MA 205: Complex Analysis
Tutorial Solutions

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Contents

Note: Many of these solutions are either inspired by, or in some cases directly taken from Aryaman Maithani's tutorial solutions for last year's offering of this course.

§1. Week 1

3rd August, 2021

<u>Notation</u>: We use $\mathbb{C}[x]$ to denote the set of all polynomials in x with complex coefficients. $\mathbb{R}[x]$ is defined similarly.

1. Show that a real polynomial that is irreducible has degree at most two, i.e, if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$. ($a_n \neq 0$, of course)

Solution. We consider two cases. First, suppose $f(x) \in \mathbb{R}[x]$ has a real root, x_0, and let $h(x) := (x - x_0)$. Since $x_0 \in \mathbb{R}$, $h(x) \in \mathbb{R}[x]$. Moreover, we can write

$$f(x) = g(x)h(x)$$

for some $g(x) \in \mathbb{R}[x]$. (Why must g be a real polynomial?) Also, since $\deg f(x) \geq 3$ and $\deg h(x) = 1$, we have that $\deg g(x) \geq 2$. Thus, g and h are two non-constant real polynomials satisfying $f(x) = g(x)h(x)$.

Now, suppose that $f(x)$ has no real root. We may also view $f(x)$ as a polynomial in $\mathbb{C}[x]$. By FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we have that $x_0 \notin \mathbb{R}$, and thus $x_0 \neq \overline{x_0}$.

> **Claim.** $f(\overline{x_0}) = 0$.
>
> *Proof.* We have
>
> $$\begin{aligned} f(\overline{x_0}) &= a_0 + a_1\overline{x_0} + \cdots + a_n(\overline{x_0})^n && \\ &= a_0 + a_1\overline{x_0} + \cdots + a_n\overline{x_0^n} && \overline{z^n} = \bar{z}^n \\ &= \overline{a_0} + \overline{a_1}\ \overline{x_0} + \cdots + \overline{a_n}\overline{x_0^n} && a_i \in \mathbb{R} \text{ and thus, } a_i = \overline{a_i} \\ &= \overline{f(x_0)} && \overline{z_1 z_2 + z_3} = \overline{z_1}\ \overline{z_2} + \overline{z_3} \\ &= \bar{0} \\ &= 0. \end{aligned}$$
>
> □

Thus, x_0 and $\overline{x_0}$ are two distinct roots of $f(x)$. Define $g(x) := (x - x_0)(x - \overline{x_0})$. A priori, we have $g(x) \in \mathbb{C}[x]$. However, note that

$$(x - x_0)(x - \overline{x_0}) = x^2 - (2\Re x_0)x + |x_0|^2 \in \mathbb{R}[x].$$

Thus, $g(x)$ is in fact a real polynomial. Since x_0 and $\overline{x_0}$ are distinct, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Why?) Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. Again, since $f(x)$ and $g(x)$ are both real polynomials, so is $f = h(x)$. Moreover, since $\deg f(x) \geq 3$ and $\deg g(x) = 2$, we have $\deg h(x) \geq 1$, and we are done. $\square$

2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 with complex coefficients, has infinitely many roots in $\mathbb{C}^2$.

   Solution. Before we prove this, we first prove the following useful Lemma.

   > **Lemma.** A complex polynomial of degree n has exactly n roots, counted with multiplicity. In particular, all nonzero complex polynomials have finitely many roots.
   >
   > *Proof.* Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n. We prove this via induction on n. When $n = 1$, $f(x) = a_0 + a_1 x$ for some $a_0, a_1 \in \mathbb{C}$ with $a_1 \neq 0$. We have
   >
   > $$\begin{aligned} & f(x) = 0 \\ \iff\ & a_0 + a_1 x = 0 \\ \iff\ & a_1 x = -a_0 \\ \iff\ & x = -\frac{a_0}{a_1}. \end{aligned}$$
   >
   > Thus, $f(x)$ has exactly 1 root.
   >
   > We now assume that an n-degree polynomial $g(x) \in \mathbb{C}[x]$ has exactly n roots (counted with multiplicity). Let $f(x) \in \mathbb{C}[x]$ have degree $n+1$. By FTA, $f(x)$ has a root $x_0 \in \mathbb{C}$. We may thus write
   >
   > $$f(x) = (x - x_0)g(x),$$
   >
   > for some n-degree polynomial $g(x) \in \mathbb{C}[x]$. Now, we have
   >
   > $$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$
   >
   > By assumption, the latter happens for exactly n values of x. Thus, $f(x)$ has exactly $n+1$ roots counted with multiplicity. The second statement follows from the fact that any polynomial has finite degree. $\square$

