State Variable Modelling:Cannonical Forms

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Learning Outcomes

After completion of this module, the students should have learned the following:

- Prove that state variable models are not unique
- Able to representation a linear system in different forms
  - Controllable Canonical Form
  - Observable Canonical Form
  - Diagonal Canonical Form
  - Jordan Canonical Form
- How to Diagonalize a System

Non uniqueness of State Variable Representation

The state variable representation of a system is not unique

Proof:

- Consider the system described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{1}$$

- Let us define another set of state variables z which is related to x by the relation

$$x = Pz; \qquad Note: \quad \Longrightarrow \quad z = P^{-1}x \tag{2}$$

where P is a constant non-singular matrix

- Since P is a constant matrix, differentiating (2) gives

$$\dot{x} = P\dot{z} \tag{3}$$

- Now substituting these into the system equation (1) gives

$$P\dot{z} = \dot{x} = Ax + Bu = A[Pz] + Bu = APz + Bu \tag{4}$$

- Pre multiplying P^{-1} in (4) gives

$$\dot{z} = P^{-1}APz + P^{-1}Bu = A_z z + B_z u; \; A_z = P^{-1}AP, B_z = P^{-1}B \tag{5}$$

- The output equation modifies to

$$y = Cx + Du = CPz + Du = C_z z + D_z u; \quad C_z = CP, D_z = D \tag{6}$$

Summary of Similarity Transformation

Original System:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$

Transformed System

$$\begin{aligned}\dot{z} &= P^{-1}APz + P^{-1}Bz, \quad y = CPz + Du \\ &= A_z z + B_z u, \quad y = C_z z + Du, \quad where \quad x = Pz\end{aligned}$$

Transfer function computed from the original System matrices is equal to that computed from transformed system matrices i.e.

$$C\left(sI - A\right)^{-1} B + D = C_z\left(sI - A_z\right)^{-1} B_z + D$$

Compute the transfer function of the system

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

Transform this system to a new state vector

$$z = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix} x$$

- Compute the transfer function from both the systems and compare.

MATLAB Code

```
A=[1 3;-4 6]; B=[1;3],C=[1 4];D=0;
[num den]=ss2tf(A,B,C,D);
sys1=tf(num,den);
```

Transformed System

```
A=[1 3;-4 6]; B=[1;3],C=[1 4];D=0;
Pinv=[3 -2;1 -4]; P=inv(P);
Az=Pinv*A*P; Bz=Pinv*B;Cz=C*P;
[num1 den1]=ss2tf(Az,Bz,Cz,D);
sys2=tf(num1,den1);
```

Canonical Forms

- Different forms of transformation matrix P gives different forms of state space models. We are only interested on the canonical forms
- Canonical forms are the standard forms of state space models.
- Each of these canonical form has specific advantages which makes it convenient for use in particular design technique.
- There are five canonical forms of state space models.

1. Phase variable canonical form (No Zeros in Transfer function)
2. Controllable Canonical form(Transfer Function has zeros)
3. Observable Canonical form(Dual of Controllable Form)
4. Diagonal Canonical form (System has Distinct Poles)(Full Diagonalisation or Decoupling)
5. Jordan Canonical Form (System has poles of different multiplicity > 1)(Partial Diagonalisation or Decoupling)

Note: The dynamics properties of system remain unchanged whichever the type of representation is used.

Why to Represent a System in Different Forms ?

- Each of the forms offers certain advantages over other forms.
- For example
  - The mathematical complexity associated with the design of state feedback controller by pole placement will significantly be reduced if the system is represented in controllable canonical form
  - Similarly if we want to estimate the unknown states from output measurements using an observer, it is preferable to represent the system in observable canonical form
  - If we want to gain better physical insight , it is preferable to represent the system in diagonal canonical form.

State Variable Models from Transfer Function Models

Case-1 : System has **no zeros**

- **Phase Variable Canonical form**

Case-2 : System has **zeros**

a. **Controllable Canonical form**
b. **Observable Canonical form**
c. **Diagonal Canonical form**
   d **Jordan Canonical form**

State Equations from Transfer Function : Phase Variable Canonical Form

Case-1:

Transfer Function does not have any zeros

- Consider a system with the transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 5s^2 + 9s + 8}$$

- This corresponds to the differential equation

$$\dddot{y} + 5\ddot{y} + 9\dot{y} + 8y = u$$

- Define $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \ddot{y}$.
- **Note:** Each state variable is defined to be the derivative of the previous state variable. Such choice of state variables are called phase variables.

