Modern Control

Formulation of state space averaged models for control purposes.

Example using DC-DC Converters

Lecture Overview

Averaged Models

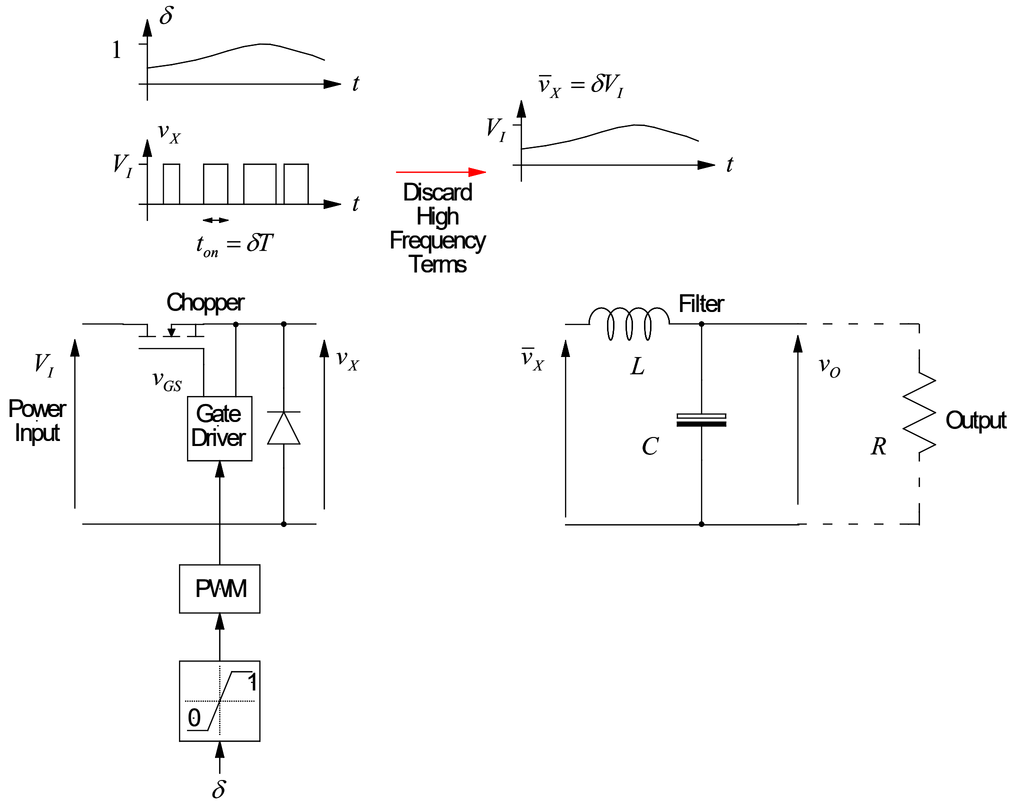
Many applications need to throttle the flow of energy from one source to create a new output. E.g. reducing the rate of fluid flow, but this can create losses.

In electronics we often want to create a different output voltage (either larger or smaller) from what is available at the input. In doing this we want minimum losses, and also smooth control with filtering to limit disturbances.

Linear voltage regulators create large voltage drops across them to create a lower voltage and are therefore lossy. An alternative approach is called a switched converter.

Operation of a Switching Converter.

Most commonly Pulse Width Modulation (PWM) controllers output a duty cycle to a power switch (or set of switches), with the objective of controlling the output voltage (or inductor current and output voltage).

