Stochastic Optimisation

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Abstract

Lecture Notes for the course EE 736 : Stochastic Optimisation taught in Spring 2022 by Prof. Vivek Borkar.

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Chapter 1

Stochastic Approximation

§1.1. The Robbins-Monro Algorithm

The basic problem we consider is to solve $h(\mathbf{x}) = 0$ given noisy measurements of h. That is, we are given access to a black box that, on input $\mathbf{x} \in \mathbb{R}^d$, gives as output $h(\mathbf{x}) + \text{noise}$. To this end, we have the *Robbins-Monro algorithm.*

Robbins-Monro Algorithm. Starting with $\mathbf{x}_0 \in \mathbb{R}^d$, do:

$$\mathbf{x}(n+1) := \mathbf{x}(n) + a(n)\left[h(\mathbf{x}(n)) + M(n+1)\right], \quad n \geq 0.$$

Here, the (non-negative) stepsize sequence (or learning parameter) $\{a(n)\}$ satisfies

$$\sum_n a(n) = \infty \quad \text{and} \quad \sum_n a(n)^2 < \infty.$$

A typical example of such a stepsize sequence is $\frac{1}{n}, \frac{1}{n \log n}, \frac{1}{n^{2/3}}$, and so on. Further, we make the following assumptions.

1. $h \colon \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz, that is, $\exists L \geq 0$ such that

   $$\|h(\mathbf{x}) - h(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

2. $\{M(n)\}$ is a square integrable martingale difference sequence. That is, for

   $$\mathcal{F}_n := \sigma\left(\mathbf{x}_0, M_m, m \leq n\right), n \geq 0$$

   we have

   $$\mathbb{E}\left[\|M(n)\|^2\right] < \infty$$

and in addition, we have that it is uncorrelated with the past. That is,

$$\mathbb{E}\left[M_i(n+1) \mid \mathcal{F}_n\right] = 0 \quad \forall i.$$

Furthermore, we assume that for some $K > 0$,

$$\mathbb{E}\left[\|M(n+1)\|^2 \mid \mathcal{F}_n\right] \leq K(1 + \|\mathbf{x}(n)\|^2) \quad \forall n \geq 0.$$

In particular, if

$$\sup_n \|\mathbf{x}(n)\| < \infty \quad \text{a.s.},$$

then,

$$\sup_n \mathbb{E}\left[\|M(n+1)\|^2 \mid \mathcal{F}_n\right] < \infty \quad \text{a.s.}.$$

This algorithm is actually more general than it appears. Suppose the algorithm is

$$\mathbf{x}(n+1) := \mathbf{x}(n) + a(n) f\left(\mathbf{x}(n), \xi(n+1)\right), \quad n \geq 0$$

where $\{\xi(n)\}$ are i.i.d. This is often how most recursive algorithms are stated. The above algorithm can be put into the form of Robbins-Monro algorithm by choosing

$$\begin{aligned} h(\mathbf{x}) &:= \mathbb{E}\left[f(\mathbf{x}, \xi(n))\right] \\ &= \mathbb{E}\left[f(\mathbf{x}(n), \xi(n+1)) \mid \mathbf{x}(n) = \mathbf{x}\right] \\ &= \mathbb{E}\left[f(\mathbf{x}(n), \xi(n+1)) \mid \mathcal{F}_n\right] \end{aligned}$$

and

$$M(n+1) := f(\mathbf{x}(n), \xi(n+1)) - h(\mathbf{x}(n)).$$

A common example of Robbins-Monro algorithm is stochastic gradient descent, where we set $h = -\nabla f$. Robbins-Monro algorithm also finds uses in many reinforcement learning algorithms. Some advantages of the Robbins-Monro algorithm are listed as follows.

1. It typically requires a small amount of computation and memory per iterate.
2. It is incremental in nature, that is, it makes only a small change in the current iterate at each step.
3. The slowly decreasing stepsize $\{a(n)\}$ captures the exploration-exploitation trade-off.
4. It averages out the noise, which can be thought of as a generalisation of the Strong Law of Large Numbers.

Another common approach to solving the same problem is the ODE (Ordinary Differential Equation) approach, which treats the iterate as a noisy discretisation of the ODE

$$\dot{\mathbf{x}}(t) = h(\mathbf{x}(t)).$$

Recall the Euler scheme for solving this ODE:

$$\mathbf{x}(n+1) := \mathbf{x}(n) + ah(\mathbf{x}(n)), \quad n \geq 0,$$

where $a > 0$ is a small discrete time step. Thus the Robbins-Monro algorithm can be viewed as a Euler scheme to approximate the ODE with slowly decreasing time steps $\{a(n)\}$ and measurement noise. With this in mind, we have the following interpretation of the Robbins-Monro conditions on the step size $\{a(n)\}$.

