Transient Circuit Analysis – First-Order Circuits

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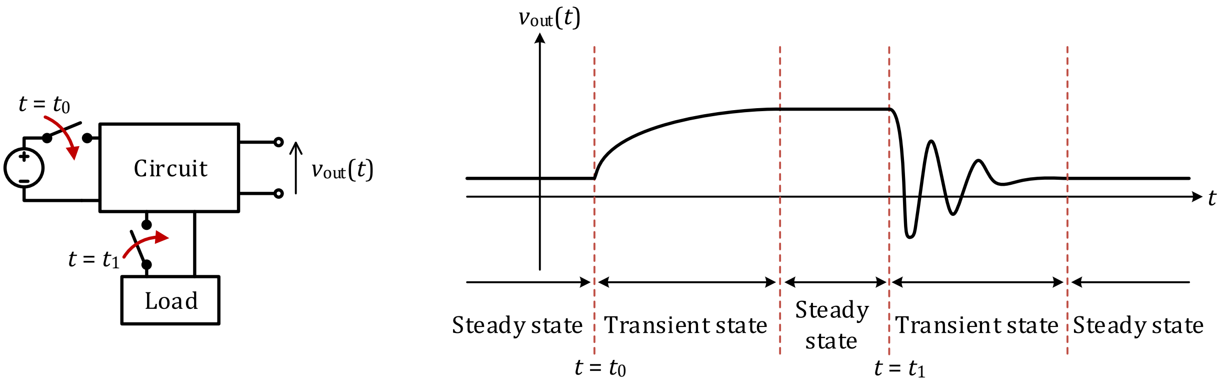
Intended Learning Outcomes

- Be able to recognise and identify a first-order circuit based on its topology, and the time-domain equations governing the underlying behaviour of the electrical quantities within the circuit
- Be able to derive the time-domain differential equations, and infer the corresponding responses describing first-order behaviours of electrical quantities from the underlying circuit
- Be able to determine the initial and final conditions necessary to characterise first-order circuits, and audit the analytical behaviours they exhibit
- Be able to interpret and relate the time-domain differential equations and responses governing the electrical behaviours of first-order circuits to the underlying application they result from

When a sudden change or interruption is made to an electric circuit, it necessarily takes a finite amount of time for the change to propagate through before all responses (currents and voltages) in the circuit settle at their new stable state – the *steady state*. The period between which the circuit transitions from its prior state (before the change) to its final steady-state is known as the *transient* interval, during which we say the circuit is in a *transient state*.



Transients are not observed (or rather, neglected) in resistive circuits because of their relatively negligible durations – signal *reflections*[1] occur in a matter of femtoseconds before the circuit reaches equilibrium, and so all results obtained from our DC analyses thus far are that of the steady-state behaviour. Transients only become apparent in circuits containing elements such as *capacitors* or *inductors* since their terminal characteristics forbid the voltages or currents from changing *instantly*. Consequently, in these circuits, the steady state is only reached after a sufficient period of time, exemplifying the transient-state behaviours.

While transient responses may seem like an inconsequential artefact of the underpinning physics governing the operations, the study of these circuit transients is important because it enables us to *understand*, from a circuit's view, why the responses behave the way they do, this in turn allows us to *predict* circuit behaviours in situations where they matter, and ultimately facilitates in the *design* of applications that make use of such behaviours.

[1] During the transient, voltage and current signals 'bounces' back and forth throughout the circuit before reaching an equilibrium. This bouncing effect is not noticeable when the fundamental wavelength of the input change made to the circuit is significantly larger than the physical length of the circuit paths the signals have to travel through – the reflections settle down on a time-scale much faster than the signals change. Details of signal reflections are studied at a later stage in ELECTENG 204.

It should be clear that the duration of the transient for all responses in a given circuit has to be the same[2], thus if a response is in a transient-state at a given time instant, all other responses must also be undergoing transient transitions at that instant, and if any response is at steady-state, then the entire circuit must be at equilibrium. Said differently, any temporal features (e.g., overshoots, oscillations, convergence, etc.) observed in a response during (and after) a transient must also be directly observable in all other responses of the same circuit – **the solution *form* of *any* response in a given circuit must be the same.**

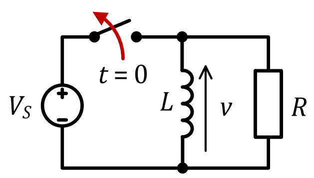
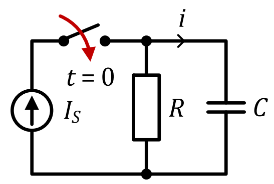
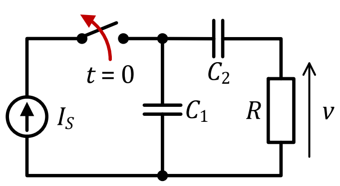
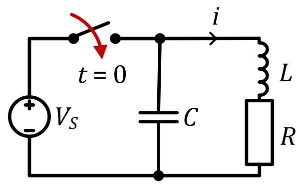
First-Order Circuits

The observation that the transient behaviour is *independent* of the individual responses within a circuit reveals that it is a property of the effectuating circuit, subject to its topology and constituents. This immediately points at a direct relationship to the *number* of capacitors and inductors in effect. We will start by examining circuits that contain, or has an equivalent[3] that contain, **one** capacitor or inductor in effect, these are known as *first-order circuits*.

As its name suggests, the responses in a first-order circuit is necessarily characterised by a first-order ordinary differential equation (ODE) of the form

$$a_1 \frac{dy}{dt} + a_0 y = x(t)$$

for some constants $a_0, a_1\ (\neq 0)$ that depend on the circuit parameters (i.e., resistances, capacitances or inductances) in effect, and the external excitation $x(t)$ (which could be zero) driving the response $y(t)$. Given that first-order circuits contains only one equivalent energy storage element, there are only two types of first-order circuits – *resistor-capacitor* (RC) and *resistor-inductor* (RL) circuits[4].

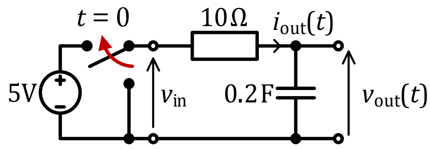
Examples of 1st-order Circuits (for $t > 0$)	
 $\frac{dv}{dt} + \frac{R}{L} v = 0$	 $\frac{di}{dt} + \frac{1}{RC} i = \frac{dI_S}{dt}$
 $\frac{dv}{dt} + \frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) v = 0$	 $L \frac{di}{dt} + Ri = V_S$

[2] Because all voltages and currents are interrelated by (1) Kirchhoff's laws, and (2) the terminal characteristics of the circuit elements, any deviation in one quantity must cause a consequential displacement in the others.

