MATH-UA 120 Section 5

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Goldbach conjecture

Every even integer greater than two is the sum of two primes.

Proposition 1

The sum of two even integers is even.

We show that if x and y are even integers, then $x + y$ is an even integer. Let x and y be even integers. Since x is even, we know $2 \mid x$ by definition. Likewise, since y is even, $2 \mid y$. Since $2 \mid y$, we know that there is an integer a such that $x = 2a$ by definition. Likewise, since $2 \mid y$, there is an integer b such that $y = 2b$. Observe that $x + y = 2 + 2b = 2(a + b)$. Therefore there is an integer c (namely, $a + b$) such that $x + y = 2c$. Therefore $2|(x + y)$. Therefore $x + y$ is even.

Proposition 2

Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Suppose a, b, and c are integers with $a \mid b$ and $b \mid c$. Since $a \mid b$, there is an integer x such that $b = ax$. Likewise there is an integer y such that $c = by$. Let $z = xy$. Then $az = a(xy) = (ax)y = by = c$.

Therefore there is an integer z such that $c = az$. Therefore $a \mid c$.

Proposition 3

Let x be an integer and suppose $x > 1$. Note that $x^3 + 1 = (x + 1)(x^2 - x + 1)$. Because x is an integer, both $x + 1$ and $x^2 - x + 1$ are integers. Therefore $(x + 1) \mid (x^3 + 1)$.

Since $x > 1$, we have $x + 1 > 1 + 1 = 2 > 1$.

Also, $x > 1$ implies that $x^2 > x$, and since $x > 1$, we have $x^2 > 1$. Multiplying both sides by x again yields $x^3 > x$. Adding 1 to both sides gives $x^3 + 1 > x + 1$.

Thus $x + 1$ is an integer with $1 < x + 1 < x^3 + 1$.

Since $x + 1$ is a divisor of $x^3 + 1$ and $1 < x + 1 < x^3 + 1$, we have that $x^3 + 1$ is composite.

Proposition 4

Let x be an integer. Then x is even if and only if $x + 1$ is odd.

Suppose x is even. This means that $2 \mid x$. Hence there is an integer a such that $x = 2a$. Adding 1 to both sides gives $x + 1 = 2a + 1$. By the definition of *odd*, $x + 1$ is odd. Suppose $x+1$ is odd. So there is an integer b such that $x+1 = 2b+1$. Subtracting 1 from both sides gives $x = 2b$. This shows that $2 \mid x$ and therefore x is even.

Proposition 5

Let a, b, c, and d be integers. If $a \mid b$, $b \mid c$, and $c \mid d$, then $a \mid d$.

Since $a \mid b$, there is an integer x such that $ax = b$.

Since $b \mid c$, there is an integer y such that $by = c$.

Since $c \mid d$, there is an integer z such that $cz = d$.

Note that $a(xyz) = (ax)(yz) = b(yz) = (by)z = cz = d$.

Therefore there is an integer $w = xyz$ such that $aw = d$.

Therefore $a \mid d$.

1 The sum of two odd integers is even

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that x is odd and y is odd. Since x is odd, there exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Since y is odd, there exists $b \in \mathbb{Z}$ such that $y = 2b + 1$. Observe

$$\begin{aligned} x + y &= (2a + 1) + (2b + 1) \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1). \end{aligned}$$

Let $c = a + b + 1 \in \mathbb{Z}$. There exists $c \in \mathbb{Z}$ such that $x + y = 2c$. Therefore, $x + y$ is even. $\blacksquare$

2 The sum of an odd integer and an even integer is odd

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that x is even and y is odd. Since x is even, $2 \mid x$. Thus, there exists $a \in \mathbb{Z}$ such that $x = 2a$. Since y is odd, there exists $b \in \mathbb{Z}$ such that $y = 2b + 1$. Observe

$$\begin{aligned} x + y &= (2a) + (2b + 1) \\ &= 2(a + b) + 1. \end{aligned}$$

Let $c = a + b \in \mathbb{Z}$. There exists $c \in \mathbb{Z}$ such that $x + y = 2c + 1$. Therefore, $x + y$ is odd. $\blacksquare$

3 If n is an odd integer, then $-n$ is also odd

Let $n \in \mathbb{Z}$ such that n is odd. Since n is odd, there exists $a \in \mathbb{Z}$ such that $n = 2a + 1$. Observe

$$\begin{aligned} -n &= -(2a+1) \\ &= -2a - 1 \\ &= 2(-a) - 1 \\ &= 2(-a-1) + 1. \end{aligned}$$

