(Grover algorithm) (Class Notes)
April 8, 2024

Linear Search $O(n)$ List is unsorted

Binary " $O(\log_2 n)$ " " Sorted

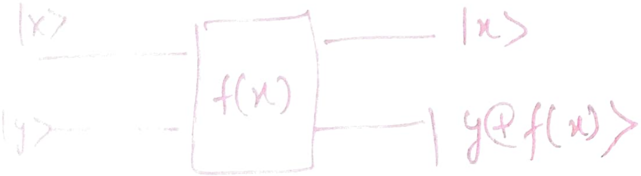
Grover algorithm $O(\sqrt{n})$ List is unsorted.

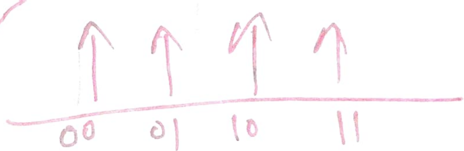
One Condition of Grover algorithm you Need some Oracle.
Black Box that will Guide you towards Solution.

$$f(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{else} \end{cases}$$

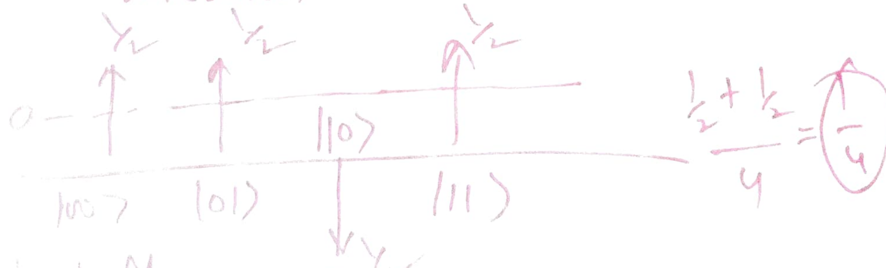
We are Searching for $x_0$ if $x_0$ is found oracle return 1.

Deutsch




Two Steps - (1) phase Inversion



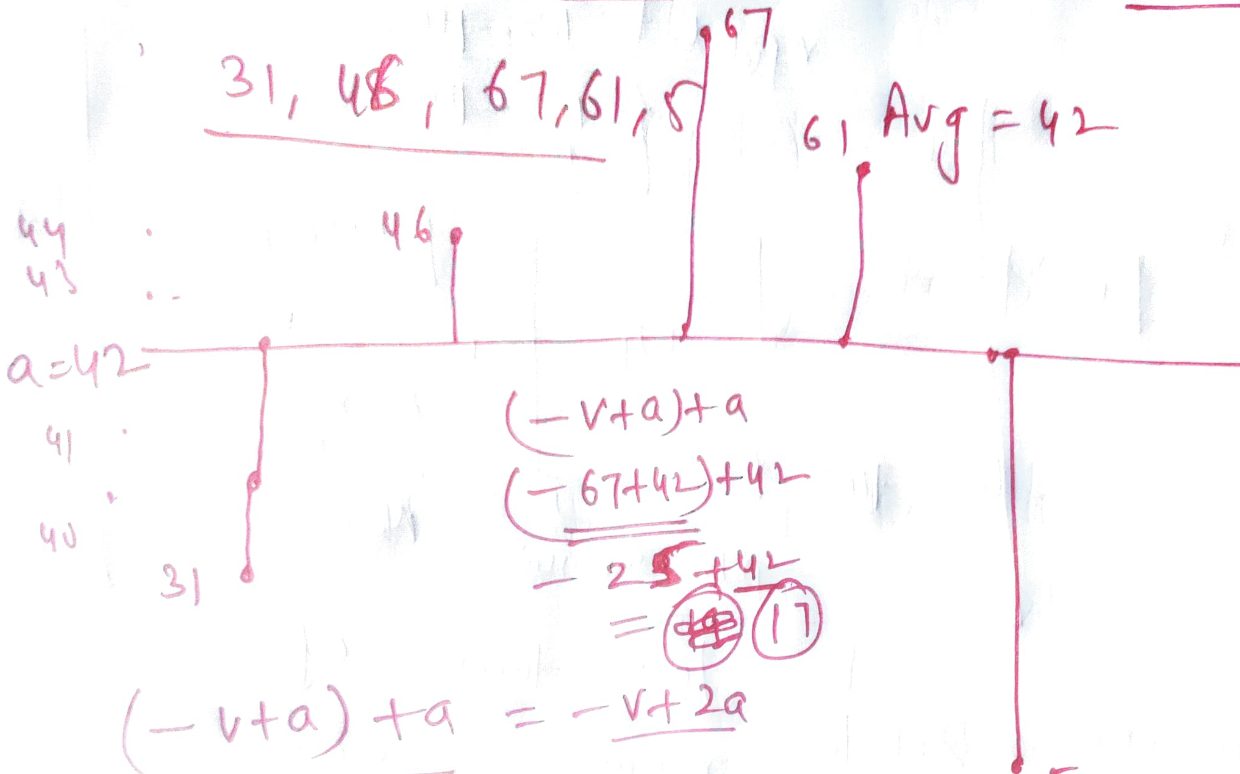

(2) Inversion about Mean

$$-V + 2a = -\frac{1}{2} + 2 \cdot \frac{1}{4} = 0$$

for $V = -\frac{1}{2}$ $\quad -V + 2a = \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

