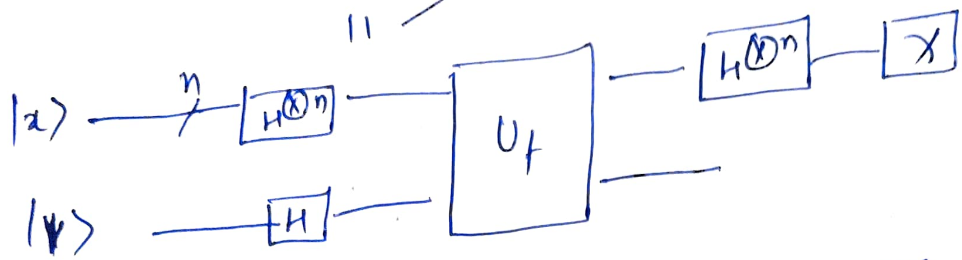
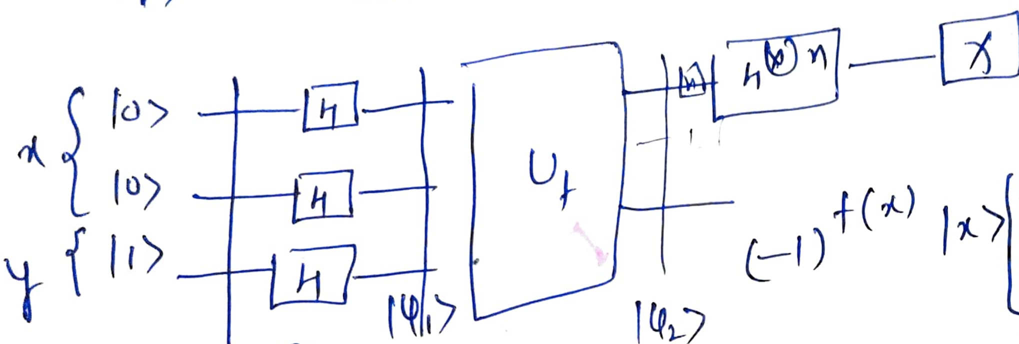
(April 16, 2024) ①

$\{0,1\}^n \longrightarrow \{0,1\}$

$n=2$

$$\begin{matrix} 00 & \\ 01 & \longrightarrow 0 \\ 10 & \\ 11 & \end{matrix}$$





$$(-1)^{f(x)}|x\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$|\varphi_0\rangle = |0\rangle|0\rangle|1\rangle$$

$$|\varphi_1\rangle = \underbrace{\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]}\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$|\varphi_1\rangle = \frac{1}{2}\Big[|00\rangle+|01\rangle+|10\rangle+|11\rangle\Big]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$$

$$|\varphi_2\rangle = \frac{1}{2}\Big[\underbrace{(-1)^{f(00)}}_{(-1)^0}|00\rangle+\underbrace{(-1)^{f(01)}}_{(-1)^0}|01\rangle+\underbrace{(-1)^{f(10)}}_{(-1)^0}|10\rangle+\underbrace{(-1)^{f(11)}}_{(-1)^0}|11\rangle\Big]|-\rangle$$

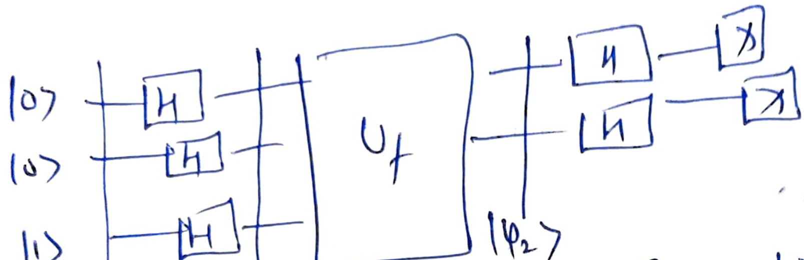
$$= \underbrace{\frac{1}{2}\Big[|00\rangle+|01\rangle+|10\rangle+|11\rangle\Big]}|-\rangle$$

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$H^{\otimes 2}\,\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\right)\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}4\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle = 0$$

If $\left.\begin{array}{l} f(00) \\ f(01) \\ f(10) \\ f(11) \end{array}\right\} \rightarrow 1$



$|0\rangle|0\rangle|1\rangle$

$$|\psi_1\rangle = \underbrace{\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$= \left[\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}\right]|-\rangle$$

$$|\psi_2\rangle = \left[\frac{(-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle}{2}\right]|-\rangle$$