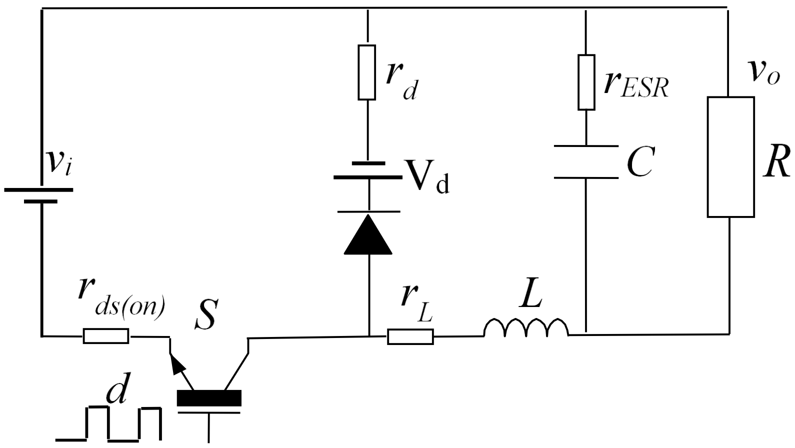


The example shows a buck converter (or chopper). Here a constant DC voltage is used at the input, and ideally we want to control the voltage at the output. This is done by regularly operating a switch at a known frequency f_s (the period of which is T_s). We then turn on and off a switch (sometimes called a chopper) for part of each T_s. The on-time (t_{on}) is related to the duty cycle (d or δ) and creates a train of pulses. The "average value" of this train of pulses can be considered as the input voltage to a filter.

As shown the control is undertaken using duty cycle that has a minimum value of 0 and a maximum value of 1.

The Buck Converter

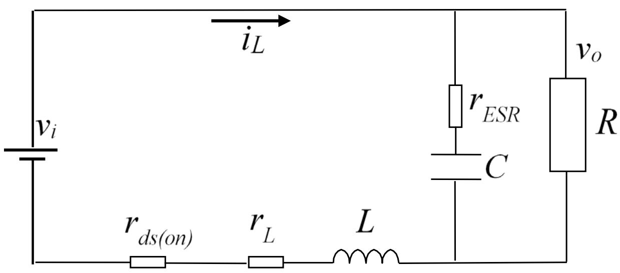
A practical circuit of a buck converter is shown below.



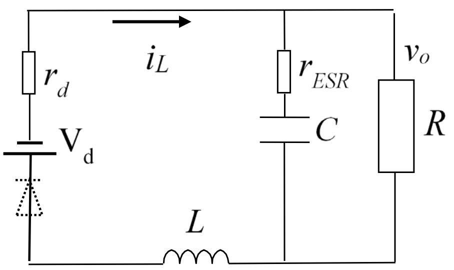
As shown this model includes the practical losses shown as:

- Series resistances in the inductor and capacitor (r_L and r_{ESR}).
- An on-state switch resistance ($r_{ds(on)}$).
- A DC voltage drop (V_d) and loss element (r_d) in the diode when it conducts

As shown it operates in two states based defined in terms of the state of the active switch.



Switch on



Switch off (diode conducting)

Many are useful to keep for control analysis, as they improve the damping of the circuit.

Developing Switching Converter Models

As shown below in Figure 10-19(b), ideally linearized models are desired to enable controller design.

The nature of these models and the way they are constructed using linearization techniques means that they are valid only with small changes about a known operating point. Non-linearites present in the system are completely neglected. Given that the controllers are usually reasonably fast and can control any perturbations, then this is acceptable. For large scale variations SPICE simulations are required to evaluate the full operation of the converter and the controller design.

