Polished Proof 2

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LEMMA.

Claim. Let $A_0 = \left\{\frac{1}{2}, 1\right\}$. For all $n \in \mathbb{N}$, let

$$A_{n+1} = \{ab : a, b \in A_n\} \cup \left\{\frac{a+b}{2} : a, b \in A_n\right\}.$$

Let $a_n \in A_n$. Then $0 \leq a_n \leq 1$.

Proof. Let $A_0 = \left\{\frac{1}{2}, 1\right\}$. For all $n \in \mathbb{N}$, let $A_{n+1} = \{ab : a, b \in A_n\} \cup \left\{\frac{a+b}{2} : a, b \in A_n\right\}$. Let $a_n \in A_n$. We will demonstrate that $0 \leq a_n \leq 1$ by induction on n.

Basis case. Consider $n = 0$. Then

$$A_n = A_0 = \left\{\frac{1}{2}, 1\right\}.$$

Note $0 \leq \frac{1}{2} \leq 1$, and $0 \leq 1 \leq 1$. Therefore, for all $a_0 \in A_0$, we have $0 \leq a_0 \leq 1$.

Inductive hypothesis. Let $k \in \mathbb{N}$. Consider $n = k$. Assume that for all $a_k \in A_k$, we have $0 \leq a_k \leq 1$.

Inductive step. Consider $n = k + 1$. We have

$$A_{k+1} = \{ab : a, b \in A_k\} \cup \left\{\frac{a+b}{2} : a, b \in A_k\right\}.$$

Let $a_{k+1} \in A_{k+1}$. Thus $a_{k+1} \in \{ab : a, b \in A_k\}$ or $a_{k+1} \in \left\{\frac{a+b}{2} : a, b \in A_k\right\}$.

Suppose $a_{k+1} \in \{ab : a, b \in A_k\}$. Then there exists $x_1, y_1 \in A_k$ such that $a_{k+1} = x_1 y_1$. Since $x_1 \in A_k$ and $y_1 \in A_k$, we have $0 \leq x_1 \leq 1$ and $0 \leq y_1 \leq 1$ by the inductive hypothesis. Since $x_1 \geq 0$ and $y_1 \geq 0$, we have $x_1 y_1 \geq 0$. Since $x_1 \geq 0$, $y_1 \geq 0$, $x_1 \leq 1$, and $y_1 \leq 1$, we have $x_1 y_1 \leq 1$. Thus $0 \leq x_1 y_1 \leq 1$. Therefore $0 \leq a_{k+1} \leq 1$.

Suppose $a_{k+1} \in \left\{\frac{a+b}{2} : a, b \in A_k\right\}$. Then there exists $x_2, y_2 \in A_k$ such that $a_{k+1} = \frac{x_2+y_2}{2}$. Since $x_2 \in A_k$ and $y_2 \in A_k$, we have $0 \leq x_2 \leq 1$ and $0 \leq y_2 \leq 1$ by the inductive hypothesis. Since $x_2 \geq 0$ and $y_2 \geq 0$, we have $x_2 + y_2 \geq 0$. Thus $\frac{x_2+y_2}{2} \geq 0$. Since $x_2 \leq 1$ and $y_2 \leq 1$, we have $\frac{x_2}{2} \leq \frac{1}{2}$ and $\frac{y_2}{2} \leq \frac{1}{2}$. Note $\left(\frac{x_2}{2} + \frac{y_2}{2}\right) \leq \left(\frac{1}{2} + \frac{1}{2}\right)$. So $\frac{x_2+y_2}{2} \leq 1$. Thus $0 \leq \frac{x_2+y_2}{2} \leq 1$. Therefore $0 \leq a_{k+1} \leq 1$.

In all cases, for all $a_{k+1} \in A_k$, we have $0 \leq a_{k+1} \leq 1$, thus completing the inductive step.

Hence, for all $n \in \mathbb{N}$, for all $a_n \in A_n$, we have $0 \leq a_n \leq 1$.

PROPOSITION.

Claim. Let $A_0 = \left\{\frac{1}{2}, 1\right\}$. For all $n \in \mathbb{N}$, let

$$A_{n+1} = \{ab : a, b \in A_n\} \cup \left\{\frac{a+b}{2} : a, b \in A_n\right\}.$$

Let

$$A = \bigcup_{j=0}^{\infty} A_j.$$

If $x \in A$, then $0 \leq x \leq 1$.

Proof. Let $A_0 = \left\{\frac{1}{2}, 1\right\}$. For all $n \in \mathbb{N}$, let $A_{n+1} = \{ab : a, b \in A_n\} \cup \left\{\frac{a+b}{2} : a, b \in A_n\right\}$. Let $A = \bigcup_{j=0}^{\infty} A_j$. Let $x \in A$. Of course,

$$x \in (A_0 \cup A_1 \cup A_2 \cup \dots).$$

So there exists $j \in \mathbb{N}$ such that $x \in A_j$. Since $j \in \mathbb{N}$ and $x \in A_j$, we have $0 \leq x \leq 1$ by lemma.

Hence if $x \in A$, then $0 \leq x \leq 1$. $\square$