

# MATH-UA 120 Section 3

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## Integer

The *integers* are the positive whole numbers, the negative whole numbers, and zero. That is, the set of integers, denoted by the letter  $\mathbb{Z}$ , is

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

## Divisible

Let  $a$  and  $b$  be integers. We say that  $a$  is *divisible* by  $b$  provided there is an integer  $c$  such that  $bc = a$ . We also say  $b$  *divides*  $a$ , or  $b$  is a *factor* of  $a$ , or  $b$  is a *divisor* of  $a$ . The notation for this is  $b|a$ .

## Even

An integer is called *even* provided it is divisible by two.

## Odd

An integer  $a$  is called *odd* provided there is an integer  $x$  such that  $a = 2x + 1$ .

## Prime

An integer  $p$  is called *prime* provided that  $p > 1$  and the only positive divisors of  $p$  are 1 and  $p$ .

## Composite

A positive integer  $a$  is called *composite* provided there is an integer  $b$  such that  $1 < b < a$  and  $b|a$ .

## Natural number

The set of *natural numbers* (the nonnegative integers), denoted by the letter  $\mathbb{N}$ , is

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

## Rational number

The set of *rational numbers* (the numbers formed by dividing two integers), denoted by the letter  $\mathbb{Q}$ , is

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

## Perfect

An integer  $n$  is called *perfect* provided it equals the sum of all its divisors that are both positive and less than  $n$ . For example, 28 is perfect because the positive divisors of 28 are 1, 2, 4, 7, 14, and 28. Note that  $1 + 2 + 4 + 7 + 14 = 28$ .

## 1 Determine

Please determine which of the following are true and which are false.

- a.  $3 \nmid 100$ . We want to find an integer  $c$  such that  $100 = 3c$ . No such  $c \in \mathbb{Z}$  exists. Therefore,  $3 \nmid 100$ .
- b.  $3 \mid 99$ . We want to find an integer  $c$  such that  $99 = 3c$ . Let  $c = 33$ . Thus,  $99 = 3 \times 33$ . Therefore,  $3 \mid 99$ .
- c.  $-3 \mid 3$ . We want to find an integer  $c$  such that  $3 = -3c$ . Let  $c = -1$ . Thus,  $3 = -3 \times -1$ . Therefore,  $-3 \mid 3$ .
- d.  $-5 \mid -5$ . We want to find an integer  $c$  such that  $-5 = -5c$ . Let  $c = 1$ . Thus,  $-5 = -5 \times 1$ . Therefore,  $-5 \mid -5$ .
- e.  $-2 \nmid -7$ . We want to find an integer  $c$  such that  $-7 = -2c$ . No such  $c \in \mathbb{Z}$  exists. Therefore  $-2 \nmid -7$ .
- f.  $0 \nmid 4$ . We want to find an integer  $c$  such that  $4 = 0c$ , or  $4 = 0$ . This statement is absurd. Therefore,  $0 \nmid 4$ .
- g.  $4 \mid 0$ . We want to find an integer  $c$  such that  $0 = 4c$ . Let  $c = 0$ . Thus,  $0 = 4 \times 0$ . Therefore,  $4 \mid 0$ .
- h.  $0 \mid 0$ . We want to find an integer  $c$  such that  $0 = 0c$ . Let  $c = 1$ . Thus,  $0 = 0 \times 1$ . Therefore, by our definition of *divisible*,  $0 \mid 0$ .

## 2 A possible alternative

The alternative definition of *divisible* states that “ $a$  is divisible by  $b$  provided  $\frac{a}{b}$  is an integer.” This alternative definition is different from the definition above because it involves division instead of multiplication. Division enforces an additional restriction on the inputs: The divisor must not be zero. According to the original definition,  $0 \mid 0$  because an integer  $c$  exists such that  $0 = 0c$  (for example, letting  $c = 1$  yields  $0 = 0 \times 1$ ). However, the alternative definition disagrees:  $0 \nmid 0$  because  $\frac{0}{0}$  is undefined.

## 3 None

None of the following numbers is prime. Explain why. Which of the following are composite?

- a. 21 is composite. To demonstrate that 21 is prime, want to show that the only positive divisors of 21 are 1 and 21 and that 21 is greater than 1.  $21 > 1$  and  $3 \mid 21$ . To illustrate that  $3 \mid 21$ , we must find an integer  $c$  such that  $21 = 3c$ . Let  $c = 7$ . Thus,  $21 = 3 \times 7$ . Therefore,  $3 \mid 21$ . However, 3 is neither 1 nor 21, so 21 is not prime. For 21 to be composite, there must be an integer  $b$  such that  $1 < b < 21$  and  $b \mid 21$ . Let  $b = 3$ .  $1 < 3 < 21$  and  $3 \mid 21$ , so 21 is composite.
- b. 0 is not prime. By definition, prime numbers must be greater than 1.  $0 < 1$ . Therefore, 0 is not prime. For 0 to be composite, there must be an integer  $b$  such that  $1 < b < 0$ , which is absurd. Therefore, 0 is not composite.
- c.  $\pi$  is not prime. By definition, prime and composite numbers must be integers.  $\pi \notin \mathbb{Z}$ . Therefore,  $\pi$  is neither prime nor composite.
- d.  $\frac{1}{2}$  is not prime. By definition, prime and composite numbers must be integers.  $\frac{1}{2} \notin \mathbb{Z}$ . Therefore,  $\frac{1}{2}$  is neither prime nor composite.
- e. -2 is not prime. By definition, prime numbers must be greater than 1.  $-2 < 1$ . Therefore -2 is not prime. For -2 to be composite, there must be an integer  $b$  such that  $1 < b < -2$ , which is absurd. Therefore, -2 is not composite.
- f. -1 is not prime. By definition, prime numbers must be greater than 1.  $-1 < 1$ . Therefore -1 is not prime. For -1 to be composite, there must be an integer  $b$  such that  $1 < b < -1$ , which is absurd.

