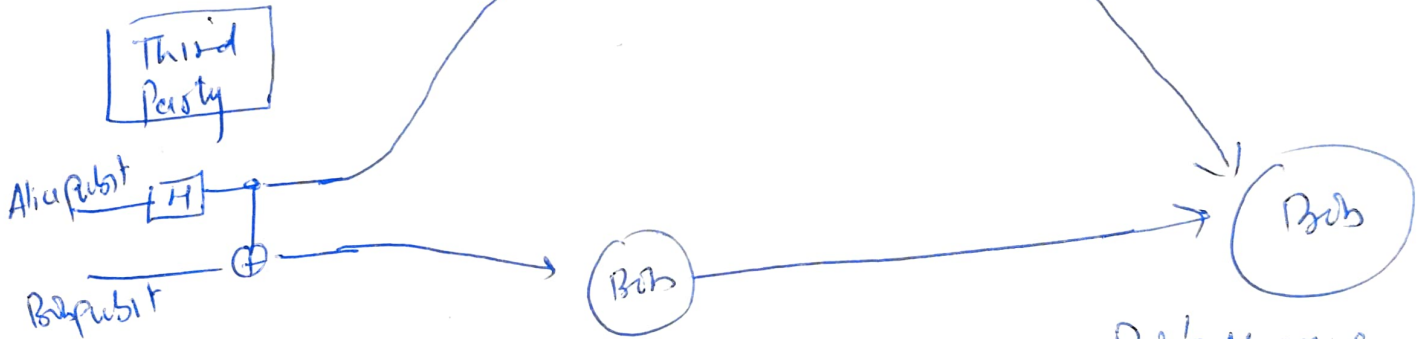# (Superdense Coding)

Alice

Bob

Want to transmit
two classical bits

$00 - I$
$01$
$10$
$11$

She transmit a
qubit to Bob

& Bob Retrieve
the information
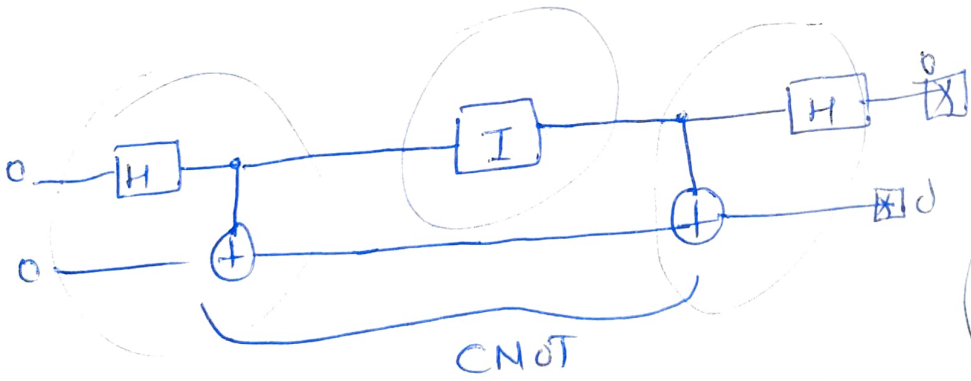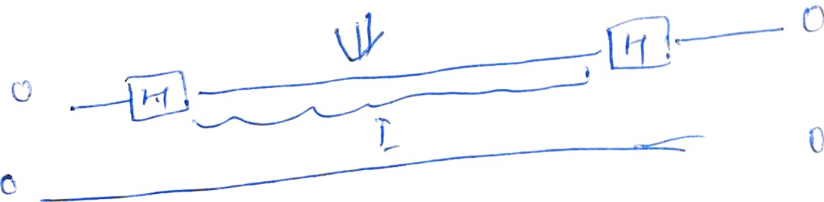
$00 - I$
$01 - X$
$10 - Z$
$11 - ZX$

