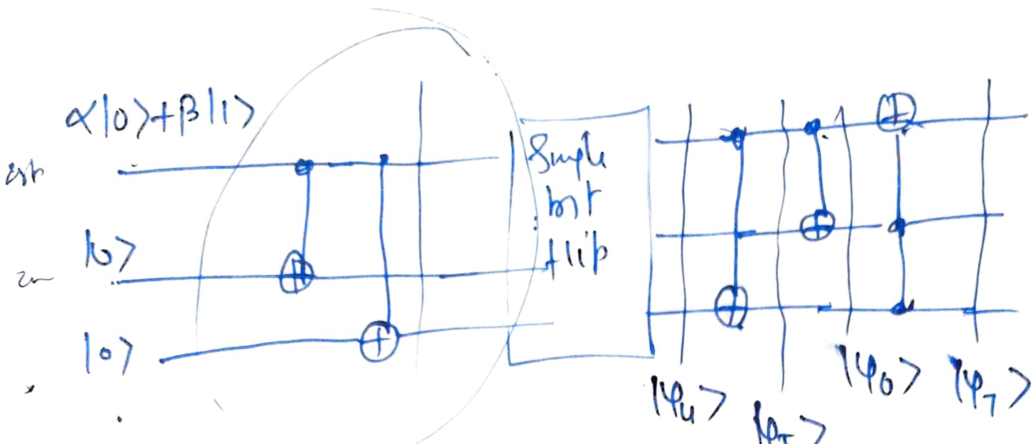
(Error detection and Correction)

E1
L



$\alpha|000\rangle+\beta|111\rangle$

First qubit → first qubit

No Error

$|\psi_4\rangle = \alpha|000\rangle+\beta|111\rangle$

$|\psi_5\rangle = \alpha|000\rangle+\beta|110\rangle$

$|\psi_6\rangle = \alpha|000\rangle+\beta|100\rangle$

$|\psi_7\rangle = (\alpha|0\rangle+\beta|1\rangle)|00\rangle$

Syndrome

First qubit

$|\psi_4\rangle = \alpha|100\rangle+\beta|011\rangle$

$|\psi_5\rangle = \alpha|101\rangle+\beta|011\rangle$

$|\psi_6\rangle = \alpha|111\rangle+\beta|011\rangle$

$= (\alpha|1\rangle+\beta|0\rangle)|\underline{11}\rangle$

$|\psi_7\rangle = (\alpha|0\rangle+\beta|1\rangle)|11\rangle$

2nd qubit

$|\psi_4\rangle = \alpha|010\rangle+\beta|101\rangle$

$|\psi_5\rangle = \alpha|010\rangle+\beta|100\rangle$

$|\psi_6\rangle = \alpha|010\rangle+\beta|110\rangle$

$= (\alpha|0\rangle+\beta|1\rangle)|10\rangle$

$|\psi_7\rangle = (\alpha|0\rangle+\beta|1\rangle)|10\rangle$

3rd qubit

$|\psi_4\rangle = \alpha|001\rangle+\beta|110\rangle$

$|\psi_5\rangle = \alpha|001\rangle+\beta|111\rangle$

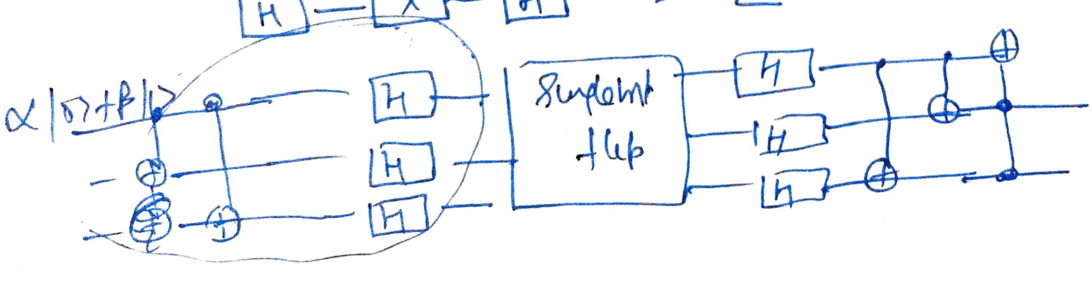
$|\psi_6\rangle = \alpha|001\rangle+\beta|101\rangle$