Let $b = -a - 1 \in \mathbb{Z}$. There exists $b \in \mathbb{Z}$ such that $-n = 2b + 1$. Therefore, $-n$ is odd. ■

4 The product of two even integers is even

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that x is even and y is even. Since x is even, $2 \mid x$. Thus, there exists $a \in \mathbb{Z}$ such that $x = 2a$. Since y is even, $2 \mid y$. Thus, there exists $b \in \mathbb{Z}$ such that $y = 2b$. Observe

$$\begin{aligned} xy &= (2a)(2b) \\ &= 2(2ab). \end{aligned}$$

Let $c = 2ab \in \mathbb{Z}$. There exists $c \in \mathbb{Z}$ such that $xy = 2c$. Thus, $2 \mid xy$. Therefore, xy is even. ■

5 The product of an even integer and an odd integer is even

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that x is even and y is odd. Since x is even, $2 \mid x$. Thus, there exists $a \in \mathbb{Z}$ such that $x = 2a$. Since y is odd, there exists $b \in \mathbb{Z}$ such that $y = 2b + 1$. Observe

$$\begin{aligned} xy &= (2a)(2b+1) \\ &= 2(2ab + a). \end{aligned}$$

Let $c = 2ab + a \in \mathbb{Z}$. There exists $c \in \mathbb{Z}$ such that $xy = 2c$. Thus, $2 \mid xy$. Therefore, xy is even. ■

6 The product of two odd integers is odd

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ such that x is odd and y is odd. Since x is odd, there exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Since y is odd, there exists $b \in \mathbb{Z}$ such that $y = 2b + 1$. Observe

$$\begin{aligned} xy &= (2a+1)(2b+1) \\ &= 2a + 2b + 4ab + 1 \\ &= 2(a + b + 2ab) + 1. \end{aligned}$$

Let $c = a + b + 2ab \in \mathbb{Z}$. There exists $c \in \mathbb{Z}$ such that $xy = 2c + 1$. Therefore, xy is odd. ■

7 The square of an odd integer is odd

Let $x \in \mathbb{Z}$ such that x is odd. Since x is odd, there exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Observe

$$\begin{aligned} x^2 &= (2a+1)^2 \\ &= (2a+1)(2a+1) \\ &= 4a^2 + 4a + 1 \\ &= 2(2a^2 + 2a) + 1. \end{aligned}$$

Let $b = 2a^2 + 2a \in \mathbb{Z}$. There exists $b \in \mathbb{Z}$ such that $x^2 = 2b + 1$. Therefore, x^2 is odd. ■

8 The cube of an odd integer is odd

Let $x \in \mathbb{Z}$ such that x is odd. Since x is odd, there exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Observe

$$\begin{aligned} x^3 &= (2a+1)^3 \\ &= 8a^3 + 12a^2 + 6a + 1 \\ &= 2(4a^3 + 6a^2 + 3a) + 1. \end{aligned}$$

Let $b = 4a^3 + 6a^2 + 3a \in \mathbb{Z}$. There exists $b \in \mathbb{Z}$ such that $x^3 = 2b + 1$. Therefore, x^3 is odd. ■

9 Given $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$ such that $a \mid b$ and $a \mid c$. Since $a \mid b$, there exists $x \in \mathbb{Z}$ such that $b = ax$. Since $a \mid c$, there exists $y \in \mathbb{Z}$ such that $c = ay$. Observe

$$\begin{aligned} b + c &= ax + ay \\ &= a(x + y). \end{aligned}$$

Let $z = x + y \in \mathbb{Z}$. There exists $z \in \mathbb{Z}$ such that $b + c = az$. Therefore, $a \mid (b + c)$. ■

10 Given $a, b, c \in \mathbb{Z}$, if $a \mid b$, then $a \mid bc$

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $c \in \mathbb{Z}$ such that $a \mid b$. Since $a \mid b$, there exists $x \in \mathbb{Z}$ such that $b = ax$. Note $bc = acx$. Let $y = cx \in \mathbb{Z}$. There exists $y \in \mathbb{Z}$ such that $bc = ay$. Therefore, $a \mid bc$. ■

