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Differential Equations

An equation involving the independent variable and dependent variable and also the derivatives of the dependent variable with respect to independent variable is known as a differential equation.

e.g., $x\dfrac{dy}{dx} + xy\dfrac{d^2y}{dx^2} + 4 = 0$ is a differential equation.

Order of a Differential Equation

The order of a differential equation is the order of the highest derivative (differential coefficient) involved in its expression.

e.g., differential equation $\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} + xy\dfrac{d^2y}{dx^2} + 4 = 0$ is of order 3. The differential equation $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = 0$ is of order 2.

Degree of Differential Equation

The highest exponent of the highest derivative is called degree of a differential equation provided exponent of each derivative and the unknown variable appearing in the differential equation is a non-negative integer.

e.g., (i) $$\frac{d^2y}{dx^2} + \sqrt{\frac{dy}{dx}} + x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{d^2y}{dx^2} + x\right)^2$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) + 2x\frac{d^2y}{dx^2} + x^2 = 0$$

Here, degree is 2.

(ii) $$\left(\frac{d^3y}{dx^3}\right)^{2/3} + x + y = 0$$

$$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)^2 = (-x-y)^3$$

So, degree is 2.

Example 1. The order and degree of the differential equation $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = k\left(\dfrac{d^2y}{dx^2}\right)$ are respectively

(a) 2, 2
(b) 2, 3
(c) 3, 4
(d) 1, 5

Solution (a) The given equation is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

This shows that the degree and order of the given differential equation are 2 and 2.

Formation of a Differential Equation

The differential equation of a family of curves of one parameter is a differential equation of the first order. The differential equation of a family of curves of two parameters is a differential equation of the order two and the differential equation of a family of curves of n parameters is a differential equation of n order.

If the family of curves have one parameter, then we differentiate it once and eliminate parameter using equation of family of curves and equation, we get after differentiation. *e.g.*, $x^2 + y^2 = a^2$ (a is a parameter), represents family of concentric circles.

$$x^2 + y^2 = a^2 \quad \text{...(i)}$$

Differentiate Eq. (i), we get

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \text{...(ii)}$$

$\Rightarrow \quad x\,dx + y\,dy = 0$ is a differential equation.

Family of curves of two parameters will be differentiated twice to get a relation independent of any parameter.

Similarly, for family of curves of n parameters will be differentiated n times and then eliminate all the parameters.

e.g., $\quad y = a \sin \mu x + b \cos \mu x$

(where a and b are parameters)

$$\Rightarrow \quad \frac{dy}{dx} = a\mu \cos \mu x - b\mu \sin \mu x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -a\mu^2 \sin \mu x - b\mu^2 \cos \mu x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\mu^2 (a \sin \mu x + b \cos \mu x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \mu^2 y = 0 \text{ is a differential equation}$$

Example 2. The differential equation corresponding to $y^2 = m(a^2 - x^2)$ is

(a) $x\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right] = y\dfrac{dy}{dx}$ (b) $2x\dfrac{dy}{dx} = y$

(c) $x^2\dfrac{dy}{dx} = 1$ (d) None of these

Solution (a) Given differential equation is

$$y^2 = m(a^2 - x^2) \quad \text{...(i)}$$

On differentiating wrt x, we get

$$\Rightarrow \quad 2y\frac{dy}{dx} = m(-2x) \quad \text{...(ii)}$$

$$\Rightarrow \quad y\frac{dy}{dx} = -mx$$

Again, differentiating wrt x, we get

$$\Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$$

- Here, parameters are also called arbitrary constants.

Solution of a Differential Equation

Any relation connecting the variables of an equation and not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation, is called a solution of the differential equation.

General Solution

The solution which contains a number of arbitrary independent constants equal to the order of the differential equation is called the general solution or the complete primitive of the equation.

Particular Solution

The solution obtained from the general solution by assigning particular values to one or more of the arbitrary constants are called particular solutions.

Different Forms of First Order and First Degree Differential Equations

Variable Separable Differential Equations

$$f(x)\,dx = g(y)\,dy$$

Method Integrate it on both sides, we get

$$\int f(x)\,dx = \int g(y)\,dy + C$$

Example 3. The solution of $\dfrac{dy}{dx} = e^{x+y}$ is

(a) $x = e^y + C$ (b) $-e^{-y} = e^x + C$

(c) $y = e^x + C$ (d) None of these

Solution (b) $\because \quad \dfrac{dy}{dx} = e^{x+y}$

$$\Rightarrow \quad \frac{dy}{dx} = e^x \cdot e^y$$

Separating the variables, we get

$$\Rightarrow \quad e^{-y}dy = e^x\,dx$$

On integrating both sides, we get

$$\Rightarrow \quad -e^{-y} = e^x + C$$

Reducible to Variable Separable Differential Equation

Some times differential equation does not take directly form of the type $f(x)\,dx = g(y)dy$ but after some substitution, we get this form.

e.g., $\quad \dfrac{dy}{dx} = x + y$

Put $\quad x + y = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dt}{dx} - 1$$

So, $\quad \dfrac{dt}{dx} - 1 = t \Rightarrow \dfrac{dt}{t+1} = dx$

Now, this reduces to variable separable differential equation.