1. $\sum_n a(n) = \infty$ ensures that the entire time axis is covered. This is essential because we want to track the asymptotic behaviour of the ODE.
2. $\sum_n a(n)^2 < \infty$ ensures that the approximation of the ODE gets better with time. In particular, $a(n) \to 0$ ensures that errors due to discretisation are asymptotically zero, and $\sum_n a(n)^2 < \infty$ ensures that errors due to the martingale difference noise are asymptotically zero almost surely, since multiplication by $a(n)$ reduces the conditional variance of the noise.

As an example, consider an initially empty urn to which one ball, either red or blue, is added at each time step. Let

$$\xi(n) := \mathbb{I}\{n^{\text{th}} \text{ ball is red}\} = \begin{cases} 1 & \text{if the } n^{\text{th}} \text{ ball is red, and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $S(n) := \sum_{m=1}^{n} \xi(m)$ be the total number of red balls at time n, and let $x(n) := \frac{S(n)}{n}$ be the fraction of red balls at time n. Then, we have

$$\begin{aligned} x(n+1) &= \frac{1}{n+1} \sum_{m=1}^{n+1} \xi(m) \\ &= \frac{1}{n+1} \sum_{m=1}^{n} \xi(m) + \frac{\xi(n+1)}{n+1} \\ &= \left(\frac{n}{n+1}\right) \frac{\sum_{m=1}^{n} \xi(m)}{n} + \frac{\xi(n+1)}{n+1} \\ &= \left(1 - \frac{1}{n+1}\right) x(n) + \frac{\xi(n+1)}{n+1} \\ &= x(n) + a(n)(\xi(n+1) - x(n)) \end{aligned}$$

for $a(n) := \frac{1}{n+1}$ which satisfies the Robbins-Monro conditions. Now, suppose that

$$\mathbb{P}\left(\xi(n+1) = 1 \mid \xi(m), m \le n\right) = p(x(n))$$

for some continuously differentiable function $p\colon [0,1] \to [0,1]$. Then, we have

$$\begin{aligned} x(n+1) &= x(n) + a(n)(\xi(n+1) - x(n)) \\ &= x(n) + a(n)\left[(p(x(n)) - x(n)) + (\xi(n+1) - p(x(n)))\right] \\ &= x(n) + a(n)\left[h(x(n)) + M(n+1)\right] \end{aligned}$$

for $h(x) := p(x) - x$, and $M(n+1) := \xi(n+1) - p(x(n))$. Since $\mathbb{E}\left[\xi(n+1) \mid \xi(m), m \le n\right] = p(x(n))$ for all n, we have that $\{M(n)\}$ is a martingale difference sequence. Since $|M(n)| \le 2$, the bound on conditional second moment is free. The limiting ODE is

$$\dot{x}(t) = p(x(t)) - x(t).$$

Under our hypothesis of continuous differentiability of p, this has a unique solution for any initial condition. Set $x(0) = x_0 \in [0,1]$. We have $p(0) - 0 \ge 0$, and $p(1) - 1 \le 0$. Since $x(t) \in [0,1]$ for all $t \ge 0$, $x(t)$ must converge to a point in $[0,1]$. If at x_0, we have that $p(x_0) = x_0$, then we are already at equilibrium. If not, suppose that $p(x_0) > x_0$, then $x(t)$ is increasing but bounded by 1, so it must converge. A similar argument works for $p(x_0) < x_0$. But does an equilibrium exist? The answer is yes. Since $p(0) - 0 \ge 0$, and $p(1) - 1 \le 0$, we have by continuity that there exists $x \in [0,1]$ such that $p(x) = x$. In fact, there can be more than one equilibria. An equilibrium x^* satisfies $p(x^*) = x^*$ and is stable if $p'(x^*) < 1$ and unstable if $p'(x^*) > 1$. Under some additional technicalities, we can show that $x(t)$ converges to one of the stable equilibria almost surely, and the probability of convergence to any stable equilibrium is strictly positive.

§1.2. Ordinary Differential Equations

We consider the ODE in $\mathbb{R}^d$, $d \ge 1$, given by

$$\dot{\mathbf{x}}(t) = h(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

A problem is said to be *well-posed* if

1. it has a solution,
2. the solution is unique, and
3. the solution depends continuously on problem parameters.

For ODEs, this translates to the ODE having a unique solution for all time that depends continuously on the initial condition.

