MA109 Additional Questions

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December 18, 2020

(Remark : The stuff in blue is only meant to give you intuition and feel)

Tutorial Sheet 2

1. Show that all the roots of the cubic polynomial $x^3 - 6x + 3$ are real.

   Solution. Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be given as $f(x) = x^3 - 6x + 3$. Consider the four points $-3, -1, 1$ and 3. We have

   $$\begin{aligned} f(-3) &= -6 \\ f(-1) &= 8 \\ f(1) &= -2 \\ f(3) &= 12 \end{aligned}$$

   Also, f is continuous at it is a polynomial. By IVT, there is a root of f in each of the intervals $(-3, -1)$, $(-1, 1)$ and $(1, 3)$. Since f has at most three roots, we get that all three of its roots are real. □

   How did I guess these points? A good place to start would be the stationary points of f (points where f' is zero). These are $\pm\sqrt{2}$. Observe that f goes to $+\infty$ as x goes to $+\infty$ and f goes to $-\infty$ as x goes to $-\infty$. Also observe that $f(\sqrt{2}) < 0$ and $f(-\sqrt{2}) > 0$. This gives you three roots in the three intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \infty)$. To make calculations easier and nicer for me, I chose the four points that I mentioned above.

12 Give an example of a function $f\colon (0, 1) \mapsto \mathbb{R}$ that is

   (i) strictly increasing and convex.

   Solution. Consider $f\colon (0, 1) \mapsto \mathbb{R}$ defined as $f(x) = x^2$. Since f is a polynomial, it is twice differentiable. Further, we have $f'(x) = 2x$. Since $f'(x) > 0$ for all $x \in (0, 1)$, f is strictly increasing. We also have $f''(x) = 2$. Thus, $f''(x) > 0$ for all $x \in (0, 1)$ and hence f is also strictly convex. □

   (ii) strictly increasing and concave.

   Solution. Consider $f\colon (0, 1) \mapsto \mathbb{R}$ defined as $f(x) = \sqrt{x}$. f is twice differentiable on $(0, 1)$ and we have $f'(x) = \frac{1}{2\sqrt{x}}$. Thus, $f'(x) > 0$ for all $x \in (0, 1)$ and thus f is strictly increasing. We also have $f''(x) = -\frac{1}{4x^{3/2}}$. Thus, $f''(x) < 0$ for all $x \in (0, 1)$ and f is thus also strictly convex. □

(iii) strictly decreasing and convex.

Solution. Consider $f\colon (0,1) \mapsto \mathbb{R}$ defined as $f(x) = \frac{1}{x}$. f is twice differentiable on $(0,1)$. We have $f'(x) = -\frac{1}{x^2}$. Thus, $f'(x) < 0$ for all $x \in (0,1)$ and hence, f is strictly decreasing. We also have $f''(x) = \frac{2}{x^3}$. Thus, $f''(x) > 0$ for all $x \in (0,1)$ and hence, f is strictly convex. □

(iv) strictly decreasing and concave.

Solution. Consider $f\colon (0,1) \mapsto \mathbb{R}$ defined as $f(x) = -x^2$. f is twice differentiable since it's a polynomial. We have $f'(x) = -2x$. Thus, $f'(x) < 0$ for all $x \in (0,1)$ and hence, f is strictly decreasing. We also have $f''(x) = -2$. Thus, $f''(x) < 0$ for all $x \in (0,1)$ and hence f is also strictly concave. □

Interesting thing to note (and may be useful) : if f is increasing, then $-f$ is decreasing. If f is convex, then $-f$ is concave, Notice how I used these two facts to construct the last example from the first. In fact, you just need to solve part (i) and (ii). For part (iii) and (iv), just use the negative of those functions. So, $-\sqrt{x}$ works for (iii) as well.

Another interesting thing / exercise : the four answers can also just be given by $\pm e^{\pm x}$. Can you tell me which one is the solution to which part? (Hint : you again just need to solve two!)