UNIT IV

CHAPTER-5

CURVE TRACING AND RECTIFICATION OF CURVES

5.1 INTRODUCTION

A mathematical function can be better described or understood by its plot or graphical representation. Till now students are well acquainted with curves like straight line, parabola, hyperbola and ellipse. Standard mathematical equations of these curves and their properties are well known at this stage. Aim of this chapter is to introduce the students, with general principles of curve plotting, so that many more curves with their mathematical equations are well understood. Mathematical studies of various kinds of regions with different boundaries can then be taken-up. Curve tracing principles taken-up in this chapter enable us to know the approximate shape of the curve, without plotting large number of points on the curve. Method to trace a curve depends upon the representation of its equation in cartesian, polar or parametric form. Study of this chapter is based upon these three categories of curves along with certain common principles.

After discussing the methods to trace the curve, measurements of the lengths of the arcs of the curves are taken-up. After covering the formulae to measure the arcs of the curves, large number of problems are solved to demonstrate the use of the formulae and methods to trace the curve. This chapter forms the basis of the chapter describing many more applications of integration.

5.2 BASIC DEFINITIONS

1. **Convex Upwards :** If the portion of the curve on both sides of 'A' lies below the tangent at A, then the curve is *convex upwards* (or *concave downwards).*

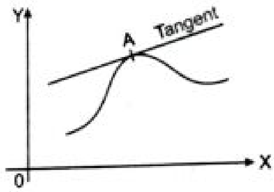


Fig. 5.1

2. **Convex Downwards :** If the portion of the curve on both sides of 'A' lies above the tangent at A, then the curve is *convex downwards* (or *concave upwards).*

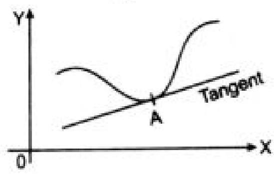


Fig. 5.2

3. **Singular Points :** An unusual point on a curve is called a *singular point* such as, a point of inflexion, a double point, a multiple point, cusp, node or a conjugate point.

4. **Point of Inflexion :** The point that separates the convex part of a continuous curve from the concave part is called the *point of inflexion* of the curve.

   It is obvious that at the point of inflexion the tangent line, if it exists, **cuts** the curve, because on one side the curve lies **under** the tangent and on the other side, **above** it.

   In other words, a point where the curve unusually crosses its tangent is called a *point of inflexion.*

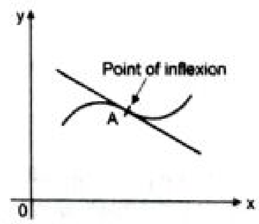


Fig. 5.3

5. **Multiple Point :** A point through which more than one branches of a curve pass is called a *multiple point* of the curve.

6. **A Double Point :** A point on a curve is called a *double point,* if two branches of the curve pass through it.

A *triple point,* if three branches pass through it. If r branches pass through a point, the point is called a *multiple point* of r^{th} order.

7. **Node :** A double point is called *node* if the branches of curve passing through it are real and the tangents at the common point of intersection are real and distinct (not coincident).

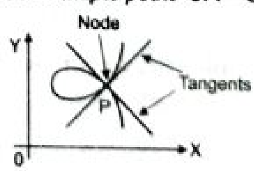


Fig. 5.4

8. **Cusp :** A double point is called *cusp* if the tangents at it to the two branches of the curve are coincident.

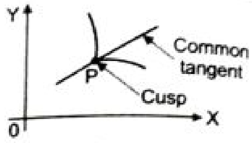


Fig. 5.5

9. **A Conjugate Point :** P is called a *conjugate point* on the curve if there are no real points on the curve in the vicinity of the point P. It is also sometimes called an *isolated point.*

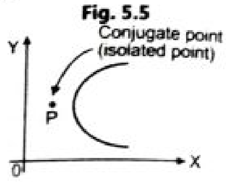


Fig. 5.6

For illustrating methods of tracing, we divide the curves into five types as :

Type 1 : Curves given by Cartesian equations (explicit relations).

Type 2 : Curves given by parametric equations.

Type 3 : Curves given by polar equations.

Type 4 : Curves given by polar equations of the type $r = a \sin n\theta$ or $r = a \cos n\theta$.

Type 5 : Curves given by Cartesian equations (implicit relations).

TYPE 1 : CURVES GIVEN BY CARTESIAN EQUATIONS (EXPLICIT RELATIONS)

5.3 TRACING OF CARTESIAN CURVES

The following rules will help in tracing a Cartesian curve together with the definitions stated in article 5.2.

Rule 1 : Symmetry :

(a) Symmetry About X-Axis : If the equation of the curve is such that the powers of y are even everywhere, then the curve is symmetrical about X-axis.

For example, $y^2 = 4ax$.

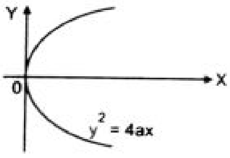


Fig. 5.7

(b) Symmetry About Y-Axis : If the equation of the curve is such that the powers of x are even everywhere, then the curve is symmetrical about Y-axis.

For example, $x^2 = 4ay$

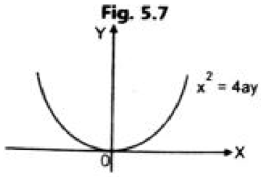


Fig. 5.8

(c) Symmetry About Both X and Y Axes : If the equation of the curve is such that the powers of x and y both are even everywhere, then the curve is symmetrical about both the axes.

For example, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

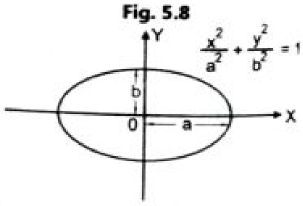


Fig. 5.9

(d) Symmetry in Opposite Quadrants : A curve is symmetrical in opposite quadrants if its equation remains unchanged if x and y are changed to – x and – y simultaneously.

For example, $y = x^3$

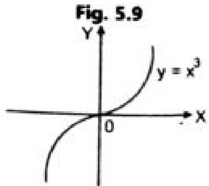


Fig. 5.10

(e) Symmetry About the Line y = x : If the equation of the curve remains unaltered when x is changed to y and y to x, the curve is said to be symmetrical about the line $y = x$.

For example, $xy = c^2$

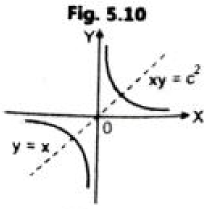


Fig. 5.11

(f) **Symmetry About the Line $y = -x$:** If the equation of the curve remains unaltered when x is changed to $-y$ and y to $-x$, the curve is said to be symmetrical about the line $y = -x$.

Rule 2 : Points of Intersection :

(a) **Origin :** Find out whether the curve passes through the origin. It will pass through the origin if the equation is satisfied by (0, 0). In other words, if the equation of the curve does not contain any absolute constant, then it passes through the origin.

(b) **Tangents at the Origin :** To investigate the nature of a multiple point, it is necessary to find the tangent or tangents at that point.

Newton's Method : If a curve is given by a rational integral algebraic equation and passes through the origin; **the equation of the tangent or tangents at the origin,** can be obtained by equating to zero, the lowest degree terms taken together in the equation of the curve.

(c) **Intersections with the Co-ordinate Axes :** If possible, try to express the given equation in the explicit form say $y = f(x)$ or $x = f(y)$.

To find the intersection with X-axis, put $y = 0$.

To find the intersection with Y-axis, put $x = 0$.

Find the tangents at these points, if necessary and the position of the curve relative to these lines.

(d) If a curve is symmetrical about the line $y = x$ or $y = -x$, find the points of intersection of the curve with these lines and also the tangents at that point because *the tangent leads the curve.*

Rule 3 : Asymptotes :

Asymptotes are the tangents to the curve at infinity. Find the asymptotes and the position of the curve with respect to them.

Asymptotes Parallel to Co-ordinate Axes :

(a) To find the equation of the *asymptote parallel to X-axis*, equate to zero the coefficient of highest degree terms in x.

(b) To find the equation of the *asymptote parallel to Y-axis*, equate to zero the coefficient of highest degree terms in y.

Oblique Asymptotes : Asymptotes which are not parallel to co-ordinate axes are called as *oblique asymptotes.*

(c) **Method 1 :** Let $y = mx + c$ be the asymptote. The points of intersection with the curve $f(x, y) = 0$ are given by $f(x, mx + c) = 0$. Equate to zero the coefficients of two successive highest powers of x, giving equations to determine m and c.

(d) **Method 2 :**

(i) Let $y = mx + c$ be the equation of the asymptote.

(ii) Find ϕ_n (m) by putting $x = 1$ and $y = m$ in the highest degree (n) terms of the equation.

(iii) Similarly, find $\phi_{n-1}(m)$.

(iv) Solve $\phi_n(m) = 0$ to determine m.

(v) Find 'c' by the formula $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$.

(vi) If the roots of m are equal, then find c by $\dfrac{c^2}{2}\phi''(m) + c\,\phi''(m) + \phi_{n-2}(m) = 0$.

(For details see solved example 1, Type 5)

(e) Find more finite points where asymptote meets any other branch of the curve anywhere.

Rule 4 : Special points on the curve :

(a) Find out such points on the curve whose presence can be readily detected.

(b) Find $\dfrac{dy}{dx}$ and the points where the tangent is parallel to either axes.

(c) If $\left(\dfrac{dy}{dx}\right)_{P(x_1, y_1)} = 0$ then the tangent at $P(x_1, y_1)$ is parallel to X-axis.

(d) If $\left(\dfrac{dy}{dx}\right)_{P(x_1, y_1)} = \infty$ then the tangent at $P(x_1, y_1)$ is parallel to Y-axis.

(e) Also find point of inflexion, if any, double or multiple points, node, cusp, conjugate or (isolated) point (see basic definitions, refer article 5.2).

Rule 5 : Region of absence of the curve :

(a) If possible, express the equation in the explicit form say $y = f(x)$ and examine how y varies as x varies continuously.

(b) For $y = f(x)$, if y becomes imaginary for some value of $x > a$ (say), then no part of the curve exists beyond $x = a$.

(c) For $x = f(y)$, if x becomes imaginary for some value of $y > b$ (say), then no part of the curve exists beyond $y = b$.

Some Useful Remarks :

(a) When we have to solve for $y = f(x)$, put $x = 0$ and see what is y. Also, observe how y varies as x increases from 0 to $+\infty$, paying special attention to the values of y for which $y = 0$ or $y \to +\infty$.

Also, observe how y behaves as x becomes negative and $x \to -\infty$; being more careful for y becoming zero or tending to $-\infty$.

(b) If $y \to \infty$ as $x \to a$ then $x = a$ must be an asymptote parallel to Y-axis.

If $x \to \infty$ as $y \to b$ then $y = b$ must be an asymptote parallel to X-axis.

(c) If we observe that $y \to \infty$ as $x \to \infty$ and there is approximately a linear relation between x and y for larger values of x, we may expect an oblique asymptote.

(d) If the curve is symmetrical about X-axis or in the opposite quadrants then only positive value of y may be considered. We may draw the curve for negative values of y by symmetry.

If the curve is symmetrical about Y-axis, we need not consider negative values of x.

At this time, we may observe the existence of a loop, which is the part of the curve.

Note : If there are two points of the curve on any axis and curve ceases to exist on either side of the two points, then there is *always a loop between the two points.*

(e) If necessary, we change the equation to polar form by using polar transformations $x = r\cos\theta$, $y = r\sin\theta$ to get the equation $r = f(\theta)$.

(f) Sometimes we change the equation to parametric form.

(g) Use only as many steps are necessary for tracing the curve.

(h) If possible prepare a table for certain values of x, y, $\frac{dy}{dx}$ which helps us to draw a rough shape of the curve.

ILLUSTRATIONS ON TYPE 1 : Cartesian Curves (Explicit Relations)

Ex. 1 : *Trace the curve $y^2(2a - x) = x^3$.* **(Nov./Dec. 2019, Dec. 2010, 2007, 2004)**

Sol. : This curve is known as *"The Cissoid of Diocle"*.

The given equation of the curve can be written as $y^2 = \frac{x^3}{2a - x}$. ... (1)

1. Equation of the curve contains only even powers of y, therefore, it is symmetrical about X-axis.
2. Equation does not contain any absolute constant therefore, it passes through the origin.
3. Tangents at the origin are obtained by equating to zero the lowest degree terms in the equation.

   From (1), we have

   $y^2(2a - x) = x^3$ i.e. $2ay^2 - xy^2 - x^3 = 0$

   $\therefore$ $2ay^2 = 0 \Rightarrow y^2 = 0,\ y = 0$ is a double point.

   $\therefore$ X-axis is a tangent at origin.
4. Since the two tangents at origin are coincident, therefore the origin is a *cusp*.

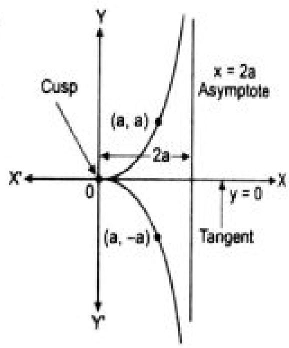


Fig. 5.12

5. For intersection with X-axis, we put $y = 0 \therefore \frac{x^3}{2a - x} = 0 \Rightarrow x = 0$ and for intersection with Y-axis, we put $x = 0$

   $\therefore$ $y^2 = 0 \Rightarrow y = 0$.

   Thus the curve meets the co-ordinate axis only at (0, 0).
6. The asymptote parallel to Y-axis can be obtained by equating to zero the coefficient of highest powers of y.

   $y^2(2a - x) - x^3 = 0$ i.e. $2a - x = 0 \Rightarrow x = 2a$ is the asymptote parallel to Y-axis.
7. From the equation of the curve, we observe that for $x < 0$ and $x > 2a$, y^2 becomes negative, hence y becomes imaginary therefore the curve does not exist for $x < 0$ and $x > 2a$.

   A rough sketch of the curve is as shown above.

Ex. 2 : *Trace the curve $x(x^2 + y^2) = a(x^2 - y^2)$ where $a > 0$.* **(Dec. 2007, 2018, May 2005)**

Sol. : This curve is known as *"strophoid"*. The given equation can be written as

$$y^2 = \frac{x^2(a - x)}{(a + x)} \quad ... (1)$$

1. Equation of the curve contains only even powers of y, therefore, it is symmetrical about X-axis.

2. Equation of the curve does not contain any absolute constant, therefore, it passes through the origin.
3. Tangents at origin are obtained by equating to zero the lowest degree terms in the equation. i.e. $ay^2 = ax^2 \Rightarrow y^2 = x^2$.

   Hence $y = x$, $y = -x$ are tangents at the origin.
4. Since tangents to the curve at origin are $y = x$, $y = -x$ which are real and different, origin is a node.
5. For intersection with X-axis, we put $y = 0$,

   $\therefore$ $x = 0$ or $x = a$.

   For intersection with Y-axis, we put $x = 0$,

   $\therefore$ $y = 0$. Thus the curve meets the co-ordinate axes at (0, 0), (a, 0).

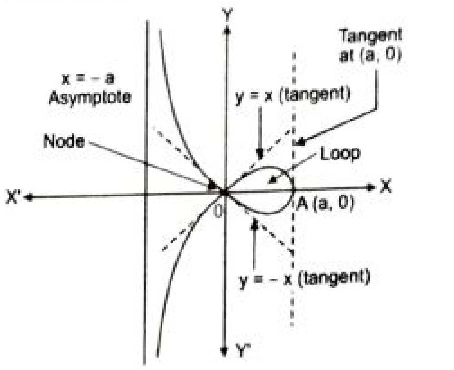


Fig. 5.13

6. The asymptotes parallel to Y-axis can be obtained by equating to zero the coefficient of highest powers of y i.e. $a + x = 0 \Rightarrow x = -a$ is the asymptote parallel to Y-axis.
7. For $x < -a$ and $x > a$, y becomes imaginary, therefore, the curve does not exist for $x < -a$ and $x > a$.
8. Since the curve passes through the origin and no branch of the curve exists to the right of $x = a$, therefore, there exists a loop between (0, 0) and (a, 0).

   A rough sketch of the curve is as shown above.

Ex. 3 :** Trace the curve $xy^2 = a^2(a - x)$.* ***(May 2011, 2010, 2005, 2018; Dec. 2005, 2013)

Sol. : This curve is known as "*Witch of Agnesi*".

1. **Symmetry** about X-axis (powers of y even).
2. **Origin :** Curve does not pass through origin but cuts X-axis at (a, 0). If we transfer the origin to the point (a, 0), the new equation becomes $y^2(x + a) + a^2x = 0$.

   Hence tangent at the new origin is $x = 0$, that is new Y-axis.

   Hence at (a, 0), tangent is parallel to old Y-axis.

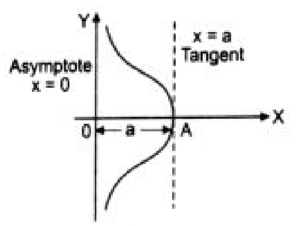


Fig. 5.14

3. **Asymptote** $y^2 = \dfrac{a^2(a - x)}{x}$.

   As $x \to 0$, $y \to \infty$, hence the only asymptote is the line $x = 0$, i.e. Y-axis.
4. **Region of Absence :** If we solve for y, $y = a\sqrt{\dfrac{a - x}{x}}$.

(i) If $x < 0$, y becomes imaginary. Hence no part of curve lies to the left of Y-axis.

(ii) As x increases from $x = 0$ to $x = a$, remaining positive y decreases from $y = \infty$ to $y = 0$ for the upper part. Also if $x > a$, y becomes again imaginary, so curve is absent beyond $x = a$ to the right.

***Ex. 4 :** Trace the curve $x = (y - 1)(y - 2)(y - 3)$.*

Sol. :

1. **Symmetry :** No symmetry at all.
2. **Origin :** It does not pass through (0, 0).
3. It is difficult to solve for y. But it is already solved for x. Here we shall regard y as independent variable and start tracing.
4. **Special Points :** By observation we can easily find that the points (0, 1), (0, 2) and (0, 3) do lie on the curve. All the points lie on Y-axis.
5. Also when $y = 0$, $x = -6$. Hence it passes through $(-6, 0)$.

