Answers to Problem Set 8

Ishan Pranav
MATH-UA 120 Discrete Mathematics

due December 8, 2023

These are to be written up and turned in to Gradescope.

Assigned Problems

1.

(a) Find $d = \gcd(29341, 1739)$*, and integers* x *and* y *such that* $29341x + 1739y = d$*.*

(b) Prove that 7 cannot be expressed as an integral linear combination of 29341 and 1739.

Answer.

(a) Let $d = \gcd(29341, 1739)$. Observe

$$\begin{aligned} d &= \gcd(29341, 1739) \\ &= \gcd(1739, 29341 \bmod 1739) && (\text{note } 29341 - 16 \cdot 1739 = 1517) \\ &= \gcd(1517, 1739 \bmod 1517) && (\text{note } 1739 - 1 \cdot 1517 = 222) \\ &= \gcd(222, 1739 \bmod 222) && (\text{note } 1517 - 6 \cdot 222 = 185) \\ &= \gcd(185, 222 \bmod 185) && (\text{note } 222 - 1 \cdot 185 = 37) \\ &= \gcd(37, 185 \bmod 37) && (\text{note } 185 - 5 \cdot 37 = 0) \\ &= 37. \end{aligned}$$

Consider $37 = 29341x + 1739y$. Observe

$$\begin{aligned} 0 \cdot 29341 + 1 \cdot 1739 &= 1739 \\ 1 \cdot 29341 - 16 \cdot 1739 &= 1517 \\ -1 \cdot 29341 + 17 \cdot 1739 &= 222 \\ 7 \cdot 29341 - 118 \cdot 1739 &= 185 \\ 29341x + 1739y &= 37. \end{aligned}$$

Thus $x = -8$ and $y = 135$ is a possible solution.

(b) *Claim.* 7 cannot be expressed as an integral linear combination of 29341 and 1739.

Proof. Assume, for the sake of contradiction, that 7 can be expressed as an integral linear combination of 29341 and 1739. Note $\gcd(29341, 1739) = 37$ is the smallest positive integer expressible as a linear combination of $29341 \in \mathbb{Z}$ and $1739 \in \mathbb{Z}$. However, $7 \in \mathbb{Z}$ and $0 < 7 < 37$, which is a contradiction. Ergo our assumption is false: 7 is not expressible as a linear combination of 29341 and 1739. □

2.

(a) Disprove: There exist integers a and b such that $a+b = 100$ and $\gcd(a, b) = 8$.

(b) Prove: There exist infinitely many pairs of integers (a, b) such that $a+b = 87$ and $\gcd(a, b) = 3$.

Answer.

(a) LEMMA.

Claim. There do not exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ where $a+b = 100$ and $\gcd(a, b) = 8$.

Proof. Assume, for the sake of contradiction, there exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ where $a+b = 100$ and $\gcd(a, b) = 8$. Since $\gcd(a, b) = 8$, we know $8 \mid a$ and $8 \mid b$. Thus there exist $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$ where $a = 8k_1$ and $b = 8k_2$. Observe

$$\begin{aligned} a + b &= 8k_1 + 8k_2 \\ &= 8(k_1 + k_2) \\ &= 100. \end{aligned}$$

There exists $(k_1 + k_2) \in \mathbb{Z}$ where $100 = 8(k_1 + k_2)$. Thus $8 \mid (a + b)$. However, $a+b = 100$ and, of course, $8 \nmid 100$. This is a contradiction. Thus our assumption is false: There do not exist $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ where $a + b = 100$. □

PROPOSITION.

Claim. Disprove: There exist integers a and b such that $a + b = 100$ and $\gcd(a, b) = 8$.

Proof. The claim is false by lemma. □

(b) *Claim.* There exist infinitely many pairs of integers (a, b) such that $a + b = 87$ and $\gcd(a, b) = 3$.

Proof. Let $X = \{x \in \mathbb{Z} : 29 \nmid x\}$.

Let $Y = \{(a, b) \in \mathbb{Z}^2 : a + b = 87 \text{ and } \gcd(a, b) = 3\}$.

