Unit 4 Chapter 7

Errors and Approximations, Maxima and Minima

7.1 Errors and Approximation

Consider a circle of radius 10 cm as shown in Fig. 7.1.1.

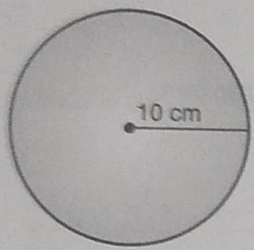

Fig. 7.1.1

$\therefore$ Area of circle $= A = \pi r^2 = \pi (10)^2$

$$A = 100\,\pi\ \text{cm}^2$$

Now, there can be a human error while calculating the radius of circle, it can be positive or negative. Lets say, radius calculated is 10.1 cm instead of 10 cm.

Now for this human error in measurement of radius will cause an error in measurement of area of circle.

$$\therefore \quad A = \pi r^2$$

$$= \pi (10.1)^2$$

$$A = 102.01\,\pi\ \text{cm}^2$$

$\therefore$ For $(10.1 - 10)$ cm $= 0.1$ cm error in radius there will corresponding $(102.01\,\pi - 100\,\pi) = 2.01\,\pi\ \text{cm}^2$ error in area.

And this is called as error.

Approximation :

If dr is the actual error made in measurement of radius 'r', then the dA is the approximate error introduced in area 'A'. It is called as approximate error since we calculate area (dA) using respective formulae.

$\frac{100\,dr}{r}$ and $\frac{100\,dA}{A}$ are known as % error in radius and area respectively.

Where $\quad dr =$ Actual error in radius

$dA =$ Approximate error in Area

$\frac{dr}{r} =$ relative error in radius

7.1.1 Solved Examples on Errors and Approximations

Example 7.1.1 **Dec. 2009, May 2011, 2012, 2015**

Find the percentage error in the area of an ellipse when an error of 2% and 3% is made in measuring its major and minor axis.

Solution :

Refer Fig. P. 7.1.1

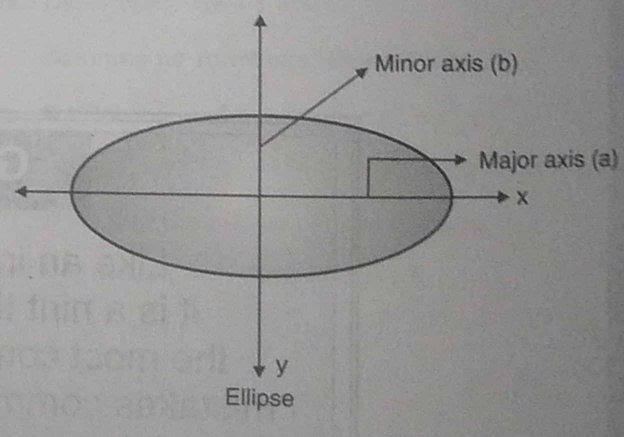

Fig. P. 7.1.1

Given :

$$\frac{100\,da}{a} = 2\ ; \qquad \frac{100\,db}{b} = 3$$

Area of an ellipse is given by,

$$\therefore \quad A = \pi ab$$

Taking log on both sides

$$\log A = \log (\pi ab)$$

$$\therefore \quad \log A = \log \pi + \log a + \log b$$

$$\because \log (abc) = \log a + \log b + \log c$$

Differentiating

$$\frac{1}{A}dA = 0 + \frac{1}{a}da + \frac{1}{b}db$$

Multiplying by 100

$$\therefore \quad \frac{100\,dA}{A} = \frac{100\,da}{a} + \frac{100\,db}{b} = 2 + 3 \quad \text{...(Given)}$$

$$\frac{100\,dA}{A} = 5$$

$\therefore$ **% error in area A = 5 %**

Example 7.1.2

In calculating volume of right circular ***cylinder****, errors of 2% and 1 % are found in measuring height and radius of base respectively. Find the percentage error in calculating volume of the cylinder.*

Solution :

Let $\quad r$ = radius of base

and $\quad h$ = height

Given :

$$\frac{100\,dh}{h} = 2$$

and

$$\frac{100\,dr}{r} = 1$$

Now, volume (V) of right circular cylinder is given by,

$$V = \pi r^2 h$$

Taking log on both sides,

$$\log V = \log(\pi r^2 h)$$

$$\log V = \log \pi + \log r^2 + \log h$$

by $\log m^n = n \log m$

$$\therefore \quad \log V = \log \pi + 2\log r + \log h$$

Differentiating,

$$\frac{1}{V}dV = 0 + 2\frac{1}{r}dr + \frac{1}{h}dh$$

Multiplying by 100

$$\frac{100\,dV}{V} = 2\frac{100\,dr}{r} + \frac{100\,dh}{h}$$

$$\frac{100\,dV}{V} = 2(1) + 2 \quad \text{...(Given)}$$

$$\frac{100\,dV}{V} = 4$$

$\therefore$ **% error in volume V = 4 %.**

Example 7.1.3 **May 2007, 10, 13, Dec. 2004, 15**

In calculating the volume of a right circular cone, errors of 2% and 1 % are made in measuring the height and radius of base respectively. Find the error in calculated volume.

Solution :

Let $\quad h$ = height

$\quad r$ = radius of base

Given :

$$\frac{100\,dh}{h} = 2; \quad \frac{100\,dr}{r} = 1$$

Volume V of right circular cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$

Taking log on both sides,

$$\log V = \log\left[\frac{1}{3}\pi r^2 h\right]$$

$$\log V = \log\frac{1}{3} + \log \pi + \log r^2 + \log h$$

by $\quad \log m^n = n \log m$

$$\log V = \log\frac{1}{3} + \log \pi + 2\log r + \log h$$

Differentiating,

$$\frac{1}{V}dV = 0 + 0 + 2\frac{1}{r}dr + \frac{1}{h}dh$$

Multiplying by 100

$$\frac{100\,dV}{V} = 2\frac{100\,dr}{r} + \frac{100\,dh}{h}$$

$$\therefore \quad \frac{100\,dV}{V} = 2(1) + 2$$

$$\frac{100\,dV}{V} = 4$$

$\therefore$ **% error in Volume V = 4 %**

Example 7.1.4 **May 2014**

The resonant frequency in a series electrical circuit is given by $f = \frac{1}{2\pi\sqrt{LC}}$. If the measurements in L and C are in error by + 2% and – 1% respectively, find the percentage error in f.

Solution :

Given :

$$\frac{100\,dL}{L} = 2 \quad \text{and} \quad \frac{100\,dC}{C} = -1$$

and $$f = \frac{1}{2\pi\sqrt{LC}}$$

$$f = \frac{1}{2\pi\sqrt{L}\sqrt{C}}$$

Taking log on both sides

$$\log f = \log\left(\frac{1}{2\pi\sqrt{L}\sqrt{C}}\right)$$

by $\log\left(\frac{A}{B}\right) = \log A - \log B$

$$\log f = \log 1 - \log(2\pi\,L^{1/2}\,C^{1/2})$$

$$\log f = \log 1 - [\log 2 + \log \pi + \log L^{1/2} + \log C^{1/2}]$$

by $\log m^n = n \log m$

$$\log f = \log 1 - \log 2 - \log \pi - \frac{1}{2}\log L - \frac{1}{2}\log C$$

Differentiating,

$$\frac{1}{f}\,df = 0 - 0 - 0 - \frac{1}{2}\frac{1}{L}\,dL - \frac{1}{2}\frac{1}{C}\,dC$$

Mutiplying by 100

$$\frac{100\,df}{f} = -\frac{1}{2}\frac{100\,dL}{L} - \frac{1}{2}\frac{100\,dC}{C}$$

$$\therefore \quad \frac{100\,df}{f} = -\frac{1}{2}(2) - \frac{1}{2}(-1) \quad \text{...(Given)}$$

$$= -1 + \frac{1}{2}$$

$$\frac{100\,df}{f} = -\frac{1}{2}$$

$\therefore$ **% error in f is – 0.5 %.**

Example 7.1.5 **Dec 2005, May 2008**

The H.P. required to propel a steamer varies as cube of the velocity and square of length. If there is 3% increase in velocity and 4% increase in length find the % increase in H.P.

