MA 205: Complex Analysis
Tutorial Solutions

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Contents

Note: Many of these solutions are either inspired by, or in some cases directly taken from Aryaman Maithani's tutorial solutions for last year's offering of this course.

§1. Week 1

3rd August, 2021

Notation: We use $\mathbb{C}[x]$ to denote the set of all polynomials in x with complex coefficients. $\mathbb{R}[x]$ is defined similarly.

1. Show that a real polynomial that is irreducible has degree at most two, i.e, if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$. ($a_n \neq 0$, of course)

Solution. We consider two cases. First, suppose $f(x) \in \mathbb{R}[x]$ has a real root, x_0, and let $h(x) := (x - x_0)$. Since $x_0 \in \mathbb{R}$, $h(x) \in \mathbb{R}[x]$. Moreover, we can write

$$f(x) = g(x)h(x)$$

for some $g(x) \in \mathbb{R}[x]$. (Why must g be a real polynomial?) Also, since $\deg f(x) \geq 3$ and $\deg h(x) = 1$, we have that $\deg g(x) \geq 2$. Thus, g and h are two non-constant real polynomials satisfying $f(x) = g(x)h(x)$.

Now, suppose that $f(x)$ has no real root. We may also view $f(x)$ as a polynomial in $\mathbb{C}[x]$. By FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we have that $x_0 \notin \mathbb{R}$, and thus $x_0 \neq \overline{x_0}$.

Claim. $f(\overline{x_0}) = 0$.

Proof. We have

$$\begin{aligned} f(\overline{x_0}) &= a_0 + a_1\overline{x_0} + \cdots + a_n(\overline{x_0})^n && \overline{z^n} = \bar{z}^n \\ &= a_0 + a_1\overline{x_0} + \cdots + a_n\overline{x_0^n} && a_i \in \mathbb{R} \text{ and thus, } a_i = \overline{a_i} \\ &= \overline{a_0} + \overline{a_1}\ \overline{x_0} + \cdots + \overline{a_n}\overline{x_0^n} && \overline{z_1 z_2 + z_3} = \overline{z_1}\ \overline{z_2} + \overline{z_3} \\ &= \overline{f(x_0)} \\ &= \bar{0} \\ &= 0. \end{aligned}$$

□

Thus, x_0 and $\overline{x_0}$ are two distinct roots of $f(x)$. Define $g(x) := (x - x_0)(x - \overline{x_0})$. A priori, we have $g(x) \in \mathbb{C}[x]$. However, note that

$$(x - x_0)(x - \overline{x_0}) = x^2 - (2\Re x_0)x + |x_0|^2 \in \mathbb{R}[x].$$

Thus, $g(x)$ is in fact a real polynomial. Since x_0 and $\overline{x_0}$ are distinct, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Why?) Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. Again, since $f(x)$ and $g(x)$ are both real polynomials, so is $f = h(x)$. Moreover, since $\deg f(x) \geq 3$ and $\deg g(x) = 2$, we have $\deg h(x) \geq 1$, and we are done. □

2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 with complex coefficients, has infinitely many roots in $\mathbb{C}^2$.

   Solution. Before we prove this, we first prove the following useful Lemma.

   > **Lemma.** A complex polynomial of degree n has exactly n roots, counted with multiplicity. In particular, all nonzero complex polynomials have finitely many roots.
   >
   > *Proof.* Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n. We prove this via induction on n. When $n = 1$, $f(x) = a_0 + a_1 x$ for some $a_0, a_1 \in \mathbb{C}$ with $a_1 \neq 0$. We have
   >
   > $$\begin{aligned} & f(x) = 0 \\ \iff & a_0 + a_1 x = 0 \\ \iff & a_1 x = -a_0 \\ \iff & x = -\frac{a_0}{a_1}. \end{aligned}$$
   >
   > Thus, $f(x)$ has exactly 1 root.
   >
   > We now assume that an n-degree polynomial $g(x) \in \mathbb{C}[x]$ has exactly n roots (counted with multiplicity). Let $f(x) \in \mathbb{C}[x]$ have degree $n+1$. By FTA, $f(x)$ has a root $x_0 \in \mathbb{C}$. We may thus write
   >
   > $$f(x) = (x - x_0)g(x),$$
   >
   > for some n-degree polynomial $g(x) \in \mathbb{C}[x]$. Now, we have
   >
   > $$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$
   >
   > By assumption, the latter happens for exactly n values of x. Thus, $f(x)$ has exactly $n+1$ roots counted with multiplicity. The second statement follows from the fact that any polynomial has finite degree. □

