Answers to Problem Set 5

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MATH-UA 120 Discrete Mathematics

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These are to be written up in LaTeX and turned in to Gradescope.

Assigned Problems

1. *Prove the following statement by contrapositive:*
For all $n \in \mathbb{N}$, if $2^n < n!$, then $n > 3$.

Answer.

Claim. Let $n \in \mathbb{N}$. If $2^n < n!$, then $n > 3$.

Proof. Let $n \in \mathbb{N}$. We demonstrate the validity of the contrapositive of the claim. Since $n \in \mathbb{N}$, and $n \leq 3$, we have $n = 0$, $n = 1$, $n = 2$, or $n = 3$.

Suppose $n = 0$. Then $2^0 = 1$, and $0! = 1$. Since $1 = 1$, we have $2^n \geq n!$.
Suppose $n = 1$. Then $2^1 = 2$, and $1! = 1$. Since $2 > 1$, we have $2^n \geq n!$.
Suppose $n = 2$. Then $2^2 = 4$, and $2! = 2$. Since $4 > 2$, we have $2^n \geq n!$.
Suppose $n = 3$. Then $2^3 = 8$, and $3! = 6$. Since $8 > 6$, we have $2^n \geq n!$.

We have for all cases of $n \leq 3$ that $2^n \geq n!$. Hence if $2^n < n!$, then $n > 3$. □

2. *Prove the following by contradiction:*
Let A, B, C be sets. If $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Answer.

Claim. Let A, B, and C be sets. If $A \subseteq B$, and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Proof. Let A, B, and C be sets. Suppose $A \subseteq B$, and $B \cap C = \emptyset$. Assume, for the sake of contradiction, that there exists $x \in A \cap C$. Then $x \in A$, and $x \in C$. Since $A \subseteq B$, we have $x \in B$. However, $B \cap C = \emptyset$, even while $x \in B$ and $x \in C$—which is absurd. Ergo, our assumption is false. There exists no $x \in A \cap C$. Hence $A \cap C = \emptyset$. □

3. *Prove the following statement by contradiction:*
Let $x, y \in \mathbb{Z}$. Then $x^2 - 4y - 3 \neq 0$.

Answer.

Claim. Let $x, y \in \mathbb{Z}$. Then $x^2 - 4y - 3 \neq 0$.

Proof. Let $x, y \in \mathbb{Z}$. Assume, for the sake of contradiction, that $x^2 - 4y - 3 = 0$. Note $x^2 = 4y + 3$. Since $x \in \mathbb{Z}$, we have $x^2 \geq 0$. Thus $4y + 3 \geq 0$. Now $y \geq -\frac{3}{4}$. However, y is an arbitrary integer, so y is not necessarily greater than $-\frac{3}{4}$. We have $y \geq -\frac{3}{4}$, even while y is any integer—which is absurd. Ergo, our assumption is false. Hence $x^2 - 4y - 3 \neq 0$. $\square$

4. *Prove the following by smallest counterexample:*
Let $n \in \mathbb{N}$. If $n \geq 1$, then $4 \mid (5^n - 1)$.

Answer.

Claim. Let $n \in \mathbb{N}$. If $n \geq 1$, then $4 \mid (5^n - 1)$.

Proof. Let $n \in \mathbb{N}$. Suppose $n \geq 1$. Assume, for the sake of contradiction, that $4 \nmid (5^n - 1)$. Let $X = \{n \in \mathbb{N} : n \geq 1 \text{ and } 4 \nmid (5^n - 1)\}$. Then $X \neq \emptyset$. By the well-ordering principle, there exists $x \in X$ such that x is the least element of X. Note $\left(5^1 - 1\right) = 4$, and $4 \mid 4$, so $x \neq 1$. Then $x - 1 \in \mathbb{N}$, but, $x - 1 \notin X$. Thus $4 \mid \left(5^{x-1} - 1\right)$, so there exists $k \in \mathbb{Z}$ such that $\left(5^{x-1} - 1\right) = 4k$. Observe

$$\begin{aligned}
\left(5^{x-1} - 1\right) &= 4k \\
5\left(5^{x-1} - 1\right) &= 5(4k) \\
(5^x - 5) &= 20k \\
(5^x - 1) &= 20k + 4 \\
(5^x - 1) &= 4(5k + 1).
\end{aligned}$$

