Unit 2 **Chapter 4**

Fourier Series

4.1 Introduction

In many engineering and physical problems, particularly those connected with vibration and co[...] heat, it is more useful to be able to express a real valued function in a series of sines and cosines withi[...] range of variables. Such a series is known as *Fourier Series.*

Thus any function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed in the fourier series as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, a_0, a_n, b_n are constants provided in the interval and $f(x)$ must follow the conditions given be[...]

(i) $f(x)$ is defined and single valued in the given interval also $\int_{c_1}^{c_2} f(x)\,dx$ exists.

(ii) $f(x)$ may have finite number of discontinuities.

(iii) $f(x)$ has finite number of maxima and minima.

These conditions are known as **Dirichlet conditions.**

4.2 Fourier Series (Definition)

Let $f(x)$ be a periodic function of period 2L defined in the interval $c \le x \le c + 2L$ i.e. $(c, c + 2L)$ and **Dirichlet's conditions**, then $f(x)$ can be expressed as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where a_0, a_n, b_n are called as fourier coefficients and are given by,

$$a_0 = \frac{1}{L}\int_{c}^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Note : Depending upon the values of c and L we get various types of fourier series, which are explained in this cha[pter]

Type 1	Interval	$0 \le x \le 2\pi$	**Type 2**	Interval	$-\pi \le x \le \pi$
Type 3	Interval	$0 \le x \le 2L$	**Type 4**	Interval	$-L \le x \le L$

4.3 Periodic Functions

A function f (x) is said to be periodic if it is defined for all real x and if there is some positive number T such that.

f (x + T) = f (x) for all x, then T is called as period of f (x).

Note : 1. sin x, cos x, sec x and cosec x are periodic functions with fundamental period 2π

i.e. $f(x + 2\pi) = f(x)$

e.g. $\sin\left(\frac{\pi}{4} + 2\pi\right) = \sin\left(\frac{\pi}{4}\right)$

$\sin(45^\circ + 360) = \sin(45)$

$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

2. tan x and cot x are periodic functions with fundamental period π.

3. The constant function f (x) = c is also a periodic function.

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Bernaulli's Rule

$$\int uv\, dx = (u)\, [v_1] - (u')\, [v_2] + (u'')\, [v_3] - \ldots\ldots$$

where, u = polynomial i.e. power of x whose successive derivative becomes zero.

dash = derivative

suffix = Integration

Note : Bernaulli's Rule is a special case of integration by parts. If integration of multiplication of two terms involves one polynomial function then we can use Bernaulli's Rule

i.e. $\int \underbrace{x^2}_{\text{polynomial}} \; \underbrace{\sin x}_{\text{trigonometric}} \, dx$ = **Bernaulli's Rule is applicable.**

but $\int \underbrace{e^x}_{\text{Exponential}} \; \underbrace{\sin x}_{\text{trigonometric}} \, dx$ = **Bernaulli's Rule is not applicable.**

4.4 Type 1 : Interval $0 \le x \le 2\pi$

For the interval $0 \le x \le 2\pi$, fourier series can be expressed as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$	$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx$	$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$

4.4.1 Solved Examples on Fourier Series Expansion in the Interval $0 \le x \le 2\pi$

Example 4.4.1 May 2

Find the fourier series of the function : $f(x) = x^2$, $0 \le x \le 2\pi$ and $f(x + 2\pi) = f(x)$.

Solution :

To find the fourier series of the given function, follow the steps given below

Step 1 : To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx$$

by $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$

$$\therefore \quad a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - \frac{(0)^3}{3}\right] = \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right]$$

$$a_0 = \frac{8\pi^2}{3}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{\underset{\downarrow}{x^2}}\ \underset{v}{\underset{\downarrow}{\cos(nx)}}\,dx$$

By Bernaulli's Rule, $\int uv\,dx = (u)[v_1] - (u')[v_2] + (u'')[v_3] - \ldots$

Also, $\int \cos nx\,dx = \dfrac{\sin nx}{n}$; $\int \sin nx\,dx = -\dfrac{\cos nx}{n}$

$$\therefore \quad a_n = \frac{1}{\pi}\left\{(x^2)\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right\}_0^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{\pi}\left\{\left[\frac{(2\pi)^2\sin(n2\pi)}{n} + \frac{(2)(2\pi)\cos(n2\pi)}{n^2} - \frac{2\sin(n2\pi)}{n^3}\right] - \left[\frac{(0)^2\sin(0)}{n} + \frac{2(0)\cos(0)}{n^2} - \frac{2\sin(0)}{n^3}\right]\right\}$$

but $\sin 2n\pi = 0$; $\cos 2n\pi = 1$

$\sin 0 = 0$; $\cos 0 = 1$

$$\therefore \quad a_n = \frac{1}{\pi}\left\{\left[0 + \frac{4\pi}{n^2} - 0\right] - [0 + 0 - 0]\right\} = \frac{4}{n^2}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{\underset{\downarrow}{x^2}}\ \underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$$

By Bernaulli's Rule.

$$b_n = \frac{1}{\pi}\left\{(x)^2\left[\frac{-\cos nx}{n}\right]-(2x)\left[\frac{-\sin nx}{n^2}\right]+(2)\left[\frac{\cos nx}{n^3}\right]\right\}_0^{2\pi} = \frac{1}{\pi}\left\{\frac{-x^2\cos(nx)}{n}+\frac{2x\sin(nx)}{n^2}+\frac{2\cos(nx)}{n^3}\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left[\left[\frac{-(2\pi)^2\cos(n2\pi)}{n}+\frac{2(2\pi)\sin(n2\pi)}{n^2}+\frac{2\cos(n2\pi)}{n^3}\right]-\left[\frac{-(0)^2\cos(0)}{n}+\frac{2(0)\sin(0)}{n^2}+\frac{2\cos(0)}{n^3}\right]\right]$$

but $\sin(2n\pi) = 0$; $\cos(2n\pi) = 1$

$\sin 0 = 0$; $\cos 0 = 1$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-4\pi^2}{n}+0+\frac{2}{n^3}\right]-\left[-0+0+\frac{2}{n^3}\right]\right\} = \frac{1}{\pi}\left\{\frac{-4\pi^2}{n}+\frac{2}{n^3}-\frac{2}{n^3}\right\} = \frac{1}{\pi}\left\{-\frac{4\pi^2}{n}\right\}$$

$$b_n = -\frac{4\pi}{n}$$

Step 4 : Fourier Series expansion :

$$f(x) = \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$x^2 = \frac{1}{2}\left(\frac{8\pi^2}{3}\right)+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx+\left(\frac{-4\pi}{n}\right)\sin nx\right]$$

$$\therefore \quad x^2 = \frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{4\cos nx}{n^2}-\frac{4\pi\sin nx}{n}\right]$$

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We have, $\sin 0 = 0$; $\sin(2n\pi) = 0$,

Therefore, while applying the limits, sine term can be ignored. i.e. we can directly write its value as zero, without any calculations.

Example 4.4.2

Find the fourier series expansion of the function $f(x) = \pi^2 - x^2$ in the interval $0 \le x \le 2\pi$ and $f(x+2\pi) = f(x)$.

Solution :

Given : $f(x) = \pi^2 - x^2 \qquad 0 \le x \le 2\pi$

To find the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi^2 - x^2)\,dx$$

Note : Derivative (constant) = 0 i.e. $\frac{d}{dx}$(constant) = 0

Integration (constant) = x i.e. $\int c\,dx = cx$

$$\therefore \quad a_0 = \frac{1}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left\{\left[\pi^2(2\pi) - \frac{(2\pi)^3}{3}\right] - \left[\pi^2(0) - \frac{0^3}{3}\right]\right\}$$

$$a_0 = \frac{1}{\pi}\left\{2\pi^3 - \frac{8\pi^3}{3}\right\} = \frac{1}{\pi}\left\{-\frac{2\pi^3}{3}\right\}$$

$$\mathbf{a_0 = \frac{-2\pi^2}{3}}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{\underset{\downarrow}{(\pi^2 - x^2)}}\ \underset{v}{\underset{\downarrow}{\cos(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule

$$a_n = \frac{1}{\pi}\left\{(\pi^2 - x^2)\left[\frac{\sin(nx)}{n}\right] - (0-2x)\left[\frac{-\cos(nx)}{n^2}\right] + (-2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left\{\frac{(\pi^2 - x^2)\sin(nx)}{n} - \frac{2x\cdot\cos(nx)}{n^2} + \frac{2\sin(nx)}{n^3}\right\}_0^{2\pi}$$

Note : As explained in the previous Gurukey, we will write 0 in place of sine terms, while applying limits.

$$\therefore \quad a_n = \frac{1}{\pi}\left\{\left[0 - \frac{2(2\pi)\cos(n2\pi)}{n} + 0\right] - \left[0 - \frac{2(0)\cos(0)}{n^2} + 0\right]\right\} = \frac{1}{\pi}\left\{\frac{-4\pi}{n}\right\} \qquad \left\{\because \cos 2n\pi = \text{ and } \cos 0 = \right.$$

$$\mathbf{a_n = -\frac{4}{n}}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{\underset{\downarrow}{(\pi^2 - x^2)}}\ \underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule

$$b_n = \frac{1}{\pi}\left\{(\pi^2 - x^2)\left[\frac{-\cos(nx)}{n}\right] - (0-2x)\left[\frac{-\sin(nx)}{n^2}\right] + (-2)\left[\frac{\cos(nx)}{n^3}\right]\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{-\frac{(\pi^2 - x^2)\cos(nx)}{n} - \frac{2x\sin(nx)}{n^2} - \frac{2\cos(nx)}{n^3}\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left[-\frac{(\pi^2 - 4\pi^2)\cos(n2\pi)}{n} - 0 - \frac{2\cos(n2\pi)}{n^3}\right] - \left[-\frac{(\pi^2 - 0)\cos(0)}{n} - 0 - \frac{2\cos(0)}{n^3}\right]\right\}$$

but $\cos(2n\pi) = 1$; $\cos 0 = 1$

$$\therefore \quad b_n = \frac{1}{\pi}\left\{\left[\frac{-(-3\pi^2)}{n} - \frac{2}{n^3}\right] - \left[-\frac{\pi^2}{n} - \frac{2}{n^3}\right]\right\} = \frac{1}{\pi}\left\{\frac{3\pi^2}{n} - \frac{2}{n^3} + \frac{\pi^2}{n} + \frac{2}{n^3}\right\} = \frac{1}{\pi}\left\{\frac{4\pi^2}{n}\right\}$$

$$\mathbf{b_n = \frac{4\pi}{n}}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

$$\pi^2 - x^2 = \frac{1}{2}\left(-\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left\{\frac{-4}{n}\cos(nx) + \frac{4\pi}{n}\sin(nx)\right\}$$

$$\pi^2 - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{-4\cos(nx)}{n} + \frac{4\pi\sin(nx)}{n}\right]$$

Example 4.4.3 **May 2012**

Obtain fourier series expansion for the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x) = f(x + 2\pi)$.

