Modern Control

Introduction to State Space Control.
Suplimentary notes with Modules 1-3

Lecture Overview

Learning Outcomes for Week 1:

1. State variable modelling of physical systems (noting it is not unique, so there are many solutions)
2. Canonical forms
3. Diagonal canonical forms
4. Computation of the transfer function (TF) from the state variable models and the solution of state equations.

There are 3 practice Quizzes related to this material
The first includes basics you should know coming into the course.

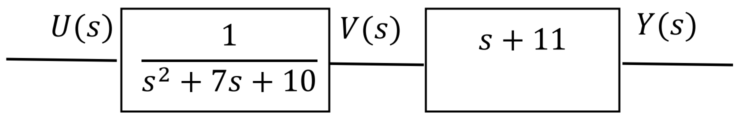
2. Controllable, Observable and Diagonal forms

Here the TF has zeros and the state variables are not taken as the output. This is common in electrical circuits.

Controllable Form

Focus on examples (proofs are in appendix p48). (p21-22)

Essentially split the system using an intermediate result $V(s)$;



Thus: $$\frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \cdot \frac{V(s)}{U(s)}$$

$\frac{V(s)}{U(s)} = \frac{1}{s^2+7s+10}$ is a system without zeros.

Follow the phase variable method to determine A and B
Second order, so use 2 states as $x_1 = v$, $x_2 = \dot{v}$.

$$\dot{x}_1 = x_2$$

and
$$\ddot{v} + 7\dot{v} + 10v = u$$
$$\ddot{v} = -10v - 7\dot{v} + u$$
$$\dot{x}_2 = -10x_1 - 7x_2 + u$$

Output $Y(s)$ comes from $\frac{Y(s)}{V(s)}$ based on the zeros of the TF.

$$Y(s) = (s + 11)V(s)$$
$$y = \dot{v} + 11v$$
$$y = x_2 + 11x_1 = 11x_1 + x_2$$

C matrix is now clear also.

Example 2: (p23)

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 2s + 5}{s^3 + 9s^2 + 11s + 15}$$

Can write by inspection from the denominator:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -11 & -9 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And from the zeros in the numerator

$$C = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}$$

Example 3:

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 5s^2 + 6s + 9}{s^4 + 7s^3 + 12s^2 + 17s + 19}$$

Write down A, B, C

We won't focus on these but the two other forms so that you are aware are shown: These will be discussed further by Akshya on controller design later and were helpful when there were no computers to help solve mathematically.

Observable Form (p24)

The observable matrices can be derived from the original matricies with examples provided on p55-56

Diagonal Cannonical form (p26)

Partial fraction expansion is used to derive this with examples provided p57-59.

Example p28 shows all three derived from the original.

However have a Diagonal form of the space matricies is very useful and this will be discussed.

Diagonalisation of the System A Matrix
(p33)

For diagonlisation work it is important to be able to compute the *Eigenvalues* and *Eigenvectors* of a Matrix.

Write these on (p60)

Example: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Eigenvalues:
Are the poles (or natural frequencies) of the system.

We solve using: $|\lambda I - A| = 0$,
here uses the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0$$

Applying the determinant to find the solutions:

$$\Rightarrow \ \lambda(\lambda + 3) + 2 = 0$$
$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 1)(\lambda + 2) = 0$$
$$\lambda_1 = -1, \lambda_2 = -2$$

Eigenvectors:
For each eigenvalue λ_i we can determine an eigenvector x_i

$$Ax_i = \lambda_i x_i$$

Thus

$$Ax_{[1]} = \lambda_1 x_{[1]}$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For $\lambda_1 = -1$ and solving means:

$$x_2 = -x_1$$

and

$-2x_1 - 3x_2 = -x_2 \quad \Rightarrow -2x_1 = 2x_2 \quad \Rightarrow x_1 = -x_2$

So the eigenvector corresponding to eigenvalue $\lambda_1 = -1$ is a vector that satifies the condition $x_1 = -x_2$

For $\lambda_2 = -2$ the solution is:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So that there is a vector satisfying the conditions

$$x_2 = -2x_1$$

and again:

