Unit 3 Chapter 5

Partial Differential Equations

5.1 Introduction

Till now, we are familier with the concepts of simple differentiation. In this chapter we will study another concept of differentiation known as Partial differentiation.

The difference between the simple differentiation and partial differentiation is explained below.

5.1.1 Simple Differentiation

Case 1 : Consider a function involving only one variable (x).

e.g. $$u = f(x)$$

$$u = 2 \sin x$$

Differentiating w.r.t. x

$$\frac{d}{dx}(u) = \frac{d}{dx}(2 \sin x)$$

Since '2' is constant we will take 2 outside the differentiation operator.

$$\therefore \quad \frac{du}{dx} = 2\frac{d}{dx}(\sin x)$$

$$\mathbf{\frac{du}{dx} = 2 \cos x}$$

Case 2 : Consider a function involving more than one variable. (x, y).

$$v = f(x, y)$$

$$v = xy$$

Differentiating w.r.t. x,

$$\frac{dv}{dx} = \frac{d}{dx}(xy)$$

As both x and y are variables, we derive it by using uv rule of derivative.

$$\therefore \quad \frac{dv}{dx} = x\frac{d(y)}{dx} + y\frac{d}{dx}(x)$$

$$\therefore \quad \frac{dv}{dx} = x\frac{d(y)}{dx} + y$$

Similarly, $$v = f(x, y)$$

$$v = xy$$

Differentiating w.r.t. y,

$$\frac{dv}{dy} = x\frac{d}{dy}(y) + y\frac{d}{dy}(x)$$

$$\frac{dv}{dy} = x + y\frac{dx}{dy}$$

5.1.2 Partial Differentiation :

Case 1 : Consider a function involving one variable (x)

e.g. $$u = f(x)$$

$$u = 2 \sin x$$

Differentiating w.r.t. x partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2 \sin x)$$

Since '2' is constant, we will take 2 outside the differentiation operator.

$$\therefore \quad \frac{\partial u}{\partial x} = 2\frac{\partial}{\partial x}(\sin x)$$

$$\mathbf{\frac{\partial u}{\partial x} = 2 \cos x}$$

Case 2 : Consider a function involving one variable (x, y)

e.g. $$v = f(x, y)$$

$$v = xy$$

Differentiating w.r.t. x partially,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(xy)$$

As we are differentiating with respect to x, partially, y becomes constant.

$$\therefore \quad \frac{\partial v}{\partial x} = y\frac{\partial}{\partial x}(x)$$

$$\frac{\partial v}{\partial x} = y\,(1)$$

$$\frac{\partial v}{\partial x} = y$$

Similarly, $v = xy$

Differentiating w.r.t. y partially, i.e. x becomes constant.

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(xy)$$

$$\frac{\partial v}{\partial y} = x\frac{\partial}{\partial y}(y)$$

$$\frac{\partial v}{\partial y} = x\,(1)$$

$$\frac{\partial v}{\partial y} = x$$

As we have seen, in the examples, the difference between simple differentiation and partial differentiation is given in the Table 5.1.

Table 5.1 : Difference between simple differentiation and partial differentiation

Simple differentiation	Partial differentiation
1. $\frac{d}{dx}(2\sin x) = 2\cos x$	1. $\frac{\partial}{\partial x}(2\sin x) = 2\cos x$
2. $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$	2. $\frac{\partial}{\partial x}(xy) = y$
3. $\frac{d}{dy}(xy) = x + y\frac{dx}{dy}$	3. $\frac{\partial}{\partial y}(xy) = x$

Conclusion : The method of finding derivative (E.g. $\frac{\partial}{\partial x}$) w.r.t. one variable only (i.e. treating other as constant) in a function of two or more than two variables is known as partial differentiation.

However, in simple differentiation all the given variables are treated as variables while finding derivatives.

5.2 Some Standard Derivatives

1. $\frac{\partial}{\partial x}(\text{constant}) = 0$
2. $\frac{\partial}{\partial x}(x) = 1$
3. $\frac{\partial}{\partial x}(x^n) = nx^{n-1}$
4. $\frac{\partial}{\partial x}(a^x) = a^x \log a$
5. $\frac{\partial}{\partial x}e^{ax} = ae^{ax}$
6. $\frac{\partial}{\partial x}\sqrt{x} = \frac{1}{2\sqrt{x}}$
7. $\frac{\partial}{\partial x}\frac{1}{x} = -\frac{1}{x^2}$
8. $\frac{\partial}{\partial x}\log x = \frac{1}{x}$
9. $\frac{\partial}{\partial x}\sin x = \cos x$
10. $\frac{\partial}{\partial x}\cos x = -\sin x$
11. $\frac{\partial}{\partial x}\tan x = \sec^2 x$
12. $\frac{\partial}{\partial x}\cot x = -\text{cosec}^2 x$
13. $\frac{\partial}{\partial x}\sec x = \sec x \cdot \tan x$
14. $\frac{\partial}{\partial x}\text{cosec}\, x = -\text{cosec}\, x \cdot \cot x$
15. $\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$
16. $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$
17. $\frac{\partial}{\partial x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
18. $\frac{\partial}{\partial x}\tan^{-1} x = \frac{1}{1+x^2}$

5.3 Type 1 : Solved Examples on Direct Differentiation

Example 5.3.1 **May 2012, 2018, Dec. 2018**

If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$$

Solution : Given : $u = \log(x^3 + y^3 - x^2y - xy^2)$

$$u = \log[(x^3 - x^2y) + (y^3 - xy^2)]$$

$$u = \log[x^2(x-y) + y^2(y-x)]$$

but $y - x = -(x-y)$

$$u = \log[x^2(x-y) - y^2(x-y)]$$

$$u = \log[(x-y)(x^2-y^2)]$$

$$u = \log[(x-y)(x-y)(x+y)]$$

$$u = \log[(x-y)^2(x+y)]$$

by $\log(AB) = \log A + \log B$

$$u = \log(x-y)^2 + \log(x+y)$$

By $\log m^n = n \log m$

$$\therefore \quad \mathbf{u = 2\log(x-y) + \log(x+y)} \qquad \ldots(1)$$

Part 1 : To Find $\frac{\partial^2 u}{\partial x^2}$

We have, $u = 2\log(x-y) + \log(x+y)$

Differentiating w.r.t. x, partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[2\log(x-y) + \log(x+y)]$$

$$= 2\frac{\partial}{\partial x}\log(x-y) + \frac{\partial}{\partial x}\log(x+y)$$

$$= 2\frac{1}{x-y}\frac{\partial}{\partial x}(x-y) + \frac{1}{x+y}\frac{\partial}{\partial x}(x+y)$$

$$= 2\frac{1}{x-y}(1-0) + \frac{1}{x+y}(1+0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{x-y} + \frac{1}{x+y}$$

Differentiating w.r.t. x, again partially,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = 2\frac{\partial}{\partial x}\frac{1}{x-y} + \frac{\partial}{\partial x}\frac{1}{x+y}$$

but $\frac{\partial}{\partial x}\frac{1}{x} = -\frac{1}{x^2}$

$$= -2\frac{1}{(x-y)^2}\frac{\partial}{\partial x}(x-y) - \frac{1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} = \frac{-2}{(\mathbf{x-y})^2} - \frac{1}{(\mathbf{x+y})^2}$$

Part 2 : To find $\frac{\partial^2 u}{\partial y^2}$:

From Equation (1),

$$u = 2\log(x-y) + \log(x+y)$$

Differentiating w.r.t. y partially,

$$\frac{\partial u}{\partial y} = 2\frac{\partial}{\partial y}\log(x-y) + \frac{\partial}{\partial y}\log(x+y)$$

$$= 2\frac{1}{x-y}\frac{\partial}{\partial y}(x-y) + \frac{1}{x+y}\frac{\partial}{\partial y}(x+y)$$

$$= 2\frac{1}{x-y}(0-1) + \frac{1}{x+y}(0+1)$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{y}} = \frac{-2}{\mathbf{x-y}} + \frac{1}{\mathbf{x+y}} \qquad \ldots(2)$$

Differentiating w.r.t. y partially again,

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -2\frac{\partial}{\partial y}\frac{1}{x-y} + \frac{\partial}{\partial y}\frac{1}{x+y}$$

$$= -2\frac{(-1)}{(x-y)^2}\frac{\partial}{\partial y}(x-y) + \frac{(-1)}{(x+y)^2}\frac{\partial}{\partial y}(x+y)$$

$$= \frac{2}{(x-y)^2}(0-1) - \frac{1}{(x+y)^2}(0+1)$$

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \frac{-2}{(\mathbf{x-y})^2} - \frac{1}{(\mathbf{x+y})^2}$$

Part 3 : To find $\frac{\partial^2 u}{\partial x \partial y}$:

From Equation (2)

$$\frac{\partial u}{\partial y} = \frac{-2}{x-y} + \frac{1}{x+y}$$

Differentiating w.r.t. x partially,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -2\frac{\partial}{\partial x}\frac{1}{x-y} + \frac{\partial}{\partial x}\frac{1}{x+y}$$

$$= -2\frac{(-1)}{(x-y)^2}(1-0) + \frac{(-1)}{(x+y)^2}(1+0)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

Now,

$$\text{L.H.S.} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2\left[\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}\right] + \left[\frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2}\right]$$

$$= \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} + \frac{4}{(x-y)^2} - \frac{2}{(x+y)^2}$$

$$- \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$= \frac{-4}{(x+y)^2} = \text{R.H.S.} \qquad \ldots\text{Hence proved}$$

Example 5.3.2

If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$. *Find* u_{xy}

Solution :

Given : $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Differentiating w.r.t. y, partially,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2 \tan^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial y}\left[y^2 \tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= x^2 \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$-\left\{y^2 \frac{\partial}{\partial y} \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \frac{\partial}{\partial y}(y^2)\right\}$$

$$= x^2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$-\left\{y^2 \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right) \cdot 2y\right\}$$

$$= x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$-\left\{\frac{y^2}{1+\frac{x^2}{y^2}} \cdot x \cdot \left(-\frac{1}{y^2}\right) + 2y \tan^{-1}\left(\frac{x}{y}\right)\right\}$$

$$= \frac{x}{\frac{x^2+y^2}{x^2}} - \left\{\frac{-x}{\frac{y^2+x^2}{y^2}} + 2y \tan^{-1}\left(\frac{x}{y}\right)\right\}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial y} = x - 2y \tan^{-1}\left(\frac{x}{y}\right)$$

