Engineering Mathematics-II

Unit - 1

First Order Ordinary Differential Equations

❖ Topics to be covered

- Order and degree of differential equations
- Solution of differential equations (General solution and particular solution)
- Formation of ordinary differential equations
- Differential equations in Variable Separable Form
- Homogeneous and Non-Homogeneous differential equations
- Exact differential equations
- Equations reducible to exact form by using Integrating Factor
- Linear Differential Equations of the first order
- Equations reducible to linear form (Bernoulli's Differential Equations)

❖ Applications of Differential Equations

1. Population growth and decay
2. Spread of epidemics
3. Newton’s law of cooling
4. Analysis of electrical networks
5. Glucose absorption by body
6. Exponential decay of radioactive material
7. In studying the blood flow through the various organs of the body

*Equation

An **equation** is a mathematical sentence that has two equal sides separated by an equal sign.

*Differential equation

An equation which contains a derivative is called differential equation.

Differential equation involves dependent variable,an independent variable and differential coefficients of various orders.

*Some examples of differential equations

1. $\frac{dy}{dx} + 5y = 0$

Here, y is the dependent variable, x is an independent variable and $\frac{dy}{dx}$ is a derivative.

2. $\frac{dy}{dx} + \frac{2}{x}y = x^2$

3. $\sqrt{2 + \frac{dy}{dx}} = \frac{d^2y}{dx^2}$

4. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

5. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Differential Equations

Ordinary Differential Equations

- It involves only one independent variable and one or more dependent variables
- Examples 1,2,3

Partial Differential Equations

- It involves two or more independent variable and one or more dependent variables
- Examples 4,5

* We have to study First Order Ordinary Differential Equations *

- Order of a Differential Equation

The order of a differential equation is the order of the highest derivative appears in the equation.

- Degree of a Differential Equation

The degree of a differential equation is the degree of the highest order derivative, provided the derivatives are free from radicals and fractions.

* Some examples to find Order and Degree of differential equation

1. $\frac{dy}{dx} + 1 = x^2$ Order = 1 , Degree = 1

2. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$ Order = 2 , Degree = 1

3. $\left(\frac{dy}{dx}\right)^3 + 3y = \left(\frac{d^2y}{dx^2}\right)^2$ Order = 2 , Degree = 2

4. $\sqrt{1 + \frac{dy}{dx}} = \frac{d^2y}{dx^2}$

We have to remove square root.Hence,squaring on both sides,we get

$$1 + \frac{dy}{dx} = \left(\frac{d^2y}{dx^2}\right)^2$$

Order = 2 , Degree = 2

5. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 5\,\frac{d^2y}{dx^2}$

We have to remove fraction $\frac{3}{2}$. Hence,squaring on both sides,we get

$$\left\{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\right\}^2 = 25\left(\frac{d^2y}{dx^2}\right)^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 25\left(\frac{d^2y}{dx^2}\right)^2$$

Now the above equation is free from fraction.

Order = 2 , Degree = 2

Note:The degree of a differential equation can be found only when it is free from radicals and fractions.

6. $(3x - 2y + 5)dx + (x - 3y + 2)dy = 0$

$$(x - 3y + 2)dy = -(3x - 2y + 5)dx$$

$$\frac{dy}{dx} = \frac{-(3x - 2y + 5)}{(x - 3y + 2)}$$

Order = 1 , Degree = 1

7. $\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 = \frac{\partial z}{\partial y}$

Order = 2 , Degree = 1

❖ Solution of a Differential Equation

Solution (also known as primitive) of a differential equation is any relation between the dependent and independent variables. Solution is always free from derivatives and it satisfies the given differential equation.

Solution

General solution / complete integral

- It contains arbitrary constants
- Number of arbitrary constants is equal to the order of differential equation

Particular solution/Particular integral

- It is obtained by assigning particular values to the arbitrary constants

Consider the differential equation of order 1

$$\frac{dy}{dx} = x^2 \longrightarrow \quad (1)$$

To find the solution of (1), integrate both sides with respect to x,

$\int \frac{dy}{dx} dx = \int x^2 + C$ where C is the only one arbitrary constant.

$$y = \frac{x^3}{3} + C \longrightarrow \quad (2)$$

This is general solution of (1) which contains one arbitrary constant.