$= (\alpha|0\rangle+\beta|1\rangle)|01\rangle$

$|\psi_7\rangle = (\alpha|0\rangle+\beta|1\rangle)|01\rangle$ ✓

Phase-flip

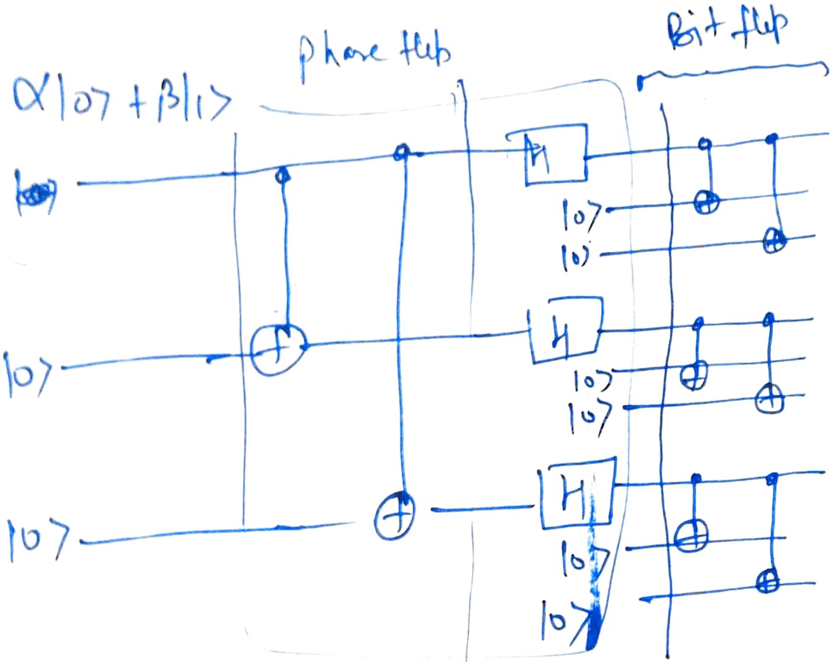
$\boxed{H} - \boxed{X} - \boxed{H} = Z$



(Shor 9's bit Code)

[Peter Shor
MIT]

Shor's Algorithm.



$$\alpha|0_1 0_4 0_7\rangle + \beta|1_1 1_4 1_7\rangle$$

$$\alpha\left[\frac{1}{\sqrt{2}}(|0_1\rangle+|1_1\rangle)|00_{23}\rangle \frac{1}{\sqrt{2}}(|0_4\rangle+|1_4\rangle)|00\rangle \frac{1}{\sqrt{2}}(|0_7\rangle+|1_7\rangle)|00\rangle\right]$$

$$+\beta\left[\frac{1}{\sqrt{2}}(|0_1\rangle-|1_1\rangle)|00\rangle \frac{1}{\sqrt{2}}(|0_4\rangle-|1_4\rangle)|00\rangle \frac{1}{\sqrt{2}}(|0_7\rangle-|1_7\rangle)|00\rangle\right]$$

$$= \alpha\left[\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle+|1_1 0_2 0_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle+|1_4 0_5 0_6\rangle) \frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle+|1_7 0_8 0_9\rangle\right]$$

$$+\beta\left[\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle-|1_1 0_2 0_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle-|1_4 0_5 0_6\rangle) \frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle-|1_7 0_8 0_9\rangle)\right]$$

Now we show step by step how encoder works

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At this point of time 2,3,5,6,8 and 9 are 0 only