   Since $f(z_1, z_2)$ is non-constant at least one of z_1 or z_2 must "appear" in $f(z_1, z_2)$. Without loss of generality, suppose that z_2 appears in $f(z_1, z_2)$. We may write

   $$f(z_1, z_2) = \sum_{k=0}^{n} f_k(z_1) \cdot z_2^k$$

   where $n \geq 1$ and $f_k(z_1) \in \mathbb{C}[z_1]$. Moreover, $f_n \neq 0$, and thus, $f_n(z_1)$ has only finitely many roots (possibly zero). Thus, there are infinitely many $\alpha \in \mathbb{C}$ such that $f_n(\alpha) \neq 0$. Since, $n \geq 1$, we have that $f(\alpha, z_2) \in \mathbb{C}[z_2]$ is non-constant for all these infinitely many α. By FTA, for each such α, there exists $\beta \in \mathbb{C}$ such that $f(\alpha, \beta) = 0$. Thus, there are infinitely many roots of $f(z_1, z_2)$ in $\mathbb{C}^2$ (since it contains all these pairs (α, β) as α takes on infinitely many values). $\square$

3. Show that the complex plane minus a countable set is path-connected.

Solution. Let $S \subset \mathbb{C}$ be countable. We must show that $\mathbb{C} \setminus S$ is path-connected. Let $z_1, z_2 \in \mathbb{C} \setminus S$ and $z_1 \neq z_2$. Let f be the line segment joining z_1 to z_2, and let g be a semicircular arc joining z_1 to z_2. For every $\lambda \in [0, 1]$, we define

$$\sigma_\lambda(t) := \lambda f(t) + (1 - \lambda) g(t) \quad \forall\, t \in [0, 1]$$

> **Claim.**
>
> (a) σ_λ is a path in $\mathbb{C}$,
>
> (b) $\sigma_\lambda(0) = z_1$ and $\sigma_\lambda(1) = z_2$ for all $\lambda \in [0, 1]$, and
>
> (c) if $\lambda_1 \neq \lambda_2$ and $t \in (0, 1)$, then $\sigma_{\lambda_1}(t) \neq \sigma_{\lambda_2}(t)$.
>
> *Proof.* We leave the proof for (a) and (b) as simple exercises. To show (c), we first note that for $t \in (0, 1)$, $f(t) \neq g(t)$. Now, let $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 \neq \lambda_2$. Suppose $\sigma_{\lambda_1}(t) = \sigma_{\lambda_2}(t)$. We then have
>
> $$\begin{aligned} &\lambda_1 f(t) + (1 - \lambda_1) g(t) = \lambda_2 f(t) + (1 - \lambda_2) g(t) \\ \implies &(\lambda_1 - \lambda_2) f(t) = (\lambda_1 - \lambda_2) g(t). \end{aligned}$$
>
> Since $\lambda_1 \neq \lambda_2$, we get $f(t) = g(t)$, a contradiction. Intuitively, this means that the images of all these paths are disjoint, barring the start and end points. $\square$

Since $[0, 1]$ is uncountable (we assume this without proof), and the images are disjoint (by claim (c)), we have that the set $\{\sigma_\lambda \mid \lambda \in [0, 1]\}$ is uncountable. Since the set S is only countable, there exists some $\lambda_0 \in [0, 1]$ such that $\sigma_{\lambda_0}(t) \notin S$ for all $t \in [0, 1]$. In other words, σ_{λ_0} is a path in $\mathbb{C} \setminus S$ starting at z_1 and ending at z_2. Since z_1, z_2 were arbitrary, we are done. $\square$

4. Check for real differentiability and holomorphicity:

   (a) $f(z) = c$

   (b) $f(z) = z$

   (c) $f(z) = z^n$, $n \in \mathbb{Z}$

   (d) $f(z) = \Re z$

   (e) $f(z) = |z|$

   (f) $f(z) = |z|^2$

(g) $f(z) = \overline{z}$

(h) $f(z) = \begin{cases} \frac{z}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Solution. Some of these are trivial and hence omitted.