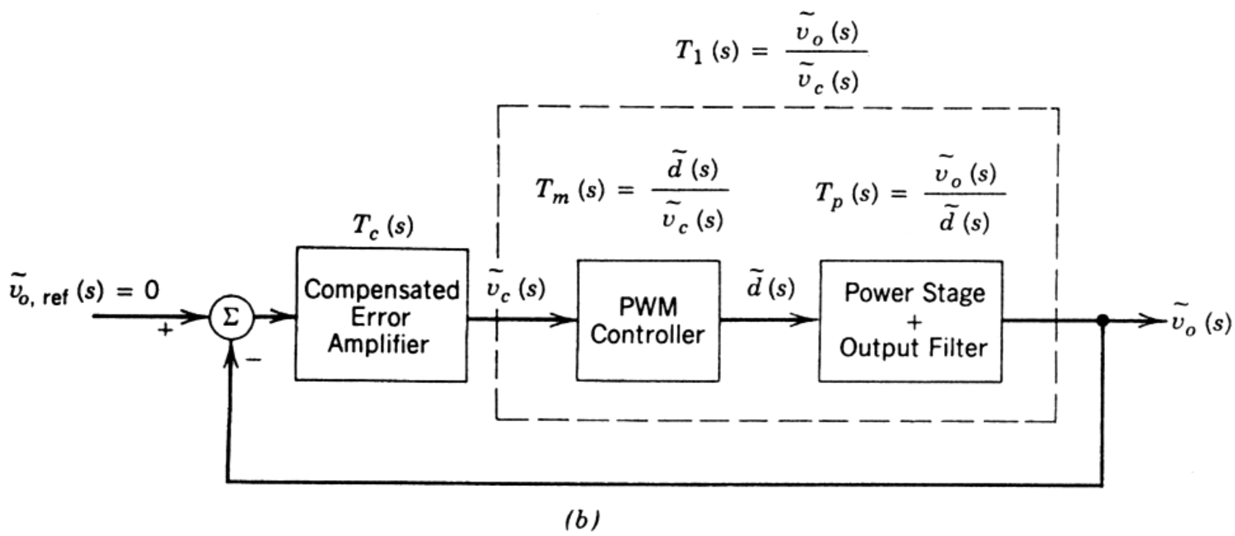


Figure 10-19 Voltage regulation: (*a*) feedback control system; (*b*) linearized feedback control system.

State Space Averaging of DC-DC Converters

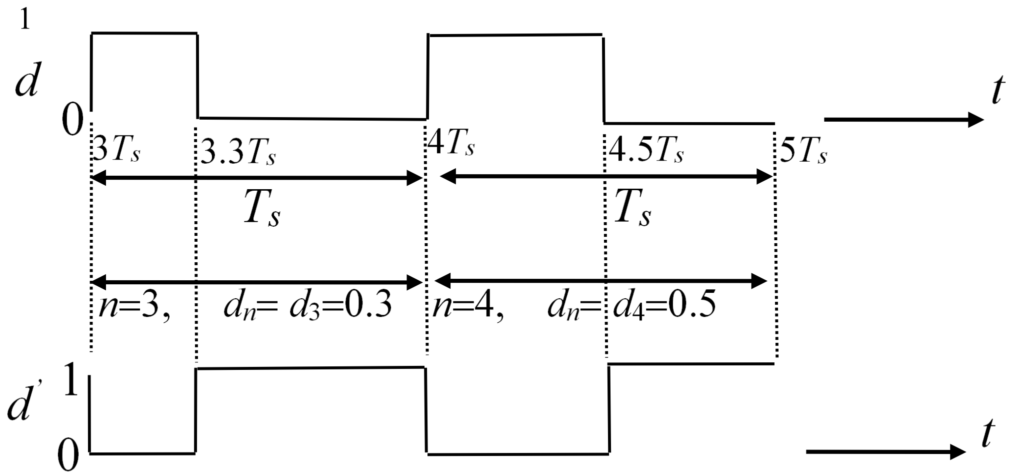
The assumption is that the converter operates under continuous conduction in one of two states, where each switch state relates to the duty cycle.

Mathematically we can write:

$$\dot{\underset{\sim}{x}} = A_1\underset{\sim}{x} + B_1 u \qquad nT_s < t < (n + d_n)T_s$$
$$\dot{\underset{\sim}{x}} = A_2\underset{\sim}{x} + B_2 u \quad (n + d_n)T_s < t < (n + 1)T_s$$

$$n = -1,\ 0,\ 1,\ 2\ \ldots\ \text{etc}$$

A_1 and A_2 are square matrices describing the circuit and B_1 and B_2 are vectors relating the state variables described in vector ($\underset{\sim}{x}$) to the input voltage, and d(t) and d'(t) are defined as follows:

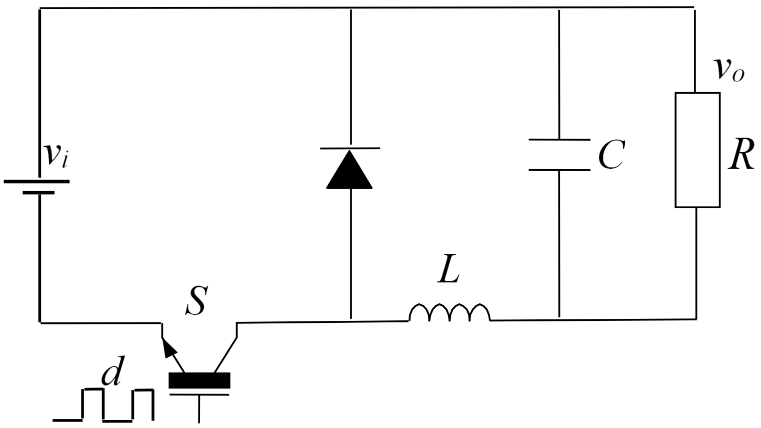


Example: An ideal buck converter

To keep the analysis really simple we begin with an ideal no-loss buck-converter.

