Homework 1

Instructor: Rotem Oshman
Name: Ishan Pranav

References

No collaborators or outside sources were referenced.

Question 1

Prove the following equality using induction on n:[1]

$$(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n \text{ for all } n \in \mathbb{N} \,. \tag{1}$$

1. Check the base case ($n = 1$).

> **Solution:** *Basis.* Consider $n = 1$. Note $(1-r)(r^{1-1}) = (1-r) = (1-r^1)$. Therefore $(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n$ for $n = 1$.

2. Prove the inductive step.

> **Solution:** *Hypothesis.* Let $k \in \mathbb{N}$. Of course, $k \geq 1$. Consider $n = k$. Assume the result is true for $n = k$; that is, assume $(1-r)(1+r+r^2+\cdots+r^{k-1}) = 1-r^k$.
>
> *Inductive step.* Consider $n = k+1$. By the inductive hypothesis, $(1-r)(1+r+r^2+\cdots+r^{k-1}) = 1-r^k$. Observe
>
> $$\begin{aligned} (1-r)(1+r+r^2+\cdots+r^{k-1}) &= 1-r^k \\ (1-r)r^k + (1-r)(1+r+r^2+\cdots+r^{k-1}) &= 1-r^k+(1-r)r^k \\ (1-r)(1+r+r^2+\cdots+r^{(k+1)-1}) &= 1-r^k+r^k-r^{k+1} \\ &= 1-r^{k+1}. \end{aligned}$$
>
> Hence, by the principle of mathematical induction, for all $n \in \mathbb{N}$, we have $(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n$. $\square$

3. Using Eq. (1), evaluate the following sum:

$$\sum_{i=0}^{n} 2^i \cdot 3^{n-i} = 3^n + 2\cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + \cdots + 2^n = ???$$

---

[1] Recall that $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$.

Solution: Observe

$$\begin{aligned}
\sum_{i=0}^{n} 2^i \cdot 3^{n-i} &= 3^n \sum_{i=0}^{n} 2^i \cdot 3^{-i} \\
&= 3^n \sum_{i=0}^{n} \left(\frac{2}{3}\right)^i \\
&= 3^n \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{(n+1)-1}\right] \\
&= 3^n \left[\frac{\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{(n+1)-1}\right)\left(1 - \frac{2}{3}\right)}{1 - \frac{2}{3}}\right].
\end{aligned}$$

Using the result from Eq. (1), we have

$$\begin{aligned}
\sum_{i=0}^{n} 2^i \cdot 3^{n-i} &= 3^n \left[\frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}}\right] \\
&= 3^{n+1} \left[1 - \left(\frac{2}{3}\right)^{n+1}\right] \\
&= 3^{n+1} - 2^{n+1}.
\end{aligned}$$

Thus $\sum_{i=0}^{n} 2^i \cdot 3^{n-i} = 3^{n+1} - 2^{n+1}$. $\square$

Question 2

Find a flaw in the following "proof by induction" (see Figure 1 for an illustration). In particular, state why the inductive step is incorrect: **Claim:** For all $n \in \mathbb{N}$, and any set of n horses, all horses in the set have the same color.

1. Base Case ($n = 1$): If there is just one horse in the set, obviously all horses have the same color.

2. Inductive Step: Suppose the induction hypothesis holds for all $1, 2, \ldots, n$. Our goal is to prove the statement for sets of $n + 1$ horses. So take any such set. Now exclude one horse, call this horse A, and look at the set of n remaining horses. By the induction hypothesis, they all have the same color. Now exclude a different horse, call it B, and look at the set of n remaining horses, which includes horse A. Then, all horses in this set must also have the same color. This implies that A and B also have the same color. Hence, we obtain that all $n+1$ horses in our set have the same color, "proving" the claim.

Solution: The error is in the inductive step. First, the author takes a set of $n + 1$. The author must show that all horses in this set have the same color. To do this, the author removes horse A. By the strong induction hypothesis, the remaining n horses have the same color. To complete the proof, the author's objective is to show that A has the same color as the remaining n horses.

Then, the author removes another horse B where $B \neq A$ and groups the other n horses, including A. This is an error because it presumes that the induction hypothesis applies to a group of n horses which includes A, even though that fact remains unproven.

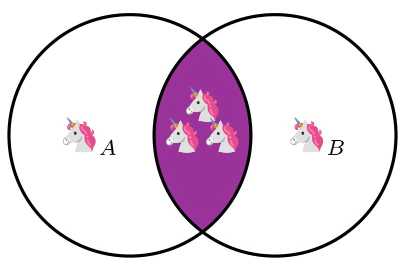

Figure 1: Horses A and B, with all the rest of the horses lying in the violet region common to both sets.

Question 3

Recall the INSERTION-SORT algorithm discussed in the lecture:

```
INSERTION-SORT(A, n)
for j = 2 to n
    key = A[j]
    i = j - 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]
        i = i - 1
    A[i + 1] = key
```

In the lecture, you have seen a correctness proof of the algorithm based on the following loop invariant for the (outer) for loop:

- Let $A_j[1, \ldots, n]$ denote the array at the beginning of iteration j (end of iteration $j-1$). We have that $A_j[1, \ldots, j-1]$ stores the same values as $A[1, \ldots, j-1]$ but in sorted order, while $A_j[\ell] = A[\ell]$ for $j \leq \ell \leq n$.