No. of constants = Order of diff.equation

Now,substitute x=1 y=2 and in equation (2)

$$2 = \frac{1}{3} + C \quad => \quad C = 2 - \frac{1}{3} \quad => \quad C = \frac{5}{3}$$

$$y = \frac{x^3}{3} + \frac{5}{3}$$

This is the particular solution of (1)

❖ Formation of Ordinary Differential Equations

- General solution involving n arbitrary constants is given. We need to find Differential equation of which it is the solution.
- This differential equation is obtained by eliminating arbitrary constants
- Differentiate general solution n times with respect to the independent variable.

* Examples on formation of Differential Equation

Ex.1 Obtain the differential equation whose general solution is $y = \sqrt{5x + C}$,where C is arbitrary constant

Sol. Given , $y = \sqrt{5x + C} \longrightarrow (1)$

It consists of only one constant C.

To eliminate C , differentiate equation (1) on both sides with respect to x

$$\frac{dy}{dx} = \frac{5}{2\sqrt{5x+C}}$$

(Use formula: If $y = \sqrt{x}$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$)

$$\frac{dy}{dx} = \frac{5}{2y} \longrightarrow \text{From equation (1)}$$

$$2y\frac{dy}{dx} = 5$$

$$\boxed{2y\frac{dy}{dx} - 5 = 0}$$

This is the required differential equation having order =1 and degree=1

Ex.2 Obtain the differential equation whose general solution is $y = 4(x - A)^2$,where A is arbitrary constant

Sol.

Given , $y = 4(x - A)^2 \longrightarrow (1)$

Here , A is arbitrary constant which needs to be eliminated.

To eliminate A , differentiate equation (1) on both sides with respect to x

$$\frac{dy}{dx} = 4 \times 2(x - A) = 8(x - A)$$

Squaring both the sides,

$$\left(\frac{dy}{dx}\right)^2 = 64 \times (x - A)^2$$

$$(x - A)^2 = \frac{\left(\frac{dy}{dx}\right)^2}{64}$$

Substitute value of $(x - A)^2$in equation (1)

$$y = 4 \times \frac{\left(\frac{dy}{dx}\right)^2}{64} = \frac{\left(\frac{dy}{dx}\right)^2}{16}$$

$$\boxed{\left(\frac{dy}{dx}\right)^2 = 16y}$$

This is the required differential equation having order =1 and degree=2

Ex.3 Form the diff. equation of which a general solution is $y = A\cos(\log x) + B\sin(\log x)$

Sol.

Given, $y = A\cos(\log x) + B\sin(\log x) \longrightarrow (1)$

A and B are two arbitrary constants.We have to eliminate both the constants.

Differentiating equation (1) with respect to x on both sides, we get

$$\frac{dy}{dx} = -A\,\frac{1}{x}\,\sin(\log x) + B\,\frac{1}{x}\,\cos(\log x)$$

$$x\frac{dy}{dx} = -A\,\sin(\log x) + B\,\cos(\log x) \longrightarrow (2)$$

Differentiating equation (2) with respect to x we get,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -A\,\frac{1}{x}\,\cos(\log x) - B\,\frac{1}{x}\,\sin(\log x)$$

(Derivative of L.H.S. is obtained by using product rule of derivative)

Multiplying both sides by x,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -A \ \cos(\log x) - B \ \sin(\log x)$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \longrightarrow \text{From equation (1)}$$

$$\boxed{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0}$$

This is the required differential equation of order = 2 and degree = 1

Ex.4 Form the diff. equation of which a general solution is $y = c^2 + \frac{c}{x}$

Sol.

Given, $y = c^2 + \frac{c}{x} \longrightarrow$ (1)

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{c}{x^2} = -\frac{c}{x^2}$$

$$c = -x^2\frac{dy}{dx}$$

Substitute the value c in equation (1)

$$y = x^4\left(\frac{dy}{dx}\right)^2 - \frac{x^2\frac{dy}{dx}}{x}$$

$$\boxed{x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0}$$

This is the required differential equation of order = 1 and degree = 2

Solve:

Ex.1 Form the differential equation whose general solution is $xy = Ae^{x} + Be^{-x}$

Ex.2 Obtain the differential equation having general solution $y^2 = 4ax$

Ex.3 Obtain the differential equation having general solution $y = Ae^{-x^2}$

❖ Differential Equations in Variable Separable Form (V.S.form)

- The differential equation which can be written in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ is said to be present in the variable separable form.
- Variables x and y are separated so that x appears only on one side of the equation and y appears only on the other side.
- The solution is obtained by integrating both the sides of equation $\int g(y)\, dy = \int f(x) dx + C$, where C is constant of integration

Ex. 1 Solve: $\frac{dy}{dx} + x = 0$

Sol.