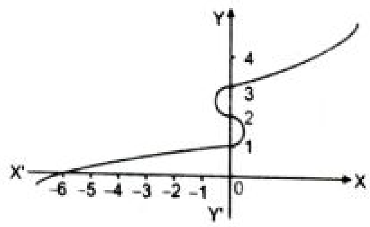


Fig. 5.15

6. When $y = 1$, $x = 0$. Between $y = 0$ and $y = 1$, x is negative.
7. When y lies between 1 and 2, x is positive and x is again zero at $y = 2$.
8. Between $y = 2$ and $y = 3$, x is negative again. Again at $y = 3$, $x = 0$.

   When $y > 3$, x is positive and remains positive continuously. As $y \to \infty$, $x \to \infty$. For very large value of y, x is almost equal to y^3. Hence no linear asymptote for this branch.
9. x is negative when y is negative. As $y \to -\infty$, $x \to -\infty$. No linear asymptote for this branch too.
10. When $y = \dfrac{3}{2}$, $x = \dfrac{3}{8}$ and when $y = \dfrac{5}{2}$, $x = \dfrac{-3}{8}$. Hence the curve is as traced above in the Fig. 5.15.

Ex. 5 : *Trace the curve $x^2 y^2 = a^2 (y^2 - x^2)$.* **(Dec. 2009, 2004, May 20**

Sol. : It is symmetrical about both axes and in the opposite quadrants also. It passes through origin and $y = \pm x$ are tangents at (0, 0). It intersects the co-ordinate axes only at (0, 0).

Given equation can be written as $a^2 y^2 - x^2 y^2 = a^2 x^2$ or $y^2 = \frac{a^2 x^2}{a^2 - x^2}$.

$x = \pm a$ are the asymptotes to the curve. For $x > a$ and $x < -a$, the curve does not exist. A rough sketch of the curve is as shown below.

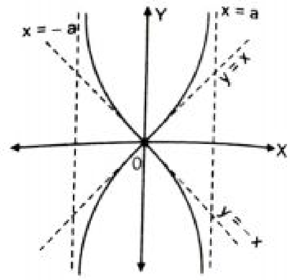


Fig. 5.16

Ex. 6 : *Trace the curve $y^2 (a^2 + x^2) = a^2 x^2$.* **(May 20**

Sol. : It is symmetrical about both the axes and in the opposite quadrants also. It passes through origin and $y = \pm x$ are tangents origin. It intersects the co-ordinate axes only at (0, 0).

To find the equation of asymptote parallel to X-axis, we have $a^2 y^2 = a^2 x^2 - x^2 y^2$

or $x^2 (a^2 - y^2) = a^2 y^2$. $\therefore x^2 = \frac{a^2 y^2}{a^2 - y^2}$. Therefore $y = \pm a$ are the asymptotes to the curve.

For $y > a$ and $y < -a$, $x^2 = \frac{a^2 y^2}{a^2 - y^2}$ becomes negative, hence the curve does not exist for $y > a$, $y < -a$.

A rough sketch of the curve is as shown in the Fig. 5.17.

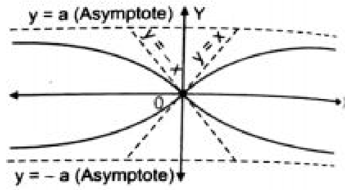


Fig. 5.17

Ex. 7 : *Trace the curve $y (1 + x^2) = x$.* **(May 200**

Sol. : It is symmetrical in the opposite quadrants. It passes through origin and $y = x$ is the tangent at origin. It intersects th co-ordinate axes only at origin $y = 0$ i.e. X-axis is an asymptote to the given curve. Because $y (1 + x^2) - x = 0$. Equate coefficient highest power of x to zero i.e. $y = 0$ is asymptote. The curve does not exist in the second and fourth quadrants.

We have $$y = \frac{x}{1 + x^2}$$

x	0	1	2	3	4	5
y	0	0.5	0.4	0.3	0.24	0.9

From the table it is clear that as x increases, y decreases.

A rough sketch of the curve is as shown in Fig. 5.18.

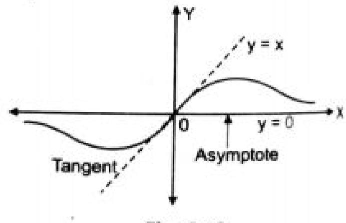


Fig. 5.18

Ex. 8 : *Trace the curve $a^2 y^2 = x^2 (2a - x) (x - a)$.*

Sol. : It is symmetrical about X-axis and passes through origin. Here origin is a conjugate point.

Intersection with X-axis are (0, 0), (2a, 0), (a, 0).

For $x < 0, x > 2a$ curve does not exist.

For $0 < x < a$ curve does not exist.

For $a < x < 2a$ curve exists.

For $x = \frac{3a}{2}$, $y = \pm \frac{3a}{4}$

No asymptote to the curve. A rough sketch of the curve is as shown in the Fig. 5.19.

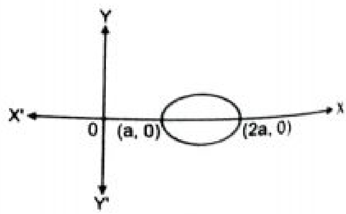


Fig. 5.19

Ex. 9 : *Trace the curve $y (x^2 + 4a^2) = 8a^3$.*

Sol. : It is symmetrical about Y-axis. It does not pass through origin. It intersects Y-axis at (0, 2a).

We have $$x^2 + 4a^2 = \frac{8a^3}{y} \qquad \therefore \qquad 2x \frac{dx}{dy} = -\frac{8a^3}{y^2}$$

$$\frac{dx}{dy} = -\frac{4a^3}{xy^2} \qquad \therefore \qquad \left(\frac{dx}{dy}\right)_{(0,\ 2a)} = -\infty$$

Therefore tangent at (0, 2a) is parallel to X-axis.

$y = 0$ i.e. X-axis is asymptote to the given curve.

For $y < 0$, $y > 2a$, the curve does not exist.

For $y = a$, $x = \pm 2a$, $(2a, a)$ $(-2a, a)$ are the points on the curve.

A rough sketch of the curve is as shown in the Fig. 5.20.

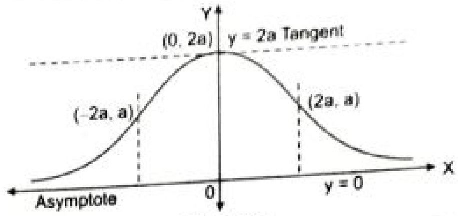


Fig. 5.20

Ex. 10 : *Trace the curve $y^2(a^2 - x^2) = a^3 x$.* **(May 2019, May 2008, 2004)**

Sol. : We have $y^2 = \dfrac{a^3 x}{a^2 - x^2}$

It is symmetrical about X-axis, passes through origin $x = 0$ i.e. Y-axis is tangent at origin. $x = \pm a$ are the asymptotes to the curve. Also $y = 0$ i.e. X-axis is an asymptote to the given curve.

For	$x > a$	the curve does not exist.
For	$x < -a$	the curve exists.
For	$0 < x < a$	the curve exists.
For	$-a < x < 0$	the curve does not exist.

A rough sketch of the curve is as shown below.

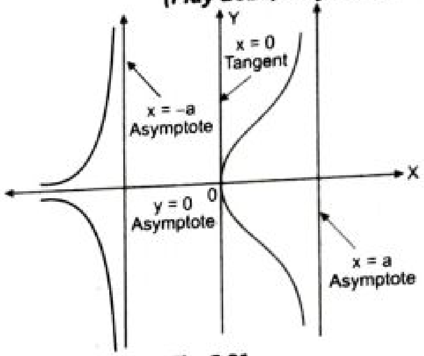


Fig. 5.21

Ex. 11 : *Trace the curve $x^{1/2} + y^{1/2} = a^{1/2}$.* **(May 2007)**

Sol. : It is symmetrical about the line $y = x$. It does not pass through the origin.

It intersects X-axis at (a, 0) and Y-axis at (0, a). It also intersects the line $y = x$ at $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$.

The curve entirely lies in the first quadrant because x and y cannot be negative.

Also, $\dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}y^{-1/2}\dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$

$\therefore$ At (a, 0), $\dfrac{dy}{dx} = 0 \Rightarrow$ tangent at (a, 0) is X-axis itself.

At (0, a), $\dfrac{dy}{dx} = -\infty \Rightarrow$ tangent at (0, a) is Y-axis itself.

A rough sketch of the curve is as shown in the following figure.

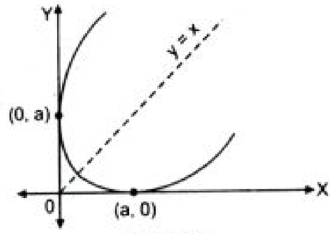


Fig. 5.22

Ex. 12 : *Trace the following curves :*

(i) $x^2(x^2 - 4a^2) = y^2(x^2 - a^2)$ **(May 2008)**

(ii) $a^2x^2 = y^3(2a - y)$ **(May 2009) (Dec. 2011)**

Sol. : (i) $y^2 = \dfrac{x^2(x^2 - 4a^2)}{x^2 - a^2}$

A rough sketch of the curve is as shown in the following figure.

It is symmetrical about both x and y axes and also in the opposite quadrants. It passes through origin and $y = \pm 2x$ are tangents at origin. Intersection with the co-ordinate axes is at (0, 0), (2a, 0), (– 2a, 0). Also $x = \pm a$ are the asymptotes to the given curve.

For $a < x < 2a$, $-2a < x < -a$, the curve does not exist.

For $x > 2a$, $x < -2a$, the curve exists.

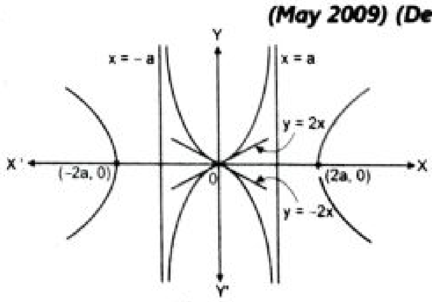


Fig. 5.23

(ii) $a^2x^2 = y^3(2a - y)$

It is symmetrical about Y-axis. It passes through the origin and $x = 0$ is the tangent at (0, 0). It meets Y-axis at (0, 0), (0, 2a) and X-axis at (0, 0).

$$2a^2 \cdot x \cdot \frac{dx}{dy} = 6ay^2 - 4y^3 \qquad \therefore \quad \frac{dx}{dy} = \frac{y^2(3a - 2y)}{a^2 x}$$

$$\left(\frac{dx}{dy}\right)_{(0,\,2a)} = -\infty \Rightarrow \text{tangent at (0, 2a) is parallel to X-axis.}$$

There exists a cusp at (0, 0). When $y = a$, $x = \pm a$. For $y < 0$, $y > 2a$, x^2 becomes negative, therefore, curve does not exist for $y > 2a$ and $y < 0$.

A rough sketch of the curve is as shown in the following figure.

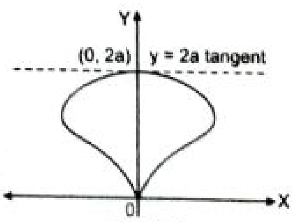


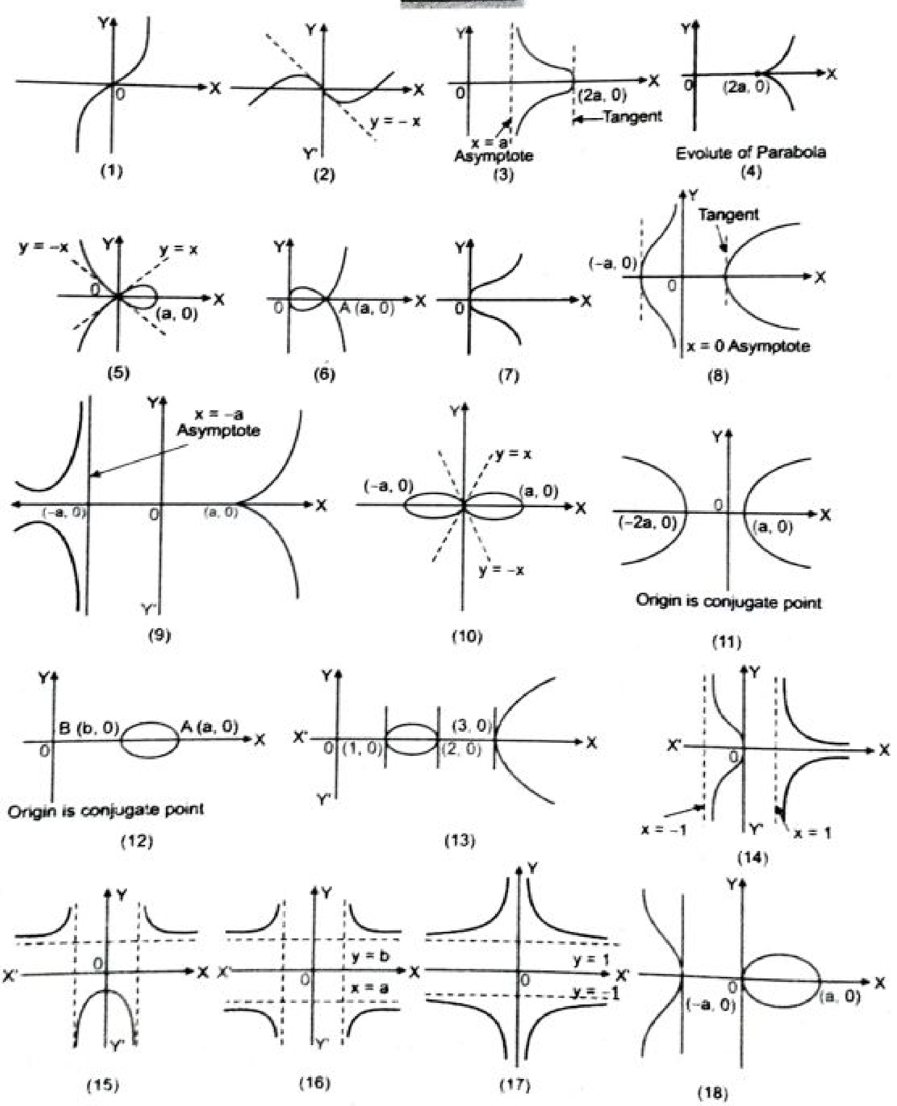
Fig. 5.24

EXERCISE 5.1

Trace the following curves :

1. $y = x^3$
2. $y = x(x^2 - 1)$
3. $y^2(x - a) = x^2(2a - x)$ **(Dec. 2009)**
4. $27ay^2 = 4(x - 2a)^3$ **(Dec. 06, 08)**
5. $ay^2 = x^2(a - x)$
6. $3ay^2 = x(x - a)^2$ **(May 06, 13, Nov. 15)**
7. $ay^2 = x(a^2 + x^2)$
8. $xy^2 = a(x^2 - a^2)$
9. $y^2(a + x) = (x - a)^3$ **(Dec. 2010)**
10. $a^2y^2 = x^2(a^2 - x^2)$
11. $a^2y^2 = x^2(x + 2a)(x - a)$ **(May 2010)**
12. $a^2y^2 = x^2(a - x)(x - b)$ where $b < a$
13. $y^2 = (x - 1)(x - 2)(x - 3)$ **(Dec. 2008, 2006)**
14. $y^2(x^2 - 1) = x$ **(May 2011, 2017)**
15. $y(x^2 - 1) = x^2 + 1$
16. $(x^2 - a^2)(y^2 - b^2) = a^2b^2$
17. $x^2y^2 = x^2 + 1$
18. $ay^2 = x(a^2 - x^2)$ **(Dec. 2011)**
19. $ay^2 = (x - a)(x - 5a)^2$
20. $y^2 = x^5(2a - x)$ **(May 2015)**
21. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$
22. $y^2(4 - x) = x(x - 2)^2$ **(May 2006, 2010)**
23. $y^2(a - x) = x^3$. **(Dec. 2011, 2017)**

ANSWERS



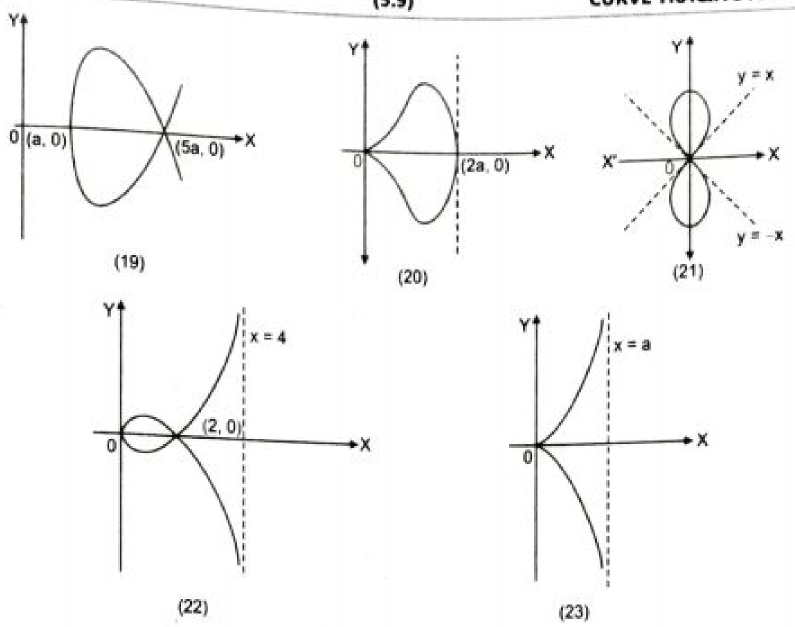


Fig. 5.25

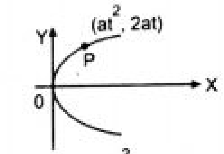
TYPE 2 : CURVES GIVEN BY PARAMETRIC EQUATIONS

5.4 TRACING OF PARAMETRIC CURVES

Rule : (If possible chart to Cartesian and then trace).