Now, Differentiating w.r.t. x partially,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}\left[2y \tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = 1 - 2y \frac{\partial}{\partial x} \tan^{-1}\left(\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{\frac{y^2+x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2y^2}{y^2+x^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\, \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

$$\therefore \quad u_{xy} = \frac{x^2-y^2}{x^2+y^2}$$

Example 5.3.3 **Dec. 2017**

If $z^3 - zx - y = 0$, *prove that,*

$$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{(3z^2+x)}{(3z^2-x)^3}$$

Solution :

Given : $z^3 - zx - y = 0$

Differentiating w.r.t. y partially,

$$\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial y}(y) = 0$$

$$3z^2 \cdot \frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1 = 0$$

$$\frac{\partial z}{\partial y}(3z^2 - x) = 1$$

$$\therefore \quad \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

Now, differentiating w.r.t. x, partially,

$$\frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}\frac{1}{3z^2-x}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{(-1)}{(3z^2-x)^2}\frac{\partial}{\partial x}(3z^2-x)$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{-1}{(3z^2-x)^2} \cdot \left[6z\frac{\partial z}{\partial x} - 1\right] \quad ...(1)$$

But to find $\frac{\partial z}{\partial x}$ we have,

$$z^3 - zx - y = 0$$

Differentiating w.r.t. x, partially,

$$\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial x}(y) = 0$$

$$\therefore \quad 3z^2 \frac{\partial z}{\partial x} - \left[z \frac{\partial}{\partial x}(x) + x \frac{\partial z}{\partial x} \right] - 0 = 0$$

$$\therefore \quad 3z^2 \frac{\partial z}{\partial x} - z - x \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(3z^2 - x) = z$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

From Equation (1),

$$\therefore \quad \frac{\partial^2 z}{\partial x\, \partial y} = \frac{-1}{(3z^2 - x)^2} \left[6z \left(\frac{z}{3z^2 - x} \right) - 1 \right]$$

$$= \frac{-1}{(3z^2 - x)^2} \left[\frac{6z^2 - 3z^2 + x}{3z^2 - x} \right]$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3} \quad \text{...Hence proved.}$$

Example 5.3.4 **May 2015, Dec. 2016**

Find the value of n for which $u = A e^{-gx} \sin(nt - gx)$ satisfies the partial differential equation.

$$\frac{\partial u}{\partial t} = m \frac{\partial^2 u}{\partial x^2}, \text{ where A, g are constants.}$$

Solution :

Part 1 : To find $\frac{\partial u}{\partial t}$

Given : $u = A e^{-gx} \sin(nt - gx)$

$$\therefore \quad e^{gx} u = A \sin(nt - gx)$$

Differntiating w.r.t. t, partially,

$$\frac{\partial}{\partial t}[e^{gx} u] = \frac{\partial}{\partial t}[A \sin(nt - gx)]$$

$$e^{gx} \frac{\partial u}{\partial t} = A \cos(nt - gx) \frac{\partial}{\partial t}(nt - gx)$$

$$\therefore \quad e^{gx} \frac{\partial u}{\partial t} = nA \cos(nt - gx)$$

$$\therefore \quad \frac{\partial u}{\partial t} = e^{-gx} nA \cos(nt - gx) \quad \text{...(A)}$$

Part 2 : To find $\frac{\partial^2 u}{\partial x^2}$

We have, $u = A e^{-gx} \sin(nt - gx)$

$$e^{gx} u = A \sin(nt - gx) \quad \text{...(1)}$$

Differentiating w.r.t. x, partially,

$$\frac{\partial}{\partial x}[e^{gx} u] = \frac{\partial}{\partial x} A \sin(nt - gx)$$

$$e^{gx} \frac{\partial u}{\partial x} + u \frac{\partial}{\partial x} e^{gx} = A \cos(nt - gx) \frac{\partial}{\partial x}(nt - gx)$$

$$e^{gx} \frac{\partial u}{\partial x} + u \cdot e^{gx} \cdot g = A \cos(nt - gx)(0 - g)$$

$$e^{gx}\left[\frac{\partial u}{\partial x} + ug\right] = -g A \cos(nt - gx) \quad \text{...(2)}$$

Differentiating w.r.t. x, partially again,

$$e^{gx}\left[\frac{\partial^2 u}{\partial x^2} + g \frac{\partial u}{\partial x}\right] + \left[\frac{\partial u}{\partial x} + gu\right] e^{gx} \cdot g$$

$$= g A \sin(nt - gx)(-g)$$

$$\therefore \quad e^{gx}\left[\frac{\partial^2 u}{\partial x^2} + g \frac{\partial u}{\partial x} + g \frac{\partial u}{\partial x} + g^2 u\right]$$

$$= -g^2 \underbrace{A \sin(nt - gx)}$$

From Equation (1)

$$e^{gx}\left[\frac{\partial^2 u}{\partial x^2} + 2g \frac{\partial u}{\partial x} + g^2 u\right] = -g^2 e^{gx} u$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + 2g \frac{\partial u}{\partial x} + g^2 u + g^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + 2g \frac{\partial u}{\partial x} + 2g^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + 2g\left[\frac{\partial u}{\partial x} + gu\right] = 0$$

But from Equation (2),

$$\frac{\partial u}{\partial x} + gu = -g A e^{-gx} \cos(nt - gx)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + 2g[-g A e^{-gx} \cos(nt - gx)] = 0$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = 2g^2 A e^{-gx} \cos(nt - gx) \quad \text{...(B)}$$

Part 3 : To find n

Given, $\frac{\partial u}{\partial t} = m \frac{\partial^2 u}{\partial x^2}$

From Equations (A) and (B)

$$e^{-gx} nA \cos(nt - gx) = m\, 2g^2 A \cos(nt - gx)\, e^{-gx}$$

$$n = 2mg^2$$

Example 5.3.5 **Dec. 2010, May 2014**

Find the value of n so that $u = r^n (3\cos^2\theta - 1)$ satisfies the partial differential equation.

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial u}{\partial\theta}\right) = 0$$

Solution :

Part 1 : To find $\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$

Given : $u = r^n (3\cos^2\theta - 1)$

Differentiating w.r.t. r, partially,

$$\therefore \quad \frac{\partial u}{\partial r} = n r^{n-1} (3\cos^2\theta - 1)$$

Multiplying by r^2 on both sides,

$$\therefore \quad r^2 \frac{\partial u}{\partial r} = n r^2 r^{n-1} (3\cos^2\theta - 1)$$

By, $x^m \cdot x^n = x^{m+n}$

$$r^2 \frac{\partial u}{\partial r} = n r^{n+1} (3\cos^2\theta - 1)$$

Differentiating w.r.t. r, partially again

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = n(3\cos^2\theta - 1)\frac{\partial}{\partial r}\; r^{n+1}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = n(3\cos^2\theta - 1)(n+1) r^n$$

But $r^n (3\cos^2\theta - 1) = u$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = n(n+1)\,u \qquad \ldots(1)$$

Part 2 : To find $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)$

We have, $u = r^n (3\cos^2\theta - 1)$

Differentiating w.r.t. θ partially,

$$\frac{\partial u}{\partial\theta} = r^n \frac{\partial}{\partial\theta}(3\cos^2\theta - 1)$$

$$\frac{\partial u}{\partial\theta} = r^n [-6\cos\theta\sin\theta - 0]$$

$$\frac{\partial u}{\partial\theta} = -6 r^n \cos\theta \cdot \sin\theta$$

Multiplying by sin θ on both sides,

$$\therefore \quad \sin\theta \frac{\partial u}{\partial\theta} = -6 r^n \cos\theta \cdot \sin^2\theta$$

Differentiating w.r.t. θ partially again,

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6 r^n \frac{\partial}{\partial\theta}[\cos\theta \cdot \sin^2\theta]$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6 r^n [\cos\theta \cdot (2\sin\theta \cdot \cos\theta) + \sin^2\theta(-\sin\theta)]$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6 r^n [2\sin\theta \cdot \cos^2\theta - \sin^3\theta]$$

$$= -6 r^n \sin\theta [2\cos^2\theta - \sin^2\theta]$$

But $\cos^2\theta + \sin^2\theta = 1$,
$\therefore \cos^2\theta - 1 = -\sin^2\theta$

$$= -6 r^n \sin\theta (2\cos^2\theta + \cos^2\theta - 1)$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6 r^n \sin\theta (3\cos^2\theta - 1)$$

But $r^n (3\cos^2\theta - 1) = u$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6\sin\theta \cdot u$$

Divide by sin θ on both sides,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6u \qquad \ldots(2)$$

But given differential equation is,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0$$

$\therefore$ From Equations (1) and (2),

$$n(n+1)\,u + (-6u) = 0$$

$$\therefore \quad u\,[n^2 + n - 6] = 0$$

$$(n^2 + n - 6) = 0 \qquad \ldots(\because u \neq 0)$$

$$\therefore \quad n = 2, -3$$

Example 5.3.6 **May 2016, 2017**

Find the value of n for which $z = t^n e^{\frac{-r^2}{4t}}$ satisfies the partial differential equation.

