STAT-UB 103 Homework 4

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1 Find

$$\sigma_X^2 = 25.$$

$$\sigma_Y^2 = 16.$$

$$\rho_{X,Y} = -0.5.$$

a.

$$\sigma_{X,Y} = \rho_{X,Y}\sigma_X\sigma_Y.$$

$$\sigma_{X,Y} = -0.5\sqrt{(25)(16)} = -10.$$

b.

$$\sigma_{2X,3Y} = (\rho_{X,Y})(2\sigma_X)(3\sigma_Y).$$

$$\sigma_{2X,3Y} = (-0.5)(2)(3)(4)(5) = -60.$$

c.

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{X,Y}.$$

$$\sigma_{X+Y}^2 = 25 + 16 - 2(10) = 21.$$

d.

$$\sigma_{2X+3Y}^2 = (2\sigma_X)^2 + (3\sigma_Y)^2 + 2\sigma_{2X,3Y}.$$

$$\sigma_{2X+3Y}^2 = 4\sigma_X^2 + 9\sigma_Y^2 + 2\sigma_{2X,3Y}.$$

$$\sigma_{2X+3Y}^2 = 4(25) + 9(16) - 2(60) = 124.$$

2 A motel

a. Let it be assumed each instance of a missed reservation is completely independent of all other instances. Each guest either arrives or does not arrive to claim their reservation. The number of guests (n) will be at most 20 per day. Since there are a fixed number of independent binary trials, each with a known probability of success (π), the probability that the number of guests (X) is equal to a given value (k) can be modeled with a binomial random variable.

$$\pi = 1 - 0.2 = 0.8.$$

$$P(X = k) = \binom{n}{k} \pi^k (1 - \pi)^{n-k}.$$

The number of guests (X) is a non-negative integer.

$$P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} \pi^i (1 - \pi)^{n-i}.$$

$$P(X \leq 15) = \sum_{i=0}^{15} \binom{20}{i} (0.8)^i (0.2)^{20-i} \approx 0.3704\ldots$$

This gives the probability of no overbooking since the number of guests is less than or equal to the number of rooms.

Let C represent the compensation that the hotel must pay.

$$C = (\$100)(X - 15).$$

$$X = \frac{C}{\$100} + 15$$

If there is no overbooking, then the compensation is zero. The compensation is non-negative.

$$P(X \leq 15) = P(C \leq 0) = P(C = 0) \approx 0.3704\ldots$$

If there are sixteen guests, then the compensation is \$100.

$$P(X = 16) = P(C = \$100) = \binom{20}{16} (0.8)^{16} (0.2)^4 \approx 0.2182\ldots$$

This gives the generalized probability density function for a given compensation c.

$$P(C = c) = \begin{cases} \sum_{i=0}^{15} \binom{20}{i} (0.8)^i (0.2)^{20-i}, & c = 0 \\ \\ \binom{20}{\frac{c}{\$100}+15} (0.8)^{\frac{c}{\$100}+15} (0.2)^{5-\frac{c}{\$100}}, & c > 0. \end{cases}$$

$$\begin{array}{rcccl} P(C=0) & = & P(X \leq 15) & \approx & 0.3704\ldots \\ P(C=\$100) & = & P(X=16) & \approx & 0.2182\ldots \\ P(C=\$200) & = & P(X=17) & \approx & 0.2054\ldots \\ P(C=\$300) & = & P(X=18) & \approx & 0.1369\ldots \\ P(C=\$400) & = & P(X=19) & \approx & 0.0576\ldots \\ P(C=\$500) & = & P(X=20) & \approx & 0.0115\ldots \end{array}$$

b. There will be between 0 and 5 overbookings on a given day. The number of overbookings (i) is a non-negative integer between 1 and 5. If there are no overbookings, then the compensation is \$0, leaving the expected value unchanged. For each overbooking, the compensation is \$100.

$$\mu_X = \sum_{i=0}^{n-1} \pi_{x_i} x_i.$$

$$c = \$100i.$$

$$\mu_C = \sum_{i=1}^{5} (\$100i) \binom{20}{\frac{\$100i}{\$100}+15} (0.8)^{\frac{\$100i}{\$100}+15} (0.2)^{5-\frac{\$100i}{\$100}}.$$

$$\mu_C = \$100 \sum_{i=1}^{5} \binom{20}{i+15} (0.8)^{i+15} (0.2)^{5-i} (i) \approx \$132.7886\ldots$$

The expected compensation is about \$132.79.

$$\sigma_X = \sqrt{\sum_{i=0}^{n-1} (\pi_{x_i})(x_i - \mu_X)^2}.$$

$$\begin{aligned} \sigma_C &= \sqrt{\sum_{i=0}^{15} (\pi_{c_i})(c_i - \mu_C)^2 + \sum_{i=16}^{20} (\pi_{c_i})(c_i - \mu_C)^2}, \\ &= \sqrt{\mu_C^2 \sum_{i=0}^{15} \binom{20}{i} (0.8)^i (0.2)^{20-i} + \sum_{i=16}^{20} \binom{20}{i} (0.8)^i (0.2)^{20-i} \left(\$100(i-15) - \mu_C\right)^2}, \\ &\approx \$131.1155\ldots \end{aligned}$$

The population standard deviation is about \$131.12.