(a) Real differentiable and holomorphic.

(b) Real differentiable and holomorphic.

(c) For $n \geq 0$, real differentiable and holomorphic. Since holomorphicity implies real differentiability, we only check for holomorphicity. Let $z_0 \in \mathbb{C}$ be arbitrary. We must check for the existence of the following limit:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

For $z \neq z_0$, we know that

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

Since the limit of the RHS exists as $z \to z_0$, we are done.

For $n < 0$, the function is defined on $\mathbb{C} \setminus \{0\}$. On $\mathbb{C} \setminus \{0\}$, $f(z)$ is non-zero. Thus, $\frac{1}{f}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ by the first case since $\frac{1}{f(z)} = z^{-n}$ and $-n > 0$. Thus, $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

(d) Real differentiable but not holomorphic. We may write f as

$$f(x + \iota y) = x + 0\iota.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$. f is clearly real differentiable since all the partial derivatives (of u and v) exist everywhere and are continuous. However, since $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$, the CR equations do not hold. Hence, f is complex differentiable nowhere, and thus, not holomorphic.

(e) $|z|$ is real differentiable precisely on $\mathbb{C} \setminus \{0\}$ and complex differentiable nowhere. We may write

$$f(x + \iota y) = \sqrt{x^2 + y^2} + 0\iota$$

giving us $u(x, y) = \sqrt{x^2 + y^2}$, and $v(x, y) = 0$. On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous, whereas u_x and u_y fail to exist at $(0, 0)$. Thus, $f(z)$ is real differentiable on $\mathbb{C} \setminus \{0\}$. Moreover, this shows that $f(z)$ is not complex differentiable at 0 since it's not even real differentiable there. Everywhere else, $v_x = v_y = 0$, but at least one of u_x, u_y is non-zero, violating the CR equations. Thus, $f(z)$ is complex differentiable nowhere.

(f) $|z|^2$ is real differentiable everywhere and complex differentiable precisely at 0. As a result, it is holomorphic nowhere. As before, we have $u(x,y) = x^2 + y^2$, and $v(x,y) = 0$. Since all partial derivatives exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. Note that

$$\begin{aligned} u_x(x,y) &= 2x \quad & u_y(x,y) &= 2y \\ v_x(x,y) &= 0 \quad & v_y(x,y) &= 0 \end{aligned}$$

Thus, the CR equations hold precisely at 0.

(g) For $f(z) = \overline{z}$, we may write

$$f(x + \iota y) = x - \iota y,$$

which gives us $u(x,y) = x$ and $v(x,y) = -y$. Since all partials exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. However, note that

$$\begin{aligned} u_x(x,y) &= 1 \quad & u_y(x,y) &= 0 \\ v_x(x,y) &= 0 \quad & v_y(x,y) &= -1 \end{aligned}$$

Since $u_x(x,y) \neq v_y(x,y)$ for all $(x,y) \in \mathbb{R}^2$, we see that the CR equations do not hold anywhere and $f(z)$ is complex differentiable nowhere.

(h) f is real differentiable precisely on $\mathbb{C} \setminus \{0\}$, and complex differentiable nowhere. We may multiply and divide by $\overline{z}$ to obtain

$$u(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad v(x,y) = \frac{2xy}{x^2 + y^2}$$

for $(x,y) \neq (0,0)$, and $u(0,0) = v(0,0) = 0$. Since u and v are not continuous at $(0,0)$ (recall MA109), neither is f. Hence, f is neither real differentiable, nor complex differentiable at $0 \in \mathbb{C}$. At all other points, all partials exist and are continuous. Hence, f is real differentiable there. However, one may explicitly compute those partial derivatives and verify that the CR equations hold nowhere. Thus, f is complex differentiable nowhere. $\square$

§2. Week 2

10th August, 2021

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions. (This is true iff u and v are defined on open, path-connected sets)

   Solution. Since v is a harmonic conjugate of u, we have

   $$u_X = v_Y \quad \text{and} \quad u_Y = -v_X.$$

   Since we also have that u is a harmonic conjugate of v, we get

   $$v_X = u_Y \quad \text{and} \quad v_Y = -u_X.$$

   Note that the above equalities hold for each point in the domain. Thus, we have

   $$u_X = u_Y = v_X = v_Y \equiv 0,$$

   identically. Since the domain is connected, this implies that u and v are constant.