Hence, deduce that, (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

Solution :

Given : $f(x) = \left(\frac{\pi - x}{2}\right)^2$ By using $(a - b)^2 = a^2 - 2ab + b^2$ $\therefore$ $f(x) = \frac{\pi^2 - 2\pi x + x^2}{4}$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi^2 - 2\pi x + x^2}{4}dx = \frac{1}{4\pi}\left[\pi^2 x - \frac{2\pi x^2}{2} + \frac{x^3}{3}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{4\pi}\left\{\left[\pi^2(2\pi) - \pi(2\pi)^2 + \frac{(2\pi)^3}{3}\right] - \left[\pi^2(0) - \pi(0)^2 + \frac{(0)^3}{3}\right]\right\}$$

$$a_0 = \frac{1}{4\pi}\left\{\left[2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right] - [0 + 0 + 0]\right\} = \frac{1}{4\pi}\left\{\frac{2\pi^3}{3}\right\}$$

$$a_0 = \frac{\pi^2}{6}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi^2 - 2\pi x + x^2)}{4}\cos(nx)\,dx$$

$$a_n = \frac{1}{4\pi}\int_0^{2\pi}\underbrace{(\pi^2 - 2\pi x + x^2)}_{u}\ \underbrace{\cos(nx)}_{v}\,dx$$

$\therefore$ By Bernaulli's Rule

$$a_n = \frac{1}{4\pi}\left\{(\pi^2 - 2\pi x + x^2)\left[\frac{\sin(nx)}{n}\right] - (0 - 2\pi + 2x)\left[\frac{-\cos(nx)}{n^2}\right] + (0+2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{4\pi}\left\{\frac{(\pi^2 - 2\pi x + x^2)\sin(nx)}{n} + \frac{(-2\pi + 2x)\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right\}_0^{2\pi}$$

$$a_n = \frac{1}{4\pi}\left\{\left[0 + \frac{(-2\pi + 4\pi)\cos(n2\pi)}{n^2} - 0\right] - \left[0 + \frac{(-2\pi + 0)\cos 0}{n^2} - 0\right]\right\}$$

but $\cos(2n\pi) = 1$ and $\cos 0 = 1$

$$a_n = \frac{1}{4\pi}\left\{\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right\}$$

$$\mathbf{a_n = \frac{1}{n^2}}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi^2 - 2\pi x + x^2)}{4}\sin(nx)\,dx$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi}\underset{u}{\underset{\downarrow}{(\pi^2 - 2\pi x + x^2)}}\ \underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule

$$b_n = \frac{1}{4\pi}\left\{(\pi^2 - 2\pi x + x^2)\left[\frac{-\cos(nx)}{n}\right] - (0 - 2\pi + 2x)\left[\frac{-\sin(nx)}{n^2}\right] + (2)\left[\frac{\cos(nx)}{n^3}\right]\right\}$$

$$b_n = \frac{1}{4\pi}\left\{-\frac{(\pi^2 - 2\pi x + x^2)\cos(nx)}{n} + \frac{(-2\pi + 2x)\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\right\}$$

$$b_n = \frac{1}{4\pi}\left\{\left[\frac{(-\pi^2 - 4\pi^2 + 4\pi^2)\cos(n2\pi)}{n} + 0 + \frac{2\cos(n2\pi)}{n^3}\right] - \left[\frac{-(\pi^2 - 0 + 0)\cos(0)}{n} + 0 + \frac{2\cos(0)}{n^3}\right]\right\}$$

but $\cos(2n\pi) = 0$ and $\cos 0 = 1$

$$b_n = \frac{1}{4\pi}\left\{\left[-\frac{\pi^2}{n} + \frac{2}{n^3}\right] - \left[\frac{-\pi^2}{n} + \frac{2}{n^3}\right]\right\} = \frac{1}{4\pi}\left\{-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right\}$$

$$b_n = \frac{1}{4\pi}(0)$$

$$\mathbf{b_n = 0}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

$$\therefore \quad \left(\frac{\pi - x}{2}\right)^2 = \frac{1}{2}\,\frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(nx) + 0\cdot\sin(nx)\right]$$

$$\therefore \quad \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Step 5 : Deductions :

(i) Put $x = 0$ in Equation (A)

$$\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos 0}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots = \frac{\pi^2}{6} \qquad \dots(1)$$

(ii) Put $x = \pi$ in Equation (A)

$$\therefore \quad 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2} = -\frac{\pi^2}{12}$$

but $\cos(n\pi) = (-1)^n$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\therefore \quad \frac{(-1)^1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \frac{(-1)^4}{4^2} \dots = \frac{-\pi^2}{12}$$

but $(-1)^{even} = 1$ and $(-1)^{odd} = -1$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots\dots = -\frac{\pi^2}{12}$$

Multiplying by '–' sign on both sides,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots = \frac{\pi^2}{12} \qquad \dots(2)$$

Adding Equation (1) and Equation (2), we get,

$$2\left[\frac{1}{1^2}\right] + 2\left[\frac{1}{3^2}\right] + 2\left[\frac{1}{5^2}\right] + \dots = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots = \frac{\pi^2}{8}$$

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For such deductions in any example always put (i) $x = 0$ (ii) $x = \pi$ in fourier expansion of f (x).

Example 4.4.4

Find the fourier series of the function $f(x) = e^{-x}$, $0 \le x \le 2\pi$ and $f(x + 2\pi) = f(x)$

Solution : Given $f(x) = e^{-x}$, $0 \le x \le 2\pi$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{-1}\right]-\left[\frac{e^{0}}{-1}\right]\right\}$$

but $e^0 = 1$

$$a_0 = \frac{1}{\pi}(-e^{-2\pi} + 1)$$

$$a_0 = \frac{(1 - e^{-2\pi})}{\pi}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos(nx)\,dx$$

by Using $\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

Here $a = -1$, $b = n$

$$a_n = \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}[-\cos nx + n\sin nx]\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1^2+n^2}(-\cos(n2\pi) + n\sin(n2\pi))\right]-\left[\frac{e^{0}}{1^2+n^2}(-\cos 0 + n\sin 0)\right]\right\}$$

but	$\cos 2n\pi = 1$	$\sin 2n\pi = 0$
	$\cos 0 = 1$	$\sin 0 = 0$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-1+0)\right]-\left[\frac{1}{1+n^2}(-1+0)\right]\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{-e^{-2\pi}}{1+n^2}+\frac{1}{1+n^2}\right\}$$

$$a_n = \frac{1}{\pi(1+n^2)}(-e^{-2\pi}+1)$$

$$a_n = \frac{(1 - e^{-2\pi})}{\pi(1+n^2)}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)\,dx$$

$$\text{but } \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

here $a = -1$ and $b = n$

$$b_n = \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}[-\sin(nx) - n\cos(nx)]\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1^2+n^2}[-\sin(n2\pi) - n\cos(n2\pi)]\right] - \left[\frac{e^0}{1^2+n^2}(-\sin 0 - n\cos(0))\right]\right\}$$

but $\sin 2n\pi = 0$ $\quad$ $\cos 2n\pi = 1$

$\sin 0 = 0$ $\quad$ $\cos 0 = 1$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(0-n)\right] - \left[\frac{1}{1+n^2}(0-n)\right]\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{-ne^{-2\pi}}{1+n^2} + \frac{n}{1+n^2}\right\}$$

$$b_n = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

$$e^{-x} = \frac{(1-e^{-2\pi})}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(1-e^{-2\pi})}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin(nx)\right]$$

Example 4.4.5

Obtain the fourier series expansion of the function $f(x) = x\sin x$ in the interval $0 \le x \le 2\pi$.

Solution :

Given, $\quad f(x) = x\sin x \qquad 0 \le x \le 2\pi$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{\substack{\downarrow\\ u}}{x}\;\underset{\substack{\downarrow\\ v}}{\sin x}\,dx$$

By Bernaulli's Rule

$$a_0 = \frac{1}{\pi}\{(x)[-\cos x] - (1)[-\sin x]\}_0^{2\pi} = \frac{1}{\pi}[-x\cos x + \sin x]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\{[-2\pi\cdot\cos(2\pi)+\sin(2\pi)]-[-0\cdot\cos 0+\sin 0]\}$$

but $\cos 2\pi = 1$ $\quad \sin 2\pi = 0$

$\cos 0 = 1$ $\quad \sin 0 = 0$

$$\therefore \quad a_0 = \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cdot\cos(nx)\,dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\cos(nx)\sin x\,dx$$

By using $2\cos A\cdot\sin B = \sin(A+B)-\sin(A-B)$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(nx+x)-\sin(nx-x)]\,dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \underset{\underset{u}{\downarrow}}{x}\;\underbrace{[\sin(n+1)x-\sin(n-1)x]}_{v}dx$$

$\therefore$ By Bernaulli's Rule

$$a_n = \frac{1}{2\pi}\left\{(x)\left[\frac{-\cos(n+1)x}{(n+1)}+\frac{\cos(n-1)x}{(n-1)}\right]-(1)\left[\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{2\pi}$$

for $n \neq 1$, because for $n = 1$, denominator $= 0$, which will give ∞.

$$a_n = \frac{1}{2\pi}\left\{\frac{-x\cdot\cos(n+1)x}{n+1}+\frac{x\cos(n-1)x}{n-1}+\frac{\sin(n+1)x}{(n+1)^2}-\frac{\sin(n-1)x}{(n-1)^2}\right\}_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{-2\pi\cdot\cos(n+1)2\pi}{n+1}+\frac{2\pi\cdot\cos(n-1)2\pi}{n-1}+\frac{\sin(n+1)2\pi}{(n+1)^2}-\frac{\sin(n-1)2\pi}{(n-1)^2}\right]\right.$$
$$\left.-\left[-0+0+\frac{\sin 0}{(n+1)^2}-\frac{\sin 0}{(n-1)^2}\right]\right\}$$

but $\cos(n\pm 1)2\pi = \cos(2n\pi\pm 2\pi) = 1$

and $\sin(n\pm 1)2\pi = \sin(2n\pi\pm 2\pi) = 0$

$$a_n = \frac{1}{2\pi}\left\{\left[\frac{-2\pi}{n+1}+\frac{2\pi}{n-1}+0-0\right]-[0-0]\right\} = \frac{1}{2\pi}(2\pi)\left[-\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$a_n = \frac{-n+1+n+1}{(n+1)(n-1)}$$

$$a_n = \frac{2}{n^2-1} \quad n > 1$$

Again, we have,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \cos(nx)\, dx$$

Put n = 1

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \cdot \cos x\, dx$$

But $2\sin x \cdot \cos x = \sin 2x$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} \underset{\underset{u}{\downarrow}}{x}\ \underset{\underset{v}{\downarrow}}{\sin 2x}\, dx$$

∴ By Bernaulli's Rule

$$a_1 = \frac{1}{2\pi}\left\{(x)\left[\frac{-\cos 2x}{2}\right]-(1)\left[\frac{-\sin 2x}{4}\right]\right\}_0^{2\pi} = \frac{1}{2\pi}\left\{\frac{-x\cos 2x}{2}+\frac{\sin 2x}{4}\right\}_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left\{\left[\frac{-2\pi\cos(4\pi)}{2}+\frac{\sin(4\pi)}{4}\right]-\left[-0+\frac{\sin 0}{0}\right]\right\}$$

but $\cos(4\pi) = \cos(4\times 180) = 1$

$\sin(4\pi) = \sin(4\times 180) = 0$

$\sin 0 = 0$

$$\therefore \quad a_1 = \frac{1}{2\pi}\left\{\frac{-2\pi}{2}\right\}$$

$$\therefore \quad a_1 = \frac{-1}{2}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin(nx)\, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin(nx)\sin x\, dx$$

By using, $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(nx - x) - \cos(nx + x)]\, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \underset{\underset{u}{\downarrow}}{x}\ \underbrace{[\cos(n-1)x - \cos(n+1)x]}_{v}\, dx$$

∴ By Bernaulli's Rule

$$b_n = \frac{1}{2\pi}\left\{(x)\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right] - (1)\left[\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right\}_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left\{\frac{x\cdot\sin(n-1)x}{(n-1)} - \frac{x\cdot\sin(n+1)x}{n+1} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right\}_0^{2\pi}$$

but $\sin(n-1)2\pi = 0$

and $\sin(n+1)2\pi = 0$

$$b_n = \frac{1}{2\pi}\left\{\left[0-0+\frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos(n+1)2\pi}{(n+1)^2}\right] - \left[0-0+\frac{\cos 0}{(n-1)^2} - \frac{\cos 0}{(n+1)^2}\right]\right\}$$

but $\cos(n\pm 1)2\pi = 1$

$$b_n = \frac{1}{2\pi}\left\{\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right\}$$

$$b_n = \frac{1}{2\pi}\{0\}$$

$$\mathbf{b_n = 0 \qquad n > 1}$$

Again, $$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cdot\sin(nx)\,dx$$

Put n = 1

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cdot\sin^2 x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cdot\left(\frac{1-\cos 2x}{2}\right)dx$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} \underset{u}{\underset{\downarrow}{x}}\ \underbrace{(1-\cos 2x)}_{v}\,dx$$

∴ By Bernaulli's Rule

$$b_1 = \frac{1}{2\pi}\left\{(x)\left[x - \frac{\sin 2x}{2}\right] - (1)\left[\frac{x^2}{2} + \frac{\cos 2x}{4}\right]\right\}_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left\{x^2 - \frac{x\sin 2x}{2} - \frac{x^2}{2} - \frac{\cos 2x}{4}\right\}_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left\{\left[4\pi^2 - \frac{2\pi\cdot\sin(4\pi)}{2} - \frac{4\pi^2}{2} - \frac{\cos(4\pi)}{4}\right] - \left[0-0-0-\frac{\cos 0}{4}\right]\right\}$$