State Equations from Transfer Function : Phase Variable Canonical Form

- In terms of state variables, this can be expressed as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_3 - 9x_2 - 8x_1 + u \\ &= -8x_1 - 9x_2 - 5x_3 + u \end{aligned}$$

- In matrix form, this is written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -9 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Phase Variable Representation (General Case)

- Consider an n-th order system with no zeros having the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n}$$

- The phase variable canonical model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Phase Variable Cannonical Form

Summary:

1. The elements of last row of matrix-A (begining from column-1 to column-n) consists of negative of denominator coefficients in ascending powers of s (Right to left).
2. The elements in all rows of matrix-B are zero except the last element which equals to 1.
3. The elements in all columns of matrix-C are zero except the first element which equals to 1.

Controllable, Observable and Diagonal Cannonical Forms

Consider a system represented by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{s+11}{s^2+7s+10}$$

- The differential equation model of this system can be expressed as:

$$\ddot{y}(t) + 7\dot{y}(t) + 10y(t) = \dot{u}(t) + 11u(t)$$

- Note that the system has zeros i.e. the dynamics contain the derivative of the input.
- It is possible to represent this system in many different forms.

Controllable Cannonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 11 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Controllable, Observable and Diagonal Cannonical Forms

Observable Cannonical Form:

- This can be derived from the controllable cannonical form as follows:
  1. The system matrix of observable form equals to the transpose of the system matrix of controllable form. Thus $\boxed{\mathbf{A_{obs}} = \mathbf{A^T_{cont}}}$
  2. The input matrix of observable form equals the output matrix of controllable form. Thus $\boxed{\mathbf{B_{obs}} = \mathbf{C_{cont}}}$
  3. The output matrix of observable form equals the input matrix of controllable form. Thus $\boxed{\mathbf{C_{obs}} = \mathbf{B_{cont}}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 11 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State Space representation in Diagonal Canonical Forms

Diagonal Cannonical Form

- Represent the transfer function using partial fraction expansion as:

$$\frac{Y(s)}{U(s)} = \frac{s+11}{s^2+7s+10} = \frac{s+11}{(s+2)(s+5)} = \frac{3}{s+2} - \frac{2}{s+5}$$

This can further be expressed as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For more Explanation see APPENDIX

Example:

Consider a system given by

$$\frac{Y(s)}{U(s)} = \frac{7s^2 + 38s + 47}{s^3 + 9s^2 + 23s + 15} = \frac{2}{s+1} + \frac{1}{s+3} + \frac{4}{s+5}$$

Controllable Cannonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -23 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \; y = \begin{bmatrix} 47 & 38 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable Cannonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -15 \\ 1 & 0 & -23 \\ 0 & 1 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 47 \\ 38 \\ 7 \end{bmatrix} u, \; y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Diagonal Cannonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u, \; y = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State Space representation Jordan Canonical Forms (Non distinct Roots)

- Consider an n-th order system described by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

- Assume that this system contains non-distinct roots. As an example assume that $p_1 = p_2 = p_3$ and $p_4 = p_5$, $p_6, \ldots p_n$ are distinct.The partial fraction expansion gives

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \\ &= b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{(s+p_1)} \\ &+ \frac{c_4}{(s+p_4)^2} + \frac{c_5}{s+p_4} + \frac{c_6}{s+p_5} + \frac{c_7}{s+p_7} \ldots + \frac{c_n}{s+p_n}\end{aligned}$$

Jordan Canonical form (continued)

- In matrix form it is expressed as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \vdots\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & . & . & 0 & 0\\ 0 & -p_1 & 1 & 0 & . & . & 0 & 0\\ 0 & 0 & -p_1 & 0 & 0 & . & . & 0\\ 0 & . & 0 & -p_4 & 1 & 0 & . & 0\\ 0 & . & 0 & 0 & -p_4 & 0 & . & 0\\ 0 & 0 & . & . & 0 & -p_6 & 0 & 0\\ 0 & . & . & . & . & . & . & .\\ 0 & . & . & . & . & . & . & -p_n \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\\ 0\\ 1\\ 1\\ \vdots\\ 1\end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_{n-1} & c_n \end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ x_3\\ .\\ .\\ x_n\end{bmatrix} + b_0 u$$

Diagonalizing a System Matrix

Summary: Canonical Forms

- Different forms of representation offer different advantages:
  - Controllable canonical form of model used for designing controller
  - Observable canonical form is used for designing observer
  - Controlling becomes easy if system is represented in diagonal canonical form

The next question is

Given a system which is not in diagonal form, can we represent this system in diagonal form preserving the characteristics ?

The answer is **YES:**

- By suitably selecting the transformation matrix P it is possible to get different state space description of system
- If the **columns of P matrix are selected to be the eigenvectors of A**, the resulting system matrix will be diagonal, with the eigenvalues of the system along the diagonal.