Let f be the relation $\{(x, (3x, 87 - 3x)) : x \in X\}$.

(Function) Let $(x_0, y_1), (x_0, y_2) \in f$. Then $y_1 = (3x_0, 87 - 3x_0)$ and $y_2 = (3x_0, 87 - 3x_0)$. Thus $y_1 = y_2$. Therefore f is a function.

(Domain) Note $\operatorname{dom} f = X$ by construction.

(Image) Let $(x, (a, b)) \in f$. Note also $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ so $(a, b) \in \mathbb{Z}^2$. Note

$$(a + b) = (3x + (87 - 3x)) = 87.$$

Note also

$$\gcd(a, b) = \gcd(3x, 87 - 3x) = \gcd(3x, 3(29 - x)).$$

Since $x \in X$, we have $x \in \mathbb{Z}$. Since $a = 3x$, we have $3 \mid a$. Since there exists $(29 - x) \in \mathbb{Z}$ where

$$b = (87 - 3x) = 3(29 - x)$$

we have $3 \mid b$. Therefore, 3 is a common divisor of a and b.

Since every integer can be expressed as a product of integer divisors, and because every pair of integers has a common divisor, there exist $k_1, k_2, k_3 \in \mathbb{Z}$ where $x = k_1 k_2$ and $(29 - x) = k_1 k_3$. Note

$$(x - (29 - x)) = (k_1 k_2 - k_1 k_3) = k_1(k_2 - k_3).$$

So $k_1 \mid (x - (29 - x))$. Since $(x - (29 - x)) = (2x - 29)$, we have $k_1 \mid (2x - 29)$. Note $k_1 \mid x$, so $k_1 \mid 2x$. So it must be that $k_1 \mid 29$. However, since 29 is prime, $k_1 = 1$ or $k_1 = 29$. But since $x \in X$, we have $29 \nmid x$. Thus $k_1 \neq 29$. Therefore $k_1 = 1$. The only common divisor between x and $(29 - x)$ is 1. Thus $\gcd(x, 29 - x) = 1$. Since $a = 3x$ and $b = 3(29 - x)$, the common divisors of a and b are 1 and 3. Since $1 < 3$, we have $\gcd(a, b) = 3$.

Since, $(a, b) \in \mathbb{Z}^2$ with $a + b = 87$ and $\gcd(a, b) = 3$, we have $(a, b) \in Y$. Therefore $\operatorname{im} f \subseteq Y$.

(Injective) Let $x_1, x_2 \in X$. Suppose $f(x_1) = f(x_2)$. Then $(3x_1, 87 - 3x_1) = (3x_2, 87 - 3x_2)$. Since $3x_1 = 3x_2$, we have $x_1 = x_2$. Therefore f is injective.

(Surjective) Let $(a', b') \in Y$. Note $3 \mid a'$. We can construct $x' = \frac{a'}{3} \in \mathbb{Z}$ where $f(x') = (a', b')$. So $\operatorname{im} f = Y$. Therefore f is surjective.

Hence, $f : X \to Y$ is bijective. Note X is an infinite set. Since f is bijective, Y is an infinite set. Ergo there are infinitely many pairs of integers (a, b) where $a + b = 87$ and $\gcd(a, b) = 3$. □

3.

(a) *Let $a, b, c \in \mathbb{Z}$ such that a and b are relatively prime. Prove that if $a \mid c$ and $b \mid c$, then $(ab) \mid c$.*

(b) *Explain why part (a) is false if a and b are not relatively prime.*

Answer.

(a) *Claim.* Let $a, b, c \in \mathbb{Z}$ where a and b are relatively prime. If $a \mid c$ and $b \mid c$, then $ab \mid c$.