# Inversion about Mean or Inversion about Average

31, 46, 67, 61, 5   Avg = 42




$(-v+a)+a$

$(-67+42)+42$

$= -25+42$

$= 17$

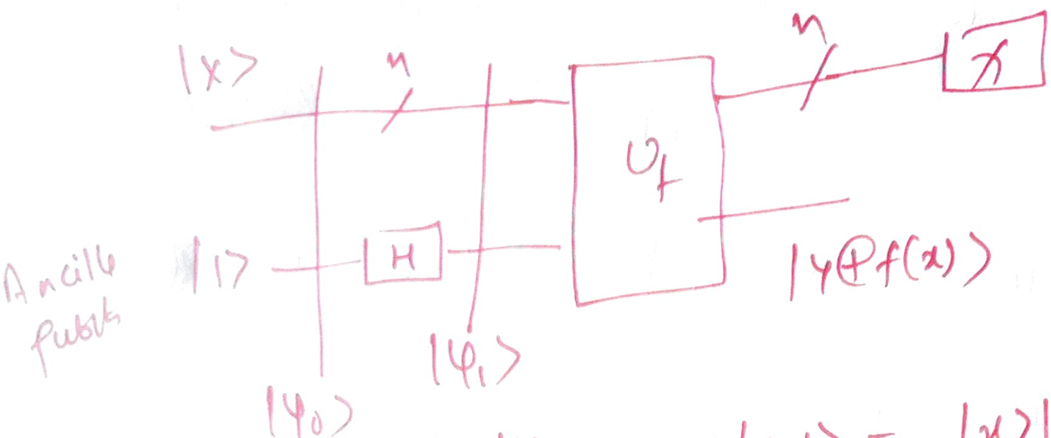
$(-v+a)+a = -v+2a$

$v=31 \quad a=42$

$(-31+42)+42 = 11+42 = 53$

$v=46 \quad (-46+42)+42 = -4+42 = 38$

# Phase Inversion




$$|\psi_0\rangle = |x,1\rangle = |x\rangle|1\rangle = |x\rangle \otimes |1\rangle$$

$$|\psi_1\rangle = |x\rangle \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = \frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}}$$

