MA 205: Complex Analysis
Tutorial Solutions

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Contents

Note: Many of these solutions are either inspired by, or in some cases directly taken from Aryaman Maithani's tutorial solutions for last year's offering of this course.

§1. Week 1

3rd August, 2021

<u>Notation</u>: We use $\mathbb{C}[x]$ to denote the set of all polynomials in x with complex coefficients. $\mathbb{R}[x]$ is defined similarly.

1. Show that a real polynomial that is irreducible has degree at most two, i.e, if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$. ($a_n \neq 0$, of course)

Solution. We consider two cases. First, suppose $f(x) \in \mathbb{R}[x]$ has a real root, x_0, and let $h(x) := (x - x_0)$. Since $x_0 \in \mathbb{R}$, $h(x) \in \mathbb{R}[x]$. Moreover, we can write

$$f(x) = g(x)h(x)$$

for some $g(x) \in \mathbb{R}[x]$. (Why must g be a real polynomial?) Also, since $\deg f(x) \geq 3$ and $\deg h(x) = 1$, we have that $\deg g(x) \geq 2$. Thus, g and h are two non-constant real polynomials satisfying $f(x) = g(x)h(x)$.

Now, suppose that $f(x)$ has no real root. We may also view $f(x)$ as a polynomial in $\mathbb{C}[x]$. By FTA, we know that $f(x)$ has a complex root $x_0 \in \mathbb{C}$. By assumption, we have that $x_0 \notin \mathbb{R}$, and thus $x_0 \neq \overline{x}_0$.

> **Claim.** $f(\overline{x}_0) = 0$.
>
> *Proof.* We have
>
> $$\begin{aligned} f(\overline{x}_0) &= a_0 + a_1\overline{x}_0 + \cdots + a_n(\overline{x}_0)^n && \overline{z^n} = \overline{z}^n \\ &= a_0 + a_1\overline{x}_0 + \cdots + a_n\overline{x_0^n} && a_i \in \mathbb{R} \text{ and thus, } a_i = \overline{a_i} \\ &= \overline{a_0} + \overline{a_1}\,\overline{x}_0 + \cdots + \overline{a_n}\,\overline{x_0^n} && \overline{z_1 z_2 + z_3} = \overline{z}_1\,\overline{z}_2 + \overline{z}_3 \\ &= \overline{f(x_0)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$
>
> $\square$

Thus, x_0 and $\overline{x}_0$ are two distinct roots of $f(x)$. Define $g(x) := (x - x_0)(x - \overline{x}_0)$. A priori, we have $g(x) \in \mathbb{C}[x]$. However, note that

$$(x - x_0)(x - \overline{x}_0) = x^2 - (2\Re x_0)x + |x_0|^2 \in \mathbb{R}[x].$$

Thus, $g(x)$ is in fact a real polynomial. Since x_0 and $\overline{x}_0$ are distinct, we see that $g(x)$ divides $f(x)$ in $\mathbb{C}[x]$. (Why?) Thus,

$$f(x) = g(x)h(x)$$

for some $h(x) \in \mathbb{C}[x]$. Again, since $f(x)$ and $g(x)$ are both real polynomials, so is $f = h(x)$. Moreover, since $\deg f(x) \geq 3$ and $\deg g(x) = 2$, we have $\deg h(x) \geq 1$, and we are done. $\square$

2. Show that a non-constant polynomial $f(z_1, z_2)$ in complex variables z_1 and z_2 with complex coefficients, has infinitely many roots in $\mathbb{C}^2$.

Solution. Before we prove this, we first prove the following useful Lemma.

Lemma. A complex polynomial of degree n has exactly n roots, counted with multiplicity. In particular, all nonzero complex polynomials have finitely many roots.

Proof. Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree n. We prove this via induction on n. When $n = 1$, $f(x) = a_0 + a_1 x$ for some $a_0, a_1 \in \mathbb{C}$ with $a_1 \neq 0$. We have

$$\begin{aligned} & f(x) = 0 \\ \iff & a_0 + a_1 x = 0 \\ \iff & a_1 x = -a_0 \\ \iff & x = -\frac{a_0}{a_1}. \end{aligned}$$

Thus, $f(x)$ has exactly 1 root.

We now assume that an n-degree polynomial $g(x) \in \mathbb{C}[x]$ has exactly n roots (counted with multiplicity). Let $f(x) \in \mathbb{C}[x]$ have degree $n+1$. By FTA, $f(x)$ has a root $x_0 \in \mathbb{C}$. We may thus write

$$f(x) = (x - x_0) g(x),$$

for some n-degree polynomial $g(x) \in \mathbb{C}[x]$. Now, we have

$$f(x) = 0 \iff x = x_0 \text{ or } g(x) = 0.$$

By assumption, the latter happens for exactly n values of x. Thus, $f(x)$ has exactly $n+1$ roots counted with multiplicity. The second statement follows from the fact that any polynomial has finite degree. $\square$

Since $f(z_1, z_2)$ is non-constant at least one of z_1 or z_2 must "appear" in $f(z_1, z_2)$. Without loss of generality, suppose that z_2 appears in $f(z_1, z_2)$. We may write

$$f(z_1, z_2) = \sum_{k=0}^{n} f_k(z_1) \cdot z_2^k$$

where $n \geq 1$ and $f_k(z_1) \in \mathbb{C}[z_1]$. Moreover, $f_n \neq 0$, and thus, $f_n(z_1)$ has only finitely many roots (possibly zero). Thus, there are infinitely many $\alpha \in \mathbb{C}$ such that $f_n(\alpha) \neq 0$. Since, $n \geq 1$, we have that $f(\alpha, z_2) \in \mathbb{C}[z_2]$ is non-constant for all these infinitely many α. By FTA, for each such α, there exists $\beta \in \mathbb{C}$ such that $f(\alpha, \beta) = 0$. Thus, there are infinitely many roots of $f(z_1, z_2)$ in $\mathbb{C}^2$ (since it contains all these pairs (α, β) as α takes on infinitely many values). $\square$

3. Show that the complex plane minus a countable set is path-connected.

   Solution. Let $S \subset \mathbb{C}$ be countable. We must show that $\mathbb{C} \setminus S$ is path-connected. Let $z_1, z_2 \in \mathbb{C} \setminus S$ and $z_1 \neq z_2$. Let f be the line segment joining z_1 to z_2, and let g be a semicircular arc joining z_1 to z_2. For every $\lambda \in [0, 1]$, we define

   $$\sigma_\lambda(t) := \lambda f(t) + (1 - \lambda)g(t) \quad \forall\, t \in [0, 1]$$

   > **Claim.**
   >
   > (a) σ_λ is a path in $\mathbb{C}$,
   >
   > (b) $\sigma_\lambda(0) = z_1$ and $\sigma_\lambda(1) = z_2$ for all $\lambda \in [0, 1]$, and
   >
   > (c) if $\lambda_1 \neq \lambda_2$ and $t \in (0, 1)$, then $\sigma_{\lambda_1}(t) \neq \sigma_{\lambda_2}(t)$.
   >
   > *Proof.* We leave the proof for (a) and (b) as simple exercises. To show (c), we first note that for $t \in (0, 1)$, $f(t) \neq g(t)$. Now, let $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 \neq \lambda_2$. Suppose $\sigma_{\lambda_1}(t) = \sigma_{\lambda_2}(t)$. We then have
   >
   > $$\begin{aligned} &\lambda_1 f(t) + (1 - \lambda_1)g(t) = \lambda_2 f(t) + (1 - \lambda_2)g(t) \\ \implies &(\lambda_1 - \lambda_2)f(t) = (\lambda_1 - \lambda_2)g(t). \end{aligned}$$
   >
   > Since $\lambda_1 \neq \lambda_2$, we get $f(t) = g(t)$, a contradiction. Intuitively, this means that the images of all these paths are disjoint, barring the start and end points. $\square$

   Since $[0, 1]$ is uncountable (we assume this without proof), and the images are disjoint (by claim (c)), we have that the set $\{\sigma_\lambda \mid \lambda \in [0, 1]\}$ is uncountable. Since the set S is only countable, there exists some $\lambda_0 \in [0, 1]$ such that $\sigma_{\lambda_0}(t) \notin S$ for all $t \in [0, 1]$. In other words, σ_{λ_0} is a path in $\mathbb{C} \setminus S$ starting at z_1 and ending at z_2. Since z_1, z_2 were arbitrary, we are done. $\square$

4. Check for real differentiability and holomorphicity:

   (a) $f(z) = c$

   (b) $f(z) = z$

   (c) $f(z) = z^n$, $n \in \mathbb{Z}$

   (d) $f(z) = \Re z$

   (e) $f(z) = |z|$

   (f) $f(z) = |z|^2$

(g) $f(z) = \overline{z}$

(h) $f(z) = \begin{cases} \frac{z}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Solution. Some of these are trivial and hence omitted.