Given $\frac{dy}{dx} + x = 0$

$$\frac{dy}{dx} = -x$$

$$dy = -xdx$$

This is v.s. form. Integrating both the sides,we get

$$\int dy = \int -xdx + C'$$

$$y = \frac{-x^2}{2} + C'$$

$$2y = -x^2 + 2C'$$

$$2y = -x^2 + C \longrightarrow 2C' = C \text{ , constant}$$

$$\boxed{x^2 + 2y = C}$$

This is the required General Solution.

Ex. 2 Solve: $ydx + xdy = 0$

Sol.

Given , $ydx + xdy = 0$

Dividing both sides by xy we get

$$\frac{y}{xy}dx + \frac{x}{xy}dy = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy = 0$$

This is v.s. form. Integrating both the sides,we get

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy = \log C$$

$$\log x + \log y = \log C$$

$$\log xy = \log C$$

$$\boxed{xy = C}$$

This is the General Solution.

Note: An arbitrary constant may be written in such a form as to make the answer simple.

Ex. 3 Solve: $\frac{dy}{dx} + \tan x = 0$

Sol.

Given, $\frac{dy}{dx} + \tan x = 0$

$$\frac{dy}{dx} = -\tan x$$

$$dy = -\tan x \;\; dx$$

This is v.s. form. Integrating both the sides,we get

$$\int dy = \int -\tan x \, dx + C$$

$$y = \log \cos x + C$$

$$\boxed{y - \log \cos x = C}$$

This is the required General Solution.

Ex. 4 Solve: $(4+e^{2x})\frac{dy}{dx} = ye^{2x}$

Sol.

Given, $(4+e^{2x})\frac{dy}{dx} = ye^{2x}$

$$\frac{dy}{y} = \frac{e^{2x}}{4+e^{2x}}dx$$

This is v.s. form. Integrating both the sides,we get

$$\int \frac{1}{y}dy = \frac{1}{2}\int \frac{2e^{2x}}{4+e^{2x}}\,dx + \log C$$

Use formula : $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$

$$\log y = \frac{1}{2}\log(4+e^{2x}) + \log C$$

$$2\log y = \log(4+e^{2x}) + 2\log C$$

$$\log y^2 = \log(4 + e^{2x}) + \log C^2$$

$$\log y^2 = \log[(4 + e^{2x})C^2]$$

$$\boxed{y^2 = (4 + e^{2x})C_1} \longrightarrow \quad C^2 = C_1, constant$$

This is the required General Solution.

Solve : $\frac{dy}{dx} = e^{x-y} + 3x^2e^{-y}$

Solve : $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0 \longrightarrow$ Use formula: $\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$

❖ Homogeneous Differential Equation

- The differential equation in the form $M(x,y)dx + N(x,y)dy = 0$ is said to be homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous functions in x and y of the same degree.
- It can also be represented as $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f(x,y) and g(x,y) are homogeneous functions in x and y of the same degree.
- To solve these equations , we can substitute $y = ux$ and $\frac{dy}{dx} = u + x\frac{du}{dx}$ so that these equations will reduced to variable separable form and can be solved further.

Examples:

1) $\frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y}$ 2) $(x^4 + y^4)dx - 2x^3y\,dy = 0$ 3) $x\frac{dy}{dx} + \frac{y^2}{x} = y$ 4) $xdy - ydx = \sqrt{x^2+y^2}\,dx$

❖ Non-Homogeneous Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is known as non-homogeneous differential equation.

❖ Exact differential equations

- Let $M(x,y)dx + N(x,y)dy = 0$ be the differential equation.
- It is said to be an exact differential equation if there exists a function $f(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = df$$

- The condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- When the condition for exactness is satisfied , the general solution is given by

$$\int_{y=constant} M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x]dy = C$$

OR sometimes we may use

$$\int_{x=constant} N dy \quad + \quad \int [Terms\ of\ M\ not\ containing\ y]dx = C$$

Ex. 1. Solve: $(xy^2 + 3x^2y)dx + (x^3 + x^2y)dy = 0$

Sol.