1. **Limitations of the Curve :** Let the parametric equation be given by $x = f(t)$, $y = g(t)$. If possible find the greatest and least values of x and y for a proper value of t and therefore the boundary lines parallel to x and y axes between which the curve lies.

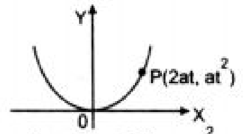
2. **Symmetry :**



Curve : $x = at^2$, $y = 2at$

Fig. 5.26

(a) If f(t) be even function of t and g(t) an odd, the curve is symmetrical to X-axis. As the parabola $x = at^2$, $y = 2at$ is symmetrical about X-axis.



Curve : $x = 2at$, $y = at^2$

Fig. 5.27

(b) If f(t) be odd and g(t) an even function, symmetry about Y-axis. For example, the parabola $x = 2at$, $y = at^2$ has symmetry about Y-axis.

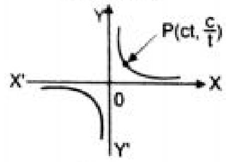


Fig. 5.28

(c) If both f(t) and g(t) are odd, the curve is symmetrical in opposite quadrants.

For example, $x = ct$, $y = \frac{c}{t}$, the rectangular hyperbola.

(d) Also we note that for values of t and –t, x remains unchanged but y has equal and opposite values, therefore, the curve is symmetrical about X-axis. For example, $x = at^2$, $y = 2at$.

(e) Also we note that for values of t and $\pi - t$, y remains unchanged but x has equal and opposite values, hence the curve is symmetrical about Y-axis.

For example, $x = a\cos^3 t$, $y = a\sin^3 t$.

3. **Origin :** If on putting $x = 0$, we obtain $y = 0$ for some value of t, then the curve passes through origin.
Also find the points of intersection of the curve and the axes.
Find asymptotes if any.

4. **Special Points :** Try to find few points on the curve by observation and also those points where $\frac{dy}{dx} = 0$ or ∞. Here $x = f(t)$, $y = g(t)$, hence use the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

5. Region of Absence of Curve :

(a) Find those regions where curve does not exist.

(b) Make a table of values of t, x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

(c) If both x and y are periodic functions of t, with a common period, we need to study the position of the curve for one period only.

6. Cycloid :

When a circle rolls in a plane along given straight line, the locus traced out by a fixed point on the circumference of rolling circle is called as *cycloid*.

There are four types of equations of the curve depending upon choice of axes and position of line along which the circle rolls.

The sketching of the cycloid from its equation depends on the values of x, y and $\frac{dy}{dx}$ at $t = 0$.

ILLUSTRATION ON TYPE 2 : Parametric Curves

Ex. 1 : *Trace the cycloid, $x = a(t + \sin t)$, $y = a(1 - \cos t)$.* ***(May 2011, Dec. 2017)***

Sol. : The curve is known as *cycloid*.

1. **Limitations :** The curve lies between the lines $y = 0$ and $y = 2a$, because the greatest value of y is 2a and least is 0.
2. **Symmetry :** $x = a(t + \sin t)$, being odd function of t and $y = a(1 - \cos t)$ an even function, the curve is symmetrical to Y-axis.
3. **Origin :** When $t = 0$, $x = 0$ and $y = 0$, hence curve passes through the origin. The curve cuts X-axis (putting $y = 0$) when $y = a(1 - \cos t) = 0$, or $t = 0$ and then $x = 0$ i.e. at (0, 0) only. Similarly, it cuts Y-axis also at (0, 0).
4. **No Asymptotes.**
5. **Special points :** We have after differentiation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \tan\frac{t}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 \text{ when } \frac{t}{2} = 0 \text{ or when } t = 0$$

$$\text{and} \quad \frac{dy}{dx} = \infty \text{ at } \tan\frac{t}{2} = \infty \text{ or at } t = \pi$$

Hence at (0, 0), tangent is parallel to X-axis (rather X-axis itself) and at $(a\pi, 2a)$ tangent is parallel to Y-axis.

6. **Region of absence (a) :** Also $y = 2a \sin^2\frac{t}{2}$ or $\sin\frac{t}{2} = \sqrt{y/2a}$, hence when y is negative, t is imaginary, i.e. no part of the curve lies below X-axis (i.e. in 3rd and 4th quadrants).
7. The table of values of t, x, y, $\frac{dy}{dx}$ is as follows.

Table 5.1

t	0	$\pi/2$	π	2π	$-\pi/2$	$-\pi$
x	0	$a(\pi/2 + 1)$	$a\pi$	$2a\pi$	$-a(\pi/2 + 1)$	$-a\pi$
y	0	a	2a	0	a	2a
$\frac{dy}{dx}$	0	1	∞	0	-1	$-\infty$

Hence some of the special points that lie on the curve are (0, 0), $\left[a\left(\frac{\pi}{2} + 1\right), a\right]$, $(a\pi, 2a)$, $(2a\pi, 0)$, $\left[-a\left(\frac{\pi}{2} + 1\right), a\right]$ and $(-a\pi, 2a)$.

From the table it is clear that as x increases from 0 to $a\pi$, y also increases from 0 to 2a. But as x further increases to $2a\pi$, y decreases again to zero.

When t is positive and varies from 0 to 2π, we can sketch the portion OAC of the curve in the first quadrant.

As x increases from $-a\pi$ to 0, y is found to decrease from 2a to 0. Also the portion DBO can be traced by symmetry.

The curve consists of congruent arches extending to infinity in both directions of X-axis.

The curve is as traced below.

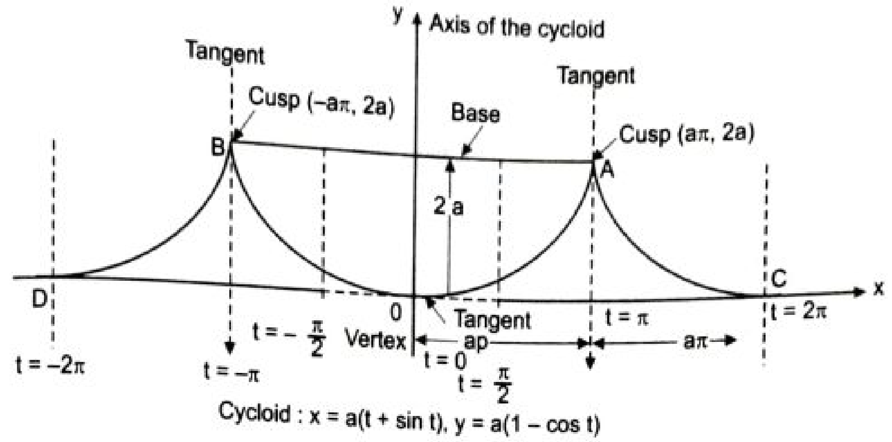


Fig. 5.29

The points A, B are called the *cusps* of the cycloid.

The line OY (i.e. Y-axis) about which the curve is symmetrical is called the *axis of the cycloid.*

The line AB joining the cusps is called the *base* of the cycloid.

The point O is called the *vertex* of the cycloid.

Ex. 2 : *Trace the cycloid $x = a\ (t - \sin t),\ y = a\ (1 - \cos t)$.* **(May 2009, 2004; Dec. 2005, May 2014, Nov. 2014, Dec. 2016)**

Sol. : The curve is known as cycloid.

The curve lies between the lines y = 0 and y = 2a because the greatest value of y is 2a and least is 0. When t = 0, x = 0 and y = 0, hence the curve passes through the origin. It does not contain any asymptote.

We have $y = a\ (1 - \cos t) = a \cdot 2 \sin^2 \left(\frac{t}{2}\right)$

$\therefore\ \sin \frac{t}{2} = \sqrt{\frac{y}{2a}}$. Hence when y is negative, t is imaginary i.e. no part of the curve lies below X-axis (i.e. in 3rd and 4th quadrants).

Also,
$$\frac{dy}{dx} = \frac{a \sin t}{a\ (1 - \cos t)} = \frac{2 \sin\frac{t}{2} \cos\frac{t}{2}}{2 \sin^2\frac{t}{2}} = \cot\left(\frac{t}{2}\right)$$

$$\frac{dy}{dx} = \infty \quad \text{when} \quad t = 0$$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad t = \pi$$

When t = 0, x = 0, y = 0 and when t = π, x = aπ, y = 2a. Therefore at (0, 0), tangent is parallel to Y-axis and at (aπ, 2a), tangent is parallel to X-axis (rather X-axis itself).

Table 5.2

t	0	π/2	π	2π
x	0	a (π/2 – 1)	aπ	2aπ
y	0	a	2a	0
$\frac{dy}{dx}$	∞	1	0	–∞

From the table it is clear that the curve is symmetrical about the line x = aπ.

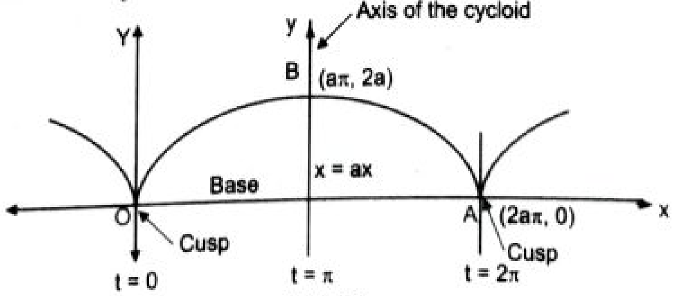


Fig. 5.30

As t increases from 0 to π, x increases from 0 to $a\pi$ and y also increases from 0 to 2a. But as x further increases to $2a\pi$, y decreases again to zero. The points O and A are called the *cusps* of the cycloid. The line $x = a\pi$ about which the curve is symmetrical is called the axis of the cycloid. The line OA joining the cusps is called the *base* of the cycloid. The curve is as traced in the figure.

Ex. 3 : *Trace the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. ***(May 05; Dec. 2011, 2010, 2009, 2008, 2006)***

Sol. : This famous curve is called as *Astroid* (or *star-shaped curve*).

The parametric equations of the Astroid are $x = a\cos^3 t$, $y = a\sin^3 t$.

(Students are advised to remember these parametric equations of the Astroid.)

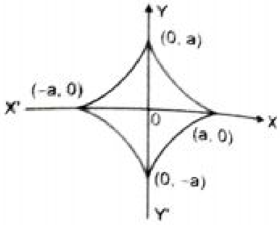


Fig. 5.32

1. $\therefore$ $a\cos^3 t$ is even and $a\sin^3 t$ is odd.

$\therefore$ Symmetry about X-axis. Also we note that for t and –t, x has same value but y has equal and opposite values, hence curve is symmetrical about X-axis.

2. $\therefore$ for t and $\pi - t$, y has same value but x has opposite values. Therefore, symmetry about Y-axis.

3. The table of values of x, y, t is as follows :

t	0	$\pi/2$	π	$3\pi/2$	2π
x	a	0	–a	0	a
y	0	a	0	–a	0

As t ranges from 0 to $\frac{\pi}{2}$, cos t decreases and sin t increases i.e. x decreases, y increases.

4. For any value of t $|\cos t| < 1$ and $|\sin t| < 1$, the values of $|x| < a$ and $|y| < a$ i.e. values of x, y numerically cannot exceed a.

5. The curve cuts X-axis at $(\pm a, 0)$ and Y-axis at $(0, \pm a)$. A rough sketch of the curve is as shown in the figure.

Ex. 4 : *Trace the curve* $x = a\left[\cos t + \frac{1}{2}\log\left(\tan^2\frac{t}{2}\right)\right]$, $y = a\sin t$. ***(May 2009)***

Sol. : This curve is known as "The Tractrix".

1. Limitations of the curve : Since $y = a\sin t$ and $-1 < \sin t < 1$, the greatest value of y is a and the least value of y is – a. x can change from $-\infty$ to $+\infty$.

2. Symmetry : $x = a\left[\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right]$, being even function and $y = a\sin t$, odd hence symmetry about X-axis.

Also we note that for values of t and –t, x remains unchanged but y has equal and opposite values. Thus the curve is symmetric about X-axis.

For $\pi - t$, $\quad \sin(\pi - t) = \sin t$ i.e. for t and $\pi - t$, value of y remains same.

Again for $\pi - t$,

$$x = a\left[\cos(\pi - t) + \frac{1}{2}\log\tan^2\left(\frac{\pi}{2} - \frac{t}{2}\right)\right]$$

$$= a\left[-\cos t - \frac{1}{2}\log\tan^2\frac{t}{2}\right] = -a\left[\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right]$$

Thus for t and $\pi - t$, values of x are equal and opposite. Hence, the curve is symmetrical about Y-axis.

3. Origin : Origin does not lie on the curve. But from the table we see that the curve concretely passes through the points (0, a) and (0, –a).

t	0	$\pi/2$	π	$3\pi/2$
x	$-\infty$	0	∞	0
y	0	a	0	–a

$$y = a\sin t, \quad \frac{dy}{dt} = a\cos t, \quad \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cos^2 t/\sin t} = \tan t$$

$$\frac{dy}{dx} = \tan\psi \Rightarrow \tan\psi = \tan t \Rightarrow \psi = t$$

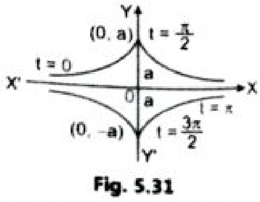


Fig. 5.31

For $\quad t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{dy}{dx} = \pm\infty$ i.e. y-axis is tangent to the curve.

The curve is as shown above.

EXERCISE 5.2

Trace the following curves :

1. $x = a (t + \sin t),\ y = a (1 + \cos t)$ **(May 2006, 2008, 2010; Dec. 2007, May 2016)**
2. $x = a (t - \sin t),\ y = a (1 + \cos t)$ **(May 2007)**
3. $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ **(Dec. 2007, May 2008)**
4. $x = t^2,\ y = t - \frac{t^3}{3}$ **(May 2007, Dec. 04, 05, 07)**
5. $x = at,\ y = \frac{a}{t}$
6. $ay^2 = x^3$

ANSWERS 5.2

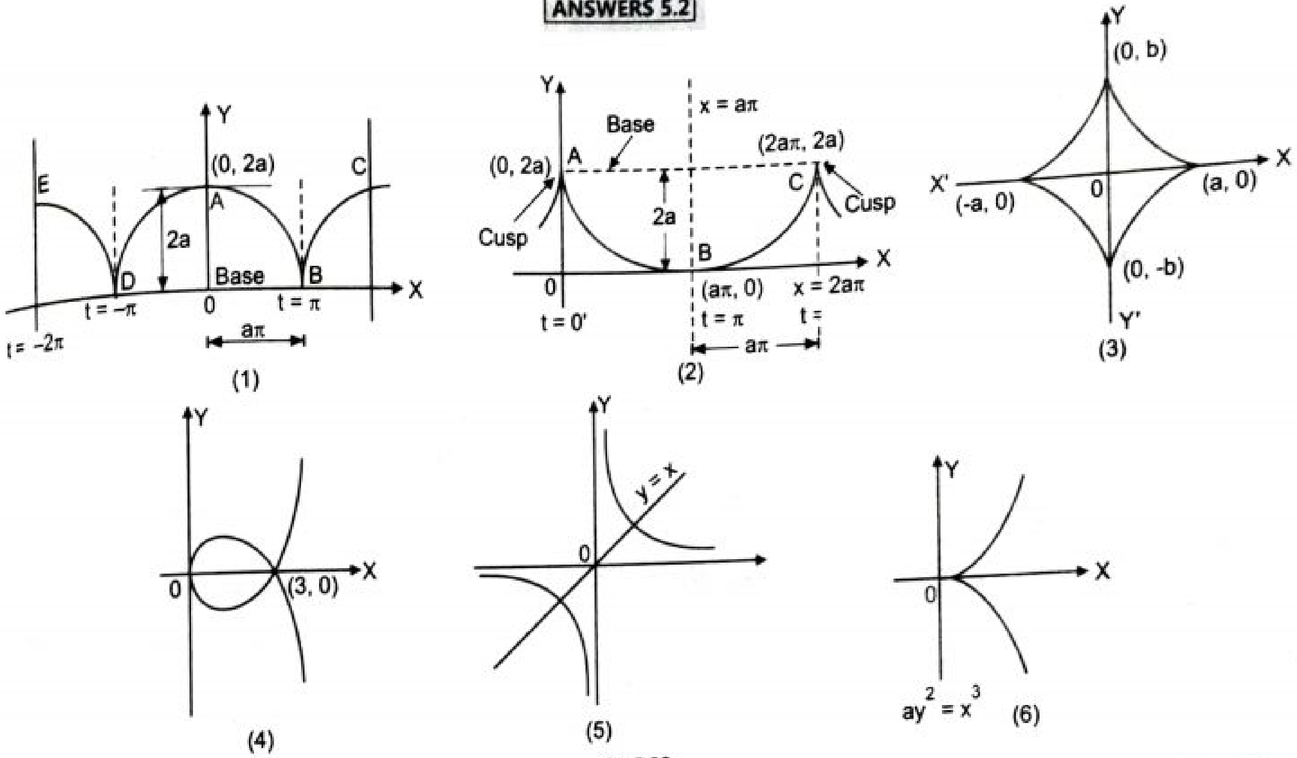


Fig. 5.33

TYPE 3 : CURVES GIVEN BY POLAR CO-ORDINATES

5.5 TRACING OF POLAR CURVE

Introduction : In polar system, the fixed point 'O' is called pole or origin. From fixed point O (pole), draw a straight line in any direction, say OX (X-axis), then this fixed straight line OX is called as initial line.

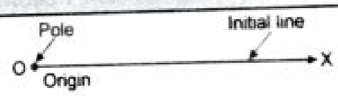


Fig. 5.34

Let P be any given point.

We note that the distance OP = r is called radius vector.

OP makes an angle θ with the initial line and is called vectorial angle measured positive in anticlockwise sense $\therefore \angle XOP = \theta$.

OP = radius vector, any point is $P(r, \theta)$

Co-ordinates of P in polar co-ordinates are (r, θ) and $r = f(\theta)$ is referred as polar equation of the curve.