$-2x_1 - 3x_2 = -2x_2 \quad \Rightarrow -2x_1 = x_2$

So the eigenvector corresponding to eigenvalue $\lambda_2 = -2$ is a vector that satifies $x_1 = -\frac{1}{2}x_2$

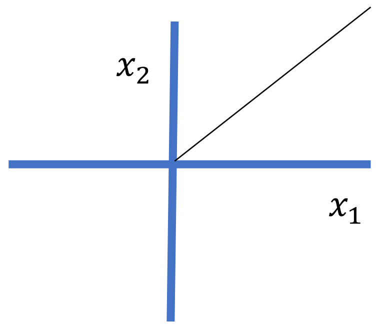
As an example:

Suppose I have a vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

I can multiple this vector by any constant (for example 5) and it also satisifies the condition.

So $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ is also ok.

Just select the easiest as its used in subsequent computation



Ex2: If $x_1 = -x_2$ solutions could be

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; or \begin{bmatrix} 3 \\ -3 \end{bmatrix} \; or \; \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

(p33) Suppose we have:

Dependent system 1

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_{11}u$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_{21}u$$

Independent system 2

$$\dot{x}_1 = a_{11}x_1 + b_{11}u$$
$$\dot{x}_2 = a_{22}x_2 + b_{21}u$$

The easiest to control is the second .. .why?
Because the dynamics of x_n only depend on itself in the second.

e.g. If I ask you "please get me a coffee", and if your decision is dependent on someone else then is if more difficult than if you make it yourself

In matrix form these are:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_{11}\\ b_{21}\end{bmatrix}u$$ **not decoupled**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}a_{11} & 0\\ 0 & a_{22}\end{bmatrix}}_{\text{Diagonal}}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_{11}\\ b_{21}\end{bmatrix}u$$ is a **decoupled** system

Example of practical is decoupling in IPT.
Allows independent design of primary and secondary.

Decoupling (p34)
Previously we have shown that the space vector model is not unique, and we can represent it in many different ways – the key is to represent it in the best way to control it.

If we don't have a decoupled (diagonal A matrix) is it possible to represent it as Diagonal? YES !

Diadonalizing:

$$\dot{x} = Ax + Bu \text{ and } y = Cx + Du$$

With $x = Pz$,

$$\dot{z} = P^{-1}APz + P^{-1}Bu \text{ and } \quad y = CPx + Du$$

Choose P so that the columns are the eigenvectors of A
Then $P^{-1}AP$ will be diagonal. This is called the Modal matrix.

Using Ex p36, first find the eigenvalues (p38), then the eigenvectors (p40 and p42)

The solutions are show as: $\lambda_1 = -2$, and $\lambda_2 = -4$
using these the eignvectors can be found:
$\lambda_1$we can prove $x_1 = x_2$ so can choose any vector $x = \begin{bmatrix} c \\ c \end{bmatrix}$
$\lambda_2$we can prove $x_1 = -x_2$ and choose any vector $x = \begin{bmatrix} d \\ -d \end{bmatrix}$

Thus the P matrix can be:

$$P = \begin{bmatrix} c & d \\ c & -d \end{bmatrix}$$

Make the easiest choice. E.g. here let $c = d = 1$

As shown $P^{-1}AP$ (p44) is diagonal, then we have decoupled the system. It is now easier to design and apply a controller

Module 3

Finding the Transfer function: $T(s) = \frac{Y(s)}{U(s)}$ (p3).

Note that for scalar x and a

$$sx(s) - ax(s) = (s-a)x(s)$$

For vector X and matrix A

$$sX(s) - AX(s) = (sI - A)X(s)$$

derivation on (p3) and (p5) and ex follows on (p7-11)

NB:
recall calculation of an inverse of a 2x2 (p9) – this is most common, and we will expect you can do this in exams.

Example.
If $A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$ then $A^{-1} = \frac{1}{5\times 3 - 2\times 4}\begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & -2 \\ -4 & 5 \end{bmatrix}$

Recall the dynamics of the system are defined by the A matrix.

The roots of $|sI - A| = 0$ are the eigenvalues of the system matrix.

The eigenvalues of A are the poles of the system, if they are not in the LHP then the system is unstable.