Alice

Bob

Third
Party

Alice qubit $\boxed{H}$

Bob qubit

Bob

Bob's Measure
both qubit &
get the Information



$0 — \boxed{H}$

$0$

$\boxed{I}$

$\boxed{H} — \boxed{X}$ $0$

$\boxed{X}$ $0$

CNOT

$0 — \boxed{H}$ $\Downarrow$ $\boxed{H} — 0$

$I$

$0$ $0$

|  | first CNOT |  | second CNOT |
|---|---|---|---|
| 10 | $\to$ | 11 | $\to$ 10 |
| 11 | $\to$ | 10 | $\to$ 11 |
| 00 | $\to$ | 00 | $\to$ 00 |
| 01 | $\to$ | 01 | $\to$ 01 |

Third Party · · · · Alice · · · · · · Bob
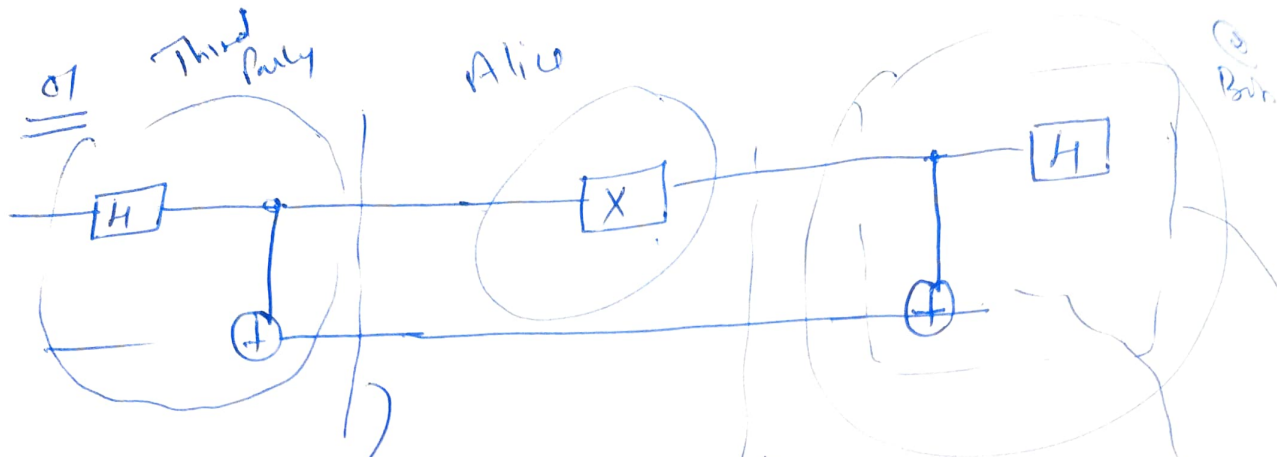


$$\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

$$\frac{1}{\sqrt{2}}|0_A 0_B\rangle + \frac{1}{\sqrt{2}}|1_A 1_B\rangle$$

$$=$$

$$|0\rangle - \boxed{H} \xrightarrow{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)} \boxed{H} - |0\rangle$$

$$\alpha|0\rangle + \beta|1\rangle - \boxed{X} \xrightarrow{\beta|0\rangle + \alpha|1\rangle}$$

$$\frac{1}{\sqrt{2}}|1_A 0_B\rangle + |0_A 1_B\rangle$$
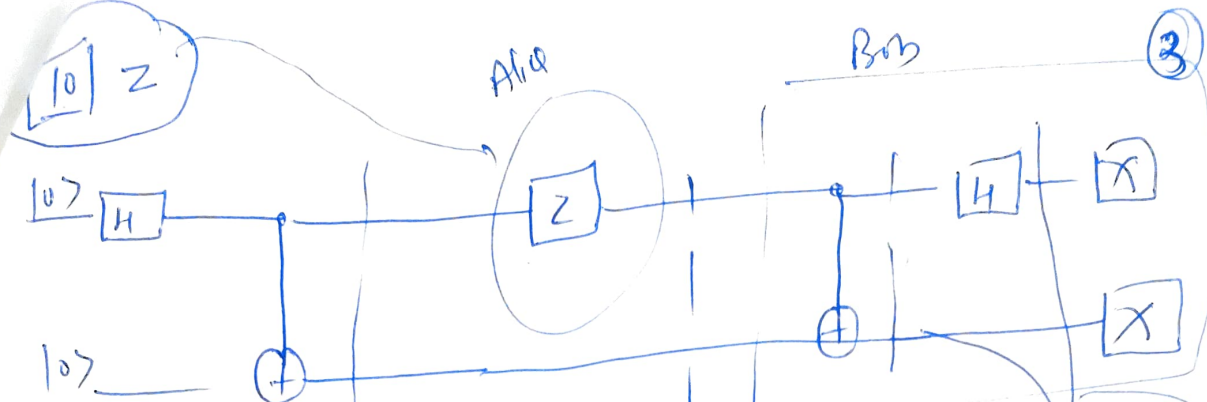
$$\frac{1}{\sqrt{2}}|1_A 1_B\rangle + |0_A 1_B\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|1_A + 0_A\rangle\right)|1_B\rangle$$