(a) Real differentiable and holomorphic.

(b) Real differentiable and holomorphic.

(c) For $n \geq 0$, real differentiable and holomorphic. Since holomorphicity implies real differentiability, we only check for holomorphicity. Let $z_0 \in \mathbb{C}$ be arbitrary. We must check for the existence of the following limit:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

For $z \neq z_0$, we know that

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

Since the limit of the RHS exists as $z \to z_0$, we are done.

For $n < 0$, the function is defined on $\mathbb{C} \setminus \{0\}$. On $\mathbb{C} \setminus \{0\}$, $f(z)$ is non-zero. Thus, $\frac{1}{f}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ by the first case since $\frac{1}{f(z)} = z^{-n}$ and $-n > 0$. Thus, $f(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

(d) Real differentiable but not holomorphic. We may write f as

$$f(x + \iota y) = x + 0\iota.$$

Thus, $u(x, y) = x$ and $v(x, y) = 0$. f is clearly real differentiable since all the partial derivatives (of u and v) exist everywhere and are continuous. However, since $u_x(x_0, y_0) = 1$ and $v_y(x_0, y_0) = 0$ for all $(x_0, y_0) \in \mathbb{R}^2$, the CR equations do not hold. Hence, f is complex differentiable nowhere, and thus, not holomorphic.

(e) $|z|$ is real differentiable precisely on $\mathbb{C} \setminus \{0\}$ and complex differentiable nowhere. We may write

$$f(x + \iota y) = \sqrt{x^2 + y^2} + 0\iota$$

giving us $u(x, y) = \sqrt{x^2 + y^2}$, and $v(x, y) = 0$. On $\mathbb{R}^2 \setminus \{(0, 0)\}$, all partial derivatives exist and are continuous, whereas u_x and u_y fail to exist at $(0, 0)$. Thus, $f(z)$ is real differentiable on $\mathbb{C} \setminus \{0\}$. Moreover, this shows that $f(z)$ is not complex differentiable at 0 since it's not even real differentiable there. Everywhere else, $v_x = v_y = 0$, but at least one of u_x, u_y is non-zero, violating the CR equations. Thus, $f(z)$ is complex differentiable nowhere.

(f) $|z|^2$ is real differentiable everywhere and complex differentiable precisely at 0. As a result, it is holomorphic nowhere. As before, we have $u(x, y) = x^2 + y^2$, and $v(x, y) = 0$. Since all partial derivatives exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. Note that

$$\begin{aligned} u_x(x, y) &= 2x \quad & u_y(x, y) &= 2y \\ v_x(x, y) &= 0 \quad & v_y(x, y) &= 0 \end{aligned}$$

Thus, the CR equations hold precisely at 0.

(g) For $f(z) = \overline{z}$, we may write

$$f(x + \iota y) = x - \iota y,$$

which gives us $u(x, y) = x$ and $v(x, y) = -y$. Since all partials exist everywhere and are continuous, $f(z)$ is real differentiable everywhere. However, note that

$$\begin{aligned} u_x(x, y) &= 1 \quad & u_y(x, y) &= 0 \\ v_x(x, y) &= 0 \quad & v_y(x, y) &= -1 \end{aligned}$$

Since $u_x(x, y) \neq v_y(x, y)$ for all $(x, y) \in \mathbb{R}^2$, we see that the CR equations do not hold anywhere and $f(z)$ is complex differentiable nowhere.

(h) f is real differentiable precisely on $\mathbb{C} \setminus \{0\}$, and complex differentiable nowhere. We may multiply and divide by $\overline{z}$ to obtain

$$u(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad v(x, y) = \frac{2xy}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$, and $u(0, 0) = v(0, 0) = 0$. Since u and v are not continuous at $(0, 0)$ (recall MA109), neither is f. Hence, f is neither real differentiable, nor complex differentiable at $0 \in \mathbb{C}$. At all other points, all partials exist and are continuous. Hence, f is real differentiable there. However, one may explicitly compute those partial derivatives and verify that the CR equations hold nowhere. Thus, f is complex differentiable nowhere. $\square$

§2. Week 2

10th August, 2021

1. If $u(X, Y)$ and $v(X, Y)$ are harmonic conjugates of each other, show that they are constant functions. (This is true iff u and v are defined on open, path-connected sets)

   Solution. Since v is a harmonic conjugate of u, we have

   $$u_X = v_Y \quad \text{and} \quad u_Y = -v_X.$$

   Since we also have that u is a harmonic conjugate of v, we get

   $$v_X = u_Y \quad \text{and} \quad v_Y = -u_X.$$

   Note that the above equalities hold for each point in the domain. Thus, we have

   $$u_X = u_Y = v_X = v_Y \equiv 0,$$

   identically. Since the domain is connected, this implies that u and v are constant.

   The following is another alternative.

   > **Lemma.** Let u be a harmonic function defined on an open, path connected set. Then, the harmonic conjugate of u is unique up to a constant.
   >
   > *Proof.* Let v and v' be two harmonic conjugates of u. It suffices to show that $(v - v')$ is a constant function. By definition, $u + \iota v$ and $u + \iota v'$ are both holomorphic, and hence satisfy the Cauchy-Riemann equations. Thus, we have
   >
   > $$u_x = v_y,\ v_x = -u_y \quad \text{and} \quad u_x = v'_y,\ v'_x = -u_y.$$
   >
   > It thus follows that
   >
   > $$(v - v')_x = (v - v')_y \equiv 0,$$
   >
   > identically. Since the domain is path-connected, this implies that $(v - v')$ is constant. □

   Now, since $v(X, Y)$ is a harmonic conjugate of $u(X, Y)$, we have that $-u(X, Y)$ is a harmonic conjugate of $v(X, Y)$ (Why?). Since we also have that $u(X, Y)$ is a harmonic conjugate of $v(X, Y)$, it follows that u and $-u$ differ only by a constant, and hence u must itself be constant. The same holds for v. □

2. Show that $u = XY - 3X^2Y - Y^3$ is harmonic and find its harmonic conjugate.

   Solution. Consider the function

   $$f(Z) = \frac{1}{2}Z^2 + Z^3,$$

defined on $\mathbb{C}$. Writing $Z = X + \iota Y$, where $X, Y \in \mathbb{R}$, we see that the function $u(X, Y)$ is the *imaginary* part of $f(Z)$. Since $f(Z)$ is holomorphic on $\mathbb{C}$, u is harmonic. Moreover, its harmonic conjugate is give by

$$v(X, Y) = -\Re f(Z) = \frac{1}{2}(Y^2 - X^2) + 3XY^2 - X^3.$$

Note that we require a minus sign since we obtained that $u(X, Y)$ was the imaginary, and not the real, part of a holomorphic function.

