EMERGENCE

CENCIARELLI

1. A Social Game System

We shall present a game model called *Social Game System* (SGS). Players belong to a set $\mathcal{U}$, called *universe*, which we assume numerable, and ranged over by the metavariables d, $e \ldots$. At any time only a finite subset of players are active in the game, each connected with a subset of the currently active players, called the player's *acquaintances*. Acquaintance is a reflexive, symmetrical relation on players, not necessarily transitive. A player d's acquaintances form an array of length k, called d's *neighborhood*, the indices of which are called *positions*. A player may hold more than one position in another player's neighborhood. A position may be held by at most one player, but it may also be *empty*, in which case we write $\perp$ its value. The first position of d's neighborhood is, by convention, always held by d itself. The length k, which is the same for all players, is therefore an upper bound to the number of acquaintances a player may have at any time. In the following we shall take $\mathcal{U}$ and k as given.

1.1. **Boards and profiles.**

In the following, $\mathcal{P}(\mathcal{A})$ denotes the powerset of a set A. We write $\mathcal{N}$ the set of natural numbers. We write $dom\,(f) \subseteq A$ the *domain* of a partial function $f : A \rightharpoonup B$. Given $k \in \mathcal{N}$ and a set A, we write $A^k_\perp$ the set $(A + \{\perp\})^k$ of *partial* k-tuples of elements of A. Clearly, $A \subseteq B$ implies $A^k_\perp \subseteq B^k_\perp$. We say that $a \in A^k_\perp$ and $b \in B^k_\perp$ have *equal structure* when $a_i = \perp$ if and only if $b_i = \perp$, for $1 \leq i \leq k$. Moreover, when $A \subseteq B$, we write $a \sqsubseteq b$ when $a_i \neq \perp$ implies $\perp \neq b_i = a_i$, for all i as above.

A *relational board* (of dimension k on $\mathcal{U}$) is a partial function

$$\beta : \mathcal{U} \rightharpoonup (\mathcal{U} \times \mathcal{N})^k_\perp$$

such that, if $\beta(d) = v$ is defined and $v_i = (e, j)$, then $\beta(e) = u$ is defined and $u_j = (d, i)$. We also postulate that $v_1 = (d, 1)$. A board represents the net of acquaintances of the active players at a certain time of the game, the domain of β being the set of currently active players. When d is active $\beta(d) = < v_1, \ldots v_k >$ is the neighborhood of d in the board. When, for some $1 \leq i \leq k$, the i-th position in the neighborhood is not empty and it is held by some player e, that is $\perp \neq v_i = (e, j)$, then, as specified above, d must hold position j in the neighborhood of e. In this sense we meant acquaintance to be *symmetric*. Similarly the condition $v_1 = (d, 1)$, requiring d to belong to its own neighborhood, makes acquaintance *reflexive*.

A *profile* of a board β consists of a partial function

$$\pi : \mathcal{U} \rightharpoonup \mathcal{N}_\perp^k$$

with same domain $D \subseteq \mathcal{U}$ as β and such that $\pi(d)$ and $\beta(d)$ have equal structure, for all $d \in D$. The numbers in the k-tuple $\pi(d)$, when defined, are called *messages*: the value $\pi(d)_i$ is meant as a message received by d from its i-th neighbor. For example, the message $\pi(d)_1$ received by d from itself, may be used to represent d's *wealth* or payoff. When considering the *dynamics* of an SGS (see section 1.2), $\pi(d)$ can be viewed as the *state* of player d at some moment of the game, so we let $\mathcal{N}_\perp^k$ be ranged over by the metavariable s. A profile represents a state of the whole board.

A game *configuration* is a pair (β, π), where β is a board and π a profile of β. Configurations evolve as specified below.

1.2. **SGS dynamics.** Players are born, interact, possibly make children and possibly die. They interact by exchanging messages with their acquaintances. New acquaintances can be made during a player's life.