Proof. Let $a, b, c \in \mathbb{Z}$ where a and b are relatively prime. Suppose $a \mid c$ and $b \mid c$. Since a and b are relatively prime, $\gcd(a, b) = 1$. Since $\gcd(a, b) = 1$ is the smallest positive integer of which a and b can be expressed as an integer linear combination, there exist $x, y \in \mathbb{Z}$ where $ax + by = 1$. Since $a \mid c$, there exists $k_1 \in \mathbb{Z}$ where $c = ak_1$. Since $b \mid c$, there exists $k_2 \in \mathbb{Z}$ where $c = bk_2$. Observe

$$\begin{aligned}
ax + by &= 1 \\
c(ax + by) &= c \\
cax + cby &= c \\
bk_2ax + ak_1by &= c \\
ab(k_2x + k_1y) &= c.
\end{aligned}$$

There exists $(k_2x + k_1y) \in \mathbb{Z}$ where $c = (k_2x + k_1y)ab$. Hence, $ab \mid c$. $\square$

(b) Let $a, b, c \in \mathbb{Z}$ where $a \mid c$ and $b \mid c$ where a and b are not relatively prime. If a and b are not relatively prime, then there exist cases where $ab > c$, and thus, $ab \nmid c$. We know c can be expressed as a product of n prime factors where q_i represents a unique prime factor and a_i represents its count for $i \in \mathbb{N}$ where $i \leq n$:

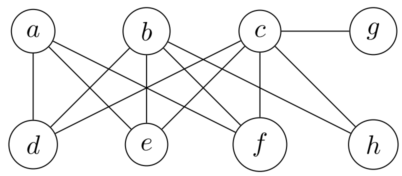
$$c = \prod_{i=1}^{n} q_i^{a_i}.$$

If a and b are not relatively prime then, letting $d = \gcd(a, b)$, we have $d > 1$. This means that a and b share at least one prime factor. There exists $q^* \in \mathbb{Z}$ where $q^* \mid a$ and $q^* \mid b$ and q^* is prime.

Now there exist $k_1, k_2 \in \mathbb{Z}$ where $a = k_1q^*$ and $b = k_2q^*$. Since $a \mid c$, we have $k_1q^* \mid c$. Since $b \mid c$, we have $k_2q^* \mid c$. Thus $k_1k_2q^* \mid c$. However, we have no guarantee that $k_1k_2(q^*)^2 \mid c$. That is, we do not know if there is more than one instance of q^* in the prime factorization of c. Thus we cannot be certain that $ab \mid c$.

Using this technique, we can construct a counterexample. Consider $4, 6, 12 \in \mathbb{Z}$. Note $4 \mid 12$ and $6 \mid 12$. Note also $\gcd(4, 6) = 2$, so 4 and 6 are not relatively prime. However, $4 \cdot 6 = 24$ and of course $24 \nmid 12$. Hence disproven.

4. *Consider the following graph G.*



(a) Write out the ordered pair $G = (V, E)$.

(b) What is the order of G?

(c) What is the size of G?

(d) What is $N(b)$?

(e) What is $d_G(c)$?

(f) What is $\Delta(G)$?

(g) What is $\delta(G)$?

(h) What elements are the vertex a adjacent to?

(i) What elements are the vertex f incident with?

(j) Is G regular? Briefly explain why or why not.

Answer.

(a)

$$\begin{aligned} G = (V, E) = (\{a, b, c, d, e, f, g, h\}, \{ \\ \{a, d\}, \{a, e\}, \{a, f\}, \\ \{b, d\}, \{b, e\}, \{b, f\}, \{b, h\}, \\ \{c, d\}, \{c, e\}, \{c, f\}, \{c, h\}, \{c, g\}\}). \end{aligned}$$

An undirected graph is an ordered pair where the first element is the set of vertices and the second element is the set of edges, where each edge is two-element set of vertices.

(b) The order of G is the number of vertices in G. Thus the order of G is $|V| = 8$.

(c) The size of G is the number of edges in G. Thus the order of G is $|E| = 12$.

(d) The neighborhood of b, $N(b)$, represents the set of vertices adjacent to vertex b. Thus $N(b) = \{d, e, f, h\}$.