We say that $h\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies a (global) Lipschitz condition if for some $L > 0$, we have

$$\|h(\mathbf{x}) - h(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

h is locally Lipschitz if for all $R > 0$, there exists an $L_R > 0$ such that

$$\|h(\mathbf{x}) - h(\mathbf{y})\| \leq L_R\|\mathbf{x} - \mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{B}_R := \left\{\mathbf{z} \in \mathbb{R}^d \mid \|\mathbf{z}\| \leq R\right\}$$

Lemma 1.2.1: Gronwall's Inequality

Suppose $0 \leq y\colon [0,T]\colon \mathbb{R}$ is differentiable and satisfies

$$y(t) \leq C + K\int_0^t y(s)\,\mathrm{d}s, \quad t \in [0,T]$$

for some $C, K > 0$. Then,

$$y(t) \leq Ce^{Kt}, \quad t \in [0,T].$$

Proof. Let $z(t) := \int_0^t y(s)\,\mathrm{d}s$, $t \geq 0$. Then,

$$\begin{aligned}
&\dot{z}(t) = y(t) \leq C + Kz(t) \\
\implies &e^{-Kt}(\dot{z}(t) - Kz(t)) \leq Ce^{-Kt} \\
\implies &\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-Kt}z(t)\right) \leq Ce^{-Kt},\ z(0) = 0.
\end{aligned}$$

Integrating both sides from 0 to t, we get

$$\begin{aligned}
&e^{-Kt}z(t) \leq \frac{C}{K}(1 - e^{-Kt}) \\
\implies &z(t) \leq \frac{C}{K}(e^{Kt} - 1)
\end{aligned}$$

Now, we have

$$\begin{aligned}
&y(t) \leq C + Kz(t) \leq C + C(e^{Kt} - 1) \\
\implies &y(t) \leq Ce^{Kt}.
\end{aligned}$$

□

Theorem 1.2.2

If h is Lipschitz, then the ODE $\{\dot{\mathbf{x}}(t) = h(\mathbf{x}(t)), \mathbf{x}(0) = \hat{\mathbf{x}}\}$ is well-posed.

Proof. We first show existence. Fix $T \in (0, 1/L)$ and a continuous function $\mathbf{x}_0 \colon [0,T] \to \mathbb{R}^d$ with $\mathbf{x}_0(0) = \hat{\mathbf{x}}$. Recursively define

$$\mathbf{x}_{n+1}(t) := \hat{\mathbf{x}} + \int_0^t h(\mathbf{x}_n(s))\,\mathrm{d}s, \quad t \in [0,T]. \tag{$\dagger$}$$

These are called *Picard iterations.* Then for $n \geq 1$, we have

$$\begin{aligned} \|\mathbf{x}_{n+1}(t) - \mathbf{x}_n(t)\| &= \left\| \int_0^t \left(h(\mathbf{x}_n(s)) - h(\mathbf{x}_{n-1}(s)) \right) \mathrm{d}s \right\| \\ &\leq \int_0^t \|h(\mathbf{x}_n(s)) - h(\mathbf{x}_{n-1}(s))\|\,\mathrm{d}s \\ &\leq L \int_0^t \|\mathbf{x}_n(s) - \mathbf{x}_{n-1}(s)\|\,\mathrm{d}s \\ &\leq LT \max_{s \in [0,T]} \|\mathbf{x}_n(s) - \mathbf{x}_{n-1}(s)\| \end{aligned}$$

Thus,

$$\max_{t \in [0,T]} \|\mathbf{x}_{n+1}(t) - \mathbf{x}_n(t)\| \leq LT \max_{t \in [0,T]} \|\mathbf{x}_n(t) - \mathbf{x}_{n-1}(t)\|$$

Applying this repeatedly, we get

$$\begin{aligned} \max_{t \in [0,T]} \|\mathbf{x}_{n+1}(t) - \mathbf{x}_n(t)\| &\leq (LT)^n \max_{t \in [0,T]} \|\mathbf{x}_1(t) - \mathbf{x}_0(t)\| \text{ for } n \geq 0 \\ &\implies \sum_{n=0}^{\infty} \max_{t \in [0,T]} \|\mathbf{x}_{n+1}(t) - \mathbf{x}_n(t)\| < \infty. \end{aligned}$$

Thus, $\mathbf{x}_n(t) = \mathbf{x}_0(t) + \sum_{m=0}^{n-1} (\mathbf{x}_{m+1}(t) - \mathbf{x}_m(t))$ converges to some $\mathbf{x}(t)$ uniformly in $t \in [0,T]$. Passing to the limit as $n \uparrow \infty$ in ($\dagger$), we have

$$\mathbf{x}(t) := \hat{\mathbf{x}} + \int_0^t h(\mathbf{x}(s))\,\mathrm{d}s, \quad t \in [0,T]$$

Thus, $\mathbf{x}$ satisfies the ODE with $\mathbf{x}(0) = \hat{\mathbf{x}}$. We repeat the above procedure for $[T, 2T], [2T, 3T]$, and so on.