## 4 Create definitions

### 4.1 Less than or equal to

Let  $x$  and  $y$  be integers. We say that  $x$  is *less than or equal to*  $y$  (denoted  $x \leq y$ ) provided  $y - x \in \mathbb{N}$ .

## 4.2 Less than

Let  $x$  and  $y$  be integers. We say that  $x$  is *less than*  $y$  (denoted  $x < y$ ) provided  $x \leq y$  and  $x \neq y$ .

## 4.3 Greater than or equal to

Let  $x$  and  $y$  be integers. We say that  $x$  is *greater than or equal to*  $y$  (denoted  $x \geq y$ ) provided that  $x$  is not less than  $y$ .

## 4.4 Greater than

Let  $x$  and  $y$  be integers. We say that  $x$  is *greater than*  $y$  (denoted  $x > y$ ) provided that  $x$  is not less than or equal to  $y$ .

# 5 Explain

Every integer is a rational number, but not every rational number is an integer because every integer  $k$  can be expressed as a ratio  $\frac{k}{1}$ , but there are rational numbers that are not integers, such as  $\frac{1}{2}$ .

# 6 Define perfect square

An integer  $x$  is called a *perfect square* provided there is an integer  $y$  such that  $y^2 = x$ .

# 7 Define square root

A number  $x$  is a *square root* of a number  $y$  provided that  $x^2 = y$ .

# 8 Define perimeter of a polygon

The *perimeter* of a polygon is the total length of its boundary.

# 9 Define midpoint of a line segment

The *midpoint* of a line segment  $\overline{AB}$  is a point  $C$  on the segment such that the distance from  $A$  to  $C$  equals the distance from  $C$  to  $B$ .

# 10 Try writing definitions

## 10.1 Teenager

A person  $p$  is called a *teenager* provided that  $p$ 's age is between thirteen and nineteen.

## 10.2 Grandmother

Person  $A$  is the *grandmother* of person  $B$  provided  $A$  is female and  $A$  is the parent of one of  $B$ 's parents.

## 10.3 Leap year

A year  $Y$  with the number  $y$  in the Gregorian Calendar is called a *leap year* provided  $y$  is divisible by four hundred or else divisible by four but not one hundred.

## 10.4 Dime

A coin  $c$  is called a *dime* provided that its face value is five cents.

## 10.5 Palindrome

A word  $w$  is called a *palindrome* provided its spelling is identical when characters are read from left to right as when read from right to left.

## 10.6 Homophone

A pair of words,  $w$  and  $x$ , are called *homophones* if they are spelled differently but share the same pronunciation.

## 10.7 Counting problems

- a. 8 has four positive divisors: 1, 2, 4, 8.
- b. 32 has six positive divisors: 1, 2, 4, 8, 16, 32.
- c.  $2^n$  has  $n + 1$  positive divisors, where  $n$  is a positive integer.
- d. 10 has 4 positive divisors: 1, 2, 5, 10.
- e. 100 has 9 positive divisors: 1, 2, 4, 5, 10, 20, 25, 50, 100.
- f. 1,000,000 has 49 positive divisors.
- g.  $10^n$  has  $n^2$  positive divisors, where  $n$  is a positive integer.
- h.  $30 = 2 \times 3 \times 5$  has 8 positive divisors: 1, 2, 3, 5, 6, 10, 15, 30.
- i.  $42 = 2 \times 3 \times 7$  has 8 positive divisors: 1, 2, 3, 6, 7, 14, 21, 42. Thirty and forty-two have the same number of positive divisors because the product of the counts of the prime factors, adding one to each, is the same:  $2 \times 2 \times 2 = 2 \times 2 \times 2 = 8$ .
- j.  $2310 = 2 \times 3 \times 5 \times 7 \times 11$  has  $2^5 = 32$  factors.
- k.  $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$  can be expressed as  $2^7 \times 3^2 \times 5 \times 7$ . Thus it has  $8 \times 3 \times 2^2 = 96$  factors.
- l. 0 has uncountably many positive divisors.

## 11 Find

- a. There is a perfect number smaller than 28: 6 is perfect because the positive divisors of 6 are 1, 2, 3, and 6. Note that  $1 + 2 + 3 = 6$ .
- b. A computer program to find the next perfect number after 28:

```
#include <limits.h>
#include <stdio.h>

/**
 * Computes the first perfect number greater than the given number.
 *
 * @param min the previous perfect number from which to begin the search
 * @return The next perfect number, or the given min if no such number exists.
 */

static int getPerfectAbove(int min)
{
    for (int value = min + 1; value < INT_MAX; value++)
    {
        int sum = 0;

        for (int factor = 1; factor < value; factor++)
        {
            if (value % factor == 0)
            {
                sum += factor;
            }
        }

        if (value == sum)
        {
            return value;
        }
    }

    return min;
}

/**
 * The main entry point for the application.
 *
 * @return An exit code.
 */
```

```
int main()
{
    printf("%d", getPerfectAbove(28));

    return 0;
}
```

## 12 The mathematician

Mathematicians rely on definitions. In the story, the mathematician does not deviate from the definition.