3 The Lindell Corporation

The number of printers in a production run (N) is 20. The number of non-defective printers in the production run ($N - K$) is 18, so there are 2 defective printers in the run. A defective printer is defined as a positive result. The number of printers shipped (n) is 10. The trials are correlated due to sampling without replacement.

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

$$P(X = 0) = \frac{\binom{2}{0}\binom{18}{10}}{\binom{20}{10}} = \frac{\binom{18}{10}}{\binom{20}{10}} \approx 0.2368\ldots$$

The number of defective printers shipped (X) s non-negative.

$$P(X \geq 0) = P(X \neq 0) = 1 - \frac{\binom{18}{10}}{\binom{20}{10}} \approx 0.7632\ldots$$

The probability that at least one defective printer shipped is about 76.32%.

4 The number of users of an ATM machine

a. It is almost inconceivable for the number of people who use the ATM in a single 15-minute interval (X) to be between 26 and 28 people. The probability is near zero.

$$r = \frac{5}{10\text{ min}} = \frac{1}{2\text{ min}}.$$

$$t = 15\text{ min}.$$

$$\lambda = rt = \frac{15\text{ min}}{2\text{ min}} = 7.5.$$

$$P(X = k) = \frac{\lambda^k e^{-k}}{k!}.$$

$$P(26 \leq X \leq 28) = \sum_{i=26}^{28} \frac{7.5^i e^{-7.5}}{i!} \approx 1.047\ldots \times 10^{-7}.$$

b.

$$\lambda = \sigma_X^2.$$

$$t = 20\text{ min}.$$

$$\lambda = rt = \frac{20 \text{ min}}{2 \text{ min}} = 10.$$

$$\sigma_X^2 = \sqrt{\lambda} = \sqrt{10} \approx 3.1623\ldots$$

5 A shipment of fruit crates

The number of fruit crates (N) is 100. The number of crates in which the fruit shows signs of spoilage (K) is 11. A spoiled crate is defined as a positive result. The number of crates inspected (n) is 8. The trials are correlated due to sampling without replacement.

$$\lambda = \sigma_X^2.$$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

$$P(X = 2) = \frac{\binom{11}{2}\binom{100-11}{8-2}}{\binom{100}{8}} = \frac{\binom{11}{2}\binom{89}{6}}{\binom{100}{8}} \approx 0.1718\ldots$$

This is the probability that the number of spoiled crates in the sample (X) is 2.

6 The probability that an audit of a retail business will turn up irregularities in the collection of state sales tax

Let it be assumed each instance of an irregularity is completely independent of all other instances. Each audit either discovers or does not discover an irregularity. The number of audits (n) is 16. Since there is a fixed number of independent binary trials, each with a known probability of success (π), the probability that the number of audits with irregularities (X) is equal to a given value (k) can be modeled with a binomial random variable.

$$\pi \approx 0.316.$$

a.

$$P(X = k) = \binom{n}{k}\pi^k(1-\pi)^{n-k}.$$

$$P(X = 5) \approx \binom{16}{5}(0.316)^5(1-0.316)^{11} \approx 0.211.$$

b.

$$P(X \geq k) = \sum_{i=k}^{n}\binom{n}{i}\pi^i(1-\pi)^{n-i}.$$

$$P(X \geq 5) \approx \sum_{i=5}^{16}\binom{16}{i}(0.316)^i(1-0.316)^{16-i} \approx 0.605.$$

c. The number of audits with irregularities is a non-negative integer.

$$P(X < 5) = P(\overline{X \geq 5}) = 1 - P(X \geq 5) \approx 0.395.$$

d.

$$P(X \geq 5) - P(X = 5) \approx 0.394.$$

7 A machine shop

The number of bolts (N) is 200. A defective bolt is defined as a positive result. The number of bolts inspected (n) is 12. The trials are correlated due to sampling without replacement.

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

a. The number of defective bolts (K) is 20.

$$P(X = 0) = \frac{\binom{20}{0}\binom{200-20}{12-0}}{\binom{200}{12}} = \frac{\binom{180}{12}}{\binom{200}{12}} \approx 0.2718\ldots$$

b. The number of defective bolts (K) is 10. The number of defective bolts discovered (X) is not negative.

$$P(X > 0) = P(X \neq 0) = 1 - \frac{\binom{10}{0}\binom{200-10}{12-0}}{\binom{200}{12}} = 1 - \frac{\binom{190}{12}}{\binom{200}{12}} \approx 0.4693\ldots$$