Deep Learning for Vision (EN4553) - Assignment 1

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The following definitions (along with other basic theorems of probability theory) will be useful in completing this assignment. Note that, although we consider only continuous random variables below, analogous definitions exist for discrete random variables as well.

Definition 1. *(Expected value and variance): Let X be a continuous random variable with the probability density function $f(x)$. Then the expected value (or the mean) of X is*[1]*,*

$$E[X] = \mu_X = \int_{-\infty}^{\infty} x f(x) dx.$$

The expected value of a function $g(X)$ of X is,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x) dx.$$

The variance of X is,

$$V[X] = \sigma_X^2 = E[(X - \mu_X)^2].$$

Definition 2. *(Expected value - multivariate case): Let $g(X_1, X_2, \ldots, X_k)$ be a function of continuous random variables, $X_1, X_2, \ldots, X_k$, which have the joint probability density function $f(x_1, x_2, \ldots, x_k)$. Then the expected value of $g(X_1, X_2, \ldots, X_k)$ is,*

$$E[g(X_1, X_2, \ldots, X_k)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots, x_k) f(x_1, x_2, \ldots, x_k) dx_1 dx_2 \ldots dx_k.$$

Definition 3. *(Covariance and correlation) Let X_1 and X_2 be random variables with means μ_1 and μ_2, respectively. Then the covariance of X_1 and X_2 is,*

$$\mathrm{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)].$$

The correlation coefficient of X_1 and X_2 is,

$$\rho_{X_1, X_2} = \frac{\mathrm{cov}(X_1, X_2)}{\sigma_1 \sigma_2},$$

where σ_1 and σ_2 are the standard deviations of X_1 and X_2, respectively.

[1] Here and everywhere else, without explicitly stating it, we assume that the integral converges absolutely.

Q1: Prove the following, where X, Y are random variables, a, b are constants, and g, h are functions. It is sufficient to prove the results for continuous X and Y although they hold for the discrete case as well. You can use previously proven results when proving the subsequent ones (2 marks each).

1. $V[X] = E[X^2] - E[X]^2$.
2. $E[aX + b] = aE[X] + b$.
3. $V[aX + b] = a^2 V[X]$.
4. $E[X + Y] = E[X] + E[Y]$.
5. $V[X + Y] = V[X] + V[Y] + 2\,\mathrm{cov}(X, Y)$.
6. $E[g(X)\,h(Y)] = E[g(X)]\,E[h(Y)]$, when X and Y are independent.
7. $\mathrm{cov}(X, Y) = E[XY] - E[X]E[Y]$.
8. cov(X, Y) vanishes when X and Y are independent (*i.e.*, independent random variables are uncorrelated).

Q2 Consider two continuous random variables X and Y. Sketch example scatter plots for the following cases, where the horizontal axis represents X and the vertical axis represents Y.

1. X and Y are positively correlated (2 marks).
2. X and Y are negatively correlated (2 marks).
3. X and Y are uncorrelated (2 marks).
4. X and Y are independent (2 marks).
5. X and Y are uncorrelated but not independent (4 marks).

Q3: Nimal has a car and a van. Let X_1 and X_2 denote the filled proportion of the capacity of the fuel tank at the start of a day, for the car and the van, respectively. Nimal finds out that the joint probability distribution of X_1 and X_2 can be modeled by:

$$f(x_1, x_2) = \begin{cases} k(x_1 + x_2), & 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

1. Find the value of the constant k (2 marks).
2. Find $P(X_1 + X_2 \leq 1)$ (2 marks).
3. Find the marginal density functions for X_1 and X_2 (4 marks).
4. Find the probability that the car's fuel tank is more than 75% full on a random morning (2 marks).
5. Find the conditional density function for X_1 given that X_2 is x_2' (2 marks).
6. The van's fuel tank is 25% full on a given morning. Find the probability that the car's fuel tank is more than 75% full (4 marks).

Q4: There are two random variables $X_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2)$. Define a new random variable Y. In each realization of Y, we sample from X_1's distribution with a probability p, and X_2's distribution with a probability $(1 - p)$.

1. Show that the mean of Y is given by (2 marks):

$$E[Y] = p\mu_1 + (1 - p)\mu_2.$$

2. Show that the variance of Y is given by (4 marks):

$$V[Y] = p\,\sigma_1^2 + (1-p)\,\sigma_2^2 + p(1-p)\,(\mu_1 - \mu_2)^2.$$

3. Roughly sketch Y's probability density function for $\mu_1 = 0, \sigma_1 = 1, \mu_2 = 10, \sigma_2 = 1, p = 0.2$ (2 marks).

4. Using the library functions `numpy.random.randn()`, and `numpy.random.uniform()` write code to simulate $10,000$ realizations of Y. Then use `seaborn.distplot()` to plot the distribution to verify that your sketch above is accurate. Include your code and the plot in the answer sheet. (4 marks)

   Hint: For the first two parts, you could define a Bernoulli-distributed random variable I such that $Y = IX_1 + (1-I)X_2$, while noting that I is independent of both X_1 and X_2 and that $I(1-I) = 0$. Alternatively, you could work out the expected values of functions of Y with the help of a tree diagram.

Q5: Consider a coin flip game with a friend where the coin is biased, having a 55% probability of landing heads up. If it lands heads up, you gain \$ 100 from your friend; if it lands tails up, you owe your friend \$ 100. These are the only possible outcomes per flip. Your friend offers two gameplay options:

Game A: The coin is flipped once, and the results determine the monetary exchange. Let X be the random variable representing your net gain in this game.

Game B: The coin is flipped 100 times consecutively, and after each flip, you settle the dues based on the outcome. Let Y be the random variable representing your net gain in this game.

1. In expectation, how much money would you make in Game A (this is equal to μ_X)? (2 marks)
2. What is the standard deviation of your earnings from Game A (this is equal to σ_X)? (2 marks)
3. When evaluating the profitability of a money-making game represented by a random variable X, we talk about the Sharpe ratio defined as $\mathrm{SR}(X) = \frac{\mu_X}{\sigma_X}$. Explain why a higher Sharpe ratio indicates a better game. (2 marks)
4. Calculate the Sharpe ratios for Game A and Game B. Which game would you prefer to play? (4 marks)
5. What is the probability of you losing money in Game A. (2 marks)
6. What is the probability of you losing money in Game B. (4 marks) (Hint: Use the normal approximation to the binomial distribution.)

References

You can further improve your skills in probability theory and statistics by studying these books:

- Mathematical Statistics with Applications. Dennis Wackerly, William Mendenhall, Richard L. Scheaffer.
- Fifty Challenging Problems in Probability with Solutions. Frederick Mosteller.
- 40 Puzzles and Problems in Probability and Mathematical Statistics. Wolfgang Schwarz.