We now prove uniqueness. Consider $\dot{\mathbf{x}}(t) = h(\mathbf{x}(t))$, $\dot{\mathbf{y}}(t) = h(\mathbf{y}(t))$, $t \geq 0$ with $\mathbf{x}(0) = \mathbf{y}(0)$. Then,

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq L \int_0^t \|\mathbf{x}(s) - \mathbf{y}(s)\| \,\mathrm{d}s \implies \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0 \quad \forall t \geq 0,$$

where the last implication follows from Gronwall's inequality. This concludes uniqueness. In general, for $\mathbf{x}(0) = \hat{\mathbf{x}}$ and $\mathbf{y}(0) = \hat{\mathbf{y}}$, we have

$$\begin{aligned} &\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\| + L \int_0^t \|\mathbf{x}(s) - \mathbf{y}(s)\| \,\mathrm{d}s \\ \implies &\|\mathbf{x}(t) - \mathbf{y}(t)\| \leq e^{Lt} \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|, \end{aligned}$$

by the Gronwall's inequality implying continuous dependence on the initial condition. Hence, the ODE is well-posed. $\square$

A few remarks:

1. Picard iteration is not a good computational scheme. In practice, Euler scheme is the most basic choice. Suppose h is bounded and let $a := \frac{T}{N}$ where $N \gg 1$. Now, let

   $$\mathbf{X}_N((n+1)a) := \mathbf{X}_N(na) + ah(\mathbf{X}_N(na)), \quad 0 \leq n < N.$$

   We interpolate linearly to get

   $$\mathbf{X}_N(t) := \mathbf{X}_N(na) + (t - na)h(\mathbf{X}_N(na)), \quad t \in [na, (n+1)a].$$

   Then, as $N \uparrow \infty$, $\mathbf{X}_N(t), t \in [0, T]$ converges to a solution of the ODE uniformly on $[0, T]$. This too proves the existence of a solution and needs only the continuity of h. However, uniqueness may fail. In numerical analysis, more sophisticated discretisations are available.

2. A local Lipschitz condition on h gives local well-posedness for a small time interval, but the solution may not exist for all time.

3. The linear growth condition shown below suffices for a solution to exist for all time:

   $$\|h(\mathbf{x}\| \leq K(1 + \|\mathbf{x}\|)$$

   for some $K > 0$. Then, we have

   $$\begin{aligned} &\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| + \left\| \int_0^t h(\mathbf{x}(s) \,\mathrm{d}s \right\| \leq \|\mathbf{x}(0)\| + \int_0^t K(1 + \|\mathbf{x}(s)\|) \,\mathrm{d}s \\ \implies &\|\mathbf{x}(t)\| \leq (\|\mathbf{x}(0)\| + KT)e^{Kt}, \quad t \in [0, T], \end{aligned}$$

   by Gronwall's inequality. We further note that the Lipschitz condition implies linear growth. A proof of this is left as an exercise for the reader.

4. A symmetric well-posedness theory can be developed for $t \leq 0$. Thus, for Lipschitz h, there is a unique solution for all $t \in \mathbb{R}$.

5. We also have a *discrete* Gronwall's inequality, which is proved similarly. Let $x_n \geq 0, a_n \geq 0, n \geq 0$, and $C, K > 0$ such that

$$x_{n+1} \leq C + K \sum_{m=0}^{n} a_m x_m \quad \forall n \geq 0.$$

   Then, $x_{n+1} \leq C e^{K \sum_{m=0}^{n} a_m}$ for all $n \geq 0$.

We now take a more qualitative look at ODEs. Assume well-posedness. There are two broad ways of thinking about ODEs.

1. We can think of the ODE as the graph of $t \mapsto \mathbf{x}(t) \in \mathbb{R}^d$, that is, we think of $\mathbf{x}(t)$ as a function of time. The component-wise time derivative at t is $h(\mathbf{x}(t))$.

2. We can think of the ODE as a trajectory, or a curve $\mathbf{x}(\cdot)$ in $\mathbb{R}^d$ with t as a running parameter. This is also called a phase portrait. The tangent at point $\mathbf{x}$ on the curve is $h(\mathbf{x})$. One often flips this picture around and imagines a vector $h(\mathbf{x})$ at each point $\mathbf{x}$ (a vector field) and think of trajectories as curves drawn that are tangent to the vector field at all points (integral curves).

Definition 1.2.3: Limit Sets

The ω-limit set of a trajectory $\mathbf{x}(\cdot)$ is the set of all points $\mathbf{x}$ such that $\exists\, t_n$ *uparrow*∞ such that $\mathbf{x}(t_n) \to \mathbf{x}$, that is, the set of limit points of $\mathbf{x}(t)$ as $t \uparrow \infty$. One can show that this set is closed but can be empty. The α-limit set is defined similarly for $t_n \downarrow -\infty$.

Definition 1.2.4: Invariance

A set $A \subseteq \mathbb{R}^d$ is said to be *positively invariant* if $\mathbf{x}(0) \in A \implies \mathbf{x}(t) \in A \quad \forall t \geq 0$. Negative invariance is defined similarly. A set that is both positively and negatively invariant is said to be *invariant*.

Proposition 1.2.5

The ω- and α-limit sets are invariant.