Solution :

Let H.P. = Horse power = H

Velocity = V

Lenght = l

Now, $H \propto V^3 l^2$

$$H = K V^3 l^2$$

Where K = *constant of proportionality*

Taking log on both sides

$$\log H = \log(KV^3 l^2)$$

$$\log H = \log K + \log V^3 + \log l^2$$

by $\log m^n = n \log m$

$$\log H = \log K + 3\log V + 2\log l$$

Differentiating,

$$\frac{1}{H}\,dH = 0 + 3\frac{1}{V}\,dV + 2\frac{1}{l}\,dl$$

Multiplying by 100,

$$\frac{100\,dH}{H} = 3\,\frac{100\,dV}{V} + 2\,\frac{100\,dl}{l}$$

But given,

$$\frac{100\,dV}{V} = 3\ ; \quad \frac{100\,dl}{l} = 4$$

$$= 3(3) + 2(4)$$

$$\frac{100\,dH}{H} = 17$$

$\therefore$ **% increase in H.P. is 17 %.**

Example 7.1.6 **May 2004, Dec. 2008, [illegible]**

The area of triangle ABC, is calculated from the formula $\Delta = \frac{1}{2}bc \sin A$. Errors of 1 %, 2 % and 3 % respectively are made in measuring b, c, A. If the correct value of A is 45°, find the % error in calculated value of Δ.

Solution :

Given :

$$\frac{100\,db}{b} = 1\ ; \quad \frac{100\,dc}{c} = 2\ ; \quad \frac{100\,dA}{A} = 3$$

and $$\Delta = \frac{1}{2}bc\sin A$$

Taking log on both sides

$$\log \Delta = \log\left[\frac{1}{2} bc \sin A\right]$$

$$\log \Delta = \log\frac{1}{2} + \log b + \log c + \log (\sin A)$$

Differnetiating

$$\frac{1}{\Delta} d\Delta = 0 + \frac{1}{b} db + \frac{1}{c} dc + \frac{1}{\sin A}\cos A \cdot dA$$

$$\boxed{\text{But } \frac{\cos A}{\sin A} = \cot A}$$

Multiplying by 100

$$\frac{100\, d\Delta}{\Delta} = \frac{100\, db}{b} + \frac{100\, dc}{c} + \cot A \cdot (100\, dA)$$

but $\frac{100\, dA}{A} = 3 \Rightarrow 100\, dA = 3A$

$$\therefore \quad \frac{100\, d\Delta}{\Delta} = 1 + 2 + \cot A\,(3A)$$

But correct value of A is $\left[45^\circ = \left(\frac{\pi^c}{4}\right)\right]$

$$\therefore \quad \frac{100\, d\Delta}{\Delta} = 3 + \frac{3\pi}{4}\cot\left(\frac{\pi}{4}\right)$$

But $\cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = \frac{1}{\tan (45)} = \frac{1}{1} = 1$

$$\therefore \quad \frac{100\, d\Delta}{\Delta} = 3 + \frac{3\pi}{4}$$

$\therefore$ **% error in Δ is $\left(3 + \frac{3\pi}{4}\right)$ %.**

Example 7.1.7 **May 2005, 2017**

Find the possible percentage error in calculating parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, if r_1, r_2, r_3 are each in error by plus 1.2 %.

Solution : Given :

$$\frac{100\, dr_1}{r_1} = 1.2\,;\ \frac{100\, dr_2}{r_2} = 1.2\,;\ \frac{100\, dr_3}{r_3} = 1.2 \qquad ...(1)$$

and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Differentiating,

$$-\frac{1}{r^2} dr = -\frac{1}{r_1^2} dr_1 - \frac{1}{r_2^2} dr_2 - \frac{1}{r_3^2} dr_3 \qquad \left\{\because \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}\right\}$$

$$-\frac{1}{r}\frac{dr}{r} = -\left[\frac{1}{r_1}\frac{dr_1}{r_1} + \frac{1}{r_2}\frac{dr_2}{r_2} + \frac{1}{r_3}\frac{dr_3}{r_3}\right]$$

Multiplying by 100 on both sides

$$\frac{1}{r}\frac{100\, dr}{r} = \frac{1}{r_1}\frac{100\, dr_1}{r_1} + \frac{1}{r_2}\frac{100\, dr_2}{r_2} + \frac{1}{r_3}\frac{100\, dr_3}{r_3}$$

$$\therefore \quad \frac{1}{r}\frac{100\, dr}{r} = \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$

$$\frac{1}{r}\frac{100\, dr}{r} = \left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right](1.2)$$

But from Equation (1)

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

So $\frac{1}{r}\frac{100\, dr}{r} = \frac{1}{r}(1.2)$

$$\therefore \quad \frac{100\, dr}{r} = 1.2$$

$\therefore$ **% Error in r is 1.2 %.**

Example 7.1.8 **May 2009, Dec. [illegible]**

The focal length of a mirror is found from the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. Find the percentage error in f if u and v are both in error by p % each.

Solution :

$$\frac{100\, du}{u} = p \text{ and } \frac{100\, dv}{v} = p \qquad ...(\text{Given})$$

And $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$ ...(1)

Differentiating,

$$-\frac{1}{v^2} dv + \frac{1}{u^2} du = -\frac{2}{f^2} df$$

$$-\left(\frac{1}{v}\frac{dv}{v} - \frac{1}{u}\frac{du}{u}\right) = -\frac{2}{f}\frac{df}{f}$$

Multiplying by 100

$$\frac{1}{v}\frac{100\, dv}{v} - \frac{1}{u}\frac{100\, du}{u} = \frac{2}{f}\frac{100\, df}{f}$$

$$\frac{1}{v}(p) - \frac{1}{u}(p) = \frac{2}{f}\frac{100\, df}{f} \qquad ...(\text{Given})$$

$$p\left(\frac{1}{v} - \frac{1}{u}\right) = \frac{2}{f}\frac{100\, df}{f}$$

But from Equation (1),

$$\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$$

So, $$p\left(\frac{2}{f}\right) = \frac{2}{f} \cdot 100 \frac{df}{f}$$

$$\therefore \quad \frac{100\, df}{f} = p$$

∴ **% error in f is p %.**

Example 7.1.9 **Dec. 2007**

If the kinetic energy is calculated by the formula $T = \frac{1}{2} mv^2$ and 'm' changes from 49 to 49.5 and 'v' changes from 1600 to 1590.

Find (i) Percentage error in kinetic energy 'T'

(ii) Approximate change in 'T'.

