MATH-UA 120 Section 3

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Integer

The *integers* are the positive whole numbers, the negative whole numbers, and zero. That is, the set of integers, denoted by the letter $\mathbb{Z}$, is

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

Divisible

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. We also say b *divides* a, or b is a *factor* of a, or b is a *divisor* of a. The notation for this is $b|a$.

Even

An integer is called *even* provided it is divisible by two.

Odd

An integer a is called *odd* provided there is an integer x such that $a = 2x + 1$.

Prime

An integer p is called *prime* provided that $p > 1$ and the only positive divisors of p are 1 and p.

Composite

A positive integer a is called *composite* provided there is an integer b such that $1 < b < a$ and $b|a$.

Natural number

The set of *natural numbers* (the nonnegative integers), denoted by the letter $\mathbb{N}$, is

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

Rational number

The set of *rational numbers* (the numbers formed by dividing two integers), denoted by the letter $\mathbb{Q}$, is

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

Perfect

An integer n is called *perfect* provided it equals the sum of all its divisors that are both positive and less than n. For example, 28 is perfect because the positive divisors of 28 are 1, 2, 4, 7, 14, and 28. Note that $1 + 2 + 4 + 7 + 14 = 28$.

1 Determine

Please determine which of the following are true and which are false.

a. $3 \nmid 100$. We want to find an integer c such that $100 = 3c$. No such $c \in \mathbb{Z}$ exists. Therefore, $3 \nmid 100$.

b. $3 \mid 99$. We want to find an integer c such that $99 = 3c$. Let $c = 33$. Thus, $99 = 3 \times 33$. Therefore, $3 \mid 99$.

c. $-3 \mid 3$. We want to find an integer c such that $3 = -3c$. Let $c = -1$. Thus, $3 = -3 \times -1$. Therefore, $-3 \mid 3$.

d. $-5 \mid -5$. We want to find an integer c such that $-5 = -5c$. Let $c = 1$. Thus, $-5 = -5 \times 1$. Therefore, $-5 \mid -5$.

e. $-2 \nmid -7$. We want to find an integer c such that $-7 = -2c$. No such $c \in \mathbb{Z}$ exists. Therefore $-2 \nmid -7$.

f. $0 \nmid 4$. We want to find an integer c such that $4 = 0c$, or $4 = 0$. This statement is absurd. Therefore, $0 \nmid 4$.

g. $4 \mid 0$. We want to find an integer c such that $0 = 4c$. Let $c = 0$. Thus, $0 = 4 \times 0$. Therefore, $4 \mid 0$.

h. $0 \mid 0$. We want to find an integer c such that $0 = 0c$. Let $c = 1$. Thus, $0 = 0 \times 1$. Therefore, by our definition of *divisible*, $0 \mid 0$.

2 A possible alternative

The alternative definition of *divisible* states that "a is divisible by b provided $\frac{a}{b}$ is an integer." This alternative definition is different from the definition above because it involves division instead of multiplication. Division enforces an additional restriction on the inputs: The divisor must not be zero. According to the original definition, $0 \mid 0$ because an integer c exists such that $0 = 0c$ (for example, letting $c = 1$ yields $0 = 0 \times 1$). However, the alternative definition disagrees: $0 \nmid 0$ because $\frac{0}{0}$ is undefined.

3 None

None of the following numbers is prime. Explain why. Which of the following are composite?

a. 21 is composite. To demonstrate that 21 is prime, want to show that the only positive divisors of 21 are 1 and 21 and that 21 is greater than 1. $21 > 1$ and $3 \mid 21$. To illustrate that $3 \mid 1$, we must find an integer c such that $21 = 3c$. Let $c = 7$. Thus, $21 = 3 \times 7$. Therefore, $3 \mid 21$. However, 3 is neither 1 nor 21, so 21 is not prime. For 21 to be composite, there must be an integer b such that $1 < b < 21$ and $b \mid 21$. Let $b = 3$. $1 < 3 < 21$ and $3 \mid 21$, so 21 is composite.

b. 0 is not prime. By definition, prime numbers must be greater than 1. $0 < 1$. Therefore, 0 is not prime. For 0 to be composite, there must be an integer b such that $1 < b < 0$, which is absurd. Therefore, 0 is not composite.

c. π is not prime. By definition, prime and composite numbers must be integers. $\pi \notin \mathbb{Z}$. Therefore, π is neither prime nor composite.

d. $\frac{1}{2}$ is not prime. By definition, prime and composite numbers must be integers. $\pi \notin \mathbb{Z}$. Therefore, π is neither prime nor composite.

e. -2 is not prime. By definition, prime numbers must be greater than 1. $-2 < 1$. Therefore -2 is not prime. For -2 to be composite, there must be an integer b such that $1 < b < -2$, which is absurd. Therefore, -2 is not composite.

f. -1 is not prime. By definition, prime numbers must be greater than 1. $-1 < 1$. Therefore -2 is not prime. For -1 to be composite, there must be an integer b such that $1 < b < -1$, which is absurd.