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right)\right] = \frac{\partial z}{\partial t}$$

Solution :

Part 1 : To find $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right)\right]$

Given : $z = t^n e^{\frac{-r^2}{4t}}$

Differentiating w.r.t. r, partially,

$$\frac{\partial z}{\partial r} = t^n \frac{\partial}{\partial r}\left(e^{\frac{-r^2}{4t}}\right)$$

$$\therefore \quad \frac{\partial z}{\partial r} = t^n e^{\frac{-r^2}{4t}} \frac{\partial}{\partial r}\left(\frac{-r^2}{4t}\right)$$

$$= t^n e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right)$$

$$\boxed{\text{But, } \frac{x^m}{x^n} = x^{m-n}}$$

$$\frac{\partial z}{\partial r} = \frac{-r\,t^{n-1}}{2}\, e^{\frac{-r^2}{4t}}$$

Multiplying by r^2 on both sides,

$$r^2 \frac{\partial z}{\partial r} = -\frac{t^{n-1}}{2}\, r^3 e^{\frac{-r^2}{4t}}$$

Differentiating w.r.t. r, partially again,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = -\frac{t^{n-1}}{2} \frac{\partial}{\partial r}\left[r^3 e^{\frac{-r^2}{4t}}\right]$$

$$= -\frac{t^{n-1}}{2}\left[r^3 e^{\frac{-r^2}{4t}}\left(\frac{-2r}{4t}\right) + e^{\frac{-r^2}{4t}}(3r^2)\right]$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = -\frac{t^{n-1}}{2}\, r^2 e^{\frac{-r^2}{4t}}\left[\frac{-2r^2}{4t} + 3\right]$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = -\frac{t^{n-1}}{2}\, e^{\frac{-r^2}{4t}}\left(\frac{-2r^2}{4t} + 3\right) \quad \ldots(1)$$

Part 2 : To find $\frac{\partial z}{\partial t}$

Given: $z = t^n e^{-\frac{r^2}{4t}}$

Differentiating w.r.t t partially,

$$\frac{\partial z}{\partial t} = t^n e^{-\frac{r^2}{4t}}\left[-\frac{r^2}{4} \times \left(\frac{-1}{t^2}\right)\right] + e^{-\frac{r^2}{4t}}\, n t^{n-1}$$

$$\frac{\partial z}{\partial t} = t^n e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t^2}\right) + e^{-\frac{r^2}{4t}}\, n t^{n-1}$$

$$\frac{\partial z}{\partial t} = \frac{r^2 t^{n-1} e^{-\frac{r^2}{4t}}}{4t} + n t^{n-1}\, e^{-\frac{r^2}{4t}}$$

$$\frac{\partial z}{\partial t} = t^{n-1} e^{-\frac{r^2}{4t}}\left[\frac{r^2}{4t} + n\right]$$

But given differential equation is,

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right)\right] = \frac{\partial z}{\partial t}$$

$\therefore$ From Equations (1) and (2),

$$-\frac{t^{n-1}}{2}\, e^{-\frac{r^2}{4t}}\left[-\frac{2r^2}{4t} + 3\right] = t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{4t} + n\right)$$

$$\frac{-1}{2}\left[\frac{-r^2}{2t} + 3\right] = \frac{r^2}{4t} + n$$

$$\frac{r^2}{4t} - \frac{3}{2} = \frac{r^2}{4t} + n$$

$$\therefore \quad n = -\frac{3}{2}$$

Example 5.3.7 **Dec. 2015, May [illegible]**

If $z = \tan(y + ax) + (y - ax)^{3/2}$, find the value of

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$$

Solution :

Part 1 : To find $\frac{\partial^2 z}{\partial x^2}$

Given : $z = \tan(y + ax) + (y - ax)^{3/2}$

Differentiating w.r.t. x, partially,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\tan(y + ax) + \frac{\partial}{\partial x}(y - ax)^{3/2}$$

$$= \sec^2(y + ax)(0 + a) + \frac{3}{2}(y - ax)^{\frac{3}{2}-1}(0 - a)$$

$$\frac{\partial z}{\partial x} = a\sec^2(y + ax) - \frac{3a}{2}(y - ax)^{\frac{1}{2}}$$

Differentiating w.r.t. x, again partially,

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = a\frac{\partial}{\partial x}[\sec^2(y + ax)] - \frac{3a}{2}\frac{\partial}{\partial x}(y - ax)^{\frac{1}{2}}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = a \cdot 2\sec(y + ax)\sec(y + ax) \cdot \tan(y + ax)(0 + a) - \frac{3a}{2}\left(\frac{1}{2}\right)(y - ax)^{\frac{1}{2}-1}(0 - a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2 \sec^2 (y + ax) \tan (y + ax)$$

$$+ \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}} \qquad ...(1)$$

Part 2 : To find $a^2 \frac{\partial^2 z}{\partial y^2}$

Given : $z = \tan (y + ax) + (y - ax)^{3/2}$

Differentiating w.r.t. y, partially,

$$\frac{\partial z}{\partial y} = \sec^2 (y + ax) (1 + 0) + \frac{3}{2} (y - ax)^{\frac{1}{2}} (1 - 0)$$

$$\frac{\partial z}{\partial y} = \sec^2 (y + ax) + \frac{3}{2} (y - ax)^{\frac{1}{2}}$$

Differentiating w.r.t. y, partially again,

$$\frac{\partial^2 z}{\partial y^2} = 2 \sec^2 (y + ax) \tan (y + ax) + \frac{3}{4} (y - ax)^{-\frac{1}{2}}$$

Multiplying by a^2 on both sides,

$$a^2 \frac{\partial^2 z}{\partial y^2} = 2a^2 \sec^2 (y + ax) \tan (y + ax)$$

$$+ \frac{3a^2}{4} (y - ax)^{-\frac{1}{2}} \qquad ...(2)$$

Subtracting Equation (1) and Equation (2)

$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Example 5.3.8 **Dec. 2017**

If $z^3 - xz - y = 4$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution :

Part 1 : To find $\frac{\partial z}{\partial x}$

Given : $z^3 - xz - y = 4$

Differentiating w.r.t. x partially,

$$\frac{\partial}{\partial x} (z^3) - \frac{\partial}{\partial x} (xz) - \frac{\partial}{\partial x} (y) = \frac{\partial}{\partial x} (4)$$

$$3z^2 \frac{\partial z}{\partial x} - \left[x \frac{\partial z}{\partial x} + z (1) \right] - 0 = 0$$

$$\therefore \quad 3z^2 \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial x} = z$$

$$\frac{\partial z}{\partial x} (3z^2 - x) = z$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

Part 2 : To find $\frac{\partial z}{\partial y}$

Given : $z^3 - xz - y = 4$

Differentiating w.r.t. y partially,

$$\frac{\partial}{\partial y} (z^3) - \frac{\partial}{\partial y} (xz) - \frac{\partial}{\partial y} (y) = \frac{\partial}{\partial y} (4)$$

$$\therefore \quad 3z^2 \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} - 1 = 0$$

$$\frac{\partial z}{\partial y} (3z^2 - x) = 1$$

$$\therefore \quad \frac{\partial z}{\partial y} = \frac{1}{3z^2 - x}$$

Example 5.3.9

Prove that a point of the surface $x^x y^y z^z = C$ where $x = y = z$.

$$\frac{\partial^2 z}{\partial x \, \partial y} = -(x \log e x)^{-1}$$

Solution :

Given : $x^x y^y z^z = C$

Taking log on both sides,

$$\log [x^x y^y z^z] = \log C$$

By $\log (ABC) = \log A + \log B + \log C$

$$\therefore \log x^x + \log y^y + \log z^z = \log C$$

By $\log m^n = n \log m$

$$\therefore \quad x \log x + y \log y + z \log z = \log C \qquad ...(1)$$

Differentiating w.r.t. y partially,

$$\frac{\partial}{\partial y} (x \log x) + \frac{\partial}{\partial y} (y \log y) + \frac{\partial}{\partial y} (z \log z) = \frac{\partial}{\partial z} (\log C)$$

$$0 + y \cdot \frac{1}{y} + \log y (1) + z \cdot \frac{1}{z} \frac{\partial z}{\partial y} + \log z \frac{\partial z}{\partial y} = 0$$

$$\therefore 1 + \log y + \frac{\partial z}{\partial y} (1 + \log z) = 0$$

$$\therefore \quad \frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)}$$

Differentiating w.r.t. x partially,

$$\frac{\partial}{\partial x} \frac{\partial z}{\partial y} = -\frac{\partial}{\partial x} \left[\frac{1 + \log y}{1 + \log z} \right]$$

$$\therefore \quad \frac{\partial^2 z}{\partial x \, \partial y} = -(1 + \log y) \frac{\partial}{\partial x} \frac{1}{1 + \log z}$$

By $\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$= -(1+\log y)\frac{(-1)}{(1+\log z)^2}\cdot\left[0+\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{(1+\log y)}{(1+\log z)^2}\cdot\frac{1}{z}\frac{\partial z}{\partial x} \quad \text{...(2)}$$

Now to find $\frac{\partial z}{\partial x}$ consider Equation (1)

$$\therefore \quad x\log x + y\log y + z\log z = \log C$$

Differentiating w.r.t. x, partially,

$$\frac{\partial}{\partial x}(x\log x) + \frac{\partial}{\partial x}(y\log y) + \frac{\partial}{\partial x}(z\log z) = \frac{\partial}{\partial x}\log C$$

$$x\cdot\frac{1}{x} + \log x\,(1) + 0 + z\cdot\frac{1}{z}\frac{\partial z}{\partial x} + \log z\cdot\frac{\partial z}{\partial x} = 0$$

$$\therefore \quad 1+\log x + \frac{\partial z}{\partial x}(1+\log z) = 0$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Put this value in Equation (2),

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{(1+\log y)}{z\,(1+\log z)^2}\cdot\left[-\frac{(1+\log x)}{(1+\log z)}\right]$$

But $x = y = z$

$\therefore$ Converting R.H.S. in terms of x

$$= \frac{-(1+\log x)(1+\log x)}{x\,(1+\log x)^2\,(1+\log x)} = -\frac{1}{x\,(1+\log x)}$$

But $\log e = 1$

$$= -\frac{1}{x\,(\log e + \log x)}$$

By $\log A + \log B = \log(AB)$

$$= -\frac{1}{x\log ex}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = -(x\log e\,x)^{-1} \quad \text{...Hence proved}$$

Example 5.3.10 **May 2016**

If $u = \log(x^3+y^3+z^3-3xyz)$, *prove that,*

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution :

Given : $\quad u = \log(x^3+y^3+z^3-3xyz)$

Differentiating w.r.t. x, partially,

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\times(3x^2+0+0-3yz)$$

$$\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$

and $\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$

Now, L.H.S.