   The following is another alternative.

   > **Lemma.** Let u be a harmonic function defined on an open, path connected set. Then, the harmonic conjugate of u is unique up to a constant.
   >
   > *Proof.* Let v and v' be two harmonic conjugates of u. It suffices to show that $(v - v')$ is a constant function. By definition, $u + \iota v$ and $u + \iota v'$ are both holomorphic, and hence satisfy the Cauchy-Riemann equations. Thus, we have
   >
   > $$u_x = v_y,\ v_x = -u_y \quad \text{and} \quad u_x = v'_y,\ v'_x = -u_y.$$
   >
   > It thus follows that
   > $$(v - v')_x = (v - v')_y \equiv 0,$$
   > identically. Since the domain is path-connected, this implies that $(v - v')$ is constant. □

   Now, since $v(X, Y)$ is a harmonic conjugate of $u(X, Y)$, we have that $-u(X, Y)$ is a harmonic conjugate of $v(X, Y)$ (Why?). Since we also have that $u(X, Y)$ is a harmonic conjugate of $v(X, Y)$, it follows that u and $-u$ differ only by a constant, and hence u must itself be constant. The same holds for v. □

2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

   Solution. Consider the function

   $$f(Z) = \frac{1}{2}Z^2 + Z^3,$$

defined on $\mathbb{C}$. Writing $Z = X + \imath Y$, where $X, Y \in \mathbb{R}$, we see that the function $u(X, Y)$ is the *imaginary* part of $f(Z)$. Since $f(Z)$ is holomorphic on $\mathbb{C}$, u is harmonic. Moreover, its harmonic conjugate is give by

$$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$

Note that we require a minus sign since we obtained that $u(X, Y)$ was the imaginary, and not the real, part of a holomorphic function.

Note that the above method required us to intelligently guess the function $f(Z)$. However, if this is difficult to observe, we have the following 'standard' way of solving this problem. Some simple calculations give us

$$u_{XX}(X_0, Y_0) = 6Y_0 \quad \text{and} \quad u_{YY}(X_0, Y_0) = -6Y_0,$$

which gives us that $u_{XX} + u_{YY} \equiv 0$, verifying that u is harmonic. Note that $u_X = v_Y$, giving us $v_Y = Y + 6XY$. Integrating with respect to Y gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function g. We also have the relation $v_X = -u_Y$. Computing each individually gives us

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, we get

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} nz^n$,

   (b) $\displaystyle\sum_{p \text{ prime}} z^p$,

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$.

Solution. We shall use the ratio test in the first and third parts, and the root test in the second part.

(a) Note that we have

$$\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$$

and thus,

$$R = \alpha^{-1} = 1.$$

(b) We may rewrite the series as

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

Since there are infinitely many primes, given any $n \in \mathbb{N}$, there exists $m \geq n$ with $a_m = 1$. Thus, we clearly have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

Thus, the root test gives us

$$R = \alpha^{-1} = 1.$$

(c) We have

$$a_n = \frac{n!}{n^n}.$$

Thus,

$$\begin{aligned} \alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} \\ &= \frac{1}{e}. \end{aligned}$$

Since the above limit exists, we may apply the ratio test to get

$$R = \alpha^{-1} = e.$$

□

4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

Solution. Consider the sequence (a_n) defined by

$$a_{2n} = \frac{1}{n^2} \quad \text{and} \quad a_{2n-1} = \frac{1}{n^3}$$

Since $\sum n^{-2}$ and $\sum n^{-3}$ converge (via the integral test), we have that $\sum a_n$ converges. However, note that

$$L = \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \limsup_{n\to\infty} \left| \frac{a_{2n}}{a_{2n-1}} \right| = \limsup_{n\to\infty} n = \infty.$$

Thus $L > 1$ clearly, but the series is convergent. Hence, we have showed that even $L = \infty$ is not sufficient to conclude the divergence of a series. $\square$

5. Construct an infinitely differentiable function $f: \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

Solution. We saw in the lectures that the function $g: \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0 \end{cases}$$

is infinitely differentiable. Using this function, we construct $f: \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := g(x)g(1-x).$$

f is clearly infinitely differentiable. Moreover, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$. Thus, f vanishes outside the bounded set $(0,1)$. It remains to show that f is non-zero. Indeed, we have that

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

Suppose $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic function which vanishes outside some bounded set K. We now show that f is identically zero. For this, recall the Identity Theorem:

> **Theorem**
>
> Let $\Omega \subset \mathbb{C}$ be a domain. If $f: \Omega \to \mathbb{C}$ is analytic, then either f is identically zero, or the zeros of f form a discrete set.