$$|0_A\rangle |1_B\rangle$$

$|0\rangle$ Z

$|0\rangle$ — H — ● — Z — ● — H — X

$|0\rangle$ — ⊕ — ⊕ — X

$$\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

$$\frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 0_B\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0_A\rangle - |1_A\rangle\right)|0_B\rangle$$

$|1\rangle$ — H — — H — $|1\rangle$

$$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|1_A\rangle|0_B\rangle\right)$$

$|0_A\rangle$

⑪

$Z X$  $| iY$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ④

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$iY = \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
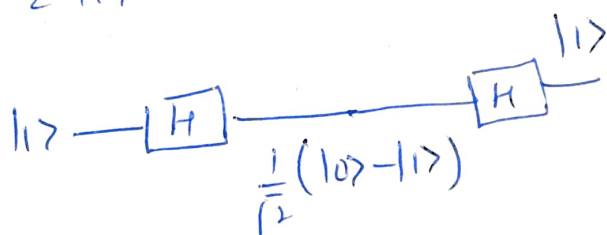


$\frac{1}{\sqrt{2}} ( |0_A 0_B\rangle + |1_A 1_B\rangle )$

$\frac{1}{\sqrt{2}} ( |0_A 0_B\rangle - |1_A 1_B\rangle )$
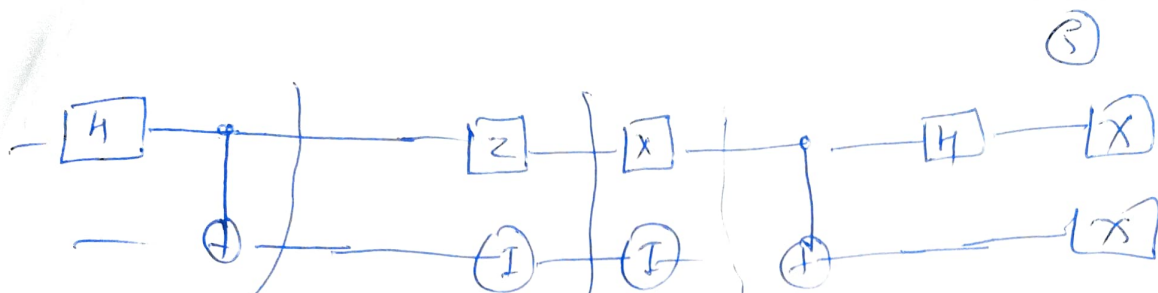
$-\frac{1}{\sqrt{2}} |0_A 0_B\rangle + |1_A 0_B\rangle$

$-\frac{1}{\sqrt{2}} ( |0_A 1_B\rangle + |1_A 1_B\rangle ) |1_B\rangle$

$( -\frac{1}{\sqrt{2}} |0_A\rangle + |1_A\rangle ) \Rightarrow -( \frac{1}{\sqrt{2}} |0_A\rangle - |1_A\rangle ) |1_B\rangle$

$\Downarrow$

$-|1_A\rangle |1_B\rangle$

$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$Z \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} ( -|0_A 1_B\rangle + |1_A 0_B\rangle )$
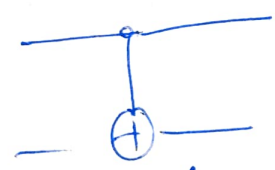
$$\frac{1}{\sqrt{2}}\left(\,|0_A\,0_B\rangle + |1_A 1_B\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$Z\otimes I = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right)$$

$$\begin{bmatrix}1\\0\\\end{bmatrix}$$

$$X\otimes I = \begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\left(-|0_A 1_B\rangle + |1_A 0_B\rangle\right)$$