[3] The notion of equivalence here is identical to that used in defining *equivalent resistances* – two capacitors/inductors in series or parallel can be, equivalently, reduced to *one* equivalent capacitor/inductor in terms of their terminal characteristics. Similarly, when we say two circuits are equivalent, we mean that both circuits produce identical response(s) of interest.

[4] Circuits containing only capacitors or inductors are trivial first-order circuits – their behaviours are governed directly by the respective terminal characteristics, $i = C\, dv/dt$ or $v = L\, di/dt$.

Example 1. For the circuit shown, determine the voltage across the capacitor, $v_{\text{out}}(t)$, and the current $i_{\text{out}}(t)$ for $t > 0$.



As an initial probe, let us consider what happens at the two steady-states: prior to $t = 0$, and well after $t = 0$, as $t \to \infty$:

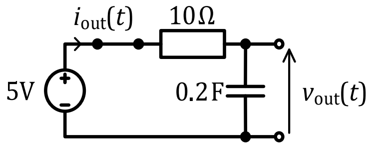
Before $t = 0$, it is reasonable to assume that, in the absence of further details, the circuit has been in its given state indefinitely, and is thus already in steady-state, under which[5] $v_{\text{out}}(t) = 0$ V for $t < 0$, and in particular, *just before* $t = 0$, $v_{\text{out}}(0^-) = 0$ V.

After the 5V source is connected at $t = 0$, as $t \to \infty$ the capacitor will eventually settle at its new steady-state, $v_{\text{out}}(\infty) = 5$ V – the capacitor has *charged* to what is possible at where it is placed.

From the behaviour of a capacitor we know that *just after* $t = 0$, $v_{\text{out}}(0^+) = v_{\text{out}}(0^-) = 0$ V.

The transient occurs in between the two steady-states (i.e., $0 < t < \infty$), to determine the behaviour of v_{out} during this interval, we analyse the circuit as usual using, say, KCL and the terminal characteristics of the elements in effect to obtain the first-order differential equation

$$\frac{5 - v_{\text{out}}}{R} = C\frac{dv_{\text{out}}}{dt} \qquad \left(\text{or} \quad RC\frac{dv_{\text{out}}}{dt} + v_{\text{out}} = 5\right)$$

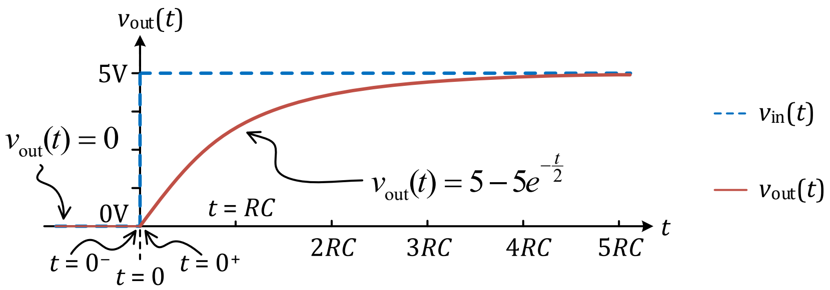


solving which[6]

$$v_{\text{out}}(t) = 5 + Ke^{-\frac{t}{RC}} \quad \text{for } t > 0,$$

and using the initial condition found above we have $v_{\text{out}}(t) = 5 - 5e^{-\frac{t}{2}}$ for $t > 0$.

The differential equation shows that with the capacitor initially uncharged, immediately after connecting the 5V source, the current through the resistor (and capacitor) is $i(0^+) = (5 - 0)/R$. This current results in a positive change in the capacitor voltage v_{out} since $i = C\,dv_{\text{out}}/dt$ and causes it to rise. This in turn reduces the current through the resistor $i = (5 - v_{out})/R$ and slows down the rate at which v_{out} rises until eventually the rate of change reaches zero. At this point $v_{\text{out}}(\infty) = 5$ V as depicted by the analytical description of $v_{\text{out}}(t)$ – the differential equation provides the physical *interpretation* behind the underlying behaviour, while its solution gives a precise *description* of it.



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[5] Why? Because if the circuit is at a constant equilibrium, the capacitor current has to be zero, and acts like an open-circuit. Applying Ohm's law and KVL then immediately gives the result.

[6] Either by separation of variables, integrating factor, or trial substitutions – methods learnt in ENGSCI 111 & 211.

$i_{\text{out}}(t)$

With knowledge of $v_{\text{out}}(t)$, to find $i_{\text{out}}(t)$ for $t > 0$ we note that $i_{\text{out}}(t)$ is the current through the 10Ω, and so by Ohm's law

$$i_{\text{out}}(t) = \frac{5 - v_{\text{out}}(t)}{R} = 0.5e^{-\frac{t}{2}}.$$

Sanity Check for $i_{\text{out}}(t)$

Alternatively, $i_{\text{out}}(t)$ is also the capacitor current satisfying $\quad i_{\text{out}}(t) = C\frac{dv_{\text{out}}}{dt} = 0.5e^{-\frac{t}{2}}.$

Of course, we could have, instead, found $i_{\text{out}}(t)$ from scratch without knowledge of $v_{\text{out}}(t)$. Upon inspection, assuming the circuit is in steady-state prior to $t = 0$, it should be clear that *just before* $t = 0$, $i_{\text{out}}(0^-) = 0$ A. However, note that this has **no** connection to what the current $i_{\text{out}}(0^+)$ *just after* $t = 0$ is – the terminal behaviour of the capacitor only dictates its voltage be preserved in the event of a sudden change.

Similarly, when the capacitor settles at its new steady-state as $t \to \infty$, $i_{\text{out}}(\infty) = 0$ A.

To determine the initial condition $i_{\text{out}}(0^+)$ *just after* $t = 0$, we note that since $v_{\text{out}}(0^+) = 0$ V, by KVL and Ohm's law we can deduce that $i_{\text{out}}(0^+) = (5 - 0)/10 = 0.5$ A.