$0_2 \; 0_3 \; 0_5 \; 0_6 \; 0_8 \; 0_9$

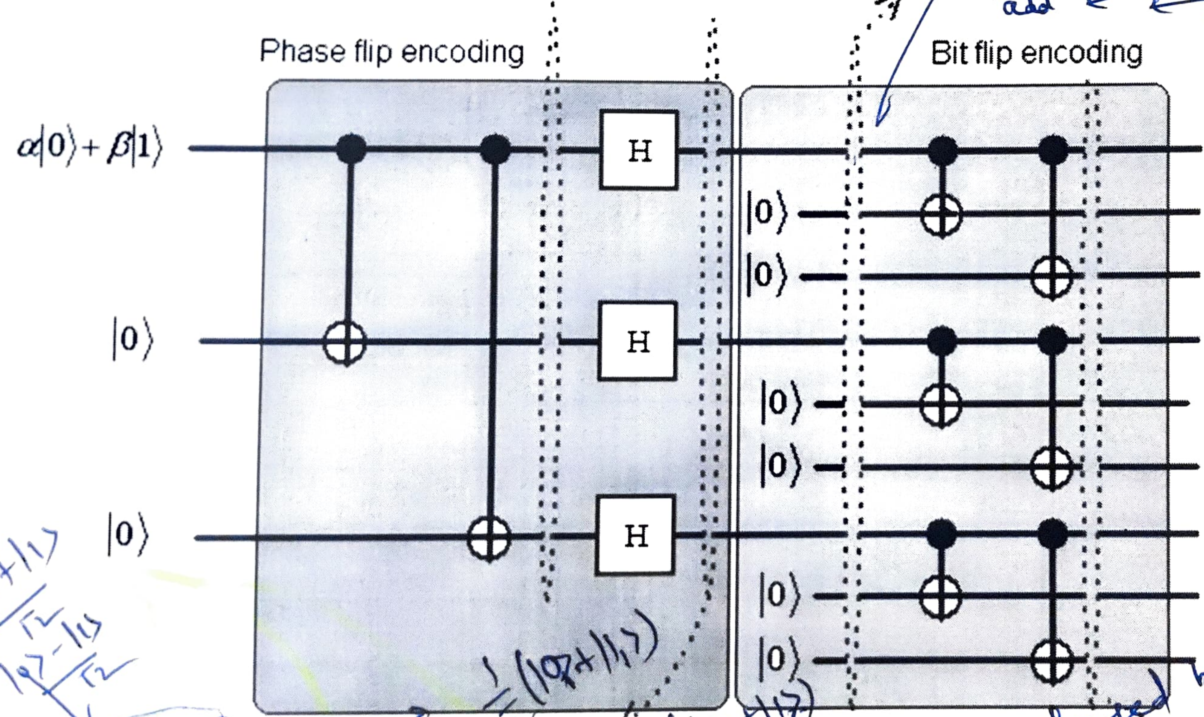
add add

$$\alpha\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle+|1_1 0_2 0_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle+|1_4 0_5 0_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle+|1_7 0_8 0_9\rangle)$$

$$+\beta\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle-|1_1 0_2 0_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle-|1_4 0_5 0_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle-|1_7 0_8 0_9\rangle)$$

add

$\alpha|000\rangle+\beta|111\rangle$



Tensor product of results of Hadamards with zeros

Xoring in second and third bits with 1 from first bit

$H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$H|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}(|0_1\rangle+|1_1\rangle)$

$\frac{1}{\sqrt{2}}(|0_2\rangle+|1_2\rangle)$

$$\alpha\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$+\beta\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

changed bit

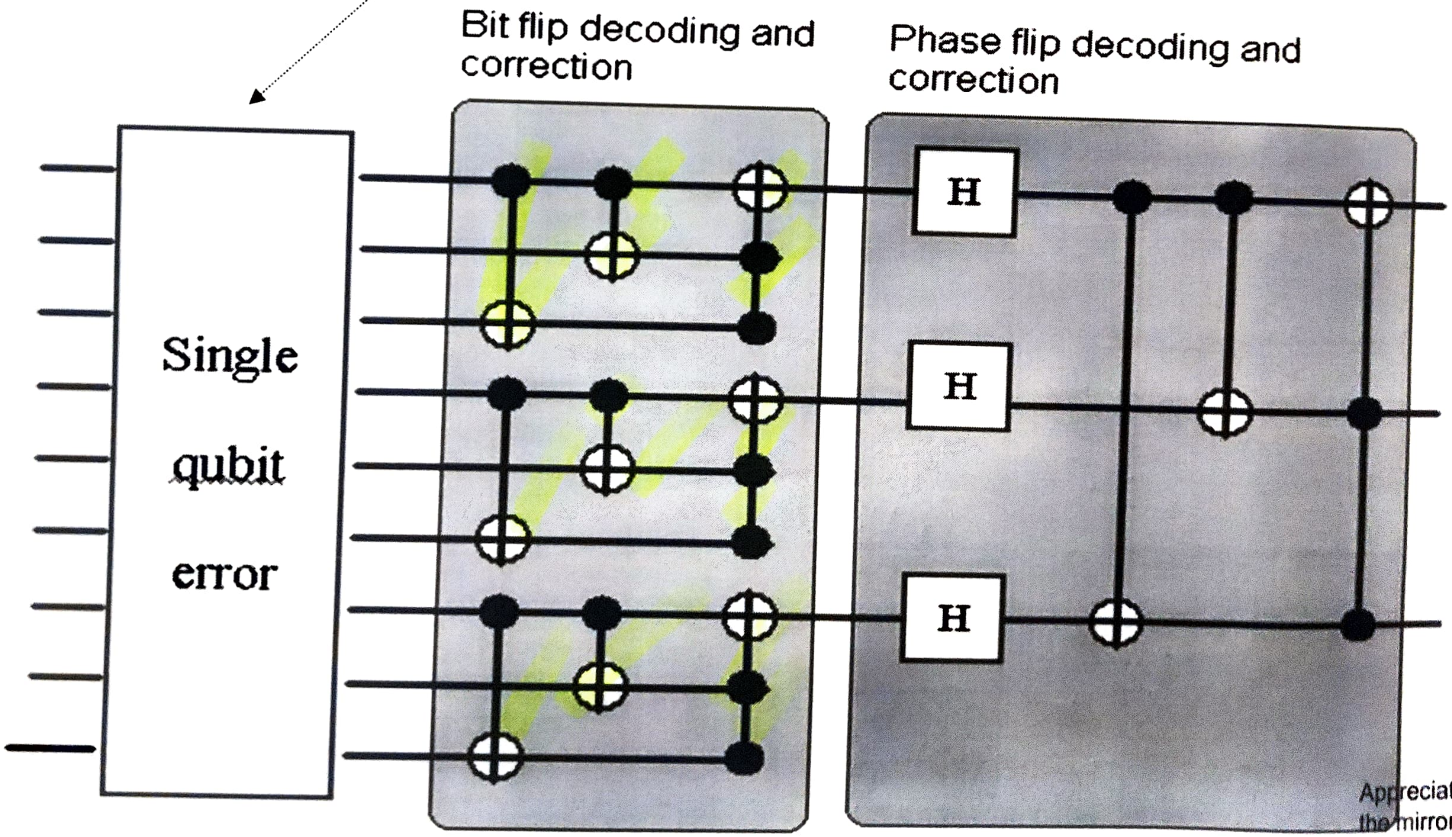
$$\alpha\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle+|1_1 1_2 1_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle+|1_4 1_5 1_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle+|1_7 1_8 1_9\rangle)$$