Here, $\mathrm{M} = (xy^2 + 3x^2y)$ and $\mathrm{N} = (x^3 + x^2y)$

$$\frac{\partial M}{\partial y} = 2xy + 3x^2 \qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, given equation is exact.

Hence,general solution is

$$\int_{y = constant} M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x] dy = C$$

$$\int_{y = constant} (xy^2 + 3x^2y) dx \quad + \quad \int 0 dy = C$$

$$\int xy^2 dx + \int 3x^2 y\, dx + \int 0\, dy = C$$

$$y^2 \int x\, dx + y \int 3x^2 dx + 0 = C$$

$$y^2 \frac{x^2}{2} + 3y \frac{x^3}{3} = C$$

$$\boxed{\frac{x^2 y^2}{2} + x^3 y = C}$$

This is the general solution.

Ex.2 Solve: $(3 + 2y\cos x)dx + (2\sin x - 4y^3)dy = 0$

Sol.

Here , $M = 3 + 2y\cos x$ and $N = 2\sin x - 4y^3$

$$\frac{\partial M}{\partial y} = 2\cos x \qquad \frac{\partial N}{\partial x} = 2\cos x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, given equation is exact.

Hence,general solution is

$$\int M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x] dy = C$$

$y = constant$

$$\int (3 + 2y \cos x\ ) dx \ + \ \int -4y^3 dy = C$$

$y = constant$

$$\int 3\ dx + 2y \int \cos x\ dx \ - 4 \int y^3 dy = C$$

$$\boxed{3x + 2ysinx - y^4 = C}$$

This is the general solution.

Ex. 3 Solve: $(2x + e^x \log y)dx + e^x dy = 0$

Sol.

Here, $\mathrm{M} = (2x + e^x \log y)$ and $\mathrm{N} = (e^x)$

$$\frac{\partial M}{\partial y} = \frac{1}{y} e^x \qquad \frac{\partial N}{\partial x} = e^x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, given equation is exact.

Hence,general solution is

$$\int M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x] dy = C$$

$y = constant$

$$\int (2x + e^x \log y) dx \quad + \quad \int 0 dy = C$$

$y = constant$

$$\int 2x\,dx + \log y \int e^x\,dx + \int 0\,dy = C$$

$$2\frac{x^2}{2} + e^x \log y = C$$

$$\boxed{x^2 + e^x \log y = C}$$

This is the general solution.

Ex.4 Solve : $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Sol.

Here, $\mathrm{M} = (x^2 - 4xy - 2y^2)$ and $\mathrm{N} = (y^2 - 4xy - 2x^2)$

$$\frac{\partial M}{\partial y} = -4x - 4y \qquad \frac{\partial N}{\partial x} = -4y - 4x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, given equation is exact.

Hence,general solution is

$$\int Mdx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x]dy = C$$

$$y = constant$$

$$\int (x^2 - 4xy - 2y^2)dx \quad + \quad \int y^2 dy = C$$

$$y = constant$$

$$\int x^2 dx - 4y \int xdx - 2y^2 \int dx + \int y^2 dy = C$$

$$\frac{x^3}{3} - 4y\frac{x^2}{2} - 2y^2\mathrm{x} + \frac{y^3}{3} = C$$

$$\boxed{x^3 - 6x^2y - 6y^2x + y^3 = \mathrm{C}}$$

This is the general solution.

❖ Equations Reducible to Exact form by using Integrating Factor

Integrating Facor is a multiplying factor by which the equation can be made exact.

* Steps to find solution of non-exact differential equation by finding integrating factor

1. Consider the equation $Mdx + Ndy = 0$ is Non-Exact Differential Equation. It means $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

2. Find Integrating Factor by applying appropriate rule.

3. Multiply given Non-Exact equation throughout by an integrating Factor. Due to this multiplication the equation becomes Exact.

