MA 205: Complex Analysis
Tutorial Solutions

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Contents

Note: Many of these solutions are either inspired by, or in some cases directly taken from Aryaman Maithani's tutorial solutions for last year's offering of this course.

§1. Week 1

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Notation: We use $\mathbb{C}[x]$ to denote the set of all polynomials in x with complex coefficients. $\mathbb{R}[x]$ is defined similarly.

1. Show that a real polynomial that is irreducible has degree at most two, i.e, if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$. ($a_n \neq 0$, of course)

Solution. We consider two cases. First, suppose $f(x) \in \mathbb{R}[x]$ has a real root, x_0, and let $h(x) := (x - x_0)$. Since $x_0 \in \mathbb{R}$, $h(x) \in \mathbb{R}[x]$. Moreover, we can write

$$f(x) = g(x)h(x)$$

for some $g(x) \in \mathbb{R}[x]$. (Why must g be a real polynomial?) Also, since $\deg f(x) \geq 3$ and $\deg h(x) = 1$, we have that $\deg g(x) \geq 2$. Thus, g and h are two non-constant real polynomials satisfying $f(x) = g(x)h(x)$.

Now, suppose that $f(x)$ has no real root. We may also view $f(x)$ as a polynomial in $\mathbb{C}[x]$. By FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we have that $x_0 \notin \mathbb{R}$, and thus $x_0 \neq \overline{x_0}$.

Claim. $f(\overline{x_0}) = 0$.

Proof. We have

$$\begin{aligned} f(\overline{x_0}) &= a_0 + a_1\overline{x_0} + \cdots + a_n(\overline{x_0})^n & &\overline{z^n} = \bar{z}^n \\ &= a_0 + a_1\overline{x_0} + \cdots + a_n\overline{x_0^n} & &a_i \in \mathbb{R} \text{ and thus, } a_i = \overline{a_i} \\ &= \overline{a_0} + \overline{a_1}\ \overline{x_0} + \cdots + \overline{a_n}\overline{x_0^n} & &\overline{z_1 z_2 + z_3} = \overline{z_1}\ \overline{z_2} + \overline{z_3} \\ &= \overline{f(x_0)} \\ &= \bar{0} \\ &= 0. \end{aligned}$$

□

Thus, x_0 and $\overline{x_0}$ are two distinct roots of $f(x)$. Define $g(x) := (x - x_0)(x - \overline{x_0})$. A priori, we have $g(x) \in \mathbb{C}[x]$. However, note that

$$(x - x_0)(x - \overline{x_0}) = x^2 - (2\Re x_0)x + |x_0|^2 \in \mathbb{R}[x].$$

Thus, $g(x)$ is in fact a real polynomial. Since x_0 and $\overline{x_0}$ are distinct, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Why?) Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. Again, since $f(x)$ and $g(x)$ are both real polynomials, so is $f(x)$. Moreover, since $\deg f(x) \geq 3$ and $\deg g(x) = 2$, we have $\deg h(x) \geq 1$, and we are done.

$\square$

2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 with complex coefficients, has infinitely many roots in $\mathbb{C}^2$.

Solution. Before we prove this, we first prove the following useful Lemma.

> **Lemma.** A complex polynomial of degree n has exactly n roots, counted with multiplicity. In particular, all complex polynomials have finitely many roots.
>
> *Proof.* Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n. We prove this via induction on n. When $n = 1$, $f(x) = a_0 + a_1 x$ for some $a_0, a_1 \in \mathbb{C}$ with $a_1 \neq 0$. We have
>
> $$\begin{aligned} & f(x) = 0 \\ \iff & a_0 + a_1 x = 0 \\ \iff & a_1 x = -a_0 \\ \iff & x = -\frac{a_0}{a_1}. \end{aligned}$$
>
> Thus, $f(x)$ has exactly 1 root.
>
> We now assume that an n-degree polynomial $g(x) \in \mathbb{C}[x]$ has exactly n roots (counted with multiplicity). Let $f(x) \in \mathbb{C}[x]$ have degree $n+1$. By FTA, $f(x)$ has a root $x_0 \in \mathbb{C}$. We may thus write
>
> $$f(x) = (x - x_0)g(x),$$
>
> for some n-degree polynomial $g(x) \in \mathbb{C}[x]$. Now, we have
>
> $$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$
>
> By assumption, the latter happens for exactly n values of x. Thus, $f(x)$ has exactly $n+1$ roots counted with multiplicity. The second statement follows from the fact that any polynomial has finite degree. $\square$