$$+\beta\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle-|1_1 1_2 1_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle-|1_4 1_5 1_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle-|1_7 1_8 1_9\rangle)$$

Rece... from lin...

Now we show step by step how DECODER works

- Assuming at most 1 qubit error and the error is just as likely to affect any qubit.
- The decoding and correction circuit:



Appreciate please the mirror like symmetry

Single Qubit Error $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$, $\alpha|0\rangle+\beta|1\rangle \to \alpha|1\rangle+\beta|0\rangle$

Phase flip Error $|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$, $\alpha|0\rangle+\beta|1\rangle \to \alpha|0\rangle-\beta|1\rangle$

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Bit & phase flip error $|0\rangle \to |1\rangle$, $|1\rangle \to -|0\rangle$, $\alpha|0\rangle+\beta|1\rangle \to \alpha|1\rangle-\beta|0\rangle$

Detailed analysis of an error

- Example: Assume encoded qubit damaged such that:

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $-i\sigma_y = -i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sigma_x\sigma_z$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$\alpha\frac{1}{\sqrt{2}}\left(|0_1 0_2 0_3\rangle+|1_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle+|1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle+|1_7 1_8 1_9\rangle\right)$$
$$+\beta\frac{1}{\sqrt{2}}\left(|0_1 0_2 0_3\rangle-|1_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle-|1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle-|1_7 1_8 1_9\rangle\right)$$