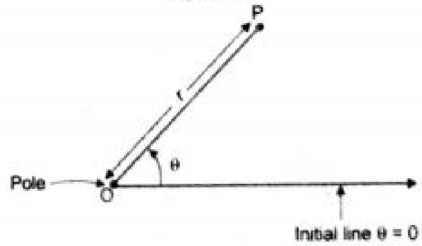


Fig. 5.35

The initial line OX represents $\theta = 0$. We draw $\theta = \frac{\pi}{2}$ as a line perpendicular to initial line and passing through the pole.

The following rules given below should enable the students to sketch a curve given by polar equation in simple cases.

Rule 1 : Symmetry :

In polar co-ordinates, the curve is often given by the equation $r = f(\theta)$.

(a) If the equation to the curve remains unchanged by changing θ to –θ, it will be symmetrical to the initial line [viz. $r = a (1 + \cos \theta)$].

(b) If the equation of the curve remains unchanged by changing r to – r, the curve is symmetrical to the pole. In such a case, only even power of r will occur in the equation [$r^2 = a^2 \cos 2\theta$].

(c) If the equation remains unchanged by changing θ to – θ and r to – r, at the same time, the curve is symmetrical to the line through the pole, perpendicular to the initial line (i.e. Y-axis) [$r^2 = a^2 \cos 2\theta$].

The same symmetry also exists if the equation remains unchanged when θ is changed to π – θ, as for example, the curve $r = (1 + \sin\theta)$.

Rule 2 : Pole :

(a) The pole will lie on the curve if for some value of θ, r becomes zero.

(b) Then find the equation of tangent or tangents at the pole. If we put r = 0, the value of θ gives the tangent at the pole.

Care should be taken of the points where the curve cuts the initial line and the line $\theta = \frac{\pi}{2}$.

Rule 3 : The table showing values of r for different values of θ is very useful in plotting a polar curve. Also find the values of θ at which r = 0 or r = ∞.

Rule 4 : Angle between the radius vector and tangent [ϕ] :

Use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ and also the points where ϕ = 0 or ∞.

i.e. find the points where the tangent coincides with the radius vector or is perpendicular to it.

Rule 5 : Asymptotes : Find asymptotes if any.

Rule 6 : Region of absence of the curve :

(a) Solve the equation for r and consider how r varies as θ increases from 0 to + ∞ and also when θ decreases from 0 to – ∞. If necessary form a table of values of r and θ.

(b) If for some values of θ, say α and β, r^2 is negative, i.e. r imaginary, this means that no branch of the curve exists between lines θ = α and θ = β.

(c) If the maximum numerical value of r is a, the entire curve will lie within a circle of radius a (i.e. r = a). If least numerical value of r is b, the curve will lie outside the circle r = b.

(d) In most of the polar equations, only periodic functions sin θ and cos θ occur and hence values of θ from 0 to 2π should only be considered. The remaining values of θ, give no new branch of the curve.

Note : In some problems, changing to Cartesian proves more convenient.

ILLUSTRATIONS ON TYPE 3 : Polar Curves

Ex. 1 : *Trace the curve $r^2 = a^2 \cos 2\theta$.* ***(May 2009, 2004; Dec. 2011, 2010, 2005)***

Sol. : This curve is known as "*Lemniscate of Bernoulli*".

1. The curve is symmetrical to the initial line.
2. The curve is symmetrical to the pole.
3. The curve is symmetrical to the line perpendicular to the initial line passing through the pole.
4. When θ = π/4 or 3π/4, r becomes zero, hence the curve passes through the pole.
5. Tangents at pole are obtained by putting r = 0, we have cos 2θ = 0.

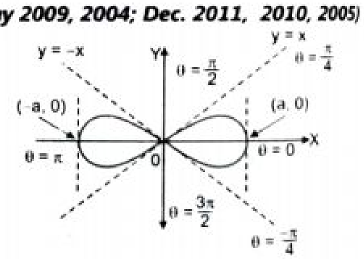


Fig. 5.36

$$\therefore \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \quad \text{i.e.} \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$$

6. Variation of r corresponding to θ is tabulated as follows :

θ	0	π/4	3π/4	π	5π/4	7π/4	2π
r	a	0	0	–a	0	0	a

Thus we see that maximum value of r is a (since maximum of cos 2θ is +1).

When θ = 0, r = ± a i.e. curve passes through the points (a, 0) and (–a, 0). Minimum value of r is 0.

7. As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, r^2 remains negative, hence r becomes imaginary for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ and the curve does not exist in this region. Similarly curve does not exist in $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$.

8. Angle ϕ : We use the formula $\tan\phi = r\frac{d\theta}{dr}$ given $r^2 = a^2\cos 2\theta$, differentiating with respect to θ, $\therefore\ 2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$

$$\text{i.e.} \qquad \frac{dr}{d\theta} = -\frac{a^2\sin 2\theta}{r} \quad \therefore \quad r\frac{d\theta}{dr} = \frac{-r^2}{a^2 \sin 2\theta} = -\frac{a^2\cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = \tan\left(\frac{\pi}{2} + 2\theta\right) \text{ i.e. } \phi = \frac{\pi}{2} + 2\theta$$

Hence at $\theta = 0$, $\phi = \frac{\pi}{2}$ i.e. tangent is perpendicular to the initial line at the points (a, 0) and (–a, 0).

9. The complete curve lies within the circle
$r = a$ since $r^2 \le a^2$.

A rough sketch of the curve is as shown in Fig. 5.36.

Ex. 2 : *Trace the curve $r = a + b \cos \theta$ for $a > b$, $a < b$ and $a = b$.* **(May 2011, 2017, Nov. 2015)**

Sol. : Case I : $r = a + b \cos \theta$, where $a > b$. The curve is known as *Pascal's Limacon.*

1. The curve is symmetrical about the initial line.
2. It does not pass through the pole.
3. Maximum value of r is a + b at $\theta = 0$ and minimum value of r is a – b at $\theta = \pi$.
4. Table of values of r corresponding to θ :

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	a + b	$a + b/\sqrt{2}$	a	$a - b/\sqrt{2}$	a – b

5. Since $-1 \le \cos \theta \le 1$ and a > b, for this case r is never negative.

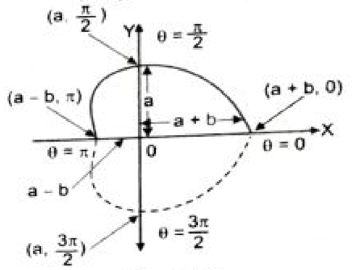


Fig. 5.37

6. We trace the curve for values of θ between 0 and π. Remaining portion of the curve (i.e. for the values of θ between π and 2π) can be traced by symmetry about initial line.

Note :

(i) Since a and b are arbitrary, approximate shape is as shown in Fig. 5.37.

(ii) Dotted portion represents the curve between $\theta = \pi$ and $\theta = 2\pi$ by symmetry about initial line.

Case II : $r = a + b \cos \theta$ when a < b. This curve is known as *Pascal's Limacon.* **(May 2008, 2006)**

1. The curve is symmetrical about the initial line.
2. Since a < b, the curve passes through the pole.
[e.g. if we set a = 1, b = 2 i.e. $r = 1 + 2 \cos \theta$ then $r = 0 \Rightarrow 2 \cos \theta = -1$.
$\cos \theta = -\frac{1}{2}$ giving $\theta = \frac{2\pi}{3}$ $\therefore$ $r = 1 + 2 \cos \theta$ passes through the pole.]
3. Also note that for some value of θ, r can become negative also because for a < b, a – b < 0 hence when $\theta = \pi$, r = a – b is negative. Hence we get an inner loop.
4. Table of values of r corresponding to θ. Maximum value of r is a + b, when $\theta = 0$ and minimum value of r is a – b at $\theta = \pi$.

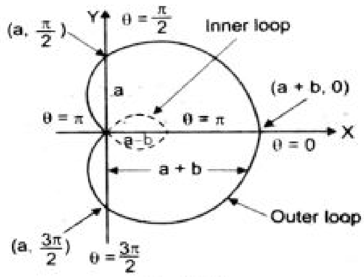


Fig. 5.38

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	a + b	$a + b/\sqrt{2}$	a	$a - b/\sqrt{2}$	a – b

The shape of the curve $r = a + b \cos \theta$, (a < b) is as shown in Fig. 5.38.

Note that the negative r = a – b for $\theta = \pi$ is denoted in the opposite direction.

Case III : $r = a + b \cos \theta$ where a = b. For a = b, the curve becomes the *cardioide*

$$r = a(1 + \cos \theta). \textit{ (Heart-shaped curve).}$$

1. The curve is symmetrical about the initial line.
2. The curve passes through the pole because for $\theta = \pi$, r = 0.
3. Tangent at the pole is obtained by putting r = 0 in the equation.

$$\therefore \quad 0 = a(1 + \cos \theta)$$

$$\text{i.e. } \cos \theta = -1 \text{ i.e. } \theta = \pi$$

Hence tangent at the pole is the initial line itself.

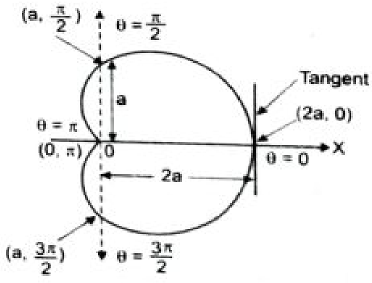


Fig. 5.39

4. If $\theta = 0$, r = 2a, hence the curve cuts the initial line also at (2a, 0).
5. Table of values of r corresponding to θ is as given below :

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	2a	a	0	a	2a

From table we get the information that, as θ increases from $\theta = 0$ to $\theta = \pi$, r decreases from 2a to 0 and also as θ increases from $\theta = \pi$ to $\theta = 2\pi$, r increases from 0 to 2a. Maximum value of r is 2a and minimum value of r is 0.

6. Angle between radius vector r and tangent to the curve [ϕ] :

We use the formula $\tan\phi = r\dfrac{d\theta}{dr}$.

Now, given $r = a(1+\cos\theta) \therefore \dfrac{dr}{d\theta} = -a\sin\theta$

Hence $$\tan\phi = r\cdot\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

or $$\tan\phi = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

So that $$\phi = \frac{\pi}{2}+\frac{\theta}{2}$$

$\therefore$ at $\theta = 0$, $$\phi = \frac{\pi}{2}$$

Hence the tangent is perpendicular to the initial line at (2a, 0).

Note : The shape of the curve from $\theta = \pi$ to $\theta = 2\pi$ may also be traced by symmetry.

Spirals : There is an important class of curves called *spirals*.

A spiral is a curve usually given by an equation of the type $r = a\theta^m$.

In such curves, as θ increases indefinitely, r either goes on increasing indefinitely or decreases continuously, i.e. the curve goes on winding and winding round the pole. We shall trace some important spirals only.

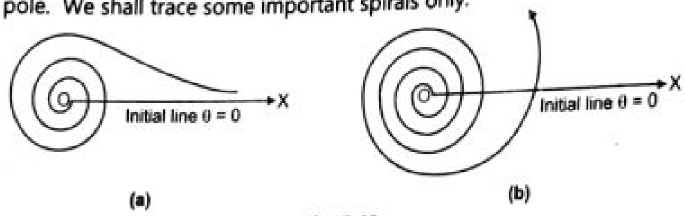


Fig. 5.40

Ex. 3 : *Trace the curve $r = ae^{m\theta}$, where a and m are positive (Equiangular spiral).*

Sol. : The curve is known as Equiangular spiral.

1. **Symmetry :** No symmetry of any type.
2. **Pole :** Curve does not pass through the pole, because $r \neq 0$ for any value of θ.
3. **The angle ϕ :** $\dfrac{dr}{d\theta} = ame^{m\theta} = mr$.

$$\Rightarrow \quad \tan\phi = r\frac{d\theta}{dr} = \frac{1}{mr} = \frac{1}{m} = \text{constant}.$$

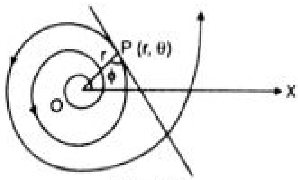


Fig. 5.41

Hence ϕ is a constant angle; i.e. the tangent always makes a constant angle with the radius vector.

4. **Region :**
   (a) When $\theta = 0$, $r = a$, i.e. the point (a, 0) lies on the curve.
   (b) As θ increases from 0 to ∞, r also increases from a to ∞, but remains positive.
   (c) In fact, r remains always positive.
   (d) If θ decreases from 0 to $-\infty$, r while remaining positive tends to 0. Hence the curve is as shown above.

Note : If we put $m = \cot\alpha$ in the curve $r = a\,e^{m\theta}$, it becomes the equiangular spiral $r = ae^{m\cot\alpha}$ and then $\phi = \alpha$. Hence the spiral where tangent m takes a fixed angle α with the radius vector always. This is also known as *logarithmic spiral.*

Ex. 4 : *Trace the curve $r = a\theta$.*

Sol. : This curve is known as the *"Spiral of Archimidies"*.

$$r = a\theta \qquad \text{... (1)}$$

1. **Symmetry :** If θ and r have negative signs simultaneously, equation (1) does not change, hence symmetry about the line $\theta = \dfrac{\pi}{2}$.
2. **Pole :** If $\theta = 0$, $r = 0$, hence pole lies on the curve.
3. As θ increases, r also increases and as $\theta \to \infty$, $r \to \infty$. But when $\theta \to -\infty$, $r \to -\infty$. Hence the curve starting from the pole goes round and round the pole for an infinite number of times.

The dotted lines show the tracing for negative θ. The curve is as traced above.

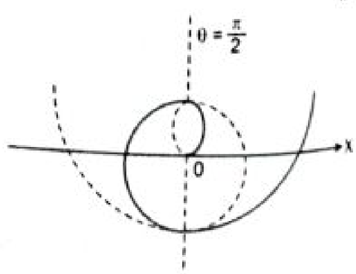


Fig. 5.42

EXERCISE 5.3

Trace the following curves :

1. $r = a(1 + \sin\theta)$ **(Dec. 2008, 2006, 2018)**
2. $r = a(1 - \sin\theta)$ **(May 2009)**
3. $r(1 + \sin\theta) = 2a$ **(May 2011, 2007, 2005, 2004)**
4. $r(1 - \sin\theta) = 2a$
5. $r = a \operatorname{cosec}\theta \pm b$
6. $r^2\theta = a^2$
7. $r\theta = a, \quad a > 0$

On type : $r = a + b\cos\theta$

8. $r = a\left(\frac{\sqrt{3}}{2} + \cos\frac{\theta}{2}\right)$
9. $r = a\left(\sqrt{3} + 2\cos\theta\right)$
10. $r = a(1 + 2\cos\theta)$ **(May 2010, 2004)**
11. $r = \frac{a}{2}(1 + \cos\theta)$ **(Dec. 2005)**
12. $r = 2a\cos\theta$

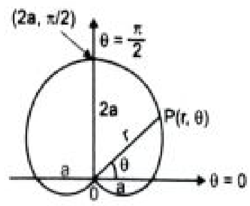
On Type : $r^n = a^n \cos n\theta$: For $n = \pm 1, \pm 2, \pm 1/2$

13. $r\cos\theta = a$
14. $r^2 = a^2\cos 2\theta$
15. $r^2\cos 2\theta = a^2$
16. $r = \frac{1}{2}a(1 + \cos\theta)$
17. $r(1 + \cos\theta) = 2a$
18. $r = \frac{a}{2}(1 - \cos\theta)$
19. $r = 2a\sin\theta$

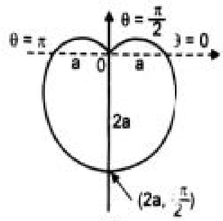
On Type : $r^n = a^n \sin n\theta$: For $n = \pm 1, \pm 2, \pm\frac{1}{2}$

20. $r\sin\theta = a$
21. $r^2 = a^2\sin 2\theta$
22. $r^2\sin 2\theta = a^2$
23. $r(1 - \cos\theta) = 2a$

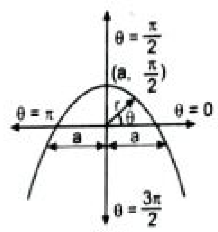
ANSWERS



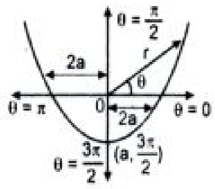
(1)



(2)

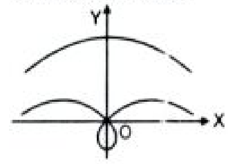


(3)

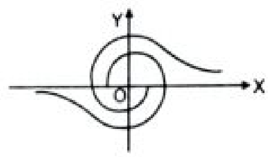


(4)

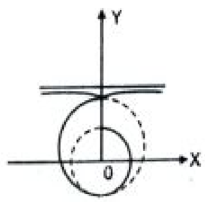
Conicoid of Archimides



(5)

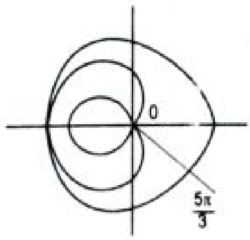


(6)

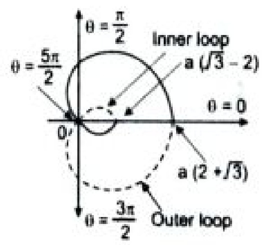


Hyperbolic spiral

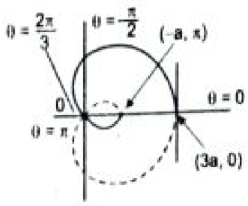
(7)



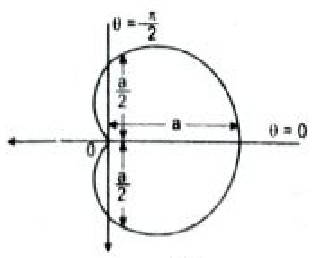
(8)



(9)



(10)



