Modern Control

Introduction to State Space Control.
Suplimentary notes with Modules 1-3

Lecture Overview

Learning Outcomes for Week 1:

1. State variable modelling of physical systems (noting it is not unique, so there are many solutions)
2. Canonical forms
3. Diagonal canonical forms
4. Computation of the transfer function (TF) from the state variable models and the solution of state equations.

There are 3 practice Quizzes related to this material
The first includes basics you should know coming into the course.

2 Weeks: State Space Averaging
Practical aspects of state space looking at buck, boost and buck-boost and how to design the controllers - Assignment 1

Following 2 weeks:
Patrick: Non-linear systems behaviour – Assignment 2

Remainder course
Akshya: Design of state feedback controllers in continuous and discrete time – Assignment 3

Module 1

Transfer function models.

Classical control is based on transfer function models of a system

In modern control we have state variable models and need to be able to find this new model of the system.

Most of the control system analysis and controller design are essentially model based analysis and model based controller design.

The physics of model building is based on storage and dissipation (p6), with various orders (p8-p16).

Examples: RL, RC, and RLC (p18-21)

Note that if we can cascade these, the order of the over all model increases by summing up the order of the storage elements within it.

(p22-23):

Time domain	Freq. domain from Laplace
$\frac{dy(t)}{dt}$ or $\dot{y}$	$sY(s)$
$\frac{d^2y(t)}{dt^2}$ or $\ddot{y}$	$s^2Y(s)$
$\frac{d^3y(t)}{dt^3}$ or $\dddot{y}$	$s^3Y(s)$
$\frac{d^ny(t)}{dt^n}$	$s^nY(s)$

Transfer Function in s domain (p24), which can be checked in own time using matlab scripts (p27-30).

Finding a time-based alternative to the T.F. model. (p33-34):

As an example, suppose we have the following expression

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_1 u$$

This is third order so we define three variables of interest (or states):

$$x_1 = y\ , x_2 = \dot{y}, x_3 = \ddot{y}$$

This means that: $\dot{x}_3 = \dddot{y}$

The original third order expression can now be defined as 3 first order differential equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$

And by substituting these into the original equation (written in terms of $\dddot{y}$) we have:

$$\dot{x}_3 = -a_3 x_1 - a_2 x_3 - a_1 x_3 + b_1 u$$

Now look at examples (p35) onwards.

Example 1: (p35)

The general matrix form is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Note general definitions:

A is the system matrix,
B is the input matrix,
C is the output matrix,
D is the transmission matrix

Where y are the outputs and u are the inputs (vectors)

Example 2: (p36-37)

$$m\ddot{x} + f\dot{x} + kx = u$$

Can also be written for freq domain

$$ms^2X(s) + fsX(s) + kX(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + fs + k}$$

Choose states: $x_1 = x$, $x_2 = \dot{x}$
And two first order differential equations can result.

Note: Output y is written in terms of x here but if we want to see $\dot{x}$ instead then we can change it to:

$$y = x_2$$

Thus $$y = Cx = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(p39)
Adding numbers, the poles of the system can be computed.

Note: The Eigenvalues of the system matrix are the poles of the system. These represent the natural frequencies of the system.

How to evaluate these is shown on (p41).

CHECK familiarity with matrix manipulation

Example 3: (p43 - 47) See notes.

On (p47) the capacitor voltage was chosen as the output.

But if current is wanted at the output then simply write:

$$y = x_2$$

And
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note: you can choose anything as the output, and even a combination of multiple states (This just changes the C matrix)

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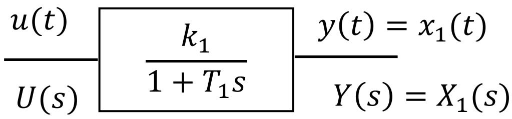
Extra examples

Ex.1 

The order of the system determines the number of states required. Here only 1, so only 1 state is needed.

Laplace:

$$Y(s) = \frac{k_1}{1+T_1 s}.U(s) \Rightarrow X_1(s) = \frac{k_1}{1+T_1 s}.U(s)$$

$$\Rightarrow X_1(s)(1 + T_1 s) = k_1 U(s)$$

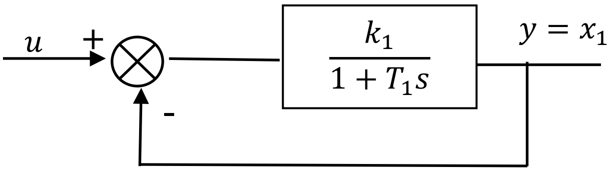
Recall from Laplace (page 22 & 24) that $sX(s) = \dot{x}$ in time domain, so writing in time domain.

$$x_1 + T_1 \dot{x}_1 = k_1 u$$

Rearranging gives:

$$\dot{x}_1 = \frac{-1}{T_1} x_1 + \frac{k_1}{T_1} u$$

Ex. 2: with feedback controller:



Because the objective is a time domain description often (as in Akshya notes) we don't write down $X_1(s)$ and simply write x_1 down in the time domain form directly. Thus here:

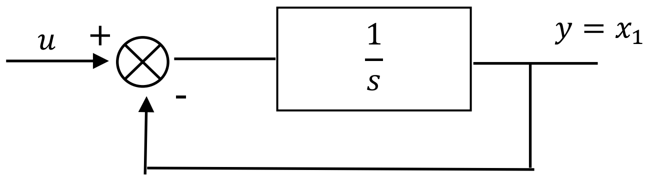
$$y = x_1 = \frac{k_1}{1 + T_1 s}.(u - x_1)$$