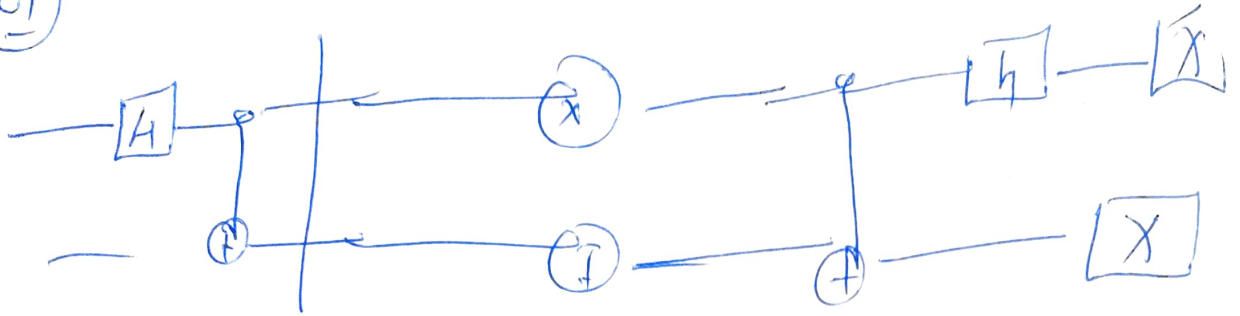$$CNOT\ \frac{1}{\sqrt{2}}\left(-|0_A 1_B\rangle + |1_A 0_B\rangle\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(-|0_A 1_B\rangle + |1_A 1_B\rangle\right) = \frac{1}{\sqrt{2}}\left(-|0_A\rangle + |1_A\rangle\right)|1_B\rangle$$

$$\overset{e^{i\pi}}{\Rightarrow}\ \overset{=}{(-1)}\frac{1}{\sqrt{2}}\left(|0_A\rangle - |1_A\rangle\right)|1_B\rangle$$

$$\underbrace{\qquad}_{H}$$

$$|1_A\rangle|1_B\rangle$$

$$\frac{1}{\sqrt{2}}\left( |0_A 0_B\rangle + |1_A 1_B\rangle \right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\left( |0_A 1_B\rangle + |1_A 0_B\rangle \right)$$
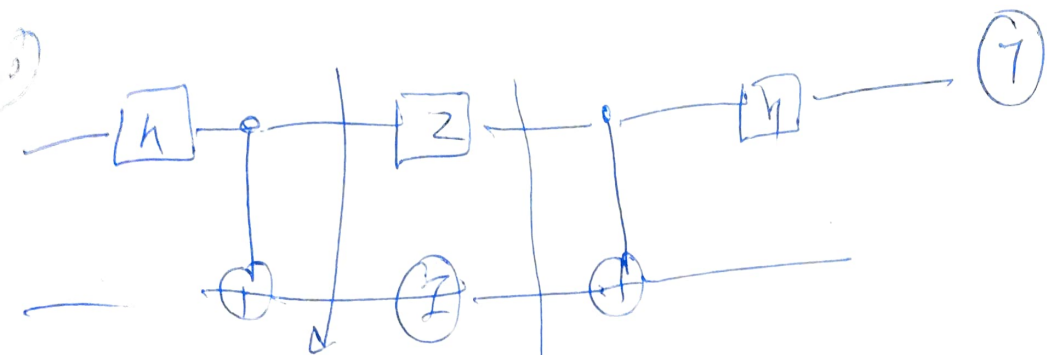
$$\Downarrow CNOT$$

$$\frac{1}{\sqrt{2}}\left[ |0_A 1_B\rangle + |1_A 1_B\rangle \right]$$

$$\frac{1}{\sqrt{2}}\left[ |0_A\rangle + |1_A\rangle \right] |1_B\rangle$$

$$\Downarrow H$$

$$|01\rangle$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Z \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle - |1_A 1_B\rangle\right)$$

$$\boxed{XZ = -ZX}$$