   Since $f(z_1, z_2)$ is non-constant at least one of z_1 or z_2 must "appear" in $f(z_1, z_2)$. Without loss of generality, suppose that z_2 appears in $f(z_1, z_2)$. We may write

   $$f(z_1, z_2) = \sum_{k=0}^{n} f_k(z_1) \cdot z_2^k$$

   where $n \geq 1$ and $f_k(z_1) \in \mathbb{C}[z_1]$. Moreover, $f_n \neq 0$, and thus, $f_n(z_1)$ has only finitely many roots (possibly zero). Thus, there are infinitely many $\alpha \in \mathbb{C}$ such that $f_n(\alpha) \neq 0$. Since, $n \geq 1$, we have that $f(\alpha, z_2) \in \mathbb{C}[z_2]$ is non-constant for all these infinitely many α. By FTA, for each such α, there exists $\beta \in \mathbb{C}$ such that $f(\alpha, \beta) = 0$. Thus, there are infinitely many roots of $f(z_1, z_2)$ in $\mathbb{C}^2$ (since it contains all these pairs (α, β) as α takes on infinitely many values). $\square$

3. Show that the complex plane minus a countable set is path-connected.

   Solution. Let $S \subset \mathbb{C}$ be countable. We must show that $\mathbb{C} \setminus S$ is path-connected. Let $z_1, z_2 \in \mathbb{C} \setminus S$ and $z_1 \neq z_2$. Let f be the line segment joining z_1 to z_2, and let g be a semicircular arc joining z_1 to z_2. For every $\lambda \in [0, 1]$, we define

   $$\sigma_\lambda(t) := \lambda f(t) + (1 - \lambda) g(t) \quad \forall\, t \in [0, 1]$$

   > **Claim.**
   >
   > (a) σ_λ is a path in $\mathbb{C}$,
   >
   > (b) $\sigma_\lambda(0) = z_1$ and $\sigma_\lambda(1) = z_2$ for all $\lambda \in [0, 1]$, and
   >
   > (c) if $\lambda_1 \neq \lambda_2$ and $t \in (0, 1)$, then $\sigma_{\lambda_1}(t) \neq \sigma_{\lambda_2}(t)$.
   >
   > *Proof.* We leave the proof for (a) and (b) as simple exercises. To show (c), we first note that for $t \in (0, 1)$, $f(t) \neq g(t)$. Now, let $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 \neq \lambda_2$. Suppose $\sigma_{\lambda_1}(t) = \sigma_{\lambda_2}(t)$. We then have
   >
   > $$\begin{aligned} &\lambda_1 f(t) + (1 - \lambda_1) g(t) = \lambda_2 f(t) + (1 - \lambda_2) g(t) \\ \implies &(\lambda_1 - \lambda_2) f(t) = (\lambda_1 - \lambda_2) g(t). \end{aligned}$$
   >
   > Since $\lambda_1 \neq \lambda_2$, we get $f(t) = g(t)$, a contradiction. Intuitively, this means that the images of all these paths are disjoint, barring the start and end points. $\square$

   Since $[0, 1]$ is uncountable (we assume this without proof), and the images are disjoint (by claim (c)), we have that the set $\{\sigma_\lambda \mid \lambda \in [0, 1]\}$ is uncountable. Since the set S is only countable, there exists some $\lambda_0 \in [0, 1]$ such that $\sigma_{\lambda_0}(t) \notin S$ for all $t \in [0, 1]$. In other words, σ_{λ_0} is a path in $\mathbb{C} \setminus S$ starting at z_1 and ending at z_2. Since z_1, z_2 were arbitrary, we are done. $\square$

4. Check for real differentiability and holomorphicity:

   (a) $f(z) = c$

   (b) $f(z) = z$

   (c) $f(z) = z^n$, $n \in \mathbb{Z}$

   (d) $f(z) = \Re z$

   (e) $f(z) = |z|$

   (f) $f(z) = |z|^2$

(g) $f(z) = \overline{z}$

(h) $f(z) = \begin{cases} \frac{z}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Solution. Some of these are trivial and hence omitted.