So

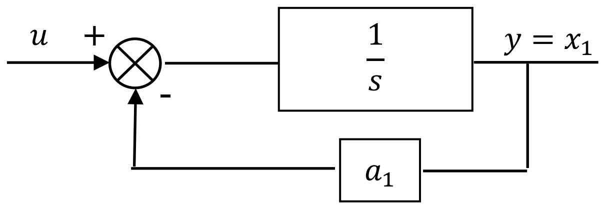
$$x_1(1+T_1 s) = k_1(u - x_1)$$
$$x_1 + T_1\dot{x}_1 = k_1(u - x_1)$$

Other examples to try:

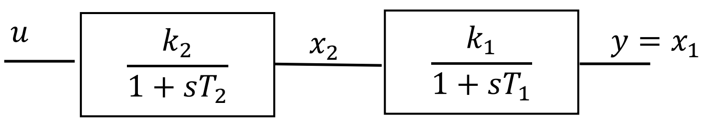
1. Given the following show that $\dot{x}_1 = -x_1 + u$



2. Given the following show that $\dot{x}_1 = -a_1 x_1 + u$



3. Given the system and the defined states (which are always given)



working from left to right we can write:

$x_1 = \frac{k_1}{1+sT_1} x_2$ and $x_2 = \frac{k_2}{1+sT_2} u$

Show that: $\dot{x}_1 = -\frac{1}{T_1} x_1 + \frac{k_1}{T_1} x_2$ and $\dot{x}_2 = -\frac{1}{T_2} x_2 + \frac{k_2}{T_2} u$

Example 5: (p55 – 62) Self review, starting from the right.

Example 4: (p49- 53) This shows a non-linear system cannot be represented in standard linear system form but still can be written in first order differential equations.

Another non-linear example. Suppose:

$$\dddot{y} + a_1\ddot{y}^2 + a_2\dot{y}^3 + a_3y^5 = yu$$

This is third order, so three states are sufficient, resulting in three first order differential expressions.
Let:

$$x_1 = y\ ,\ x_2 = \dot{y},\ x_3 = \ddot{y} \text{ and } \dot{x}_3 = \dddot{y}$$

Then:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -a_1\ddot{y}^2 - a_2\dot{y}^3 - a_3y^5 + yu$$
$$\dot{x}_3 = -a_1{x_3}^2 - a_2{x_2}^3 - a_3{x_1}^5 + x_1u$$

This again is the simplest form we can create.

Module 2

Is control simply mathematics unrelated to physical relevance? No ! Later we will show how to express real systems into these forms and derive controllers.

We start with generic models and system expressions. Once parameters known solve with pure mathematics.

Before proceeding –recall the relationship between scalar and matrix manipulation.

So for for **scalar** case (one variable):

$ax = b \Rightarrow \quad x = \frac{b}{a}$ providing $a \neq 0$ (a is scalar)

Also $\quad x = a^{-1}b$

For the **vector** case (using two variable case):

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

In matrix form this is:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Or $\quad AX = B$

So again:

$X = \frac{B}{A} = A^{-1}B$ provided $det(A) \neq 0$

Which means A is a non singular matrix
A is rank sufficient meaning the inverse exists.

A State Space Model is not Unique. (p3)

Given a T.F. system model we can create several state space models.

But given a system, for a particular input and output the TF is unique.

Proof (p3): Eqn defines one TF model.

If P is constant matrix and non-singular

Then $\quad x = Pz$

then $\quad \dot{x} = P\dot{z} + \dot{P}z$ but $\dot{P} = 0$ because P is constant

expand equations (p3)

Note that because *P* is a constant matrix

$$\dot{z} = P^{-1}AP + P^{-1}Bz$$

has the same transfer function as the original system, meaning there are many solutions

Example code is given on (p7) to run in your own time.

Canonical Forms. (p9)
There are many ways in which we can represent the state space model and all are the same.

Our focus will normally will forms 2 - 4. We wont cover 5.

1. Phase Variable (system has no zeros) so as a result (p14)

Here we **take the output as the state variable** and each state variable is defined as the derivitave of the previous state variable.

Ex 14. Select first state variable as output y, and rest as derivitaves (third order therefore 3 states)
$x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$ then write expression for $\dddot{y}$

General case shown p18 and p22.

Note:
Phase variable form cannot be used in practical electrical circuits:

e.g. If a system has three inductors and two capacitors. The 5 states are chosen as the 3 inductor currents and 2 capacitor voltages as they are easier to observe and control, not the derivatives of the output.

Office hour: Many other generatic examples are possible.
Encourage class to do Quizz examples.

1. $\frac{Y(s)}{U(s)} = \frac{b_0}{s^2+a_1s+a_2}$ can be written as $\ddot{y} + a_1\dot{y} + a_2y = b_ou$

Choose $x_1 = y$, $x_2 = \dot{y}$ and write down for $\ddot{y}$

To show $A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ b_o \end{bmatrix}$, $C = [1 \quad 0]$

2. $\frac{Y(s)}{U(s)} = \frac{1}{s^4+a_1s^3+a_2s^2+a_3s+a_4}$

Choose $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, $x_4 = \dddot{y}$

Show that $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

3. $\frac{Y(s)}{U(s)} = \frac{10}{s^4+2s^3+9s+7}$

show $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & -9 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$