Choose state variables as the variables of interest in the circuit. In this case the inductor current and the capacitor voltage.

$$\underset{\sim}{x} = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$



To create an averaged state model of the circuit (to assess its averaged operation) we can consider how it behaves in the two working modes of the circuit.

In this ideal model the assumptions are:

- The switch is ideal when the switch is on and when it is off with no loss.

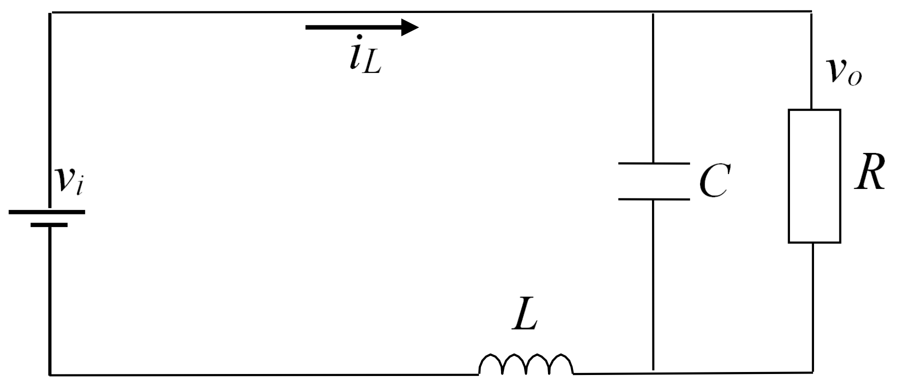
- The diode is also considered ideal:
  When conducting it is like a switch fully on with no loss,
  When not conducting it is considered an open switch

NB: More complicated models are also ok to create but take a little more care to properly define the matricies. But they will follow the exact same process.

1. Buck Converter State Variable Descriptions

Write down a state variable description of the circuit for each state

With ideal switch *S* turned on then the circuit becomes:



We can write the dynamics of this RLC circuit as we did before. Here in this ideal circuit: $v_o = v_c$

$$\frac{v_i - v_c}{L} = \frac{di_L}{dt}$$

$$i_L = C\frac{dv_c}{dt} + \frac{v_c}{R}$$

As before (*page 19 State variable models A.S. notes*)

We write the equations in the following form.

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$
$$y = \mathbf{C}x$$

The input *u* in this ideal circuit is simply the voltage v_i,
The output *y* is v_o

Thus:

$$\dot{\underset{\sim}{x}} = A_1 \underset{\sim}{x} + B_1 u$$

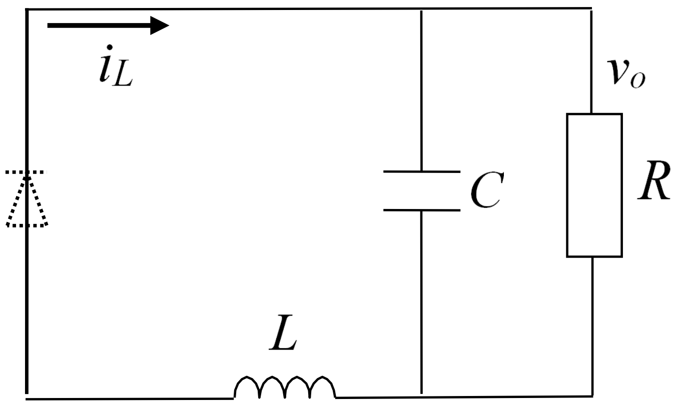
Where

$$\underset{\sim}{x} = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$

Results in:

and as $v_o = v_c$

With S off then:



Following the same process the diode is conducting and is "on".

Thus:

$$-\frac{v_c}{L} = \frac{di_L}{dt} \qquad\qquad i_L = C\frac{dv_c}{dt} + \frac{v_c}{R}$$

Again:

$$\dot{\underset{\sim}{x}} = A_2\underset{\sim}{x} + B_2u$$

where

and again

2. *State Variable Averaging*

To solve the circuit, average the state variable description using the duty cycle over one full period T of operation.

$$d' = 1 - d$$

Averaging over one period *T*:

$$\dot{\underset{\sim}{x}} = (dA_1 + d'A_2)\underset{\sim}{x} + (dB_1 + d'B_2)u$$

$$y = (dC_1 + d'C_2)\underset{\sim}{x} \qquad \ldots(1)$$

3. *Steady State DC and AC Perturbations*

Small ac perturbations are introduced to all variables to evaluate circuit average behaviour and its response to changes in the control variable (duty cycle).