4. Now,solve this Exact differential equation by using the formula

$$\int_{y = constant} Mdx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x]dy = C$$

❖ Rules for finding integrating factors (I.F)

Rule 1 : Let the equation $Mdx + Ndy = 0$ be Non-Exact but Homogeneous , and if $xM + yN \neq 0$ then I.F.= $\frac{1}{xM+yN}$

Rule 2 : Let the equation $Mdx + Ndy = 0$ be Non-Exact but has the form $yf_1(xy)dx + xf_2(xy)dy = 0$, and if and if $xM - yN \neq 0$ then I.F.= $\frac{1}{xM-yN}$

Rule 3 : Let the equation $Mdx + Ndy = 0$ be Non-Exact and if $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$ then I.F.=$e^{\int f(x)dx}$

Here, $f(x)$ indicates some function of x only

Rule 4 : Let the equation $Mdx + Ndy = 0$ be Non-Exact and if $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = g(y)$ then I.F.=$e^{\int g(y)dy}$

Here,$g(y)$indicates some function of y only

❖ Examples on Differential Equations Reducible to Exact form by using Integrating Factor

Ex 1. Solve : $(x^2 - 3xy + 2y^2)dx + (3x^2 - 2xy)dy = 0 \longrightarrow$ (1)

Sol.

Here, $\frac{\partial M}{\partial y} = -3x + 4y$, $\frac{\partial N}{\partial x} = 6x - 2y$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ hence, equation is non-exact.

But equation is homogeneous.Also, $xM + yN = x^3 - 3x^2y + 2y^2x + 3x^2y - 2xy^2 = x^3 \neq 0$

Hence,by Rule 1 , I.F.=$\frac{1}{xM+yN} = \frac{1}{x^3}$

Now multiply given equation (1) by $\frac{1}{x^3}$

$$\left(\frac{1}{x} - \frac{3y}{x^2} + \frac{2y^2}{x^3}\right)dx + \left(\frac{3}{x} - \frac{2y}{x^2}\right)dy = 0 \longrightarrow \quad (2)$$

Equation (2) is now exact. (No need to check condition of exactness)

Solution of equation (2) is $\int\limits_{y=constant} M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x] dy = C$

$$\int \frac{1}{x} dx - 3y \int \frac{1}{x^2} dx + 2y^2 \int \frac{1}{x^3} dx + \int 0 dy = C$$

$$\boxed{\log x + 3y\frac{1}{x} - 2y^2\frac{1}{2x^2} = C}$$

This is general solution.

Ex. 2 Find Integrating factor of $(y^2 - 2xy)dx + (2x^2 + 3xy)dy = 0$

Sol. Given equation is non-exact but homogeneous.

Also , $xM + yN = y^2x - 2x^2y + 2x^2y + 3y^2x = 4y^2x \neq 0$

Hence ,by Rule 1 $\boxed{\text{I.F.} = \frac{1}{xM+yN} = \frac{1}{4y^2x}}$

Ex. 3 Solve: $(1+xy)ydx+(1-xy)xdy=0 \longrightarrow (1)$

Sol.

Here, $M=y+xy^2$, $N=x-x^2y$, $\frac{\partial M}{\partial y}=1+2xy$, $\frac{\partial N}{\partial x}=1-2xy$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ hence, equation is non-exact.

Given equation can be written as $(x^0y^0+xy)ydx+(x^0y^0-xy)xdy=0$

Also, $xM-yN=xy+x^2y^2-xy+x^2y^2=2x^2y^2\neq 0$

Hence,by Rule 2 , I.F.=$\frac{1}{xM-yN}=\frac{1}{2x^2y^2}$

Now,multiply equation (1) by $\frac{1}{2x^2y^2}$

$$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2y^2x}-\frac{1}{2y}\right)dy=0 \longrightarrow (2)$$

Equation (2) is now exact. (No need to check condition of exactness)

Solution of equation (2) is $\underset{y = constant}{\int M dx} + \int [Terms\ of\ N\ not\ containing\ x] dy = C$

$$\frac{1}{2y}\int \frac{1}{x^2}dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{y}dy = C$$

$$\boxed{-\frac{1}{2yx} + \frac{1}{2}\log x - \frac{1}{2}\log y = C}$$

This is general solution.

Ex. 4 Find solution of $(x^2y^2 + 2)ydx + (2 - 2x^2y^2)xdy = 0$ with integrating factor $\frac{1}{3x^3y^3}$

Sol.

Multiply given equation by $\frac{1}{3x^3y^3}$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$$

This is now exact. (No need to check condition of exactness)

Solution is $\int\limits_{y=constant} M dx + \int [Terms\ of\ N\ not\ containing\ x] dy = C$

$$\int \frac{1}{3x} dx + \frac{2}{3y^2} \int \frac{1}{x^3} dx - \frac{2}{3} \int \frac{1}{y} dy = C$$

$$\boxed{\log x - \frac{1}{y^2 x^2} - 2 \log y = 3C = C_1}$$

This is general solution.