Definition 1.2.6: Liapunov Stable

An equilibrium $\mathbf{x}^*$ is said to be *Liapunov stable* if given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta \implies \|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon \quad \forall t \geq 0.$$

Definition 1.2.7: Asymptotically Stable

An equilibrium $\mathbf{x}^*$ is said to be *asymptotically stable* if it is Liapunov stable and there exists an open neighbourhood $\mathcal{O}$ of $\mathbf{x}^*$ such that

$$\mathbf{x}(0) \in \mathcal{O} \implies \mathbf{x}(t) \to \mathbf{x}^*.$$

Definition 1.2.8: Domain of Attraction

The largest positively invariant open set $\mathcal{D}$ such that

$$\mathbf{x}(0) \in \mathcal{D} \implies \mathbf{x}(t) \to \mathbf{x}^*$$

is called the *domain of attraction* of $\mathbf{x}^*$.

One sufficient condition for the above is that there exist a continuously differentiable $V \colon \mathcal{D} \to [0, \infty)$ such that

$$\lim_{\mathbf{x} \to \partial\mathcal{D}} V(\mathbf{x}) = \infty, \text{ and}$$

$$\langle \nabla V(\mathbf{x}), h(\mathbf{x}) \rangle < 0 \quad \forall \mathbf{x} \in \mathcal{D}, \mathbf{x} \neq \mathbf{x}^*.$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{x}(t)) = \langle \nabla V(\mathbf{x}(t)), h(\mathbf{x}(t)) \rangle < 0 \quad \text{when } \mathbf{x}(t) \neq \mathbf{x}^*,$$

that is, V decreases along the trajectory. Since $V \geq 0$, $\mathbf{x}(t) \to \mathbf{x}^*$. Further, if we consider $\mathcal{B}_c(\mathbf{x}^*) := \{\mathbf{x} \mid V(\mathbf{x}) \leq c\} \subseteq \mathcal{D}$ for a suitable $c > V(\mathbf{x}^*$, then

$$\mathbf{x}(0) \in \mathcal{B}_c(\mathbf{x}^*) \implies \mathbf{x}(t) \in \mathcal{B}_c(\mathbf{x}^*) \quad \forall t \geq 0.$$

Note that $\mathbf{x}^* \in \mathcal{B}_c(\mathbf{x}^*)$ and $\mathcal{B}_c(\mathbf{x}^*)$ shrinks to $\{\mathbf{x}^*\}$ as $c \downarrow 0$. In particular, any ϵ-neighbourhood of $\mathbf{x}^*$ contains $\mathcal{B}_c(\mathbf{x}^*)$ for sufficiently small c. Thus, $\mathbf{x}^*$ is Liapunov stable and hence asymptotically stable. In this case, V is called a *Liapunov function*. Conversely, if $\mathbf{x}^*$ is asymptotically stable, then such a V exists and can be taken to satisfy $V(\mathbf{x}) \to \infty$ as $\mathbf{x} \to \partial\mathcal{D}$.

More generally, we have the *LaSalle Invariance Prinsciple* which states that $\mathbf{x}(t)$ converges to the largest invariant set contained in $\mathcal{A} := \{\mathbf{x} \mid \langle V(\mathbf{x}, h(\mathbf{x})\rangle = 0\}$.

If there exists some continuously differentiable $V\colon \mathcal{D} \to [0, \infty)$ such that

$$\lim_{\mathbf{x}\to\partial\mathcal{D}} V(\mathbf{x}) = \infty, \text{ and}$$
$$\langle \nabla V(\mathbf{x}), h(\mathbf{x})\rangle < 0 \quad \forall \mathbf{x} \notin \mathcal{C}$$

for some bounded set $\mathcal{C}$, then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} V(\mathbf{x}(t)) = \langle \nabla V(\mathbf{x}(t)), h(\mathbf{x}(t))\rangle < 0 \quad \text{when } \mathbf{x}(t) \notin \mathcal{C}, \\ &\implies \mathbf{x}(t) \to \mathcal{C}. \end{aligned}$$

In particular, the trajectories remain bounded.

We now consider the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ for some $\mathbf{A} \in \mathbb{R}^{d\times d}$. Then the origin, $\mathbf{0}$, is an equilibrium. If $\mathbf{A}$ is non-singular, it is the only equilibrium. It is asymptotically stable if all eigenvalues of $\mathbf{A}$ are in the left half plane. If not, suppose there are no eigenvalues on the imaginary axis. Suppose there are $m < d$ eigenvalues in the left half plane. We can write $\mathbb{R}^d = \mathcal{S} \oplus \mathcal{U}$ where $\mathcal{S}$ is the m-dimensional stable subspace corresponding to the eigenvectors of eigenvalues in the left half plane, and $\mathcal{U}$ is the $(d-m)$-dimensional stable subspace corresponding to the eigenvectors of eigenvalues in the right half plane. Then, $\mathbf{x}(0) \in \mathcal{S} \implies \mathbf{x}(t) \to \mathbf{0}$ and $\mathbf{x}(0) \in \mathcal{U}$ implies that $\mathbf{x}(t)$ moves away from $\mathbf{0}$. More importantly, if $\mathbf{x}(0) \notin \mathcal{S}$, $\mathbf{x}(t)$ eventually moves away from $\mathbf{0}$. That is,