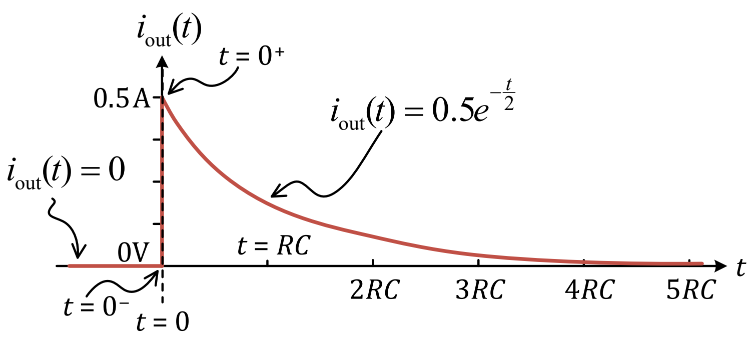
The differential equation governing i_{out} during the transient can be shown to be[7]

$$R\frac{di_{\text{out}}}{dt} = -\frac{i_{\text{out}}}{C} \qquad \left(\text{or} \quad RC\frac{di_{\text{out}}}{dt} + i_{\text{out}} = 0\right),$$

solving which

$$i_{\text{out}}(t) = Ke^{-\frac{t}{RC}} \quad \text{for t} > 0,$$

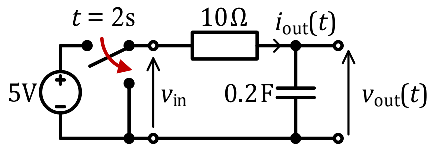
and with the initial condition also gives $i_{\text{out}}(t) = 0.5e^{-\frac{t}{2}}$ for $t > 0$ as expected.



Notice in order to obtain an unique description of $v_{\text{out}}(t)$ and $i_{\text{out}}(t)$ for the circuit (i.e., to find the unknown constants K) we require the initial conditions $v_{\text{out}}(0^+)$ and $i_{\text{out}}(0^+)$ *just after* the abrupt change, and this *depends* on the state of the circuit *just before* the change (i.e., we need $v_{\text{out}}(0^-)$). This is a common theme you will need to be acquainted with in transient analysis – **knowledge of initial conditions just after an abruption *requires* information on the circuit behaviour at the instant just before**.

[7] For example, since $i_{\text{out}} = C\,dv_{\text{out}}/dt$ and $v_{\text{out}} = 5 - Ri_{\text{out}}$ by KVL, the result follows upon substituting for v_{out}.

Example 2. For the same circuit in **Example 1**, the switch is set to the original position after $t = 2\,\mathrm{s}$, determine the capacitor voltage $v_{out}(t)$ and current $i_{\text{out}}(t)$ for $t > 2$.



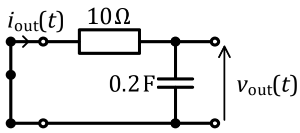
As before, we start by making some observations regarding the behaviour of v_{out} *before* and *after* the abrupt change at $t = 2\text{s}$.

Prior to the abrupt change at $t = 2$, from **Example 1** we know the description of $v_{\text{out}}(t)$ is given by[8] $v_{\text{out}}(t) = 5 - 5e^{-0.5t}$ and so $v_{\text{out}}(2^-) \approx 3.16\ \text{V}$ which means *just after* the change, $v_{\text{out}}(2^+) \approx 3.16\ \text{V}$ since the capacitor voltage cannot change instantly[9].

Well after $t = 2$, if we waited long enough, the circuit will eventually reach steady-state at which the capacitor current will be zero, the voltage across the resistor, and hence the capacitor voltage, will be $v_{\text{out}}(\infty) = 0$ – the capacitor has *discharged* completely.

Between $2 < t < \infty$ the transient behaviour of the circuit in effect can be determined via standard analysis, for example, by KCL to obtain the first-order differential equation

$$C\frac{dv_{\text{out}}}{dt} = -\frac{v_{\text{out}}}{R} \qquad \left(\text{or} \quad RC\frac{dv_{\text{out}}}{dt} + v_{\text{out}} = 0\right)$$

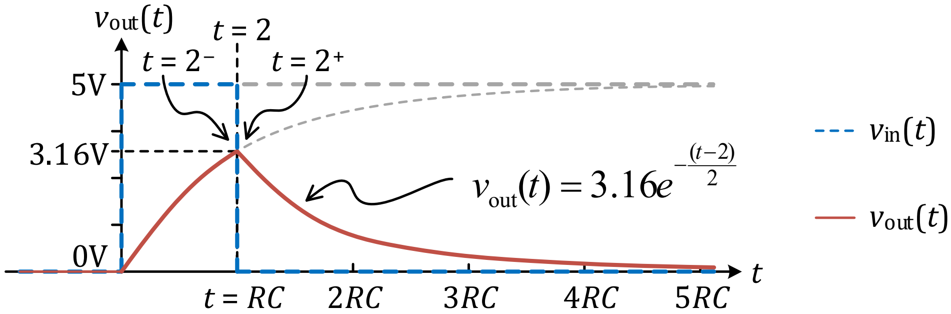


solving over the domain $t > 2$ we get

$$v_{\text{out}} = Ke^{-\frac{(t-2)}{RC}} \quad \text{for } t > 2,$$

and using the initial condition found above we have $\quad v_{\text{out}}(t) = 3.16e^{-\frac{(t-2)}{2}} \quad \text{for } t > 2.$

The differential equation makes perfect sense because immediately after the change, the current through the resistor and hence the circuit is $i_{\text{out}}(2^+) = -v_{\text{out}}(2^+)/R$ and this current results in a negative change in the capacitor voltage v_{out} since $i_{\text{out}} = C\,dv_{\text{out}}/dt$ causing v_{out} to drop. This in turn decreases the current through the circuit $i_{\text{out}} = -v_{\text{out}}/R$ and reduces the rate at which v_{out} falls until the rate reaches zero. At this point $v_{\text{out}}(\infty) = 0$ as depicted by the analytical solution.



[8] In this case we have prior knowledge of the circuit for $t < 2$, and so we have not assumed it to be in steady-state.
[9] Note that the argument is not that v_{out} cannot change instantly, it is the voltage *across* the capacitor that cannot change instantly due to its terminal behaviour. But in this case, v_{out} is designated as that across the capacitor.