(e) $d_G(c)$ represents the degree of vertex c in G. Thus $d_G(c) = |N(c)| = 5$.

(f) $\Delta(G)$ represents the maximum degree of G. Thus $\Delta(G) = d_G(c) = 5$.

(g) $\delta(G)$ represents the minimum degree of G. Thus $\delta(G) = d_G(g) = 1$.

(h) Vertex a is adjacent to vertices d, e, and f because for graph $G = (V, E)$, we have $a, d, e, f \in V$ and $\{a, d\}, \{a, e\}, \{a, f\} \in E$.

(i) Vertex f is incident with edges $\{a, f\}$, $\{b, f\}$, and $\{c, f\}$ because for graph $G = (V, E)$, we have $f \in V$ and $\{a, f\}, \{b, f\}, \{c, f\} \in E$.

(j) No, G is not regular. In a regular graph (V', E'), there exists a number k where for all vertices $v \in V'$, we have $d_{(V',E')}(v) = k$. Consider $b, c \in G$. Note $d_G(c) = 5$ and $d_G(g) = 1$. However, $5 \neq 1$. Thus G is not regular.

Put another way, in a regular graph (V, E), we have $\Delta((V, E)) = \delta((V, E))$. However $(\Delta(G) = 5) \neq (\delta(G) = 1)$. Thus G is not regular.

5. *Let $G = (V, E)$ be a graph. Prove by induction:*

The sum of the degrees of the vertices in G is twice the number of edges.

Answer. *Claim.* Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} d_G(v) = 2|E|.$$

Proof. Let $G = (V, E)$ be a graph. We will demonstrate $\sum_{v \in V} d_G(v) = 2|E|$ by induction on $|E|$.

Basis case. Consider a graph $G_0 = (V_0, E_0)$ where $|E_0| = 0$. Since there are no edges in E_0, for all $v_0 \in V_0$, we have $d_{G_0}(v_0) = 0$. Therefore

$$\sum_{v_0 \in V_0} d_{G_0}(v_0) = \sum_{v_0 \in V_0} 0 = 2(0).$$

Inductive hypothesis. Let $k \in \mathbb{N}$. Consider a graph $G_k = (V_k, E_k)$ where $|E_k| = k$. Assume that

$$\sum_{v \in V_k} d_{G_k}(v) = 2k.$$

Induction step. Consider a graph $G^* = (V^*, E^*)$ where $|E^*| = k + 1$. Note $k + 1 > 0$ so $|E^*| > 0$. Thus $E^* \neq \emptyset$. By omitting an edge $e \in E^*$, we can construct a graph $G' = (V^*, E^* - \{e\})$. Of course

$$|E^* - \{e\}| = ((k + 1) - 1) = k.$$

By the induction hypothesis, we have

$$\sum_{v \in V^*} d_{G'}(v) = 2|E^* - \{e\}| = 2k.$$

Recall that $e \in E^*$. There exist vertices $u, v \in V^*$ where $e = \{u, v\}$. Since e is a set, $u \neq v$. Note that $d_{G^*}(u) = d_{G'}(u)+1$ because in graph G^*, vertex u is connected to vertex v by edge e. Symmetrically, $d_{G^*}(v) = d_{G'}(v) + 1$. Let $W = V^* - \{u, v\}$. The inclusion or omission of e has no effect on the degree of any other vertex $w \in W$. Thus

$$\begin{aligned}
\sum_{v \in V^*} d_{G^*}(v) &= (d_{G'}(u) + 1) + (d_{G'}(v) + 1) + \sum_{w \in W} d_{G'}(w) \\
&= 1 + 1 + d_{G'}(u) + d_{G'}(v) + \sum_{w \in W} d_{G'}(w) \\
&= 1 + 1 + \sum_{v \in V^*} d_{G'}(v) \\
&= 2 + 2k \\
&= 2(k + 1).
\end{aligned}$$

This completes the inductive step.

Hence, by the principle of mathematical induction, for any graph $G = (V, E)$, we have

$$\sum_{v \in V} d_G(v) = 2|E|.$$

□