(a) Real differentiable and holomorphic.

(b) Real differentiable and holomorphic.

(c) For $n \geq 0$, real differentiable and holomorphic. Since holomorphicity implies real differentiability, we only check for holomorphicity. Let $z_0 \in \mathbb{C}$ be arbitrary. We must check for the existence of the following limit:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

For $z \neq z_0$, we know that

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

Since the limit of the RHS exists as $z \to z_0$, we are done.

For $n < 0$, the function is defined on $\mathbb{C} \setminus \{0\}$. On $\mathbb{C} \setminus \{0\}$, $f(z)$ is non-zero. Thus, $\frac{1}{f}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ by the first case since $\frac{1}{f(z)} = z^{-n}$ and $-n > 0$. Thus, $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

(d) Real differentiable but not holomorphic. We may write f as

$$f(x + \iota y) = x + 0\iota.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$. f is clearly real differentiable since all the partial derivatives (of u and v) exist everywhere and are continuous. However, since $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$, the CR equations do not hold. Hence, f is complex differentiable nowhere, and thus, not holomorphic.

(e) $|z|$ is real differentiable precisely on $\mathbb{C} \setminus \{0\}$ and complex differentiable nowhere. We may write

$$f(x + \iota y) = \sqrt{x^2 + y^2} + 0\iota$$

giving us $u(x, y) = \sqrt{x^2 + y^2}$, and $v(x, y) = 0$. On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous, whereas u_x and u_y fail to exist at $(0, 0)$. Thus, $f(z)$ is real differentiable on $\mathbb{C} \setminus \{0\}$. Moreover, this shows that $f(z)$ is not complex differentiable at 0 since it's not even real differentiable there. Everywhere else, $v_x = v_y = 0$, but at least one of u_x, u_y is non-zero, violating the CR equations. Thus, $f(z)$ is complex differentiable nowhere.

(f) $|z|^2$ is real differentiable everywhere and complex differentiable precisely at 0. As a result, it is holomorphic nowhere. As before, we have $u(x, y) = x^2 + y^2$, and $v(x, y) = 0$. Since all partial derivatives exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. Note that

$$\begin{aligned} u_x(x, y) &= 2x & u_y(x, y) &= 2y \\ v_x(x, y) &= 0 & v_y(x, y) &= 0 \end{aligned}$$

Thus, the CR equations hold precisely at 0.

(g) For $f(z) = \overline{z}$, we may write

$$f(x + \iota y) = x - \iota y,$$

which gives us $u(x, y) = x$ and $v(x, y) = -y$. Since all partials exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. However, note that

$$\begin{aligned} u_x(x, y) &= 1 & u_y(x, y) &= 0 \\ v_x(x, y) &= 0 & v_y(x, y) &= -1 \end{aligned}$$

Since $u_x(x, y) \neq v_y(x, y)$ for all $(x, y) \in \mathbb{R}^2$, we see that the CR equations do not hold anywhere and $f(z)$ is complex differentiable nowhere.

(h) f is real differentiable precisely on $\mathbb{C} \setminus \{0\}$, and complex differentiable nowhere. We may multiply and divide by $\overline{z}$ to obtain

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{2xy}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$, and $u(0, 0) = v(0, 0) = 0$. Since u and v are not continuous at $(0, 0)$ (recall MA109), neither is f. Hence, f is neither real differentiable, nor complex differentiable at $0 \in \mathbb{C}$. At all other points, all partials exist and are continuous. Hence, f is real differentiable there. However, one may explicitly compute those partial derivatives and verify that the CR equations hold nowhere. Thus, f is complex differentiable nowhere. $\square$

§2. Week 2

10th August, 2021

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions. (This is true iff u and v are defined on open, path-connected sets)

   Solution. Since v is a harmonic conjugate of u, we have

   $$u_X = v_Y \quad \text{and} \quad u_Y = -v_X.$$

   Since we also have that u is a harmonic conjugate of v, we get

   $$v_X = u_Y \quad \text{and} \quad v_Y = -u_X.$$

   Note that the above equalities hold for each point in the domain. Thus, we have

   $$u_X = u_Y = v_X = v_Y \equiv 0,$$

   identically. Since the domain is connected, this implies that u and v are constant.