$i_{\text{out}}(t)$

We note that for $t > 2$ the voltage across the 10Ω equals that across the capacitor, thus the current through 10Ω is, by Ohm's law,

$$i_{\text{out}}(t) = \frac{0 - v_{\text{out}}(t)}{R} = -0.316e^{-\frac{(t-2)}{2}}.$$

Sanity Check for $i_{\text{out}}(t)$

Alternatively, i_{out} is also the capacitor current satisfying $\quad i_{\text{out}}(t) = C\frac{dv_{\text{out}}}{dt} = -0.316e^{-\frac{(t-2)}{2}}.$

On the other hand, to find i_{out} from scratch, we note that *just before* $t = 2$, $i_{\text{out}}(2^-) \approx 0.184$ A, however *just after* $t = 2$ we have $i_{\text{out}}(2^+) = -v_{\text{out}}(2^+)/10 = -0.316$ A. As $t \to \infty$, the circuit settles and $i_{\text{out}}(\infty) = 0$ A.

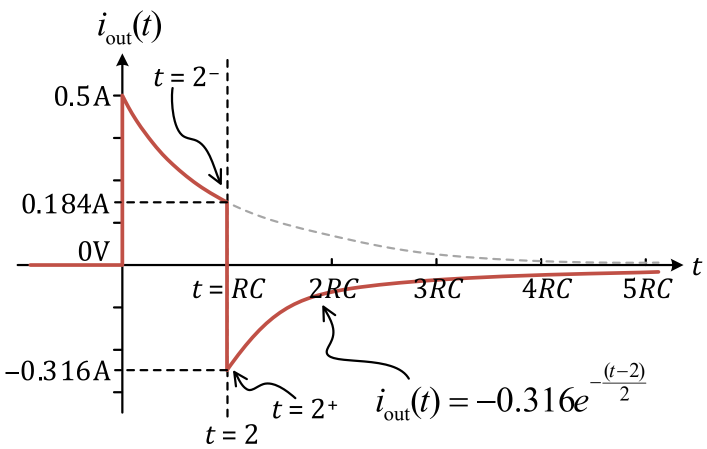
The differential equation governing i_{out} during the transient (for $t > 2$) can be shown to be[10]

$$R\frac{di_{\text{out}}}{dt} = -\frac{i_{\text{out}}}{C} \qquad \left(\text{or} \quad RC\frac{di_{\text{out}}}{dt} + i_{\text{out}} = 0\right),$$

solving which

$$i_{\text{out}}(t) = Ke^{-\frac{(t-2)}{RC}} \quad \text{for t} > 2,$$

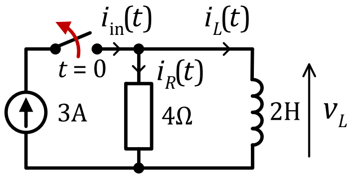
and with the initial condition also gives $i_{\text{out}}(t) = -0.316e^{-\frac{(t-2)}{2}}$ for $t > 2$ as expected.



Notice as before, the description of $v_{\text{out}}(t)$ and $i_{\text{out}}(t)$ for $t > 2$ requires the initial conditions $v_{\text{out}}(2^+)$ and $i_{\text{out}}(2^+)$ *just after* the sudden change, and this *depends* on the circuit behaviour *just before* the interruption (i.e., $v_{\text{out}}(2^-)$).

[10] For example, since $i_{\text{out}} = C\,dv_{\text{out}}/dt$ and $v_{\text{out}} = -Ri_{\text{out}}$ by Ohm's law, the result follows upon substituting for v_{out}.

Example 3. For the circuit shown, determine the inductor current $i_L(t)$, and the current through the resistor $i_R(t)$ for $t > 0$.



Prior to $t = 0$, one can reasonably assume that the circuit has already reached steady-state, and so we can deduce[11] that in particular $i_R(0^-) = 0$, and the inductor current $i_L(0^-) = 3$ A.

After the current source is disconnected for $t \geq 0$, as $t \to \infty$ the series circuit in effect will once again reach its new steady-state at which the inductor voltage $v_L(\infty) = 0$, and so by Ohm's law $i_R(\infty) = -i_L(\infty) = 0$.

The initial conditions *just after* the abrupt change to the circuit can be deduced by noting that the terminal behaviour of the inductor dictates that $i_L(0^-) = i_L(0^+) = 3$ A downwards, which means $i_R(0^+) = -i_L(0^+) = -3$ A.

To, arbitrarily, find $i_R(t)$ first, we apply standard analysis to the circuit for $t > 0$ and determine the corresponding differential equation describing $i_R(t)$. KVL and Ohm's law reveals

$$v_L = Ri_R,$$

since $v_L = L\,di_L/dt$, and $i_L(t) = -i_R(t)$,

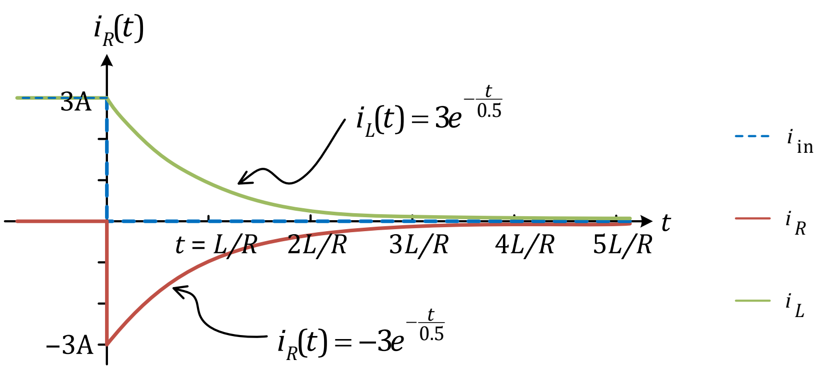
$$L\frac{di_R}{dt} = -Ri_R \qquad \left(\text{or } \frac{L}{R}\frac{di_R}{dt} + i_R = 0\right),$$

solving which gives

$$i_R(t) = Ke^{-\frac{t}{L/R}} \text{ for } t > 0.$$

Using the initial condition for $i_R(t)$ we obtain $\quad i_R(t) = -3e^{-\frac{t}{0.5}}$ for $t > 0$.

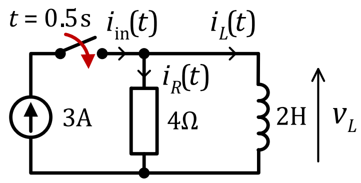
Since the two elements are in series, we have[12] $\quad i_L(t) = -i_R(t) = 3e^{-\frac{t}{0.5}}$.