Note that the above method required us to intelligently guess the function $f(Z)$. However, if this is difficult to observe, we have the following 'standard' way of solving this problem. Some simple calculations give us

$$u_{XX}(X_0, Y_0) = 6Y_0 \quad \text{and} \quad u_{YY}(X_0, Y_0) = -6Y_0,$$

which gives us that $u_{XX} + u_{YY} \equiv 0$, verifying that u is harmonic. Note that $u_X = v_Y$, giving us $v_Y = Y + 6XY$. Integrating with respect to Y gives us

$$v = \frac{1}{2}Y^2 + 3XY^2 + g(X)$$

for some function g. We also have the relation $v_X = -u_Y$. Computing each individually gives us

$$3Y^2 + g'(X) = -X - 3X^2 + 3Y^2.$$

Thus, up to a constant, we get

$$g(X) = -\frac{1}{2}X^2 - X^3.$$

Finally, we get

$$v = \frac{1}{2}Y^2 + 3XY^2 - \frac{1}{2}X^2 - X^3.$$

□

3. Find the radius of convergence of the following power series:

   (a) $\displaystyle\sum_{n=0}^{\infty} nz^n$,

   (b) $\displaystyle\sum_{p \text{ prime}} z^p$,

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$.

Solution. We shall use the ratio test in the first and third parts, and the root test in the second part.

(a) Note that we have

$$\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$$

and thus,

$$R = \alpha^{-1} = 1.$$

(b) We may rewrite the series as

$$\sum_{n=1}^{\infty} a_n z^n,$$

where

$$a_n := \begin{cases} 0 & n \text{ is not a prime,} \\ 1 & n \text{ is a prime.} \end{cases}$$

Since there are infinitely many primes, given any $n \in \mathbb{N}$, there exists $m \geq n$ with $a_m = 1$. Thus, we clearly have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

Thus, the root test gives us

$$R = \alpha^{-1} = 1.$$

(c) We have

$$a_n = \frac{n!}{n^n}.$$

Thus,

$$\begin{aligned} \alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} \\ &= \frac{1}{e}. \end{aligned}$$

Since the above limit exists, we may apply the ratio test to get

$$R = \alpha^{-1} = e.$$

□

4. Show that $L > 1$ in the ratio test (Lecture 3 slides) does not necessarily imply that the series is divergent.

Solution. Consider the sequence (a_n) defined by

$$a_{2n} = \frac{1}{n^2} \quad \text{and} \quad a_{2n-1} = \frac{1}{n^3}$$

Since $\sum n^{-2}$ and $\sum n^{-3}$ converge (via the integral test), we have that $\sum a_n$ converges. However, note that

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq \limsup_{n\to\infty} \left|\frac{a_{2n}}{a_{2n-1}}\right| = \limsup_{n\to\infty} n = \infty.$$

Thus $L > 1$ clearly, but the series is convergent. Hence, we have showed that even $L = \infty$ is not sufficient to conclude the divergence of a series. $\square$

5. Construct an infinitely differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ which is non-zero but vanishes outside a bounded set. Show that there are no holomorphic functions which satisfy this property.

Solution. We saw in the lectures that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & x \leq 0, \\ e^{-1/x} & x > 0 \end{cases}$$

is infinitely differentiable. Using this function, we construct $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := g(x)g(1-x).$$

f is clearly infinitely differentiable. Moreover, $f(x) = 0$ if $x \leq 0$ or $x \geq 1$. Thus, f vanishes outside the bounded set $(0, 1)$. It remains to show that f is non-zero. Indeed, we have that

$$f\left(\frac{1}{2}\right) = \left(g\left(\frac{1}{2}\right)\right)^2 = e^{-4} \neq 0.$$

Suppose $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function which vanishes outside some bounded set K. We now show that f is identically zero. For this, recall the Identity Theorem:

> **Theorem**
>
> Let $\Omega \subset \mathbb{C}$ be a domain. If $f\colon \Omega \to \mathbb{C}$ is analytic, then either f is identically zero, or the zeros of f form a discrete set.

Although the above theorem is for analytic functions, we shall show later in the course that holomorphic functions are indeed analytic. Since the set K is bounded, there exists $M > 0$ such that

$$|z| \leq M \text{ for all } z \in K.$$

Choosing the point $z_0 = M + 2$, we see that f vanishes in a neighbourhood of radius 1 around z_0. Since $\mathbb{C}$ is open and path-connected (and hence a domain), and since any open disc is not discrete, we conclude from the above theorem that f must be identically zero on $\mathbb{C}$. $\square$

6. Show that $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto.

   Solution. Let $z_0 \in \mathbb{C}^\times$. It suffices to show that $\exp(z) = z_0$ for some $z \in \mathbb{C}$. Since z_0 is non-zero, $r := |z_0| \neq 0$. Thus,
   $$w := \frac{z_0}{r}$$
   has modulus 1. Thus,
   $$w = x_0 + \iota y_0$$
   for some $(x_0, y_0) \in \mathbb{R}^2$ satisfying $x_0^2 + y_0^2 = 1$. Hence, $x_0 = \cos\theta$ and $y_0 = \sin\theta$ for some $\theta \in [0, 2\pi)$. We now define
   $$z := \log(r) + \iota\theta,$$
   where the above log is the real-valued log. Thus, we have
   $$\begin{aligned}\exp(z) = \exp(\log(r) + \iota\theta) &= \exp(\log(r)) \cdot \exp(\iota\theta)\\ &= r \cdot (\cos\theta + \iota\sin\theta)\\ &= r \cdot w = z_0.\end{aligned}$$
   Thus, $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is onto. □

7. Show that $\sin, \cos\colon \mathbb{C} \to \mathbb{C}$ are surjective. (In particular, note the difference with real sine and real cosine which were bounded by 1).

   Solution. We prove that cos is surjective. A similar method works for sin. Recall that
   $$\cos(z) = \frac{1}{2}\left(e^{\iota z} + e^{-\iota z}\right).$$
   Let $z_0 \in \mathbb{C}$. As before, it suffices to show that $\cos(z) = z_0$ for some $z \in \mathbb{C}$. Consider the quadratic equation
   $$\frac{1}{2}\left(t + \frac{1}{t}\right) = z_0 \qquad (\dagger)$$
   Rearranging this gives us
   $$t^2 - 2z_0 t + 1 = 0.$$
   Since the above is a (non-constant) complex polynomial, it has a complex root t_0 (by FTA). Moreover, note that $t_0 \neq 0$. By the previous question, there exists $z' \in \mathbb{C}$ satisfying $e^{z'} = t_0$. Considering $z = z'/\iota$, we see that $e^{\iota z} = t_0$. Plugging $t_0 = e^{\iota z}$ in ($\dagger$) gives us
   $$\cos(z) = z_0,$$
   as desired. □

8. Show that for any complex number z, $\cos^2(z) + \sin^2(z) = 1$.

Solution. Consider the function $f\colon \mathbb{C} \to \mathbb{C}$ defined as

$$f(z) = \cos^2(z) + \sin^2(z) - 1.$$

Note that f is holomorphic, and hence analytic. Since f vanishes on $\mathbb{R}$ and $\mathbb{R}$ is not discrete, f must vanish everywhere, by the Identity Theorem. $\square$

§3. Week 3

17th August, 2021

1. Show that the Cauchy-Riemann equations take the form

$$u_r = \frac{1}{r}v_\theta \text{ and } v_r = -\frac{1}{r}u_\theta$$

in polar coordinates.