Example 4. The solution of $\dfrac{dy}{dx} = \cos(x+y)$ is

(a) $\sin(x+y) = C$ (b) $\cos(x^2+y^2) = 2C$

(c) $\tan\dfrac{(x+y)}{2} = x + C$ (d) None of these

Solution (c) Given, $\dfrac{dy}{dx} = \cos(x+y)$

Let $x + y = t$

$\Rightarrow \quad 1+\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow \frac{dy}{dx}=\frac{dt}{dx}-1$

$\therefore \quad \frac{dt}{dx}-1=\cos t$

$\Rightarrow \quad \frac{dt}{dx}=1+\cos t$

Separating the variables, we get

$\Rightarrow \quad \frac{1}{2}\sec^2\frac{t}{2}\,dt=dx$

On integrating both sides, we get

$$\tan\frac{t}{2}=x+C$$

$$\Rightarrow \quad \tan\frac{(x+y)}{2}=x+C$$

Homogeneous Differential Equation

A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$, where $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of x, y and of the same degree is said to be homogeneous differential equation.

e.g., $x^2+y^2\cdot\frac{dy}{dx}+xy=0$ is a homogeneous differential equation.

Method $\quad \frac{dy}{dx}=f\left(\frac{y}{x}\right)$

Now, put $\quad \frac{y}{x}=t \Rightarrow y=tx$

or $\quad \frac{dy}{dx}=t+x\frac{dt}{dx}$

So, $\quad t+x\frac{dt}{dx}=f(t)$

$\Rightarrow \quad \frac{dt}{f(t)-t}=\frac{dx}{x}$

This reduces to variable separable differential equation.

Example 5. The solution of $(x+y)\,dy-(x-y)\,dx=0$ is

(a) $x^2+y^2-2x=C$

(b) $(x^2-2xy-y^2)^{-1/2}=C$

(c) $x^2+y^2=C$

(d) None of the above

Solution (b) Given, $(x+y)\,dy-(x-y)\,dx=0$

$\Rightarrow \quad \frac{dy}{dx}=\frac{x-y}{x+y}$

Put $\quad y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore \quad v+x\frac{dv}{dx}=\frac{1-v}{1+v}$

$\Rightarrow \quad x\frac{dv}{dx}=\frac{1-v-v-v^2}{1+v}$

$\Rightarrow \quad \frac{1+v}{(1-2v-v^2)}dv=\frac{1}{x}dx$

$\Rightarrow \quad \int\frac{1+v}{2-(1+v)^2}dv=\int\frac{1}{x}dx$

$\Rightarrow \quad -\frac{1}{2}\log[2-(1+v)^2]=\log x+\log C$

$\Rightarrow \quad -\frac{1}{2}\log\left(1-\frac{2y}{x}-\frac{y^2}{x^2}\right)=\log x+\log C$

$\Rightarrow \quad -\frac{1}{2}\log(x^2-2xy-y^2)=\log C$

$\Rightarrow \quad (x^2-2xy-y^2)^{-1/2}=C$

Reducible to Homogeneous Differential Equation

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2} \quad \left(\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right)$$

Put $x=X+h$ and $y=Y+k$ (h, k are constants)

$$\frac{dy}{dx}=\frac{dY}{dX}$$

$$\Rightarrow \quad \frac{dY}{dX}=\frac{a_1X+b_1Y+c_1+a_1h+b_1k}{a_2X+b_2Y+c_2+a_2h+b_2k}$$

Put $\quad a_1h+b_1k+c_1=0 \quad$ ...(i)

$\quad a_2h+b_2k+c_2=0 \quad$ ...(ii)

Solving Eqs. (i) and (ii) for h and k

Now, $\quad \frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$

So, this reduces to homogeneous differential equation and if $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, then put $a_1x+b_1y=t$ and then it reduces to variable separable differential equation.

Example 6. The solution of $(2x-y+4)\,dy+(x-2y+5)\,dx=0$ is

(a) $(x+y-1)^3=A(x-y+3)$

(b) $x+y=A$

(c) $x^2+y^2=3A$

(d) None of the above

Solution (a) $(2x-y+4)\,dy+(x-2y+5)\,dx=0 \quad$ ...(i)

The given equation is non-homogeneous and

$$a_1b_2-a_2b_1=-3\neq 0$$

On solving, $2x-y+4=0$, $x-2y+5=0$,

We get, $\quad x=-1$ and $y=2$

Now, substitute $x=-1+u$, $y=2+v$ in Eq. (i)

and $\quad dx=du,\ dy=dv$

$\therefore$ Eq. (i) reduce to

$$(2u-v)\,dv+(u-2v)\,du=0 \quad \text{...(ii)}$$

This is a homogeneous equation in u and v.