Steady state DC quantities are represented by large letters while small AC pertubations are represented here by ^.

Thus:

$$u = U + \hat{u}, \qquad \underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{\hat{x}}, \qquad y = Y + \hat{y},$$

$$d = D + \hat{d}, \qquad d' = D' - \hat{d}$$

Note that $\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{\hat{x}}}$ since the steady state term $\dot{\underset{\sim}{X}} = 0$

Hence:

$$\dot{\underset{\sim}{x}} = \left((D+\hat{d})A_1 + (D'-\hat{d})A_2\right)(\underset{\sim}{X} + \hat{\underset{\sim}{x}}) + \left((D+\hat{d})B_1 + (D'-\hat{d})B_2\right)(U+\hat{u})$$

$$=$$

$$=$$

Now let:

$$A = DA_1 + D'A_2$$
$$B = DB_1 + D'B_2$$

Resulting in:

$$\dot{\underset{\sim}{x}} =$$

The nonlinear terms represented by $\hat{\underset{\sim}{x}}\hat{d}$ and $\hat{u}\hat{d}$ are both extremely small and can be neglected without affecting the accuracy of the analysis. This is because each ac pertubation is assumed to be small and much less than the steady state value. So the product of two small perturbations is even smaller.

Thus:

$$\dot{\underset{\sim}{x}} = (A\underset{\sim}{X} + BU) + (A\hat{\underset{\sim}{x}} + B\hat{u}) + \left((A_1 - A_2)\underset{\sim}{X} + (B_1 - B_2)U\right)\hat{d} \quad ...(2)$$

The output equation can also be expressed as:

$$Y + \hat{y} = \left(\left(D + \hat{d}\right)C_1 + \left(D' - \hat{d}\right)C_2\right)(\underset{\sim}{X} + \underset{\sim}{\hat{x}})$$

$$=$$

Let: $C = DC_1 + D'C_2$ and again ignore $\underset{\sim}{\hat{x}}\hat{d}$

Results in:

$$Y + \hat{y} = C\underset{\sim}{X} + C\underset{\sim}{\hat{x}} + (C_1 - C_2)\underset{\sim}{X}\hat{d} \qquad \ldots (3)$$

4. The Steady State Transfer Function

The steady state equations can be determined from (2) and (3) by setting all perturbations to zero.

$$0 = (A\underset{\sim}{X} + BU) \qquad \Longrightarrow \qquad \underset{\sim}{X} = -A^{-1}BU \qquad \ldots (4)$$

$$Y = C\underset{\sim}{X} \qquad \ldots (5)$$

Putting (4) into (5) and recognising that *Y* and *U* are scalars then:

$$Y = -CA^{-1}BU$$

Thus: $$\frac{Y}{U} = -CA^{-1}B \qquad \ldots (6)$$

Here A^{-1} is the inverse matrix of *A*.

5. Small Signal Transfer Functions

Here we **IGNORE** the steady state terms and only look at the small signal outputs.

Thus (2) and (3) become:

$$\dot{\underset{\sim}{x}} = (A\hat{\underset{\sim}{x}} + B\hat{u}) + \big((A_1 - A_2)\underset{\sim}{X} + (B_1 - B_2)U\big)\hat{d} \qquad \ldots (7)$$

$$\hat{y} = C\hat{\underset{\sim}{x}} + (C_1 - C_2)\underset{\sim}{X}\hat{d} \qquad \ldots (8)$$

Two common small signal transfer functions can be determined:

- The input to output transfer function (ignore $\hat{d}$)
- The duty cycle to output transfer function (ignore $\hat{u}$)

5.1 ***Small signal input to output transfer function***

Put $\hat{d} = 0$ so that (7) and (8) become:

$$\dot{\underset{\sim}{x}} = (A\underset{\sim}{\hat{x}} + B\hat{u}) \qquad \text{and} \qquad \hat{y} = C\underset{\sim}{\hat{x}}$$

To solve, the above transfer functions need to be transfered into the *s*-domain using the Lapalace transformation.