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} + \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz} + \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}\right]$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3\,(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}\right]$$

But, $x^3+y^3+z^3-3xyz$
$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3}{x+y+z}\right]$$

$$= \frac{\partial}{\partial x}\frac{3}{x+y+z} + \frac{\partial}{\partial y}\frac{3}{x+y+z} + \frac{\partial}{\partial z}\frac{3}{x+y+z}$$

But $\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$= \frac{(-3)}{(x+y+z)^2} + \frac{(-3)}{(x+y+z)^2} + \frac{(-3)}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

$= \text{R.H.S.}$ ...Hence proved

Example 5.3.11

If $u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$ *then show that,*

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$

Solution :

Given : $$u = \tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$$

Differentiating w.r.t. y partially,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$$

But $\frac{\partial}{\partial x}\tan^{-1} x = \frac{1}{1+x^2}$

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$$

By $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2 y^2}{1+x^2+y^2}} \cdot \left[\frac{\sqrt{1+x^2+y^2}\,\frac{\partial}{\partial y}(xy) - xy\,\frac{\partial}{\partial y}(\sqrt{1+x^2+y^2})}{(\sqrt{1+x^2+y^2})^2}\right]$$

But $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$= \frac{1}{\frac{1+x^2+y^2+x^2y^2}{1+x^2+y^2}} \cdot \left[\frac{x\sqrt{1+x^2+y^2} - xy\cdot\frac{1}{2\sqrt{1+x^2+y^2}}\cdot(0+0+2y)}{1+x^2+y^2}\right]$$

But $\sqrt{a}\sqrt{a} = a$

By $\frac{a}{b/c} = \frac{ac}{b}$ and $\frac{a/b}{c} = \frac{a}{bc}$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\left[\frac{x(1+x^2+y^2) - xy^2}{(1+x^2+y^2)\sqrt{1+x^2+y^2}}\right]$$

$$\therefore \frac{\partial u}{\partial y} = \frac{x(1+x^2+y^2-y^2)}{[1+x^2+y^2(1+x^2)]\sqrt{1+x^2+y^2}}$$

$$= \frac{x(1+x^2)}{(1+x^2)(1+y^2)\sqrt{(1+x^2+y^2)}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}$$

Now, differentiating w.r.t. x partially,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left[\frac{x}{(1+y^2)\sqrt{1+x^2+y^2}}\right]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{1+y^2}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]$$

$$= \frac{1}{1+y^2}\left[\frac{\sqrt{1+x^2+y^2}\,(1) - x\frac{1}{2\sqrt{1+x^2+y^2}}(0+2x+0)}{(\sqrt{1+x^2+y^2})^2}\right]$$

$$= \frac{1}{1+y^2}\left[\frac{\frac{\sqrt{1+x^2+y^2}\sqrt{1+x^2+y^2} - x^2}{\sqrt{1+x^2+y^2}}}{(1+x^2+y^2)}\right]$$

But $\sqrt{a}\sqrt{a} = a$ & $\frac{a/b}{c} = \frac{a}{bc}$

$$= \frac{1}{1+y^2}\left[\frac{1+x^2+y^2-x^2}{\sqrt{1+x^2+y^2}\,(1+x^2+y^2)}\right]$$

$$= \frac{1}{(1+y^2)}\cdot\left[\frac{1+y^2}{(1+x^2+y^2)^{1/2}(1+x^2+y^2)^1}\right]$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$$ **...Hence proved.**

Example 5.3.12

If $v = (x^2 - y^2) f(xy)$ then show that,

$$V_{xx} + V_{yy} = (x^4 - y^4)\, f''(xy)$$

Solution :

Let, $v = (x^2 - y^2) f(xy)$

Differentiating w.r.t. x, partially,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[(x^2 - y^2) f(xy)]$$

$$= (x^2 - y^2)\frac{\partial}{\partial x}[f(xy)] + f(xy)\frac{\partial}{\partial x}(x^2 - y^2)$$

$$= (x^2 - y^2) f'(xy)\frac{\partial}{\partial x}(xy) + f(xy)\cdot(2x - 0)$$

$$\frac{\partial v}{\partial x} = y(x^2 - y^2) f'(xy) + 2x f(xy)$$

Differentiating w.r.t. x, partially again,

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[y(x^2 - y^2) f'(xy)] + \frac{\partial}{\partial x}[2x f(xy)]$$

$$= y\frac{\partial}{\partial x}[(x^2 - y^2) f'(xy)] + 2\frac{\partial}{\partial x}[x f(xy)]$$

$$\frac{\partial^2 v}{\partial x^2} = y\,[(x^2-y^2)\,f''(xy)\cdot y + f'(xy)\,(2x-0)]$$

$$+\,2\,[x\,f'(xy)\,y + f(xy)\,(1)]$$

$$V_{xx} = y^2(x^2-y^2)\,f''(xy) + 2xy\,f'(xy)$$

$$+\,2xy\,f'(xy) + 2\,f(xy)$$

$$V_{xx} = y^2(x^2-y^2)\,f''(xy) + 4\,xy\,f'(xy)$$

$$+\,2f(xy) \qquad \ldots(1)$$

Again, $V = (x^2-y^2)\,f(xy)$

Differentiating w.r.t. y partially,

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\,[(x^2-y^2)\,f(xy)]$$

$$\frac{\partial v}{\partial y} = (x^2-y^2)\,\frac{\partial}{\partial y}f(xy) + f(xy)\,\frac{\partial}{\partial y}(x^2-y^2)$$

$$\therefore \quad \frac{\partial v}{\partial y} = (x^2-y^2)\,f'(xy)\,\frac{\partial}{\partial y}(xy) + f(xy)\,(0-2y)$$

$$\frac{\partial v}{\partial y} = x\,(x^2-y^2)\,f'(xy) - 2y\,f(xy)$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 v}{\partial y^2} = x\,[(x^2-y^2)\,f''(xy)\,x + f'(xy)\,(0-2y)]$$

$$-\,2\,[y\,f'(xy)\cdot x + f(xy)\,(1)]$$

$$= x^2(x^2-y^2)\,f''(xy) - 2xy\,f'(xy)$$

$$-\,2xy\,f'(xy) - 2\,f(xy)$$

$$V_{yy} = x^2(x^2-y^2)\,f''(xy) - 4xy\,f'(xy)$$

$$-\,2f(xy) \qquad \ldots(2)$$

Adding Equation (1) and Equation (2)

$$\therefore \quad V_{xx} + V_{yy} = x^2(x^2-y^2)\,f''(xy)$$

$$+\,y^2(x^2-y^2)\,f''(xy)$$

$$= (x^2-y^2)\,f''(xy)\,[x^2+y^2]$$

$$= f''(xy)\,(x^2-y^2)\,(x^2+y^2)$$

But $(a-b)(a+b) = a^2-b^2$

$$\therefore \quad V_{xx} + V_{yy} = (x^4-y^4)\,f''(xy)$$

...Hence Proved

Exercise 5.1

1. If $v = (1-2xy+y^2)^{-1/2}$ then show that,
   (i) $x\,V_x - y\,V_y = y^2\,v^3$
   (ii) $\frac{\partial}{\partial x}\,[(1-x^2)\,V_x] + \frac{\partial}{\partial y}\,[y^2\,V_y] = 0$

2. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2\,\partial y} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial x}$

3. If $v = \frac{c}{\sqrt{t}}\,e^{\frac{-x^2}{4a^2t}}$ then show that,
   $$\frac{\partial V}{\partial t} = a^2\,\frac{\partial^2 V}{\partial x^2}$$

4. If $z = u(x, y)\,e^{ax+by}$ where $u(x, y)$ is such that $\frac{\partial^2 u}{\partial x\,\partial y} = 0$, find the constants a, b such that
   $$\frac{\partial^2 z}{\partial x\,\partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$$

Ans.: a = 1 , b = 1

5. Show that $z = f(x+at) + \phi(x-at)$ is a solution of $a^2\,\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$ for all f, ϕ and a being constant.

6. If $u = x^m\,y^n$, show that, $\frac{\partial^3 u}{\partial x\,\partial y\,\partial x} = \frac{\partial^3 u}{\partial y\,\partial x^2}$

7. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ in the following cases :
   (a) $u = ax^2 + 2hxy + by^2$
   (b) $u = x^y + y^x$
   (c) $x^3 y - \sin z + z^3 = 0$.

8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where
   (a) $z = \frac{x}{x^2+y^2}$
   (b) $z = \log(x^2+y^2)$

9. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
   Find $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}$ and $\frac{\partial \phi}{\partial x}$ in terms of r, θ, ϕ.
   Hint : Use $r^2 = x^2+y^2+z^2$,
   $$\phi = \tan^{-1}\frac{y}{x},$$
   $$\theta = \tan^{-1}\sqrt{\frac{(x^2+y^2)}{z^2}}$$

Ans.: $\sin\theta\cos\phi$, $\frac{\cos\theta\cos\phi}{r}$, $\frac{-\sin\phi}{r\sin\theta}$

10. Find the value of n for which $u = kt^{-1/2} \cdot e^{\frac{-x^2}{4a^2t}}$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Ans.: $n = 4$.

11. If $u = t^n e^{\frac{-r^2}{4kt}}$ find n for which

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \quad (k = \text{constant})$$

12. If $u = e^{xyz}$ then show that

$$\frac{\partial^3 u}{\partial y\, \partial x\, \partial y} = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}.$$

13. If $u = x^p y^q$, show that $\frac{\partial^3 u}{\partial x\, \partial y^2} = \frac{\partial^3 u}{\partial y\, \partial x\, \partial y}$.

14. If $u = e^{x - pt} \cos(x - pt)$, prove that $\frac{\partial^2 u}{\partial t^2} = p^2 \frac{\partial^2 u}{\partial x^2}$

15. If $u = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x\, u_x + \sin 2y\, u_y + \sin 2z\, u_z = 2$.

16. If $u = (x^2 - y^2)\, f(r)$ where $r = xy$ then show that $u_{xy} = (x^2 - y^2)\,[3 f'(r) + r f''(r)]$.

17. Prove that $u = \frac{1}{r}[f(ct + r) + \phi(ct - r)]$ satisfies the partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] \text{ where } c = \text{constant}.$$

18. If $u + iv = f(x + iy)$, prove that $u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$.

19. If $z(x + y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

20. If $u = x^y$ show that $\frac{\partial^3 u}{\partial x^2\, \partial y} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z}$

21. If $\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1$, show that $(u_x)^2 + (u_y)^2 + (u_z)^2 = 2\,(x u_x + y u_y + z u_z)$.