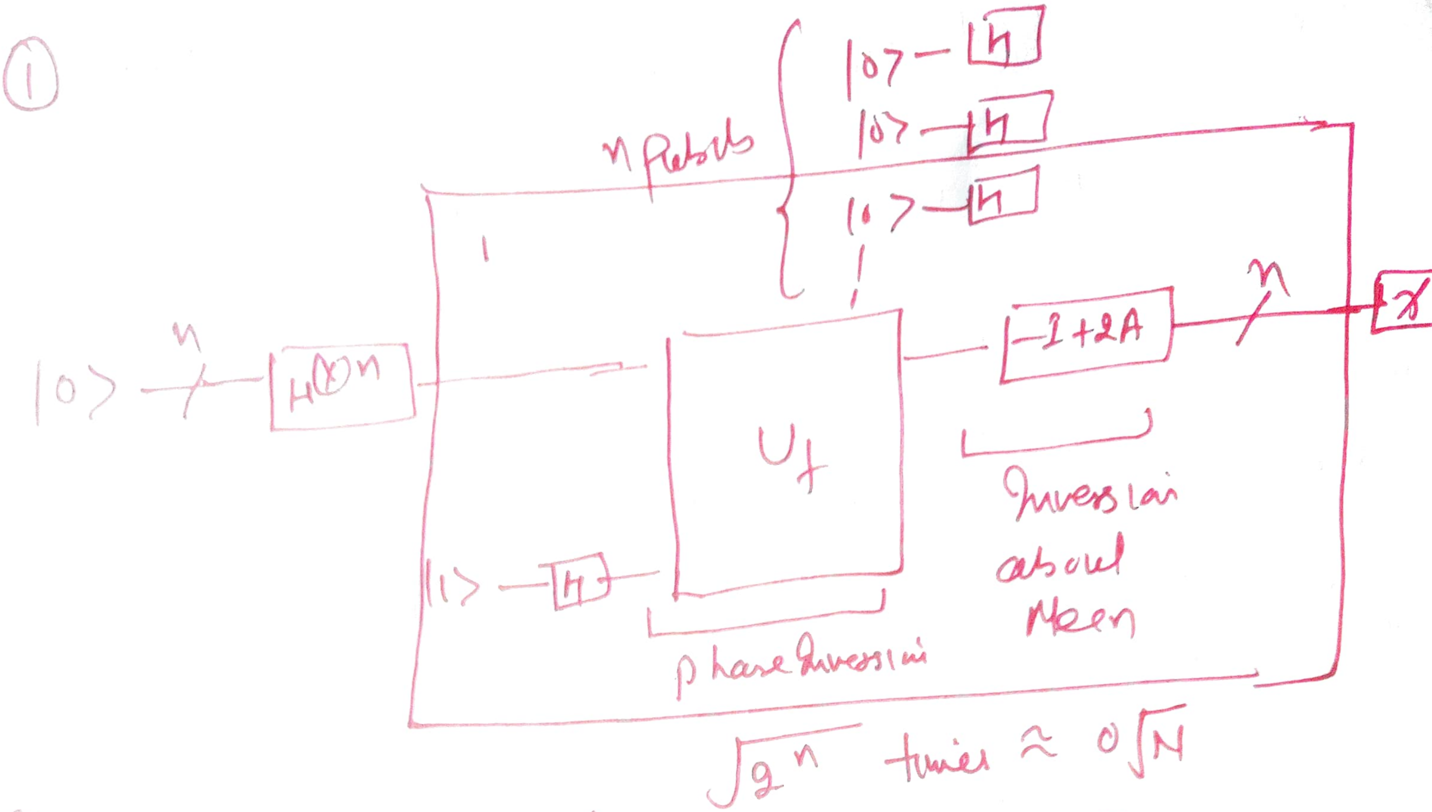
$$|\psi_2\rangle = |x\rangle \left[\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}}\right]$$

$$\Rightarrow |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \quad \text{If } f(x) = 0$$

$$|x\rangle \left[\frac{|1\rangle - |0\rangle}{\sqrt{2}}\right] \quad \text{If } f(x) = 1$$

$$(-1)^{f(x)} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

(1)




1. Start with a State $\underbrace{|0\rangle}$ n qubits in 0 zeros $|00\_\_ n \text{ times}\rangle = |0\rangle$
2. Apply $H^{\otimes n}$ $\quad H^{\otimes 2} = H \otimes H$
3. Repeat $\sqrt{2^n}$ times
   - 3(a) Apply phase Inversion operator $U_f(I \otimes H)$
   - 3(b) Apply the Inversion about mean $-I+2A$
4. Measure the Qubits.

Example: String = 101

$f(101) = 1$  Rest $\left.\begin{matrix} f(000) \\ \vdots \\ f(111) \end{matrix}\right\} \Rightarrow 0$

$$|\psi_1\rangle = \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array}$$
$$\begin{bmatrix} \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} \end{bmatrix}^T |1\rangle$$

$$= \begin{bmatrix} \frac{1}{\sqrt8} \\ \frac{1}{\sqrt8} \\ \frac{1}{\sqrt8} \\ \vdots \\ \frac{1}{\sqrt8} \end{bmatrix}$$

$$|\psi_2\rangle = \qquad \frac{|0\rangle - |1\rangle}{\sqrt2}$$

101 is Solution


$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array}$$
$$\begin{bmatrix} \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} & -\frac{1}{\sqrt8} & \frac{1}{\sqrt8} & \frac{1}{\sqrt8} \end{bmatrix}$$

$$\text{Mean} = \frac{7 \cdot \frac{1}{\sqrt8} - \frac{1}{\sqrt8}}{8} = \frac{6}{8} \cdot \frac{1}{\sqrt8} = \frac{3}{4\sqrt8}$$

$$-V + 2a = -\frac{1}{\sqrt8} + 2 \cdot \frac{3}{4\sqrt8} = \frac{1}{2\sqrt8}$$

$$V = \frac{1}{\sqrt8}$$

$$V = -\frac{1}{\sqrt8} \qquad -V + 2a$$

$$\frac{1}{\sqrt8} + \frac{2 \cdot 3}{4\sqrt8} = \frac{5}{2\sqrt8}$$

$$\begin{bmatrix} \frac{1}{2\sqrt8} & \frac{1}{2\sqrt8} & \frac{1}{2\sqrt8} & \frac{1}{2\sqrt8} & \frac{1}{2\sqrt8} & \frac{5}{2\sqrt8} & \frac{1}{2\sqrt8} & \frac{1}{2\sqrt8} \end{bmatrix}$$
$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array}$$