4 Create definitions

4.1 Less than or equal to

Let x and y be integers. We say that x is *less than or equal to* y (denoted $x \leq y$) provided $y - x \in \mathbb{N}$.

4.2 Less than

Let x and y be integers. We say that x is *less than* y (denoted $x < y$) provided $x \leq y$ and $x \neq y$.

4.3 Greater than or equal to

Let x and y be integers. We say that x is *greater than or equal to* y (denoted $x \geq y$) provided that x is not less than y.

4.4 Greater than

Let x and y be integers. We say that x is *greater than* y (denoted $x > y$) provided that x is not less than or equal to y.

5 Explain

Every integer is a rational number, but not every rational number is an integer because every integer k can be expressed as a ratio $\frac{k}{1}$, but there are rational numbers that are not integers, such as $\frac{1}{2}$.

6 Define perfect square

An integer x is called a *perfect square* provided there is an integer y such that $y^2 = x$.

7 Define square root

A number x is a *square root* of a number y provided that $x^2 = y$.

8 Define perimeter of a polygon

The *perimeter* of a polygon is the total length of its boundary.

9 Define between

10 Define midpoint of a line segment

The *midpoint* of a line segment $\overline{AB}$ is a point C on the segment such that the distance from A to C equals the distance from C to B.

11 Try writing definitions

11.1 Teenager

A person p is called a *teenager* provided that p's age is between thirteen and nineteen.

11.2 Grandmother

Person A is the *grandmother* of person B provided A is female and A is the parent of one of B's parents.

11.3 Leap year

A year Y with the number y in the Gregorian Calendar is called a *leap year* provided y is divisible by four hundred or else divisible by four but not one hundred.

11.4 Dime

A coin c is called a *dime* provided that its face value is five cents.

11.5 Palindrome

A word w is called a *palindrome* provided its spelling is identical when characters are read from left to right as when read from right to left.

11.6 Homophone

A pair of words, w and x, are called *homophones* if they are spelled differently but share the same pronunciation.

11.7 Counting problems

a. 8 has four positive divisors: 1, 2, 4, 8.

b. 32 has six positive divisors: 1, 2, 4, 8, 16, 32.

c. 2^n has $n+1$ positive divisors, where n is a positive integer.

d. 10 has 4 positive divisors: 1, 2, 5, 10.

e. 100 has 9 positive divisors: 1, 2, 4, 5, 10, 20, 25, 50, 100.

f. 1,000,000 has 49 positive divisors.

g. 10^n has n^2 positive divisors, where n is a positive integer.

h. $30 = 2 \times 3 \times 5$ has 8 positive divisors: 1, 2, 3, 5, 6, 10, 15, 30.

i. $42 = 2 \times 3 \times 7$ has 8 positive divisors: 1, 2, 3, 6, 7, 14, 21, 42. Thirty and forty-two have the same number of positive divisors because the product of the counts of the prime factors, adding one to each, is the same: $2 \times 2 \times 2 = 2 \times 2 \times 2 = 8$.

j. $2310 = 2 \times 3 \times 5 \times 7 \times 11$ has $2^5 = 32$ factors.

k. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$ can be expressed as $2^7 \times 3^2 \times 5 \times 7$. Thus it has $8 \times 3 \times 2^2 = 96$ factors.

l. 0 has uncountably many positive divisors.

12 Find

a. There is a perfect number smaller than 28: 6 is perfect because the positive divisors of 6 are 1, 2, 3, and 6. Note that $1 + 2 + 3 = 6$.

b. A computer program to find the next perfect number after 28:

```
#include <limits.h>
#include <stdio.h>

/**
 *  Computes the first perfect number greater than the given number.
 *
 * @param  min the previous perfect number from which to begin the search
 * @return The next perfect number.
*/

static int getPerfectAbove(int min)
{
   for (int value = min + 1; value < INT_MAX; value++)
   {
      int sum = 0;

      for (int factor = 1; factor < value; factor++)
      {
         if (value % factor == 0)
         {
            sum += factor;
         }
      }

      if (value == sum)
      {
         return value;
      }
    }
}
```

```
/**
 * The main entry point for the application.
 *
 * @return An exit code.
*/

int main()
{
   printf("%d", getPerfectAbove(28));

   return 0;
}
```

13 The mathematician

Mathematicians rely on definitions. In the story, the mathematician does not deviate from the definition.