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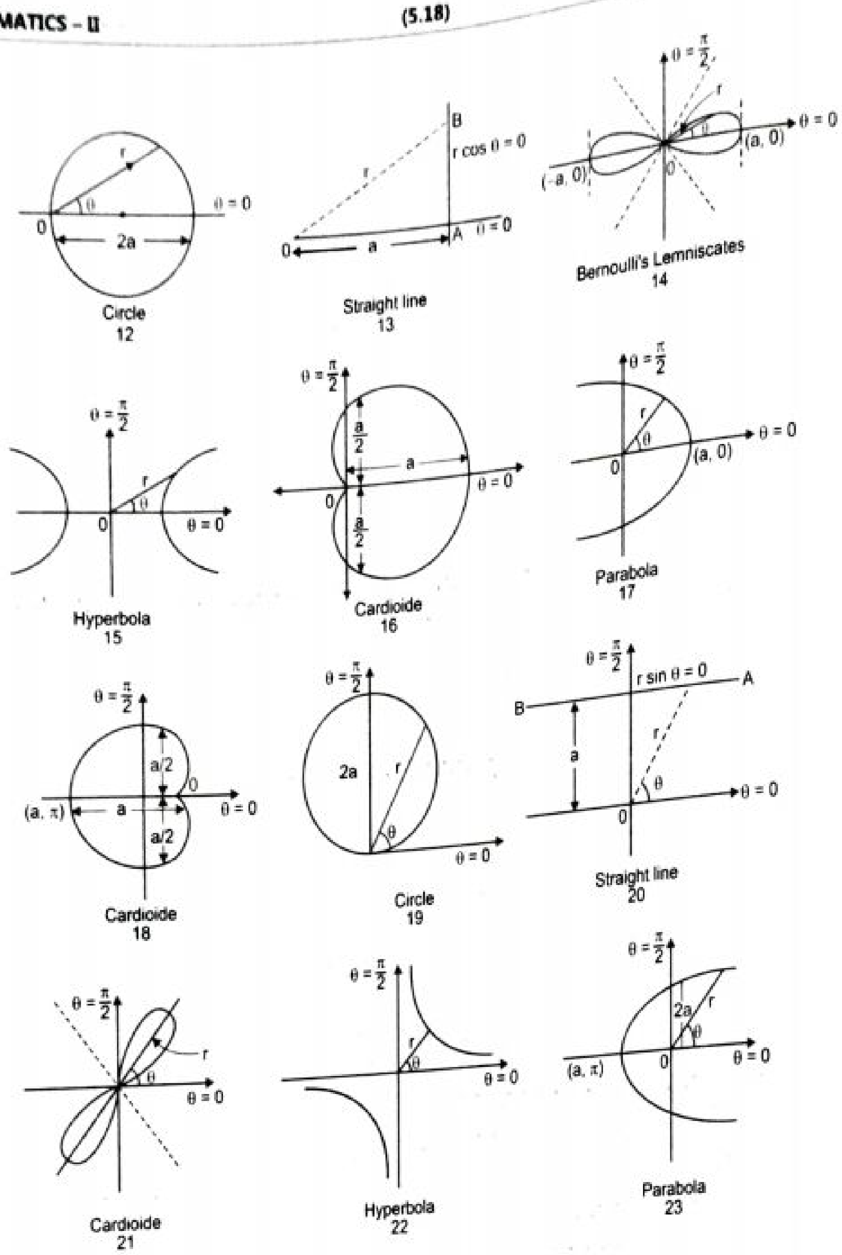


Fig. 5.43

TYPE 4 : Curves given by Polar equations of the type $r = a \sin n\theta$ or $r = a \cos n\theta$.

5.6 TRACING OF ROSE CURVES

The following rules will help in tracing a polar curve given by $r = a \sin n\theta$ or $r = a \cos n\theta$.

Rule 1 : Symmetry :

(a) If the substitution of $-\theta$ for θ in the equation leaves the equation unaltered, the curve is symmetrical about the initial line. For example, $r = a \cos 2\theta$.

(b) If the equation remains unchanged by changing θ to $-\theta$ and r to $-r$, at the same time, then the curve is symmetrical about the line $\theta = \pi/2$ through the pole perpendicular to the initial line. e.g. $r = a \sin 3\theta$.

Rule 2 : Find in particular values of θ which give $r = 0$.

Rule 3 : If pole lies on the curve then find the equation of tangent or tangents at the pole. Put $r = 0$, the value of θ gives the tangent at the pole.

Rule 4 : For $r = a \sin n\theta$ or $r = a \cos n\theta$, the maximum numerical value of r is a, hence the entire curve will lie within a circle of radius a (i.e. $r = a$).

Rule 5 : Since $\sin \theta$ and $\cos \theta$ are periodic functions, values of θ from 0 to 2π should only be considered. Values of $\theta > 2\pi$ give no new branch of the curve.

Rule 6 : The curve $r = a \sin n\theta$ or $r = a \cos n\theta$ consists of

(a) n equal loops if n is odd.

(b) 2n equal loops if n is even.

Rule 7 : For drawing the loops, divide each quadrant into 'n' equal parts.

For $r = a \sin n\theta$,

(a) First loop is drawn along $\theta = \frac{\pi/2}{n}$.

(b) If n is even, draw loops in two sectors consecutively from $\theta = 0$ to $\theta = 2\pi$.

(c) If n is odd, draw loops in two sectors alternatively keeping two sectors between the loops vacant.

Rule 8 : For drawing the loops, divide each quadrant into 'n' equal parts.

For $r = a \cos n\theta$,

(a) First loop is drawn along $\theta = 0$.

(b) If n is even, draw loops in two sectors consecutively from $\theta = 0$ to $\theta = 2\pi$.

(c) If n is odd, draw loops in two sectors alternately keeping two sectors between the loops vacant.

Rule 9 : Angle between the radius vector and the tangent $[\phi]$:

Use the formula $\tan\phi = r\frac{d\theta}{dr}$ and find ϕ and also the points where $\phi = 0$ or ∞ i.e. find the points where the tangent coincides with the radius vector or is perpendicular to it.

Rule 10 : Prepare the table of values of r and θ and observe how r varies as θ increases from 0 to 2π.

Note :

1. $$\sin n\theta = 0 \quad \text{for } n\theta = 0, \pi, 2\pi, 3\pi, 4\pi \ldots\ldots$$

$$\therefore \quad \theta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}, \frac{5\pi}{n}, \ldots$$

2. $$\cos n\theta = 0 \quad \text{for} \quad n\theta = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2} \ldots$$

$$\therefore \quad \theta = \frac{-\pi}{2n}, \frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}, \frac{9\pi}{2n}, \frac{11\pi}{2n}$$

3. $r = -a$ means radial distance to be taken in the opposite direction.

4. Since the shapes of the curves given by $r = a \sin n\theta$ or $r = a \cos n\theta$ are like rose, these curves are known as *Rose curves*.

Ex. 1 : *Trace the curve $r = a \sin 3\theta$* **(Dec. 2009, 2006; May 2013 Nov. 2014)**

Sol. : This curve is known as *Three leaved rose*.

1. The curve $r = a \sin 3\theta$ consists of three equal loops (since the curve $r = a \sin n\theta$ consists of n equal loops if n is odd.)
2. We see that the equation remains unchanged by changing θ to $-\theta$ and r to $-r$, at the same time. Hence the curve is symmetrical to line $\theta = \frac{\pi}{2}$ through the pole perpendicular to the initial line (i.e. Y-axis in cartesian co-ordinates).
3. The pole lies on the curve (since for $\theta = 0, r = 0$).
4. To obtain equation of tangents at pole we put $r = 0$,

   we get $\sin 3\theta = 0$. i.e. $3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots$ Hence $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots\ldots$

   Draw these lines and place equal loops in alternate division, the first loop between $\theta = 0, \theta = \frac{\pi}{3}$, second between $\theta = \frac{2\pi}{3}$ and $\theta = \pi$ and third between $\theta = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.
5. Hence we divide each quadrant into 3 parts and as such we have following table of values.

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{\pi}{2}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{3\pi}{2}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

From the table we see that r is never greater than a. Hence the curve lies within the circle of radius a.

6. Angle between radius vector and tangent to the curve [ϕ]. We use the formula :

$$\tan\phi = r\frac{d\theta}{dr}, \frac{dr}{d\theta} = 3a\cos 3\theta$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{a\sin 3\theta}{3a\cos 3\theta} = \frac{1}{3}\tan 3\theta$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{1}{3}\tan 3\theta$$

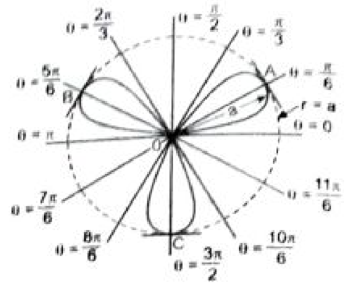


Fig. 5.44

For $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$ r = 0 (from table) and at these points as $\phi = 0$ [from (1)], the tangent and radius vectors are coincident.

For $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$, the values of r (from table) are $\pm$ a (i.e. points A, B, C) and ϕ is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Hence at points A, B, C, tangent is perpendicular to the radius vector.

7. Here we consider the value of θ from 0 to 2π since sin θ is periodic function and $\theta > 2\pi$ gives no new branch of the curve.

8. When θ increases from 0 to $\frac{\pi}{6}$, r remains positive and increases from 0 to a. Further when θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r remains positive but decreases from a to 0. Thus we get first loop of the curve between $\theta = 0$ to $\theta = \frac{\pi}{3}$ i.e. the loop OAO.

When θ increases from $\frac{2\pi}{6}$ to $\frac{\pi}{2}$ $\left(\text{i.e. } \frac{\pi}{3} \text{ to } \frac{\pi}{2}\right)$, r remains negative and numerically increases from 0 to a (i.e. portion of the curve below initial line). As θ increases from $\frac{\pi}{2}$ to $\frac{4\pi}{6}$ $\left(\text{i.e. } \frac{\pi}{2} \text{ to } \frac{2\pi}{3}\right)$, r is negative and numerically decreases from a to 0.

Now, we get the second loop of the curve OCO. When θ increases from $\theta = \frac{2\pi}{3}$ to $\theta = \frac{5\pi}{6}$, r remains positive and increases from 0 to a. Further, when θ increases from $\frac{5\pi}{6}$ to π, r remains positive but decreases from a to 0. Finally, we get the third loop of the curve OBO. A rough sketch of the curve is as shown above.

Ex. 2 : *Trace the curve r = a cos 2θ.* ***(May 2019, May 2010, 2015, 2018; Dec. 2013)***

Sol. : This curve is known as *Four Leaved Rose.*

1. The curve r = a cos 2θ consists of four equal loops.
2. The curve has only symmetry about initial line.
3. The pole will lie on the curve (or curve passes through the pole).
4. Equations of tangents (or tangent) at the pole are obtained by putting r = 0, we have

   $\cos 2\theta = 0$ i.e. $2\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$. Hence $\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots\ldots$

   Draw these lines and place equal loops in each division. The first loop between $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$, second loop between $\theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$, third loop between $\theta = \frac{3\pi}{4}$ to $\theta = \frac{5\pi}{4}$ and fourth loop between $\theta = \frac{5\pi}{4}$ to $\theta = \frac{7\pi}{4}$.

5. Hence we divide each quadrant into two (since n = 2) parts and as such we have following table of values.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	a	0	–a	0	a	0	–a	0	a

Thus we see that r is never greater than a (i.e. maximum value of r is a) and the curve lies wholly within a circle of radius a.

6. Angle between radius vector r and tangent to the curve [ϕ] :

We use the formula $\tan\phi = r\dfrac{d\theta}{dr}$, where value of ϕ will indicate the direction of tangent.

Given $r = a\cos 2\theta$

Differentiating with respect to θ, we have,

$$\frac{dr}{d\theta} = -2a\sin 2\theta$$

$$r\frac{d\theta}{dr} = \frac{a\cos 2\theta}{-2a\sin 2\theta} = -\frac{1}{2}\cot 2\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{1}{2}\tan\left(\frac{\pi}{2}+2\theta\right) \qquad \dots (1)$$

$\tan\phi = 0$ when $r = 0$ i.e. when

$$\theta = -\frac{\pi}{4}\left(\text{or } \frac{7\pi}{4}\right), \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$$

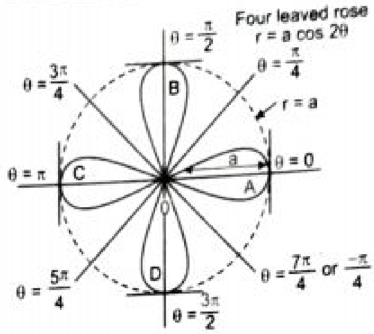


Fig. 5.45

Thus at points $\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ $(r = 0)$, the tangents are coincident with radius vectors. Also $\tan\phi$ is infinite when $r = \pm a$ i.e. when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Thus at points $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ $(r = \pm a)$, the tangents are perpendicular to radius vectors. Hence at each of the points A, B, C, D, tangent is perpendicular to the radius vector.

7. A rough sketch of the curve is as shown above.

EXERCISE 5.4

Trace the following curves :

1. $r = a\cos 5\theta$
2. $r = a\sin 4\theta$ **(May 2008)**
3. $r = a\cos 4\theta$
4. $r = a\cos 3\theta$ **(Nov./Dec. 2019, Dec. 2010, 2008, 2004; May 2007, 2006, 2016)**
5. $r = a\sin 5\theta$
6. $r = a\sin 2\theta$ **(May 2005, Dec. 2007)**

ANSWERS

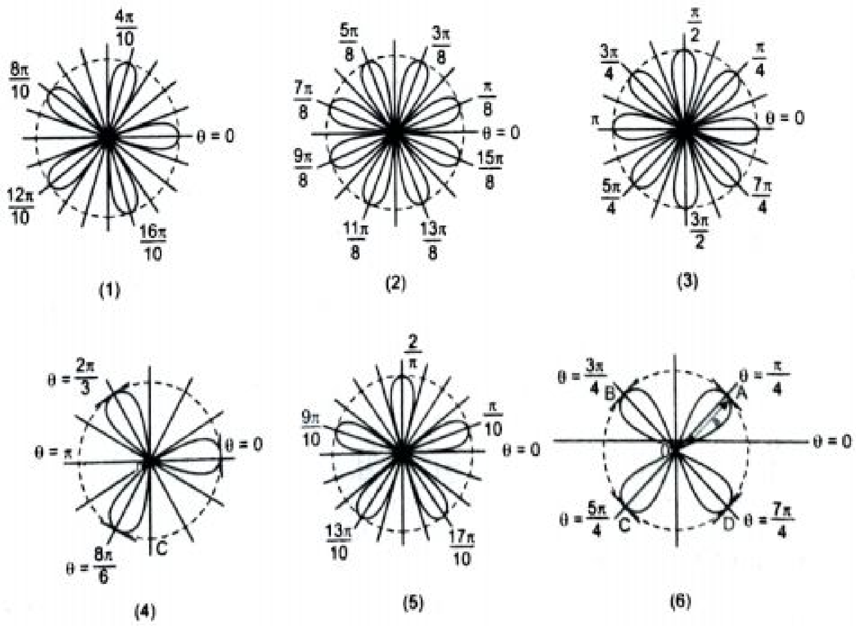


Fig. 5.46

TYPE 5 : CURVES GIVEN BY CARTESIAN EQUATIONS (IMPLICIT RELATIONS)

5.7 TRACING OF CARTESIAN CURVES (WITH IMPLICIT RELATIONS)

For the curves expressed in implicit form, first we will apply all the rules already discussed under cartesian equation of the cur Now, we transform the equation to polar form by using $x = r\cos\theta$, $y = r\sin\theta$ getting the polar equation of the curve in explicit fo $r = f(\theta)$.

We note that if power of r is even then the curve is symmetrical about the pole. Also find the oblique asymptotes by using meth discussed earlier. The transformation to polar helps us to determine that region where the part of the curve does not exist.

Hence to trace the curves given by implicit equations, it is convenient to make use of polar as well as cartesian forms of th equations.

***Ex. 1 :** Trace the curve $x^3 + y^3 = 3axy$.* **(May 2007, 2005; Dec. 2010, 2009, 2008, 2006)**

Sol. : This curve is known as *"Foulium of Descartes"*.

1. The curve has only symmetry about line $y = x$.
2. The curve passes through the origin.
3. Tangents at the origin are $x = 0$, $y = 0$ (i.e. X-axis and Y-axis). Hence we may expect a node at the origin.
4. The curve cuts the axes only at (0, 0). It also cuts the line $y = x$ at (0, 0) and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.
5. For the equation $x^3 + y^3 = 3axy$ if x and y both become negative then R.H.S. becomes positive but L.H.S. remains negati which is absurd. Hence x and y both cannot be negative at the same time. This means curve has no branch in the th quadrant.
6. Oblique asymptote : $x^3 + y^3 - 3axy = 0$.

   Here third degree term $= x^3 + y^3$, second degree term $= -3axy$.

   We put $x = 1$, $y = m$

   Let $\phi_3(m) = 1 + m^3$ and $\phi_2(m) = -3am$

   Solve $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$

   $(m + 1)(m^2 - m + 1) = 0$ giving only real value $m = -1$ because the remaining two roots are imaginary.

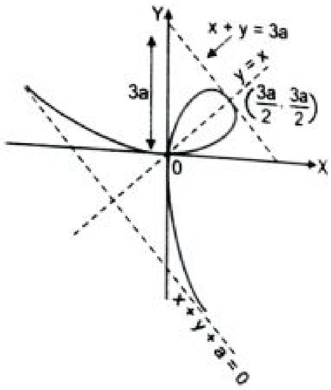


Fig. 5.47

To find c, we use the formula $c = -\frac{\phi_2(m)}{\phi_3'(m)}$

Here $\phi_3(m) = 1 + m^3 \quad \therefore \quad \phi_3'(m) = \frac{d}{dm}(1 + m^3) = 3m^2 \quad \therefore \quad c = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m}$

Now put $m = -1 \quad \therefore \quad c = -a$

By putting $m = -1$, $c = -a$ in $y = mx + c$, we have $y = -x - a$ i.e. $x + y + a = 0$ is an oblique asymptote to the curve.