Recall

$$\dot{\underset{\sim}{\hat{x}}} \rightarrow s\,\underset{\sim}{\hat{x}}(s) - \underset{\sim}{\hat{x}}(0)$$

Assuming all initial conditions are zero, then (following *AS module 3 notes pg 3*):

$$s\underset{\sim}{\hat{x}}(s) = A\underset{\sim}{\hat{x}}(s) + B\hat{u}(s)$$

$$(sI - A)\underset{\sim}{\hat{x}}(s) = B\hat{u}(s)$$

$\Rightarrow$

$$\frac{\underset{\sim}{\hat{x}}(s)}{\hat{u}(s)} = (sI - A)^{-1}B \qquad \ldots (9)$$

Here *I* is the identity matrix

and

$$\hat{y}(s) = C\underset{\sim}{\hat{x}}(s) \qquad \ldots (10)$$

Combining (9) and (10) results in:

$$\frac{\hat{y}(s)}{\hat{u}(s)} = C(sI - A)^{-1}B \qquad \ldots (11)$$

5.2 *Small signal duty cycle to output transfer function*

Put $\hat{u} = 0$ so that (7) and (8) become:

$$\dot{\underset{\sim}{x}} = A\hat{\underset{\sim}{x}} + \left((A_1 - A_2)\underset{\sim}{X} + (B_1 - B_2)U\right)\hat{d}$$

$$\hat{y} = C\hat{\underset{\sim}{x}} + (C_1 - C_2)\underset{\sim}{X}\hat{d}$$

In the *s*-domain, again assuming initial conditions are zero, these become:

… (12)

… (13)

Combining (12) and (13) results in:

$$\hat{y}(s) = C(sI - A)^{-1}\left((A_1 - A_2)\underset{\sim}{X} + (B_1 - B_2)U\right)\hat{d}(s) + (C_1 - C_2)\underset{\sim}{X}\hat{d}(s)$$

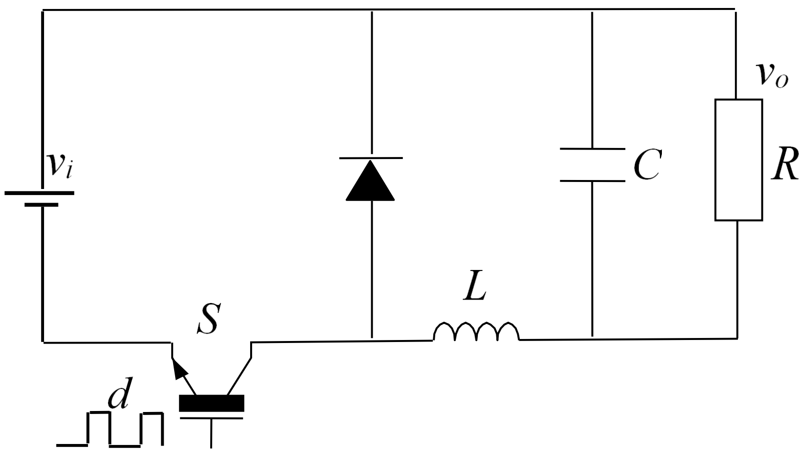
Thus:

$$\frac{\hat{y}(s)}{\hat{d}(s)} = C(sI - A)^{-1}\left((A_1 - A_2)\underset{\sim}{X} + (B_1 - B_2)U\right) + (C_1 - C_2)\underset{\sim}{X} \quad \ldots (14)$$

6. *Ideal Buck Converter Transfer Functions*

Recall that here: $u = v_i$, and $y = v_o$

6.1 *Buck converter steady state transfer function*



Determining the inverse matrix (using a 3×3 example):

$$A^{-1} = \frac{Adj(A)}{|A|} = \frac{(cofactorA)^T}{|A|}$$

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the cofactor matrix of A is:

$$cofactorA = \begin{bmatrix} m_{11} & -m_{12} & m_{13} \\ -m_{21} & m_{22} & -m_{23} \\ m_{31} & -m_{32} & m_{33} \end{bmatrix}$$

Here m_{11} is the minor element of a_{11} given by:

$$m_{11} = a_{22} \cdot a_{33} - a_{23} \cdot a_{33}$$

m_{12} is the minor element of a_{12} given by:

$$m_{12} = a_{21} \cdot a_{33} - a_{23} \cdot a_{31}$$

m_{13} is the minor element of a_{13} given by:

$$m_{13} = a_{21} \cdot a_{32} - a_{22} \cdot a_{31}$$

Thus the Adjugate of the matrix is therefore:

$$AdjA = \begin{bmatrix} m_{11} & -m_{21} & m_{31} \\ -m_{12} & m_{22} & -m_{32} \\ m_{13} & -m_{23} & m_{33} \end{bmatrix}$$

The determinant of matrix *A* is a ***scalar*** given by:

$$|A| = a_{11} \cdot m_{11} - a_{12} \cdot m_{12} + a_{13} \cdot m_{13}$$

Generally the above process needs to be followed carefully, given we are dealing with expressions.