   The following is another alternative.

   > **Lemma.** Let u be a harmonic function defined on an open, path connected set. Then, the harmonic conjugate of u is unique up to a constant.
   >
   > *Proof.* Let v and v' be two harmonic conjugates of u. It suffices to show that $(v - v')$ is a constant function. By definition, $u + \iota v$ and $u + \iota v'$ are both holomorphic, and hence satisfy the Cauchy-Riemann equations. Thus, we have
   >
   > $$u_x = v_y,\ v_x = -u_y \quad \text{and} \quad u_x = v'_y,\ v'_x = -u_y.$$
   >
   > It thus follows that
   > $$(v - v')_x = (v - v')_y \equiv 0,$$
   > identically. Since the domain is path-connected, this implies that $(v - v')$ is constant. □

   Now, since $v(X, Y)$ is a harmonic conjugate of $u(X, Y)$, we have that $-u(X, Y)$ is a harmonic conjugate of $v(X, Y)$ (Why?). Since we also have that $u(X, Y)$ is a harmonic conjugate of $v(X, Y)$, it follows that u and $-u$ differ only by a constant, and hence u must itself be constant. The same holds for v. □

2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

   Solution. Consider the function

   $$f(Z) = \frac{1}{2}Z^2 + Z^3,$$

defined on $\mathbb{C}$. Writing $Z = X + \iota Y$, where $X, Y \in \mathbb{R}$, we see that the function $u(X, Y)$ is the *imaginary* part of $f(Z)$. Since $f(Z)$ is holomorphic on $\mathbb{C}$, u is harmonic. Moreover, its harmonic conjugate is give by

$$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$

Note that we require a minus sign since we obtained that $u(X, Y)$ was the imaginary, and not the real, part of a holomorphic function.

Note that the above method required us to intelligently guess the function $f(Z)$. However, if this is difficult to observe, we have the following 'standard' way of solving this problem. Some simple calculations give us

$$u_{XX}(X_0, Y_0) = 6Y_0 \quad \text{and} \quad u_{YY}(X_0, Y_0) = -6Y_0,$$

which gives us that $u_{XX} + u_{YY} \equiv 0$, verifying that u is harmonic. Note that $u_X = v_Y$, giving us $v_Y = Y + 6XY$. Integrating with respect to Y gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function g. We also have the relation $v_X = -u_Y$. Computing each individually gives us

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, we get

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} nz^n$,

   (b) $\displaystyle\sum_{p \text{ prime}} z^p$,

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$.

Solution. We shall use the ratio test in the first and third parts, and the root test in the second part.

(a) Note that we have

$$\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$$

and thus,

$$R = \alpha^{-1} = 1.$$

(b) We may rewrite the series as

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

Since there are infinitely many primes, given any $n \in \mathbb{N}$, there exists $m \geq n$ with $a_m = 1$. Thus, we clearly have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

As before, the root test gives us

$$R = \alpha^{-1} = 1.$$

(c) We have

$$a_n = \frac{n!}{n^n}.$$

Thus,

$$\begin{aligned} \alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} \\ &= \frac{1}{e}. \end{aligned}$$

Since the above limit exists, we may apply the ratio test to get

$$R = \alpha^{-1} = e.$$

□

4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

   Solution. Consider the sequence (a_n) defined by

$$a_{2n} = \frac{1}{n^2} \quad \text{and} \quad a_{2n-1} = \frac{1}{n^3}$$

   Since $\sum n^{-2}$ and $\sum n^{-3}$ converge (via the integral test), we have that $\sum a_n$ converges. However, note that

$$L = \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \limsup_{n\to\infty} \left| \frac{a_{2n}}{a_{2n-1}} \right| = \limsup_{n\to\infty} n = \infty.$$