Send to line

Check first slide

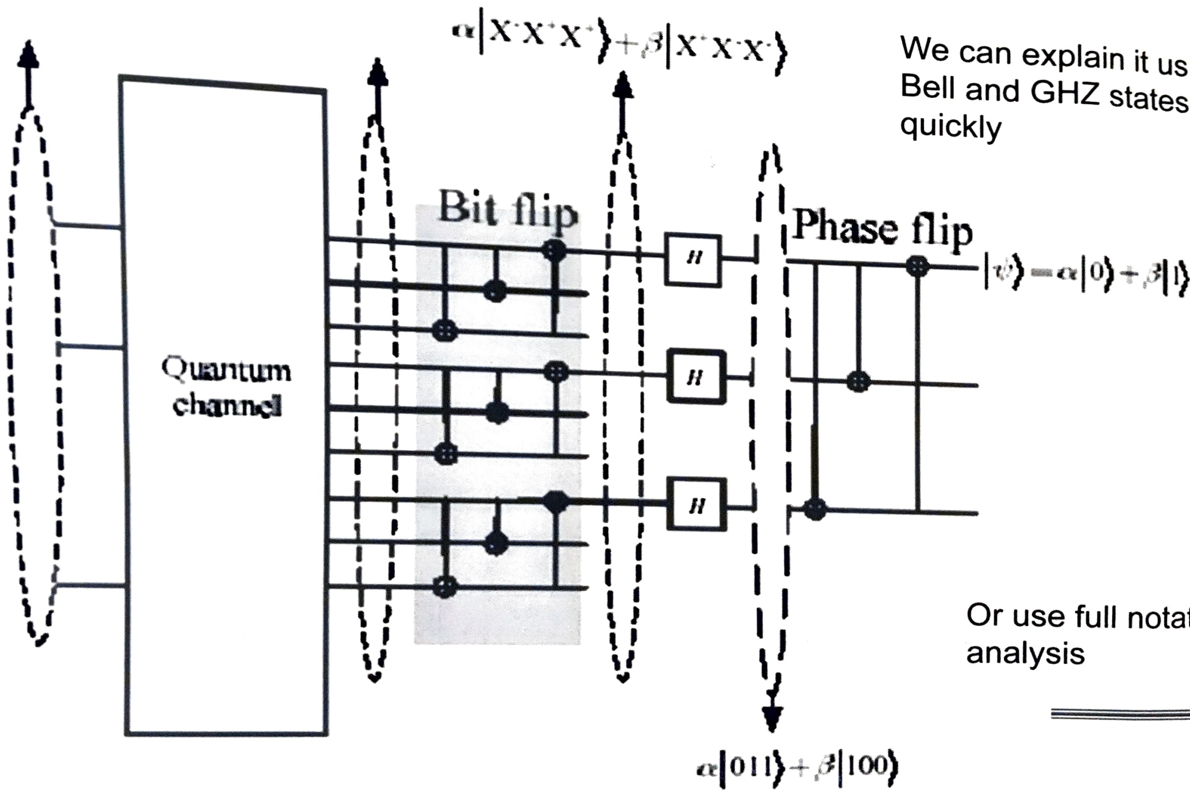
Phase and bit flip on 1st qubit $(-i\sigma_y)$

$$\alpha\frac{1}{\sqrt{2}}\left(|1_1 0_2 0_3\rangle-|0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle+|1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle+|1_7 1_8 1_9\rangle\right)$$
$$+\beta\frac{1}{\sqrt{2}}\left(|1_1 0_2 0_3\rangle+|0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle-|1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle-|1_7 1_8 1_9\rangle\right)$$

Received from line

As we see the red error is in phase and bit flip of first qubit

Shor code –Decoding



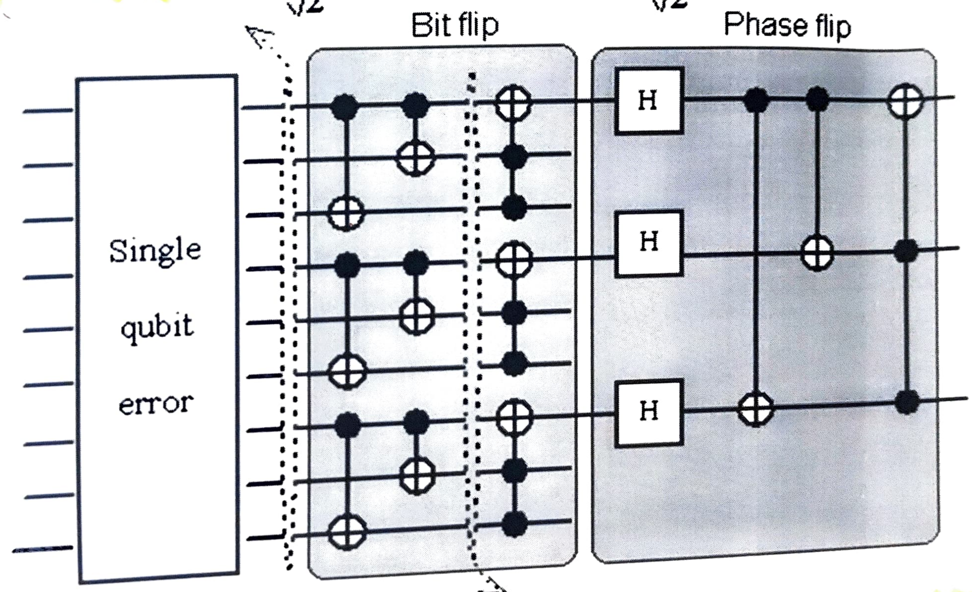
We can explain it using Bell and GHZ states quickly