However for a 2×2 matrix (as in this example) the inverse is reasonably easy as follows (see also *AS notes module 3 pg 9*):

$$A^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{(a_{11} \cdot a_{22} - a_{12} \cdot a_{21})},$$

Thus for the ideal buck converter:

Thus using (6) results in:

This is a well known result and suggests we have done something right !

6.2 *The buck converter voltage transfer function*

Applying this to the buck converter then:

$$(sI - A) = \begin{bmatrix} s & \frac{1}{L} \\ -\frac{1}{C} & s + \frac{1}{CR} \end{bmatrix}$$

and

Thus:

6.3 *Buck converter output voltage to duty cycle TF*

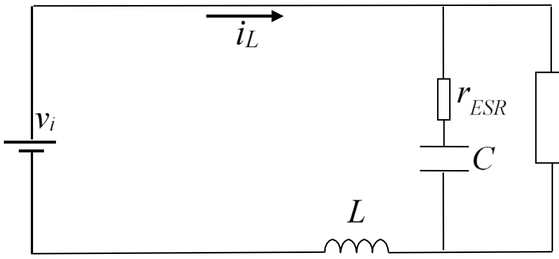
From above then:

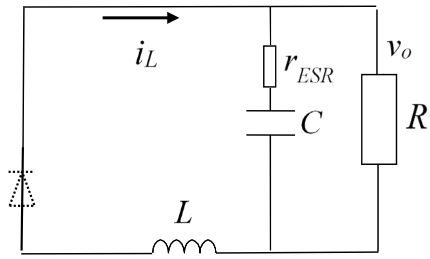
and

Combining the above results for the buck converter into (14) we get:

7. *Impact of Loss Components*

How does including r_{ESR} impact the C matrix?





Switch on **Switch off**

With this included then in both the on-state and off-state we can write down equations of the currents and voltages in the output loop:

$$v_o = i_c r_{ESR} + v_c$$
$$i_L = i_c + i_o \quad \text{where } i_C = C\frac{dv_c}{dt} \quad \text{and } i_o = \frac{v_o}{R}$$

As before we want $y = v_o = C\underset{\sim}{x}$
The C matrix sets the output in terms of the states i_L and v_c

Thus combining the above two expressions (replacing i_C)

$$v_o = (i_L - i_o)r_{ESR} + v_c$$

And replacing i_o:

$$v_o = \left(i_L - \frac{v_o}{R}\right)r_{ESR} + v_c$$

Rearranging $\quad v_o\left(1 + \frac{r_{ESR}}{R}\right) = i_L r_{ESR} + v_c$

$$v_o = \left(\frac{R}{R + r_{ESR}}\right)(x_1 r_{ESR} + x_2)$$

Let $a = \left(\frac{R}{R+r_{ESR}}\right)$

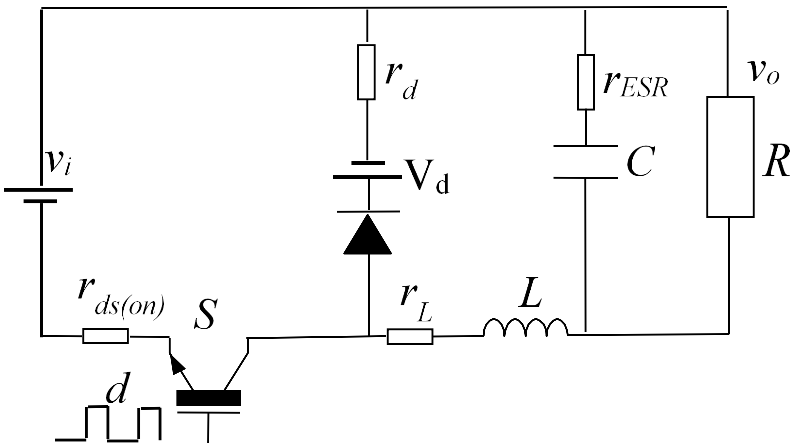
Then $\quad v_o = [a r_{ESR} \quad a]\,\underset{\sim}{x}$