Each player $d \in \mathcal{U}$ is endowed with a feedback function:

$$\phi_d : \mathcal{N}_\perp^k \rightharpoonup \mathcal{N}_\perp^k.$$

In the dynamics of a game, given a configuration (β, π), ϕ_d is applied to the state $s = \pi(d)$ the current profile assigns to d. If $\phi_d(s)$ is defined and equal to $< s'_1, \dots s'_k >$, each s'_i is fed back to d's i-th neighbor to form the profile of the next configuration. Function ϕ_d is required to *preserve* the structure of its input, that is, if defined, $\phi_d(s)$ must have identical structure as s: no messages are sent to (nor received from) empty neighborhood positions. Since feedback functions are *partial*, $\phi_d(s)$ may also be undefined, in which case s is *terminal* for d, that is: if s reached, d dies and becomes inactive. The set of players who *survive*, that is do not die, in a given configuration (β, π) is written:

$$\downarrow (\beta, \pi) = \{d \in dom(\beta) \mid \phi_d(\pi(d)) \text{ is defined}\}.$$

A player d is also endowed with it's own *spawn* function:

$$\sigma_d : \mathcal{N}_\perp^k \to \mathcal{P}(\mathcal{U}).$$

The set $\sigma_d(s) \subseteq \mathcal{U}$ is the set of children d may *possibly* spawn in state s.

A *run* of a game is a sequence of configurations, which evolve as result of players births, deaths and interactions. Formally: a game *transition* is a pair of configurations, written $(\beta_1, \pi_1) \to (\beta_2, \pi_2)$ such that:

(1) for all $d \in \downarrow (\beta_1, \pi_1)$, $\beta_2(d)$ is defined and $\beta_1(d) \sqsubseteq \beta_2(d)$;
(2) for all $e \in dom(\beta_2)$, if $e \not\in \downarrow (\beta_1, \pi_1)$ then $e \in \sigma_d(\pi_1(d))$ for some $d \in dom(\beta_1)$;
(3) for all d and e in $\downarrow (\beta_1, \pi_1)$, if $\beta_1(d)_i = (e, j)$ then $\pi_2(e)_j = \phi_d(\pi_1(d))_i$.

Condition (1) ensures that all players who do not perish in (β_1, π_1) are still active in β_2, that is $\downarrow(\beta_1, \pi_1) \subseteq dom(\beta_2)$, and with a possibly larger neighborhood. Condition (2) says that new players in β_2 are the children of the players in β_1. Note that not all of the players in $\sigma_d(\pi_1(d))$ are forced by this condition to come to life in β_2. Condition (3) states that the messages assigned to player $e \in \downarrow(\beta_1, \pi_1)$ by π_2 are indeed the ones meant for it by the feedback functions ϕ_d of each of its neighbors d applied to the former state $\pi_1(d)$.

2. Social Game Systems vs. noncooperative games

In the following, *"game"* stands for noncooperative game, defined in game theory as follows [Wei95]: A set $I = \{1, \dots k\}$ of *players* is given. Each player $i \in I$ is endowed with a set $S_i = \{1, \dots n_i\}$ of *pure strategies*. The cartesian product $S = \Pi_{i \in I} S_i$ is called the space of pure strategies of the game and its tuples $(s_1, \dots s_k)$, $s_i \in S_i$, are called *pure strategy profiles*. Each player i is also endowed with a *payoff function* $P_i : S \to \mathcal{R}$ mapping profiles to reals. A game is a triple $G = (I, S, P)$, where P is the vector of all players' payoff functions.

With each game $G = (I, S, P)$ we associate an SGS $\mathcal{S}_G$ defined as follows. In game theory payoffs $P_i(s_1, \dots s_k)$ are meant as *outcomes* for player i of the combination $(s_1, \dots s_k)$ of strategis chosen by each of the k players involved in the game. In social game systems each player $d \in \mathcal{U}$ is endowed with a feedback function $\phi_d(v_1, \dots v_k)$ of a combination of messages v_i received by d from each player d_i in d's neighborhood. It is therefore natural to take the universe $\mathcal{U}$ of $\mathcal{S}_G$ as the set I of G's players, a system's board of dimension k (the cardinality if I) forming the *complete* graph of the players (each player having all players as neighbors) and the strategies of G as messages in $\mathcal{S}_G$. Given this correspondence, however, the two notions of *profile*, in G and in $\mathcal{S}_G$, do not quite match still, as in $\mathcal{S}_G$ a profile π assigns to each player d a possibly different view $\pi(d)$ of other players' strategies. Hence, we force all players in $\mathcal{S}_G$ to share, as they do in game theory, the same global view: a profile $s = (s_1, \dots s_k)$ of G corresponds to the profile π of $\mathcal{S}_G$ in which, for any two players d and e, if a player d_i holds position j in d's neighborhood and position h in e's, then $\pi(d)_j = \pi(e)_h = s_i$.

References

[Wei95] J. W. Weibull. *Evolutionary game theory.* MIT Press, 1995.