Ex. 5 Solve: $(x^2 + y^2 + 1)dx - 2xydy = 0$

Sol.

Here, $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -2y$ $\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ hence, equation is non-exact.

Also, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-2y)}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$

Hence,by Rule 3 , I.F.$= e^{\int f(x)dx} = e^{\int -\frac{2}{x} dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Now,multiply given equation by $\frac{1}{x^2}$

$$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx-\frac{2y}{x}dy=0$$

This is an Exact equation.

Solution is $\int_{y=constant} M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x]dy = C$

$$\int 1dx + y^2\int\frac{1}{x^2}dx+\int\frac{1}{x^2}dx-\int 0dy = C$$

$$\boxed{x-\frac{y^2}{x}-\frac{1}{x}=C}$$

This is general solution.

Ex.6 Find integrating factor of differential equation $\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx+\left(\frac{x+xy^2}{4}\right)dy=0$

Sol.

Here, $\frac{\partial M}{\partial y}=1+y^2$, $\frac{\partial N}{\partial x}=\frac{1+y^2}{4}$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1+y^2-\left(\frac{1+y^2}{4}\right)}{\left(\frac{x+xy^2}{4}\right)}=\frac{\frac{3}{4}(1+y^2)}{\frac{x}{4}(1+y^2)}=\frac{3}{x}=f(x)$$

Hence,by Rule 3 , I.F.=$e^{\int f(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\log x}=e^{\log x^3}=x^3$

$$\boxed{\text{I.F.}= x^3}$$

Ex. 7 Find integrating factor of equation $y\log y dx+(x-\log y)dy=0$

Sol.

Here, $M=y\log y$ and $N=x-\log y$

Here, $\frac{\partial M}{\partial y}=y.\frac{1}{y}+\log y=1+\log y$, $\frac{\partial N}{\partial x}=1$

Also, $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{1-1-\log y}{y\log y}=\frac{-\log y}{y\log y}=-\frac{1}{y}=g(y)$

Hence,by Rule 4 , I.F.=$e^{\int g(y)dy}=e^{\int\frac{-1}{y}dx}=e^{-1\log y}=e^{\log y^{-1}}=y^{-1}=\frac{1}{y}$

$$\boxed{\text{I.F.}=\frac{1}{y}}$$

Ex.8 Solve : $y(2xy+e^x)dx-e^x dy=0$

Sol.

$M=2xy^2+ye^x, N=-e^x$, $\frac{\partial M}{\partial y}=4xy+e^x$, $\frac{\partial N}{\partial x}=-e^x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ hence, equation is non-exact.

Also, $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{-e^x-(4xy+e^x)}{2xy^2+ye^x} = \frac{-2e^x-4xy}{2xy^2+ye^x} = \frac{-2(e^x+2xy)}{y(2xy+e^x)} = \frac{-2}{y} = g(y)$

Hence,by Rule 4 , I.F.=$e^{\int g(y)dy} = e^{\int \frac{-2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

Now,multiply given equation by $\frac{1}{y^2}$

$$\left(2x + \frac{e^x}{y}\right) dx - \frac{e^x}{y^2} dy = 0$$

This is an Exact equation.

Solution is $\int\limits_{y\,=\,constant} M dx \quad + \quad \int [Terms\ of\ N\ not\ containing\ x] dy = C$

$$\int 2x dx + \frac{1}{y}\int e^x dx - \int 0 dy = C$$

$$\boxed{x^2 + \frac{1}{y} e^x = C}$$

This is general solution.

Solve

1. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \ \ Hint: Rule\ 1$

2. $(1 + xy)ydx + (x^2y^2 + xy + 1)xdy = 0 \ \ Hint: Rule\ 2$

3. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0 \ \ Hint: Rule\ 4$

4. $(x^2 + y^2 + x)dx + xydy = 0 \ \ Hint: Rule\ 3$

❖ Linear Differential Equations of the First Order

A differential equation is said to be linear if

- The degree (or power) of dependent variable and its derivatives is 1.
- No term involves product of derivatives and dependent variables.
- General form of the linear differential equation is given by,

  $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x or constants.
- Similarly, a linear differential equation can also be given by

  $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y)$ and $Q(y)$ are functions of y or constants.
- Coefficient of $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is 1.