11 Given $a, b, d, x, y \in \mathbb{Z}$, if $d \,|\, a$ and $d \,|\, b$, then $d \,|\, (ax + by)$

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $d \in \mathbb{Z}$, $x \in \mathbb{Z}$, and $y \in \mathbb{Z}$ such that $d \,|\, a$ and $d \,|\, b$. Since $d \,|\, a$, there exists $c \in \mathbb{Z}$ such that $a = cd$. Since $d \,|\, b$, there exists $z \in \mathbb{Z}$ such that $b = dz$. Observe

$$\begin{aligned} ax + by &= (cd)(x) + (dz)(z) \\ &= d(cx + yz). \end{aligned}$$

Let $r = cx + yz \in \mathbb{Z}$. There exists $r \in \mathbb{Z}$ such that $ax + by = dr$. Therefore, $d \,|\, (ax + by)$. ■

12 Given $a, b, c, d \in \mathbb{Z}$, if $a \,|\, b$ and $c \,|\, d$, then $ac \,|\, bd$

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $c \in \mathbb{Z}$, and $d \in \mathbb{Z}$ such that $a \,|\, b$ and $c \,|\, d$. Since $a \,|\, b$, there exists $x \in \mathbb{Z}$ such that $b = ax$. Since $c \,|\, d$, there exists $y \in \mathbb{Z}$ such that $d = cy$. Note $bd = acxy$. Let $z = xy \in \mathbb{Z}$. There exists $z \in \mathbb{Z}$ such that $bd = acz$. Therefore $ac \,|\, bd$. ■

13 Given $x \in \mathbb{Z}$, x is odd $\iff$ $x + 1$ is even

Let $x \in \mathbb{Z}$.

First, we will prove that if $x+1$ is even, then x is odd. Suppose $x+1$ is even. Since $x + 1$ is even, $2 \,|\, (x + 1)$. Thus, there exists $y \in \mathbb{Z}$ such that $x + 1 = 2y$. Observe

$$\begin{aligned} x + 1 &= 2y \\ x &= 2y - 1 \\ x &= 2(y - 1) + 1. \end{aligned}$$

Let $z = y - 1 \in \mathbb{Z}$. There exists $z \in \mathbb{Z}$ such that $x = 2z + 1$. Therefore, x is odd if $x + 1$ is even.

Next, we will prove that if x is odd, then $x + 1$ is even. Suppose x is odd. Since x is odd, there exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Observe

$$\begin{aligned} x &= 2a + 1 \\ x + 1 &= 2a + 2 \\ x + 1 &= 2(a + 1). \end{aligned}$$

Let $b = a + 1 \in \mathbb{Z}$. There exists $b \in \mathbb{Z}$ such that $x + 1 = 2b$. Therefore, $x + 1$ is even if x is odd.

We conclude that x is odd if and only if $x + 1$ is even. ■

14 x is odd $\iff$ $\exists\, b \in \mathbb{Z}$ such that $x = 2b - 1$

Let $x \in \mathbb{Z}$.

First, we will prove that if there exists $b \in \mathbb{Z}$ such that $x = 2b - 1$, then x is odd. Suppose there exists $b \in \mathbb{Z}$ such that $x = 2b - 1$. Observe

$$x = 2b - 1 = 2(b - 1) + 1.$$

Let $a = b - 1 \in \mathbb{Z}$. There exists $a \in \mathbb{Z}$ such that $x = 2a + 1$. Therefore, x is odd.

Next, we will prove that if x is odd, then there exists $b \in \mathbb{Z}$ such that $x = 2b - 1$. Suppose x is odd. Since x is odd, there exists $c \in \mathbb{Z}$ such that $x = 2c + 1$. Observe

$$x = 2c + 1 = 2(c + 1) - 1.$$

Let $b = c + 1 \in \mathbb{Z}$. Therefore, there exists $b \in \mathbb{Z}$ such that $x = 2b - 1$.

We conclude that x is odd if and only if there exists $b \in \mathbb{Z}$ such that $x = 2b - 1$. ■

15 Given $x \in \mathbb{Z}$, $0 \mid x \iff x = 0$

Let $x \in \mathbb{Z}$.

First, we will prove that if $x = 0$, then $0 \mid x$. We want to find $n \in \mathbb{Z}$ such that $x = 0n$. Note $x = 0n = 0$ for all integers n (including 0). Let $n = 0$. Therefore $0 \mid x$.

Next, we will prove that if $0 \mid x$, then $x = 0$. Since $0 \mid x$, there exists $n \in \mathbb{Z}$ such that $x = 0n = 0$. Therefore $x = 0$.

We conclude that $0 \mid x$ if and only if $x = 0$. ■