Although the above theorem is for analytic functions, we shall show later in the course that holomorphic functions are indeed analytic. Since the set K is bounded, there exists $M > 0$ such that

$$|z| \leq M \text{ for all } z \in K.$$

Choosing the point $z_0 = M + 2$, we see that f vanishes in a neighbourhood of radius 1 around z_0. Since $\mathbb{C}$ is open and path-connected (and hence a domain), and since any open disc is not discrete, we conclude from the above theorem that f must be identically zero on $\mathbb{C}$. $\square$

6. Show that $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto.

   Solution. Let $z_0 \in \mathbb{C}^\times$. It suffices to show that $\exp(z) = z_0$ for some $z \in \mathbb{C}$. Since z_0 is non-zero, $r := |z_0| \neq 0$. Thus,
   $$w := \frac{z_0}{r}$$
   has modulus 1. Thus,
   $$w = x_0 + \iota y_0$$
   for some $(x_0, y_0) \in \mathbb{R}^2$ satisfying $x_0^2 + y_0^2 = 1$. Hence, $x_0 = \cos\theta$ and $y_0 = \sin\theta$ for some $\theta \in [0, 2\pi)$. We now define
   $$z := \log(r) + \iota\theta,$$
   where the above log is the real-valued log. Thus, we have
   $$\begin{aligned}\exp(z) = \exp(\log(r) + \iota\theta) &= \exp(\log(r)) \cdot \exp(\iota\theta) \\ &= r \cdot (\cos\theta + \iota\sin\theta) \\ &= r \cdot w = z_0.\end{aligned}$$
   Thus, $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto. □

7. Show that $\sin, \cos\colon \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and real cosine which were bounded by 1).

   Solution. We prove that cos is surjective. A similar method works for sin. Recall that
   $$\cos(z) = \frac{1}{2}\left(e^{\iota z} + e^{-\iota z}\right).$$
   Let $z_0 \in \mathbb{C}$. As before, it suffices to show that $\cos(z) = z_0$ for some $z \in \mathbb{C}$. Consider the quadratic equation
   $$\frac{1}{2}\left(t + \frac{1}{t}\right) = z_0 \qquad (\dagger)$$
   Rearranging this gives us
   $$t^2 - 2z_0 t + 1 = 0.$$
   Since the above is a (non-constant) complex polynomial, it has a complex root t_0 (by FTA). Moreover, note that $t_0 \neq 0$. By the previous question, there exists $z' \in \mathbb{C}$ satisfying $e^{z'} = t_0$. Considering $z = z'/\iota$, we see that $e^{\iota z} = t_0$. Plugging $t_0 = e^{\iota z}$ in ($\dagger$) gives us
   $$\cos(z) = z_0,$$
   as desired. □

8. Show that for any complex number z, $\cos^2(z) + \sin^2(z) = 1$.

Solution. Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \cos^2(z) + \sin^2(z) - 1.$$

Note that f is holomorphic, and hence analytic. Since f vanishes on $\mathbb{R}$ and $\mathbb{R}$ is not discrete, f must vanish everywhere, by the Identity Theorem. $\square$

§3. Week 3

17th August, 2021

1. Show that the Cauchy-Riemann equations take the form
$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$
in polar coordinates.