Since $f(z_1, z_2)$ is non-constant at least one of z_1 or z_2 must "appear" in $f(z_1, z_2)$. Without loss of generality, suppose that z_2 appears in $f(z_1, z_2)$. We may write

$$f(z_1, z_2) = \sum_{k=0}^{n} f_k(z_1) \cdot z_2^k$$

where $n \geq 1$ and $f_k(z_1) \in \mathbb{C}[z_1]$. Moreover, $f_n \neq 0$, and thus, $f_n(z_1)$ has only finitely many roots (possibly zero). Thus, there are infinitely many $\alpha \in \mathbb{C}$ such that

$f_n(\alpha) \neq 0$. Since, $n \geq 1$, we have that $f(\alpha, z_2) \in \mathbb{C}[z_2]$ is non-constant for all these infinitely many α. By FTA, for each such α, there exists $\beta \in \mathbb{C}$ such that $f(\alpha, \beta) = 0$. Thus, there are infinitely many roots of $f(z_1, z_2)$ in $\mathbb{C}^2$ (since it contains all these pairs (α, β) as α takes on infinitely many values). $\square$

3. Show that the complex plane minus a countable set is path-connected.

   Solution. Let $S \subset \mathbb{C}$ be countable. We must show that $\mathbb{C} \setminus S$ is path-connected. Let $z_1, z_2 \in \mathbb{C} \setminus S$ and $z_1 \neq z_2$. Let f be the line segment joining z_1 to z_2, and let g be the circular arc joining z_1 to z_2. For every $\lambda \in [0, 1]$, we define

   $$\sigma_\lambda(t) := \lambda f(t) + (1 - \lambda) g(t) \quad \forall\, t \in [0, 1]$$

   Claim.

   (a) σ_λ is a path in $\mathbb{C}$,

   (b) $\sigma_\lambda(0) = z_1$ and $\sigma_\lambda(1) = z_2$ for all $\lambda \in [0, 1]$, and

   (c) if $\lambda_1 \neq \lambda_2$ and $t \in (0, 1)$, then $\sigma_{\lambda_1}(t) \neq \sigma_{\lambda_2}(t)$.

   Proof. We leave the proof for (a) and (b) as simple exercises. To show (c), we first note that for $t \in (0, 1)$, $f(t) \neq g(t)$. Now, let $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 \neq \lambda_2$. Suppose $\sigma_{\lambda_1}(t) = \sigma_{\lambda_2}(t)$. We then have

   $$\begin{aligned} &\lambda_1 f(t) + (1 - \lambda_1) g(t) = \lambda_2 f(t) + (1 - \lambda_2) g(t) \\ \implies &(\lambda_1 - \lambda_2) f(t) = (\lambda_1 - \lambda_2) g(t). \end{aligned}$$

   Since $\lambda_1 \neq \lambda_2$, we get $f(t) = g(t)$, a contradiction. Intuitively, this means that the images of all these paths are disjoint, barring the start and end points. $\square$

   Since $[0, 1]$ is uncountable (we assume this without proof), and the images are disjoint (by claim (c)), we have that the set $\{\sigma_\lambda \mid \lambda \in [0, 1]\}$ is uncountable. Since the set S is only countable, there exists some $\lambda_0 \in [0, 1]$ such that $\sigma_{\lambda_0}(t) \notin S$ for all $t \in [0, 1]$. In other words, σ_{λ_0} is a path in $\mathbb{C} \setminus S$ starting at z_1 and ending at z_2. Since z_1, z_2 were arbitrary, we are done.