5.4 Type 2 : Solved Examples on Verification of $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$

Example 5.4.1 **May 2010**

If $u = \log(x^2 + y^2)$, verify $u_{xy} = u_{yx}$

Solution :

Part 1 : To find u_{xy} i.e. $\frac{\partial^2 u}{\partial x\, \partial y}$

$$u = \log(x^2 + y^2)$$

Differentiating w.r.t. y partially,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \log(x^2 + y^2)$$

but $\frac{d}{dx} \log x = \frac{1}{x}$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}\frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}(0 + 2y)$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

Now, differentiating w.r.t. x, partially,

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left[\frac{2y}{x^2 + y^2}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\, \partial y} = 2y \frac{\partial}{\partial x}\frac{1}{x^2 + y^2}$$

but $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$

$$= 2y\left[\frac{-1}{(x^2 + y^2)}\right]\frac{\partial}{\partial x}(x^2 + y^2)$$

$$= 2y\left[\frac{-1}{(x^2 + y^2)^2}\right](2x + 0)$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{-4xy}{(x^2 + y^2)^2} \quad \ldots(1)$$

Part 2 : To find $\frac{\partial^2 u}{\partial x\, \partial y} = u_{yx}$

Given : $u = \log(x^2 + y^2)$

Differentiating w.r.t. x partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x+0)$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$$

Now, differentiating w.r.t. y, partially,

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial y\,\partial x} = 2x\frac{\partial}{\partial y}\left[\frac{1}{x^2+y^2}\right]$$

$$= 2x\left[\frac{-1}{(x^2+y^2)^2}\right]\frac{\partial}{\partial y}(x^2+y^2)$$

$$= 2x\left[\frac{-1}{(x^2+y^2)^2}\right](0+2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{-4xy}{(x^2+y^2)^2} \qquad \ldots(2)$$

From Equation (1) and Equation (2),

$$\therefore \quad u_{xy} = u_{yx}$$

Exercise 5.2

1. If $u = x^y$ then verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Ans.: $x^{y-1} + y\,x^{y-1}\log x$

2. If $z = x^y + y^x$, then verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Ans.: $x^{y-1} + yx^{y-1}\log x + xy^{x-1}\log y + y^{x-1}$

3. If $u = \tan(y+ax) - (y-ax)^{3/2}$ then verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Ans.: $2a\sec^2(y+ax)\tan(y+ax) + \frac{a^3}{4}(y-ax)^{-1/2}$

4. If $u = 3xy - y^3 + (y^2-2x)^{3/2}$ then verify $u_{xy} = u_{yx}$

Ans.: $3 - 3y(y^2-2x)^{-1/2}$

5. If $u = \log\left(\frac{x^2+y^2}{xy}\right)$ then verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Ans.: $\frac{-4xy}{(x^2+y^2)^2} - \frac{1}{x}$

5.5 Derivatives of Composite Functions

1. If $u = f(r)$ and $r = f(x, y)$

then $\frac{\partial u}{\partial x} = \frac{du}{dr}\times\frac{\partial r}{\partial x}$

and $\frac{\partial u}{\partial y} = \frac{du}{dr}\times\frac{\partial r}{\partial y}$

2. Similarly If $u = f(r)$ and $r = f(x, y, z)$

then, $\frac{\partial u}{\partial x} = \frac{du}{dr}\times\frac{\partial r}{\partial x}$

$\frac{\partial u}{\partial y} = \frac{du}{dr}\times\frac{\partial r}{\partial y}$

$\frac{\partial u}{\partial z} = \frac{du}{dr}\times\frac{\partial r}{\partial z}$

Note : 1. If $r^2 = x^2+y^2$

then $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$

2. If $r^2 = x^2+y^2+z^2$

then $\frac{\partial r}{\partial x} = \frac{x}{r};\ \frac{\partial r}{\partial y} = \frac{y}{r};\ \frac{\partial r}{\partial z} = \frac{z}{r}$

5.5.1 Type 3 : Solved Examples on Partial Derivatives of Composite Functions

Example 5.5.1 May 2018

If $u = f(r)$ where $r = \sqrt{x^2+y^2}$ then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

OR $\quad u_{xx} + u_{yy} = \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr}$

Solution :

Step 1 :

Here, $r = \sqrt{x^2+y^2}$

Differentiating w.r.t. x, partially,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x+0)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \because r = \sqrt{x^2+y^2}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

Step 2 : To find $\frac{\partial^2 u}{\partial x^2}$; $\frac{\partial^2 u}{\partial y^2}$

Given : $u = f(r)$

Differentiating w.r.t. x, partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = f'(r) \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \frac{x}{r}$$

Note : Now before differentiating 2^{nd} time, we will take $\frac{1}{r}$ to numerator as r^{-1}, so as to avoid $\frac{u}{v}$ rule of derivative and use uv rule, which is more simpler.

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r) \times r^{-1}$$

Differentiating w.r.t. x, partially, again,

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} [f'(r) \times r^{-1}]$$

By $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f'(r)(x)\frac{\partial}{\partial x}(r^{-1}) + f'(r)\, r^{-1}\frac{\partial}{\partial x}(x) + x\, r^{-1}\frac{\partial}{\partial x}[f'(r)]$$

$$= f'(r)\, x\,(-r^{-2})\frac{\partial r}{\partial x} + f'(r)\, r^{-1}(1) + x\, r^{-1} f''(r)\frac{\partial r}{\partial x}$$

$$= -\frac{f'(r)\, x}{r^2}\cdot\left(\frac{x}{r}\right) + \frac{f'(r)}{r} + \frac{x\, f''(r)}{r}\left(\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{f'(r)\, x^2}{r^3} + \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2} \quad \text{...(1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{f'(r)\, y^2}{r^3} + \frac{f'(r)}{r} + \frac{y^2 f''(r)}{r^2} \quad \text{...(2)}$$

Step 3 : L.H.S. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\therefore$ From Equation (1) and (2)

$$= \left[-\frac{f'(r)\, x^2}{r^3} + \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2}\right] + \left[-\frac{f'(r)\, y^2}{r^3} + \frac{f'(r)}{r} + \frac{y^2 f''(r)}{r^2}\right]$$

$$= -\frac{f'(r)}{r^3}(x^2+y^2) + \frac{2f'(r)}{r} + \frac{f''(r)}{r^2}(x^2+y^2)$$

but $x^2 + y^2 = r^2$

$$= -\frac{f'(r)}{r^3} r^2 + \frac{2f'(r)}{r} + \frac{f''(r)}{r^2}(r^2)$$

$$= -\frac{f'(r)}{r} + \frac{2}{r} f'(r) + f''(r)$$

$$= f''(r) + \frac{1}{r} f'(r)$$

$$= \text{R.H.S.}$$

...Hence proved.

Example 5.5.2

If $u = f(r)$, where $r = \sqrt{x^2+y^2+z^2}$, then prove that, $u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$.

Solution :

Step 1 :

Here, $r = \sqrt{x^2+y^2+z^2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Step 2 : To find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$

Given : $u = f(r)$

Differentiating w.r.t. x, partially,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r)$$

$$= f'(r) \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = f'(r) \frac{x}{r}$$

Note : Now before differentiating 2nd time, we will take $\frac{1}{r}$ to numerator as r^{-1}, so as to avoid $\frac{u}{v}$ rule of derivative and use uv rule, which is more simpler.

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r) \times r^{-1}$$

Differentiating w.r.t. x, partially, again,

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} [f'(r) \times r^{-1}]$$

By $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f'(r)(x)\frac{\partial}{\partial x}(r^{-1}) + f'(r)\, r^{-1}\frac{\partial}{\partial x}(x)$$

$$+ x\, r^{-1}\frac{\partial}{\partial x}[f'(r)]$$

$$= f'(r)(x)(-r^{-2})\frac{\partial r}{\partial x} + f'(r)\, r^{-1}(1)$$

$$+ x\, r^{-1} f''(r)\frac{\partial r}{\partial x}$$

$$= -\frac{f'(r)\, x}{r^2}\cdot\left(\frac{x}{r}\right) + \frac{f'(r)}{r} + \frac{x\, f''(r)}{r}\left(\frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{f'(r)}{r^3}x^2 + \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2} \quad \ldots(1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{f'(r)}{r^3}y^2 + \frac{f'(r)}{r} + \frac{y^2 f''(r)}{r^2} \quad \ldots(2)$$

and

$$\frac{\partial^2 u}{\partial z^2} = -\frac{f'(r)}{r^3}z^2 + \frac{f'(r)}{r} + \frac{z^2 f''(r)}{r^2} \quad \ldots(3)$$

Step 3 : Equation (1) + Equation (2) + Equation (3)

$$\text{LHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \left[-\frac{f'(r)}{r^3}x^2 + \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2}\right]$$

$$+ \left[-\frac{f'(r)}{r^3}y^2 + \frac{f'(r)}{r} + \frac{y^2 f''(r)}{r^2}\right]$$

$$+ \left[-\frac{f'(r)}{r^3}z^2 + \frac{f'(r)}{r} + \frac{z^2 f''(r)}{r^2}\right]$$

$$= -\frac{f'(r)}{r^3}(x^2+y^2+z^2) + \frac{3f'(r)}{r}$$

$$+ \frac{f''(r)}{r^2}(x^2+y^2+z^2)$$

But $(x^2+y^2+z^2) = r^2$ (Given)

$$= -\frac{f'(r)}{r^3}(r^2) + \frac{3f'(r)}{r} + \frac{f''(r)}{r^2}(r^2)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

$$= \text{R.H.S.}$$

...Hence Proved

Example 5.5.3

If $u^2(x^2+y^2+z^2) = 1$ then prove that, $u_{xx} + u_{yy} + u_{zz} = 0$.