Solution. We use the same method shown in the slides while deriving the (original) Cauchy-Riemann equations. We first write
$$f(r, \theta) = f(re^{\iota\theta}) = u(r, \theta) + \iota v(r, \theta).$$
Suppose that f is differentiable at $z_0 = r_0 e^{\iota\theta_0} \neq 0$. Then, we know that the limit
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists. We shall calculate it in two ways:

(a) Fix $\theta = \theta_0$ and let $r \to r_0$. Then, we get
$$\begin{aligned} f'(z_0) &= \lim_{r \to r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{\iota\theta_0}(r - r_0)} + \iota \frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{\iota\theta_0}(r - r_0)} \right\} \\ &= e^{-\iota\theta_0} \lim_{r \to r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + \iota \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right\} \\ &= e^{-\iota\theta_0} \left(u_r(r_0, \theta_0) + \iota v_r(r_0, \theta_0) \right). \qquad (*) \end{aligned}$$

(b) Fix $r = r_0$ and let $\theta \to \theta_0$. Then, we get
$$\begin{aligned} f'(z_0) &= \lim_{\theta \to \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})} + \iota \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})} \right\} \\ &= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} + \iota \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} \right\} \qquad (**) \end{aligned}$$

We concentrate on the first term of the limit. Note that
$$\begin{aligned} &\lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} \\ &= \lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{\iota\theta} - e^{\iota\theta_0}}. \end{aligned}$$

In the product, the first term is clearly $u_\theta(r_0, \theta_0)$, after taking the limit. The second term can be calculated to be

$$\frac{1}{\iota e^{\iota\theta_0}}.$$

(Write $e^{\iota\theta}$ in terms of sin and cos, differentiate, and put it back.) A similar argument holds for the v term as well. Thus, $(**)$ transforms to

$$f'(z_0) = \frac{e^{-\iota\theta_0}}{r_0}\left(-\iota u_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)\right).$$

Equating the above with $(*)$, cancelling $e^{-\iota\theta_0}$, and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0} v_\theta(r_0, \theta_0) \quad \text{and} \quad v_r(r_0, \theta_0) = -\frac{1}{r_0} u_\theta(r_0, \theta_0),$$

as desired. $\square$

2. Prove Cauchy's Theorem assuming Cauchy Integral Formula.

   Solution. Let γ be a simple closed contour (oriented positively) and let Ω be an open set containing γ as well as its interior. Let f be holomorphic everywhere on Ω. Let z_0 be interior to γ. Now, we define

   $$g(z) := (z - z_0) \cdot f(z).$$

   Since f is holomorphic on Ω, so is g. Moreover, $g(z_0) = 0$. Applying the Cauchy Integral Formula to g, we have

   $$g(z_0) = 0 = \frac{1}{2\pi\iota} \int_\gamma \frac{g(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi\iota} \int_\gamma \frac{(z - z_0) \cdot f(z)}{z - z_0}\,\mathrm{d}z$$

   Since z_0 is interior to γ, $z - z_0$ is non-zero on all of γ. Thus, we get

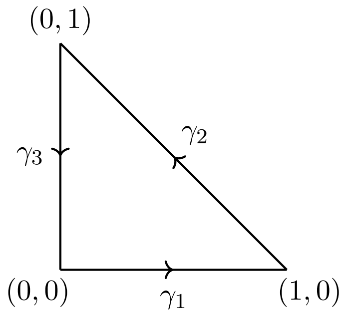
   $$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

   which is what Cauchy's Theorem tells us. $\square$

3. Let γ be the boundary of the triangle $\{0 < y < 1 - x; 0 \leq x \leq 1\}$ taken with the anticlockwise orientation. Evaluate

   (a) $\int_\gamma \Re(z)\,\mathrm{d}z$,

   (b) $\int_\gamma z^2\,\mathrm{d}z$.



Solution.

(a) Note that we may compute the integrals along γ_1, γ_2, and γ_3 individually and then add them. Along γ_3, we have

$$\int_{\gamma_3} \Re(z)\,\mathrm{d}z = \int_{\gamma_3} 0\,\mathrm{d}z = 0.$$

Along γ_1, we parameterise the curve as

$$\gamma_1(t) = t + 0\iota, \quad \text{for } t \in [0, 1].$$

Then, $\gamma_1'(t) = 1 + 0\iota$. Thus,

$$\begin{aligned}\int_{\gamma_1} \Re(z)\,\mathrm{d}z &= \int_0^1 \Re(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}t \\ &= \int_0^1 t\,\mathrm{d}t \\ &= \frac{1}{2}.\end{aligned}$$