Solution :

Given :

m changes from 49 to 49.5

49 ↓ initial value, 49.5 ↓ final value

$$\therefore \quad dm = \text{final value} - \text{initial value}$$

$$dm = 49.5 - 49$$

$$dm = 0.5 \quad \text{and} \quad m = 49 \rightarrow \text{original value}$$

and v changes from 1600 to 1590

1600 ↓ initial value, 1590 ↓ final value

$$\therefore \quad dv = \text{final value} - \text{initial value}$$

$$= 1590 - 1600$$

$$dv = -10 \quad \text{and} \quad v = 1600 \rightarrow \text{original value}$$

(i) To find percentage error in T :

We have, $$T = \frac{1}{2} mv^2$$

Taking log on both sides

$$\log T = \log\left[\frac{1}{2} mv^2\right]$$

$$\therefore \quad \log T = \log \frac{1}{2} + \log m + \log v^2$$

By $\log m^n = n \log m$

$$\log T = \log \frac{1}{2} + \log m + 2 \log v$$

Differentiating,

$$\frac{1}{T} dT = 0 + \frac{1}{m} dm + 2 \frac{1}{v} dv$$

$$\therefore \quad \frac{1}{T} dT = \frac{1}{m} dm + 2 \frac{1}{v} dv \quad \text{...(1)}$$

Multiplying by 100

$$\frac{100\, dT}{T} = \frac{100\, dm}{m} + 2 \frac{100\, dv}{v}$$

$$\frac{100\, dT}{T} = \frac{100\,(0.5)}{49} + \frac{(2)(100)(-10)}{1600}$$

...(From given values)

$$\therefore \quad \frac{100\, dT}{T} = -0.2295$$

∴ **% error in T is – 0.2295 %**

(ii) To find approximate change in T (dT)

From Equation (1)

$$\frac{dT}{T} = \frac{dm}{m} + \frac{2dv}{v}$$

$$dT = T\left[\frac{dm}{m} + \frac{2dv}{v}\right]$$

but $$T = \frac{1}{2} mv^2$$

$$dT = \frac{1}{2} mv^2 \left[\frac{dm}{m} + \frac{2dv}{v}\right]$$

$$= \frac{1}{2} (49)(1600)^2 \left[\frac{0.5}{49} + \frac{(2)(-10)}{1600}\right]$$

$$dT = -144000$$

∴ **Approximate change in T is – 144000.**

Example 7.1.10 **Dec. 2004, 2005**

The period of simple pendulum with small oscillations is $T = 2\pi\sqrt{\frac{l}{g}}$. If T is calculated using l and g are 8.05 ft and 32.01 ft/sec² respectively, find % error in T if the true values of l and g are 8 ft and 32 ft/sec² respectively.

Solution :

Given : $$dl + l = 8.05$$

and $$dg + g = 32.01$$

But $l = 8$, $g = 32$

$$\therefore \quad dl = 8.05 - 8 \quad \text{and} \quad dg = 32.01 - 32$$

$$dl = 0.05 \qquad dg = 0.01$$

Now, $T = 2\pi\sqrt{\frac{l}{g}}$

$$T = 2\pi\frac{\sqrt{l}}{\sqrt{g}}$$

Taking log on both sides

$$\log T = \log\left[2\pi\frac{\sqrt{l}}{\sqrt{g}}\right]$$

$$= \log(2\pi\sqrt{l}) - \log\sqrt{g}$$

$$= \log 2 + \log\pi + \log l^{1/2} - \log g^{1/2}$$

$$\log T = \log 2 + \log\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

Differentiating

$$\frac{1}{T}dT = 0 + 0 + \frac{1}{2}\frac{1}{l}dl - \frac{1}{2}\frac{1}{g}dg$$

Multiplying by 100

$$\frac{100\,dT}{T} = \frac{1}{2}\frac{100\,dl}{l} - \frac{1}{2}\frac{100\,dg}{g}$$

$$= \frac{1}{2}\frac{(100)(0.05)}{(8)} - \frac{1}{2}\frac{(100)(0.01)}{(32)}$$

$$\frac{100\,dT}{T} = 0.2968$$

% error in T is 0.2968 %

Example 7.1.11 **Dec– 2013**

The resistance R of a circuit was found by using the formula $I = \frac{E}{R}$. If there is an error of 0.1 amp in reading I and 0.5 in E, find the corresponding possible % error in R when readings are I = 15 amp and E = 100 volts.

Solution :

Given : $dI = 0.1 \Rightarrow$ error in I

$dE = 0.5 \Rightarrow$ error in E

$I = 15$

$E = 100$

We have,

$$I = \frac{E}{R}$$

Important Step

$$R = \frac{E}{I}$$

Taking log on both sides,

$$\therefore \quad \log R = \log\left(\frac{E}{I}\right)$$

$$\therefore \quad \log R = \log E - \log I$$

Differentiating ,

$$\frac{1}{R}dR = \frac{1}{E}dE - \frac{1}{I}dI$$

Multiplying by 100,

$$\frac{100\,dR}{R} = \frac{100\,dE}{E} - \frac{100\,dI}{I}$$

$$\frac{100\,dR}{R} = \frac{(100)(0.5)}{100} - \frac{(100)(0.1)}{15} \quad \text{...(Given)}$$

$$\frac{100\,dR}{R} = -0.166$$

$\therefore$ **% error in R is – 0.166 %.**

Example 7.1.12 **Dec. 2015, 2016**

In estimating the cost of a pile of bricks measured 2m × 15 m × 1.2 m. The tape is stretched 1% beyond the standard length. If the count is 450 bricks per cubic meter and bricks cost Rs. 130 per thousand, find the approximate error in cost. Is the brick seller gaining or losing ?

Solution :

Let l = length, b = breadth and h = height of the pile

$\therefore$ Also, $V = 2 \times 15 \times 1.2$

$V = 36$ cubic meters

Now, $V = l \times b \times h$

Taking log on both sides

$$\log V = \log(l\,bh)$$

$$\therefore \quad \log V = \log l + \log b + \log h$$

Differentiating

$$\frac{1}{V}dV = \frac{1}{l}dl + \frac{1}{b}db + \frac{1}{h}dh$$

Multiplying by 100

$$\frac{100\,dV}{V} = \frac{100\,dl}{l} + \frac{100\,db}{b} + \frac{100\,dh}{h}$$

But tape is stretched 1% beyond the standard length.

$$\therefore \quad \frac{100\,dl}{l} = \frac{100\,db}{b} = \frac{100\,dh}{h} = 1$$

$$\therefore \quad \frac{100\,dV}{V} = 1 + 1 + 1$$

$$\therefore \quad \frac{100\,dV}{V} = 3$$

$$\therefore \quad dV = \frac{3V}{100}$$

$$dV = \frac{3 \times 36}{100}$$

$$dV = 1.08$$

This represent approximate change in volume due to 1 % stretch in tape.

Now,

$$\text{Number of bricks in error} = dV \times 450$$

$$= 1.08 \times 450$$

$$\text{Number of bricks in error} = 486$$

Given cost of bricks is Rs. 130 per thousand

$$\therefore \quad \text{Error in cost} = 486 \times \frac{130}{1000}$$

Error in cost = 63.18 Rs.

Since the calculated volume V + dV = 36 + 1.08 = 37.08 cubic meter is more than the actual volume 36 cubic meter, seller is gaining.

Example 7.1.13 **May 2010**

The voltage V, across a resistor is measured with error dV and the resistance R with error dR. Prove that the error in calculating the power $W = \frac{V^2}{R}$, generated in the resistor is $\frac{V^2}{R}(2RdV - VdR)$. If there are errors of 1 % and 2% respectively in measuring the voltage V and resistance R, find % error in calculating $W = \frac{V^2}{R}$.

Solution :

Part 1 : To find % error in W.