Solution. We use the same method shown in the slides while deriving the (original) Cauchy-Riemann equations. We first write

$$f(r, \theta) = f(re^{\iota\theta}) = u(r, \theta) + \iota v(r, \theta).$$

Suppose that f is differentiable at $z_0 = r_0 e^{\iota\theta_0} \neq 0$. Then, we know that the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. We shall calculate it in two ways:

(a) Fix $\theta = \theta_0$ and let $r \to r_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{r \to r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{\iota\theta_0}(r - r_0)} + \iota \frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{\iota\theta_0}(r - r_0)} \right\} \\ &= e^{-\iota\theta_0} \lim_{r \to r_0} \left\{ \frac{u(r, \theta_0) - u(r_0, \theta_0)}{r - r_0} + \iota \frac{v(r, \theta_0) - v(r_0, \theta_0)}{r - r_0} \right\} \\ &= e^{-\iota\theta_0} \left(u_r(r_0, \theta_0) + \iota v_r(r_0, \theta_0) \right). \end{aligned} \tag{$*$}$$

(b) Fix $r = r_0$ and let $\theta \to \theta_0$. Then, we get

$$\begin{aligned} f'(z_0) &= \lim_{\theta \to \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})} + \iota \frac{v(r_0, \theta) - v(r_0, \theta_0)}{r_0(e^{\iota\theta} - e^{\iota\theta_0})} \right\} \\ &= \frac{1}{r_0} \lim_{\theta \to \theta_0} \left\{ \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} + \iota \frac{v(r_0, \theta) - v(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} \right\} \end{aligned} \tag{$**$}$$

We concentrate on the first term of the limit. Note that

$$\begin{aligned} &\lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{e^{\iota\theta} - e^{\iota\theta_0}} \\ &= \lim_{\theta \to \theta_0} \frac{u(r_0, \theta) - u(r_0, \theta_0)}{\theta - \theta_0} \frac{\theta - \theta_0}{e^{\iota\theta} - e^{\iota\theta_0}}. \end{aligned}$$

In the product, the first term is clearly $u_\theta(r_0, \theta_0)$, after taking the limit. The second term can be calculated to be

$$\frac{1}{\iota e^{\iota\theta_0}}.$$

(Write $e^{\iota\theta}$ in terms of sin and cos, differentiate, and put it back.) A similar argument holds for the v term as well. Thus, $(**)$ transforms to

$$f'(z_0) = \frac{e^{-\iota\theta_0}}{r_0}\left(-\iota u_\theta(r_0, \theta_0) + v_\theta(r_0, \theta_0)\right).$$

Equating the above with $(*)$, cancelling $e^{-\iota\theta_0}$, and comparing the real and imaginary parts, we get

$$u_r(r_0, \theta_0) = \frac{1}{r_0}v_\theta(r_0, \theta_0) \quad \text{and} \quad v_r(r_0, \theta_0) = -\frac{1}{r_0}u_\theta(r_0, \theta_0),$$

as desired. □

2. Prove Cauchy's Theorem assuming Cauchy Integral Formula.

Solution. Let γ be a simple closed contour (oriented positively) and let Ω be an open set containing γ as well as its interior. Let f be holomorphic everywhere on Ω. Let z_0 be interior to γ. Now, we define

$$g(z) := (z - z_0) \cdot f(z).$$

Since f is holomorphic on Ω, so is g. Moreover, $g(z_0) = 0$. Applying the Cauchy Integral Formula to g, we have

$$g(z_0) = 0 = \frac{1}{2\pi\iota}\int_\gamma \frac{g(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi\iota}\int_\gamma \frac{(z - z_0)\cdot f(z)}{z - z_0}\,\mathrm{d}z$$

Since z_0 is interior to γ, $z - z_0$ is non-zero on all of γ. Thus, we get

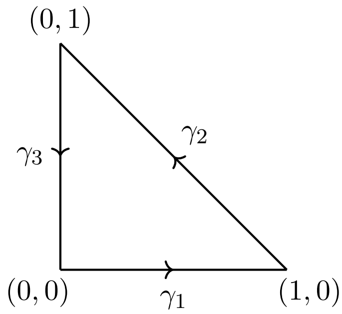
$$\int_\gamma f(z)\,\mathrm{d}z = 0,$$

which is what Cauchy's Theorem tells us. □

3. Let γ be the boundary of the triangle $\{0 < y < 1 - x; 0 \le x \le 1\}$ taken with the anticlockwise orientation. Evaluate

   (a) $\int_\gamma \Re(z)\,\mathrm{d}z$,

   (b) $\int_\gamma z^2\,\mathrm{d}z$.



Solution.

(a) Note that we may compute the integrals along γ_1, γ_2, and γ_3 individually and then add them. Along γ_3, we have

$$\int_{\gamma_3} \Re(z)\,\mathrm{d}z = \int_{\gamma_3} 0\,\mathrm{d}z = 0.$$

Along γ_1, we parameterise the curve as

$$\gamma_1(t) = t + 0\iota, \quad \text{for } t \in [0, 1].$$

Then, $\gamma_1'(t) = 1 + 0\iota$. Thus,

$$\begin{aligned}\int_{\gamma_1} \Re(z)\,\mathrm{d}z &= \int_0^1 \Re(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}t \\ &= \int_0^1 t\,\mathrm{d}t \\ &= \frac{1}{2}.\end{aligned}$$

Along γ_2, we parameterise the curve as

$$\gamma_2(t) = 1 - t + \iota t \quad \text{for } t \in [0, 1].$$

Then, $\gamma_2'(t) = -1 + \iota$. Thus,

$$\begin{aligned}\int_{\gamma_2} \Re(z)\,\mathrm{d}z &= \int_0^1 \Re(\gamma_2(t))\gamma_2'(t)\,\mathrm{d}t \\ &= \int_0^1 (1-t)(1-\iota)\,\mathrm{d}t \\ &= \frac{\iota - 1}{2}.\end{aligned}$$

Thus,

$$\int_{\gamma} \Re(z)\,\mathrm{d}z = \int_{\gamma_1} \Re(z)\,\mathrm{d}z + \int_{\gamma_2} \Re(z)\,\mathrm{d}z + \int_{\gamma_3} \Re(z)\,\mathrm{d}z = \boxed{\frac{\iota}{2}}.$$

(b) Note that z^2 admits a primitive on $\mathbb{C}$ and γ is a closed curve. Thus,

$$\int_\gamma z^2 \,\mathrm{d}z = \boxed{0}.$$

$\square$

4. Compute $\displaystyle\int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z$. (Assume that the integral is in the clockwise sense).

Solution. Note that the contour of integration does not enclose -1. Thus, we define $f\colon \mathbb{C}\setminus\{-1\} \to \mathbb{C}$ as

$$f(z) = \frac{2z-1}{z+1}.$$

Note that f is holomorphic on $\mathbb{C}\setminus\{-1\}$. Moreover, γ and its interior lie completely within $\mathbb{C}\setminus\{-1\}$. Thus, using the Cauchy integral formula, we have

$$2\pi\iota f(1) = \int_{|z-1|=1} \frac{f(z)}{z-1}\,\mathrm{d}z = \int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z,$$

which is precisely the integral we wish to calculate. Thus,

$$\int_{|z-1|=1} \frac{2z-1}{z^2-1}\,\mathrm{d}z = 2\pi\iota f(1) = \boxed{\pi\iota}.$$

$\square$

5. Show that if γ is a simple closed curve traced counterclockwise, then the integral $\displaystyle\int_\gamma \overline{z}\,\mathrm{d}z$ equals $2\iota\mathrm{Area}(\gamma)$. Evaluate $\displaystyle\int_\gamma \overline{z}^m\,\mathrm{d}z$ over a circle γ centered at the origin.