Example-1: Diagonalization

- Consider a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx$$

- Represent this system in the form

$$\dot{z} = P^{-1}APz + P^{-1}Bu$$

such that $P^{-1}AP$ is diagonal.

Solution: Example-1: Diagonalization

- Note that if the columns of P are the eigenvectors of A, then $P^{-1}AP$ becomes diagonal.
- Hence the problem of diagonalization reduces to finding the eigenvectors of system matrix A.

  Step-1: Find the eigenvalues of system matrix A.
- This is obtained by solving

$$|\lambda I - A| = 0$$

Now

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda+3 & -1 \\ -1 & \lambda+3 \end{bmatrix} \right|$$
$$= \lambda^2 + 6\lambda + 8 = 0$$

- This gives the eigenvalues at $\lambda_1 = -2$ and $\lambda_2 = -4$.

Solution: Example-1: Diagonalization (contd)

Step-2: Determine the eigenvectors corresponding to each of the distinct eigenvalues

- This is obtained by solving

$$(\lambda_i I - A)\, x_i = 0$$

- The eigenvectors corresponding to eigenvalue $\lambda_1 = -2$ is obtained from

$$Ax_i = \lambda_1 x_i \implies \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies -3x_1 + x_2 = -2x_1$$
$$x_1 - 3x_2 = -2x_2$$

- This gives $x_1 = x_2$. We may select $x = \begin{bmatrix} c \\ c \end{bmatrix}$

Solution: Example-1: Diagonalization (contd)

Step-2: Similarly,the eigenvectors corresponding to eigenvalue $\lambda_2 = -4$ is obtained from

$$Ax_i = \lambda_2 x_i \implies \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies -3x_1 + x_2 = -4x_1$$
$$x_1 - 3x_2 = -4x_2$$

- This gives $x_1 = -x_2$. We may select $x = \begin{bmatrix} c \\ -c \end{bmatrix}$

Solution: Example-1: Diagonalization (contd)

- **Step-3:**Considering one possible choice of P as:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

First Compute P^{-1}, then compute $P^{-1}AP$, $P^{-1}B$ and CP.

$$P^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \quad Thus$$

$$P^{-1}AP = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

$$CP = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

APPENDIX

Controllable Cannonical Form : Strictly Proper Transfer Function

How to represent a strictly proper transfer function in state space

- **System Having Zeros**: Consider a system with transfer function

$$\frac{Y(s)}{U(s)} = \frac{3s^3 + 7s + 15}{s^3 + 7s^2 + 14s + 8}$$

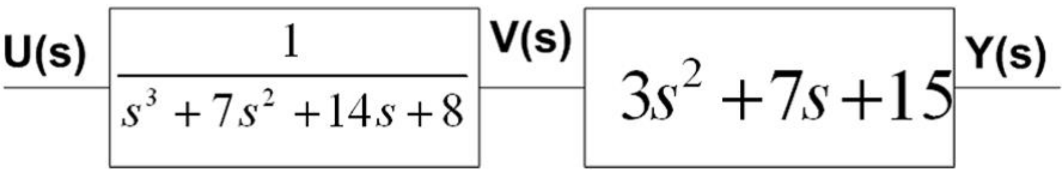
- This corresponds to the following differential equation

$$\dddot{y} + 7\ddot{y} + 14\dot{y} + 8y = 3\ddot{u} + 7\dot{u} + 15u$$

- Note that the dynamics contain derivative of the input.

Controllable Cannonical Form : Strictly Proper Transfer Function(contd)

- Let us split this function into two parts as shown
  - One with all pole parts
  - One with all zero parts



- The all pole part is written as

$$\frac{V(s)}{U(s)} = \frac{1}{s^3 + 7s^2 + 14s + 8}$$

- The all zero part is written as:

$$Y(s) = \left(3s^2 + 7s + 15\right) V(s) = 3\ddot{v} + 7\dot{v} + 15v$$

Controllable Cannonical Form : Strictly Proper Transfer Function (contd)

- Consider the all pole part of the system and define the state variables: $x_1 = v$,$x_2 = \dot{v}$ and $x_3 = \ddot{v}$
- This gives the following representation called as controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

- The output equation can be written as

$$y = \begin{bmatrix} 15 & 7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Controllable Cannonical Form: Strictly Proper Transfer Function

Summary:

1. The elements of last row of matrix-A (begining from column-1 to column-n) consists of negative of denominator coefficients in ascending powers of s(Right to left).
2. The elements in all rows of matrix-B are zero except the last element which equals to 1.
3. The elements in all columns of matrix-C consists of numerator coefficients in ascending powers of s(Right to left).

- Consider a proper transfer function where the degrees of numerator and denominator polynomial is same.