$$\mathbf{x}(t) \to \mathbf{0} \iff \mathbf{x}(0) \in \mathcal{S}.$$

However, $\mathcal{S}$ has zero volume in $\mathbb{R}^d$, and thus, for a typical initial condition, $\mathbf{x}(t)$ eventually moves away from $\mathbf{0}$. The above arguments extend to any point $\mathbf{x}^* \in \mathbb{R}^d$ if we replace the linear ODE by the following affine ODE:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\left(\mathbf{x}(t) - \mathbf{x}^*\right).$$

We now extend these ideas to the non-linear case. Suppose h is continuously differentiable and let $Dh(\mathbf{x})$ denote its Jacobian matrix at $\mathbf{x}$, that is, the (i,j) entry of of $Dh(\mathbf{x})$ is $\frac{\partial h_i}{\partial x_j}(\mathbf{x})$. By Taylor formula, for $\mathbf{x} \approx \mathbf{x}^*$, we have

$$h(\mathbf{x}) \approx h(\mathbf{x}^*) + Dh(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) = Dh(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*).$$

We now consider the affine ODE

$$\dot{\mathbf{z}}(t) = Dh(\mathbf{x}^*)\left(\mathbf{z}(t) - \mathbf{x}^*\right)$$

which is called the *linearisation* of the original ODE at $\mathbf{x}^*$.

Theorem 1.2.9: Hartman-Großman Theorem

Let $\mathbf{x}^*$ be a *hyperbolic* equilibrium, that is, $Dh(\mathbf{x}^*)$ has no eigenvalues on the imaginary axis. Then, there exist open neighbourhoods $\mathcal{O}_1, \mathcal{O}_2$ of $\mathbf{x}^*$ such that the phase portrait of the original ODE in $\mathcal{O}_1$ and its linearisation in $\mathcal{O}_2$ can be mapped to each other by a continuous and continuously invertible transformation.

Thus, 'stable subspaces' morph into 'stable manifolds' and 'unstable subspaces' morph into 'unstable manifolds'.

§1.3. Convergence Analysis

Recall that our iteration is

$$\mathbf{x}(n+1) := \mathbf{x}(n) + a(n)\left[h(\mathbf{x}(n)) + M(n+1)\right], \quad n \geq 0.$$

Since we view $a(n)$ as a discrete time step, we define the *algorithmic time scale* as

$$t_0 := 0, \quad t_n := \sum_{m=0}^{n-1} a(m), \quad n \geq 0.$$

Now, we define $\overline{\mathbf{x}}(t)$, $t \in [0, \infty)$ as follows.

$$\begin{aligned} \overline{\mathbf{x}}(t_n) &:= \mathbf{x}(n) \quad \forall n \geq 0 \text{ and} \\ \overline{\mathbf{x}}(t) &:= \overline{\mathbf{x}}(t_n) + \left(\frac{t - t_n}{t_{n+1} - t_n}\right)\left(\overline{\mathbf{x}}(t_{n+1}) - \overline{\mathbf{x}}(t_n)\right) \quad \text{for } t \in [t_n, t_{n+1}]. \end{aligned}$$

That is, we linearly interpolate on $[t_n, t_{n+1}]$. Then, $\overline{\mathbf{x}}$ is continuous and piecewise linear. We assume stability, that is,

$$\sup_{n \geq 0} \|\mathbf{x}(n)\| < \infty \quad \text{a.s.}$$

Fix $T > 0$. We shall compare $\overline{\mathbf{x}}(\cdot)$ on a sliding window $[t, t+T]$ as $t \uparrow \infty$, with the ODE trajectory on the same interval that matches with it at the beginning of the interval, that is, with $\mathbf{x}^t(s)$, $s \in [t, t+T]$, satisfying

$$\dot{\mathbf{x}}^t(s) = h(\mathbf{x}(s)),\ s \in [t, t+T],\ \mathbf{x}^t(t) = \overline{\mathbf{x}}(t).$$

It suffices to consider $t = t_n$ for some $n \geq 0$ since for the general case there is a negligible additional error which can be easily handled.