In this problem, you will fill in a bit more detail in the proof, by also introducing a loop invariant for the (inner) while loop. You will use the following loop invariant:

- Let $A_{j,i}[1, \ldots, n]$ denote the array at the beginning of iteration i of the inner loop (for $0 \leq i \leq j-1$). Then:
  - If the loop executes with value $i+1$, then
    1. $A_{j,i}[1, \ldots, i] = A_j[1, \ldots, i]$, and
    2. If $i + 2 \leq j$, then $A_{j,i}[i+2, \ldots, j] = A_j[i+1, \ldots, j-1]$ and $A_j[j] < A_{j,i}[i+2]$.
  - Otherwise (if the while loop terminates before reaching value $i+1$), then $A_{j,i} = A_{j,i+1}$.

Solve the following tasks:

1. Prove the while loop invariant above using induction over i. Start your base case at $i = j - 1$ and use backwards induction to show that the claim holds for all smaller i. (Prove this for an arbitrary value of j, where $1 \leq j \leq n$.)

Solution: Let $A_{j,i}[1, \ldots, n]$ denote the array at the beginning of iteration i of the Insertion-Sort inner loop, where $i, j \in \mathbb{Z}$ and $0 \leq i \leq j - 1$.

We will demonstrate that

- if the loop executes with value $i + 1$, then
  (a) $A_{j,i}[1, \ldots, i] = A_j[1, \ldots, i]$, and
  (b) if $i + 2 \leq j$, then $A_{j,i}[i+2, \ldots, j] = A_j[i+1, \ldots, j-1]$ and $A_j[j] < A_{j,i}[i+2]$;
- otherwise, $A_{j,i} = A_{j,i+1}$,

by induction on i.

Basis. Consider $i = j - 1$.

We check if the loop executes with value $(j-1)+1 = j$. Since $A_{j,j-1}[j] \not> A_j[j]$ the loop does not execute. Therefore, there is no change to the array, so $A_{j,j-1} = A_{j,j}$. The loop invariant holds.

Hypothesis. Let $k \in \mathbb{N}$ where $0 \leq k \leq j - 1$. Consider $i = k$. Assume that

- if the loop executes with value $k+1$, then
  (a) $A_{j,k}[1, \dots, k] = A_j[1, \dots, k]$, and
  (b) if $k+2 \leq j$, then $A_{j,k}[k+2, \dots, j] = A_j[k+1, \dots, j-1]$ and $A_j[j] < A_{j,k}[k+2]$;
- otherwise, $A_{j,k} = A_{j,k+1}$.

Inductive step. Consider $i = k - 1$.

- Suppose $A_{j,k}[k-1] > A_j[j]$. Then the inner loop body executes.
  (a) The only modification occurs in position $k+1$ of the array. Therefore the first $k-1$ elements are unchanged relative to the previous iteration. By the inductive hypothesis, we have $A_{j,k-1}[1, \dots, k-1] = A_j[1, \dots, k-1]$.
  (b) Suppose $k+1 \leq j$. By the induction hypothesis, $A_{j,k}[k+2, \dots, j] = A_j[k+1, \dots, j-1]$. Since we are running the loop on $k = (k-1)+1$, we assign from position k of the array into position $k+1$ in the loop body. Therefore, we know that the shift has completed such that $A_{j,k-1}[k+1, \dots, j] = A_j[k, \dots, j-1]$.
  Since the loop executed, we know that before execution $A_{j,k}[k-1] > A_j[j]$. In other words, the value shifted is known to be greater than the key; otherwise, the loop would not have executed. By the induction hypothesis, $A_j[j] < A_{j,k}[k+2]$. Since the element from position k was assigned into $k+1$, and because that element is known to be greater than the key $A_j[j]$, we have $A_j[j] < A_{j,k-1}[k+1]$.

  The loop invariant holds in this case.
- Suppose instead that $A_{j,k}[k-1] \leq A_j[j]$. Then the inner loop body does not execute. Thus $A_{j,k-1} = A_{j,k}$. The loop invariant holds in this case.

The loop invariant holds for all cases where $i = k - 1$.

Hence, by the principle of mathematical induction, for all i where $0 \leq i \leq j - 1$, the loop invariant holds. $\square$

2. Use the inner loop invariant to show the induction step of the outer loop invariant.

Solution: *Hypothesis.* Let $m \in \mathbb{N}$ where $2 \leq m \leq n$. Assume that the loop invariant holds for $j = m$.

Inductive step. Consider $j = m + 1$. By the induction hypothesis, we have that $A_m[1, \ldots, m - 1]$ contains the elements $A[1, \ldots, m - 1]$ in sorted order. After the inner loop executes, the inner loop invariant asserts that for all elements $a \in A[m, \ldots, n]$, we have $m < a$. Given that $A[1, \ldots, m - 1]$ it follows that $A[1, \ldots m]$. Suppose not: Then there is some element $a \in A[m, \ldots n]$ where $a > m$, which contradicts the inner loop invariant. Otherwise, m would belong in $A[1, \ldots m - 1]$ which contradicts the induction hypothesis. So m is in its sorted position and $A[1, \ldots m]$ is sorted, thus the loop invariant holds.

Since the inner loop terminates before $i = m$ and runs only from $i = m - 1$ down to $m = 0$, and, based on the inner loop body and invariant, modifies values from $i = m$ down to $i = 1$, we know $A_{m+1}[\ell] = A[\ell]$ for $m + 1 \leq \ell \leq n$.

The loop invariant holds for $j = m + 1$.