Or use full notation for analysis

From line

$$\alpha \frac{1}{\sqrt{2}}\left(|1_1 0_2 0_3\rangle - |0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle + |1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle + |1_7 1_8 1_9\rangle\right)$$

$$+\beta \frac{1}{\sqrt{2}}\left(|1_1 0_2 0_3\rangle + |0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle - |1_4 1_5 1_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle - |1_7 1_8 1_9\rangle\right)$$



$$\alpha \frac{1}{\sqrt{2}}\left(|1_1 1_2 1_3\rangle - |0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle + |1_4 0_5 0_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle + |1_7 0_8 0_9\rangle\right)$$

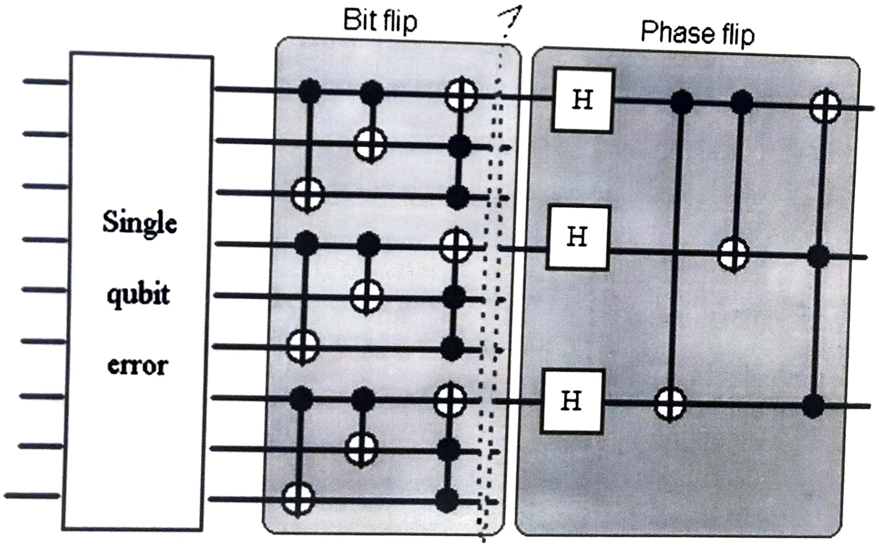
$$+\beta \frac{1}{\sqrt{2}}\left(|1_1 1_2 1_3\rangle + |0_1 1_2 1_3\rangle\right)\frac{1}{\sqrt{2}}\left(|0_4 0_5 0_6\rangle - |1_4 0_5 0_6\rangle\right)\frac{1}{\sqrt{2}}\left(|0_7 0_8 0_9\rangle - |1_7 0_8 0_9\rangle\right)$$

Before correction

After correction

$$\alpha\frac{1}{\sqrt{2}}\left(\left|0_1 1_2 1_3\right\rangle-\left|1_1 1_2 1_3\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|0_4 0_5 0_6\right\rangle+\left|1_4 0_5 0_6\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|0_7 0_8 0_9\right\rangle+\left|1_7 0_8 0_9\right\rangle\right)$$
$$+\beta\frac{1}{\sqrt{2}}\left(\left|0_1 1_2 1_3\right\rangle+\left|1_1 1_2 1_3\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|0_4 0_5 0_6\right\rangle-\left|1_4 0_5 0_6\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|0_7 0_8 0_9\right\rangle-\left|1_7 0_8 0_9\right\rangle\right)$$

$$=\alpha\left(\frac{\left(\left|0_1\right\rangle-\left|1_1\right\rangle\right)}{\sqrt{2}}\left|1_2 1_3\right\rangle\right)\left(\frac{\left(\left|0_4\right\rangle+\left|1_4\right\rangle\right)}{\sqrt{2}}\left|0_5 0_6\right\rangle\right)\left(\frac{\left(\left|0_7\right\rangle+\left|1_7\right\rangle\right)}{\sqrt{2}}\left|0_8 0_9\right\rangle\right)$$
$$+\beta\left(\frac{\left(\left|0_1\right\rangle+\left|1_1\right\rangle\right)}{\sqrt{2}}\left|1_2 1_3\right\rangle\right)\left(\frac{\left(\left|0_4\right\rangle-\left|1_4\right\rangle\right)}{\sqrt{2}}\left|0_5 0_6\right\rangle\right)\left(\frac{\left(\left|0_7\right\rangle-\left|1_7\right\rangle\right)}{\sqrt{2}}\left|0_8 0_9\right\rangle\right)$$