Solution. Suppose $\gamma(t) = x(t) + \iota y(t)$ for $t \in [a,b]$. Then,

$$\begin{aligned}
\int_\gamma \overline{z}\,\mathrm{d}z &= \int_a^b \overline{\gamma(t)}\gamma'(t)\,\mathrm{d}t \\
&= \int_a^b (x(t) - \iota y(t))(x'(t) + \iota y'(t))\,\mathrm{d}t \\
&= \int_a^b (x(t)x'(t) + y(t)y'(t))\,\mathrm{d}t + \iota\int_a^b (x(t)y'(t) - y(t)x'(t))\,\mathrm{d}t \\
&= \int_\gamma (x\mathrm{d}x + y\mathrm{d}y) + \iota\int_\gamma (x\mathrm{d}y - y\mathrm{d}x).
\end{aligned}$$

Now, we recall Green's Theorem which said that

$$\int_\gamma (M\mathrm{d}x + N\mathrm{d}y) = \iint_{\mathrm{Int}(\gamma)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,\mathrm{d}(x,y)$$

if γ is a (nice enough) closed curve oriented counterclockwise. Here, $\mathrm{Int}(\gamma)$ denotes the "interior" of γ. Thus, we have

$$\begin{aligned}\int_\gamma \overline{z}\,\mathrm{d}z &= \iint_{\mathrm{Int}(\gamma)} (0-0)\,\mathrm{d}(x,y) + \iota \iint_{\mathrm{Int}(\gamma)} (1-(-1))\,\mathrm{d}(x,y) \\ &= 2\iota \iint_{\mathrm{Int}(\gamma)} 1\,\mathrm{d}(x,y) \\ &= 2\iota\mathrm{Area}(\gamma).\end{aligned}$$

For the second part, we parameterise the circle as

$$\gamma(t) = re^{\iota t} \quad \text{for } t \in [0, 2\pi],$$

where $r > 0$ is arbitrary. We have

$$\gamma'(t) = \iota re^{\iota t} = \iota\gamma(t).$$

Thus,

$$\begin{aligned}\int_\gamma \overline{z}^m\,\mathrm{d}z &= \int_0^{2\pi} \overline{(\gamma(t))}^m \cdot \gamma'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} \overline{(\gamma(t))}^{m-1} \cdot \overline{\gamma(t)} \cdot \gamma'(t)\,\mathrm{d}t \\ &= \iota \int_0^{2\pi} \overline{(\gamma(t))}^{m-1} \cdot |\gamma(t)|^2\,\mathrm{d}t \\ &= \iota r^2 \int_0^{2\pi} r^{m-1}e^{-\iota(m-1)t}\,\mathrm{d}t\end{aligned}$$

The above integral is 0 whenever $m \neq 1$. When $m = 1$, we have

$$\int_0^{2\pi} 1\,\mathrm{d}t = 2\pi.$$

Thus,

$$\int_\gamma \overline{z}^m\,\mathrm{d}z = \begin{cases} 2\pi\iota r^2 & m = 1, \\ 0 & m \neq 1. \end{cases}$$

□

6. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ be the (strict) open right half plane. Construct a non-constant function f which is holomorphic on $\mathbb{H}$ and satisfies $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$.

Solution. We define

$$f(z) := \sin\left(\frac{\pi}{z}\right).$$

Since $0 \notin \mathbb{H}$, we conclude that f is a composition of holomorphic functions, and hence is holomorphic on $\mathbb{H}$. Moreover, for any $n \in \mathbb{N}$, we have

$$f\left(\frac{1}{n}\right) = \sin(n\pi) = 0.$$

Lastly, f is non-constant since

$$f(2) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0.$$

□

7. Let f be a holomorphic function on $\mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$. Show that f is constant.

Solution. Note that f is holomorphic and hence continuous. Thus, we have

$$\begin{aligned} f(0) &= f\left(\lim_{n\to\infty} \frac{1}{n}\right) \\ &= \lim_{n\to\infty} f\left(\frac{1}{n}\right) \\ &= \lim_{n\to\infty} 0 \\ &= 0. \end{aligned}$$

Now, we see that f is zero on

$$S := \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

However, S is not discrete. To see this, note that $0 \in S$, and given any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \delta$. Thus, for any $\delta > 0$, $B_\delta(0) \cap S$ contains a point other than 0. Now, we use the Identity Theorem to conclude that f is identically zero, and in particular, constant. □

8. Expand $\dfrac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Solution. Let $f(z)$ be the expression in the question. We may compute the power by computing $f^{(n)}(0)$. However, if we are able to find a power series by some other method, we may directly use that since power series expansion is unique. Note that

$$\frac{1}{1+2z^2} = 1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots$$

for $\left|2z^2\right| < 1$ for $|z| < \dfrac{1}{\sqrt{2}}$. Moreover, the above series diverges for $|z| > \dfrac{1}{\sqrt{2}}$. Thus, the power series of f is given by

$$\begin{aligned} f(z) &= (1+z)\left(1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots\right) \\ &= (1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots) + z(1 - 2z^2 + (2z^2)^2 - (2z^2)^3 + \cdots) \\ &= 1 + z - 2z^2 - 2z^3 + 4z^4 + 4z^5 - 8z^6 - 8z^7 + \cdots \end{aligned}$$

for $|z| < \dfrac{1}{\sqrt{2}}$. Moreover, multiplying with a non-zero finite power series does not change the radius of convergence. Thus, the radius of convergence remains $\boxed{\dfrac{1}{\sqrt{2}}}$.

More concisely, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = (-2)^{\lfloor n/2 \rfloor}$. □

Addendum.

Since a lot of alternate solutions were discussed, I am adding all of those here.

3a. *Solution.* Let $\gamma(t) = x(t) + \iota y(t)$ be a parameterisation of the entire curve, where $t \in [a, b]$. We then have

$$\begin{aligned}\int_\gamma \Re(z)\,\mathrm{d}z &= \int_a^b x(t) \cdot (x'(t) + \iota y'(t))\,\mathrm{d}t \\ &= \int_\gamma x\,\mathrm{d}x + \iota \int_\gamma x\,\mathrm{d}y \\ &= \iint_{\mathrm{Int}(\gamma)} 0\,\mathrm{d}(x, y) + \iota \iint_{\mathrm{Int}(\gamma)} 1\,\mathrm{d}(x, y) \\ &= \iota \mathrm{Area}(\gamma) = \boxed{\frac{\iota}{2}}.\end{aligned}$$

In going from the single integral to the double integral, we have used Green's Theorem. □

3a. *Solution.* Note that

$$\Re(z) = \frac{z + \overline{z}}{2}.$$

Let γ be the given curve. We then have

$$\int_\gamma \Re(z)\,\mathrm{d}z = \frac{1}{2}\int_\gamma z\,\mathrm{d}z + \frac{1}{2}\int_\gamma \overline{z}\,\mathrm{d}z.$$

Note that the first integral is 0 since z admits a primitive. Moreover, Q5 tells us that the second integral must be 2ι times the area enclosed by the curve (the triangle, in this case), which is just $\frac{1}{2}$. Thus,

$$\int_\gamma \Re(z)\,\mathrm{d}z = \boxed{\frac{\iota}{2}}.$$

□

5 We show another method for the second part. Let the circle γ have radius $r > 0$. Notice that over the circle, we have

$$\overline{z} = \frac{r^2}{z}.$$

Thus, we have

$$\int_\gamma \overline{z}^m \, \mathrm{d}z = \int_\gamma r^{2m} z^{-m} \, \mathrm{d}z$$

Moreover, for $m \neq 1$, z^{-m} admits a primitive and hence the integral is zero. For $m = 1$, one may use Cauchy Integral Formula, or simply recognise that the integral reduces to the one already computed in the first part. In either case, we have

$$\int_\gamma \overline{z}^m \, \mathrm{d}z = \begin{cases} 2\pi\iota r^2 & m = 1, \\ 0 & m \neq 1. \end{cases}$$

§4. Week 4

28th August, 2021

1. Show that there is a strict inequality

$$\left| \int_{|z|=R} \frac{z^n}{z^m - 1} \, \mathrm{d}z \right| < \frac{2\pi R^{n+1}}{R^m - 1},$$

where $R > 1$, $m \geq 1$, and $n \geq 0$.