But $\sin(4\pi) = 0$; $\cos(4\pi) = 1$; $\cos 0 = 1$

$$b_1 = \frac{1}{2\pi}\left\{4\pi^2 - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right\}$$

$$b_1 = \frac{1}{2\pi}\left\{\frac{4\pi^2}{2}\right\}$$

$$\mathbf{b_1 = \pi}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Note this step :

$$f(x) = \frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \sum_{n=2}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$x \sin x = \frac{1}{2}(-2) + \left[-\frac{1}{2}\cos x + \pi \sin x\right] + \sum_{n=2}^{\infty}\left[\frac{2}{n^2-1}\cos nx + 0 \sin nx\right]$$

$$x \sin x = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty}\left[\frac{2\cos nx}{n^2-1}\right]$$

Example 4.4.6

Determine the fourier series for the function $f(x) = \sqrt{1-\cos x}$ in the interval $0 \le x \le 2\pi$ and hence deduce that $\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$

Solution :

Given : $f(x) = \sqrt{1-\cos x}$ $\quad 0 \le x \le 2\pi$

$$f(x) = \sqrt{2\sin^2\left(\frac{x}{2}\right)} = \sqrt{2}\sin\left(\frac{x}{2}\right)$$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)dx = \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^{2\pi}$$

$$a_0 = \frac{2\sqrt{2}}{\pi}\left\{\left[-\cos\left(\frac{2\pi}{2}\right)\right] - [-\cos(0)]\right\}$$

But $\cos(\pi) = \cos(180°) = -1$ and $\cos 0 = 1$

$$a_0 = \frac{2\sqrt{2}}{\pi}(1+1) = \frac{4\sqrt{2}}{\pi}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)\cos(nx)\,dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos(nx)\sin\left(\frac{x}{2}\right)dx$$

by $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$\therefore 2\cos nx \cdot \sin\frac{x}{2} = \sin\left(nx + \frac{x}{2}\right) - \sin\left(nx - \frac{x}{2}\right)$$

$$2\cos(nx)\sin\left(\frac{x}{2}\right) = \sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x$$

$$a_n = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right]dx$$

But $\int \sin ax\, dx = -\frac{\cos ax}{a}$

$$\therefore \quad a_n = \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos\left(\frac{2n+1}{2}\right)x}{\left(\frac{2n+1}{2}\right)} + \frac{\cos\left(\frac{2n-1}{2}\right)x}{\left(\frac{2n-1}{2}\right)}\right]_0^{2\pi}$$

$$a_n = \frac{\sqrt{2}}{2\pi}\left\{\left[\frac{-\cos(2n+1)\pi}{\left(\frac{2n+1}{2}\right)} + \frac{\cos(2n+1)\pi}{\left(\frac{2n-1}{2}\right)}\right] - \left[\frac{-\cos 0}{\left(\frac{2n+1}{2}\right)} + \frac{\cos 0}{\left(\frac{2n-1}{2}\right)}\right]\right\}$$

but $\cos(2n+1)\pi = \cos(2n\pi + \pi) = \cos 2n\pi \cdot \cos\pi - \sin 2n\pi \cdot \sin\pi = (1)(-1) - (0)(0)$

$$\cos(2n+1)\pi = -1$$

Similarly, $\cos(2n-1)\pi = -1$

$$a_n = \frac{\sqrt{2}}{2\pi}\left\{\frac{1}{\frac{2n+1}{2}} - \frac{1}{\frac{2n-1}{2}} + \frac{1}{\frac{2n+1}{2}} - \frac{1}{\frac{2n-1}{2}}\right\}$$

but $\frac{a}{b/c} = \frac{ac}{b}$

$$a_n = \frac{\sqrt{2}}{2\pi}\left\{\frac{2}{2n+1} - \frac{2}{2n-1} + \frac{2}{2n+1} - \frac{2}{2n-1}\right\}$$

$$a_n = \frac{\sqrt{2}}{2\pi}\left\{\frac{4}{2n+1} - \frac{4}{2n-1}\right\} = \frac{2\sqrt{2}}{\pi}\left\{\frac{1}{2n+1} - \frac{1}{2n-1}\right\}$$

$$a_n = \frac{2\sqrt{2}}{\pi}\left\{\frac{2n-1-2n-1}{(2n+1)(2n-1)}\right\} = \frac{2\sqrt{2}}{\pi}\left[\frac{-2}{4n^2-1}\right]$$

$$a_n = -\frac{4\sqrt{2}}{\pi}\cdot\frac{1}{(4n^2-1)}$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)\sin(nx)\,dx$$

$$b_n = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin(nx)\sin\left(\frac{x}{2}\right)dx$$

but $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$$\therefore \quad 2\sin(nx)\sin\left(\frac{x}{2}\right) = \cos\left(nx - \frac{x}{2}\right) - \cos\left(nx + \frac{x}{2}\right)$$

$$2\sin(nx)\cdot\sin\left(\frac{x}{2}\right) = \cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x$$

$$\therefore \quad b_n = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x\right]dx$$

$$\text{but } \int \cos ax\,dx = \frac{\sin ax}{a}$$

$$b_n = \frac{\sqrt{2}}{2\pi}\left\{\frac{\sin\left(\frac{2n-1}{2}\right)x}{\left(\frac{2n-1}{2}\right)} - \frac{\sin\left(\frac{2n+1}{2}\right)x}{\left(\frac{2n+1}{2}\right)}\right\}_0^{2\pi}$$

$$b_n = \frac{\sqrt{2}}{2\pi}\left\{\left[\frac{\sin(2n-1)\pi}{\left(\frac{2n-1}{2}\right)} - \frac{\sin(2n+1)\pi}{\left(\frac{2n+1}{2}\right)}\right] - \left[\frac{\sin 0}{\left(\frac{2n-1}{2}\right)} - \frac{\sin 0}{\left(\frac{2n+1}{2}\right)}\right]\right\}$$

but $\sin(2n-1)\pi = 0$ and $\sin(2n+1)\pi = 0$; $\sin 0 = 0$

$$b_n = 0$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

$$\therefore \quad \sqrt{1-\cos x} = \frac{1}{2}\frac{4\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\left[\frac{-4\sqrt{2}}{\pi(4n^2-1)}\cos(nx) + 0\cdot\sin(nx)\right]$$

$$\therefore \quad \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}\cos(nx)}{\pi(4n^2-1)} \qquad \text{...(A)}$$

Step 5 : Deduction :

Put $x = 0$ in Equation (A)

$$\sqrt{1-\cos 0} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}\cos(0)}{\pi(4n^2-1)}$$

But $\cos 0 = 1$

$$\therefore \quad 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\cdot\frac{1}{4n^2-1}$$

$$\frac{-2\sqrt{2}}{\pi} = \frac{-4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{\frac{-2\sqrt{2}}{\pi}}{\frac{-4\sqrt{2}}{\pi}} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

...Hence proved.

Example 4.4.7

Find the fourier expansion of the function defined in one period by the relations : $f(x) = \begin{cases} 1 & 0 < x < \pi \\ 2 & \pi < x < 2\pi \end{cases}$

Solution :

$$\text{Given : } f(x) = \begin{cases} 1 & 0 < x < \pi \\ 2 & \pi < x < 2\pi \end{cases}$$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\} = \frac{1}{\pi}\left\{\int_0^{\pi}(1)\,dx + \int_{\pi}^{2\pi}(2)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{(x)_0^{\pi} + (2x)_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\left[(\pi - 0) + (4\pi - 2\pi)\right] = \frac{1}{\pi}(\pi + 2\pi)$$

$$a_0 = 3$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos(nx)\,dx + \int_{\pi}^{2\pi} f(x)\cos(nx)\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi}(1)\cos(nx)\,dx + \int_{\pi}^{2\pi}(2)\cos(nx)\,dx\right\} = \frac{1}{\pi}\left\{\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{2\sin(nx)}{n}\right]_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right] - \left[\frac{2\sin(n2\pi)}{n} - \frac{2\sin(\pi)}{n}\right]\right\}$$

But $\sin n\pi = 0$; $\sin 0 = 0$; $\sin(2n\pi) = 0$

$$\therefore \quad a_n = 0$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin(nx)\,dx + \int_{\pi}^{2\pi} f(x)\sin(nx)\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi}(1)\sin(nx)\,dx + \int_{\pi}^{2\pi}(2)\sin(nx)\,dx\right\} = \frac{1}{\pi}\left\{\left[\frac{-\cos nx}{n}\right]_0^{\pi} + \left[\frac{-2\cos(nx)}{n}\right]_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-\cos n\pi}{n} + \frac{\cos 0}{n}\right] + \left[\frac{-2\cos(2n\pi)}{n} + \frac{2\cos(n\pi)}{n}\right]\right\}$$

but $\cos 0 = 1$; $\cos(2n\pi) = 1$

$$b_n = \frac{1}{\pi}\left\{\frac{-\cos n\pi}{n} + \frac{1}{n} - \frac{2}{n} + \frac{2\cos(n\pi)}{n}\right\}$$

$$b_n = \frac{1}{n\pi}\left[\cos(n\pi) - 1\right]$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}(\cos n\pi - 1)\sin(nx)$$

Example 4.4.8

What is the fourier expansion of the periodic function whose definition in one period is

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$$

Solution :

Given : $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\,dx + \int_{\pi}^{2\pi}(x-\pi)\,dx\right\} = \frac{1}{\pi}\left\{(-\pi x)_0^{\pi} + \left(\frac{x^2}{2} - \pi x\right)_{\pi}^{2\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{(-\pi^2 + 0) + \left[\left(\frac{4\pi^2}{2} - 2\pi^2\right) - \left(\frac{\pi^2}{2} - \pi^2\right)\right]\right\}$$

$$a_0 = \frac{1}{\pi}\left\{-\pi^2 + 0 + 0 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left\{\frac{-2\pi^2 + \pi^2}{2}\right\}$$

$$a_0 = \frac{-\pi}{2}$$

Step 2 : To find a_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos(nx)\,dx + \int_{\pi}^{2\pi} f(x)\cos(nx)\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\cos(nx)\,dx + \int_{\pi}^{2\pi} \underset{\substack{\downarrow \\ u}}{(x-\pi)}\ \underset{\substack{\downarrow \\ v}}{\cos(nx)}\,dx\right\}$$

∴ By Bernaulli's rule,

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin(nx)}{n}\right]_0^{\pi} + \left\{(x-\pi)\left[\frac{\sin nx}{n}\right] - (1)\left[\frac{-\cos(nx)}{n^2}\right]\right\}_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin(n\pi)}{n} + \frac{\pi\sin 0}{n}\right] + \left\{\frac{(x-\pi)\sin nx}{n} + \frac{\cos(nx)}{n^2}\right\}_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{[0+0] + \left\{\left[\frac{(2\pi-\pi)\sin(2n\pi)}{n} + \frac{\cos(2n\pi)}{n^2}\right] - \left[\frac{(\pi-\pi)\sin n\pi}{n} + \frac{\cos(n\pi)}{n^2}\right]\right\}\right\}$$

But $\sin 0 = 0$; $\sin 2n\pi = 0$; $\cos(0) = 1$; $\cos(2n\pi) = 1$; $\cos n\pi = \cos n\pi$

$$a_n = \frac{1}{\pi}\left\{\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right\}$$

$$a_n = \frac{1}{n^2\pi}(1 - \cos n\pi)$$

Step 3 : To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin(nx)\,dx + \int_{\pi}^{2\pi} f(x)\sin(nx)\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} -\pi\sin(nx)\,dx + \int_{\pi}^{2\pi} \underbrace{(x-\pi)}_{u}\,\underbrace{\sin(nx)}_{v}\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{\pi\cos(nx)}{n}\right]_0^{\pi} + \left\{(x-\pi)\left[\frac{-\cos(nx)}{n}\right] - (1)\left[\frac{-\sin(nx)}{n^2}\right]\right\}_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{\pi\cos(nx)}{n} - \frac{\pi\cos 0}{n}\right] + \left\{\frac{-(x-\pi)\cos nx}{n} + \frac{\sin(nx)}{n^2}\right\}_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{\pi\cos n\pi}{n} - \frac{\pi}{n}\right] + \left\{\left[-\frac{(2\pi-\pi)\cos n2\pi}{n} + \frac{\sin(2n\pi)}{n^2}\right] - \left[-\frac{(\pi-\pi)\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]\right\}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} - \frac{\pi}{n} - \frac{\pi}{n} + 0 + 0 - 0\right\} = \frac{1}{n\pi}\,\pi(\cos n\pi - 2)$$

$$b_n = \frac{1}{n}(\cos n\pi - 2)$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n\sin nx\right]$$

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1-\cos n\pi)\cos nx + \frac{1}{n}(\cos n\pi - 2)\sin nx\right]$$

Exercise 4.1

Find the fourier expansion for the following functions in the interval $0 \le x \le 2\pi$.