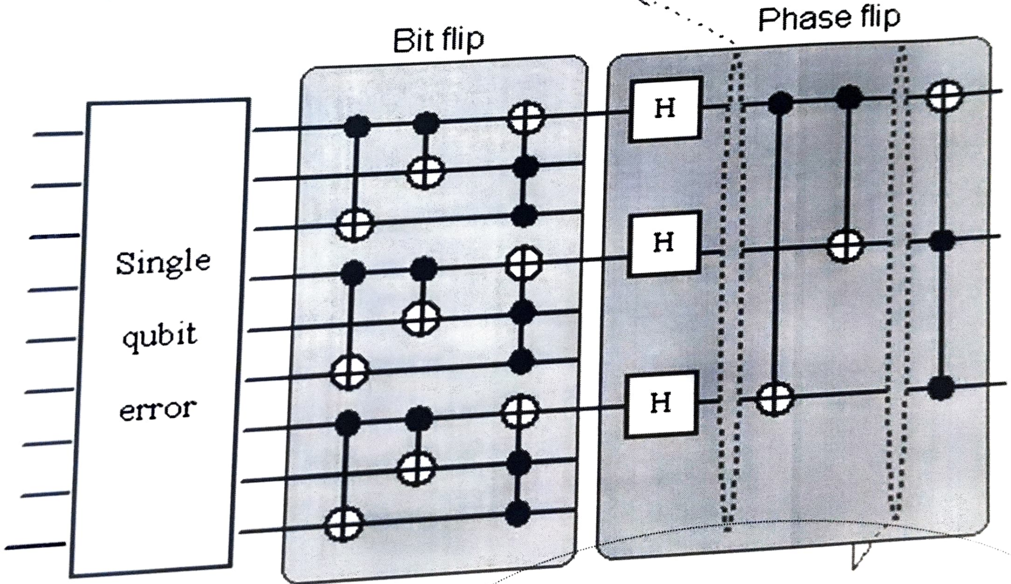
Before Hadamards, from previous slide

$$\alpha\left(\frac{\left(|0_1\rangle-|1_1\rangle\right)}{\sqrt{2}}\right)\left(\frac{\left(|0_4\rangle+|1_4\rangle\right)}{\sqrt{2}}\right)\left(\frac{\left(|0_7\rangle+|1_7\rangle\right)}{\sqrt{2}}\right)$$

$$+\beta\left(\frac{\left(|0_1\rangle+|1_1\rangle\right)}{\sqrt{2}}\right)\left(\frac{\left(|0_4\rangle-|1_4\rangle\right)}{\sqrt{2}}\right)\left(\frac{\left(|0_7\rangle-|1_7\rangle\right)}{\sqrt{2}}\right)$$

After Hadamards

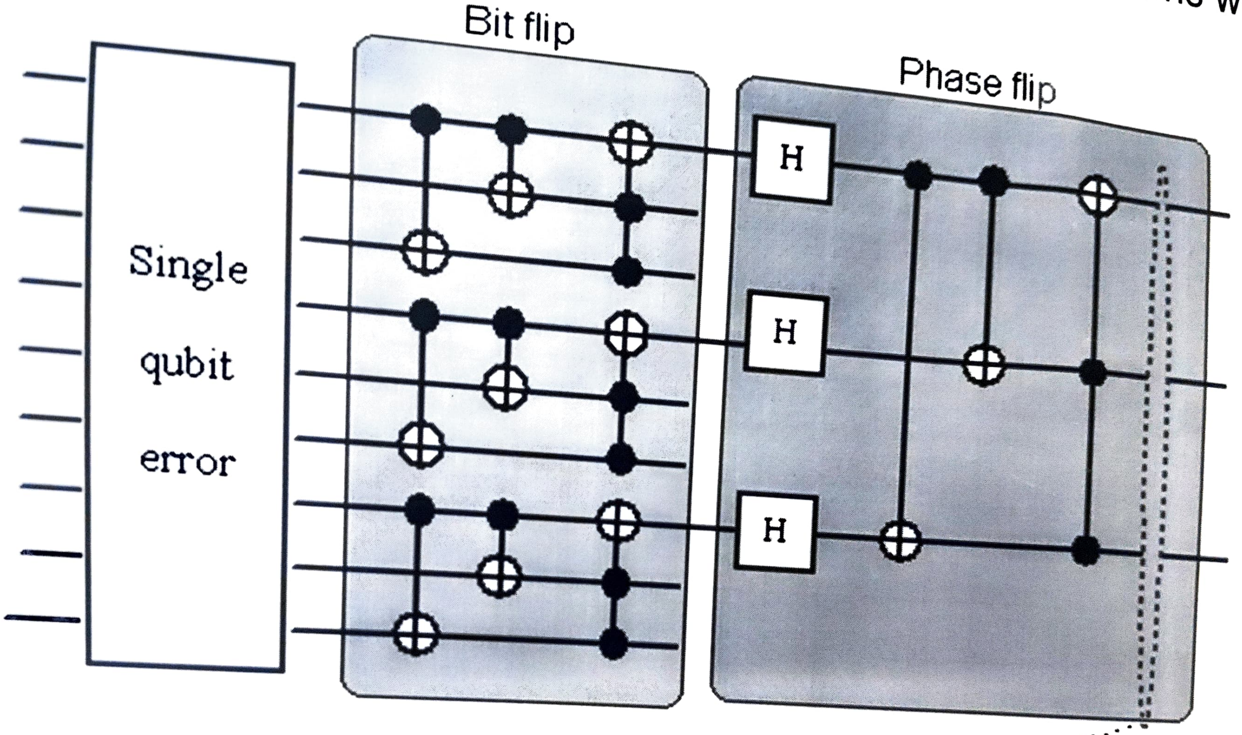
$$\rightarrow \alpha|1_1\rangle|0_4\rangle|0_7\rangle+\beta|0_1\rangle|1_4\rangle|1_7\rangle$$