Solution. We first look at a stronger version of the ML inequality.

Theorem 1: The Stronger ML Inequality

Let $f\colon \Omega \to \mathbb{C}$ be a continuous function and let $\gamma\colon [a, b] \to \Omega$ be a curve. Let $M > 0$ be such that

$$|f(\gamma(t))| \leq M, \quad \text{for all } t \in [a, b].$$

Also, suppose that $|f(\gamma(t))| < M$ for some $t \in [a, b]$. Then,

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| < ML,$$

where L denotes the length of the curve. That is, if $|f| < M$ holds even for one point, the inequality becomes strict.

Proof. Note that

$$\int_a^b [M - |f(\gamma(t))|] \left|\gamma'(t)\right| \mathrm{d}t \geq 0,$$

since the integrand is non-negative. Moreover, recall from MA109 that the integral is zero **iff** the integrand is identically zero. (We use continuity here.) Since we know that the integrand is not identically zero (here we use the fact that γ' is zero at only finitely many points, if any), it follows that

$$\int_a^b [M - |f(\gamma(t))|] \left|\gamma'(t)\right| \mathrm{d}t > 0.$$

Since

$$\int_a^b M\left|\gamma'(t)\right| \mathrm{d}t = ML,$$

the theorem follows. □

Now, we consider the function

$$f(z) = \frac{z^n}{z^m - 1}$$

defined on $\Omega := \{z \in \mathbb{C} \mid |z| > 1\}$. For a point satisfying $|z| = R$, we have

$$\begin{aligned}
\left|\frac{z^n}{z^m - 1}\right| &= \frac{R^n}{|z^m - 1|} \\
&\le \frac{R^n}{||z|^m - 1|} \\
&= \frac{R^n}{R^m - 1}.
\end{aligned}$$

Thus, we may take $M = \dfrac{R^n}{R^m - 1}$. Also, considering $z = R\exp\left(\frac{\iota\pi}{m}\right)$ shows that the inequality is indeed strict at one point. Thus, we may appeal to The Stronger ML Inequality to conclude that

$$\begin{aligned}
\left|\int_{|z|=R} \frac{z^n}{z^m - 1}\,\mathrm{d}z\right| &\le \int_{|z|=R} \left|\frac{z^n}{z^m - 1}\right| \mathrm{d}z \\
&< M(2\pi R) \\
&= \frac{2\pi R^{n+1}}{R^m - 1}.
\end{aligned}$$

□

2. A power series with center at the origin and positive radius of convergence has a sum $f(z)$. It is known that $f(\overline{z}) = \overline{f(z)}$ for all values z within the disc of convergence. What conclusions can you draw about the power series?

   Solution. Conclusion: All the coefficients of the power series are real. We now justify this.

   We show that $f^{(k)}(0)$ is real for all $k \in \mathbb{N} \cup \{0\}$. This suffices since we know that the coefficients are given by $f^{(k)}(0)/k!$. In what follows, we assume that x and x_0 are real, and within the (open) disc of convergence. For real x, we have

$$f(x) = f(\overline{x}) = \overline{f(x)}.$$

   That is, $f(x)$ is real whenever x is real. We now wish to show that $f^{(k)}(x)$ is real for real x for all $k \ge 1$. It suffices to show this for f'. (Induction!) Since we know that f' exists within the disc, we may compute the limit along the real axis. Fix a real x_0 within the disc. We note that

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \in \mathbb{R}}} \frac{f(x) - f(x_0)}{x - x_0}.$$

   Since the above expression is a quotient of two purely real expressions, we see that the limit is real. Thus, we are done. Note that we knew beforehand that all the higher derivatives of f do exist. Hence, we can apply the inductive process by computing the limit along the real axis each time. □

3. The following is called the Taylor series with remainder.

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^N}{N!}f^N(0) + \frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N f^{N+1}(tz)\,\mathrm{d}t.$$

Use this to prove the following inequalities.

(a) $\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \le \frac{|z|^{N+1}}{(N+1)!}$ where $\Re(z) \le 0$.

(b) $\left| \cos(z) - \sum_{i=0}^{N} \frac{(-1)^i z^{2i}}{2i!} \right| \le \frac{|z|^{2N+2}\cosh R}{(2N+2)!}$ where $|\Im(z)| \le R$.

Solution.

(a) Note that the sum subtracted is the first $N+1$ terms of the Taylor expansion of e^z. Thus, the quantity within the modulus is simply

$$\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N \exp(tz)\,\mathrm{d}t.$$

We have used the fact that $\exp^{(N+1)} = \exp$. Also, we have $|\exp(z)| = \exp(\Re(z))$. Thus, we get

$$\begin{aligned}\left|\int_0^1 (1-t)^N \exp(tz)\,\mathrm{d}t\right| &\le \int_0^1 \left|(1-t)^N \exp(tz)\right| \mathrm{d}t \\ &= \int_0^1 (1-t)^N \exp(t\Re(z))\,\mathrm{d}t \\ &\le \int_0^1 (1-t)^N\,\mathrm{d}t \\ &= \frac{1}{N+1}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left| e^z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| &= \left|\frac{z^{N+1}}{(N+1)!}\int_0^1 (1-t)^N \exp(tz)\,\mathrm{d}t\right| \\ &\le \frac{|z|^{N+1}}{(N+1)!}\frac{1}{N+1} \\ &\le \frac{|z|^{N+1}}{(N+1)!}.\end{aligned}$$

(b) Note that the sum subtracted is the first $2N+2$ terms of the Taylor expansion of $\cos(z)$. Thus, the quantity within the modulus is simply

$$\frac{z^{2N+2}}{(2N+2)!}\int_0^1 (1-t)^{2N+1} \cos^{(2n+2)}(tz)\,\mathrm{d}t.$$

Also, we have

$$\begin{aligned}
|\cos(z)| &= \frac{1}{2}\left|e^{\iota z} + e^{-\iota z}\right| \\
&\leq \frac{1}{2}\left(\left|e^{\iota z}\right| + \left|e^{-\iota z}\right|\right) \\
&= \frac{1}{2}\left(e^{y} + e^{-y}\right) \\
&= \cosh y.
\end{aligned}$$

Since $\cos^{(2N+2)} = \cos$ or $-\cos$, we have in either case that

$$\left|\cos^{(2N+2)}(tz)\right| \leq |\cosh ty|.$$

Note that $\cosh y$ is an increasing function of $|y|$ (for real y.) Thus, we have

$$|\cosh ty| \leq |\cosh y|$$

for all $t \in [0, 1]$. Moreover, since $|y| \leq R$, we have

$$|\cosh ty| \leq |\cosh y| \leq \cosh R$$

for all $t \in [0, 1]$. Thus, we have

$$\begin{aligned}
\left|\int_0^1 (1-t)^{2N+1} \cos^{(2n+2)}(tz)\,\mathrm{d}t\right| &\leq \int_0^1 (1-t)^{2N+1} \left|\cos^{(2n+2)}(tz)\right| \mathrm{d}t \\
&\leq \int_0^1 (1-t)^{2N+1} \cosh R \,\mathrm{d}t \\
&= \frac{\cosh R}{2N+2}.
\end{aligned}$$

As before, the desired inequality follows. □

4. By computing $\displaystyle\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\,\mathrm{d}z$, show that $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\,\mathrm{d}\theta = \frac{2\pi}{4^n} \cdot \frac{(2n)!}{(n!)^2}$.