Let $m(n) := \min\{k \geq n \colon t_k \geq t_n + T\}$. Then, $t_{m(n)} \approx t_n + T$. We shall compare $\overline{\mathbf{x}}(\cdot)$ and $\mathbf{x}^{t_n}(\cdot)$ on the interval $[t_n, t_{m(n)}]$. Now, define

$$\zeta(n) := \sum_{m=0}^{n-1} a(m)M(m+1), \quad n \geq 1.$$

This is a martingale, that is, $\mathbb{E}[\zeta(n+1) \mid \mathcal{F}_n] = \zeta(n)$ for all n. Also,

$$\begin{aligned}
\sum_n \mathbb{E}\left[\|\zeta(n+1) - \zeta(n)\|^2 \mid \mathcal{F}_n\right] &= \sum_n a(n)^2 \mathbb{E}\left[\|M(n+1)\|^2 \mid \mathcal{F}_n\right] \\
&\leq \sum_n a(n)^2 K(1 + \|\mathbf{x}(n)\|^2) \\
&\leq K(1 + \sup_n \|\mathbf{x}(n)\|^2) \cdot \sum_m a(m)^2 \\
&< \infty \quad \text{a.s.}
\end{aligned}$$

By the convergence theorem for square integrable martingales, $\zeta(n)$ converges almost surely as $n \uparrow \infty$. We now define this convergence theorem more formally.

Theorem 1.3.1

Let $(Z_n, \mathcal{F}_n)$ be a square-integrable martingale. Its *quadratic variation process* $\langle Z \rangle_n$, $n \geq 0$ is given by

$$\langle Z \rangle_n := \sum_{m=0}^{n} \mathbb{E}\left[(Z_{m+1} - Z_m)^2 \mid \mathcal{F}_n\right].$$

Then, almost surely,

$$\lim_{n \uparrow \infty} \langle Z \rangle_n < \infty \implies \langle Z \rangle_n \text{ converges.}$$

Now, let $0 \leq k \leq m(n) - n$. Then,

$$\begin{aligned}
\overline{\mathbf{x}}(t_{n+k}) &= \overline{\mathbf{x}}(t_n) + \sum_{i=0}^{k-1} a(n+i) h(\overline{\mathbf{x}}(t_{n+i})) + \sum_{l=0}^{k-1} a(n+l) M(n+l+1) \\
&= \overline{\mathbf{x}}(t_n) + \sum_{i=0}^{k-1} a(n+i) h(\overline{\mathbf{x}}(t_{n+i})) + \zeta(n+k) - \zeta(n).
\end{aligned}$$

Further, we have

$$
\begin{aligned}
\mathbf{x}^{t_n}(t_{n+k}) &= \mathbf{x}^{t_n}(t_n) + \int_{t_n}^{t_{n+k}} h(\mathbf{x}^{t_n}(s))\,\mathrm{d}s \\
&= \mathbf{x}^{t_n}(t_n) + \sum_{i=0}^{k-1} a(n+i)\cdot h(\mathbf{x}^{t_n}(t_{n+i})) + \sum_{l=n}^{n+k-1} \int_{t_l}^{t_{l+1}} \big(h(\mathbf{x}^{t_n}(s)) - h(\mathbf{x}^{t_n}(t_l))\big)\,\mathrm{d}s
\end{aligned}
$$

because for $l \geq n$, we have

$$a(l)h(\mathbf{x}^{t_n}(t_l)) = \int_{t_l}^{t_{l+1}} h(\mathbf{x}^{t_n}(t_l))\,\mathrm{d}s.$$

Thus,

$$\mathbf{x}^{t_n}(t_{n+k}) = \mathbf{x}^{t_n}(t_n) + \sum_{i=0}^{k-1} a(n+i)\cdot h(\mathbf{x}^{t_n}(t_{n+i})) + \int_{t_n}^{t_{n+k}} \big(h(\mathbf{x}^{t_n}(s)) - h(\mathbf{x}^{t_n}([t]))\big)\,\mathrm{d}s$$

where $[t] := \max\{t_m \colon t_m \leq t\}$. Note that $\overline{\mathbf{x}}(t_n) = \mathbf{x}^{t_n}(t_n) = \mathbf{x}(n)$. Thus, we have

$$\big\|\overline{\mathbf{x}}(t_{n+k}) - \mathbf{x}^{t_n}(t_{n+k})\big\| \leq \sum_{i=0}^{k-1} a(n+i)\big\|h(\overline{\mathbf{x}}(t_{n+i})) - h(\mathbf{x}^{t_n}(t_{n+i}))\big\| + \mathcal{I}_d + \mathcal{I}_n$$

where $\mathcal{I}_d$ denotes the error due to discretisation and $\mathcal{I}_n$ denotes the error due to noise.

$$\big\|\overline{\mathbf{x}}(t_{n+k}) - \mathbf{x}^{t_n}(t_{n+k})\big\| \leq L\sum_{i=0}^{k-1} a(n+i)\big\|\overline{\mathbf{x}}(t_{n+i}) - \mathbf{x}^{t_n}(t_{n+i})\big\| + \mathcal{I}_d + \mathcal{I}_n$$