  Example-1: The state space model for the system

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

Example-2: The state space model for the system

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_3 - a_1 b_0 & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

Summary : Controllable Canonical form of Strictly Proper Transfer Functions

- The state space model for **Strictly Proper Transfer Function**

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

is expressed as: $\dot{\mathbf{x}} = \mathbf{AX} + \mathbf{BU}, \quad \mathbf{y} = \mathbf{CX} + \mathbf{DU}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ldots & \ldots & \vdots & \vdots \\ \vdots & \ldots & \ldots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} b_n & b_{n-1} & \ldots & \ldots & b_2 & b_1 \end{bmatrix}$$

Summary : Controllable Canonical form of Proper Transfer Functions

- The state space model for **Strictly Proper Transfer Function**

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

is expressed as: $\dot{\mathbf{x}} = \mathbf{AX} + \mathbf{BU}, \quad \mathbf{y} = \mathbf{CX} + \mathbf{DU}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ldots & \ldots & \vdots & \vdots \\ \vdots & \ldots & \ldots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0, & \ldots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix}, \mathbf{D} = b_0$$

Summary: Observable Cannonical Form of Proper Transfer Function

- The system matrix $\mathbf{A}$,input matrix $\mathbf{B}$, the output matrix $\mathbf{C}$ and transmission matrix $\mathbf{D}$ for observable canonical form is expressed as:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 & -a_n \\ 1 & 0 & \dots & \dots & 0 & -a_{n-1} \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & -a_2 \\ 0 & 0 & \dots & \dots & 1 & -a_1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0, & \dots & 0 & 1 \end{bmatrix}, \mathbf{D} = b_0$$

1. The columns of system matrix in observable cannonical form A_{obs} equals to the rows of systtem matrix of controllable canonical form A_{cont} i.e. $A_{obs} = A_{cont}^T$.
2. $B_{obs} = C_{cont}^T$ and $C_{obs} = B_{cont}^T$

State Space representation in Diagonal Canonical Forms (Distinct Roots)

- Consider an example of a second order system described by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

- Let us assume that the two poles of the system p_1 and p_2 are distinct i.e. $p_1 \neq p_2$.
- Using partial fraction expansion, this can be represented as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} = b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2}$$

- This can further be expressed as:

$$Y(s) = b_0 U(s) + \frac{c_1}{s + p_1} U(s) + \frac{c_2}{s + p_2} U(s)$$

State Space representation in Diagonal Canonical Forms (Distinct Roots)

- Let us define the state variables as:

$$X_1(s) = \frac{1}{s+p_1}U(s) \quad or, (s+p_1)X_1(s) = U(s) \implies sX_1(s) = -p_1X_1(s) + U(s)$$
$$X_2(s) = \frac{1}{s+p_2}U(s) \quad or, (s+p_2)X_2(s) = U(s) \implies sX_2(s) = -p_2X_2(s) + U(s)$$
$$Y(s) = c_1X_1(s) + c_2X_2(s) + b_0U(s)$$

- Taking the inverse Laplace transform of these equations give

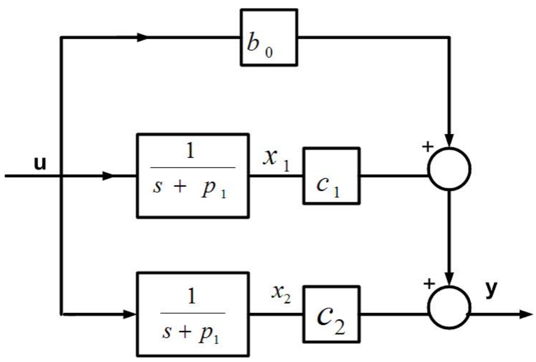
$$\begin{aligned} \dot{x}_1 &= -p_1x_1 + u \\ \dot{x}_2 &= -p_2x_2 + u \\ y &= c_1x_1 + c_2x_2 + b_0u \end{aligned}$$

State Space representation in Diagonal Canonical Forms (Distinct Roots)

- In matrix form, it is expressed as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -p_1 & 0 \\ 0 & -p_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0u$$

- The schematic is shown below



- Consider an n-th order system described by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$
$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \ldots + \frac{c_n}{s+p_n}, \quad p_i \neq p_j, \forall\ i,j = 1,.n$$

- The diagonal canonical model of the system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -p_1 & 0 & 0 & \ldots & 0 \\ 0 & -p_2 & 0 & \ldots & 0 \\ \vdots & \ldots & & \ldots & \vdots \\ 0 & 0 & \ldots & -p_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & -p_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & c_2 & \ldots & c_{n-1} & c_n \end{bmatrix} \mathbf{x} + b_0 u$$