MATH-UA 120 Section 7

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And

The operation *and*, denoted $\wedge$, is defined:

$$\top \wedge \top = \top,$$

$$\top \wedge \bot = \bot,$$

$$\bot \wedge \top = \bot,$$

$$\bot \wedge \bot = \bot.$$

Or

The operation *or*, denoted $\vee$, is defined:

$$\top \vee \top = \top,$$

$$\top \vee \bot = \top,$$

$$\bot \vee \top = \top,$$

$$\bot \vee \bot = \bot.$$

Not

The operation *not*, denoted $\neg$, is defined:

$$\neg\top = \bot,$$

$$\neg\bot = \top.$$

Proposition 6

The Boolean expressions $\neg(x \wedge y)$ and $(\neg x) \vee (\neg y)$ are logically equivalent.

x	y	$x \wedge y$	$\neg(x \wedge y)$	$\neg x$	$\neg y$	$(\neg x) \vee (\neg y)$
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\top$

Commutative property of *and*

$$x \wedge y = y \wedge x.$$

Commutative property of *or*

$$x \vee y = y \vee x.$$

Associative property of *and*

$$(x \wedge y) \wedge z = x \wedge (y \wedge z).$$

Associative property of *or*

$$(x \vee y) \vee z = x \vee (y \vee z).$$

True identity

$$x \wedge \top = x.$$

False identity

$$x \vee \bot = x.$$

Idemopotency

$$x \wedge x = x.$$

$$x \vee x = x.$$

Double negative

$$\neg(\neg x) = x.$$

Distributive property of *and*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Distributive property of *or*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Inverse element

$$x \wedge (\neg x) = \bot.$$
$$x \vee (\neg x) = \top.$$

De Morgan's laws

$$\neg(x \wedge y) = (\neg x) \vee (\neg y),$$

$$\neg(x \vee y) = (\neg x) \wedge (\neg y).$$

Implication

The *material conditional* operation (also called an *if-then* or *implication*), denoted $\rightarrow$, is defined:

x	y	$x \rightarrow y$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

Equivalence

The *material biconditional* operation (also called an *if and only if* or *equivalence*), denoted $\leftrightarrow$, is defined:

x	y	$x \leftrightarrow y$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

Proposition 7

The expressions $x \to y$ and $(\neg x) \vee y$ are logically equivalent.

x	y	$x \to y$	$\neg x$	$(\neg x) \vee y$
$\top$	$\top$	$\top$	$\perp$	$\top$
$\top$	$\perp$	$\perp$	$\perp$	$\perp$
$\perp$	$\top$	$\top$	$\top$	$\top$
$\perp$	$\perp$	$\top$	$\top$	$\top$

The columns for $x \to y$ and $(\neg x) \vee y$ are the same, and therefore these expressions are logically equivalent.

Tautology

A Boolean expression is a *tautology* if the evaluation of all possible values of its variables is true.

Contradiction

A Boolean expression is a *contradiction* if the evaluation of all possible values of its variables is false.

Contingency

A Boolean expression is a *contingency* if its evaluation is sometimes true and sometimes false.

1 Calculations

a. $\top \wedge \top \wedge \top \wedge \top \wedge \perp = \perp$.

b. $(\neg \top) \vee \top = \top$.

c. $\neg(\top \vee \top) = \perp$.

d. $(\top \vee \top) \wedge \perp = \perp$.

e. $\top \vee (\top \wedge \perp) = \top$.

2 Prove: $(x \wedge y) \vee (x \wedge \neg y) = x$

$$\begin{aligned}
(x \wedge y) \vee (x \wedge \neg y) &= x \wedge (y \vee \neg y), && \textit{distributive property;} \\
&= x \wedge \top, && \textit{inverse elements;} \\
&= x, && \textit{identity.}\ \blacksquare
\end{aligned}$$

3 Prove: $x \to y = (\neg y) \to (\neg x)$

$$
\begin{aligned}
x \to y &= (\neg x) \vee y, && \textit{Proposition 7;}\\
&= y \vee (\neg x), && \textit{commutative property;}\\
&= \neg(\neg y) \vee (\neg x), && \textit{double negative;}\\
&= (\neg y) \to (\neg x), && \textit{Proposition 7.}
\end{aligned}
$$

We conclude that an *if-then* statement $(x \to y)$ is logically equivalent to its contrapositive, $(\neg y) \to (\neg x)$. ■

4 Prove: $x \leftrightarrow y = (x \to y) \wedge (y \to x)$

$$
\begin{aligned}
x \leftrightarrow y &= ((\neg x) \vee y) \wedge ((\neg y) \vee x)\\
&= ((\neg x) \vee y) \wedge (y \to x)\\
&= (x \to y) \wedge (y \to x). \blacksquare
\end{aligned}
$$

5 Prove: $x \to y = (\neg x) \vee y$

x	y	$x \to y$	$\neg x$	$(\neg x) \vee y$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

We conclude that $x \to y$ implies $(\neg x) \vee y$. ■