[11] At a constant steady-state, there is no further change in the inductor current, di_L/dt. Since $v_L = L\,di_L/dt = 0$, the inductor acts like a short-circuit. By Ohm's law, $i_R = 0$, and so KCL reveals $i_L = 3$ A.

[12] Alternatively, this can be obtained via integrating $v_L(t) = 4i_R(t)$ from $t \geq 0$ since $v_L = L\,di_L/dt$, or from its governing differential equation $L\frac{di_L}{dt} + Ri_L = 0$.

Example 4. For the same circuit in **Example 3**, the switch is set to the original position after $t = 0.5$ s, find the inductor current i_L and the resistor current i_R for $t > 0.5$.



Before $t = 0.5$, we know from **Example 3** that $i_R(0.5^-) \approx -1.10$ A, and $i_L(0.5^-) \approx 1.10$ A. The only information we can infer from this prior state on the state *just after* the abruption is that the inductor current cannot change instantly, and so $i_L(0.5^+) = i_L(0.5^-) \approx 1.10$ A.

After $t = 0.5$, as $t \to \infty$ the circuit will reach its new steady-state where $i_R(\infty) = 0$, and $i_L(\infty) = 3$ A.

The initial conditions on the required quantities *just after* the abruption can be determined by KCL

$$i_R(0.5^+) = 3 - \underbrace{i_L(0.5^+)}_{\approx 1.10\text{ A}} \approx 1.90\text{ A}.$$

Suppose we wish to find $i_L(t)$ first, applying KVL and Ohm's law at the right loop for $t > 0.5$ we have

$$v_L = 4i_R,$$

and since $v_L = L\, di_L/dt$ and $i_R = 3 - i_L$,

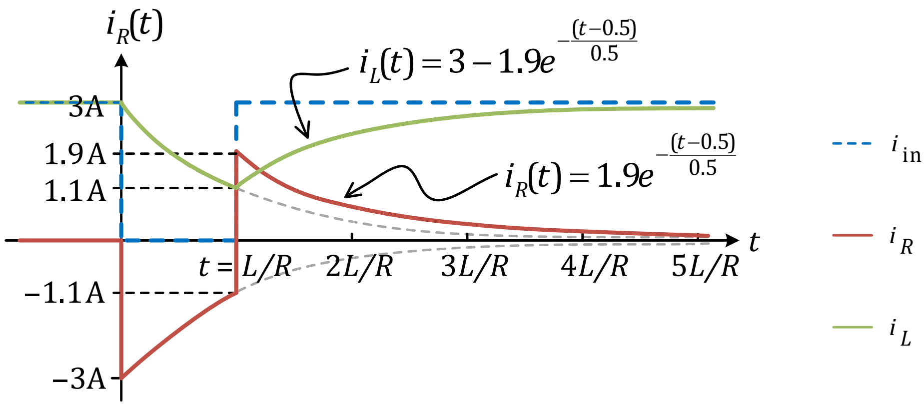
$$L\frac{di_L}{dt} = R(3 - i_L) \qquad \left(\text{or} \quad \frac{L}{R}\frac{di_L}{dt} + i_L = 3\right),$$

solving over the domain $t > 0.5$,

$$i_L(t) = 3 + Ke^{-\frac{(t-0.5)}{L/R}} \quad \text{for } t > 0.5,$$

And with the determined initial condition we have $\quad i_L(t) = 3 - 1.90e^{-\frac{(t-0.5)}{0.5}}$ for $t > 0.5$.

KCL then reveals $i_R(t)$ to be[13] $\quad i_R(t) = 3 - i_L(t) = 1.90e^{-\frac{(t-0.5)}{0.5}}$.



[13] Alternatively, we note $v_L = L\, di_L/dt$, and so $i_R = v_L/R$ by Ohm's law. The differential equation governing i_R can also be shown to be $L\frac{di_R}{dt} + Ri_R = 0$.

General Solution to First-Order Responses

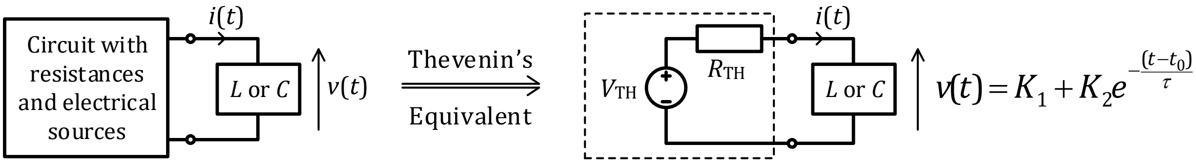
In all of the examples we have witnessed thus far, notice the transient behaviours of *all* responses when subjected to an abruption at $t = t_0$ take on the amiable form

$$y(t) = K_1 + K_2 e^{-\frac{(t-t_0)}{\tau}}$$

for some constants K_1, K_2 governed by both the initial and steady-state (final) conditions of the transient:

$$K_1 = \underbrace{y(\infty)}_{=\text{final cond.}} \quad , \quad K_2 = \underbrace{y(t_0^+)}_{=\text{initial cond.}} - y(\infty),$$

and τ (the Greek letter *tau*) which depends on the circuit elements in effect ($\tau = RC$ or $\tau = L/R$). As it turns out, *all* first-order circuits respond to a sudden change with the *exact* same exponential decay characteristic. This is because *any* first-order circuit can be equivalently reduced[14] to a canonical one that consists of a voltage source (which could be zero), a resistor, and a capacitor (or an inductor), and consequently all first-order circuits can be effectively converted to the equivalent problems in **Example 1** to **Example 4**:



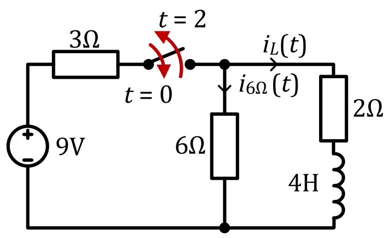
Since all responses in a given circuit must take on the same solution form, this conveniently gives us a way to bypass the need to solve differential equations every time – **any** response in an arbitrary first-order circuit subjected to a sudden change at $t = t_0$ obeys the general form

$$y(t) = \underbrace{y(\infty)}_{=\text{steady-state}} + \underbrace{[y(t_0^+) - y(\infty)]e^{-\frac{(t-t_0)}{\tau}}}_{=\text{transient}} \quad \text{for } t > t_0.$$

with $\tau = R_{TH}C$ or $\tau = L/R_{TH}$, where R_{TH} is the Thevenin resistance *seen* by the energy storage element in the circuit. Determining an arbitrary first-order response then boils down to identifying the initial, steady-state conditions, and the constant τ.