Please review basic matlab commands (p13-14) in preparation for the future assignments.

Solution of State Equations

Start with systems without an input (*Homogeneous case*)

Scalar (p15) given $\dot{x}$ find solution $x(t)$

If $\dot{x} = ax$ does it have any input? No.

Then how can we get a response without an input?

- It is possible when there are initial conditions

What is the physical significance of such initial conditions?

- They are the various energies stored in the system.

(p15) Given $X(s)$ is the Laplace transform of $x(t)$
use Inverse Laplace. Recall $\frac{1}{(s-a)} \longrightarrow e^{at}$

Vector (p17)
Following the scalar case where $L^{-1}[(s-a)^{-1}] = e^{at}x(0)$
the vector case is:

$$L^{-1}\{(sI - A)^{-1}\} = e^{At}x(0) = \Phi(t)x(0)$$

$\Phi(t)$ is called the state transition matrix.

Common examination question: define this and compute it.

The properties of $\Phi(t)$ are shown on (p22)
proofs (p23-25)

1. $\Phi(t) = e^{At}$ therefore $\Phi(0) = e^{A0} = I$
2. $\Phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \Phi(-t)$ ….. etc

We use these to help solve.

Example (p27, p29) to show how we use it in practice.

Recalling $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

When computing $(sI - A)^{-1}$ on (p31) you should be familiar with doing the partial fraction expansion.

e.g. $\frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} - \frac{1}{(s+2)}$

And if you are asked to find the inverse of the state transition matrix then simply use this property

$$\Phi^{-1}(t) = \Phi(-t)$$

And write it down.

For systems with an input (*non-homogeneous case*) (p33-36)
Simply repeat the maths but include the input.
So the response is dependent on both the initial conditions and the input.

Having found $X(s)$ can find $Y(s) = CX(s) + DU(s)$ ...

And $\quad x(t) = L^{-1}\{(sI - A)^{-1}[x(0) + BU(s)]\}$

Focus on *Example* p38.
No initial conditions and the input is a unit step. $U(s) = \frac{1}{s}$

(p40) to speed up the procedure, having found the 2x2 $(sI - A)^{-1}$ matrix just use symbols

$$(sI - A)^{-1} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

Don't waste energy computing

$$x(t) = L^{-1}\{(sI - A)^{-1}[x(0) + BU(s)]\}$$

Instead first find: $x(0) + BU(s)$

So for this then we have $x(0) + BU(s) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$

Now

$$X(s) = (sI - A)^{-1}[x(0) + BU(s)] = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{S_{12}}{S} \\ \frac{S_{22}}{s} \end{bmatrix}$$

and $Y(s) = [1 \quad 0]X(s) = \frac{s_{12}}{s}$

Only now apply the inverse laplace to get $x(t)$ and $y(t)$

Summary of Learnings:

1. Physical system modelling, there the number of states is equal to the order of the model.

2. We usually select measureable and controllable variables (e.g. in electrical systems inductor current and capacitor voltage) rather than the output and its derivatives

3. The state model is not unique, since we can choose any constant P matrix to re-represent it.

e.g.

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx + Du$$

chose

$$x = Pz \qquad\qquad z = P^{-1}x$$

thus

$$\dot{z} = P^{-1}APz + P^{-1}Bu \qquad y = CPz + Du$$

4. An independent system (diagonal) is easier to compute and control

e.g.

$$\dot{x}_1 = a_{11}x_1 + b_{11}u$$
$$\dot{x}_2 = a_{22}x_2 + b_{21}u$$

Thus decoupling is preferred, so choose a suitable transition matrix P such that the columns of P are the eigenvectors of the A matrix, so that $P^{-1}AP$ is diagonal.

5. The response is given by the state transition matrix.

Given expressions for $\dot{x}$ to determine $x(t) = \int \dot{x} dt$
Usually we don't compute $\Phi(t)$ –we do it numerically.

In matlab using the 4th and 5th order Runga-Kutta solver method (ode45) to determine the response (ex: pgs 45-50)
The ode45 function requires column vectors and as part of the syntax time t.

When pass the parameters (see p50) you have to add a gap [] else get an error.