1. $f(x) = x$ **Ans.:** $x = \pi - \sum_{n=1}^{\infty}\frac{2}{n}$

2. $f(x) = \frac{1}{2}(\pi - x)$ **Ans.:** $\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty}\frac{1}{n}$

3. $f(x) = e^x$ **Ans.:** $e^x = \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\cos nx - \frac{n}{(1+n^2)}\right.$

4. $f(x) = \begin{cases}\sin x & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi\end{cases}$

   Deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{1}{2}$ **Ans.:** $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx + \frac{1}{2}$

5. $f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$ and $f(x + 2\pi) = f(x)$ **Ans.:** $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$

6. $f(x) = \begin{cases} mx & 0 < x < \pi \\ -mx + 2m\pi & \pi < x < 2\pi \end{cases}$ and $f(x + 2\pi) = f(x)$

Prove that $f(x) = \frac{m\pi}{2} - \frac{4m}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$

7. $f(x) = \begin{cases} a, & 0 < x < \pi \\ -a, & \pi < x < 2\pi \end{cases}$ and $f(x + 2\pi) = f(x)$ **Ans.:** $f(x) = \frac{4a}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin(2n+1)x$

8. $f(x) = \begin{cases} \frac{x}{a} & 0 < x < a \\ \frac{\pi - x}{\pi - a} & a < x < 2\pi - a \\ \frac{2\pi - x}{a} & 2\pi - a < x < 2\pi \end{cases}$ and $f(x + 2\pi) = f(x)$

Show that for this function $a_n = 0$, $b_n = \frac{2 \sin na}{(\pi - a) n^2}$

9. If $f(x) = \frac{(3x^2 - 6x\pi + 2\pi^2)}{12}$ then, prove that $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and hence show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$

10. $f(x) = \cos \alpha x$. Deduce that, $\pi \cot 2\pi\alpha = \frac{1}{2\alpha} + \alpha \sum_{n=1}^{\infty} \frac{1}{\alpha^2 - n^2}$

11. Prove that, $\frac{1}{12} x(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$

12. $i = \begin{cases} I_0 \sin x & 0 \le x \le \pi \\ 0 & \pi < x < 2\pi \end{cases}$ **Ans.:** $i = \frac{I_0}{\pi}\left[1 - 2\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)} \cos 2nx + \frac{\pi}{2} \sin x\right]$

4.5 Type 2 : Interval $-\pi \le x \le \pi$

Whenever a function is defined in the interval $-a \le x \le a$ we need to check if the function is

(i) even or

(ii) odd or

(iii) neither even nor odd.

To identify above cases, we replace x by – x

and if we get,

(i) $f(x) = f(-x)$, **function is even**

(ii) $f(x) = -f(-x)$, **function is odd**

(iii) $f(x) \ne f(-x)$ **function is neither even nor odd**

e.g. (i) $f(x) = x^2$

Put $x = -x$

$f(-x) = (-x)^2$

$f(-x) = x^2$

$f(-x) = f(x)$

$\therefore$ Function is even

(ii) $f(x) = x^3$

Put $x = -x$

$f(-x) = (-x)^3$

$f(-x) = -x^3$

$f(-x) = -f(x)$

$\therefore$ Function is odd.

(iii) $f(x) = e^x$

Put $x = -x$

$f(-x) = e^{-x}$

$f(-x) \neq f(x)$

$\therefore$ Function is neither even nor

For the Interval $-\pi \leq x \leq \pi$, Fourier series can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

For Interval $-\pi \leq x \leq \pi$, apply formulae accordingly as shown in Table 4.5.1

Table 4.5.1

Even functions	Odd functions	Neither even nor odd
$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	$a_0 = 0$	$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$
$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$	$a_n = 0$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)$
$b_n = 0$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$	and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$

4.5.1 Solved Examples on Fourier Series Expansions of the functions in the interval $-\pi \leq x$

Example 4.5.1 **May 2017, De**

Find the fourier series expansion of the function $f(x) = x$ in the interval $-\pi \leq x \leq \pi$

Solution : To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd :

As the given function is defined in $(-\pi, +\pi)$ interval, we will check if the given function is even or odd.

Let, $f(x) = x$ Put $x = -x$

$f(-x) = -x$ $f(-x) = -f(x)$

$\therefore$ Function is odd.

Step 2 : To find a_0 and a_n

As the function is odd

$$a_0 = 0 \text{ and } a_n = 0$$

Step 3 : To find b_n

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{\underset{\downarrow}{x}}\ \underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$$

∴ By Bernaulli's rule,

$$b_n = \frac{2}{\pi}\left\{(x)\left[-\frac{\cos(nx)}{n}\right]-(1)\left[\frac{-\sin nx}{n^2}\right]\right\}_0^{\pi} = \frac{2}{\pi}\left\{\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right\}_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n}+\frac{\sin n\pi}{n^2}\right]-\left[-0+\frac{\sin 0}{n}\right]\right\}$$

but $\sin n\pi = 0$; $\sin 0 = 0$

Note : $\cos n\pi \neq 1$

$$\therefore \quad b_n = \frac{2}{\pi}\left\{\frac{-\pi\cos n\pi}{n}\right\}$$

$$b_n = \frac{-2\cos n\pi}{n}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \cdot \sin nx]$$

but $a_0 = 0$ and $a_n = 0$

$$x = \sum_{n=1}^{\infty}\left[-\frac{2\cos n\pi}{n}\cdot \sin nx\right]$$

Example 4.5.2 **Dec. 2009, 2012, May 2016, 2013**

Find the fourier series to represent the function $f(x) = \pi^2 - x^2$ in the interval $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$. Hence deduce that,

(i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$ *(ii)* $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$

Solution :

To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = \pi^2 - x^2$$

Put $\quad x = -x$

$$f(-x) = \pi^2-(-x)^2 = \pi^2 - x^2 = f(x)$$

∴ Function is even

Step 2 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\,dx = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^\pi$$

$$a_0 = \frac{2}{\pi}\left\{\left[\pi^3 - \frac{\pi^3}{3}\right] - \left[0 - \frac{0}{3}\right]\right\} = \frac{2}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$\mathbf{a_0 = \frac{4\pi^2}{3}}$$

Step 3 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \underset{u}{\underset{\downarrow}{(\pi^2 - x^2)}}\ \underset{v}{\underset{\downarrow}{\cos(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule

$$a_n = \frac{2}{\pi}\left\{(\pi^2 - x^2)\left[\frac{\sin nx}{n}\right] - (0 - 2x)\left[\frac{-\cos nx}{n^2}\right] + (-2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^\pi$$

$$a_n = \frac{2}{\pi}\left\{\frac{(\pi^2 - x^2)\sin nx}{n} - \frac{2x\cdot\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right\}_0^\pi = \frac{2}{\pi}\left\{\left[0 - \frac{2\pi\cos n\pi}{n^2} + 0\right] - \left[0 - \frac{2(0)\cos 0}{n^2} + \right.\right.$$

$$a_n = \frac{2}{\pi}\left\{\frac{-2\pi\cos n\pi}{n^2}\right\}$$

$$\mathbf{a_n = \frac{-4\cos n\pi}{n^2}}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

as f (x) is even, $b_n = 0$

$$\therefore\ \pi^2 - x^2 = \frac{1}{2}\left(\frac{4\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{-4\cos n\pi}{n^2}\cdot\cos nx + 0\cdot\sin nx\right]$$

$$\therefore\ \boldsymbol{\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cos n\pi\cdot\cos nx}{n^2}}$$

Step 5 : Deductions :

(i) Put $x = 0$ in Equation (A)

$$\pi^2 - 0^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4\cos n\pi\cdot\cos 0}{n^2}$$

$$\pi^2 - \frac{2\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$$

but $\cos n\pi = (-1)^n$

$$\frac{\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \frac{(-1)^1}{1^2}+\frac{(-1)^2}{2^2}+\frac{(-1)^3}{3^2}+\frac{(-1)^4}{4^2}+\frac{(-1)^5}{5^2}+$$

but $(-1)^{even} = 1$ and $(-1)^{odd} = -1$

$$\frac{-\pi^2}{12} = -\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+$$

Multiplying by – sign on both sides,

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\ldots = \frac{\pi^2}{12}$$

(ii) Put $x = \pi$ in Equation (A)

$$\pi^2-\pi^2 = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\frac{-4\cos n\pi\cdot\cos n\pi}{n^2}$$

$$0 = \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\frac{-4[\cos n\pi]^2}{n^2}$$

but $\cos n\pi = (-1)^n$; $(\cos n\pi)^2 = [(-1)^n]^2 = (-1)^{2n} = (1)^n$; $(\cos n\pi)^2 = 1$

$$\frac{-2\pi^2}{3} = -4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{-2\pi^2}{(3)(-4)} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots = \frac{\pi^2}{6} \quad (2)$$

Adding Equation (1) and Equation (2), we get

$$2\left(\frac{1}{1^2}\right)+2\left(\frac{1}{3^2}\right)+2\left(\frac{1}{5^2}\right)+\ldots\ldots = \frac{\pi^2}{12}+\frac{\pi^2}{6}$$

$$2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots = \frac{\pi^2}{8}$$

Example 4.5.3 May 2016

Find the fourier series expansion of the function $f(x) = x^3$ in the interval $-\pi \le x \le \pi$.

Solution :

Given : $f(x) = x^3$; $-\pi \le x \le \pi$

To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = x^3$$

Put $x = -x$

$$f(-x) = (-x)^3$$

$$f(-x) = -x^3$$

$$f(-x) = -f(x)$$

$\therefore$ Function is odd

Step 2 : To find a_0 and a_n

As the function is odd

$$a_0 = 0 \text{ and } a_n = 0$$

Step 3 : To find b_n

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{\underset{\downarrow}{x^3}}\ \underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule

$$b_n = \frac{2}{\pi}\left\{(x^3)\left[\frac{-\cos nx}{n}\right]-(3x^2)\left[\frac{-\sin nx}{n^2}\right]+(6x)\left[\frac{\cos nx}{n^3}\right]-(6)\left[\frac{\sin nx}{n^4}\right]\right\}_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-x^3\cos n\pi}{n}+\frac{3x^2\sin nx}{n^2}+\frac{6x\cos nx}{n^3}-\frac{6\sin nx}{n^4}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi^3\cos n\pi}{n}+0+\frac{6\pi\cos n\pi}{n^3}-0\right]-[-0+0+0-0]\right\}$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^3\cos n\pi}{n}+\frac{6\pi\cos n\pi}{n^3}\right]$$

$$b_n = \frac{2}{n}\left(-\pi^2+\frac{6}{n^2}\right)\cos n\pi$$

Step 4 : Fourier series expansion

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]$$

$$x^3 = \sum_{n=1}^{\infty}\left[\frac{2}{n}\left(-\pi^2+\frac{6}{n^2}\right)\cos n\pi\cdot\sin nx\right]$$

Example 4.5.4

Find the fourier series expansion of the function : $f(x) = x^2$ in the interval $-\pi \le x \le \pi$

Solution

Given $f(x) = x^2 \quad -\pi \le x \le \pi$

To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = x^2$$

put $\quad x = -x$

$$f(-x) = (-x)^2$$

$$f(-x) = x^2$$

Function is even.