After bit flip decoding

$$\alpha|1_1\rangle|1_4\rangle|1_7\rangle+\beta|0_1\rangle|1_4\rangle|1_7\rangle$$

Xoring in bits 2 and 3 only in alpha part lip and phase flip corrections we get:



$$\alpha|0_1\rangle|1_4\rangle|1_7\rangle+\beta|1_1\rangle|1_4\rangle|1_7\rangle = \big(\underbrace{\alpha|0_1\rangle+\beta|1_1\rangle}_{\text{1st qubit is protected qubit}}\big)\otimes|1_4\rangle|1_7\rangle$$

So we got what we wanted to get!

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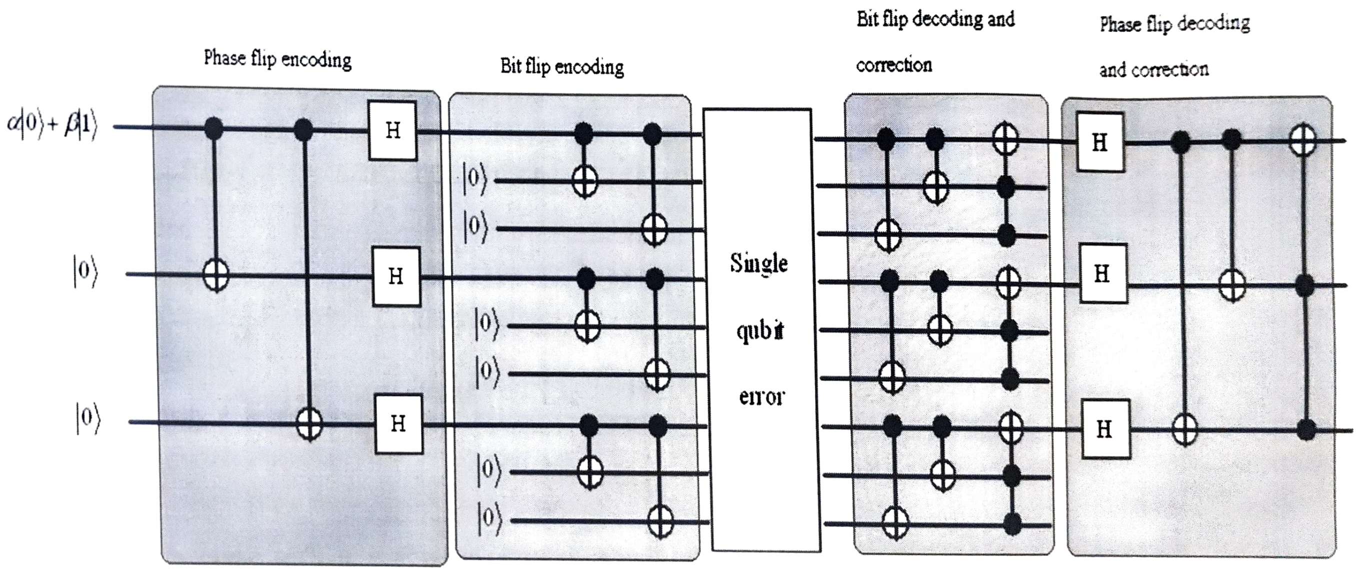
The (9,1) circuit – put all together



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