Examples :

1) $\frac{dy}{dx} + x^2 y = e^x$ is linear equation.

2) $\frac{dy}{dx} + y = 5$ is linear equation.

3) $\frac{dy}{dx} + y^2 \sin x = \cos x$ is not linear equation.

4) $\left(\frac{dy}{dx}\right)^2 - 3x^4 y + 1 = 0$ is not linear equation

5) $\frac{dx}{dy} + (1 + y)x = 3y^3$ is linear equation.

6) $\frac{dy}{dx} - \frac{(1+x^2)}{x} y = e^{-x^2}$ is linear equation.

Solution of linear differential equations

- For the equation $\frac{dy}{dx} + P(x)y = Q(x)$

  Integrating Factor(I.F.)= $I = e^{\int P(x)dx}$

  Solution is $I.y = \int I.Q(x)\,dx + C$

- For the equation $\frac{dx}{dy} + P(y)x = Q(y)$

  Integrating Factor(I.F.)= $I = e^{\int P(y)dy}$

  Solution is $I.x = \int I.Q(y)\,dy + C$

Ex.1 Solve : $\frac{dy}{dx} + (1 + 2x)y = e^{-x^2}$

Sol.

Given equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = (1 + 2x)$, $Q(x) = e^{-x^2}$

Hence, it is linear differential equation.

Integrating Factor(I.F.)= $I = e^{\int P(x)dx} = e^{\int (1+2x)dx} = e^{x+x^2}$

General solution is $I.y = \int I.Q(x)\,dx + C$

$$e^{x+x^2}.y = \int e^{x+x^2}.e^{-x^2}\,dx + C$$

$$e^{x+x^2}.y = \int e^{x}.e^{x^2}.e^{-x^2}\,dx + C$$

$$e^{x+x^2}.y = \int e^{x}.e^{x^2}.e^{-x^2}\,dx + C = e^{x} + C$$

$$\boxed{e^{x+x^2}.y = e^{x} + C}$$

This is the general solution.

Ex. 2 Find the integrating factor of the differential equation $\frac{dy}{dx} + \frac{y}{1+x^2} = x^2$

Sol.

Given equation is linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$

where $P(x) = \frac{1}{1+x^2}$ $and\ Q(x) = x^2$

Integrating Factor(I.F.)= $I = e^{\int P(x)dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1} x}$

$$\boxed{I = e^{\tan^{-1} x}}$$

Ex. 3 Find the integrating factor of the differential equation $\frac{dy}{dx} + y \cot x = \sin 2x$

Sol.

Given equation is linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$

where $P(x) = \cot x$ $and\ Q(x) = \sin 2x$

Integrating Factor(I.F.)= $I = e^{\int P(x)dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

$$\boxed{I = \sin x}$$

Ex.4 Find the general solution of $\frac{dy}{dx} + \frac{3}{x} y = \frac{e^x}{x^2}$ with integrating factor x^3.

Sol. Given equation is linear differential equation with $P = \frac{3}{x}$, $Q = \frac{e^x}{x^2}$ $and\ I = x^3$

Solution is $I.y = \int I.Q(x)\,dx + C$

$$x^3 y = \int x^3 \frac{e^x}{x^2} dx + C = \int x e^x dx + C = xe^x - e^x + C = (x-1)e^x + C$$

$$\boxed{x^3 y = (x-1)e^x + C}$$

This is the general solution.

Ex.5 Find the general solution of $\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{1}{x} \sec x$ with integrating factor $x \sec x$

Sol. Given equation is linear diff. equation with $P = \tan x + \frac{1}{x}$, $Q = \frac{1}{x} \sec x$ $and\ I = x \sec x$

Solution is $I.y = \int I.Q(x)\,dx + C$

$$x \sec x \,.\, y = \int I.Q(x)\,dx + C = \int x \sec x \,.\, \frac{1}{x} \sec x\,dx + C = \int sec^2 x dx + C = \tan x + C$$

$$\boxed{x \sec x \,.\, y = \tan x + C}$$

This is the general solution.

❖ Equations reducible to the Linear Form (Bernoulli's Differential Equation)

A differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x).\, y^n$ is called Bernoulli's Differential Equaton.