Step 2 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right]$$

$$\mathbf{a_0 = \frac{2\pi^2}{3}}$$

Step 3 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \underset{\substack{\downarrow \\ u}}{x^2}\ \underset{\substack{\downarrow \\ v}}{\cos(nx)}\,dx$$

$$a_n = \frac{2}{\pi}\left\{(x^2)\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi\cos n\pi}{n^2} - 0\right] - [0 + 0 - 0]\right\}$$

$$\mathbf{a_n = \frac{4\cos n\pi}{n^2}}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin x]$$

Since $f(x)$ is even, $b_n = 0$

$$x^2 = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\frac{4\cos n\pi}{n^2}\cos nx + 0$$

$$\mathbf{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos n\pi}{n^2}\cdot\cos nx}$$

Example 4.5.5

Obtain the fourier expansion for function : $f(x) = \begin{cases} \pi + x & \text{if } -\pi \le x \le 0 \\ \pi - x & \text{if } 0 \le x \le \pi \end{cases}$ *and* $f(x + 2\pi) = f(x)$

Solution :

Given : $f(x) = \begin{cases} \pi + x & -\pi \le x \le 0 \\ \pi - x & 0 \le x \le \pi \end{cases}$

To obtain the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = \begin{cases} \pi + x & -\pi \le x \le 0 \\ \pi - x & 0 \le x \le \pi \end{cases}$$

Put $x = -x$

$$f(-x) = \begin{cases} \pi - x & -\pi \le -x \le 0 \\ \pi + x & 0 \le -x \le \pi \end{cases}$$

Multiply by – sign to the interval.

$$f(-x) = \begin{cases} \pi - x & \pi \ge x \ge 0 \\ \pi + x & 0 \ge x \ge -\pi \end{cases}$$

$f(-x) = f(x)$ $\therefore$ **Function is even.**

Step 2 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx$$

Note : $f(x) = \pi - x$ as limits of integration are 0 to π

$$a_0 = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$\mathbf{a_0 = \pi}$$

Step 3 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underset{\underset{u}{\downarrow}}{(\pi - x)}\ \underset{\underset{v}{\downarrow}}{\cos nx}\,dx$$

$\therefore$ By Bernauli's Rule

$$a_n = \frac{2}{\pi}\left\{(\pi - x)\left[\frac{\sin nx}{n}\right] - (-1)\left[\frac{-\cos nx}{n^2}\right]\right\}_0^{\pi} = \frac{2}{\pi}\left\{\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 - \frac{\cos n\pi}{n^2}\right] - \left[0 - \frac{\cos 0}{n^2}\right]\right\} = \frac{2}{\pi}\left\{\frac{-\cos n\pi + 1}{n^2}\right\}$$

$$\mathbf{a_n = \frac{2}{n^2\pi}(1 - \cos n\pi)}$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

As $f(x)$ is even, $b_n = 0$

$$\mathbf{f(x) = \frac{1}{2}(\pi) + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(1 - \cos n\pi)\cos nx}$$

Example 4.5.6

Find the fourier series expansion for periodic function $f(x)$, if, $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Solution :

Given : $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Put $x = -x$

$$f(-x) = \begin{cases} -\pi & -\pi < -x < 0 \\ -x & 0 < -x < \pi \end{cases}$$

$$f(x) \neq f(-x)$$

$\therefore$ Function is neither even nor odd.

Step 2 : To find a_0 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\} = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right\} = \frac{1}{\pi}\left\{[-\pi x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{[-\pi(0) + (\pi)(-\pi)] + \left[\frac{\pi^2}{2} - 0\right]\right\} = \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left\{-\frac{\pi^2}{2}\right\}$$

$$a_0 = -\frac{\pi}{2}$$

Step 3 : To find a_n :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} \underset{u}{\underset{\downarrow}{x}}\ \underset{v}{\underset{\downarrow}{\cos nx}}\,dx\right\}$$

$$\therefore \quad a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin nx}{n}\right]_{-\pi}^{0} + \left\{(x)\left[\frac{\sin nx}{n}\right] - (1)\left[\frac{-\cos nx}{n^2}\right]\right\}_{0}^{\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[-0 + \frac{\pi\sin(-n\pi)}{n}\right] + \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

But $\sin(-n\pi) = -\sin(n\pi) = 0$

$$a_n = \frac{1}{\pi}\left\{[-0+0] + \left\{\left[0 + \frac{\cos n\pi}{n^2}\right] - \left[0 + \frac{\cos 0}{n^2}\right]\right\}\right\} = \frac{1}{\pi}\left\{\frac{\cos n\pi - 1}{n^2}\right\}$$

$$a_n = \frac{(\cos n\pi - 1)}{n^2\pi}$$

Step 4 : To Find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} \underset{u}{\underset{\downarrow}{x}}\ \underset{v}{\underset{\downarrow}{\sin nx}}\,dx\right\}$$

$$\therefore \quad b_n = \frac{1}{\pi}\left\{\left[\frac{\pi\cos nx}{n}\right]_{-\pi}^{0} + \left\{(x)\left[\frac{-\cos nx}{n}\right] - (1)\left[\frac{-\sin nx}{n^2}\right]\right\}_0^{\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{\pi\cos 0}{n} - \frac{\pi\cos(-n\pi)}{n}\right] + \left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

But $\cos(-n\pi) = \cos(n\pi)$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{\pi}{n} - \frac{\pi\cos(n\pi)}{n}\right] + \left\{\left[\frac{-\pi\cos n\pi}{n} + 0\right] - [0+0]\right\}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi}{n} - \frac{\pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right\}$$

$$\mathbf{b_n = \frac{1}{n}\{1 - 2\cos n\pi\}}$$

Step 5 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

$$\mathbf{f(x) = \frac{1}{2}\left(-\frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(\cos n\pi - 1)\cos nx + \frac{1}{n}(1 - 2\cos n\pi)\sin nx\right]}$$

Example 4.5.7

Find the fourier series to represent : $f(x) = e^{ax}$ in the interval $-\pi < x < \pi$

Solution :

To find the fourier series of the given function, follow the steps given below.

Step 1 : Check for even / odd

$$f(x) = e^{ax}$$

$$\text{Put } x = -x$$

$$f(-x) = e^{-ax}$$

$$f(-x) \neq f(x)$$

$\therefore$ Given function is neither even nor odd.

Step 2 : To find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{1}{\pi}\left\{\frac{e^{ax}}{a}\right\}_{-\pi}^{\pi} = \frac{1}{\pi}\left\{\frac{e^{a\pi}}{a} - \frac{e^{-a\pi}}{a}\right\}$$

$$a_0 = \frac{2}{a\pi}\left\{\frac{e^{a\pi} - e^{-a\pi}}{2}\right\}$$

but $\frac{e^{\theta} - e^{-\theta}}{2} = \sinh\theta$

$$\therefore \quad \mathbf{a_0 = \frac{2}{a\pi}\sinh(a\pi)}$$

Step 3 : To find a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$\text{by } \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Here, a = a and b = n

$$a_n = \frac{1}{\pi}\left\{\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right\}_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{a\pi}}{a^2+n^2}(a\cos n\pi + 0)\right] - \left[\frac{e^{-a\pi}}{a^2+n^2}(a\cos n\pi + 0)\right]\right\}$$

$\because \cos(-\theta) = \cos\theta$

$$\therefore \quad a_n = \frac{1}{\pi}\frac{1}{a^2+n^2}\left[e^{a\pi}a\cos n\pi - e^{-a\pi}a\cos n\pi\right]$$

$$a_n = \frac{1}{\pi}\frac{2a\cos n\pi}{(a^2+n^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$\mathbf{a_n = \frac{2a\cos n\pi}{\pi(a^2+n^2)}\sinh(a\pi)}$$

Step 4 : To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin(nx)\,dx$$

$$\text{by } \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

Here, a = a and b = n

$$b_n = \frac{1}{\pi}\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{a\pi}}{a^2+n^2}(0 - n\cos n\pi)\right] - \left[\frac{e^{-a\pi}}{a^2+n^2}(0 - n\cos n\pi)\right]\right\}$$

$$b_n = \frac{1}{\pi}\,\frac{2\,(-n\cos n\pi)}{a^2+n^2}\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right)$$

$$b_n = \frac{-2n\cos n\pi}{\pi\,(a^2+n^2)}\sinh(a\pi)$$

Step 5 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

$$e^{ax} = \frac{\sinh(a\pi)}{\pi a} + \sum_{n=1}^{\infty}\left[\frac{2a\cos(n\pi)\sinh(a\pi)}{\pi(a^2+n^2)}\cos nx + \frac{(-2)\,n\cos(n\pi)\sinh(a\pi)}{\pi(a^2+n^2)}\sin nx\right]$$

Example 4.5.8

Find the fourier series of the function $f(x) = x+\frac{x^2}{4}$ when $-\pi<x<\pi$ and $f(x+2\pi) = f(x)$.

Hence show that, $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots = \frac{\pi^2}{12}$

Solution :

Given : $f(x) = x+\frac{x^2}{4}$

$$f(x) = f_1(x)+f_2(x)$$

$\therefore$ $f_1(x) = x$

and $f_2(x) = \frac{x^2}{4}$

To find the fourier series of the given function, follow the steps given below.

Part 1 : Fourier coefficients for the function : $f_1(x) = x$

$f_1(x) = x \qquad -\pi<x<\pi$

Put $x = -x$

$f(-x) = -x$

$f(-x) = -f(x)$

$\therefore$ Function is odd

$\therefore$ $a_0 = 0$ and $a_n = 0$

and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\underset{u}{\underset{\downarrow}{x}}\;\underset{v}{\underset{\downarrow}{\sin(nx)}}\,dx$

$$b_n = \frac{2}{\pi}\left\{(x)\left[\frac{-\cos nx}{n}\right]-(1)\left[\frac{-\sin nx}{n^2}\right]\right\}_0^{\pi} = \frac{2}{\pi}\left\{\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right\}_0^{\pi} = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n}+0\right]-\right.$$

$$b_n = \frac{-2\cos n\pi}{n}$$

Part 2 : Fourier coefficients for the function :

$$f(x) = \frac{x^2}{4} \qquad -\pi < x < \pi$$

Put $x = -x$

$$f(-x) = \frac{(-x^2)}{4} = \frac{x^2}{4} = f(x)$$

∴ Function is even

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{x^2}{4}\,dx = \frac{2}{\pi}\left[\frac{x^3}{12}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{12}\right]$$

$$\mathbf{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{x^2}{4}\cos(nx)\,dx$$

$$a_n = \frac{1}{2\pi}\int_0^{\pi} \underset{u}{x^2}\ \underset{v}{\cos(nx)}\,dx = \frac{1}{2\pi}\left\{(x^2)\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right\}_0^{\pi} = \frac{1}{2\pi}\left\{\left[0 + \frac{2\pi\cos n\pi}{n^2} - 0\right] - [0+0-0]\right\}$$

$$\mathbf{a_n = \frac{\cos n\pi}{n^2}}$$

∴ Fourier series representation is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

$$\mathbf{x + \frac{x^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left[\frac{\cos n\pi}{n^2}\cdot\cos nx - \frac{2\cos n\pi}{n}\sin nx\right]}$$

Exercise 4.2

Find the fourier expansion for the following functions in the interval $-\pi \le x \le \pi$.

1. $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$ and $f(x+2\pi) = f(x)$

Ans.: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx$

2. $f(x) = |x|$

Ans.: $|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$

3. $f(x) = |\sin x|$

Ans.: $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \ldots\right)$

4. $f(x) = \begin{cases} -\frac{1}{2} & -\pi < x < 0 \\ \frac{1}{2} & 0 < x < \pi \end{cases}$ **Ans.:** $f(x) = \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \ldots\right.$

5. Prove that, $\sin ax = \frac{\sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} \ldots\right)$

6. Prove that, $\cos ax = \frac{2a\sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2 - a^2}\cos nx\right]$

7. Prove that, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n^2 - 1)}\sin nx$

8. Prove that, $x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n^2 - 1)}\cos nx$

Deduce that, $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \ldots\ldots\ldots = \frac{1}{4}(\pi - 2)$

9. $f(x) = \sqrt{1 - \cos x}$ **Ans.:** $\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2 - 1}\cos$

10. $f(x) = x - x^2$ deduce that, $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

Ans.: $x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin$

11. $f(x) = x + x^2$. Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$

Ans.: $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 c}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin$

12. $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$ **Ans.:** $a_n = -\frac{1 + \cos n\pi}{(n^2 - 1)\pi}$, $b_n = 0$, $a_1 = 0$, b_1

13. $f(x) = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$ and $f(x + 2\pi) = f(x)$ **Ans.:** $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n +$

14. $f(x) = \begin{cases} \pi + x & -\pi \le x \le -\frac{\pi}{2} \\ \frac{\pi}{2} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$ **Ans.:** $f(x) = \frac{3\pi}{8} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{3n\pi}{4}\sin\frac{n\pi}{4}}{n^2}\cos$

15. $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$

Ans.: $\frac{\pi^2}{12} - \frac{x^2}{4} = \cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x \ldots$

16. Prove that, $\frac{1}{2}(\pi - x)\sin x - \frac{1}{2} + \frac{1}{4}\cos x - \left(\frac{1}{1\cdot 3}\cos 2x + \frac{1}{2\cdot 4}\cos 3x + \ldots\right)$

17. $f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$

Ans.: $f(x) = \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) + \frac{2}{\pi}\left[\sum_{n=1}^{\infty} \frac{1}{(1-4r^2)}\cos 2rx + \frac{2}{(1-4r^2)}\sin 2rx\right]$

18. $f(x) = \begin{cases} \cos x & ; -\pi < x < 0 \\ -\cos x & ; 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$

Ans.: $f(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{2n}{(2n)^2 - 1}\sin 2nx$

19. $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

Ans.: $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

20. Prove that, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2 + a^2)}\cos nx\right]$

21. Prove that, $\sinh ax = \frac{2}{\pi}\sinh a\pi\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cdot n}{(n^2 + a^2)}\sin nx\right]$

4.6 Type 3 : Interval : $0 \le x \le 2L$

The functions considered so far had a period of 2π. In most of the engineering applications, the period of function to be expanded is not always 2π, but it has some other arbitrary interval, say 2L.