Solution :

Step 1 :

Let, $(x^2+y^2+z^2) = r^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} =$$

Step 2 :

$$u^2(x^2+y^2+z^2) = 1$$

$$u^2 r^2 = 1$$

$$u^2 = \frac{1}{r^2} = r^{-2}$$

$$u = (r^{-2})^{1/2}$$

$$u = r^{-1}$$

Differentiating w.r.t. x partially

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(r^{-1})$$

$$\frac{\partial u}{\partial x} = -r^{-2}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = -r^{-2}\frac{x}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = -r^{-3}x$$

Now, differentiating w.r.t. x partially again,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}[r^{-3}x]$$

$$= -\left[r^{-3}\frac{\partial}{\partial x}(x) + x\frac{\partial}{\partial x}(r^{-3})\right]$$

$$= -\left[r^{-3}(1) + x(-3r^{-4})\frac{\partial r}{\partial x}\right]$$

$$= -\left[\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^3} + \frac{3x^2}{r^5} \quad \ldots(1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^3} + \frac{3y^2}{r^5} \quad (2)$$

and

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{r^3} + \frac{3z^2}{r^5} \quad \ldots(3)$$

Step 3 : Equation (1) + Equation (2) + Equation (3)

$$\text{LHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \left[-\frac{1}{r^3} + \frac{3x^2}{r^5}\right] + \left[-\frac{1}{r^3} + \frac{3y^2}{r^5}\right] + \left[-\frac{1}{r^3} + \frac{3z^2}{r^5}\right]$$

$$= -\frac{1}{r^3} + \frac{3x^2}{r^5} - \frac{1}{r^3} + \frac{3y^2}{r^5} - \frac{1}{r^3} + \frac{3z^2}{r^5}$$

$$= -\frac{3}{r^3} + \frac{3}{r^5}(x^2 + y^2 + z^2)$$

But $(x^2 + y^2 + z^2) = r^2$

$$= -\frac{3}{r^3} + \frac{3}{r^5}(r^2)$$

$$= -\frac{3}{r^3} + \frac{3}{r^3}$$

$= 0 = \text{R.H.S.}$ **...Hence proved.**

Example 5.5.4 **Dec. 2012**

If $u = \log\sqrt{x^2 + y^2 + z^2}$ then prove that, $(x^2 + y^2 + z^2)(u_{xx} + u_{yy} + u_{zz}) = 1$.

Solution :

Step 1 :

Let, $x^2 + y^2 + z^2 = r^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Step 2 :

Given : $u = \log\sqrt{x^2 + y^2 + z^2}$

$u = \log\sqrt{r^2}$

$u = \log r$

Differentiating w.r.t. x partially

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log r$$

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{1}{r}\frac{x}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{r^2} = x\,r^{-2}$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}[x\,r^{-2}]$$

$$= x\frac{\partial}{\partial x}(r^{-2}) + r^{-2}\frac{\partial}{\partial x}(x)$$

$$= x(-2r^{-3})\frac{\partial r}{\partial x} + r^{-2}(1)$$

$$= \frac{-2x}{r^3}\left(\frac{x}{r}\right) + \frac{1}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x^2}{r^4} + \frac{1}{r^2} \qquad ...(1)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{-2y^2}{r^4} + \frac{1}{r^2} \qquad ...(2)$$

and $$\frac{\partial^2 u}{\partial z^2} = \frac{-2z^2}{r^4} + \frac{1}{r^2} \qquad ...(3)$$

Step 3 :

$$\text{LHS} = (x^2 + y^2 + z^2)(u_{xx} + u_{yy} + u_{zz})$$

$$= (x^2 + y^2 + z^2)\left[\frac{-2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} + \frac{1}{r^2} - \frac{2z^2}{r^4} + \frac{1}{r^2}\right]$$

$$= (x^2 + y^2 + z^2)\left[-\frac{2}{r^4}(x^2 + y^2 + z^2) + \frac{3}{r^2}\right]$$

But $x^2 + y^2 + z^2 = r^2$

$$= r^2\left[-\frac{2}{r^4}(r^2) + \frac{3}{r^2}\right] = r^2\left[-\frac{2}{r^2} + \frac{3}{r^2}\right] = r^2\left(\frac{1}{r^2}\right)$$

$= 1 = \text{R.H.S.}$ **...Hence proved.**

Example 5.5.5 **May 2013**

If $u = r^m$ where $r = \sqrt{x^2 + y^2 + z^2}$ then find the value of $u_{xx} + u_{yy} + u_{zz}$.

Solution :

Step 1 :

Here, $r = \sqrt{x^2 + y^2 + z^2}$

i.e. $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Step 2 :

Given : $u = r^m$

Differentiating w.r.t. x partially

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(r^m)$$

$$\frac{\partial u}{\partial x} = m\, r^{m-1} \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = m\, r^{m-1} \left(\frac{x}{r}\right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = m\, r^{m-2}\, x$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 u}{\partial x^2} = m \frac{\partial}{\partial x} [r^{m-2}\, x]$$

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} \frac{\partial}{\partial x}(x) + x \frac{\partial}{\partial x}(r^{m-2})\right]$$

$$\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2}(1) + x(m-2)\, r^{m-3} \frac{\partial r}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = m\, r^{m-2} + x\, m\,(m-2)\, r^{m-2} \cdot r^{-1} \frac{x}{r}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = r^{m-2}\left[m + \frac{x^2}{r^2}\, m\,(m-2)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = m\, r^{m-2}\left[1 + \frac{x^2}{r^2}(m-2)\right] \quad \text{...(1)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = m\, r^{m-2}\left[1 + \frac{y^2}{r^2}(m-2)\right] \quad \text{...(2)}$$

and

$$\frac{\partial^2 u}{\partial z^2} = m\, r^{m-2}\left[1 + \frac{z^2}{r^2}(m-2)\right] \quad \text{...(3)}$$

Step 3 : Equation (1) + Equation (2) + Equation (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= m\, r^{m-2}\left[1 + \frac{x^2}{r^2}(m-2)\right]$$

$$+ m\, r^{m-2}\left[1 + \frac{y^2}{r^2}(m-2)\right]$$

$$+ m\, r^{m-2}\left[1 + \frac{z^2}{r^2}(m-2)\right]$$

$$= m\, r^{m-2}\left\{3 + (m-2)\frac{(x^2 + y^2 + z^2)}{r^2}\right\}$$

But $x^2 + y^2 + z^2 = r^2$

$$= m\, r^{m-2}\, \{3 + m - 2\}$$

$$\therefore \quad \mathbf{u_{xx} + y_{yy} + u_{zz} = m\,(m+1)\, r^{m-2}}$$

Example 5.5.6

If $u = x \cdot \log(x + r) - r$ where $r^2 = x^2 + y^2$. Find $u_{xx} + u_{yy}$.

Solution :

Step 1 :

Here, $r^2 = x^2 + y^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

Step 2 : To find u_{xx}

Given : $u = x \log(x + r) - r$

Differentiating w.r.t. x partially

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x \log(x + r)] - \frac{\partial}{\partial x}(r)$$

$$= x \cdot \frac{\partial}{\partial x} \log(x + r)$$

$$+ \log(x + r) \frac{\partial}{\partial x}(x) - \frac{\partial r}{\partial x}$$

$$= x \cdot \frac{1}{x + r}\left(1 + \frac{\partial r}{\partial x}\right) + \log(x + r) - \frac{\partial r}{\partial x}$$

$$= \frac{x}{x + r}\left(1 + \frac{x}{r}\right) + \log(x + r) - \frac{x}{r}$$

$$= \frac{x}{x + r}\left(\frac{r + x}{r}\right) + \log(x + r) - \frac{x}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{r} + \log(x + r) - \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = \log(x + r)$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \log(x + r)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x + r}\left(1 + \frac{\partial r}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x + r}\left(1 + \frac{x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x + r}\left(\frac{r + x}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}$$

Step 3 : To find u_{yy}

Given : $u = x \log(x + r) - r$

Differentiating w.r.t. y partially,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x\log(x+r)] - \frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = x\cdot\frac{1}{x+r}\left(0+\frac{\partial r}{\partial y}\right) - \frac{y}{r}$$

$$= \frac{x}{x+r}\left(\frac{y}{r}\right) - \frac{y}{r}$$

$$= \frac{y}{r}\left[\frac{x}{x+r} - 1\right]$$

$$= \frac{y}{r}\left[\frac{x-x-r}{x+r}\right]$$

$$\frac{\partial u}{\partial y} = \frac{-y}{x+r}$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\left[\frac{y}{x+r}\right]$$

$$= -\left\{\frac{(x+r)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x+r)}{(x+r)^2}\right\}$$

$$= -\left\{\frac{(x+r)(1) - y\left(0+\frac{\partial r}{\partial y}\right)}{(x+r)^2}\right\}$$

$$= -\left\{\frac{(x+r) - y\left(\frac{y}{r}\right)}{(x+r)^2}\right\}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-x-r+\frac{y^2}{r}}{(x+r)^2} = \frac{\frac{-xr-r^2+y^2}{r}}{(x+r)^2}$$

but $\quad r^2 = x^2+y^2 \quad = \dfrac{-xr-x^2-y^2+y^2}{r(x+r)^2}$

$$= \frac{-x(r+x)}{r(x+r)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-x}{r(x+r)} \qquad \ldots(2)$$

Add: Equation (1) + Equation (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r} + \frac{-x}{r(x+r)} = \frac{1}{r}\left[1-\frac{x}{x+r}\right]$$

$$= \frac{1}{r}\left[\frac{x+r-x}{x+r}\right]$$

$$= \frac{1}{r}\left[\frac{r}{x+r}\right]$$

$$u_{xx} + u_{yy} = \frac{1}{x+r}$$

Exercise 5.3

1. If $v = e^{\frac{r-x}{l}}$ where $r^2 = x^2 + y^2$, l is constant, show that, $v_{xx} + v_{yy} = \dfrac{v}{lr} - \dfrac{2\,v_x}{l}$

2. If $\theta = e^{r-x}$ and $r = \sqrt{x^2+y^2}$ then show that $\dfrac{\partial^2\theta}{\partial y^2} = \left(\dfrac{x^2}{r^3} + \dfrac{y^2}{r^2}\right) e^{r-x}$

3. If $u = \log r$ and $r = \sqrt{(x-a)^2+(y-b)^2}$ then prove that $u_{xx} + u_{yy} = 0$, where a and b are constants.

4. If $u = \log r$ where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ then prove that, $u_{xx} + u_{yy} + u_{zz} = \dfrac{1}{r^2}$

5.6 Concept of Variables to be Treated as Constant :

If x = u tan v and y = u sec v then we have 4 variables x, y, u, v.