[14] By combining capacitances/inductances into an equivalent capacitance/inductance, and replacing the circuit seen by this equivalent with its Thevenin's equivalent.

Example 5. In the circuit shown, the switch is closed at $t = 0$ and opened at $t = 2$. Determine the current $i_{6\Omega}(t)$ through the 6Ω for $t > 0$.



Since this is a first-order circuit, instead of forming (and solving) the governing differential equation, we could use our knowledge that all first-order responses must obey the solution form

$$y(t) = y(\infty) + [y(t_0^+) - y(\infty)]e^{-\frac{t-t_0}{L/R_{TH}}} \quad \text{for } t > t_0$$

to find $i_{6\Omega}(t)$. By doing so, we only need to determine the initial and final conditions $i_{6\Omega}(t_0^+)$, $i_{6\Omega}(\infty)$, and the Thevenin resistance R_{TH} seen by the inductor for the circuit in effect over the periods of interest. Since there are two transient intervals $0 < t < 2$ and $2 < t < \infty$, we need to consider each separately (as the circuit in effect is necessarily different):

$0 < t < 2$

For $t < 0$, it is reasonable to assume the circuit is already in steady-state, thus, in particular, $i_{6\Omega}(0^-) = 0$. Since the inductor current cannot change instantly, just after the abruption at $t_0 = 0$, we know $i_L(0^+) = i_L(0^-) = 0$ and thus

$$i_{6\Omega}(0^+) = \frac{9}{3+6} = 1 \text{ A}.$$

To find $i_{6\Omega}(\infty)$ we find the steady-state values of the circuit while the switch is closed – this is the value $i_{6\Omega}$ would converge to *if* the switch remained closed indefinitely[15]:

$$i_{6\Omega}(\infty) = \underbrace{\left(\frac{9}{3+6\|2}\right)}_{=\text{supplied current}} \cdot \underbrace{\left(\frac{1/6}{1/6+1/2}\right)}_{=\text{current divider}} = 0.5 \text{ A}.$$

Lastly, the Thevenin resistance of the circuit seen by the inductor during $0 < t < 2$ is

$$R_{TH} = 2 + 3\|6 = 4\ \Omega,$$

so $\tau = L/R_{TH} = 1$. Thus[16] for $0 < t < 2$,

$$i_{6\Omega}(t) = 0.5 + (1 - 0.5)e^{-\frac{t-0}{1}} = \underbrace{0.5}_{=\text{steady-state}} + \underbrace{0.5e^{-t}}_{=\text{transient}} \text{ A}.$$

[15] Remember the $t = \infty$ instance here refers to the steady-state behaviour of the circuit in effect *during* the transient interval we are interested in, i.e., $0 < t < 2$, and **not** the actual behaviour of the circuit at $t = \infty$. Said differently, since the circuit cannot predict the future, during the interval $0 < t < 2$ it would respond as if the circuit topology remained as is forever – it does not know that the switch will be opened at $t = 2$.

[16] Determining the governing differential equation is a bit tricky in this case, but it can be shown that the corresponding differential equations for $i_{6\Omega}$ is $\frac{di_{6\Omega}}{dt} + i_{6\Omega} = \frac{1}{2}$. Convince yourself that the determined response satisfies this equation.

$2 < t < \infty$

When the switch opens at $t_0 = 2$, the inductor current cannot change instantly, $i_L(2^+) = i_L(2^-)$ where $i_L(2^-)$ can be found by[17], say, KCL

$$i_L(2^-) = \underbrace{\frac{9 - 6i_{6\Omega}(2^-)}{3}}_{=\text{supplied current}} - i_{6\Omega}(2^-) = 3 - 3i_{6\Omega}(2^-) \approx 1.30\text{ A} = i_L(2^+),$$

and since we have a series circuit for $t > 2$,

$$i_{6\Omega}(2^+) = -i_L(2^+) \approx -1.30\text{ A}.$$

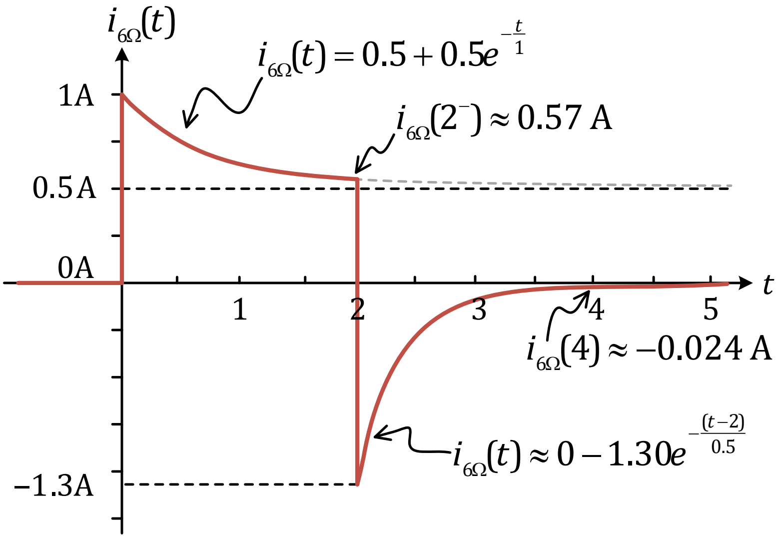
With the inductor voltage being zero at steady-state, clearly $i_{6\Omega}(\infty) = 0$ A.

The Thevenin resistance of the circuit seen by the inductor during $2 < t < \infty$ is

$$R_{TH} = 2 + 6 = 8\ \Omega,$$

and so $\tau = L/R_{TH} = 0.5$. Thus for $t > 2$,

$$i_{6\Omega}(t) = 0 + (-1.30 - 0)e^{-\frac{(t-2)}{0.5}} = \underbrace{0}_{=\text{steady-state}} - \underbrace{1.3e^{-\frac{(t-2)}{0.5}}}_{=\text{transient}}.$$



[17] Alternatively, we could determine a description of $i_L(t)$ during $0 < t < 2$ and use it to evaluate $i_L(2^-)$. Considering the initial and final conditions between $0 < t < 2$, we have $i_L(0^+) = 0$ and $i_L(\infty) = 1.5$ A, therefore $i_L(t) = 1.5 + (0 - 1.5)e^{-t}$, and so $i_L(2^-) \approx 1.30$ A as expected.