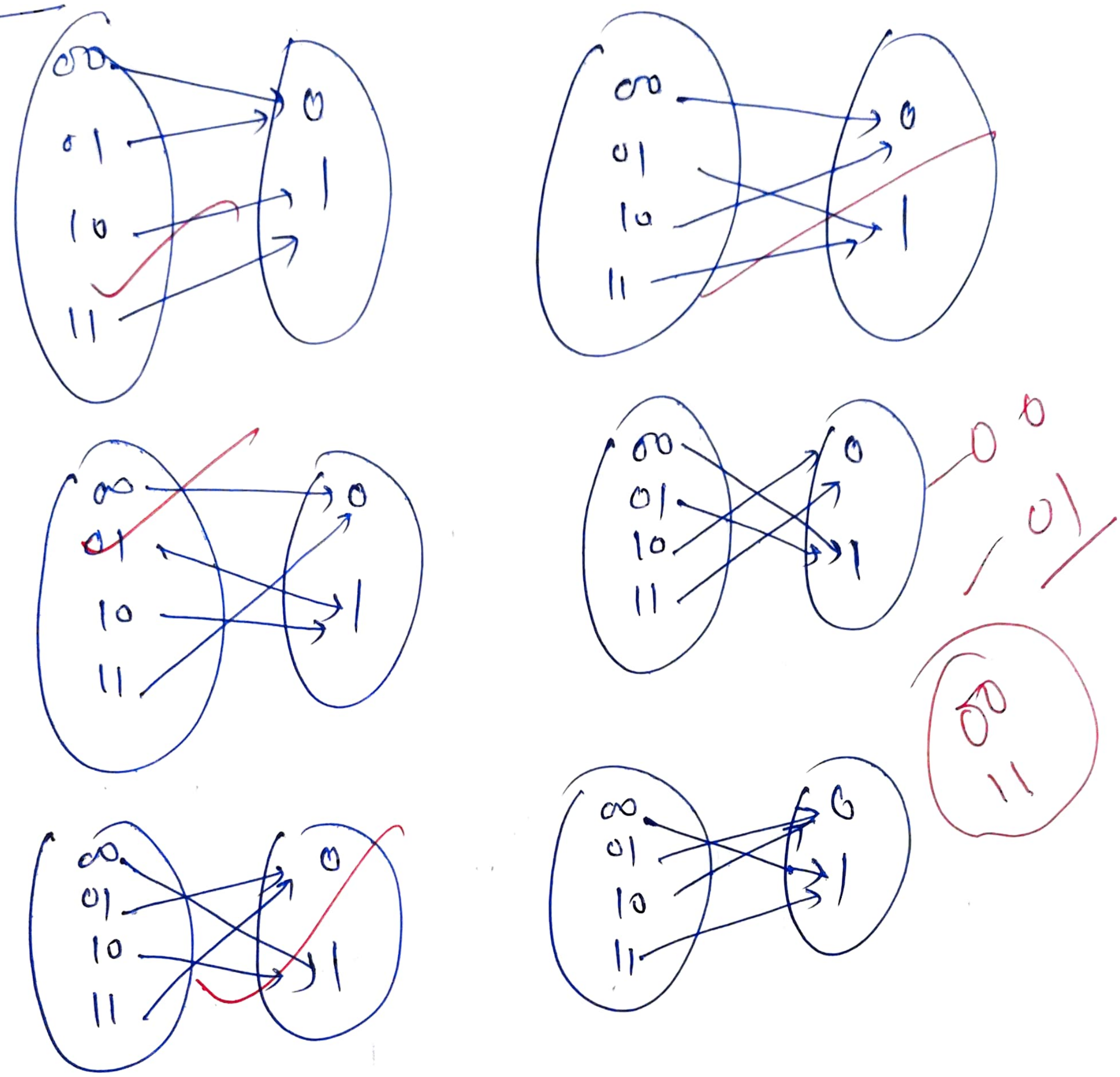
$$= \frac{-|00\rangle - |01\rangle \cancel{+} |10\rangle - |11\rangle}{2}\,|-\rangle$$

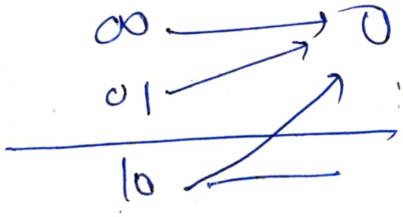
$$|\psi_3\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -|00\rangle$$