Expansion of the function f (x) in the interval $0 \le \pi \le 2L$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Where, $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$	$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{2\pi x}{L}\right)dx$

4.6.1 Solved Examples on Fourier Series Expansion of Functions in the interval $0 \le x \le 2L$

Example 4.6.1

Find the fourier series expansion of the function $f(x) = x^2$, $0 \le x \le 3$ and period is 3.

Solution :

To find the fourier series of the given function, follow the steps given below.

Step 1 : Given interval is, $0 \le x \le 3$,

Comparing with $0 \le x \le 2L$

$$2L = 3$$

$$\therefore \quad L = \frac{3}{2}$$

Step 2 : To find a_0

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{3/2}\int_0^3 x^2\,dx = \frac{2}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[\frac{3^3}{3} - \frac{0^3}{3}\right]$$

$$a_0 = \frac{2}{3}[9] = 6$$

Step 3 : To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3/2}\int_0^3 x^2\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$a_n = \frac{2}{3}\int_0^3 \underset{u}{\underset{\downarrow}{x^2}}\ \underset{v}{\underset{\downarrow}{\cos\left(\frac{2n\pi x}{3}\right)}}\ dx$$

∴ By Bernaulli's Rule

$$a_n = \frac{2}{3}$$

$$a_n = \frac{2}{3}\left\{(x^2)\left[\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right] - (2x)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right] + (2)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}_0^3$$

$$a_n = \frac{2}{3}\left\{\frac{x^2\cdot\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} + \frac{2x\cdot\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$a_n = \frac{2}{3}\left\{\left[\frac{(3)^2\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)} + \frac{2(3)\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3}\right] - \left[0 + 0 - \frac{2\pi\sin(0)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}$$

But $\cos 2n\pi = 1$; $\sin 2n\pi = 0$; $\sin 0 = 0$

$$a_n = \frac{2}{3}\left\{\left[0 + \frac{6}{\left(\frac{2n\pi}{3}\right)^2} - 0\right] - [0 + 0 - 0]\right\} = \frac{2}{3}\cdot\frac{6}{\frac{4n^2\pi^2}{9}}$$

$$\therefore \quad a_n = \frac{9}{n^2\pi^2}$$

Step 4 : To find b_n :

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3/2}\int_0^3 x^2\sin\left(\frac{n\pi x}{3/2}\right)dx = \frac{2}{3}\int_0^3 \underset{u}{\underset{\downarrow}{x^2}}\ \underset{v}{\underset{\downarrow}{\sin\left(\frac{2n\pi x}{3}\right)}}dx$$

∴ By Bernaulli's Rule

$$b_n = \frac{2}{3}\left\{(x^2)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right] - (2x)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right] + (2)\left[\frac{\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\frac{-x^2 \cdot \cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} + \frac{2x \cdot \sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\left[\frac{-(3)^2 \cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)} + \frac{2(3)\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3}\right] - \left[0 + 0 + \frac{2\cos(0)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}$$

But $\cos 2n\pi = 1$; $\sin 2n\pi = 0$; $\cos 0 = 1$

$$b_n = \frac{2}{3}\left\{\frac{-9}{\frac{2n\pi}{3}} + 0 + \frac{2}{\left(\frac{2n\pi}{3}\right)^3} - \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\} = \frac{2}{3}\left[\frac{-9(3)}{2n\pi}\right]$$

$$b_n = \frac{-9}{n\pi}$$

Step 5 : Fourier Series expansion :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$x^2 = 3 + \sum_{n=1}^{\infty}\left[\frac{9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{9}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Example 4.6.2

Find the fourier series expansion of the function : $f(x) = 2x - x^2$; $0 \le x \le 3$, and period is 3.

Solution :

To find the fourier series of the given function, follow the steps given below.

Step 1 : Given interval is $0 \le x \le 3$

Comparing with $0 \le x \le 2L$

$$\therefore \quad 2L = 3 \Rightarrow L = \frac{3}{2}$$

Step 2 : To find a_0

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{3/2}\int_0^3 (2x - x^2)\,dx$$

$$a_0 = \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left\{\left[(3)^2 - \frac{(3)^3}{3}\right] - \left[(0)^2 - \frac{(0)^3}{3}\right]\right\}$$

$$a_0 = \frac{2}{3}(0 - 0)$$

$$a_0 = 0$$

Step 3 : To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3/2}\int_0^3 (2x-x^2)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$a_n = \frac{2}{3}\int_0^3 (\underset{\underset{u}{\downarrow}}{2x-x^2})\ \underset{\underset{v}{\downarrow}}{\cos}\left(\frac{2n\pi x}{3}\right)dx$$

$\therefore$ By Bernaulli's Rule,

$$a_n = \frac{2}{3}\left\{(2x-x^2)\left[\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right]-(2-2x)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right]+(-2)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}_0^3$$

$$a_n = \frac{2}{3}\left\{\frac{(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}+\frac{(2-2x)\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}+\frac{2\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$a_n = \frac{2}{3}\left\{\left[0+\frac{(2-6)\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2}+0\right]-\left[0+\frac{(2-0)\cos 0}{\left(\frac{2n\pi}{3}\right)^2}+0\right]\right\}$$

But $\cos 2n\pi = 1$; $\cos 0 = 1$

$$a_n = \frac{2}{3}\left\{\frac{(-4)}{\left(\frac{2n\pi}{3}\right)^2}-\frac{2}{\left(\frac{2n\pi}{3}\right)^2}\right\} = \frac{2}{3}\left\{\frac{-6}{\frac{4n^2\pi^2}{9}}\right\}$$

$$a_n = \frac{-9}{n^2\pi^2}$$

Step 4 : To find b_n

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{n\pi x}{3/2}\right)dx = \frac{2}{3}\int_0^3 (\underset{\underset{u}{\downarrow}}{2x-x^2})\ \underset{\underset{v}{\downarrow}}{\sin}\left(\frac{2n\pi x}{3}\right)dx$$

$\therefore$ By Bernaulli's Rule,

$$b_n = \frac{2}{3}\left\{(2x-x^2)\left[\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right]-(2-2x)\left[\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right]+(-2)\left[\frac{+\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\frac{-(2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}+\frac{(2-2x)\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}-\frac{2\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$b_n = \frac{2}{3}\left\{\left[\frac{-(6-9)\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)} + 0 - \frac{2\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3}\right] - \left[0 + 0 - \frac{2\cos 0}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}$$

$$b_n = \frac{2}{3}\left\{\frac{3}{\frac{2n\pi}{3}} - \frac{2}{\left(\frac{2n\pi}{3}\right)^3} + \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$b_n = \frac{3}{n\pi}$$

Step 5 : Fourier series representation :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$2x - x^2 = 0 + \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Example 4.6.3

Find the fourier series expansion of the function $f(x) = 4 - x^2$ in the interval $0 < x < 2$.

Solution : To find the fourier series of the given function, follow the steps given below.

Step 1 : Given interval is, $0 < x < 2$.

Comparing with, $0 < x < 2L$.

$$\therefore \quad 2L = 2 \Rightarrow L = 1$$

Step 2 : To find a_0

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{1}\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$a_0 = \left[4(2) - \frac{(2)^3}{3}\right] - \left[4(0) - \frac{(0)^2}{2}\right]$$

$$a_0 = \frac{16}{3}$$

Step 3 : To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 \underset{u}{\underset{\downarrow}{(4 - x^2)}}\ \underset{v}{\underset{\downarrow}{\cos}}\left(\frac{n\pi x}{1}\right)dx$$

$\therefore$ By Bernaulli's Rule,

$$a_n = \left\{(4 - x^2)\left[\frac{\sin(n\pi x)}{n\pi}\right] - (-2x)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{-\sin(n\pi x)}{(n\pi)^3}\right]\right\}_0^2$$

$$a_n = \left\{\frac{(4-x^2)\sin(n\pi x)}{n\pi} - \frac{2x\cos(n\pi x)}{(n\pi)^2} + \frac{2\sin(n\pi x)}{(n\pi)^3}\right\}_0^2$$

$$a_n = \left\{\left[0 - \frac{2(2)\cos(2n\pi)}{(n\pi)^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{-4}{n^2\pi^2}$$

Step 4 : To find b_n

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 (4-x^2)\sin\left(\frac{n\pi x}{1}\right)dx = \int_0^2 \underset{u}{\underset{\downarrow}{(4-x^2)}}\ \underset{v}{\underset{\downarrow}{\sin(n\pi x)}}\,dx$$

$\therefore$ By Bernaulli's Rule.

$$b_n = \left\{(4-x^2)\left[\frac{-\cos(n\pi x)}{(n\pi)}\right] - (-2x)\left[\frac{-\sin(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{\cos(n\pi x)}{(n\pi)^3}\right]\right\}_0^2$$

$$b_n = \left\{\frac{-(4-x^2)\cos(n\pi x)}{n\pi} - \frac{2x\sin(n\pi x)}{(n\pi)^2} - \frac{2\cos(n\pi x)}{(n\pi)^3}\right\}_0^2$$

$$b_n = \left\{\left[\frac{-(4-2^2)\cos(2n\pi)}{(n\pi)} - 0 - \frac{2\cos(2n\pi)}{(n\pi)^3}\right] - \left[\frac{-(4-0^2)\cos(0)}{(n\pi)} - 0 - \frac{2\cos 0}{(n\pi)^3}\right]\right\}$$

But $\cos 2n\pi = 1$; $\cos 0 = 1$

$$b_n = \left\{\left[0 - 0 - \frac{2}{n^3\pi^3} + \frac{4}{n\pi} + \frac{2}{n^3\pi^3}\right]\right\}$$

$$b_n = \frac{4}{n\pi}$$

Step 5 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$4 - x^2 = \frac{8}{3} + \sum_{n=1}^{\infty}\left[\frac{-4}{n^2\pi^2}\cos(n\pi x) + \frac{4}{n\pi}\sin(n\pi x)\right]$$

Example 4.6.4

Find the fourier series expansion of the function : $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ 2(\pi - x) & 1 \le x \le 2 \end{cases}$ in the interval $0 \le x \le 2$ ... period 2.

Solution : To find the fourier series of the given function, follow the steps given below.

Step 1 : Given interval is, $0 \le x \le 2$.

Comparing with, $0 \le x \le 2L$.