7. Since tangents to the curve at origin are $x = 0$, $y = 0$ which are real and different, origin is a node.
8. Also $\frac{dy}{dx} = -\left(\frac{x^2 - ay}{y^2 - ax}\right)$; $\therefore \left[\frac{dy}{dx}\right]_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -1$ which shows that the tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is perpendicular to $y = x$. Now t

   equation of this tangent is $y - y_1 = m(x - x_1)$ where $m = -1$ and (x_1, y_1) is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

   $\therefore \; y - \frac{3a}{2} = -1\left(x - \frac{3a}{2}\right)$ i.e. $x + y = 3a$ is the tangent at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ which is perpendicular to $y = x$. A rough sketch of the cur is as shown above.

We also note that the parametric equations of $x^3 + y^3 = 3axy$ are

$x = \frac{3at}{1 + t^3}$, $y = \frac{3at^2}{1 + t^3}$ and polar equation is $r = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$.

Ex. 2 : Trace the curve $x^5 + y^5 - 5a^2x^2y = 0$

Sol. :

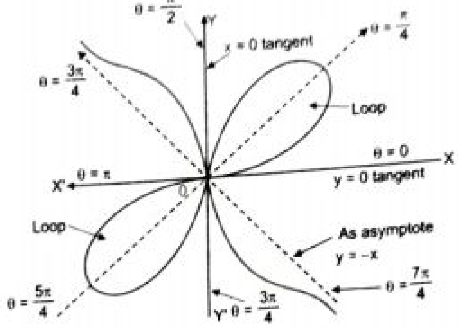


Fig. 5.48

1. It is symmetrical in the opposite quadrants.
2. It passes through the origin.
3. Tangents at origin are x = 0, y = 0.
4. Since tangents at the origin are x = 0, y = 0 which are real and different, origin is a node.
5. For oblique asymptote, fifth degree term = $x^5 + y^5$, fourth degree term = 0.

   We put x = 1, y = m.

   $\phi_5(m) = 1 + m^5$, $\phi_4(m) = 0$. Solve $\phi_5(m) = 0$ i.e. $1 + m^5 = 0$ giving only m = –1 as real value and we note that c = 0.

   y = mx + c i.e. y = – x is oblique asymptote to the curve.
6. Now, convert the equation to polar form

   $$r^5(\cos^5\theta + \sin^5\theta) = 5a^2r^3\cos^2\theta\sin\theta \quad r^2 = \frac{5a^2\cos^2\theta\sin\theta}{\cos^5\theta + \sin^5\theta}.$$

   when $\theta = 0$, r = 0, $\theta = \frac{\pi}{4}$, $r = \sqrt{5}\cdot a$.

   $\theta = \pi/2$, r = 0. Thus the curve has one loop in first quadrant symmetrical about y = x. Because of symmetry in the opposite quadrants, we get one more loop in the 3rd quadrant.

   When $\theta = \frac{3\pi}{4}$, $r \to \infty$ and for $\pi/2 < \theta < \frac{3\pi}{4}$, r^2 increases for 0 to ∞, curve exists.

   Further for $\frac{3\pi}{4} < \theta < \pi$, r^2 is negative, the curve does not exist for $\frac{3\pi}{4} < \theta < \pi$.

   Values of θ between π to 2π need not be considered because of symmetry in opposite quadrants.
7. Since power of r is even, curve is also symmetrical about pole. Hence the curve is as shown above.

EXERCISE 5.5

Trace the following curves :

1. $x^4 + y^4 = 4axy^2$
2. $x^4 + y^4 = a^2(x^2 - y^2)$
3. $x^6 + y^6 = a^2x^2y^2$
4. $x^5 + y^5 = 5ax^2y^2$ **(Dec. 2004)**
5. $x^4 + y^4 = 2a^2xy$
6. $y^4 - x^4 + xy = 0$

ANSWERS

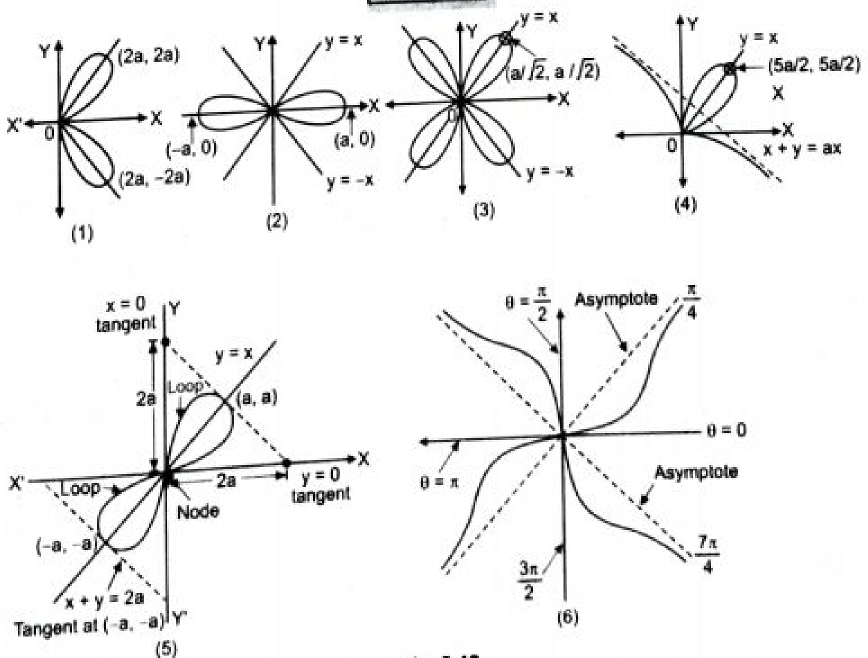


Fig. 5.49

5.8 SOME WELL-KNOWN STANDARD CURVES

Ex. 1 : *Trace the following curve* $y = c \cosh \frac{x}{c}$.

Sol. : This curve is known as *"The common catenary"*.

A curve in which a perfectly flexible and uniform string hangs under gravity is called *catenary*. The equation of the curve is $y = c \cosh \frac{x}{c}$, where c is known as *parameter* of the curve. It is symmetrical about Y-axis.

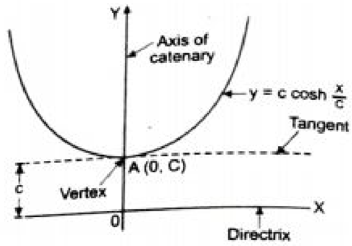


Fig. 5.50 (a)

It does not pass through the origin. When x = 0 we have y = c ($\because$ cosh 0 = 1).

It cuts Y-axis at A (0, c) and this point A (0, c) is called as *vertex* of the common catenary.

Here X-axis is called as the *directrix*. It does not cut X-axis at all.

$$\frac{dy}{dx} = c \cdot \sinh \frac{x}{c} \cdot \frac{1}{c} \left[\because \frac{d}{dx} \cosh\left(\frac{x}{c}\right) = \frac{1}{c} \sinh \frac{x}{c}\right]$$

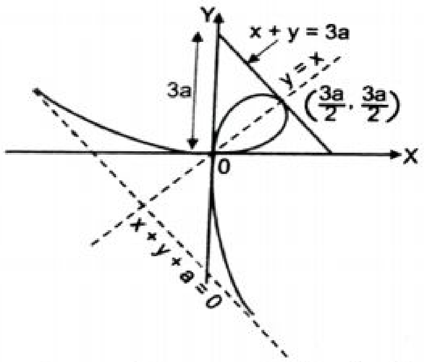
$$\frac{dy}{dx} = \sinh \frac{x}{c} > 0 \text{ for } x > 0$$

$\therefore$ y increases as x increases from 0 to ∞

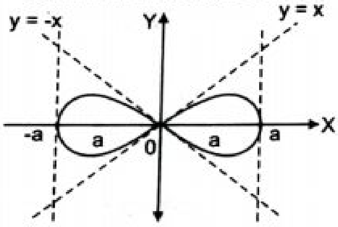
$$\text{when } x = 0, \frac{dy}{dx} = \sinh 0 = 0.$$

Therefore, the tangent at A (0, c) is parallel to X-axis. Here AY is called as the *axis of the common catenary*. A rough sketch of the curve is as shown in Fig. 5.50 (b)

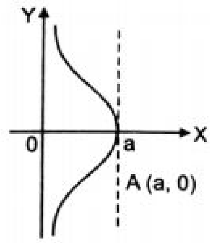
SOME WELL-KNOWN CURVES



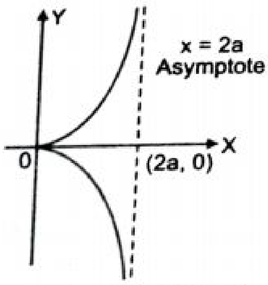
Foulium of Descart as $x^3 + y^3 = 3$ any



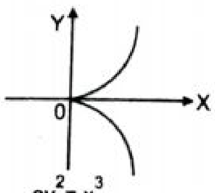
Bemoulli's Lemniscate
$r^2 = a^2 \cos 2\theta$



Witch of agnest
$y^2 = 4a^2(a - x)/x$

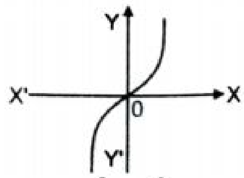


The Cissoid of Diocle
$y^2(2a - x) = x^3$



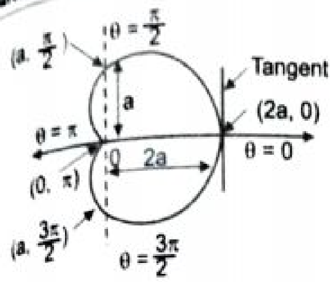
$ay^2 = x^3$

Semi-cubic parabola



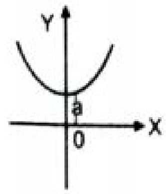
$a^2 y = x^3$

Cubic parabola



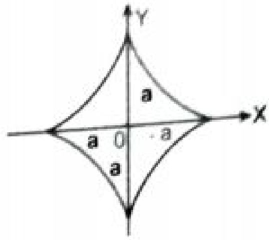
$r = a\,(1 + \cos\theta)$

cardiode



$y = a \cos h \dfrac{x}{a}$

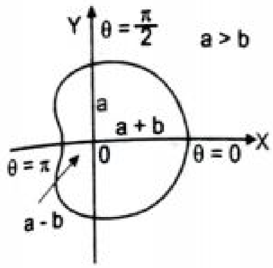
Catenary



$x^{2/3} + y^{2/3} = a^{2/3}$ or

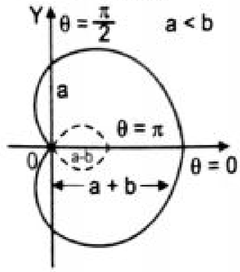
$x = a \cos^3\theta,\ y = a \sin^3\theta$

Asstroid



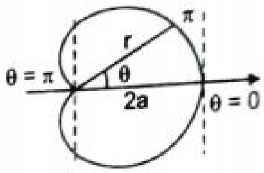
Pascal's Limacon

$r = a + b \cos\theta$



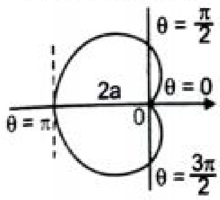
Pascal's Limacon

$r = a + b \cos\theta$



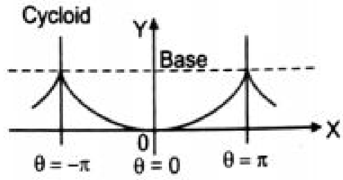
Cardioide

$r = (a + b \cos\theta),\ a = b$

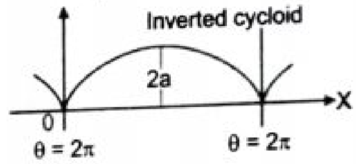


Cardioide

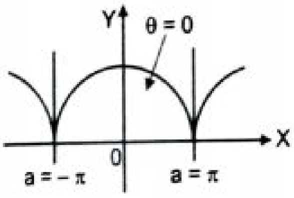
$r = a\,(1 - \cos\theta)$



$x = a\,(\theta + \sin\theta),\ y = a\,(1 - \cos\theta)$

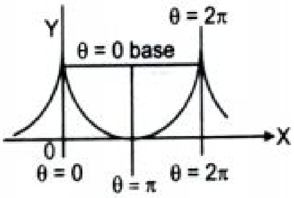


$X = a\,(\theta - \sin\theta),\ y = a(1 - \cos\theta)$



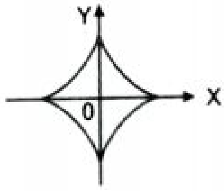
$x = a\,(\theta + \sin\theta),$

$y = a\,(1 + \cos\theta)$

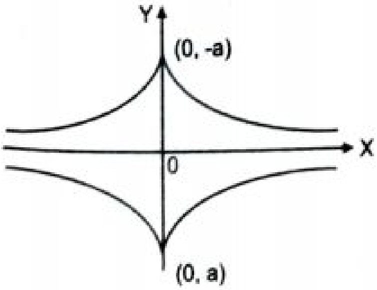


$x = a\,(\theta - \sin\theta),$

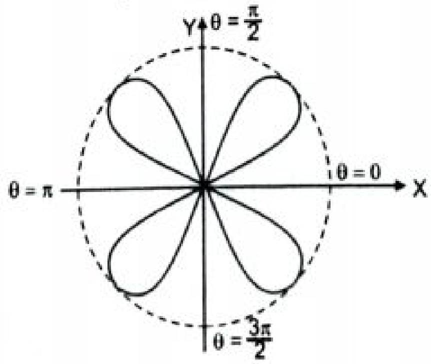
$y = a\ \ (1 - \cos\theta)$



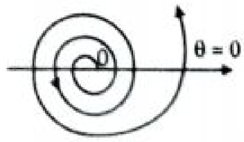
$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$



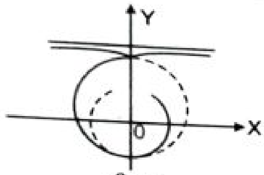
$x = a \cos\theta + \dfrac{a}{2} \log \tan^2\dfrac{\theta}{2},\ y = a \sin\theta$



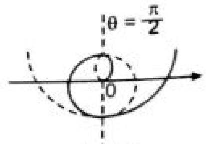
$r = a \sin 2\theta$



$r = ae^{m\theta}$



Hyperbolic spiral



Spiral of Archimidies

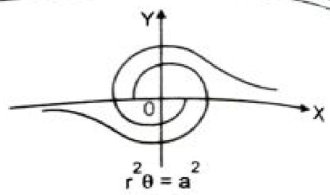


Fig. 5.50 (b)

5.9 RECTIFICATION OF CURVES

Introduction :

In this section, we shall consider the applications of integration to measure the length of arc of plane curves. This procedure is called *Rectification*.

The curves that we come across, have their equations either in cartesian, parametric or polar form. We shall first develop formulae which are used when equations of curves are given in either cartesian or parametric form.

5.10 RECTIFICATION OF PLANE CURVE FOR CARTESIAN EQUATION

1. Cartesian Equation : y = f(x) : Consider the curve C as shown in figure below. Consider two points P (x, y), Q (x + δx, y + δy), where arc PQ = δs. PM, QN are perpendiculars on X-axis and PR is perpendicular on QN.

Then $OM = x,\ ON = x + \delta x,\ MP = y,\ NQ = y + \delta y$

∴ $PR = ON - OM = \delta x$

$RQ = NQ - NR = NQ - MP = \delta y$

From right angled triangle PQR,

$$PQ^2 = PR^2 + RQ^2$$

or $$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 \qquad ...(1)$$

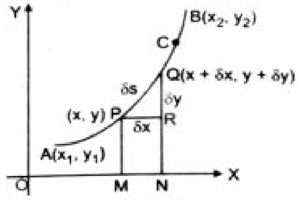


Fig. 5.51

(When Q is indefinitely closer to P, arc PQ = δs = Chord PQ)

Dividing (1) by $(\delta x)^2$, we get,

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

Taking the limit as $Q \to P$ or $\delta x \to 0$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or $$\left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

i.e. $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Integrating $$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

In particular, if we want to determine arc length AB, where $A(x_1, y_1)$, $B(x_2, y_2)$ are any two points on the curve, we substitute the limits of integration for x and we have,

$$\boxed{s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx} \qquad ...(a)$$

2. Cartesian Equation : x = f (y) : Again dividing (1) by $(\delta y)^2$ and taking limiting case as before,

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 \quad \text{or} \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

i.e. $$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Integrating between the limits y_1 to y_2,

$$\boxed{s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy} \qquad \text{...(b)}$$

Formulae (a) and (b) both give the same arc length AB.

They are to be used when equation of the curve is given in cartesian form f (x, y) = 0. Choice between (a) and (b) is to be made so that integration becomes simple.

5.11 RECTIFICATION OF PLANE CURVE FOR PARAMETRIC EQUATIONS OF THE FORM $X = \phi(t)$, $Y = \Psi(t)$

When equation of the curve is given in parametric form $x = \phi(t)$, $y = \Psi(t)$, where t is the parameter. (As t varies we get different points on the curve).

Dividing (1) by $(\delta t)^2$ and taking limit as $Q \to P$, we get,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \quad \text{or} \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

i.e.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Integrating,

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If a point A corresponds to the parameter t_1 and point B corresponds to the parameter t_2, then integrating between the limits t_1 and t_2,

$$\boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt} \qquad \text{...(c)}$$

This formula gives length of arc AB when equation of the curve is expressed in parametric form.

5.12 RECTIFICATION OF PLANE CURVE WHEN ITS EQUATION IS IN POLAR FORM $r = f(\theta)$

1. **Polar Equation : $r = f(\theta)$:** We shall now obtain two formulae which give length of arc of a curve when its equation is given in polar form $r = f(\theta)$.

P (r, θ), Q $(r + \delta r, \theta + \delta\theta)$ are two closed points on the curve C as shown in Fig. 5.53.

A (r_1, θ_1), B (r_2, θ_2) are any two points on the curve C and arc PQ = δs.

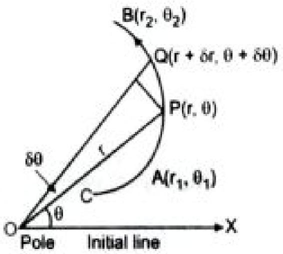


Fig. 5.52

$$OP = r,\ OQ = r + \delta r.\ \text{PR is} \perp \text{on OQ.}$$
$$OR = OP \cos \delta\theta = r \cos \delta\theta \to r \text{ as } \delta\theta \to 0$$
and
$$PR = OP \sin \delta\theta = r \sin \delta\theta \to r\,\delta\theta$$
$$RQ = OQ - OR = r + \delta r - r = \delta r$$

When P and Q are indefinitely closer to each other, arc PQ = δs = chord PQ.

