1 (Lecture - 4) (P-1)

CNOT gate (Controlled - Not gate)

If Control bit is 1 then apply Not to target bit (flip the target bit

Inputs		Outputs		2nd CNOT
Co	to	Co	to	to
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$

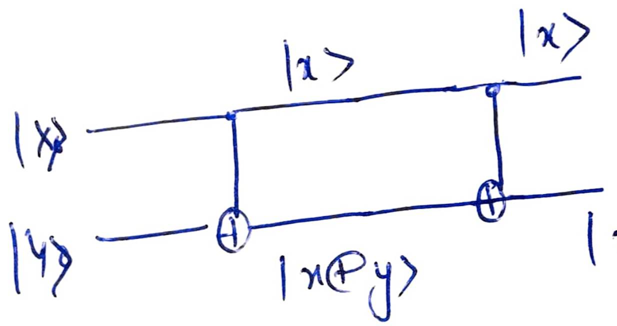
Same as Co

Ex-OR classical

(Co + to) mod 2

(1 + 1) mod 2 = 0

(0 + 0) mod 2 = 0



$|x \oplus x \oplus y\rangle = |0 \oplus y\rangle = |y\rangle$

$|x \oplus x\rangle$

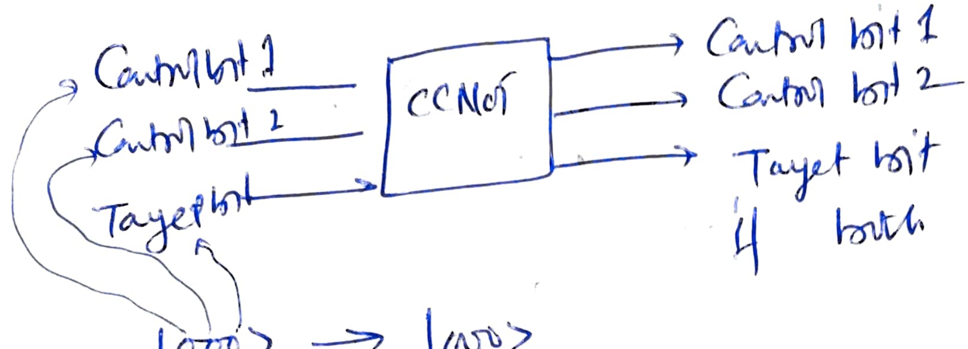
$x = 0 \quad |0 \oplus 0\rangle = 0$

$x = 1 \quad |1 \oplus 1\rangle = 0$

(*)

CCNOT (Toffoli gate)

↳ Controlled Controlled-Not gate



Target bit will flip only if both control bits are 1.

$|000\rangle \rightarrow |000\rangle$

$|001\rangle \rightarrow |001\rangle$

$\vdots$

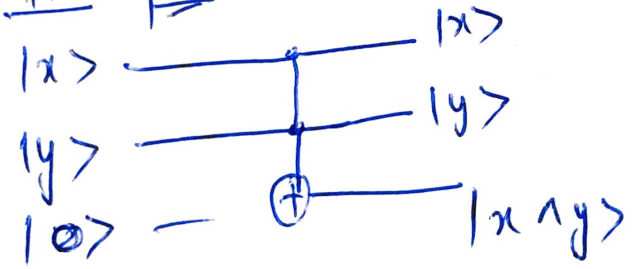
$|110\rangle \rightarrow |111\rangle$

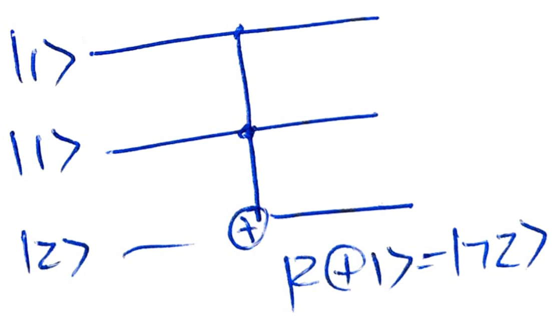
$|111\rangle \rightarrow |110\rangle$

	$\|000\rangle$	$\|001\rangle$	$\|010\rangle$	$\|011\rangle$	$\|100\rangle$	$\|101\rangle$	$\|110\rangle$	$\|111\rangle$
$\|000\rangle$	1							
$\|001\rangle$		1						
$\|010\rangle$			1					
$\|011\rangle$				1				
$\|100\rangle$					1			
$\|101\rangle$						1		
$\|110\rangle$								1
$\|111\rangle$							1	

All other entries are zeros.

Toffoli gate





NAND is a universal gate

Same way CCNOT will be a universal gate

X-Gate (Pauli-X)

Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle+\beta\|1\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$	$\beta\|0\rangle+\alpha\|1\rangle=\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$

Input

	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	0	1
$\|1\rangle$	1	0

$X(\alpha|0\rangle+\beta|1\rangle)$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \beta\begin{pmatrix}1\\ 0\end{pmatrix}+\alpha\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \beta|0\rangle+\alpha|1\rangle$$

Phase-Gate

$$\begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ e^{i\phi}\beta\end{pmatrix}$$

No change in $|0\rangle$ } Relative phase
A phase " " $|1\rangle$

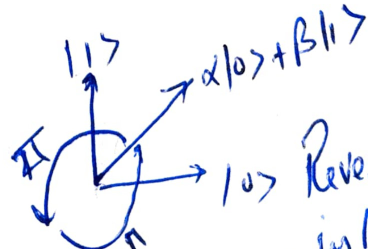
$e^{i\phi} = \cos\phi + i\sin\phi$ } Euler's formula

Z-gate $\qquad \phi=\pi$

$e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$Z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$



Reversible in Nature

$$\alpha|0\rangle+\beta|1\rangle=\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longrightarrow \boxed{Z} \xrightarrow{\begin{pmatrix}\alpha\\ -\beta\end{pmatrix}} \boxed{Z} \longrightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$Z^2 = I$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\boxed{\text{S-gate}} \Longrightarrow \quad \phi = \pi/2$

$\boxed{\begin{array}{c}\text{Apply S-gate twice}\\ \text{Means Z-gate}\end{array}}$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad \text{and } \phi = \pi/2$$

$$e^{i\phi} = \cos\phi + i\sin\phi = i$$

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$S^\dagger$ gate $\rightarrow$ dagger

$\Downarrow$

$$\phi = -\pi/2$$

$$e^{i\phi} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \;—\; \underbrace{\boxed{S} \;—\; \boxed{S^\dagger}}_{S^\dagger S} \;—\; \alpha|0\rangle + \beta|1\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -i^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\boxed{\text{T gate}}$ $\quad \phi = \pi/4 \quad e^{i\phi} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{pmatrix}$$

$T^2 = S$

$T^4 = Z$