$\therefore$ $2L = 2 \Rightarrow L = 1$

Step 2 : To find a_0

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi\left[\frac{1^2}{2} - \frac{0^2}{2}\right] + \pi\left\{\left[2(2) - \frac{2^2}{2}\right] - \left[2(1) - \frac{(1)^2}{2}\right]\right\}$$

$$a_0 = \frac{\pi}{2} + \pi\left\{2 - \frac{3}{2}\right\} = \frac{\pi}{2} + \frac{\pi}{2}$$

$$\mathbf{a_0 = \pi}$$

Step 3 : To find a_n

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_0^1 f(x)\cos(n\pi x)\,dx + \int_1^2 f(x)\cos(n\pi x)\,dx$$

$$a_n = \int_0^1 \underset{u}{\underset{\downarrow}{(\pi x)}}\ \underset{v}{\underset{\downarrow}{\cos(n\pi x)}}\,dx + \int_1^2 \underset{u}{\underset{\downarrow}{(2\pi - \pi x)}}\ \underset{v}{\underset{\downarrow}{\cos(n\pi x)}}\,dx$$

∴ By Bernaulli's Rule,

$$a_n = \left\{(\pi x)\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (\pi)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right]\right\}_0^1 + \left\{(2\pi - \pi x)\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (-\pi)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right]\right\}_1^2$$

$$a_n = \left\{\frac{\pi x \cdot \sin(n\pi x)}{(n\pi)} + \frac{\pi\cos(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)\sin(n\pi x)}{(n\pi)} - \frac{\pi\cos(n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$a_n = \left\{\left[0 + \frac{\pi\cos(n\pi)}{(n\pi)^2}\right] - \left[0 + \frac{\pi\cos(0)}{(n\pi)^2}\right]\right\} + \left\{\left[0 - \frac{\pi\cos(2n\pi)}{(n\pi)^2}\right] - \left[0 - \frac{\pi\cos(n\pi)}{(n\pi)^2}\right]\right\}$$

$$a_n = \frac{\pi\cos(n\pi)}{n^2\pi^2} - \frac{\pi}{n^2\pi^2} - \frac{\pi}{n^2\pi^2} + \frac{\pi\cos(n\pi)}{n^2\pi^2} = \frac{\pi}{n^2\pi^2}[2\cos n\pi - 2]$$

$$\mathbf{a_n = \frac{2}{n^2\pi}(\cos n\pi - 1)}$$

Step 4 : To find b_n :

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$b_n = \int_0^1 f(x)\sin(n\pi x)\,dx + \int_1^2 f(x)\sin(n\pi x)\,dx$$

$$b_n = \int_0^1 \underset{u}{\underset{\downarrow}{(\pi x)}}\ \underset{v}{\underset{\downarrow}{\sin(n\pi x)}}\,dx + \int_1^2 \underset{u}{\underset{\downarrow}{(2\pi - \pi x)}}\ \underset{v}{\underset{\downarrow}{\sin(n\pi x)}}\,dx$$

By Bernoulli's Rule

$$b_n = \left\{(\pi x)\left[\frac{-\cos(n\pi x)}{n\pi}\right] - (\pi)\left[\frac{-\sin(n\pi x)}{(n\pi)^2}\right]\right\}_0^1 + \left\{(2\pi - \pi x)\left[\frac{-\cos(n\pi x)}{n\pi}\right] - (-\pi)\left[\frac{-\sin(n\pi x)}{(n\pi)^2}\right]\right\}_1^2$$

$$b_n = \left\{\frac{-(\pi x)\cos(n\pi x)}{n\pi} + \frac{\pi\sin(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{-(2\pi - \pi x)\cos(n\pi x)}{n\pi} - \frac{\pi\sin(n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$b_n = \left\{\left[\frac{-\pi\cos(n\pi)}{n\pi} + 0\right] - [0 + 0]\right\} + \left\{[-0 + 0] - \left[\frac{-(2\pi - \pi)\cos(n\pi)}{n\pi} - 0\right]\right\}$$

$$b_n = -\frac{\pi\cos n\pi}{n\pi} + \frac{\pi\cos(n\pi)}{n\pi}$$

$$b_n = 0$$

Step 5 : Fourier series representation :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(\cos n\pi - 1)\cos(n\pi x)}$$

Exercise 4.3

Obtain the fourier series expansions of the following functions :

1. $f(x) = 2 - \frac{x^2}{2},\ 0 \le x \le 2$ **Ans.:** $2 - \frac{x^2}{2} = \frac{4}{3} - \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\cos(n\pi x) + \frac{2}{n\pi}$ s

2. $f(x) = \begin{cases} 0, & 0 \le x \le l \\ a, & l \le x \le 2l \end{cases}$ **Ans.:** $f(x) = \frac{a}{2} - \frac{2a}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\right.$

3. $f(x) = \begin{cases} 8, & 0 < x < 2 \\ -8, & 2 < x < 4 \end{cases}$ **Ans.:** $f(x) = \frac{32}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\right.$

4. $f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l \le x \le 2l \end{cases}$

Ans.: $f(x) = \frac{1}{4} + \frac{2l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right) + \frac{1}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right.$

5. $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **Ans.:** $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

6. $f(x) = \begin{cases} t & 0 < t < 1 \\ 1 - t & 1 < t < 2 \end{cases}$ **Ans.:** $f(x) = \frac{-4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi t}{(2n+1)^2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\sin$

7. $f(x) = \begin{cases} 1 + x^2 & 0 \le x \le 1 \\ 3 - x & 1 \le x \le 2 \end{cases}$

Ans.: $f(x) = \frac{17}{2} + \left[\frac{-4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi t}{(2n+1)^2} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi t}{n^2}\right] - \frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n}{(2n}$

8. $f(x) = \frac{1}{2} - x,\ 0 < x < l$, prove that, $\frac{1}{2} - x = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{l}$

9. $f(x) = \begin{cases} 2x & 0 \le x \le 3 \\ 0 & -3 < x < 0 \end{cases}$ **Ans.:** $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{6(\cos n\pi - 1)}{n^2 \pi^2} \cos \frac{n\pi x}{3} - \frac{\cos n\pi}{n\pi} \sin \frac{n\pi x}{3} \right]$

4.7 Type 4 : Interval $-L \le x \le L$

Whenever the function is defined in the the interval from $-L$ to L. We have to check if the given function is

(i) even or

(ii) odd or

(iii) neither even nor odd

and for interval $-L \le x \le L$ apply the formulae accordingly as defined in Table 4.7.1.

Table 4.7.1

Even functions	Odd functions	Neither Even nor odd
$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$	$a_0 = 0$	$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$
$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$	$a_n = 0$	$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$
$b_n = 0$	$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$	$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

4.7.1 Solved Examples on Fourier Series Expansion of the Functions in the Interval $-L \le x \le L$

Example 4.7.1 **May 2018**

Find the fourier series expansion of the function $f(x) = x^2$ in the interval, $-1 \le x \le 1$.

Solution : To find the fourier series of the given function, follow the steps given below.

Step 1 : $f(x) = x^2 \qquad -1 \le x \le 1$

Put $x = -x$

$f(-x) = (-x)^2$

$f(-x) = x^2$

$f(-x) = f(x)$ $\therefore$ Function is even.

Also, given interval is $-1 \le x \le 1$

Comparing with $-L \le x \le L$

$\therefore \quad L = 1$

Step 2 : To find a_0

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx = \frac{2}{1} \int_0^1 x^2\, dx$$

$$a_0 = 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$a_0 = \frac{2}{3}$$

Step 3 : To find a_n

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{1}\int_0^1 x^2\cos\left(\frac{n\pi x}{1}\right)dx = 2\int_0^1 \underset{u}{x^2}\,\underset{v}{\cos(n\pi x)}\,dx$$

By Bernaulli's Rule,

$$a_n = 2\left\{(x^2)\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (2x)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right] + (2)\left[\frac{-\sin(n\pi x)}{(n\pi)^3}\right]\right\}_0^1$$

$$a_n = 2\left\{\frac{x^2\sin(n\pi x)}{n\pi} + \frac{2x\cos(n\pi x)}{(n\pi)^2} - \frac{2\sin(n\pi x)}{(n\pi)^3}\right\}_0^1$$

$$a_n = 2\left\{\left[0 + \frac{2(1)\cos(n\pi)}{(n\pi)^2} - 0\right] - [0+0-0]\right\}$$

$$a_n = \frac{4\cos(n\pi)}{(n\pi)^2}$$

Step 4 : $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$

As function is even ; $b_n = 0$

$$\therefore \quad x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4\cos n\pi}{n^2\pi^2}\cos(n\pi x)$$

Example 4.7.2

Find the fourier series for the function $f(x) = x - x^2$ in the interval $-1 \le x \le 1$.

Solution : To find the fourier series of the given function, follow the steps given below.

Step 1 : $f(x) = x - x^2 \quad -1 \le x \le 1$

Put $x = -x$

$f(-x) = -x - (-x)^2$

$f(-x) = -x - x^2$

$f(-x) \ne f(x)$

$\therefore$ Function is neither even nor odd.

Also, given interval is $-1 \le x \le 1$

Comparing with $-L \le x \le L$

$L = 1$

Step 2 : To find a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\,dx$$

$$a_0 = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-1}^{1} = \left[\frac{(1)^2}{2} - \frac{(1)^3}{3}\right] - \left[\frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right]$$

$$a_0 = \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} = -\frac{2}{3}$$

Step 3 : To find a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$a_n = \int_{-1}^{1}\underset{u}{(x - x^2)}\,\underset{v}{\cos(n\pi x)}\,dx$$

By Bernaulli's Rule,

$$a_n = \left\{(x - x^2)\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (1 - 2x)\left[\frac{-\cos(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{-\sin(n\pi x)}{(n\pi)^3}\right]\right\}_{-1}^{1}$$

$$a_n = \left|\frac{(x - x^2)\sin(n\pi x)}{n\pi} + \frac{(1 - 2x)\cos(n\pi x)}{(n\pi)^2} + \frac{2\sin(n\pi x)}{(n\pi)^3}\right|_{-1}^{1}$$

$$a_n = \left\{\left[\frac{(1 - 1^2)\sin(n\pi)}{n\pi} + \frac{(1 - 2(1))\cos(n\pi)}{(n\pi)^2} + \frac{2\sin(n\pi)}{(n\pi)^3}\right]\right.$$

$$\left. - \left[\frac{((-1) - (-1)^2)\sin(-n\pi)}{n\pi} + \frac{(1 - 2(-1))\cos(n\pi)}{(n\pi)^2} + \frac{2\sin(-n\pi)}{(n\pi)^3}\right]\right\}$$

$\sin n\pi = 0$; $\sin(-n\pi) = -\sin(n\pi) = -0 = 0$

$$a_n = \left\{\left[0 - \frac{\cos n\pi}{n^2\pi^2} + 0\right] - \left[0 + \frac{3\cos(n\pi)}{(n\pi)^2} + 0\right]\right\}$$

$$a_n = \frac{-\cos n\pi}{n^2\pi^2} - \frac{3\cos(n\pi)}{n^2\pi^2}$$

$$a_n = \frac{-4\cos n\pi}{n^2\pi^2}$$

Step 4 : To find b_n

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\sin\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{1}\underset{u}{\underset{\downarrow}{(x - x^2)}}\ \underset{v}{\underset{\downarrow}{\sin(n\pi x)}}\,dx$$

∴ By Bernaulli's Rule,

$$b_n = \left\{(x - x^2)\left[\frac{-\cos(n\pi x)}{(n\pi)}\right] - (1 - 2x)\left[\frac{-\sin(n\pi x)}{(n\pi)^2}\right] + (-2)\left[\frac{\cos(n\pi x)}{(n\pi)^3}\right]\right\}_{-1}^{1}$$

$$b_n = \left|\frac{-(x - x^2)\cos(n\pi x)}{n\pi} + \frac{(1 - 2x)\sin(n\pi x)}{(n\pi)^2} - \frac{2\cos(n\pi x)}{(n\pi)^3}\right|_{-1}^{1}$$

$$b_n = \left\{\left[\frac{-(1 - 1^2)\cos(n\pi)}{n\pi} + 0 - \frac{2\cos(n\pi)}{(n\pi)^3}\right] - \left[\frac{((-1) - (-1)^2)\cos(-n\pi)}{n\pi} + 0 - \frac{2\cos(-n\pi)}{(n\pi)^3}\right]\right\}$$

but $\cos(-n\pi) = \cos n\pi$

$$b_n = \left[\frac{-2\cos n\pi}{n^3\pi^3} - \frac{2\cos(n\pi)}{n\pi} + \frac{2\cos(n\pi)}{n^3\pi^3}\right]$$

$$b_n = \frac{-2\cos(n\pi)}{n\pi}$$

Step 5 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{-4\cos(n\pi)}{n^2\pi^2}\cos(n\pi x) - \frac{2\cos(n\pi)}{n\pi}\sin(n\pi x)\right]$$

Example 4.7.3

Determine the fourier expansion for

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases} \quad ; \text{ period } 4$$

Solution : To determine the fourier series of the given function, follow the steps given below.