But here in variables to be treated as constant, we have to deal with only 3 variables at a time.

e.g. $\left(\dfrac{\partial u}{\partial y}\right)_x$ **which can be read as partial derivative of u w.r.t. y keeping x constant.**

For $\left(\dfrac{\partial u}{\partial y}\right)_x$ we require only 3 variables, u, x, y that means v needs to be eliminated from the given two equations with algebric calculations, which can be well understood with the following examples.

5.6.1 Type 4 : Solved Examples on Variables to be Treated as Constant

Example 5.6.1 **May 2010, 11, 18, Dec. 2009, 16, 15**

If $u = 2x + 3y$

$v = 3x - 2y$

then find the value of

$$\left(\frac{\partial u}{\partial x}\right)_y, \left(\frac{\partial y}{\partial v}\right)_x, \left(\frac{\partial x}{\partial u}\right)_v, \left(\frac{\partial v}{\partial y}\right)_u$$

Solution :

Given : $u = 2x + 3y$ ...(1)

$v = 3x - 2y$ ...(2)

Part 1 : To find $\left(\frac{\partial u}{\partial x}\right)_y$

From Equation (1),

$$u = 2x + 3y$$

Differentiating w.r.t. x keeping y constant,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial (2x)}{\partial x} + \frac{\partial (3y)}{\partial x}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = 2 + 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = 2$$

Part 2 : To find $\left(\frac{\partial y}{\partial v}\right)_x$

From Equation (2),

$$v = 3x - 2y$$

$$2y = 3x - v$$

$$y = \frac{3x - v}{2}$$

Differentiating w.r.t. v keeping x constant,

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{\partial}{\partial v}\left(\frac{3x - v}{2}\right) = \frac{1}{2}\left[\frac{\partial}{\partial v}(3x) - \frac{\partial}{\partial v} \cdot v\right]$$

$$= \frac{1}{2}[0 - 1]$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{2}$$

Part 3 : To find $\left(\frac{\partial x}{\partial u}\right)_v$

From Equation (1)

$$u = 2x + 3y$$

$$u - 2x = 3y$$

$$\frac{u - 2x}{3} = y$$

Put in Equation (2)

$$v = 3x - 2y = 3x - 2\left(\frac{u - 2x}{3}\right)$$

$$v = \frac{9x - 2u + 4x}{3}$$

$$3v = 13x - 2u$$

$$3v + 2u = 13x$$

$$\therefore x = \frac{3v + 2u}{13}$$

Differentiating w.r.t. u keeping v constant,

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial}{\partial u}\left(\frac{3v + 2u}{13}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{13}\left[\frac{\partial}{\partial u}(3v) + \frac{\partial}{\partial u}(2u)\right] = \frac{1}{13}(0 +$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13}$$

Part 4 : To find $\left(\frac{\partial v}{\partial y}\right)_u$

From Equation (1)

$$u = 2x + 3y$$

$$2x = u - 3y$$

$$x = \frac{u - 3y}{2}$$

Put in Equation (2)

$$v = 3x - 2y = 3\left(\frac{u - 3y}{2}\right) - 2y$$

$$v = \frac{3u - 9y - 4y}{2}$$

$$2v = 3u - 13y$$

$$v = \frac{3u - 13y}{2}$$

Differentiating w.r.t. y keeping u constant,

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{\partial}{\partial y}\left(\frac{3u - 13y}{2}\right) = \frac{1}{2}\frac{\partial}{\partial y}(3u - 13$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{1}{2}\left[\frac{\partial}{\partial y}(3u) - \frac{\partial}{\partial y}(13y)\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{1}{2}[0 - 13]$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{-13}{2}$$

Now, given differential expression is,

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u$$

$$= (2)\left(-\frac{1}{2}\right)\left(\frac{2}{13}\right)\left(-\frac{13}{2}\right) = 1$$

Example 5.6.2

If $u = ax + by$

$v = bx - ay$

then find the value of

$\left(\frac{\partial u}{\partial x}\right)_y, \left(\frac{\partial y}{\partial v}\right)_x, \left(\frac{\partial x}{\partial u}\right)_v, \left(\frac{\partial v}{\partial y}\right)_u$

Solution :

Given : $u = ax + by$ ...(1)

$v = bx - ay$ ...(2)

Part 1 : To find $\left(\frac{\partial u}{\partial x}\right)_y$

From Equation (1),

$$u = ax + by$$

Differentiating w.r.t. x keeping y constant,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}(ax + by)$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial x}(by)$$

$$\left(\frac{\partial u}{\partial x}\right)_y = a + 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = a$$

Part 2 : To find $\left(\frac{\partial y}{\partial v}\right)_x$

From Equation (2),

$$v = bx - ay$$

$$ay = bx - v$$

$$y = \frac{bx - v}{a}$$

Differentiating w.r.t. v keeping x constant,

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{\partial}{\partial v}\left(\frac{bx - v}{a}\right)$$

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{1}{a}\left[\frac{\partial}{\partial v}(bx) - \frac{\partial}{\partial v}\cdot v\right]$$

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{1}{a}[0 - 1]$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{a}$$

Part 3 : To find $\left(\frac{\partial x}{\partial u}\right)_v$

From Equation (1)

$$u = ax + by$$

$$by = u - ax$$

$$y = \frac{u - ax}{b}$$

Put in Equation (2)

$$v = bx - ay$$

$$v = bx - a\left(\frac{u - ax}{b}\right)$$

$$v = \frac{b^2 x - au - a^2 x}{b}$$

$$bv = b^2 x - au - a^2 x$$

$$bv = x(b^2 - a^2) - au$$

$$bv + au = x(b^2 - a^2)$$

$$\therefore x = \frac{bv + au}{b^2 - a^2}$$

Differnetiating w.r.t. u keeping v constant,

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial}{\partial u}\left(\frac{bv + au}{b^2 - a^2}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{b^2 - a^2}\left[\frac{\partial}{\partial u}(bv) - \frac{\partial}{\partial u}(au)\right]$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{b^2 - a^2}(0 + a)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{+a}{b^2 - a^2}$$

Part 4 : To find $\left(\frac{\partial v}{\partial y}\right)_u$

From Equation (1),

$$u = ax + by$$

$$ax = u - by$$

$$x = \frac{u - by}{a}$$

Put in Equation (2)

$$v = bx - ay$$

$$v = b\left(\frac{u - by}{a}\right) - ay$$

$$v = \frac{bu - b^2 y - a^2 y}{a}$$

$$ax = bu - b^2 y - a^2 y$$

$$ax = bu - y(a^2 + b^2)$$

$$x = \frac{bu - y(a^2 + b^2)}{a}$$

Differentiating w.r.t. y keeping u constant,

$$\left(\frac{\partial x}{\partial y}\right)_u = \frac{\partial}{\partial y}\left(\frac{bu - y(a^2+b^2)}{a}\right)$$

$$\left(\frac{\partial x}{\partial y}\right) = \frac{1}{a}\left[\frac{\partial}{\partial y}(bu) - \frac{\partial}{\partial y}y(a^2+b^2)\right]$$

$$\left(\frac{\partial x}{\partial y}\right)_u = \frac{1}{a}\{(0) - (a^2+b^2)\}$$

$$\left(\frac{\partial x}{\partial y}\right)_u = \frac{-(a^2+b^2)}{a}$$

Now, given differential expression is,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u$$

$$= (a)\left(\frac{-1}{a}\right)\left(\frac{a}{b^2+a^2}\right)\left(\frac{-a^2-b^2}{a}\right)$$

$$= 1$$

Example 5.6.3

If $ux + vy = 0$

$$\frac{u}{x} + \frac{v}{y} = 1$$

Then prove that, $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$

Solution :

Given : $ux + vy = 0$ ...(1)

$$\frac{u}{x} + \frac{v}{y} = 1 \quad ...(2)$$

Part 1 : To find $\left(\frac{\partial x}{\partial u}\right)_v$

From Equation (1)

$$ux + vy = 0$$

$$vy = -ux$$

$$y = -\frac{ux}{v}$$

Put in Equation (2)

$$\frac{u}{x} + \frac{v}{y} = 1$$

$$\frac{u}{x} + \frac{v}{\frac{-ux}{v}} = 1$$

$$\frac{u}{x} - \frac{v^2}{ux} = 1$$

$$\frac{1}{x}\left(u - \frac{v^2}{u}\right) = 1$$

$$x = u - \frac{v^2}{u}$$

Differentiating w.r.t. v keeping u constant,

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial}{\partial u}\left(u - \frac{v^2}{u}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial}{\partial u}(u) - \frac{\partial}{\partial u}\left(\frac{v^2}{u}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = 1 - v^2\left(\frac{-1}{u^2}\right)$$

$$\left(\frac{\partial x}{\partial u}\right)_v = 1 + \frac{v^2}{u^2}$$

Part 2 : To find $\left(\frac{\partial y}{\partial v}\right)_u$

From Equation (1),

$$ux + vy = 0$$

$$ux = -vy$$

$$x = -\frac{vy}{u}$$

Put in Equation (2),

$$\frac{u}{x} + \frac{v}{y} = 1$$

$$\frac{u}{\frac{-vy}{u}} + \frac{v}{u} = 1$$

$$\frac{u^2}{-vy} + \frac{v}{y} = 1$$

$$\frac{1}{y}\left(\frac{u^2}{-v} + v\right) = 1$$

$$y = -\frac{u^2}{v} + v$$

$$y = v - \frac{u^2}{v}$$

Differentiating w.r.t. v keeping u constant,

$$\left(\frac{\partial y}{\partial v}\right)_u = \frac{\partial}{\partial v}\left[v - \frac{u^2}{v}\right]$$

$$\left(\frac{\partial y}{\partial v}\right)_u = \frac{\partial}{\partial v}(v) - \frac{\partial}{\partial v}\left(\frac{u^2}{v}\right)$$

$$\left(\frac{\partial y}{\partial v}\right)_u = 1 - u^2\left(-\frac{1}{v^2}\right)$$

$$\left(\frac{\partial y}{\partial v}\right)_u = 1 + \frac{u^2}{v^2}$$

$$\text{L.H.S.} = \frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u$$

$$= \frac{u}{x}\left(1 + \frac{v^2}{u^2}\right) + \frac{v}{y}\left(\frac{u^2}{v^2} + 1\right)$$

$$= \frac{u}{x}\left(\frac{u^2+v^2}{u^2}\right) + \frac{v}{y}\left(\frac{u^2+v^2}{v^2}\right)$$

$$= \frac{u^2+v^2}{ux} + \frac{u^2+v^2}{vy}$$

But, $ux + vy = 0$ ...[From Equation (1)]

$$vy = -ux$$

$$= \frac{u^2+v^2}{ux} + \frac{u^2+v^2}{-ux}$$

$$= \frac{u^2+v^2}{ux} - \frac{u^2+v^2}{ux}$$

$= 0 = \text{R.H.S.}$ ...**Hence proved**

Example 5.6.4 **May 2016, Dec. 2014, 2010**

If $ux + vy = 0$

$$\frac{u}{x} + \frac{v}{y} = 1$$

Then prove that,

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}$$

Solution :