Time constant

Notice in **Example 5** the 'speed' at which the response $i_{6\Omega}(t)$ approaches its expected steady-state value is different for the two separate transient intervals $0 < t < 2$ and $2 < t < \infty$. For example, after **two** seconds,

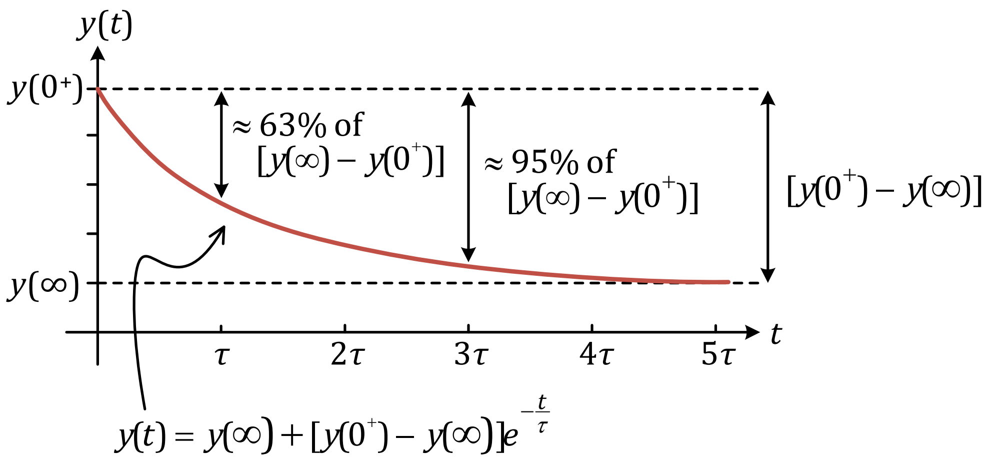
$$i_{6\Omega} \text{ have gotten } \begin{cases} \dfrac{(1-0.57)}{0.5} \approx 86\% \\[2ex] \dfrac{(1.3-0.024)}{1.3} \approx 98\% \end{cases} \text{ of the way to its new steady-state between } \begin{cases} 0 < t < 2 \\[2ex] 2 < t < \infty \end{cases}$$

The 'speed' at which a transient response reaches its steady-state is governed by the constant τ (it controls the rate at which the exponential function diminishes). As such, the constant τ is also known as the *time constant* and unsurprisingly has units of seconds[18]. A circuit with a larger τ takes *longer* to reach its new steady-state compared to the same circuit but with a smaller τ which settles more quickly.

More precisely,

$$\text{after } \begin{cases} \text{1 time constant} \\ \text{3 time constants} \\ \text{5 time constants} \end{cases} \text{ the circuit will have gotten } \begin{cases} 1 - e^{-1} \approx 63\% \\ 1 - e^{-3} \approx 95\% \\ 1 - e^{-5} \approx 99\% \end{cases} \text{ of the way to its new state,}$$

and so usually after approximately 5 time constants, for all practical purposes[19], we say the circuit has reached steady state. The time constant thus provides a good measure of "how long" the transient lasts.



Returning to **Example 5**, the time constant for the circuit in effect between $0 < t < 2$ (i.e., $\tau = 1$) is *twice* that of the circuit in effect between $2 < t < \infty$ (i.e., $\tau = 0.5$), and so the response in the former takes twice as long to settle in comparison to the latter – after two seconds, only two time constants have passed for the former circuit, while four time constants have already gone by for the latter.

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[18] Since $\tau = R_{TH}C$ or $\tau = L/R_{TH}$, and dimensionally R_{TH} has units Volts/(Coulombs per seconds), C has units Coulombs per Volts, and L has units Volts/(Coulombs per seconds^2), evaluating the dimension of τ in both cases shows that it has the units of time, seconds.
[19] Mathematically, the steady state is never reached since it is an asymptote, i.e., $y(t) \to y(\infty)$ only if $t \to \infty$.

Problems*

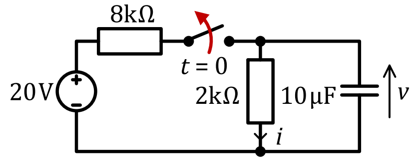
1. Consider the circuit shown, for $t > 0$ find

   (a) the differential equation governing v and i

   (b) the initial conditions $v(0^+)$ and $i(0^+)$,

   (c) the steady-state responses $v(\infty)$ and $i(\infty)$,

   (d) the responses $v(t)$ and $i(t)$ for $t > 0$.



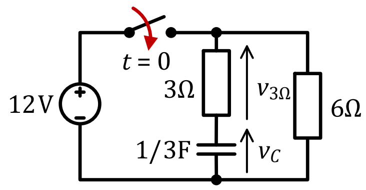
2. Consider the circuit shown, for $t > 0$ find

   (a) the differential equation governing v_C and $v_{3\Omega}$,

   (b) the initial conditions $v_C(0^+)$ and $v_{3\Omega}(0^+)$,

   (c) the steady-state responses $v_C(\infty)$ and $v_{3\Omega}(\infty)$,

   (d) the responses $v_C(t)$ and $v_{3\Omega}(t)$ for $t > 0$.



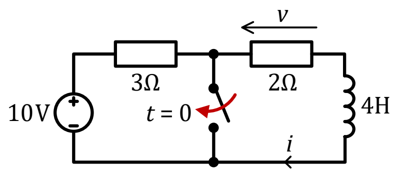
3. Consider the circuit shown, for $t > 0$ find

   (a) the differential equation governing v and i,

   (b) the initial conditions $v(0^+)$ and $i(0^+)$,

   (c) the steady-state responses $v(\infty)$ and $i(\infty)$,

   (d) the responses $v(t)$ and $i(t)$ for $t > 0$.



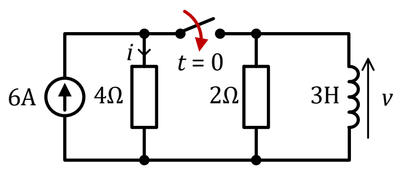
4. Consider the circuit shown, for $t > 0$ find

   (a) the differential equation governing v and i,

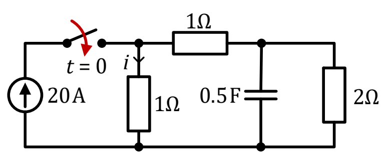
   (b) the initial conditions $v(0^+)$ and $i(0^+)$,

   (c) the steady-state responses $v(\infty)$ and $i(\infty)$,

   (d) the responses $v(t)$ and $i(t)$ for $t > 0$.