Note $5k + 1 \in \mathbb{Z}$. So we have $4 \mid (5^x - 1)$, even while $x \in X$—which is absurd. Ergo, our assumption is false. Hence if $n \geq 1$, then $4 \mid (5^n - 1)$. $\square$

5. *Let $n \in \mathbb{Z}$. Use induction to prove there are $3 \mid (n^3 + 2n)$.*

Answer.

Lemma. Let $n \in \mathbb{N}$. We demonstrate that $3 \mid \left(n^3 + 2n\right)$ by induction on n.

Base case: Consider $n = 0$. Note $0^3 + 2 \cdot 0 = 0$. Note also $0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$, so $3 \mid 0$. Therefore $3 \mid \left(n^3 + 2n\right)$ for $n = 0$.

Inductive hypothesis: Let $k \in \mathbb{N}$. Of course, $k \geq 0$. Consider $n = k$. Assume the result is true for $n = k$; that is, assume $3 \mid \left(k^3 + 2k\right)$.

Inductive step: Consider $n = k + 1$. By the inductive hypothesis, $3 \mid \big(k^3 + 2k\big)$. Thus there exists $a \in \mathbb{Z}$ such that $\big(k^3 + 2k\big) = 3a$. Observe

$$\begin{aligned}
\big(k^3 + 2k\big) &= 3a \\
\big(k^3 + 2k\big) + \big(3k^2 + 3k + 3\big) &= 3a + \big(3k^2 + 3k + 3\big) \\
\big(k^3 + 3k^2 + 3k + 1\big) + 2k + 2 &= 3a + \big(3k^2 + 3k + 3\big) \\
(k + 1)^3 + 2(k + 1) &= 3a + \big(3k^2 + 3k + 3\big) \\
(k + 1)^3 + 2(k + 1) &= 3\left(a + k^2 + k + 1\right).
\end{aligned}$$

Note $\big(a + k^2 + k + 1\big) \in \mathbb{Z}$, so $3 \mid \big((k + 1)^3 + 2(k + 1)\big)$.

Hence, by the principle of mathematical induction, for all $n \in \mathbb{N}$, we have $3 \mid \big(n^3 + 2n\big)$.

Claim. Let $x \in \mathbb{Z}$. Then $3 \mid \big(x^3 + 2x\big)$.

Proof. Let $x \in \mathbb{Z}$. Now we may demonstrate that $3 \mid \big(x^3 + 2x\big)$. Since $x \in \mathbb{Z}$, either $x < 0$ or $x \geq 0$.

Suppose $x < 0$. Then $-x \geq 0$, so $-x \in \mathbb{N}$. By the lemma, $3 \mid \big((-x)^3 + 2(-x)\big)$. So there exists $b \in \mathbb{Z}$ such that $(-x)^3 + 2(-x) = 3b$. Observe

$$\begin{aligned}
(-x)^3 + 2(-x) &= 3b \\
-x^3 - 2x &= 3b \\
-(x^3 + 2x) &= 3b \\
x^3 + 2x &= 3(-b).
\end{aligned}$$

Note $(-b) \in \mathbb{Z}$, so $3 \mid \big(x^3 + 2x\big)$.

Suppose instead $x \geq 0$. Then $x \in \mathbb{N}$. By the lemma, $3 \mid \big(x^3 + 2x\big)$.

We have for all cases of $x \in \mathbb{Z}$ that $3 \mid \big(x^3 + 2x\big)$. □