Given : $\frac{100\,dV}{V} = 1$ and $\frac{100\,dR}{R} = 2$

Now $\quad W = \frac{V^2}{R}$

Taking log on both sides

$$\log W = \log\left(\frac{V^2}{R}\right)$$

$$\log W = \log V^2 - \log R$$

$$\therefore \quad \log W = 2 \log V - \log R$$

Differentiating ,

$$\frac{1}{W}dW = 2\frac{1}{V}dV - \frac{1}{R}dR$$

Multiplying by 100

$$\frac{100\,dW}{W} = 2\frac{100\,dV}{V} - \frac{100\,dR}{R}$$

$$\therefore \quad \frac{100\,dW}{W} = 2(1) - 2$$

$$\frac{100\,dW}{W} = 0$$

$\therefore$ **% error in W = 0**

Part 2 : To prove $dW = \frac{V^2}{R}$ [2RdV – vdR]

We have

$$W = \frac{V^2}{R}$$

Differentiating w.r.t. V partially

$$\frac{\partial W}{\partial V} = \frac{2V}{R}$$

$$\text{Again } W = \frac{V^2}{R}$$

Differentiating w.r.t. R partially

$$\frac{\partial W}{\partial R} = -\frac{V^2}{R^2}$$

Now by concept of partial differentiation,

$$\partial W = \frac{\partial W}{\partial V}dV + \frac{\partial W}{\partial R}dR$$

$$\partial W = \frac{2V}{R}dV + \left(-\frac{V^2}{R^2}\right)dR$$

$$= \frac{2VRdV}{R^2} - \frac{V^2\,dR}{R^2}$$

$$\partial W = \frac{V}{R^2}[2RdV - VdR]$$

Hence proved.

Example 7.1.14 **May 2016**

A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surrounded by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the % change in the volume of a balloon.

Solution :

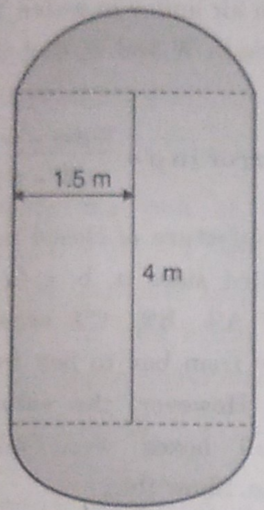

Fig. P. 7.1.14

Given : $r = 1.5$ m

$h = 4$ m $\rightarrow$ length / height

$dr = 0.01$ m

$dh = 0.05$ m

Total volume V of the balloon is,

V = Volume of cylinder + Volume of two Hemispheres

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$$

$$V = \pi r^2 h + \frac{4}{3}\pi r^3 \quad \ldots(1)$$

Differentiating,

Note : We can not take log, as terms are in addition.

$$dV = \pi\,[r^2 dh + h \cdot 2r\, dr] + \frac{4}{3}\pi 3r^2\, dr$$

$$dV = \pi r^2\, dh + 2\pi r\, h dr + 4\pi r^2\, dr$$

Divide by Equation (1)

$$\frac{dV}{V} = \frac{\pi r^2\, dh + 2\pi r\, h dr + 4\pi r^2\, dr}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$\therefore \quad \frac{dV}{V} = \frac{\pi r\,[r dh + 2h dr + 4r dr]}{\pi r\left[rh + \frac{4}{3}r^2\right]}$$

$$\frac{dV}{V} = \frac{r dh + 2h dr + 4r dr}{rh + \frac{4}{3}r^2}$$

$$\therefore \quad \frac{dV}{V} = \frac{(1.5)(0.05) + 2(4)(0.01) + 4(1.5)(0.01)}{(1.5)(4) + \frac{4}{3}(1.5)^2}$$

$$\frac{dV}{V} = 0.0238$$

$\therefore$ Multiply by 100 for % error

$$\therefore \quad \frac{100\, dV}{V} = 100 \times 0.0238$$

$$\frac{100\, dV}{V} = 2.38$$

$\therefore$ **% error in v is 2.38**

Example 7.1.15 **Dec. 2009**

Given $z = 2xy^2 - 3x^2y$, x increases at the rate of 2 cm / sec. as it passes through 3 cm. Show that if y is passing through 1 cm, y must decrease at the rate of $\frac{32}{15}$ cm / sec. in order that z shall remain constant.

Solution :

Given : $\frac{dx}{dt} = 2$ cm/sec.

Given that, z shall remain constant.

$$\therefore \quad \frac{dz}{dt} = 0$$

Also $\quad z = 2xy^2 - 3x^2y$

Differntiate w.r.t. x partially

$$\therefore \quad \frac{\partial z}{\partial x} = 2y^2 - 6xy$$

$$\left(\frac{\partial z}{\partial x}\right)_{(3,1)} = 2(1)^2 - 6(3)(1)$$

$$\left(\frac{\partial z}{\partial x}\right)_{(3,1)} = -16$$

Again, $\quad z = 2xy^2 - 3x^2y$

Differentiate w.r.t. y partially

$$\left(\frac{\partial z}{\partial y}\right) = 4xy - 3x^2$$

$$\left(\frac{\partial z}{\partial y}\right)_{(3,1)} = 4(3)(1) - 3(3)^2$$

$$\left(\frac{\partial z}{\partial y}\right)_{(3,1)} = -15$$

Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\therefore \quad 0 = (-16)(2) + (-15)\frac{dy}{dt}$$

$$\therefore \quad 32 = -15\frac{dy}{dt}$$

$$\therefore \quad \frac{dy}{dt} = -\frac{32}{15}$$

Hence y decreases at the rate of $\frac{32}{15}$ cm/sec.

Exercise 7.1

1. The deflection at the centre of a rod of length l and diameter d, supported at its ends and loaded at the centre with a weight w, varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in w, l and d of 3, 2 and 1 respectively ?

 [Hint : $D \sim wl^3 d^{-4}$]

Ans.: 5 %

2. A power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Using calculus, find the approximate percentage error in P when E is increased by 3% and R is decreaed by 2 %.

 Hint : As E is increased by 3%

 $$\therefore \quad \frac{100\, dE}{E} = 3$$

 And as R is decreased by 2%

 $$\therefore \quad \frac{100\, dR}{R} = -2$$

Ans.: % error in power = 8%

3. The area of rectangular field is calculated by measuring its length and breadth. If there is an error of 2 % in measuring the length and an error of 3 % in measuring the breadth of the field, find the approximate % error in the calculated area of the field.

Ans.: 5 %

4. Find the possible % error in calculating the parallel resistance r of two resistances r_1, say r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where r_1 and r_2 are both in error by + 2 % each.

Ans.: 2 %

5. The density ρ of a body is calculated from its weight W in air and w in water. If errors dW and dw are made in W and w, find the relative error in ρ.

Ans.: Relative error in $\rho = \frac{W\,dw - w\,dW}{(W - w)^2}$

6. In the manufacture of closed rectangular boxes with specified sides a, b, c ($a \neq b \neq c$) small changes of A%, B%, C% occurred in a, b, c respectively from box to box from the specified dimension. However, the volume and surface area of all boxes were according to the specification. Show that :

 $$\frac{A}{a\,(b-c)} = \frac{B}{b\,(c-a)} = \frac{C}{c\,(a-b)}$$

7. In estimating cost of a pile of bricks measured 2m × 15 m × 1.2 m, **the top of the pile is stretched 1 % beyond the standard length.** If the count is 450 bricks pile cubic meter and bricks cost Rs. 450 per thousand, find the approximate error in cost. **(Dec. [illegible])**

 Hint : $\frac{100\, dl}{l} = 0, \frac{100\, db}{b} = 0, \frac{100\, dh}{h} = 1$

Ans.: 72.9 Rs.