   Solution. We have the following "generalised" Cauchy integral formula, which states that

   $$\int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w = \frac{2\pi\iota}{n!} f^{(n)}(z_0)$$

   where f is a function that is holomorphic on an open disc $B_R(z_0)$ and $r < R$. In this question, we take $z_0 = 0$, and $r = 1$. We take

   $$f(z) = (z^2 + 1)^{2n}$$

which is holomorphic on all of $\mathbb{C}$. (We may thus take $R = 2$.) Using the above formula gives us

$$\begin{aligned}\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \,\mathrm{d}z &= \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}} \,\mathrm{d}z \\ &= \frac{2\pi\iota}{(2n)!} f^{(2n)}(0).\end{aligned}$$

We now wish to compute $f^{(2n)}(0)$. We know that $f^{(2n)}(0)/(2n)!$ is precisely the coefficient of z^{2n} in the expansion of $(z^2+1)^{2n}$. We use binomial expansion to see that

$$(z^2+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{n} z^{2k}.$$

Thus, the coefficient of z^{2n} is $\binom{2n}{n}$ and the integral becomes

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \,\mathrm{d}z = 2\pi\iota \binom{2n}{n}.$$

Now, we use the standard parameterisation $z(t) = e^{\iota t}$ for $t \in [0, 2\pi]$. The integral then becomes

$$\begin{aligned}\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z} \,\mathrm{d}z &= \int_0^{2\pi} (2\cos t)^{2n} \frac{1}{e^{\iota t}} \iota e^{\iota t} \,\mathrm{d}t \\ &= 4^n \iota \int_0^{2\pi} \cos^{2n} t \,\mathrm{d}t.\end{aligned}$$

Equating the two gives us the desired result. □

5. Let $f(z)$ be an entire function. Show that $f(z)$ is a polynomial of degree at most n if and only if there exists a positive real constant C such that $|f(z)| \le C|z|^n$ for all z with $|z|$ sufficiently large.

Solution. Let f be an entire function satisfying $|f(z)| \le C|z|^n$ for some positive constant C and all z with $|z| > R_0$. We note that $|f|$ is bounded on the set $\{z \in \mathbb{C} \mid |z| \le R_0\}$ since f is continuous and the latter set is compact. Let M be a bound on $|f|$ on this set. Pick $R > 0$. On $B_R(0)$, we then have

$$|f(z)| \le \max\{M, CR^n\} \le M + CR^n.$$

Now, for $m > n$, Cauchy's estimate gives us

$$\left|f^{(m)}(0)\right| \le \frac{m! \cdot (M + CR^n)}{R^m}.$$

Since the above holds for arbitrary $R > 0$, we may let $R \to \infty$. Since $m > n$, we see that $f^{(m)}(0) = 0$ for all $m > n$. Now, since f is entire we may write

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k,$$

for all $z \in \mathbb{C}$. Hence, f is a polynomial of degree at most n.

Conversely, suppose $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree at most n. Note that

$$\frac{f(z)}{z^n} \to a_n \text{ as } z \to \infty.$$

Thus, there must exist $R > 0$ such that for all z with $|z| > R$, we have

$$\left| \frac{f(z)}{z^n} \right| \leq |a_n| + 1.$$

Thus, $C := |a_n| + 1$ works. $\square$

6. Let f and g be entire, non-vanishing functions with $\left(\frac{f'}{f}\right)\left(\frac{1}{n}\right) = \left(\frac{g'}{g}\right)\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$. Show that g is a non-zero scalar multiple of f.

Solution. We define

$$h := \frac{g}{f}.$$

Since f is non-vanishing, h is defined on all of $\mathbb{C}$. We also note that h is non-vanishing since g is non-vanishing. Moreover, h is entire since f and g are entire. Now, we have

$$\frac{h'}{h} = \frac{g'f - gf'}{gf} = \frac{g'}{g} - \frac{f'}{f}.$$

Thus, we see that

$$\left(\frac{h'}{h}\right)\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Since h is non-vanishing, we get

$$h'\left(\frac{1}{n}\right) = 0 \text{ for all } n \in \mathbb{N}.$$

Utilising the result from Question 7, Week 3, we conclude that $h' \equiv 0$. Since $\mathbb{C}$ is path-connected, h is a constant, say c. We thus have

$$\frac{g}{f} = c \implies g = c \cdot f.$$

Moreover, $c \neq 0$ since g is non-vanishing. $\square$

§5. Week 5

31st August, 2021

1. Locate and classify the singularities of the following:

   (a) $\dfrac{\sin(1/z)}{1+z^4}$,

   (b) $\dfrac{z^5\sin(1/z)}{1+z^4}$,

   (c) $\dfrac{1}{\sin(1/z)}$,

   (d) $e^{\frac{1}{z}}$.

   Solution.

   (a) Note that the numerator is not defined when $z = 0$ and the denominator is not defined whenever $z^4 + 1 = 0$. Thus, the set of singularities is

$$S = \left\{0, \frac{1}{\sqrt{2}}(\pm 1 \pm \iota)\right\}.$$

   Since there are only finitely many singularities, each of them is isolated. If $z_0 \in S \setminus \{0\}$, it is easy to see that

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

   Thus, all non-zero singularities are poles. Now, we show that $z = 0$ is an *essential singularity*. That is, it is neither removable nor a pole. It suffices to show that $\lim\limits_{z\to 0} f(z)$ does not exist, neither as a finite complex number, nor as ∞.

   Approaching 0 along the positive imaginary axis, we have

$$\begin{aligned}\lim_{y\to 0^+} f(z) &= \lim_{y\to 0^+} \frac{\sin(1/\iota y)}{1+(\iota y)^4}\\ &= \frac{1}{2}\lim_{y\to 0^+}(e^{1/y} - e^{-1/y}).\end{aligned}$$

   Note that the above limit exists as ∞, so 0 is not a removable singularity. Approaching along the real axis, we have that sin is bounded and the denominator tends to 1, so 0 is not a pole either.

   (b) This follows the same approach as the first one. All the singularities (and their types) remain the same.

(c) Here, we have a problem if $z = 0$ or $\sin(1/z) = 0$. Thus, the set of singularities is given by

$$S = \{0\} \cup \left\{ \frac{1}{n\pi} \mid n \in \mathbb{Z} \setminus \{0\} \right\}.$$

Note that 0 is not an isolated singularity since every neighbourhood of 0 contains some point of the form $1/(n\pi)$. We thus do not classify 0. All other singularities, however, are isolated. To see this, let $z_0 \in S \setminus \{0\}$. Then,

$$z_0 = \frac{1}{n\pi}$$

for some $n \in \mathbb{Z} \setminus \{0\}$. Now, choose

$$\delta := \min \left\{ \left| \frac{1}{n\pi} - \frac{1}{(n+1)\pi} \right|, \left| \frac{1}{n\pi} - \frac{1}{(n-1)\pi} \right| \right\}.$$

(If $n = \pm 1$, then just choose the other value). For the above choice of δ, the punctured neighbourhood $B_\delta(z_0) \setminus \{z_0\}$ contains no other point of S.

Now, we show that all of these isolated singularities are poles. To see this, we note that

$$\lim_{z \to z_0} \frac{z - z_0}{\sin(1/z)}$$

exists (as a finite number) and is nonzero for any $z_0 \in S \setminus \{0\}$. Thus, all these points are poles.

(d) The only problematic point here is 0. We show that 0 is an essential singularity. Note that as $z \to 0$ along the negative real axis, we have that $e^{1/z} \to 0$. However, as $z \to 0$ along the positive real axis, we have $e^{1/z} \to \infty$. Thus, $\lim\limits_{z \to 0} e^{1/z}$ does not exist, neither as a finite complex number, nor as ∞.