❖ Steps to find solution of Bernoulli's equation:

1. Divide given equation by y^n
2. So it becomes $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \longrightarrow (1)$
3. Substitute $y^{1-n} = \mathrm{u}$ Hence, $(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$
4. (1) becomes, $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$ which is linear equation.
5. Integrating factor=$I.F. = I = e^{\int (1-n)P(x)dx}$
6. General solution is $I.u = \int I.(1-n)Q(x)\, dx + C$
7. Back substitute $\mathrm{u} = y^{1-n}$

- An equation $\frac{dx}{dy} + P(y)x = Q(y).x^n$ is also called Bernoulli's Differential Equaton. It can be solved in the similar way by substituting $x^{1-n} = \mathrm{u}$
  - Integrating factor=$I.F. = I = e^{\int (1-n)P(y)dy}$
  - General solution is $I.u = \int I.(1-n)Q(y)\,dy + C$
  - Back substitute $\mathrm{u} = x^{1-n}$

- An equation of the form $f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$ is reducible to the linear form.
  - Substitute $f(y) = u \quad and \quad f'(y)\frac{dy}{dx} = \frac{du}{dx}$
  - Hence,equation converts into $\frac{du}{dx} + P(x).u = Q(x)$ which is linear in u and can be solved further.

- An equation of the form $f'(x)\frac{dx}{dy} + P(y)f(x) = Q(y)$ is reducible to the linear form.
  - Substitute $f(x) = u \quad and \quad f'(x)\frac{dx}{dy} = \frac{du}{dy}$
  - Hence,equation converts into $\frac{du}{dy} + P(y).u = Q(y)$ which is linear in u and can be solved further

Ex.1 Reduce the Bernoulli's equation $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$ to the linear form.

Sol.

Given equation is of the form $\frac{dy}{dx} + P(x)y = Q(x).y^n$ where $P(x) = -x\,, Q(x) = -e^{-x^2}, n = 3$

Substitute $y^{1-n} = y^{1-3} = y^{-2} = u\ and (1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx}$ $i.e. -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$

Hence, given equation becomes $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$$\boxed{\frac{du}{dx} + 2xu = 2e^{-x^2} \text{ which is linear.}}$$

Ex.2 Find the integrating factor of $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$

Sol. Given equation $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$ is Bernoulli's equation where $P(x) = -x\,, n = 3$

$$I.F. = I = e^{\int (1-n)P(x)dx} = e^{\int 2x dx} = e^{x^2}$$

$$\boxed{I = e^{x^2}}$$

Ex. 3 Find the general solution of $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$ with integrating factor e^{x^2}.

Sol.

Given equation $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$ is Bernoulli's equation where $Q(x) = -e^{-x^2}, n = 3, I = e^{x^2}$

General solution is $I.u = \int I.(1-n)Q(x)\,dx + C$ where $u = y^{1-n} = y^{-2}$

$$e^{x^2}.u = \int e^{x^2}.2e^{-x^2}\,dx + C = \int 2dx + C = 2x + C$$

Back substitute $u = y^{-2}$

$$\boxed{e^{x^2}.y^{-2} = 2x + C}$$

This is the general solution.

Ex.4 Reduce the equation $\tan y \frac{dy}{dx} + \tan x = cos^2 x \cos y$ to the linear form..

Sol.

Multiplying given equation by $\sec y$, we get $\sec y \tan y \frac{dy}{dx} + \sec y \tan x = cos^2 x \longrightarrow$ (1)

Now substitute $\sec y = u$ so that $\sec y \tan y \frac{dy}{dx} = \frac{du}{dx}$

Equation (1) becomes, $\frac{du}{dx} + u\tan x = cos^2 x$ which is linear with $\mathrm{P} = \tan x$ and $\mathrm{Q} = cos^2 x$

* Further solution can be obtained as

$$I = e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Solution is $I.u = \int I.Q(x)\,dx + C$

$$\sec x\,.\,u = \int \sec x\,.\,cos^2 x\;dx + C = \int \cos x dx + C = \sin x + C$$

Back substitute $\mathrm{u} = \sec y$

$\sec x\,.\sec y = \sin x + C$ is the general solution.

Ex. Find the general solution of 1) $\frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}$ 2) $\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{-tan^{-1}y}}{1+y^2}$

3) $\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^3}$ 4) $\frac{dy}{dx} - y\tan x = y^4 \sec x$