8. Find $[(3.82)^2 + 2(2.1)^3]^{1/5}$ by using the theory of approximations.

Ans.: 2.012

9. At a distance 20 m from the foot of a tower, the elevation of its top is 60°. If the possible errors in measuring distance and elevation are 1 cm and 1 minute, find the approximate error in the calculated height.

Ans.: 0.0407 m

24. If $e^z = \sec x - \sec y$ and errors of magnitude h and − h are made in measuring x and y, where x and y are found to be $\frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively. Find the error in z. (May 2013)

Ans.: $dz = \frac{4\sqrt{3}}{5} h$

7.2 Maxima and Minima

A maximum is a high point and a minimum is a low point as shown in Fig 7.2.1.

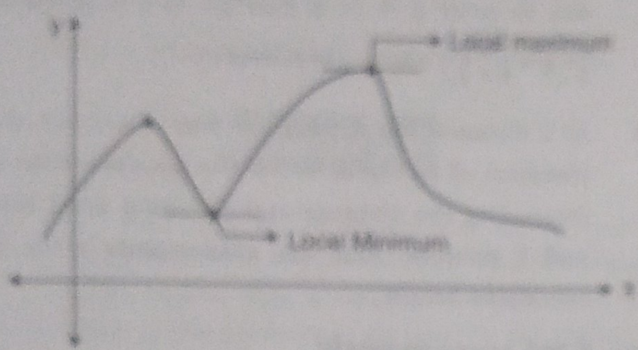

Fig. 7.2.1

A high point is called as maxima (plural of maximum).

A low point is called as Minima (plural of Minimum).

The general word for maxima or minima is extrema (plural of extremum).

Note : We say local maximum or local minimum when there may be higher or lower points else where but not nearby.

7.2.1 Working Rule for Determining the Maxima and Minima of the Function f(x, y)

Step 1 : Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero.

i.e. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Step 2 : Solve these simultaneous equations to find different values of x and y.

Let the roots are $(a_1, b_1), (a_2, b_2)$

Step 3 : Calculate

$$r = \frac{\partial^2 f}{\partial x^2} \quad \Bigg| \quad s = \frac{\partial^2 f}{\partial x \partial y} \quad \Bigg| \quad t = \frac{\partial^2 f}{\partial y^2}$$

For all roots $(a_1, b_1), (a_2, b_2)$ and so on.

Step 4 :

$(rt - s^2)_{(a_1, b_1)}$	$r_{(a_1, b_1)}$	Remark	Max / Min value
$rt - s^2 > 0$	$r < 0$	f(x, y) has max value	$f(a_1, b_1) = f_{max}$
$rt - s^2 > 0$	$r > 0$	f(x, y) has min value	$f(a_1, b_1) = f_{min}$
$rt - s^2 < 0$	–	f has neither max nor min value	$f(a_1, b_1)$ is a saddle point
$rt - s^2 = 0$	–	No conclusion	Further investigation is necessary

Note : **Saddle point** : The point which has neither maximum nor minimum value is called as saddle point.

7.2.2 Solved Examples on Maxima and Minima

Example 7.2.1 Dec. 2005, 2017, May 2016

Discuss the maxima and minima of $f(x, y) = x^2 + y^2 + 6x + 12$.

Solution :

Step 1 : Let

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$2x + 6 = 0 \qquad 2y = 0$$

$$2x = -6 \qquad y = 0$$

$$x = -3$$

Step 2 : Roots are (– 3, 0)

Step 3 : To find r, s, t

Now, $f = x^2 + y^2 + 6x + 12$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 2x + 6$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = 2 = (r)$$

Again, $f = x^2 + y^2 + 6x + 12$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 2y$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \Rightarrow (s)$$

Again, $f = x^2 + y^2 + 6x + 12$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 2y$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = 2 \Rightarrow (t)$$

Step 4: $r = 2$

Put $x = -3, y = 0$

$$(r)_{(-3,0)} = 2$$

$$s = 0$$

Put $x = -3, y = 0$

$$(s)_{(-3,0)} = 0$$

$$t = 2$$

Put $x = -3$ and $y = 0$

$$(t)_{(-3,0)} = 2$$

$$\therefore rt - s^2 = (2)(2) - (0)^2$$

$$rt - s^2 = 4$$

$$\Rightarrow rt - s^2 > 0$$

and $r = 2$

$$r > 0$$

As $rt - s^2 > 0$, $r > 0$, function has minium value at $(-3, 0)$

$\therefore$ **Minimum value**

$$f = x^2 + y^2 + 6x + 12$$

Put $x = -3, y = 0$

$$f_{min} = (-3)^2 + (0)^2 + 6(-3) + 12$$

$$f_{min} = 3$$

Example 7.2.[illegible] **Dec. 2016, May 2015, 2017**

Discuss the maxima and minima of $x^3 y^2 (1 - x - y)$.

Solution :

Here

$$f(x, y) = x^3 y^2 (1 - x - y)$$

$$f = x^3 y^2 - x^4 y^2 - x^3 y^3$$

Step 1 : Let

$$\frac{\partial f}{\partial x} = 0$$

$$\therefore \quad 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0 \quad \ldots(1)$$

And

$$\frac{\partial f}{\partial y} = 0$$

$$2x^3 y - 2x^4 y - 3x^3 y^2 = 0 \quad \ldots(2)$$

Step 2 : Values of x and y

Equation (1) is,

$$3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$$

$$\therefore \quad x^2 y^2 (3 - 4x - 3y) = 0$$

$$x = 0 \text{ OR } y = 0 \text{ OR } -4x - 3y = -3 \quad \ldots(3)$$

Equation (2) is,

$$2x^3 y - 2x^4 y - 3x^3 y^2 = 0$$

$$x^3 y (2 - 2x - 3y) = 0$$

$$x = 0 \text{ OR } y = 0 \text{ OR } -2x - 3y = -2 \quad \ldots(4)$$

Solving Equations (3) and (4) simultaneously on calculator

$$\therefore \quad x = \frac{1}{2} \text{ and } y = \frac{1}{3}$$

$\therefore$ Roots are (0, 0) and $\left(\frac{1}{2}, \frac{1}{3}\right)$

Step 3 : To find r, s, t

$$f = x^3 y^2 - x^4 y^2 - x^3 y^3$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3 \Rightarrow (r)$$

Again, $f = x^3 y^2 - x^4 y^2 - x^3 y^3$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2 \Rightarrow (s)$$

Again, $f = x^3y^2 - x^4y^2 - x^3y^3$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y \Rightarrow (t)$$

Step 4 A : For point (0, 0)

$$r = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$r = 6(0)(0)^2 - 12(0)^2(0)^2 - 6(0)(0)^3$$

$$r = 0$$

$$s = 6x^2y - 8x^3y - 9x^2y^2$$

$$s = 6(0)^2(0) - 8(0)^3(0) - 9(0)^2(0)^2$$

$$s = 0$$

$$t = 2x^3 - 2x^4 - 6x^3y$$

$$t = 2(0)^3 - 2(0)^4 - 6(0)^3(0)$$

$$t = 0$$

$\therefore \quad rt - s^2 = 0$

$\therefore$ As $rt - s^2 = 0$

No conlusion can be drawn about maxima or minima at (0, 0) and further inrestigation is needed.