□

2. Construct a meromorphic function on $\mathbb{C}$ with infinitely many poles.

Solution. We define $f : \mathbb{C} \setminus \{n\pi \mid n \in \mathbb{Z}\} \to \mathbb{C}$ as

$$f(z) := \frac{1}{\sin z}.$$

It is easy to note that f has infinitely many singularities, which are given precisely by the set $S := \{n\pi \mid n \in \mathbb{Z}\}$, and all these are isolated. Moreover, for each $z_0 \in S$, we have

$$\lim_{z \to z_0} = \frac{1}{f(z)} = \lim_{z \to z_0} \sin z = 0.$$

Thus, all the singularities are also poles. Hence, f is meromorphic on $\mathbb{C}$ with infinitely many poles. □

3. Find Laurent expansions for the function $f(z) = \dfrac{2(z-1)}{z^2-2z-3}$ valid on the annuli

   (a) $0 \leq |z| < 1$,

   (b) $1 < |z| < 3$,

   (c) $3 < |z|$.

Solution. Note that

$$\frac{2(z-1)}{z^2-2z-3} = \frac{1}{z-3} + \frac{1}{z+1}.$$

In each part, we expand each fraction as a Laurent series such that the series converges on that particular disc.

(a) Here, we may write

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n,$$

and

$$\frac{1}{z+1} = \sum_{n=0}^{\infty}(-z)^n.$$

Thus, the Laurent series in the annulus $|z| < 1$ is given as the sum of the above two.

(b) Here, we may write

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n,$$

and

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \sum_{n=0}^{\infty}(-z)^{-n-1}.$$

Thus, the Laurent series in the annulus $1 < |z| < 3$ is given as the sum of the above two.

(c) Here, we may write

$$\frac{1}{z-3} = \frac{1}{z}\frac{1}{1-\frac{3}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{-n},$$

and

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \sum_{n=0}^{\infty}(-z)^{-n-1}.$$

Thus, the Laurent series in the annulus $1 < |z| < 3$ is given as the sum of the above two.

$\square$

4. Let Ω be a domain in $\mathbb{C}$ and let $z_0 \in \Omega$. Suppose that z_0 is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighbourhood of z_0. Show that $f(z)$ has a removable singularity at z_0.

   Solution. Fix $\delta > 0$ such that f is bounded and holomorphic on the punctured disc of radius δ centered at z_0. (such a δ exists since z_0 is an isolated singularity.) Define $g(z) := f(z)(z - z_0)$ on this punctured disc. Then, g is holomorphic on this punctured disc. Moreover, we have

   $$\lim_{z \to z_0} g(z) = 0$$

   since f is bounded on the punctured disc. Thus, by RRST, we see that z_0 is a removable singularity of g. Furthermore, defining $g(z_0) := 0$ makes it holomorphic on $B_\delta(z_0)$. (This again follows from RRST.) Thus we may expand g on $B_\delta(z_0)$ as

   $$g(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

   Thus, for $z \in B_\delta(z_0) \setminus \{z_0\}$, we have that

   $$f(z) = a_1 + a_2(z - z_0) + \cdots .$$

   Thus, z_0 is a removable singularity of f since defining $f(z_0) := a_1$ makes it holomorphic on $B_\delta(z_0)$. $\square$

5. A complex-valued function $f(z)$ on $\mathbb{C}$ is called doubly periodic if there exist linearly independent vectors $v, w \in \mathbb{C}$ over $\mathbb{R}$ such that $f(z + v) = f(z)$ and $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. Show that any doubly periodic entire function is constant.

   Solution. Suppose f is doubly periodic and entire. It is easy to see that for all $z \in \mathbb{C}$, we have

   $$f(z + nv) = f(z) = f(z + mw) \quad \text{for all } m, n \in \mathbb{Z}.$$

   Since v and w are linearly independent over $\mathbb{R}$, we have that every $z \in \mathbb{C}$ can be uniquely written as $z = xv + yw$ where $x, y \in \mathbb{R}$. Let $\{x\}$ denote the fractional part of x and let $[x]$ denote its integer part. We then have

   $$\begin{aligned} f(z) &= f(xv + yw) \\ &= f([x]v + \{x\}v + [y]w + \{y\}w) \\ &= f(\{x\}v + \{y\}w). \end{aligned}$$

   Now, let $S := \{xv + yw \mid x, y \in [0, 1]\}$ denote the parallelogram with vertices $0, v, w$, and $v + w$. We note that $\{x\}v + \{y\}w \in S$ for all $x, y \in \mathbb{R}$. Hence, the set of values f takes is decided entirely by the set of values it takes on S. Since S is compact and f is continuous, f must be bounded on S, and thus bounded on all of $\mathbb{C}$. By Liouville's Theorem, we conclude that f is constant. $\square$

6. By transforming into an integral over the unit circle, show that

$$\int_0^{2\pi} \frac{1}{a^2+1-2a\cos\theta}\,\mathrm{d}\theta = -\frac{2\pi}{1-a^2},$$

where $a>1$. Also compute the value for $a<1$.

Solution. Assuming $0<a\neq 1$, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a^2-2a\cos\theta+1}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{1}{a^2-a(e^{-\iota\theta}+e^{\iota\theta})+e^{-\iota\theta}\cdot e^{\iota\theta}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{(a-e^{\iota\theta})(a-e^{-\iota\theta})}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{e^{\iota\theta}}{(a-e^{\iota\theta})(ae^{\iota\theta}-1)}\,\mathrm{d}\theta \\
&= \frac{1}{\iota}\int_0^{2\pi} \frac{\iota e^{\iota\theta}}{(a-e^{\iota\theta})(ae^{\iota\theta}-1)}\,\mathrm{d}\theta \\
&= \frac{1}{\iota}\int_{|z|=1} \frac{1}{(a-z)(az-1)}\,\mathrm{d}z \\
&= -\frac{1}{a\iota}\int_{|z|=1} \frac{1}{(z-a)(z-1/a)}\,\mathrm{d}z.
\end{aligned}$$

Note that for both cases $a>1$ and $a<1$, the integrand has exactly one pole within the unit circle. For $a>1$, the pole is at $1/a$. Using Cauchy's Integral Formula, we get

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a^2-2a\cos\theta+1}\,\mathrm{d}\theta &= -\frac{1}{a\iota}\cdot 2\pi\iota\frac{1}{1/a-a} \\
&= -\frac{2\pi}{1-a^2}.
\end{aligned}$$

For $a<1$, the pole is at a, which gives us

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a^2-2a\cos\theta+1}\,\mathrm{d}\theta &= -\frac{1}{a\iota}\cdot 2\pi\iota\frac{1}{a-1/a} \\
&= -\frac{2\pi}{a^2-1}
\end{aligned}$$

□

7. Show that if $a_1,\ldots,a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, then for each $1\leq k\leq n$, we have the formula

$$\prod_{j\neq k}(a_k-a_j) = P'(a_k).$$

Solution. Since $P(z)$ is monic and we know all its factors. we may write

$$P(z) = (z - a_1) \cdots (z - a_n).$$

Fix $k \in \{1, \ldots, n\}$. Note also that $P(a_k) = 0$. Thus, we may write

$$P(z) - P(a_k) = (z - a_1) \cdots (z - a_n).$$

If $z \neq a_k$, we may divide by $z - a_k$ to get

$$\frac{P(z) - P(a_k)}{z - a_k} = \prod_{j \neq k} (z - a_j).$$

Letting $z \to a_k$ gives us the answer. $\square$

Solution. Alternatively, we may write

$$P(z) = (z - a_k) P_k(z)$$

where

$$P_k(z) := \prod_{j \neq k} (z - a_j).$$

Differentiating both sides using the product rule, we get

$$P'(z) = (z - a_k) P_k'(z) + P_k(z).$$

Substituting $z = a_k$ gives us the desired result. $\square$