Put $u=vt$ and $du=t\,dv+v\,dt$

$\therefore$ Eq. (ii) reduce to

$$(2tv - v)\,dv + (tv - 2v)(t\,dv + v\,dt) = 0$$

$$\Rightarrow \quad (2t-1)\,dv + (t-2)\,t\,dv + v\,(t-2)\,dt = 0$$

$$\Rightarrow \quad (t^2 - 2t + 2t - 1)\,dv + v\,(t-2)\,dt = 0$$

Separating the variable v and t, we get

$$\frac{dv}{v} + \frac{t-2}{t^2-1}\,dt = 0$$

$$\therefore \quad \int \frac{dv}{v} + \int \frac{t-2}{t^2-1}\,dt = C_1$$

$$\Rightarrow \quad \ln|v| + \int \left\{ \frac{-1}{2(t-1)} + \frac{3}{2(t+1)} \right\} dt = C_1$$

$$\Rightarrow \quad \ln|v| - \frac{1}{2}\ln|t-1| + \frac{3}{2}\ln|t+1| = \ln C,$$

where $C_1 = \ln C \ (C > 0)$

Put $t = \dfrac{u}{v}$

$$\Rightarrow \quad \ln|v| - \frac{1}{2}\ln\left|\frac{u-v}{v}\right| + \frac{3}{2}\ln\left|\frac{u+v}{v}\right| = \ln C$$

$$\Rightarrow \quad \ln\left[\frac{|v|\cdot|v^{1/2}|\cdot(u+v)^{3/2}}{(u-v)^{1/2}\cdot v^{3/2}}\right] = \ln C$$

$$\therefore \quad (u+v)^{3/2} = C\,(u-v)^{1/2}$$

$$\Rightarrow \quad (u+v)^3 = A\,(u-v),\ A = C^2$$

$\Rightarrow$ $(x+y-1)^3 = A\,(x-y+3)$ is the general solution.

Linear Differential Equation

Differential equation of the form $\dfrac{dy}{dx} + Py = Q$ (where P and Q are functions of x) is known as linear differential equation.

Method Multiply it with R (a function of x)

Now, let $R\dfrac{dy}{dx} + R\cdot Py = \dfrac{d}{dx}(R\cdot y)$

$$R\frac{dy}{dx} + R\cdot Py = R\cdot\frac{dy}{dx} + y\cdot\frac{dR}{dx}$$

$$\Rightarrow \quad y\cdot\frac{dR}{dx} = R\cdot Py$$

$$\Rightarrow \quad \frac{dR}{R} = P\,dx$$

$$\Rightarrow \quad R = e^{\int P\,dx}$$

(we call it integrating factor and denote it by IF)

Now, $\dfrac{d}{dx}(R\cdot y) = R\cdot Q$

$\Rightarrow$ $R\cdot y = \int R\cdot Q\,dx$ (where, R is integrating factor)

Example 7. The solution of $\dfrac{dy}{dx} + \dfrac{y}{x} = \log x$ is

(a) $yx = \log x + C$
(b) $yx = \dfrac{x^2}{2}(\log x) - \dfrac{x^2}{4} + C$
(c) $x^2y^2 = \log x + C$
(d) None of these

Solution (b) It is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

Here, $P = \dfrac{1}{x}, Q = \log x$

$$\therefore \quad \text{IF} = e^{\int P\,dx} = e^{\int 1/x\,dx} = e^{\log x} = x$$

Hence, the general solution is

$$y\cdot(\text{IF}) = \int Q\cdot(\text{IF})\,dx + C$$

$$\Rightarrow \quad yx = \int (\log x)\,x\,dx + C$$

$$\Rightarrow \quad yx = (\log x)\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx + C$$

$$\Rightarrow \quad yx = \frac{x^2}{2}(\log x) - \frac{x^2}{4} + C$$

Reducible to Linear Differential Equation

$$T(y)\frac{dy}{dx} + P\cdot S(y) = Q$$

(where, P and Q are functions of x)

and $\dfrac{dS(y)}{dy} = T(y)$

Method Put $S(y) = z$, then

$$\frac{dz}{dx} + P\cdot z = Q$$

e.g.,

(1) $\quad 2y\dfrac{dy}{dx} + \dfrac{y^2}{x} = \sin x \quad \ldots$ (i)

Putting $y^2 = t \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Eq. (i) transforms to $\dfrac{dt}{dx} + \dfrac{t}{x} = \sin x$

(linear differential equation)

(2) $\quad \cos y\dfrac{dy}{dx} + x\sin y = x^2 \quad \ldots$(i)

Putting $\sin y = t$

$$\Rightarrow \quad \cos y\frac{dy}{dx} = \frac{dt}{dx}$$

Eq. (i) transforms to $\dfrac{dt}{dx} + xt = x^2$

(linear differential equation)

Example 8. The solution of $(1+x^2)\dfrac{dy}{dx} + 2xy - 1 = 0$ subject to initial condition is

(a) $y(1+x^2) = C$
(b) $xy^2 = C$
(c) $xy^3 = C$
(d) None of these

Solution (a) The given equation can be rewritten as

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{1+x^2}$$

On comparing with $\dfrac{dy}{dx} + Py = Q$, we get