Step 4 B : For point $\left(\frac{1}{2}, \frac{1}{3}\right)$

$$r = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$(r)_{\left(\frac{1}{2}, \frac{1}{3}\right)} = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$$

$$r = -\frac{1}{9}$$

$$s = 6x^2y - 8x^3y - 9x^2y^2$$

$$(s)_{\left(\frac{1}{2}, \frac{1}{3}\right)} = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$$

$$s = -\frac{1}{12}$$

and

$$t = 2x^3 - 2x^4 - 6x^3y$$

$$(t)_{\left(\frac{1}{2}, \frac{1}{3}\right)} = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$$

$$t = -\frac{1}{8}$$

$\therefore$ Now,

$$rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$$

$$rt - s^2 = \frac{1}{144} > 0$$

$\therefore \quad rt - s^2 > 0$

and $\quad r = -\frac{1}{9}$

$$r < 0$$

$\therefore$ For $rt - s^2 > 0$ and $r < 0$, function has maximum value at point $\left(\frac{1}{2}, \frac{1}{3}\right)$.

Maximum value

$$f(x, y) = x^3y^2 - x^4y^2 - x^3y^3$$

Put $\quad x = \frac{1}{2},\ y = \frac{1}{3}$

$$f_{max} = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^4\left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^3$$

$$f_{max} = \frac{1}{432}$$

Example 7.2.3 **Dec.2007, 2014, May 2011, [illegible]**

Find all the stationary points of the function $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Examine whether the function is maximum or minimum at those points.

Solution :

Given :

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Step 1 :

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \quad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$6xy - 30y = 0 \quad ...(2)$$

Step 2 : To find values of x and y

From Equation (2)

$$6xy - 30y = 0$$

$$6y(x-5) = 0$$

$$6y = 0 \quad \textbf{OR} \quad x - 5 = 0$$

$$y = 0 \quad x = 5$$

Put $y = 0$ in Equation (1)

$$3x^2 - 30x + 72 = 0$$

$$3x^2 - 30x + 72 = 0$$

$x = 6$ and $x = 4$

(6, 0) and (4, 0)

Put $x = 5$ in Equation (1)

$$3x^2 + 3y^2 - 30x + 72 = 0$$

$$3(5)^2 + 3y^2 - 30(5) + 72 = 0$$

$$3y^2 = 3$$

$$y^2 = 1$$

$$y = \pm 1$$

∴ **(5, 1) and (5 – 1)**

∴ Roots are (6, 0) (4, 0) (5, 1) and (5, – 1)

Step 3 : To find r, s, t

$$f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = 6x - 30 \Rightarrow (\mathbf{r})$$

$$f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x \partial y} = 6y \Rightarrow (\mathbf{s})$$

and $\quad f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 6xy - 30y$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = 6x - 30 \Rightarrow (t)$$

Step 4A : For point (6, 0)

$$r = 6x - 30$$

$$r_{(6,0)} = 6(6) - 30$$

$$\mathbf{r = 6}$$

$$s = 6y$$

$$s_{(6,0)} = 6(0)$$

$$\mathbf{s = 0}$$

$$t = 6x - 30$$

$$t_{(6,0)} = 6(6) - 30$$

$$\mathbf{t = 6}$$

∴ Now,

$$rt - s^2 = (6)(6) - 0$$

$$rt - s^2 = 36$$

$$\therefore \quad rt - s^2 > 0$$

and $\quad r = 6, r > 0$

As $rt - s^2 > 0$ and $r > 0$ function has minimum value at (6, 0)

$$\therefore \quad f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

∴ Put $x = 6$ and $y = 0$

$$f_{min} = (6)^3 + 3(6)(0)^2 - 15(6)^2 - 15(0)^2 + 72(6)$$

$$\mathbf{f_{min} = 108}$$

Step 4B : For point (4, 0)

$$r = 6x - 30$$

$$r_{(4,0)} = 6(4) - 30$$

$$\mathbf{r = -6}$$

$$s = 6y$$

$$s_{(4,0)} = 6(0)$$

$$\mathbf{s = 0}$$

$$t = 6x - 30$$
$$t_{(4,0)} = 6(4) - 30$$
$$t = -6$$
$$\therefore \quad rt - s^2 = (-6)(-6) - 0$$
$$rt - s^2 = 36$$
$$\therefore \quad rt - s^2 > 0$$
and $\quad r = -6 < 0$

As $rt - s^2 > 0$ and $r < 0$ function has maximum value at (4, 0)

$$\therefore \quad f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$\therefore$ Put $x = 4$ and $y = 0$

$$f_{max} = (4)^3 + 3(4)(0)^2 - 15(4)^2 - 15(0)^2 + 72(4)$$
$$f_{max} = 112$$

Step 4C : For point (5, 1)

$$r = 6x - 30$$
$$r_{(5,1)} = 6(5) - 30$$
$$r = 0$$
$$s = 6y$$
$$s_{(5,1)} = 6(1)$$
$$s = 6$$
$$t = 6x - 30$$
$$t_{(5,1)} = 6(5) - 30$$
$$t = 0$$
$$\therefore \quad rt - s^2 = (0)(0) - (6)^2$$
$$rt - s^2 = -36$$
$$\therefore \quad rt - s^2 < 0$$
and $\quad r = 0$

$\therefore$ Now, $r = 0$, Function has neither maximum nor minimum at (4, 0)

Step 4D : Similar by **Step 4C** function has neither maximum nor minimum at (5, –1).

Example 7.7.4

Find the extreme values of $xy(a - x - y)$.

Solution :

Here, $\quad f(x, y) = xy(a - x - y)$

$$\therefore \quad f = axy - x^2y - xy^2$$

Step 1 : $\dfrac{\partial f}{\partial x} = 0$

$$ay - 2xy - y^2 = 0$$
$$\frac{\partial f}{\partial y} = 0$$
$$ax - x^2 - 2xy = 0$$

Step 2 : To find values of x and y

From Equation (1) and (2),

$y(a - 2x - y) = 0$ and $x(a - x - 2y) = 0$

i.e.

1. $x = 0 \qquad y = 0$
2. $x = 0 \Rightarrow a - 2x - y = 0$
   $a - y = 0$
   $\therefore y = a$
3. $y = 0 \Rightarrow a - x - 2y = 0$
   $a - x = 0$
   $x = a$
4. $a - 2x - y = 0$ and $a - x - 2y = 0$
   $2x + y = a \qquad x + 2y = a$

   Solving simultaneously

$$2x + y = a$$
$$2x + 4y = 2a$$
$$(-) \quad (-) \quad (-)$$
$$-3y = -a$$
$$y = \frac{a}{3}$$

Put $y = \frac{a}{3}$ in $2x + y = a$

$$\therefore \quad 2x + \frac{a}{3} = a$$
$$2x = a - \frac{a}{3} = \frac{2a}{3}$$
$$x = \frac{a}{3}$$

$\therefore$ Roots are (0, 0), (0, a) (a, 0) and $\left(\frac{a}{3}, \frac{a}{3}\right)$

Step 3 : To find r, s, t

$$f = axy - x^2y - xy^2$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = -2y \Rightarrow (r)$$

$$f = axy - x^2y - xy^2$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y \Rightarrow (s)$$

$$f = axy - x^2y - xy^2$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = -2x \Rightarrow (t)$$

Step 4A : For point (0, 0)

$$r = 0 ; s = a ; t = 0$$

$$\therefore \quad rt - s^2 = 0 - a^2$$

$$rt - s^2 = -a^2$$

$$rt - s^2 < 0$$

$\therefore$ Function has neither maximum nor minimum at (0, 0)

Step 4B : For point (0, a)

$$r = -2a ; s = -a ; t = 0$$

$$\therefore \quad rt - s^2 = -a^2 < 0$$

$\therefore$ Function has neither maximum nor minimum at (0, a)