5. For the circuit shown, determine the current $i(t)$ for $t > 0$ [20].



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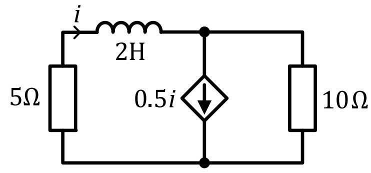
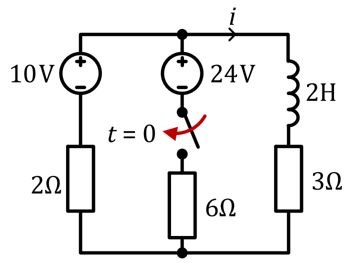
* Some of the following problems are adapted from: Alexander, C. K. & Sadiku, M. N. O. (2013). First-Order Circuits. *Fundamentals of Electric Circuits* (pp. 301-309). New York, NY: McGraw-Hill.

[20] Finding the differential equation for i is tricky. Can you show that the governing equation is $\frac{di}{dt} + 2i = 30$?

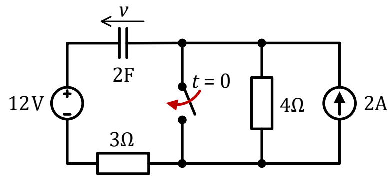
6. For the circuit shown, determine the current $i(t)$ for $t > 0$, given that $i(0) = 5$ A.



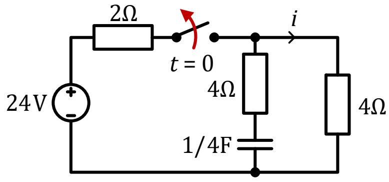
7. For the circuit shown, determine the current $i(t)$ for $t > 0$.



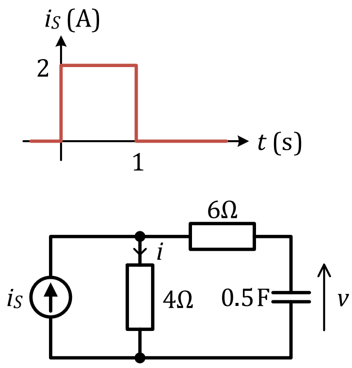
8. For the circuit shown, determine the capacitor voltage $v(t)$ for $t > 0$.



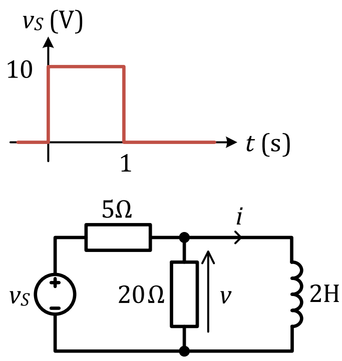
9. For the circuit shown, determine the current $i(t)$ for $t > 0$.



10. A step current is applied to the circuit shown at $t = 0$ before it is discontinued at $t = 1$. Determine the responses $v(t)$ and $i(t)$ for $t > 0$.



11. A step voltage is applied to the circuit shown at $t = 0$ before it is discontinued at $t = 1$. Determine the responses $i(t)$ and $v(t)$ for $t > 0$.



Outline Solutions

1(a). $\frac{dv}{dt}+50v=0,\quad \frac{di}{dt}+50i=0.$ 1(b). $v(0^+)=4\text{V},\quad i(0^+)=2\text{mA}.$

1(c). $v(\infty)=0\text{A},\quad i(\infty)=0\text{A}.$ 1(d). $v(t)=4e^{-50t}\text{ V},\quad i(t)=2e^{-50t}\text{ mA}, t>0.$

2(a). $\frac{dv_C}{dt}+v_C=12,\quad \frac{dv_{3\Omega}}{dt}+v_{3\Omega}=0.$ 2(b). $v_C(0^+)=0\text{V},\quad v_{3\Omega}(0^+)=12\text{V}.$

2(c). $v_C(\infty)=12\text{V},\quad v_{3\Omega}(\infty)=0\text{V}.$ 2(d). $v_C(t)=12(1-e^{-t})\text{V}, v_{3\Omega}(t)=12e^{-t}\text{V}, t>0.$

3(a). $2\frac{dv}{dt}+v=0,\quad 2\frac{di}{dt}+i=0.$ 3(b). $v(0^+)=4\text{V},\quad i(0^+)=2\text{A}.$

3(c). $v(\infty)=0\text{V},\quad i(\infty)=0\text{A}.$ 3(d). $v(t)=4e^{-0.5t}\text{V},\quad i(t)=2e^{-0.5t}\text{A}, t>0.$

4(a). $9\frac{dv}{dt}+4v=0,\quad 9\frac{di}{dt}+4i=0.$ 4(b). $v(0^+)=8\text{V},\quad i(0^+)=2\text{A}.$

4(c). $v(\infty)=0\text{V},\quad i(\infty)=0\text{A}.$ 4(d). $v(t)=8e^{-\frac{4}{9}t}\text{V},\quad i(t)=2e^{-\frac{4}{9}t}\text{A}, t>0.$

5. $i(t)=15-5e^{-2t}\text{A for } t>0.$

6. $i(t)=5e^{-5t}\text{ A for } t>0.$

7. $i(t)=3-e^{-2.25t}\text{A for } t>0.$

8. $v(t)=12-8e^{-\frac{t}{6}}\text{ V for } t>0.$

9. $i(t)=2e^{-0.5t}\text{ for } t>0.$

10. $v(t)=\begin{cases}8-8e^{-0.2t}\text{ V}, & 0<t<1\\ 1.45e^{-0.2(t-1)}\text{ V}, & t>1\end{cases},\quad i(t)=\begin{cases}2-0.8e^{-0.2t}\text{ A}, & 0<t<1\\ 0.145e^{-0.2(t-1)}\text{ A}, & t>1\end{cases}.$

11. $i(t)=\begin{cases}2-2e^{-2t}\text{ A}, & 0<t<1\\ 1.729e^{-2(t-1)}\text{ A}, & t>1\end{cases},\quad v(t)=\begin{cases}8e^{-2t}\text{ V}, & 0<t<1\\ -6.917e^{-2(t-1)}\text{ V}, & t>1\end{cases}.$