   Thus $L > 1$ clearly, but the series is convergent. Hence, we have showed that even $L = \infty$ is not sufficient to conclude the divergence of a series. $\square$

5. Construct an infinitely differentiable function $f \colon \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

   Solution. We saw in the lectures that the function $g \colon \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0 \end{cases}$$

   is infinitely differentiable. Using this function, we construct $f \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := g(x)g(1-x).$$

   f is clearly infinitely differentiable. Moreover, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$. Thus, f vanishes outside the bounded set $(0, 1)$. It remains to show that f is non-zero. Indeed, we have that

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

   Suppose $f \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function which vanishes outside some bounded set K. We now show that f is identically zero. For this, recall the Identity Theorem:

   > **Theorem**
   >
   > Let $\Omega \subset \mathbb{C}$ be a domain. If $f \colon \Omega \to \mathbb{C}$ is analytic, then either f is identically zero, or the zeros of f form a discrete set.

   Although the above theorem is for analytic functions, we shall show later in the course that holomorphic functions are indeed analytic. Since the set K is bounded, there exists $M > 0$ such that

$$|z| \leq M \text{ for all } z \in K.$$

Choosing the point $z_0 = M + 2$, we see that f vanishes in a neighbourhood of radius 1 around z_0. Since $\mathbb{C}$ is open and path-connected (and hence a domain), and since any open disc is not discrete, we conclude from the above theorem that f must be identically zero on $\mathbb{C}$. $\square$

6. Show that $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto.

   Solution. Let $z_0 \in \mathbb{C}^\times$. It suffices to show that $\exp(z) = z_0$ for some $z \in \mathbb{C}$. Since z_0 is non-zero, $r := |z_0| \neq 0$. Thus,
$$w := \frac{z_0}{r}$$
   has modulus 1. Thus,
$$w = x_0 + \iota y_0$$
   for some $(x_0, y_0) \in \mathbb{R}^2$ satisfying $x_0^2 + y_0^2 = 1$. Hence, $x_0 = \cos\theta$ and $y_0 = \sin\theta$ for some $\theta \in [0, 2\pi)$. We now define
$$z := \log(r) + \iota\theta,$$
   where the above log is the real-valued log. Thus, we have
$$\begin{aligned}\exp(z) = \exp(\log(r) + \iota\theta) &= \exp(\log(r)) \cdot \exp(\iota\theta)\\ &= r \cdot (\cos\theta + \iota\sin\theta)\\ &= r \cdot w = z_0.\end{aligned}$$
   Thus, $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto. $\square$

7. Show that $\sin, \cos\colon \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and real cosine which were bounded by 1).

   Solution. We prove that cos is surjective. A similar method works for sin. Recall that
$$\cos(z) = \frac{1}{2}\left(e^{\iota z} + e^{-\iota z}\right).$$
   Let $z_0 \in \mathbb{C}$. As before, it suffices to show that $\cos(z) = z_0$ for some $z \in \mathbb{C}$. Consider the quadratic equation
$$\frac{1}{2}\left(t + \frac{1}{t}\right) = z_0 \qquad (\dagger)$$
   Rearranging this gives us
$$t^2 - 2z_0 t + 1 = 0.$$
   Since the above is a (non-constant) complex polynomial, it has a complex root t_0 (by FTA). Moreover, note that $t_0 \neq 0$. By the previous question, there exists $z' \in \mathbb{C}$ satisfying $e^{z'} = t_0$. Considering $z = z'/\iota$, we see that $e^{\iota z} = t_0$. Plugging $t_0 = e^{\iota z}$ in ($\dagger$) gives us
$$\cos(z) = z_0,$$
   as desired. $\square$

8. Show that for any complex number z, $\cos^2(z) + \sin^2(z) = 1$.

   Solution. Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \cos^2(z) + \sin^2(z) - 1.$$

   Note that f is holomorphic, and hence analytic. Since f vanishes on $\mathbb{R}$ and $\mathbb{R}$ is not discrete, f must vanish everywhere, by the Identity Theorem. $\square$