Step 4C : For point (a, 0)

$$r = 0 ; s = -a ; t = -2a$$

$$\therefore \quad rt - s^2 = (0)(-2a) - (-a)^2$$

$$rt - s^2 = -a^2$$

$$rt - s^2 < 0$$

$\therefore$ Function has neither maximum nor minimum at (a, 0)

Step 4D : For point $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$r = -\frac{2a}{3} ; s = a - \frac{2a}{3} - \frac{2a}{3} = \frac{-a}{3} ; t = -\frac{2a}{3}$$

$$\therefore \quad rt - s^2 = \left(-\frac{2a}{3}\right)\left(\frac{-2a}{3}\right) - \left(-\frac{a}{3}\right)^2$$

$$rt - s^2 = \frac{4a^2}{9} - \frac{a^2}{9}$$

$$= \frac{3a^2}{9}$$

$$\therefore \quad rt - s^2 > 0 \text{ and } r < 0$$

$\therefore$ Function has maximum / minimum value at $\left(\frac{a}{3}, \frac{a}{3}\right)$ which depends on value of a, either positive or negative.

$$\therefore \quad f = axy - x^2y - xy^2$$

put $x = \frac{a}{3}$; $y = \frac{a}{3}$

$$\therefore \quad f_{extreme} = a\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) - \left(\frac{a}{3}\right)^2\left(\frac{a}{3}\right) - \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)^2$$

$$= \frac{a^3}{9} - \frac{a^3}{27} - \frac{a^3}{27} = \frac{3a^3 - a^3 - a^3}{27}$$

$$f_{extreme} = \frac{a^3}{27}$$

Example 7.2.5

Show that the minimum value of $xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$. Q.2.(c) ...

Solution :

$$\text{Here, } f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\therefore \quad f = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Step 1 : $\frac{\partial f}{\partial x} = 0$

$$y - \frac{a^3}{x^2} = 0 \quad \ldots(1) \quad \left\{\because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right\}$$

And $\frac{\partial f}{\partial y} = 0$

$$x - \frac{a^3}{y^2} = 0 \qquad ...(2)$$

Step 2 : To find x and y

From Equation (1) and (2)

$\therefore x^2 y = a^3$ and $xy^2 = a^3$

$$x = \frac{a^3}{y^2}$$

$$\left(\frac{a^3}{y^2}\right)^2 y = a^3$$

$$\therefore \left(\frac{a^6}{y^4}\right)(y) = a^3$$

$$a^3 = y^3$$

$$\therefore y = a$$

Put $y = a$ in $xy^2 = a^3$

$$xa^2 = a^3$$

$$\therefore x = a$$

$\therefore$ roots are (a, a)

Step 3 : To find r, s, t

$$f = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$$

$$\left\{\because \frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{d}{dx}x^{-2} = -2x^{-3}\right\}$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3} \Rightarrow \textbf{(r)}$$

$$f = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x\,\partial y} = 1 \Rightarrow \textbf{(s)}$$

$$f = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3} \Rightarrow (t)$$

Step 4 : For point (a, a)

$$\therefore r = \frac{2a^3}{x^3}$$

$$r_{(a,\,a)} = 2$$

$$s = 1$$

$$t = \frac{2a^3}{y^3}$$

$$t_{(a,\,a)} = 2$$

$$\therefore rt - s^2 = (2)(2) - (1)^2$$

$$rt - s^2 = 3 > 0$$

and $r > 0$

$\therefore$ Function has minimum value at (a, a)

Now, $f = xy + \frac{a^3}{x} + \frac{a^3}{y}$

put $x = a$ $y = a$

$$f_{min} = a \cdot a + \frac{a^3}{a} + \frac{a^3}{a}$$

$$= a^2 + a^2 + a^2$$

$$\mathbf{f_{min} = 3a^2}$$

Example 7.2.6

Find maximum and minimum value of

$x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Solution :

Here, $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Step 1 : $\therefore \frac{\partial f}{\partial x} = 0$

$$\therefore 3x^2 + 3y^2 - 6x = 0 \qquad ...(1)$$

and $\therefore \frac{\partial f}{\partial y} = 0$

$$6xy - 6y = 0 \qquad ...(2)$$

Step 2 : To find values of x and y

From equation (2),

$$6xy - 6y = 0$$
$$6y(x-1) = 0$$
$$6y = 0$$
$$y = 0$$

or

$$x - 1 = 0$$

or

$$x = 1$$

For y = 0,	**For x = 1**
Equation (1) becomes,	Equation (1) becomes,
$3x^2 - 6x = 0$	$3 + 3y^2 - 6 = 0$
$3x(x-2) = 0$	$3y^2 = 3$
$x = 0$ or $x = 2$	$y^2 = 1$
$\therefore$ (0, 0) (2, 0)	$y = \pm 1$
	$\therefore$ **(1, 1) and (1, –1)**

Roots are (0, 0) (2, 0) (1, 1) and (1, –1)

Step 3 : To find r, s, t

$$f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = 6x - 6 \Rightarrow (r)$$

$$f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 6xy - 6y$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x\,\partial y} = 6y \Rightarrow (s)$$

$$f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 6xy - 6y$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = 6x - 6 \Rightarrow (t)$$

Step 4A : For point (0, 0)

$\therefore$ $r = -6,\ s = 0,\ t = -6$

$$\therefore\ rt - s^2 = (-6)(-6) - 0^2$$
$$rt - s^2 = 36$$
$$rt - s^2 > 0 \text{ and } r < 0$$

$\therefore$ Function has maximum value at (0, 0)

Now, $f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

Put $x = 0,\ y = 0$

$$f_{max} = 4$$

Step 4B : For point (2, 0)

$r = 6x - 6$	$s = 6y$	$t = 6x - 6$
$r = 6(2) - 6$	$s = 6(0)$	$t = 6(2) - 6$
$\mathbf{r = 6}$	$\mathbf{s = 0}$	$\mathbf{t = 6}$

$$\therefore\ rt - s^2 = 6(6) - (0)^2$$
$$rt - s^2 = 36$$
$$\therefore\ rt - s^2 > 0 \text{ and } r > 0$$

$\therefore$ function has minimum value at (2, 0)

$$f = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Put $x = 2,\ y = 0$

$$f_{min} = (2)^3 + 0 - 3(2)^2 - 0 + 4$$
$$f_{min} = 0$$

Step 4C : For point (1, 1)

$r = 6x - 6$	$s = 6y$	$t = 6x - 6$
$r = 6(1) - 6$	$\mathbf{s = 6}$	$t = 6(1) - 6$
$\mathbf{r = 0}$		$\mathbf{t = 0}$

$$\therefore\ rt - s^2 = (0)(0) - (6)^2$$
$$rt - s^2 < 0$$

$\therefore$ Function has neither maximum nor minimum at (1,1)

Step 4D : For (1, –1)

$r = 6x - 6$	$s = 6y$	$t = 6x - 6$
$r = 6(1) - 6$	$s = 6(-1)$	$t = 6(1) - 6$
$\mathbf{r = 0}$	$\mathbf{s = -6}$	$\mathbf{t = 0}$

$$\therefore\ rt - s^2 = (0)(0) - (-6)^2 = -36$$
$$rt - s^2 < 0$$

$\therefore$ Function has neither maximum nor minimum at (1, 1).

Example 7.2.7 **May 2012, 2014, Dec. 2013**

Discuss the maxima and minima of $x^3 + y^3 - 3axy$.

Solution : Here, $f(x, y) = x^3 + y^3 - 3axy$

Step 1 : $\frac{\partial f}{\partial x} = 0$

$$3x^2 - 3ay = 0$$

$$x^2 - ay = 0 \quad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 3ax = 0$$

$$y^2 - ax = 0 \quad \ldots(2)$$

Step 2 : Find values of x and y.

