(Lecture - 5)

H, X, Z, CNOT, CCNOT, phase gate

$$HZH = X$$
$$HXH = Z$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0&2\\2&0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix} = X$$

$$HXH = Z$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2&0\\0&-2\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix} = Z$$

$$X^2 = I \qquad Z^2 = I$$
$$H^2 = I$$

$$\sqrt{NOT} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&+1\end{bmatrix}$$

$$\sqrt{NOT} * \sqrt{NOT} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}0&-2\\2&0\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$$ ✓

(Vectors)

Length of a vector / Norm $\|v\| = \sqrt{\underbrace{\langle v|v\rangle}}$

$\Downarrow$

$\langle v||v\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

orthogonal Vector :- Angle between two vector is 90°.

Normalized " :- Its length is 1.

orthonormal " :- Length of Both are 1 and angle between them is 90°

$|w\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|v\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\langle 0| = (1 \quad 0)$

$\langle 1| = (0 \quad 1)$

$\|w\| = \sqrt{\langle w|w\rangle}$

$= \sqrt{(1 \quad 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{1} = 1$

$\|v\| = \sqrt{\langle v|v\rangle}$

$= \sqrt{(0 \quad 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = 1 \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle 0|1\rangle = (1 \quad 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$\langle 1|0\rangle = (0 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

$|0\rangle, |1\rangle$ are orthonormal Vectors

Computational Basis

(Vectors)

Length of a vector / Norm $\quad \|v\| = \sqrt{\underbrace{\langle v|v\rangle}_{\Downarrow}}$

$\langle v||v\rangle$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

orthogonal Vector :- Angle between two vector is 90°.

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$|w\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |v\rangle = |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad \langle 0| = (1 \;\; 0)$

$\langle 1| = (0 \;\; 1)$

$\|w\| = \sqrt{\langle w|w\rangle}$

$= \sqrt{(1 \;\; 0)\begin{pmatrix}1\\0\end{pmatrix}} = \sqrt{1} = 1$

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$\langle 1|0\rangle = (0 \;\; 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$

$\underbrace{|0\rangle, |1\rangle}$ are orthonormal Vectors

Computational Basis

$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ $\quad \langle +| = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}$ 2

$$\langle +|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}(1-1) = 0$$

$$\|\underbrace{|+\rangle}_{|+\rangle}\| = \sqrt{\langle +|+\rangle}$$

$$= \sqrt{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$= \sqrt{\frac{1}{2}(2)} = 1$$

$|i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$

$\langle i| = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}$

$$\||-\rangle\| = \sqrt{\langle -|-\rangle} = \sqrt{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}$$

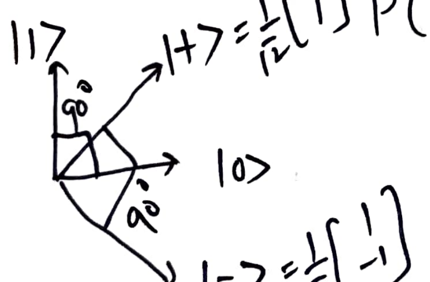
$$= \sqrt{\frac{1}{2}(2)} = 1$$

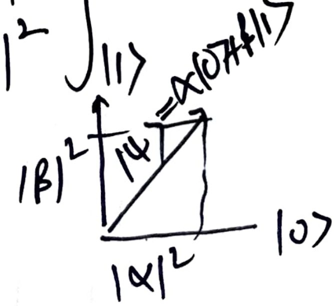
Classical $\quad |0\rangle, |1\rangle$

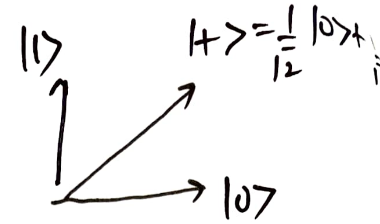
Quantum $\quad |0\rangle, |1\rangle \Rightarrow$ Computational Basis

$|+\rangle, |-\rangle \Rightarrow$ Hadamard "

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\quad \left.\begin{array}{l} P(|0\rangle) = |\alpha|^2 \\ P(|1\rangle) = |\beta|^2 \end{array}\right\}$ Computational Basis







$|\psi\rangle$ is any state
and you want to measure
$|\psi_1\rangle$ Basis

$$\underbrace{\left|\langle\psi_1|\psi\rangle\right|^2}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

Computational Basis $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $\langle 0| = \begin{bmatrix}1 & 0\end{bmatrix}$

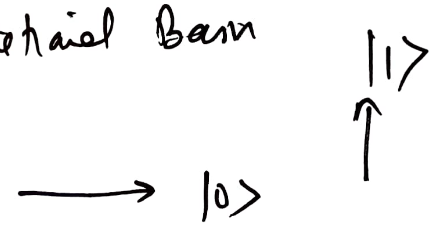
$$\left|\overrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right|^2 = \frac{1}{2}$$

$$\left|\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right|^2 = |\alpha|^2$$

$$\left|\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right|^2 = \frac{1}{2}$$

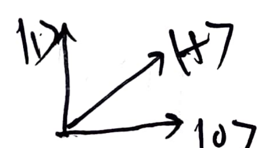
$$\left|\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right|^2 = |\beta|^2$$

$|\psi\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ Computational Basis



Measured $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$$\left|\underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix}\right|^2 = 1$$



$\langle 0|\psi\rangle$

	Measure $\vert 0\rangle$	Measure $\vert 1\rangle$	$\vert +\rangle$	$\vert -\rangle$
$\vert\psi\rangle = \vert 0\rangle$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
$\vert\psi\rangle = \vert 1\rangle$	0	1	$\frac{1}{2}$	$\frac{1}{2}$
$\vert\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$0 \to \Vert\langle -\vert +\rangle\Vert^2$
$\vert\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1

$\Vert\langle +|-\rangle\Vert^2$ $\Vert\langle +|+\rangle\Vert^2$ $\Vert\langle -|-\rangle\Vert^2$

Relative phase

Relative phase

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}$$

$$= \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi}\frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$= -1$$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$|+\rangle + |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix}$$

Increase in $|0\rangle$ amplitude } Constructive Interference
decrease " $|1\rangle$ " } destructive "

Global phase

$$|0\rangle \qquad e^{i\phi}|0\rangle = \begin{bmatrix}e^{i\phi}\\0\end{bmatrix} \qquad \boxed{|\psi\rangle \quad e^{i\phi}|\psi\rangle}$$

$$\| \langle 0|0\rangle \|^2 = \left\| \cancel{\sqrt{}} \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \right\|^2 = 1$$

α is a Complex
$\|\alpha\|^2 = \alpha\alpha^\dagger$

~~no square root~~

$$\| \langle 0|e^{i\theta}|0\rangle \|^2 = \left\| \cancel{\sqrt{}} \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}e^{i\phi}\\0\end{bmatrix} \right\|^2 = \|e^{i\theta}\|^2 = \|e^{i\theta}e^{-i\theta}\|^2$$

$$= \|e^{i\theta - i\theta}\|^2 = \|e^0\|^2$$
$$= 1$$

$|+\rangle$
$|-\rangle$ } Hadamard Basis