Homework – 2

1. A continuous-time, deterministic linear system can be described by the equations

$$\dot{x} = Ax + Bu$$
$$z = Cx$$

   where x is the state vector, u is the control vector, and z is the output vector (measurement).

   If A, B and C are constant then show that the solution to above equation is given by:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$z(t) = Cx(t)$$

   where t_0 is the initial time of the system, which is often taken to be 0, and

   $e^A = \mathrm{I} + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$ is the matrix A exponential.

   <u>Hint</u>:

$$\frac{dy}{dx} + Py = Q$$

   1st order differential equations solved using integrating factor

2. The matrix exponential e^{At} can also be computed using the inverse Laplace transform.
   a. Obtain the Laplace transform of the following continuous-time, deterministic linear system equation

$$\dot{x} = Ax + Bu$$

   b. Apply the inverse Laplace transform operator to the result of (a) and compare it with

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

   to show that $e^{At} = \mathcal{L}^{-1}[(s\mathrm{I} - A)^{-1}]$, assuming $t_0 = 0$.

3. The linear position p of an object under constant acceleration is given by

$$p = p_0 + \dot{p}t + \frac{1}{2}\ddot{p}t^2$$

where p_0 is the initial position of the object.

a. Define a state vector as $x = [p \quad \dot{p} \quad \ddot{p}]^T$ and write the given system in the form of $\dot{x} = Ax$. We also call such equations as state-space equations.
b. Find e^{At} using
    i. $e^{At} = \mathrm{I} + \frac{At}{1!} + \frac{(At)^2}{2!} + \cdots$
    ii. $e^{At} = \mathcal{L}^{-1}[(s\mathrm{I} - A)^{-1}]$
c. Show that $e^{A0} = \mathrm{I}$ from the result found in (b).
d. Discretize the system. Take the discretization step size as Δt.

4. Suppose that X is a Gaussian random variable given by

$$X \sim N(10,4) \text{ and}$$
$$Y = g(X) = 2X + 5$$

Find the probability density function of Y.

5. Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_W and σ_V, respectively. What is the standard deviation of the random variable $X = W + V$?

6. The observations of 4 random variables are given by the following vectors

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}: \quad \begin{bmatrix} 1 \\ -5 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 8 \\ 10 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \\ 7 \end{bmatrix}$$

a. Find the covariance matrix
b. Which random variable has the lowest uncertainty?
c. Which two random variables are highly correlated?