From Equation (2),

$$y^2 = ax$$

$$\therefore \quad \frac{y^2}{a} = x$$

Put this value in Equation (1),

$$\therefore \quad x^2 - ay = 0$$

$$\left(\frac{y^2}{a}\right)^2 - ay = 0$$

$$\frac{y^4}{a^2} - ay = 0$$

$$y^4 - a^3 y = 0$$

$$y(y^3 - a^3) = 0$$

$$\therefore \quad y = 0 \text{ or } y^3 = a^3$$

$$y = 0 \text{ or } y = a$$

Put $y = 0$ in Equation (1),

$$x = \frac{y^2}{a}$$

$$x = 0$$

$$(x, y) = (0, 0)$$

Put $y = a$ in Equation (1)

$$x = \frac{y^2}{a}$$

$$x = \frac{a^2}{a}$$

$$x = a$$

$$(x, y) = (a, a)$$

$\therefore$ Roots are (0, 0) and (a, a).

Step 3 : To find r, s, t

$$f = x^3 + y^3 - 3axy$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = 6x \Rightarrow \textbf{(r)}$$

$$f = x^3 + y^3 - 3axy$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x\, \partial y} = -3a \Rightarrow \textbf{(s)}$$

$$f = x^3 + y^3 - 3axy$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 3y^2 - 3a$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = 6y \Rightarrow \textbf{(t)}$$

Step 4A : For point (0, 0)

$r = 6x$		$t = 6y$
$r = 6(0)$	$s = -3a$	$t = 6(0)$
$r = 0$		$t = 0$

$$\therefore \quad rt - s^2 = (0)(0) - (-3a)^2$$

$$= -9a^2$$

$$\therefore \quad rt - s^2 < 0 \qquad (\because \text{ Square of a can't be } - \text{ve})$$

$\therefore$ Function has neither maximum nor minimum at (0, 0).

Step 4B : For point (a, a)

$r = 6a$	$s = -3a$	$t = 6a$

$$\therefore \quad rt - s^2 = (6a)(6a) - (-3a)^2$$

$$rt - s^2 = 27a^2$$

$$\therefore \quad rt - s^2 > 0$$

and $\quad r = 6a$

$\therefore$ If $a = -$ ve then $r < 0$ which gives maximum value.

& If $a = +$ ve then $r > 0$ which gives minimum value.

...7.7.0 **May 2016**

...for maxima and minima of

...x + sin y + sin (x + y)

...n :

$$f(x, y) = \sin x + \sin y + \sin (x + y)$$

...p 1 : $\frac{\partial f}{\partial x} = 0$

$$\cos x + \cos (x + y)(1 + 0) = 0$$

$$\cos x + \cos (x + y) = 0 \quad ...(1)$$

and $\frac{\partial f}{\partial y} = 0$

$$\cos y + \cos (x + y)(0 + 1) = 0$$

$$\cos y + \cos (x + y) = 0 \quad ...(2)$$

...p 2 : To find values of x and y.

Equation (1) – Equation (2),

$$\cos x - \cos y = 0$$

$$\cos x = \cos y$$

$$x = y$$

Put $y = x$ in Equation (1), we get,

$$\cos x + \cos (x + x) = 0$$

$$\cos 2x = -\cos x$$

$$\cos 2x = \cos (\pi - x)$$

$$2x = \pi - x$$

$$3x = \pi$$

$$x = \frac{\pi}{3}$$

but $y = x$

$$y = \frac{\pi}{3}$$

∴ Roots are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

...p 3 : To find r, s, t

$$f = \sin x + \sin y + \sin (x + y)$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = \cos x + \cos (x + y)$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin (x + y) \Rightarrow (r)$$

$$f = \sin x + \sin y + \sin (x + y)$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = \cos y + \cos (x + y)$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x \, \partial y} = -\sin (x + y) \Rightarrow (s)$$

$$f = \sin x + \sin y + \sin (x + y)$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = \cos y + \cos (x + y)$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin (x + y) \Rightarrow (t)$$

Step 4 : For point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

∴ $r = -\sin x - \sin (x + y)$

$$r = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$\left[\because \sin 60 = \frac{\sqrt{3}}{2}, \sin 120 = \frac{\sqrt{3}}{2}\right]$$

$$r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$r = -\sqrt{3}$$

$$s = -\sin (x + y)$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$s = -\frac{\sqrt{3}}{2}$$

$$t = -\sin y - \sin (x + y)$$

$$t = -\sin \frac{\pi}{3} - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$t = -\sqrt{3}$$

∴ $rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$

$$rt - s^2 = \frac{9}{4}$$

∴ $rt - s^2 > 0$ and $r < 0$

$\therefore$ Function has maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Now, $\quad f = \sin x + \sin y + \sin (x + y)$

put $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$

$$f_{max} = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin \left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$f_{max} = \frac{3\sqrt{3}}{2}$$

Example 7.2.9 **May 2011, 2012**

Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

Solution :

Let, 1^{st} part $= x$

2^{nd} part $= y$

3^{rd} part $= z$

$$\therefore \quad x + y + z = 120 \qquad \text{...(A)}$$

Now, sum of their products taken two at a time is,

$$f(x, y) = xy + yz + zx$$

$$= xy + y(120 - x - y) + (120 - x - y)x$$

$$= xy + 120y - xy - y^2 + 120x - x^2 - xy$$

$$f(x, y) = 120y + 120x - xy - x^2 - y^2$$

Step 1 : $\quad \frac{\partial f}{\partial x} = 0$

$$120 - y - 2x = 0$$

$$\therefore \quad 2x + y = 120 \qquad \text{...(1)}$$

$$\frac{\partial f}{\partial y} = 0$$

$$120 - x - 2y = 0$$

$$x + 2y = 120 \qquad \text{...(2)}$$

Step 2 : To find values of x and y

Solving Equation (1) and (2) simultaneously on calculator.

$$x = 40 \text{ and } y = 40$$

$\therefore$ Roots are (40, 40).

Step 3 : To find r, s, t

$$f = 120y + 120x - xy - x^2 - y^2$$

Differentiating w.r.t. x partially,

$$\frac{\partial f}{\partial x} = 120 - y - 2x$$

Differentiating w.r.t. x partially again,

$$\frac{\partial^2 f}{\partial x^2} = -2 \Rightarrow (r)$$

$$f = 120y + 120x - xy - x^2 - y^2$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 120 - x - 2y$$

Differentiating w.r.t. x partially,

$$\frac{\partial^2 f}{\partial x\, \partial y} = -1 \Rightarrow (s)$$

$$f = 120y + 120x - xy - x^2 - y^2$$

Differentiating w.r.t. y partially,

$$\frac{\partial f}{\partial y} = 120 - x - 2y$$

Differentiating w.r.t. y partially again,

$$\frac{\partial^2 f}{\partial y^2} = -2 \Rightarrow (t)$$

Step 4 :

Now, $\quad rt - s^2 = (-2)(-2) - (-1)^2$

$$rt - s^2 = 4 - 1$$

$$rt - s^2 = 3$$

$$\therefore \quad rt - s^2 > 0 \text{ and } r < 0$$

$\therefore$ Function has maximum value at (40, 40)

$\therefore$ Now, From Equation (A)

$$x + y + z = 120$$

$$\therefore \quad 40 + 40 + z = 120$$

$$\therefore \quad z = 40$$

$\therefore$ **Function has maximum value at**

$x = 40$, $y = 40$, $z = 40$