Along γ_2, we parameterise the curve as

$$\gamma_2(t) = 1 - t + \iota t \quad \text{for } t \in [0, 1].$$

Then, $\gamma_2'(t) = -1 + \iota$. Thus,

$$\begin{aligned}\int_{\gamma_2} \Re(z)\,\mathrm{d}z &= \int_0^1 \Re(\gamma_2(t))\gamma_2'(t)\,\mathrm{d}t \\ &= \int_0^1 (1 - t)(1 - \iota)\,\mathrm{d}t \\ &= \frac{\iota - 1}{2}.\end{aligned}$$

Thus,

$$\int_{\gamma} \Re(z)\,\mathrm{d}z = \int_{\gamma_1} \Re(z)\,\mathrm{d}z + \int_{\gamma_2} \Re(z)\,\mathrm{d}z + \int_{\gamma_3} \Re(z)\,\mathrm{d}z = \boxed{\frac{\iota}{2}}.$$

Alternate. Let $\gamma(t) = x(t) + \iota y(t)$ be a parameterisation of the entire curve, where $t \in [a, b]$. We then have

$$
\begin{aligned}
\int_\gamma \Re(z)\,\mathrm{d}z &= \int_a^b x(t) \cdot (x'(t) + \iota y'(t))\,\mathrm{d}t \\
&= \int_\gamma x\,\mathrm{d}x + \iota \int_\gamma x\,\mathrm{d}y \\
&= \iint_{\mathrm{Int}(\gamma)} 0\,\mathrm{d}(x, y) + \iota \iint_{\mathrm{Int}(\gamma)} 1\,\mathrm{d}(x, y) \\
&= \iota \mathrm{Area}(\gamma) = \boxed{\frac{\iota}{2}}.
\end{aligned}
$$

In going from the single integral to the double integral, we have used Green's Theorem.

(b) Note that z^2 admits a primitive on $\mathbb{C}$ and γ is a closed curve. Thus,

$$\int_\gamma z^2\,\mathrm{d}z = \boxed{0}.$$

□

4. Compute $\displaystyle\int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z$. (Assume that the integral is in the clockwise sense).

Solution. Note that the contour of integration does not enclose -1. Thus, we define $f\colon \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ as

$$f(z) = \frac{2z-1}{z+1}.$$

Note that f is holomorphic on $\mathbb{C} \setminus \{-1\}$. Moreover, γ and its interior lie completely within $\mathbb{C} \setminus \{-1\}$. Thus, using the Cauchy integral formula, we have

$$2\pi\iota f(1) = \int_{|z-1|=1} \frac{f(z)}{z-1}\,\mathrm{d}z = \int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z,$$

which is precisely the integral we wish to calculate. Thus,

$$\int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z = 2\pi\iota f(1) = \boxed{\pi\iota}.$$

□

5. Show that if γ is a simple closed curve traced counterclockwise, then the integral $\displaystyle\int_\gamma \overline{z}\,\mathrm{d}z$ equals $2\iota\mathrm{Area}(\gamma)$. Evaluate $\displaystyle\int_\gamma \overline{z}^m\,\mathrm{d}z$ over a circle γ centered at the origin.

Solution. Suppose $\gamma(t) = x(t) + \iota y(t)$ for $t \in [a, b]$. Then,

$$\begin{aligned}
\int_\gamma \overline{z}\,\mathrm{d}z &= \int_a^b \overline{\gamma(t)}\gamma'(t)\,\mathrm{d}t \\
&= \int_a^b (x(t) - \iota y(t))(x'(t) + \iota y'(t))\,\mathrm{d}t \\
&= \int_a^b (x(t)x'(t) + y(t)y'(t))\,\mathrm{d}t + \iota \int_a^b (x(t)y'(t) - y(t)x'(t))\,\mathrm{d}t \\
&= \int_\gamma (x\mathrm{d}x + y\mathrm{d}y) + \iota \int_\gamma (x\mathrm{d}y - y\mathrm{d}x).
\end{aligned}$$

Now, we recall Green's Theorem which said that

$$\int_\gamma (M\mathrm{d}x + N\mathrm{d}y) = \iint_{\mathrm{Int}(\gamma)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}(x, y)$$

if γ is a (nice enough) closed curve oriented counterclockwise. Here, $\mathrm{Int}(\gamma)$ denotes the "interior" of γ. Thus, we have

$$\begin{aligned}
\int_\gamma \overline{z}\,\mathrm{d}z &= \iint_{\mathrm{Int}(\gamma)} (0 - 0)\,\mathrm{d}(x, y) + \iota \iint_{\mathrm{Int}(\gamma)} (1 - (-1))\,\mathrm{d}(x, y) \\
&= 2\iota \iint_{\mathrm{Int}(\gamma)} 1\,\mathrm{d}(x, y) \\
&= 2\iota \mathrm{Area}(\gamma).
\end{aligned}$$