   $\square$

4. Check for real differentiability and holomorphicity:

   (a) $f(z) = c$

   (b) $f(z) = z$

   (c) $f(z) = z^n$, $n \in \mathbb{Z}$

   (d) $f(z) = \Re z$

   (e) $f(z) = |z|$

(f) $f(z) = |z|^2$

(g) $f(z) = \overline{z}$

(h) $f(z) = \begin{cases} \frac{z}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Solution. Some of these are trivial and hence omitted.

(a) Real differentiable and holomorphic.

(b) Real differentiable and holomorphic.

(c) For $n \geq 0$, real differentiable and holomorphic. Since holomorphicity implies real differentiability, we only check for holomorphicity. Let $z_0 \in \mathbb{C}$ be arbitrary. We must check for the existence of the following limit:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

For $z \neq z_0$, we know that

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

Since the limit of the RHS exists as $z \to z_0$, we are done.

For $n < 0$, the function is defined on $\mathbb{C} \setminus \{0\}$. On $\mathbb{C} \setminus \{0\}$, $f(z)$ is non-zero. Thus, $\frac{1}{f}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ by the first case since $\frac{1}{f(z)} = z^{-n}$ and $-n > 0$. Thus, $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

(d) Real differentiable but not holomorphic. We may write f as

$$f(x + \iota y) = x + 0\iota.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$. f is clearly real differentiable since all the partial derivatives exist everywhere and are continuous. However, since $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$, the CR equations do not hold. Hence, f is complex differentiable nowhere, and thus, not holomorphic.

(e) $|z|$ is real differentiable precisely on $\mathbb{C} \setminus \{0\}$ and complex differentiable nowhere. We may write

$$f(x + \iota y) = \sqrt{x^2 + y^2} + 0\iota$$

giving us $u(x, y) = \sqrt{x^2 + y^2}$, and $v(x, y) = 0$. On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous, whereas u_x and u_y fail to exist at $(0, 0)$. Thus, $f(z)$ is real differentiable on $\mathbb{C} \setminus \{0\}$. Moreover, this shows that $f(z)$ is not complex differentiable at 0 since it's not even real differentiable there. Everywhere else, $v_x = v_y = 0$, but at least one of u_x, u_y is non-zero, violating the CR equations. Thus, $f(z)$ is complex differentiable nowhere.

(f) $|z|^2$ is real differentiable everywhere and complex differentiable precisely at 0. As a result, it is holomorphic nowhere. As before, we have $u(x,y) = x^2 + y^2$, and $v(x,y) = 0$. Since all partial derivatives exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. Note that

$$\begin{aligned} u_x(x,y) &= 2x & u_y(x,y) &= 2y \\ v_x(x,y) &= 0 & v_y(x,y) &= 0 \end{aligned}$$

Thus, the CR equations hold precisely at 0.

(g) For $f(z) = \overline{z}$, we may write

$$f(x + \iota y) = x - \iota y,$$

which gives us $u(x,y) = x$ and $v(x,y) = -y$. Since all partials exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. However, note that

$$\begin{aligned} u_x(x,y) &= 1 & u_y(x,y) &= 0 \\ v_x(x,y) &= 0 & v_y(x,y) &= -1 \end{aligned}$$

Since $u_x(x,y) \neq v_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$, we see that the CR equations do not hold anywhere and $f(z)$ is complex differentiable nowhere.

(h) $f(z)$ is real differentiable precisely on $\mathbb{C} \setminus \{0\}$, and complex differentiable nowhere. We may multiply and divide by $\overline{z}$ to obtain

$$u(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad v(x,y) = \frac{2xy}{x^2 + y^2}$$

for $(x,y) \neq (0,0)$, and $u(0,0) = v(0,0) = 0$. Since u and v are not continuous at $(0,0)$ (recall MA109), neither is f. Hence, f is neither real differentiable, nor complex differentiable at $0 \in \mathbb{C}$. At all other points, all partials exist and are continuous. Hence, f is real differentiable there. However, one may explicitly compute those partial derivatives and verify that the CR equations hold nowhere. Thus, f is complex differentiable nowhere.

$\square$