By the discrete Gronwall inequality, there exists $C(T) > 0$ such that

$$\sup_{n \leq m \leq m(n)} \big\|\overline{\mathbf{x}}(t_m) - \mathbf{x}^{t_n}(t_m)\big\| \leq C(T)(\mathcal{I}_d + \mathcal{I}_n)$$

For $\infty > K \geq \sup_{t\in[t_n, t_{m(n)}]} \|h(\mathbf{x}^{t_n}(t))\| > 0$, we have

$$
\begin{aligned}
\left\|\int_{t_m}^{t_{m+1}} \big(h(\mathbf{x}^{t_n}(s)) - h(\mathbf{x}^{t_n}([t]))\big)\,\mathrm{d}s\right\| &= \left\|\int_{t_m}^{t_{m+1}} \big(h(\mathbf{x}^{t_n}(s)) - h(\mathbf{x}^{t_n}(t_m))\big)\,\mathrm{d}s\right\| \\
&\leq L\int_{t_m}^{t_{m+1}} \big\|\mathbf{x}^{t_n}(s) - \mathbf{x}^{t_n}(t_m)\big\|\,\mathrm{d}s \\
&\leq L\int_{t_m}^{t_{m+1}} \left\|\int_{t_m}^{s} h(\mathbf{x}^{t_n}(u))\,\mathrm{d}u\right\|\mathrm{d}s \\
&\leq \frac{LK}{2}(t_{m+1} - t_m)^2 \\
&= L'a(m)^2.
\end{aligned}
$$

Hence,

$$\mathcal{I}_d = \left\| \int_{t_m}^{t_{m+1}} \big(h(\mathbf{x}^{t_n}(s)) - \overline{h}(\mathbf{x}^{t_n}([t])) \big) \, \mathrm{d}s \right\| \le L' \sum_{m \ge n} a(m)^2 \downarrow 0 \text{ as } n \uparrow \infty.$$

Also,

$$\mathcal{I}_n \le \sup_{m \ge 0} \|\zeta(n+m) - \zeta(n)\| \to 0 \text{ a.s. as } n \uparrow \infty.$$

Thus, as $n \uparrow \infty$,

$$\begin{aligned} &\max_{n \le m \le m(n)} \left\|\overline{\mathbf{x}}(t_m) - \mathbf{x}^{t_n}(t_m)\right\| \to 0 \text{ a.s.} \implies \\ &\lim_{t \uparrow \infty} \max_{s \in [0,T]} \left\|\overline{\mathbf{x}}(t+s) - \mathbf{x}^{t_n}(t+s)\right\| \to 0 \text{ a.s.} \quad \forall T > 0. \end{aligned}$$

Now, let

$$\begin{aligned} \mathcal{D} :&= \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists\, 0 < s_n \uparrow \infty \text{ such that } \overline{\mathbf{x}}(s_n) \to \mathbf{x} \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^d \mid \exists\, 0 < t_k \uparrow \infty \text{ such that } \overline{\mathbf{x}}(t_k) \to \mathbf{x} \right\} \end{aligned}$$

Proposition 1.3.2

$\mathcal{D}$ is an invariant set for the ODE.

Proof. Suppose $s_n \uparrow \infty$ and $\overline{\mathbf{x}}(s_n) \to \mathbf{x}$. Then, $\mathbf{x} \in \mathcal{D}$. Let $\dot{\tilde{\mathbf{x}}}(t) = h(\tilde{\mathbf{x}}(t))$, $\tilde{\mathbf{x}}(0) = \mathbf{x}$. By the above, for $T > 0$, we have

$$\overline{\mathbf{x}}(s_n + T) - \mathbf{x}^{s_n}(s_n + T) \to 0.$$

By continuous dependence on initial condition,

$$\mathbf{x}^{s_n}(s_n) = \overline{\mathbf{x}}(s_n) \to \mathbf{x} \implies \mathbf{x}^{s_n}(s_n + T) - \tilde{\mathbf{x}}(T) \to 0.$$

Thus, $\mathbf{x}^{s_n}(s_n + T) - \tilde{\mathbf{x}}(T) \to 0$, implying $\tilde{\mathbf{x}}(T) \in \mathcal{D}$. Similar argument works for $T < 0$. Hence, $\mathcal{D}$ is invariant. $\square$

Definition 1.3.3: Internally Chain Transitive

We say that $\mathcal{D}$ is an *internally chain transitive* invariant set if given any $\epsilon, T > 0$ and points $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, we can find $n \ge 1$, and a chain of points $\mathbf{x} = \mathbf{x}_0, \ldots, \mathbf{x}_n = \mathbf{y}$ such that for $0 \le i < n$, there exists a trajectory segment of the ODE of duration at least T which starts in the ϵ-neighbourhood of $\mathbf{x}_i$ and ends in the ϵ-neighbourhood of $\mathbf{x}_{i+1}$.

Theorem 1.3.4: Benaim's Theorem

$\mathbf{x}(n)$ converges to an internally chain transitive invariant set of the ODE (a.s.).