For the second part, we parameterise the circle as

$$\gamma(t) = re^{\iota t} \quad \text{for } t \in [0, 2\pi],$$

where $r > 0$ is arbitrary. We have

$$\gamma'(t) = \iota re^{\iota t} = \iota\gamma(t).$$

Thus,

$$\begin{aligned}
\int_\gamma \overline{z}^m\,\mathrm{d}z &= \int_0^{2\pi} \overline{(\gamma(t))}^m \cdot \gamma'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \overline{(\gamma(t))}^{m-1} \cdot \overline{\gamma(t)} \cdot \gamma'(t)\,\mathrm{d}t \\
&= \iota \int_0^{2\pi} \overline{(\gamma(t))}^{m-1} \cdot |\gamma(t)|^2\,\mathrm{d}t \\
&= \iota r^2 \int_0^{2\pi} r^{m-1}e^{-\iota(m-1)t}\,\mathrm{d}t
\end{aligned}$$

The above integral is 0 whenever $m \neq 1$. When $m = 1$, we have

$$\int_0^{2\pi} 1 \, \mathrm{d}t = 2\pi.$$

Thus,

$$\int_\gamma \overline{z}^m \, \mathrm{d}z = \begin{cases} 2\pi\iota r^2 & m = 1, \\ 0 & m \neq 1. \end{cases}$$

□

6. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ be the (strict) open right half plane. Construct a non-constant function f which is holomorphic on $\mathbb{H}$ and satisfies $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$.

   Solution. We define

$$f(z) := \sin\left(\frac{\pi}{z}\right).$$

   Since $0 \notin \mathbb{H}$, we conclude that f is a composition of holomorphic functions, and hence is holomorphic on $\mathbb{H}$. Moreover, for any $n \in \mathbb{N}$, we have

$$f\left(\frac{1}{n}\right) = \sin(n\pi) = 0.$$

   Lastly, f is non-constant since

$$f(2) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0.$$

□

7. Let f be a holomorphic function on $\mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$. Show that f is constant.

   Solution. Note that f is holomorphic and hence continuous. Thus, we have

$$\begin{aligned} f(0) &= f\left(\lim_{n\to\infty} \frac{1}{n}\right) \\ &= \lim_{n\to\infty} f\left(\frac{1}{n}\right) \\ &= \lim_{n\to\infty} 0 \\ &= 0. \end{aligned}$$

   Now, we see that f is zero on

$$S := \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

   However, S is not discrete. To see this, note that $0 \in S$, and given any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \delta$. Thus, for any $\delta > 0$, $B_\delta(0) \cap S$ contains a point other than 0. Now, we use the Identity Theorem to conclude that f is identically zero, and in particular, constant. □

8. Expand $\dfrac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Solution. Let $f(z)$ be the expression in the question. We may compute the power by computing $f^{(n)}(0)$. However, if we are able to find a power series by some other method, we may directly use that since power series expansion is unique. Note that

$$\frac{1}{1+2z^2} = 1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots$$

for $\left|2z^2\right| < 1$ for $|z| < \dfrac{1}{\sqrt{2}}$. Moreover, the above series diverges for $|z| > \dfrac{1}{\sqrt{2}}$. Thus, the power series of f is given by

$$\begin{aligned} f(z) &= (1+z)\left(1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots\right) \\ &= (1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots) + z(1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots) \\ &= 1 + z - 2z^2 - 2z^3 + 4z^4 + 4z^5 - 8z^6 - 8z^7 + \cdots \end{aligned}$$

for $|z| < \dfrac{1}{\sqrt{2}}$. Moreover, multiplying with a non-zero finite power series does not change the radius of convergence. Thus, the radius of convergence remains $\boxed{\dfrac{1}{\sqrt{2}}}$.

More concisely, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = (-2)^{\lfloor n/2 \rfloor}$. □