Balance

f_1





$2^2 = 4$

$2^{n-1}+1 = 3$

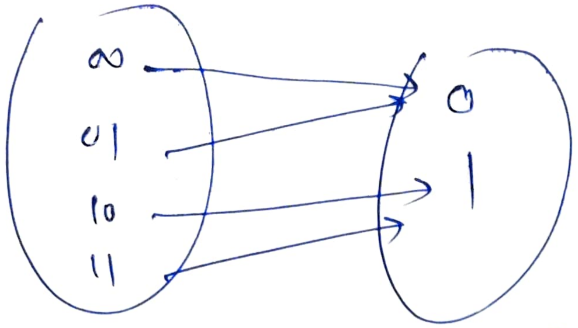
n=3

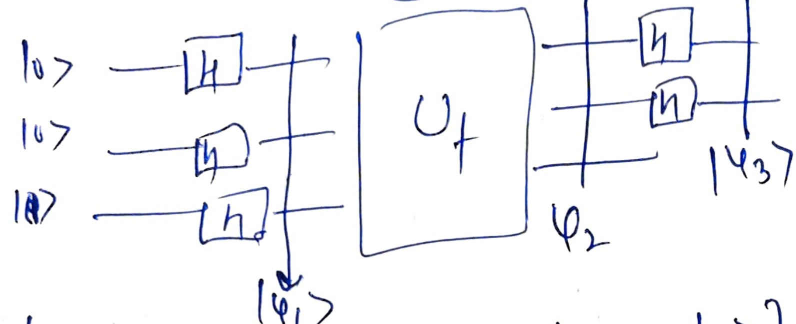
000
001
010
|
111

5

$2^{3-1}+1 = 5$

f_1:





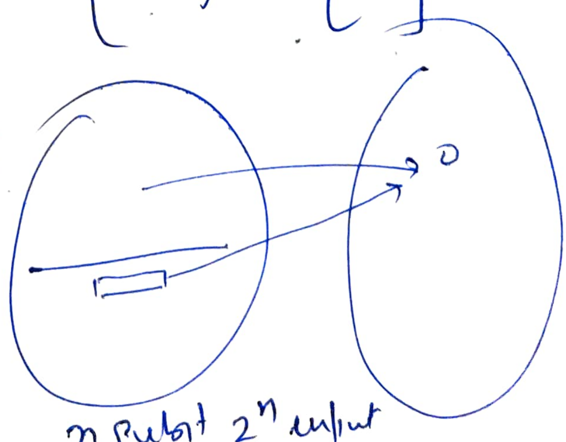
$$|\psi_1\rangle = \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]|-\rangle$$

$$|\psi_2\rangle = \frac{1}{2}\Big(\underbrace{(-1)^{f(00)}}_{(-1)^0}|00\rangle + \underbrace{(-1)^{f(01)}}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle\Big)|-\rangle$$

$$\frac{1}{2}\left[|00\rangle + |01\rangle - |10\rangle - |11\rangle\right]\underline{|-\rangle}$$

$$H^{\otimes 2}\ \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} = |10\rangle$$

<u>Balanced</u> <u>Constant</u> <u>function</u>



n qubit 2^n input

$$H^{\otimes n} \underbrace{|0\rangle} = \frac{1}{\sqrt{2^n}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

$$H^{\otimes 2} |0\rangle = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \left\{ |00\rangle + |01\rangle + |10\rangle + |11\rangle \right.$$

$$\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$H^{\otimes n} |y\rangle = \frac{1}{\sqrt{2^n}} (-1)^{\langle x, y \rangle} |x\rangle$$

$$\frac{1}{\sqrt{2^2}} \left[(-1)^{\underbrace{\langle 00, 01 \rangle}} |00\rangle + (-1)^{\langle \underset{x}{01}, \underset{y}{01} \rangle} |\underset{x}{01}\rangle + (-1)^{\langle 10, 01 \rangle} |10\rangle + (-1)^{\langle 11, 01 \rangle} |11\rangle \right]$$

$$(1 \wedge 0) \oplus (0 \wedge 1) \qquad (1 \wedge 0) \oplus (1 \wedge 1)$$

$$(0 \wedge 0) \oplus (0 \wedge 1) \qquad (0 \wedge 0) \oplus (1 \wedge 1)$$

$$= \frac{1}{2} \left[|00\rangle - |01\rangle + |10\rangle - |11\rangle \right]$$