From Δ PQR,

$$PQ^2 = RQ^2 + PR^2 \quad \text{or} \quad (\delta s)^2 = (\delta r)^2 + (r\,\delta\theta)^2 \qquad \text{... (2)}$$

Dividing by $(\delta\theta)^2$ and taking limit as $Q \to P$ or $\delta\theta \to 0$, we get,

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

or

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad \text{or} \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Integration gives,

$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

If arc length AB is to be determined then we substitute the limits of θ as θ_1 and θ_2 and we get

$$\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta} \quad \ldots (d)$$

2. Polar Equation : $\theta = f(r)$: Similarly dividing (2) by $(\delta r)^2$ and taking the limit $Q \to P$ or $\delta r \to 0$

$$\left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2 \quad \text{or} \quad \frac{ds}{dr} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}$$

Integration gives
$$s = \int \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

To determine arc length AB, substituting the limits of r from r_1 to r_2,

$$\boxed{s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr} \quad \ldots (e)$$

Formulae (d) and (e) are applicable to polar curves. Formula (d) is to be used, if integration w.r.t. θ is simple and formula (e) is to be used, if integration w.r.t. r is simple.

1. Cartesian Curves :

	Equation of Curve	Formula in Differential Calculus	Formula in Integral Calculus
1.	$y = f(x)$	$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$	$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$
2.	$x = g(y)$	$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$	$s = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$
3.	$x = f_1(t),\ y = f_2(t)$	$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$	$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$

2. Polar Curves :

	Equation of Curve	Formula in Differential Calculus	Formula in Integral Calculus
1.	$r = f(\theta)$	$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$	$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$
2.	$\theta = f(r)$	$ds = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \cdot dr$	$s = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} \cdot dr$
3.	$r = f_1(\alpha);\ \theta = f_2(\alpha)$ α is parameter	$ds = \sqrt{\left(\frac{dr}{d\alpha}\right)^2 + r^2\left(\frac{d\theta}{d\alpha}\right)^2} \cdot d\alpha$	$s = \int_{\alpha_1}^{\alpha_2} \sqrt{\left(\frac{dr}{d\alpha}\right)^2 + r^2\left(\frac{d\theta}{d\alpha}\right)^2} \cdot d\alpha$

ILLUSTRATIONS :

Ex. 1 : *Find the circumference of circle of radius 'a'.*

Sol. : Equation of circle of radius a with centre as origin as shown in the figure can be taken as,

$x^2 + y^2 = a^2$

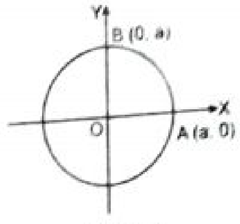


Fig. 5.53

$$\therefore \quad y = \sqrt{a^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1(-2x)}{2\sqrt{a^2 - x^2}} = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2 - x^2 + x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$$

Using formula (a), S_{AB} = length of arc AB = $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Along arc AB, x varies from a to 0, but as x decreases, s increases and hence to get positive value, limits of x are taken from x = 0 to x = a.

$$\therefore \quad S_{AB} = \int_0^a \sqrt{\frac{a^2}{a^2 - x^2}} \cdot dx = \int_0^a \frac{a}{\sqrt{a^2 - x^2}}\, dx = a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= a\left[\sin^{-1}\frac{a}{a} - \sin^{-1} 0\right] = a\left[\frac{\pi}{2} - 0\right] = \frac{\pi a}{2}$$

Circumference = $4 \times S_{AB}$

$$\boxed{\text{Circumference} = 4 \times \frac{\pi a}{2} = 2\pi a}$$

Ex. 2 : *Find the length of arc of parabola $y^2 = 4ax$ from vertex to one extremity of latus rectum.*

Sol. : Latus rectum of a parabola ($y^2 = 4ax$) is a line segment perpendicular to the axis (y = 0) of the parabola, through the focus [S(a, 0)] and whose end points are [A(a, 2a) and B(a, – 2a)] lie on the parabola. Length of latus rectum is 4a.

Required length is of arc OA. Here we use formula (b).

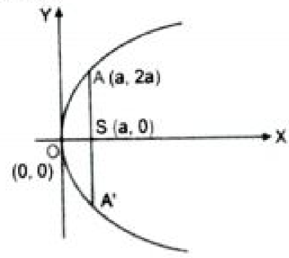


Fig. 5.54

$$S_{OA} = \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x = \frac{y^2}{4a} \quad \therefore \quad \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2} = \frac{4a^2 + y^2}{4a^2}$$

$$\therefore \quad S_{OA} = \int_0^{2a} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy = \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\, dy$$

which is in a standard form. In this example if we had used formula (a), integral would not have been directly in standard form.

$$S_{OA} = \frac{1}{2a}\left[\frac{y\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\log\left\{y + \sqrt{4a^2 + y^2}\right\}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a\sqrt{4a^2 + 4a^2}}{2} + 2a^2\log\left\{2a + \sqrt{4a^2 + 4a^2}\right\}\right] - \left[2a^2 \log\left\{\sqrt{4a^2}\right\}\right]$$

$$= \frac{1}{2a}\left[2\sqrt{2}\, a^2 + 2a^2 \log\frac{2a + 2\sqrt{2}\, a}{2a}\right]$$

$$\boxed{S_{OA} = a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]}$$

***Ex. 3 :** Show that the length of the arc of the curve $ay^2 = x^3$ from origin to the point whose abscissa is b is $\frac{1}{27\sqrt{a}}(9b+4a)^{3/2} - \frac{8a}{27}$.* **(May 2009)**

Sol. : Curve is symmetrical about X-axis as shown in the figure.

P is the point on the curve whose abscissa is b.

Differentiating the equation of the curve,

$$2ay\frac{dy}{dx} = 3x^2 \quad \therefore \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9x^4}{4a^2y^2} = \frac{9x^4}{4a\cdot x^3} = \frac{9x}{4a}$$

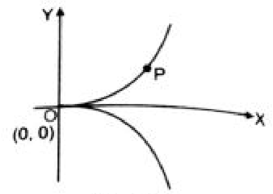


Fig. 5.55

Using the formula (a) with limits of x from 0 to b, we get,

$$S = \int_0^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^b \sqrt{1+\frac{9x}{4a}}\,dx$$

$$= \frac{1}{2\sqrt{a}}\int_0^b \sqrt{4a+9x}\,dx = \frac{1}{2\sqrt{a}}\left[\frac{(4a+9x)^{3/2}}{\frac{3}{2}\cdot 9}\right]_0^b = \frac{1}{27\sqrt{a}}\left[(4a+9b)^{3/2}-(4a)^{3/2}\right] = \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a^{3/2}}{27\sqrt{a}}$$

$$\boxed{S = \frac{(4a+9b)^{3/2}}{27\sqrt{a}} - \frac{8a}{27}}$$

***Ex. 4 :** Show that the whole length of the loop of the curve $9y^2 = (x+7)(x+4)^2$ is $4\sqrt{3}$.* **(May 2007)**

Sol. : From the equation of the curve it is clear that loop is around X-axis between x = – 7 and x = – 4. The curve is symmetrical about X-axis. Shape of the curve is as shown in Fig. 5.58.

We use formula (a), with appropriate limits to get whole length of the loop as,

$$S = 2\int_{-7}^{-4}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

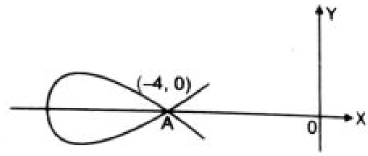


Fig. 5.56

(We multiply the integral by 2 to get the whole length of the loop.)

Differentiating equation of the curve w.r.t. 'x',

$$18y\frac{dy}{dx} = (x+4)^2 + 2(x+4)(x+7)$$

$$= (x+4)(x+4+2x+14)$$

$$\frac{dy}{dx} = \frac{3(x+4)(x+6)}{18y} = \frac{(x+4)(x+6)}{6y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+4)^2(x+6)^2}{36y^2} = \frac{(x+4)^2(x+6)^2}{4(x+7)(x+4)^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x+6)^2}{4(x+7)} = \frac{4x+28+x^2+12x+36}{4(x+7)} = \frac{x^2+16x+64}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$$

$$\therefore \quad S = \frac{2}{2}\int_{-7}^{-4}\frac{x+8}{\sqrt{x+7}}\,dx = \int_{-7}^{-4}\frac{x+7+1}{\sqrt{x+7}}\,dx = \int_{-7}^{-4}\left\{\sqrt{x+7}+(x+7)^{-1/2}\right\}dx$$

$$= \left[\frac{(x+7)^{3/2}}{3/2} + 2(x+7)^{1/2}\right]_{-7}^{-4} = \frac{2}{3}3\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$$

$$\boxed{\text{Length of the loop} = 4\sqrt{3}}$$

Ex 5 : *Find the length of the arc of the catenary $y = c \cosh \frac{x}{c}$ measured from its vertex to any point (x, y) and show that $s^2 = y^2 - c^2$*

Sol. : Here, $y = c \cosh \frac{x}{c}$ $\therefore$ $\frac{dy}{dx} = c \cdot \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 \frac{x}{c} = \cosh^2 \frac{x}{c}$$

$$\therefore \quad S = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \cosh \frac{x}{c} \cdot dx$$

$$= \left[c \sinh \frac{x}{c}\right]_0^x = c \cdot \sinh \frac{x}{c}$$

$$S^2 = c^2 \sinh^2 \frac{x}{c} = c^2\left[\cosh^2 \frac{x}{c} - 1\right] = c^2\left[\frac{y^2}{c^2} - 1\right] = y^2 - c^2$$

$$\therefore \quad \boxed{S^2 = y^2 - c^2}$$

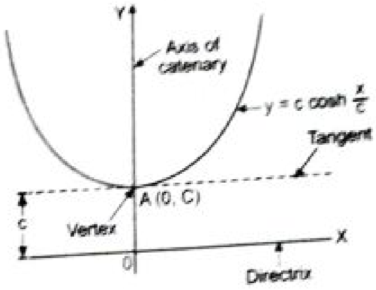


Fig. 5.57

Ex. 6 : *Show that for the curve $8a^2 y^2 = x^2 (a^2 - x^2)$, $S = \frac{a}{2\sqrt{2}} [2\theta + \sin\theta\cos\theta]$ and the perimeter of one of the loops is $\frac{\pi a}{\sqrt{2}}$*

(Dec. 2009)

Sol. : Here the curve is symmetrical about X-axis and has two loops around X-axis between x = 0 and x = a.

We first integrate between x = 0 and x = x using formula (a) i.e. we first find arc length of the curve from origin to any point (x, y) on the curve.

$$S = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

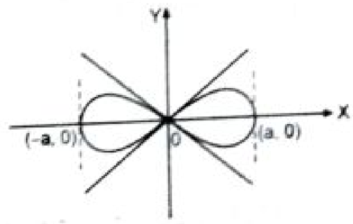


Fig. 5.58

Differentiating w.r.t. x, the equation of the curve, we get,

$$8a^2 \cdot 2y \frac{dy}{dx} = 2x(a^2 - x^2) - 2x^3$$

or
$$\frac{dy}{dx} = \frac{a^2 x - 2x^3}{8a^2 y}$$

or
$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2 (a^2 - 2x^2)^2}{64 a^4 y^2} = \frac{x^2 (a^2 - 2x^2)^2}{8a^2 \cdot x^2 (a^2 - x^2)}$$

and
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)} = \frac{8a^4 - 8a^2 x^2 + a^4 - 4a^2 x^2 + 4x^4}{8a^2 (a^2 - x^2)}$$

$$= \frac{9a^4 - 12a^2 x^2 + 4x^4}{8a^2 (a^2 - x^2)} = \frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}$$

$$\therefore \quad S = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \frac{3a^2 - 2x^2}{2\sqrt{2}\, a \sqrt{a^2 - x^2}}\, dx$$

$$= \frac{1}{2\sqrt{2}\, a} \int_0^x \frac{3a^2 - 2x^2}{\sqrt{a^2 - x^2}}\, dx$$

Putting $x = a \sin\theta$, $dx = a\cos\theta\, d\theta$. When $x = 0, \theta = 0, x = x, \theta = \theta$

$$S = \frac{1}{2\sqrt{2}\,a}\int_0^{\theta} \frac{3a^2 - 2a^2\sin^2\theta}{a\cos\theta}\cdot a\cos\theta\, d\theta$$

$$= \frac{1}{2\sqrt{2}\,a}\int_0^{\theta} [3a^2 - a^2(1-\cos 2\theta)]\, d\theta$$

$$= \frac{1}{2\sqrt{2}\,a}\int_0^{\theta} (2a^2 + a^2\cos 2\theta)\, d\theta = \frac{a}{2\sqrt{2}}\left[2\theta + \frac{\sin 2\theta}{2}\right]_0^{\theta}$$

$$\boxed{S = \frac{a}{2\sqrt{2}}[2\theta + \sin\theta\cdot\cos\theta]}$$ which is the first required result.

To get the perimeter of the loop, we put $\theta = \frac{\pi}{2}$ [When $x = a$, $\theta = \frac{\pi}{2}$ which gives length of upper half of the loop]

$$S = \frac{a}{2\sqrt{2}}\left[2\cdot\frac{\pi}{2}\right] = \frac{\pi a}{2\sqrt{2}}$$

$\therefore$
$$\boxed{\text{Length of one loop} = 2\cdot\frac{\pi a}{2\sqrt{2}} = \frac{\pi a}{\sqrt{2}}}$$

***Ex. 7 :** Find the length of the loop of the curve $x = t^2$; $y = t\left(1 - \frac{t^2}{3}\right)$.* **(Dec. 2007)**

Sol. : To trace the curve, it is advisable in this problem to convert the equation into Cartesian form.

$$y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2 \quad\text{or}\quad y^2 = x\left(1 - \frac{x}{3}\right)^2$$

Loop is around X-axis between $x = 0$ and $x = 3$ as shown in the figure. Corresponding to $x = 0$ and $x = 3$, we have $t = 0$ and $t = \sqrt{3}$.

To obtain length of loop, we use formula (c).

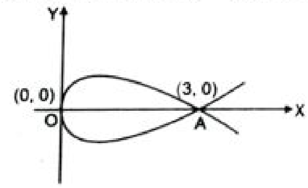


Fig. 5.59

$$S = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

which will give length of upper half of the loop

We have
$$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 1 - t^2$$

and
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4t^2 + 1 - 2t^2 + t^4 = (1+t^2)^2$$

$\therefore$
$$S = \int_0^{\sqrt{3}} \sqrt{(1+t^2)^2}\, dt = \int_0^{\sqrt{3}} (1+t^2)\, dt$$

$$\text{Total length} = 2\int_0^{\sqrt{3}} [1+t^2]\, dt = 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\left[\sqrt{3} + \frac{3\sqrt{3}}{3}\right] = 4\sqrt{3}.$$

$$\boxed{\text{Length of the loop} = 4\sqrt{3}}$$

***Ex. 8 :** Show that in the Astroid $x^{2/3} + y^{2/3} = a^{2/3}$, $S^3 \alpha x^2$. S being measured from the cusp which lies on Y-axis.*

(May 2010, 2005, 2016; Dec. 2010, 2018)

Sol. : $A(0, a)$ is a cusp on Y-axis. $P(x, y)$ is any point on the curve where arc $AP = S$ (See Fig. 5.57)

To obtain the arc length, we use parametric equations

$$x = a\cos^3\theta,\quad \frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta)$$

$$y = a\sin^3\theta,\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

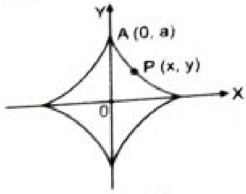


Fig. 5.60

and
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$
$$= 9a^2\sin^2\theta\cos^2\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= 9a^2\sin^2\theta\cos^2\theta$$

Corresponding to $y = a$, $\sin^3\theta = 1 \therefore \theta = \frac{\pi}{2}$

We use formula (c) with appropriate limits.

$$S = \int_{\pi/2}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\pi/2}^{\theta} 3a\sin\theta\cos\theta\, d\theta$$

$$= \frac{3a}{2}\int_{\pi/2}^{\theta} \sin 2\theta\, d\theta = \frac{3a}{2}\left[-\frac{\cos 2\theta}{2}\right]_{\pi/2}^{\theta} = -\frac{3a}{4}\,[\cos 2\theta - \cos\pi] = -\frac{3a}{4}\,[1 + \cos 2\theta]$$

$$S = -\frac{3a}{4}\,[2\cos^2\theta] = -\frac{3a}{2}\cos^2\theta$$

$$S^3 = -\frac{27a^3}{8}\cos^6\theta = -\frac{27\,a^3}{8}\,(\cos^3\theta)^2$$

$$S^3 = -\frac{27\,a^3}{8}\left(\frac{x^2}{a^2}\right) = -\frac{27\,a}{8}\,x^2$$

$$\therefore \quad \boxed{S^3 \propto x^2.}$$

Ex. 9 : *Find the length of the arc of the curve* $x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$, $y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$ *from* $\theta = 0$ *to* $\theta = \pi$

Sol. : Here equation of the curve is given in parametric form, so we use formula (c) with t replaced by θ, limits from $\theta = 0$ to $\theta = \pi$.

$$S = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Now,
$$\frac{dx}{d\theta} = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right) + e^{\theta}\left(\frac{1}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) = \frac{5}{2}e^{\theta}\cos\frac{\theta}{2}$$
$$\frac{dy}{d\theta} = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right) + e^{\theta}\left(-\frac{1}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) = -\frac{5}{2}e^{\theta}\sin\frac{\theta}{2}$$

$$\therefore \quad S = \int_0^{\pi} \sqrt{\frac{25}{4}e^{2\theta}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}\, d\theta = \frac{5}{2}\int_0^{\pi} e^{\theta}\, d\theta = \frac{5}{2}\left[e^{\theta}\right]_0^{\pi}$$

$$\boxed{S = \frac{5}{2}\,(e^{\pi} - 1)}$$

Ex. 10 : *A curve is described by the equations,*

$$x\sin\theta + y\cos\theta = f'(\theta)$$
$$x\cos\theta - y\sin\theta = f''(\theta)$$

Show that the length of arc expression is given by

$$S = f'(\theta) + f''(\theta) + c, \text{ where } c \text{ is a constant.}$$

Sol. : Consider
$$x\sin\theta + y\cos\theta = f'(\theta) \quad \text{...(1)}$$
$$x\cos\theta - y\sin\theta = f''(\theta) \quad \text{...(2)}$$

Multiplying equation (1) by $\sin\theta$, (2) by $\cos\theta$ and adding we get,

$$x = f'(\theta)\sin\theta + f''(\theta)\cos\theta \quad \text{...(3)}$$

Similarly multiplying (1) by $\cos\theta$, (2) by $\sin\theta$, and subtracting

$$y = f'(\theta) \cos\theta - f''(\theta) \sin\theta \qquad ...(4)$$

(3) and (4) are parametric equations of the curve. To get arc length expression, we use formula (c).

$$S = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

From (3) and (4), by differentiating,

$$\frac{dx}{d\theta} = f''(\theta) \sin\theta + f'(\theta) \cos\theta + f'''(\theta) \cos\theta - f''(\theta) \sin\theta = \cos\theta \{f'(\theta) + f'''(\theta)\},$$

$$\frac{dy}{d\theta} = f''(\theta) \cos\theta - f'(\theta) \sin\theta - f'''(\theta) \sin\theta - f''(\theta) \cos\theta = -\sin\theta \{f'(\theta) + f'''(\theta)\}$$

Substituting in formula for S,

$$S = \int \sqrt{\{f'(\theta) + f'''(\theta)\}^2 (\cos^2\theta + \sin^2\theta)}\, d\theta = \int \{f'(\theta) + f'''(\theta)\}\, d\theta$$

Integrating we get, $\boxed{S = f(\theta) + f''(\theta) + c}$

which is the arc length expression for the given curve.