Step 1 : As the function is defined in the interval $-2 < x < 2$.

We will first check for even / odd function.

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

Put $x = -x$

$$f(-x) = \begin{cases} 0 & -2 < -x < -1 \\ 1-x & -1 < -x < 0 \\ 1+x & 0 < -x < 1 \\ 0 & 1 < -x < 2 \end{cases}$$

Multiplying by – sign to the interval

$$f(-x) = \begin{cases} 0 & 2 < x < 1 \\ 1-x & -1 < x < 0 \\ 1+x & 0 < x < -1 \\ 0 & -1 < x < -2 \end{cases}$$

$\therefore \quad f(-x) = f(x)$

$\therefore$ Function is even.

Also, given interval is $-2 < x < 2$

Comparing with $-L < x < L$

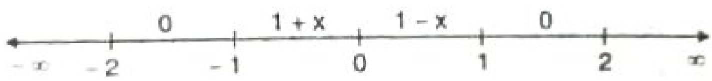
$\therefore \qquad L = 2$

Step 2 : To find a_0

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$$

x-scale of given function is,



But limits of integration are from 0 to 2

$$\underset{0}{\bullet}\xrightarrow{\quad 1-x \quad}\underset{1}{\bullet}\xrightarrow{\quad 0 \quad}\underset{2}{\bullet}\longrightarrow\underset{x}{\bullet}$$

$$\therefore \quad a_0 = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 (1-x)\,dx + 0 = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2}$$

$$a_0 = \frac{1}{2}$$

Step 3 : To find a_n

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \int_0^1 \underset{u}{\underset{\downarrow}{(1-x)}}\ \underset{v}{\underset{\downarrow}{\cos\left(\frac{n\pi x}{2}\right)}}\,dx + 0$$

By Bernaulli's Rule,

$$a_n = \left\{(1-x)\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right] - (-1)\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]\right\}_0^1 = \left\{\frac{(1-x)\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right\}_0^1$$

$$a_n = \left\{\left[\frac{(1-1)\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} - \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right] - \left[\frac{(1-0)\sin 0}{\left(\frac{n\pi}{2}\right)} - \frac{\cos(0)}{\left(\frac{n\pi}{2}\right)^2}\right]\right\}$$

$$a_n = \frac{-\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + \frac{1}{\left(\frac{n\pi}{2}\right)^2}$$

$$a_n = \frac{1}{\left(\frac{n\pi}{2}\right)^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$\therefore \quad a_n = \frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

Step 4 : Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

As the given function is even i.e. $b_n = 0$

$$\therefore \quad f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{2}\right)$$

Exercise 4.4

Obtain the fourier series expansions of the following functions.

1. $f(x) = x^2, \ -a < x < a$ **Ans.:** $x^2 = \frac{a^3}{3} + \frac{a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{a} - \frac{a^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \ldots$

2. $f(x) = x \cos\left(\frac{\pi x}{l}\right), \ -l < x < l$, prove that, $x \cos \frac{\pi x}{l} = -\frac{1}{2\pi} \sin \frac{\pi x}{l} + \frac{2l}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2 - 1} \sin \frac{n\pi x}{l}$

3. $f(x) = e^{-x}, \ (-l, l)$ **Ans.:** $\sinh l \left[\frac{1}{l} + 2l \sum_{n=1}^{\infty} \frac{(-1)^n}{(l^2 + n^2\pi^2)} \cos \frac{n\pi x}{l} + 2\pi \sum_{n=1}^{\infty} \frac{n(-1)^n}{(l^2 + n^2\pi^2)} \sin \ldots\right.$

4. $f(x) = \begin{cases} a, & -l < x < -\frac{l}{3} \\ b, & -l/3 < x < \frac{l}{3} \\ c, & \frac{l}{3} < x < +l \end{cases}$

is given by $f(x) = \frac{a+b+c}{3} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{3} \left[(2b - a - c) \cos \frac{n\pi x}{l} + 2(c - a) \sin \frac{2n\pi}{3} \sin \frac{n\pi x}{l}\right]$

5. $f(x) = \begin{cases} x, & -1 < x \le 0 \\ x + 2, & 0 < x \le 1 \end{cases}$ **Ans.:** $f(x) = 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - 2(-1)^n] \sin \ldots$

6. $f(x) = \begin{cases} 2, & -2 \le x \le 0 \\ x, & 0 < x < 2 \end{cases}$ **Ans.:** $\frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{2(\cos n\pi - 1)}{n^2\pi^2} \cos \frac{n\pi x}{2} - \frac{2}{n\pi} \sin \ldots\right.$

7. $f(x) = \begin{cases} 1 & -1 < x < 0 \\ \cos \pi x & 0 < x < 1 \end{cases}$ **Ans.:** $a_0 = 1, \ a_n = 0, \ b_{2n} = \frac{4n}{\pi(4n^2 - 1)}, \ b_{2n+1} = -\ldots$

8. $f(x) = |x|, \ -2 \le x \le 2$ **Ans.:** $|x| = 1 - \frac{8}{\pi^2} \left[\cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} \ldots\right.$

9. $f(x) = x^2 - 2, \ -2 \le x \le 2$ **Ans.:** $x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2} \left[\cos \frac{\pi x}{2} - \frac{1}{4} \cos \pi x + \frac{1}{9} \cos \frac{3\pi x}{2} \ldots\right.$

10. $f(x) = \begin{cases} e^x & -1 \le x \le 0 \\ e^{-x} & 0 \le x \le 1 \end{cases}$ **Ans.:** $f(x) = \frac{e - 1}{e} + 2 \sum_{n=0}^{\infty} \frac{e - (-1)^n}{e(n^2\pi^2 + 1)} \cos \ldots$

11. $f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$ **Ans.:** $f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n+1)\ldots}{2}}{(2n+1)}$

12. $f(x) = \begin{cases} x + 1 & -1 < x < 0 \\ x - 1 & 0 < x < 1 \end{cases}$ **Ans.:** $f(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \ldots$

13. $f(x) = \begin{cases} -x, & -4 \le x \le 0 \\ x, & 0 \le x \le 4 \end{cases}$ **Ans.:** $f(x) = 2 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)}{n^2} \cos \ldots$

14. $f(x) = 1 - x^2, \ -1 \le x \le 1.$ **Ans.:** $1 - x^2 = \frac{2}{3} + \frac{4}{\pi^2} \left[\cos \pi x - \frac{1}{2^2} \cos 2\pi x + \frac{1}{3^2} \cos 3\pi x - \ldots\right.$

- $f(x) = x - x^3, \quad -1 < x < 1$

  Ans.: $x - x^3 = \frac{12}{\pi^3}\left[\sin \pi x - \frac{1}{2^3}\sin 2\pi x + \frac{1}{3^2}\sin 3\pi x - \ldots\right]$

- $f(x) = \begin{cases} a(x - l) & -l < x < 0 \\ a(l + x) & 0 < x < l \end{cases}$ Deduce that, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots = \frac{\pi}{4}$

  Ans.: $f(x) = \frac{2al}{\pi}\left[\frac{3}{1}\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{2}{3}\sin\frac{3\pi x}{l} - \ldots\right]$

- $f(x) = \begin{cases} 0 & -3 < x < -1 \\ 1 + \cos \pi x & -1 < x < 1 \\ 0 & 1 < x < 3 \end{cases}$

  Ans.: $a_0 = \frac{1}{3}, \; a_n = \frac{-18}{\pi} \cdot \frac{1}{n(n^2 - 9)} \sin\frac{n\pi}{3}$

- $f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

  Ans.: $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \ldots\right]$

- $f(x) = e^{|x|}, \; -2 < x < 2.$

  Ans.: $e^{|x|} = \frac{e^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{4[(-1)^n e^2 - 1]}{4 + n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$

- $f(x) = \begin{cases} 0 & -\frac{T}{2} < t < 0 \\ E\sin \omega t & 0 < t < \frac{T}{2} \end{cases}$

  Ans.: $f(t) = \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)}\cos 2n\omega t + \frac{E}{2}\sin \omega t$

- $f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$

  Ans.: $f(x) = \frac{3}{2} + \frac{6}{\pi}\left[\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \ldots\right]$

- $f(x) = \sin ax, \; -l < x < l$

  Ans.: $\sin ax = 2\pi \sin al\left[\frac{1}{(\pi^2 - a^2 l^2)}\sin\frac{\pi x}{l} - \frac{2}{(2^2\pi^2 - a^2 l^2)}\sin\frac{2\pi x}{l} + \frac{3}{3^2\pi^2 - a^2 l^2}\sin\frac{3\pi x}{l} - \ldots\right]$

4.8 Half Range Expansions

In some problems it is required to obtain a fourier expansion of a function to hold for a range which is half the period of a fourier series i.e. to expand f (x) in the range $(0, \pi)$ in a fourier series of period 2π or more generally in the range (0, L) in a fourier series of period of 2L.

Now, we have divided half range expansions into two types.

1. **Half range Expansion in**
   $0 < x < \pi$ (angular interval)
2. **Half range Expansion in**
   $0 < x < L$ (arbitrary interval)

Half range expansions are again divided into two types within the interval as,

1. Half range cosine expression in, $0 < x < \pi$
2. Half range sine expression in, $0 < x < \pi$
3. Half range cosine expression in, $0 < x < L$
4. Half range sine expression in, $0 < x < L$

4.9 Half Range Expansions in the Interval $0 < x < \pi$

4.9.1 Formulae for Half Range cosine expansion in the interval $0 < x < \pi$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx]$$

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$	$b_n = 0$

[i.e. we can see these formulae are equal to formulae for even function in the interval $-\pi < x < \pi$]

4.9.2 Formulae for Half Range sine expansion in the interval $0 < x < \pi$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,dx$$

Where, $a_0 = 0$	$a_n = 0$	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

[i.e. we can see these formulae are equal to the formulae for odd function in the interval $-\pi < x < \pi$]

4.9.3 Type 5 : Solved Examples on Half range expansions in the interval $0 < x < \pi$

Example 4.9.1

Find the half range cosine series for $f(x) = x^2$ in the interval $0 \le x \le \pi$

Solution : To find the half range cosine series of the given function, follow the steps given below.

Step 1 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

Step 2 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \underset{u}{\underset{\downarrow}{x^2}}\ \underset{v}{\underset{\downarrow}{\cos(nx)}}\,dx$$

$\therefore$ By Bernaulli's Rule,

$$a_n = \frac{2}{\pi}\left\{(x^2)\left[\frac{\sin(nx)}{n}\right] - (2x)\left[\frac{-\cos(nx)}{n^2}\right] + (2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{x^2 \sin nx}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi\cos(n\pi)}{n^2} - 0\right] - [0 + 0 - 0]\right\}$$

$$a_n = \frac{4\cos(n\pi)}{n^2}$$

Step 3 : To find b_n

For half range cosine series $b_n = 0$

Step 4 : Half range cosine expansion :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos(n\pi)}{n^2}\cos(nx)$$

Example 4.9.2 **May 20**

Find the half range cosine series for $f(x) = \pi x - x^2$ in the interval $0 \le x \le \pi$

Solution : To find the half range cosine series of the given function, follow the steps given below.

Step 1 : To find a_0

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$a_0 = \frac{\pi^2}{3}$$

Step 2 : To find a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(\underset{\substack{\downarrow\\ u}}{\pi x - x^2})\,\underset{\substack{\downarrow\\ v}}{\cos(nx)}\,dx$$

$\therefore$ By Bernaulli's Rule,

$$a_n = \frac{2}{\pi}\left\{(\pi x - x^2)\left[\frac{\sin(nx)}{n}\right] - (\pi - 2x)\left[\frac{-\cos(nx)}{n^2}\right] + (-2)\left[\frac{-\sin(nx)}{n^3}\right]\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{(\pi x - x^2)\sin nx}{n} + \frac{(\pi - 2x)\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{(\pi - 2\pi)\cos n\pi}{n^2} + 0\right] - \left[0 + \frac{(\pi - 0)\cos 0}{n^2} + 0\right]\right\}$$

but $\cos 0 = 1$ and $\cos n\pi = \cos n\pi$

$$\therefore \quad a_n = \frac{2}{\pi}\left\{\frac{-\pi\cos n\pi}{n^2} - \frac{\pi}{n^2}\right\}$$

$$a_n = -\frac{2}{n^2}(1 + \cos(n\pi))$$