x)\right) \cdot v'(x)$$

Proof. First, we define $G\colon \mathbb{R} \to \mathbb{R}$ as

$$G(x) := \int_0^{x} f(t)\, \mathrm{d}t$$

Then, $G' = f$ by the Fundamental Theorem of Calculus (Part 1). Now,

$$F(x) = G\left(v(x)\right)$$

A simple application of chain rule yields

$$F'(x) = f\left(v(x)\right) \cdot v'(x)$$

as desired. □

(i) We have $v(x) = 2x$ and $f(t) = \cos(t^2)$. It thus follows from the above lemma that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \cos\left((2x)^2\right) \cdot (2x)' = \boxed{2\cos\left(4x^2\right)}$$

(ii) We have $v(x) = x^2$ and $f(t) = \cos(t)$. It thus follows from the above lemma that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \cos\left(x^2\right) \cdot \left(x^2\right)' = \boxed{2x\cos\left(x^2\right)}$$

□

6 Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}, \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \, \mathrm{d}t$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = g'(0) = 0$.

Solution. We will first make use of the identity $\sin(A-B) = \sin A \cos B - \cos A \sin B$. We have

$$\begin{aligned} g(x) &= \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) \, \mathrm{d}t \\ &= \frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) \, \mathrm{d}t \\ &= \frac{1}{\lambda} \sin \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t - \frac{1}{\lambda} \cos \lambda t \int_0^x f(t) \sin \lambda t \, \mathrm{d}t \end{aligned}$$

On applying the product rule and Fundamental Theorem of Calculus (Part 1), we get

$$\begin{aligned} g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + \cancel{\frac{1}{\lambda} \sin \lambda x \cdot f(x)} \cdot \cos \lambda x \\ + \sin \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t - \cancel{\frac{1}{\lambda} \sin \lambda x \cdot f(x)} \cdot \cos \lambda x \end{aligned}$$

$$\therefore \; g'(x) = \cos \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + \sin \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t$$

It is now easy to verify that both $g(0)$ and $g'(0)$ are indeed 0. We will differentiate g' in a similar manner to obtain

$$\begin{aligned} g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t \, \mathrm{d}t + f(x) \cos^2 \lambda x \\ &\quad + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t \, \mathrm{d}t + f(x) \sin^2 \lambda x \\ &= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) \, \mathrm{d}t \right) \\ &= f(x) - \lambda^2 g(x) \end{aligned}$$

It thus follows that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. □