Ex. 11 : *Find the arc length of the cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ from one cusp to another cusp. If S is the length of the arc from origin to point P (x, y), show that $S^2 = 8ay$.* **(Dec. 2004, May 2017)**

Sol. : Part 1 : Here

$$x = a(\theta + \sin\theta); \quad \frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$y = a(1 - \cos\theta); \quad \frac{dy}{d\theta} = a \sin\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2 (1 + \cos\theta)^2 + a^2 \sin^2\theta$$

$$= a^2 [1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= 2a^2 (1 + \cos\theta) = 2a^2 \left(2\cos^2\frac{\theta}{2}\right) = 4a^2 \cos^2\frac{\theta}{2}$$

Length of arc AB = 2 × length of OB.

$$S = 2\int_0^{\pi} 2a \cos\frac{\theta}{2} \cdot d\theta$$

$$= 4a\left[2 \sin\frac{\theta}{2}\right]_0^{\pi} = 8a$$

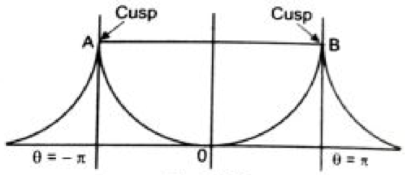


Fig. 5.61

$\boxed{S = 8a}$

which is the required length from one cusp to another cusp.

Part 2 :

$$S = \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\theta} 2a \cos\frac{\theta}{2} \cdot d\theta = 2a\left[2 \sin\frac{\theta}{2}\right]_0^{\theta}$$

$$S = 4a \sin\frac{\theta}{2}$$

$$S^2 = 16a^2 \sin^2\frac{\theta}{2} = 16a^2 \left(\frac{1 - \cos\theta}{2}\right) = 8a\ (a(1 - \cos\theta)) = 8ay$$

$\boxed{S^2 = 8ay}$

Ex. 12 : *Evaluate $\int xy\, ds$ along the arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant* **(Dec. 2005)**

Sol. : Parametric equations of the ellipse are $x = a\cos\theta$, $y = b\sin\theta$.

Along the arc in the positive quadrant, θ varies from 0 to $\frac{\pi}{2}$ as shown in Fig. 5.61.

$$\int xy\, ds = \int xy \frac{ds}{d\theta}\, d\theta = \int xy \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Substituting $x = a\cos\theta$, $y = b\sin\theta$, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$ and integrating between the limits $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$I = \int_0^{\pi/2} xy\, ds = \int_0^{\pi/2} a\cos\theta \cdot b\sin\theta \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\; d\theta$$

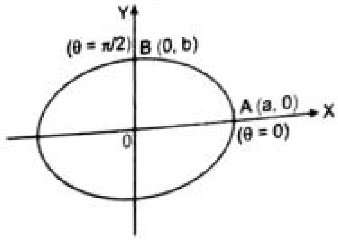


Fig. 5.62

$$= \frac{ab}{2}\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2(1-\sin^2\theta)}\; 2\sin\theta\cos\theta\, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2} \sqrt{b^2 + (a^2-b^2)\sin^2\theta}\; d(\sin^2\theta)$$

Integrating w.r.t. $\sin^2\theta$

$$I = \frac{ab}{2}\left[\frac{(b^2 + (a^2-b^2)\sin^2\theta)^{3/2}}{\frac{3}{2}\cdot(a^2-b^2)}\right]_0^{\pi/2} = \frac{ab}{3(a^2-b^2)}\left[(a^2)^{3/2} - (b^2)^{3/2}\right]$$

$$= \frac{ab}{3(a^2-b^2)}[a^3 - b^3] = \frac{ab}{3(a^2-b^2)}(a-b)(a^2+ab+b^2)$$

$$\boxed{I = \frac{1}{3}\,\frac{ab(a^2+ab+b^2)}{(a+b)}}$$

***Ex. 13 :** For the curve $x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$, $y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$*

Show that $S = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right)$ measured from $\theta = \frac{\pi b}{a}$ to θ.

Sol. : Here,

$$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

$$\frac{dx}{d\theta} = -(a+b)\sin\theta + (a+b)\sin\left(\frac{a+b}{b}\theta\right)$$

$$y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$$

$$\frac{dy}{d\theta} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\theta\right)$$

and

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a+b)^2\Big[\sin^2\theta - 2\sin\theta\sin\left(\frac{a+b}{b}\right)\theta + \sin^2\left(\frac{a+b}{b}\right)\theta + \cos^2\theta$$

$$-2\cos\theta\cos\left(\frac{a+b}{b}\right)\theta + \cos^2\left(\frac{a+b}{b}\right)\theta\Big]$$

$$= (a+b)^2\left[2 - 2\cdot\cos\left(\theta - \left(\frac{a+b}{b}\right)\theta\right)\right] = 2(a+b)^2\left[1 - \cos\left(-\frac{a\theta}{b}\right)\right] = 2(a+b)^2\left(1 - \cos\frac{a\theta}{b}\right)$$

$$= 2(a+b)^2\left(2\sin^2\frac{a\theta}{2b}\right) = 4(a+b)^2\sin^2\frac{a\theta}{2b}$$

$$\therefore\quad S = \int_{\frac{\pi b}{a}}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\; d\theta = \int_{\frac{\pi b}{a}}^{\theta} 2(a+b)\sin\left(\frac{a\theta}{2b}\right)d\theta = 2(a+b)\left[-2\frac{b}{a}\cos\left(\frac{a\theta}{2b}\right)\right]_{\frac{\pi b}{a}}^{\theta}$$

$$= \frac{4b}{a}(a+b)\left[\cos\frac{\pi}{2} - \cos\frac{a\theta}{2b}\right] = -\frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right)$$

$$\boxed{S = \frac{4b}{a}(a+b)\cos\left(\frac{a\theta}{2b}\right)}$$

Ex. 14 : *Find the perimeter of cardioide $r = a(1 + \cos\theta)$ and show that a line $\theta = \frac{\pi}{3}$ divides upper half of the cardioide.*

(May 2009, 2019, 2018, Dec. 2013)

Sol. : Curve is symmetrical about initial line OX. For upper half of the arc, θ varies from $\theta = 0$ to $\theta = \pi$.

Since the curve is given in Polar form, we use formula (d) with appropriate limits to find arc length,

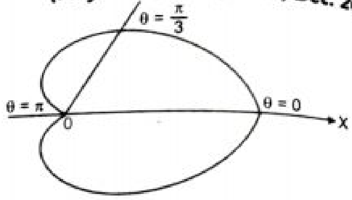


Fig. 5.63

$$S = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \text{...(1)}$$

Here

$$r = a(1 + \cos\theta), \quad \frac{dr}{d\theta} = -a\sin\theta$$

and

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= 2a^2(1 + \cos\theta)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}\, a\sqrt{1 + \cos\theta}$$

Substituting in (1),

$$S = \int_0^{\pi} \sqrt{2}\, a\sqrt{1 + \cos\theta}\, d\theta$$

$$= \sqrt{2}\, a \int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta = 2a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4a \qquad \text{... (i)}$$

This gives length of upper half. Because of symmetry of the curve, perimeter of the cardioide = 2 (4a) = 8a.

To prove second part, we integrate (1) between the limits $\theta = 0$ to $\theta = \frac{\pi}{3}$.

$$S = \sqrt{2}\, a \int_0^{\pi/3} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 2a\int_0^{\pi/3} \cos\frac{\theta}{2}\, d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3} = 4a\sin\frac{\pi}{6} = 4a\left(\frac{1}{2}\right) = 2a \qquad \text{... (ii)}$$

From (i) and (ii), $\boxed{\text{A line } \theta = \frac{\pi}{3} \text{ divides upper half of cardioiode}}$

Ex. 15 : *Find the length of the upper arc of one loop of Lemniscate $r^2 = a^2\cos 2\theta$.*

Sol. : For upper arc of the curve, θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

(Nov./Dec. 2019, Dec. 2006, May 2011, May 2014)

$$S = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad r = a\sqrt{\cos 2\theta}$$

$$\frac{dr}{d\theta} = a\cdot\frac{1(-2\sin 2\theta)}{2\sqrt{\cos 2\theta}} \qquad r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta} = \frac{a^2}{\cos 2\theta}$$

$$S = \int_0^{\pi/4} \frac{a}{\sqrt{\cos 2\theta}}\, d\theta$$

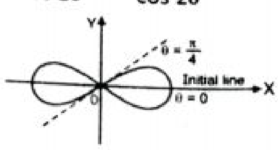


Fig. 5.64

Put $2\theta = t \therefore d\theta = \frac{1}{2}dt$ When $\theta = 0,\ t = 0;\ \theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$\therefore \quad S = \frac{a}{2}\int_0^{\pi/2} \frac{dt}{\sqrt{\cos t}} = \frac{a}{2}\int_0^{\pi/2} \sin^0 t \cos^{-1/2} t \, dt$$

$$= \frac{a}{2}\left[\frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{-1/2+1}{2}\right)}{2\,\Gamma\left(\frac{0-1/2+2}{2}\right)}\right] = \frac{a}{4}\,\frac{\sqrt{\pi}\,\Gamma(1/4)}{\Gamma(3/4)}$$

$$= \frac{a}{4}\,\frac{\sqrt{\pi}\,(\Gamma(1/4))^2}{\Gamma(1/4)\,\Gamma(3/4)} = \frac{a\sqrt{\pi}}{4}\,\frac{(\Gamma(1/4))^2}{\pi\sqrt{2}}$$

$$= \frac{a}{4\sqrt{2}}\,\frac{(\Gamma(1/4))^2}{\sqrt{\pi}} \qquad \left[\because \Gamma(1/4)\,\Gamma(3/4) = \Gamma(1/4)\,\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin \pi/4} = \pi\sqrt{2}\right]$$

$$\boxed{\text{Length of the upper arc of one loop} = \frac{a}{4\sqrt{2}}\,\frac{(\Gamma(1/4))^2}{\sqrt{\pi}}}$$

Ex. 16 : *Find the length of arc of the curve $r = a\,e^{m\theta}$ intercepted between radii vectors r_1 and r_2.* ***(Dec. 2005)***

Sol. : In this problem, it is convenient to use formula (e).

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,.\,dr$$

Differentiating equation of given curve, $r = a\,e^{m\theta}$

$$\frac{dr}{d\theta} = a\,me^{m\theta} = mr$$

$$r^2\left(\frac{d\theta}{dr}\right)^2 = r^2 \cdot \frac{1}{m^2 r^2} = \frac{1}{m^2}$$

$$\therefore \quad S = \int_{r_1}^{r_2} \sqrt{1 + \frac{1}{m^2}}\,dr = \frac{\sqrt{1+m^2}}{m}\,[r]_{r_1}^{r_2}$$

$$\boxed{\text{Length of arc of curve} = \frac{\sqrt{1+m^2}}{m}\,(r_2 - r_1)}$$

Ex. 17 : *Find the length of the cardioide $r = a\,(1 + \cos\theta)$ which lies outside the circle $r + a\cos\theta = 0$.*

Sol. : The point of intersection of the curves is given by $r = -a\cos\theta$,

$r = a\,(1 + \cos\theta) - a\cos\theta = a\,(1 + \cos\theta)$, $\cos\theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}$.

The arc length outside the circle is twice the arc length BA.

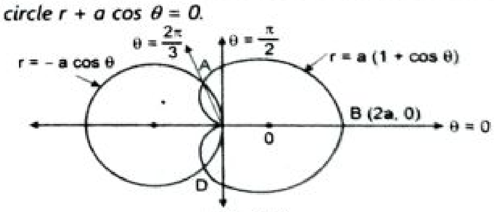


Fig. 5.64

$$\text{Required length} = L = \int ds = \int \frac{ds}{d\theta}\,d\theta.$$

Here
$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$= a^2\,[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta] = a^2\,(1 + \cos\theta) = 2a^2 \cdot 2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$$

$$\therefore \quad L = 2\int_0^{2\pi/3} 2a\cos\frac{\theta}{2}\,d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{2\pi/3} = 8a\left(\sin\frac{\pi}{3}\right) = 4\sqrt{3}\,a$$

$$\boxed{\text{Required length} = 8a\left(\sin\frac{\pi}{3}\right) = 4\sqrt{3}\,a}$$

EXERCISE 5.6

1. Show that in the catenary $y = a \cosh \frac{x}{a}$, the length of arc of the curve from the vertex to any point (x, y) is given by $S = a \sinh \frac{x}{a}$

   [Hint : Here $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \cosh \frac{x}{a}$ and limits are x = 0 to x]

2. Find the whole length of the loop of the curve $3y^2 = x(x-1)^2$. **(May 2006)**

   [Hint : Here $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1 + 3x}{2\sqrt{3}\sqrt{x}}$ and limits are x = 0 to a] **Ans. :** $\frac{4}{\sqrt{3}}$

3. Find the length of the arc of the parabola $y^2 = 4x$, cut off by the line $3y = 8x$. **Ans. :** $\log 2 + \frac{15}{16}$

   [Hint : Cutting points (0, 0) an $\left(\frac{9a}{16}, \frac{3a}{2}\right)$, Also $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{1}{2a}\sqrt{4a^2 + y^2}$ and limits are y = 0 to $\frac{3a}{2}$]

4. Show that the length of the arc of the curve $4ax = y^2 - 2a^2 \log \frac{y}{a} - a^2$ from (0, a) to any point (x, y) is given by $S = \frac{y^2}{2a} - \frac{a}{2} - x$

   [Hint : Here $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{1}{2a}\left(y + \frac{a^2}{y}\right)$ and limits are y = a to y.]

5. Find the length of the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ between the two consecutive cusps. **(Dec. 09, May 15)**

   Ans. : 8a

6. Find the length of the arc of the curve $x = e^{\theta}\cos\theta$, $y = e^{\theta}\sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$. **(May 2011, Dec. 2017)**

   [Hint : Here $\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{2}\, e^{\theta}$ and limits are $\theta = 0$ to $\frac{\pi}{2}$]

   Ans. : $\sqrt{2}\,(e^{\pi/2} - 1)$

7. Find the length of the arc of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ in the positive quadrant. **(Dec. 2011)**

   [Hint : Here $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\cos t \sin t\sqrt{a^2\cos^2 t + b^2 \sin^2 t}$; $t = 0$ to $\frac{\pi}{2}$ Put $a^2\cos^2 t + b^2\sin^2 t = t$] **Ans. :** $\frac{(a^2 + ab + b^2)}{a + b}$

8. Show that the length of the arc of the tractrix $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$ from $t = \frac{\pi}{2}$ to any point t is "a log sin t".

9. Find the complete arc length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. **(Dec. 2018) Ans. :** 6a

   [Hint : Here $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{3a}{2}\sin 2t$ and limits are $t = 0$ to $\frac{\pi}{2}$]

10. Find the length of the arc of the curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t\cos t)$ from $t = 0$ to $t = 2\pi$. **Ans. :** $2\pi^2 a$

11. Evaluate $\int y^2\, ds$ along the arc of the curve, $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$. **Ans. :** $\frac{256a^3}{15}$

12. Find the length of the arc of the cardioide $r = a(1 - \cos\theta)$ which lies outside the circle, $r = a\cos\theta$. **(May 2010, Nov. 2014)**

    Ans. : $4a\sqrt{3}$

13. Show that the length of the arc of that part of the cardioide $r = a(1 + \cos\theta)$ which lies on the side of the line $4r = 3a \sec\theta$ remote from the pole, is equal to 4a.

14. Show that the whole length of the arc of the limacon $r = a\cos\theta + b$ is equal to that of an ellipse whose semiaxes are equal in length to the maximum and minimum radii vectors of the limacon.

15. Find the length of any arc of the curve $x^{2/3} - y^{2/3} = a^{2/3}$. **Ans. :** $\left[\frac{1}{2}\left(x_2^{2/3} + y_2^{2/3}\right)^{3/2} - \left(x_1^{2/3} + y_1^{2/3}\right)^{3/2}\right]$

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