Given : $ux + vy = 0$ ...(1)

$$\frac{u}{x} + \frac{v}{y} = 1 \quad ...(2)$$

Part 1 : To find $\left(\frac{\partial u}{\partial x}\right)_y$

From Equation (1),

$$ux + vy = 0$$

$$vy = -ux$$

$$v = -\frac{ux}{y}$$

Put in Equation (2)

$$\frac{u}{x} + \frac{v}{y} = 1$$

$$\frac{u}{x} + \frac{\frac{-ux}{y}}{y} = 1$$

$$\frac{u}{x} - \frac{ux}{y^2} = 1$$

$$u\left(\frac{y^2-x^2}{xy^2}\right) = 1$$

$$u = \frac{1}{\frac{y^2-x^2}{xy^2}}$$

$$u = \frac{xy^2}{y^2-x^2}$$

Differentiating w.r.t. x keeping y constant,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}\left(\frac{xy^2}{y^2-x^2}\right)$$

$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\frac{\partial}{\partial x}\left(\frac{x}{y^2-x^2}\right)$$

$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\left\{\frac{(y^2-x^2)(1)-(x)(0-2x)}{(y^2-x^2)^2}\right\}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = y^2\left\{\frac{y^2-x^2+2x^2}{(y^2-x^2)^2}\right\}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{y^2(y^2+x^2)}{(y^2-x^2)^2}$$

Part 2 : To find $\left(\frac{\partial v}{\partial y}\right)_x$

From Equation (1),

$$ux + vy = 0$$

$$ux = -vy$$

$$u = \frac{-vy}{x}$$

Put in Equation (2)

$$\frac{u}{x} + \frac{v}{y} = 1$$

$$\frac{\frac{-vy}{x}}{x} + \frac{v}{y} = 1$$

$$\frac{-vy}{x^2} + \frac{v}{y} = 1$$

$$v\left(-\frac{1}{x^2}+\frac{1}{y^2}\right) = \frac{1}{y}$$

$$v\left(\frac{-y^2+x^2}{x^2y^2}\right) = \frac{1}{y}$$

$$v = \frac{x^2 y}{x^2-y^2}$$

Differentiating w.r.t. y keeping x constant,

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{\partial}{\partial y}\left(\frac{x^2 y}{x^2-y^2}\right)$$

$$\left(\frac{\partial v}{\partial y}\right)_x = x^2\frac{\partial}{\partial y}\left(\frac{y}{x^2-y^2}\right)$$

$$\left(\frac{\partial v}{\partial y}\right)_x = x^2\left[\frac{(x^2-y^2)(1)-(y)(0-2y)}{(x^2-y^2)^2}\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_x = x^2\left[\frac{(x^2-y^2)+2y^2}{(x^2-y^2)^2}\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2(x^2+y^2)}{(x^2-y^2)}$$

$\therefore$ Now, L.H.S. $= \dfrac{y^2(y^2+x^2)}{(y^2-x^2)^2} - \dfrac{x^2(x^2+y^2)}{(x^2-y^2)^2}$

$$= \frac{y^2(y^2+x^2)}{(y^2-x^2)^2} - \frac{x^2(y^2+x^2)}{(y^2-x^2)^2}$$

$$= \frac{(y^2+x^2)(y^2-x^2)}{(y^2-x^2)^2} = \frac{y^2+x^2}{y^2-x^2}$$

= R.H.S. ...**Hence Proved**

Example 5.6.5 **Dec. 2013, May 2012**

If $x = u\tan v$; $y = u\sec v$, then prove that,

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial v}{\partial y}\right)_x$$

Solution :

Given : $x = u\tan v$ ...(1)

$y = u\sec v$ ...(2)

$\therefore$ Equation (1) ÷ Equation (2)

$$\frac{x}{y} = \frac{u\tan v}{u\sec v}$$

$$\frac{x}{y} = \frac{\sin v}{\cos v}\times\cos v$$

$$\frac{x}{y} = \sin v \qquad ...(3)$$

By, $1+\tan^2 v = \sec^2 v$

$$1+\left(\frac{x}{u}\right)^2 = \left(\frac{y}{u}\right)^2$$

$$1+\frac{x^2}{u^2} = \frac{y^2}{u^2}$$

Multiply by u^2 $\quad u^2+x^2 = y^2$

Part 1 : To find $\left(\frac{\partial u}{\partial x}\right)_y$

From Equation (4),

$$u^2+x^2 = y^2$$

$$u^2 = y^2-x^2$$

Differentiating w.r.t. x keeping y constant,

$$\frac{\partial}{\partial x}(u^2) = \frac{\partial}{\partial x}(y^2)-\frac{\partial}{\partial x}(x^2)$$

$$2u\frac{\partial y}{\partial x} = -2x$$

$$\left(\frac{\partial u}{\partial x}\right)_y = -\frac{x}{u}$$

Part 2 : To find $\left(\frac{\partial v}{\partial x}\right)_y$

From Equation (3),

$$\sin v = \frac{x}{y}$$

Differentiating w.r.t. x keeping y constant,

$$\frac{\partial}{\partial x}(\sin v) = \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\cos v\frac{\partial v}{\partial x} = \frac{1}{y}$$

$$\left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{y\cos v}$$

Part 3 : To find $\left(\frac{\partial u}{\partial y}\right)_x$

From Equation (4),

$$u^2+x^2 = y^2$$

$$u^2 = y^2-x^2$$

Differentiating w.r.t. y keeping x constant,

$$\frac{\partial}{\partial y}(u^2) = \frac{\partial}{\partial y}(y^2)-\frac{\partial}{\partial y}(x^2)$$

$$2u\frac{\partial u}{\partial y} = 2y-0$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u}$$

Part 4 : To find $\left(\frac{\partial v}{\partial y}\right)_x$

From Equation (3),

$$\frac{x}{y} = \sin v$$

$$\sin v = \frac{x}{y}$$

Differentiating w.r.t. y keeping x constant,

$$\frac{\partial}{\partial y}(\sin v) = x\frac{\partial}{\partial y}\left(\frac{1}{y}\right)$$

$$(\cos v)\frac{\partial v}{\partial y} = x\left(-\frac{1}{y^2}\right)$$

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2 \cos v}$$

Now,

$$\text{LHS} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = -\frac{x}{u}\left(\frac{1}{y(\cos v)}\right)$$

$$\text{LHS} = \frac{-x}{uy(\cos v)}$$

$$\text{RHS} = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \left(\frac{y}{u}\right)\left(\frac{-x}{y^2 \cos v}\right)$$

$$\text{RHS} = \frac{-x}{uy \cos v}$$

L.H.S. = R.H.S. ...**Hence Proved**

Example 5.6.6 **May 2017**

If $x^2 = a\sqrt{u} + b\sqrt{v}$

$y^2 = a\sqrt{u} - b\sqrt{v}$

then prove that, $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$

Solution :

Given : $x^2 = a\sqrt{u} + b\sqrt{v}$ ...(1)

$y^2 = a\sqrt{u} - b\sqrt{v}$ ...(2)

Part 1 : To find $\left(\frac{\partial u}{\partial x}\right)_y$

From Equation (1),

$$x^2 = a\sqrt{u} + b\sqrt{v}$$

$$b\sqrt{v} = x^2 - a\sqrt{u}$$

$$\sqrt{v} = \frac{x^2 - a\sqrt{u}}{b}$$

Put in Equation (2)

$$y^2 = a\sqrt{u} - b\sqrt{v}$$

$$y^2 = a\sqrt{u} - b\left(\frac{x^2 - a\sqrt{u}}{b}\right)$$

$$y^2 = a\sqrt{u} - x^2 + a\sqrt{u}$$

$$x^2 + y^2 = 2a\sqrt{u}$$

$$\sqrt{u} = \frac{x^2 + y^2}{2a}$$

Differentiating w.r.t. x keeping y constant,

$$\frac{\partial}{\partial x}\sqrt{u} = \frac{\partial}{\partial x}\left(\frac{x^2 + y^2}{2a}\right)$$

$$\frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x} = \frac{1}{2a}\left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2)\right]$$

$$\frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x} = \frac{1}{2a}(2x) = \frac{x}{a}$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{2\sqrt{u}x}{a}$$

Part 2 : To find $\left(\frac{\partial x}{\partial u}\right)_v$

From Equation (1),

$$x^2 = a\sqrt{u} + b\sqrt{v}$$

Differentiating w.r.t. u keeping v constant,

$$\frac{\partial}{\partial u}x^2 = \frac{\partial}{\partial u}(a\sqrt{u} + b\sqrt{v})$$

$$2x\frac{\partial x}{\partial u} = \frac{a}{2\sqrt{u}}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{4\sqrt{u}x}$$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{2\sqrt{u}x}{a} \times \frac{a}{4\sqrt{u}x}$$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2}$$

Part 3 : To find $\left(\frac{\partial v}{\partial y}\right)_x$

From Equation (1),

$$x^2 = a\sqrt{u} + b\sqrt{v}$$

$$x^2 - b\sqrt{v} = a\sqrt{u}$$

$$\sqrt{u} = \frac{x^2 - b\sqrt{v}}{a}$$

Put in Equation (2),

$$y^2 = a\sqrt{u} - b\sqrt{v}$$