8. *Tutorial Problems*

8.1 *Adding loss elements*

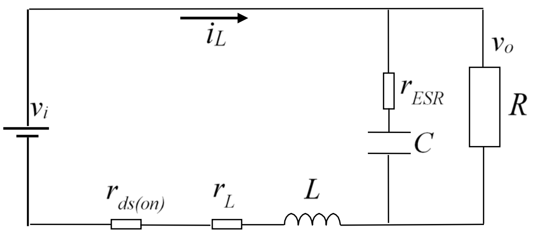
Evaluate the Buck converter matricies with practical loss elements included in the circuit.

As stated earlier, the actual circuit looks like:

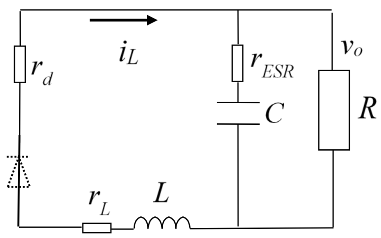


Include the resitive loss terms: $r_{ds(on)}$, r_d , r_L , and r_{ESR} .

Ignoring the Vd drop



Switch on



Switch off (diode conducting)

Derive the new matricies.

Hint, first find a replacement for i_c in terms of the states

Using: $\quad a = (\frac{R}{R+r_{ESR}})$

$$i_c = i_L - \frac{v_o}{R} = i_L - \frac{i_c r_{ESR} + v_c}{R} \quad \text{gives:} \quad i_c = a i_L - \frac{a}{R} v_c$$

First step to the answer (show the following):

Then write down the equations to find the derivatives of the states

- inductor voltage to find $\frac{di_L}{dt}$
- capacitor current to find $\frac{dv_C}{dt}$

show:

$$A_1 = \begin{bmatrix} \frac{-(r_{ds(on)}+r_L+ar_{ESR})}{L} & \frac{-a}{L} \\ \frac{a}{C} & \frac{-a}{RC} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \frac{-(r_d+r_L+ar_{ESR})}{L} & \frac{-a}{L} \\ \frac{a}{C} & \frac{-a}{RC} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And from earlier:

$$C_1 = C_2 = [ar_{ESR} \quad a]$$

8.2 *A more realistic transfer function*

Simplify the above analysis by assuming $r_{ds(on)}$ and r_d are both zero.

Prove. the following final steady state and transient transfer function to show that it has a zero in the equation (relating to the ESR resitance of the output capacitor).

With just r_{ESR} and r_L losses included:

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i(r_{ESR}Cs + 1)}{\frac{LC}{a}s^2 + \left(\frac{L}{R} + r_{ESR}C + \frac{r_L}{a}C\right)s + \left(1 + \frac{r_L}{R}\right)}$$

Note it is helpful to realise that:

$$a\left(1 + \frac{r_{ESR}}{R}\right) = 1$$

As can be seen, the final DC gain $s \to 0$ is $\frac{V_i}{\left(1+\frac{r_L}{R}\right)}$

Thius for a duty cycle step of 0.05 then $\hat{d}(s) = \frac{0.05}{s}$

And the steady state will approximate to an open-loop step change of $\frac{0.05V_i}{\left(1+\frac{r_L}{R}\right)}$

Note: Recall this is only valid for small adjustments in *duty cycle* and ignores that in practice $0 \leqq d \leqq 1$

9. *Time Domain Performance using Matlab*

Matlab code can be created to evaluate the time domain performance of this using the ode45 function.
See similar examples *AS notes module 3 pp.45-50*

Lets assume we have the following circuit parameters:

Inductor $L = 100\ \mu H$, Capacitor $C = 100\mu F$
Load $R = 1\ \Omega$

In addition the following typical circuit losses exist:
$r_{ds(on)} = 0.2\ \Omega$
$r_L = 0.1\ \Omega$
$r_d = 0.02\ \Omega$
$r_{ESR} = 0.01\ \Omega$
Here the diode voltage drop is also included
(nb: it changes the B*u matrix in the off state)
$V_d = 0.8V$

All initial conditions are zero.
The input voltage and dutycycle at time zero are:

Vin = 20V
D = 0.25

The switching frequency defines f = 1/Ts (note that while it is important for looking at the time span, it is critical for spice analysis validations).

f = 200kHz

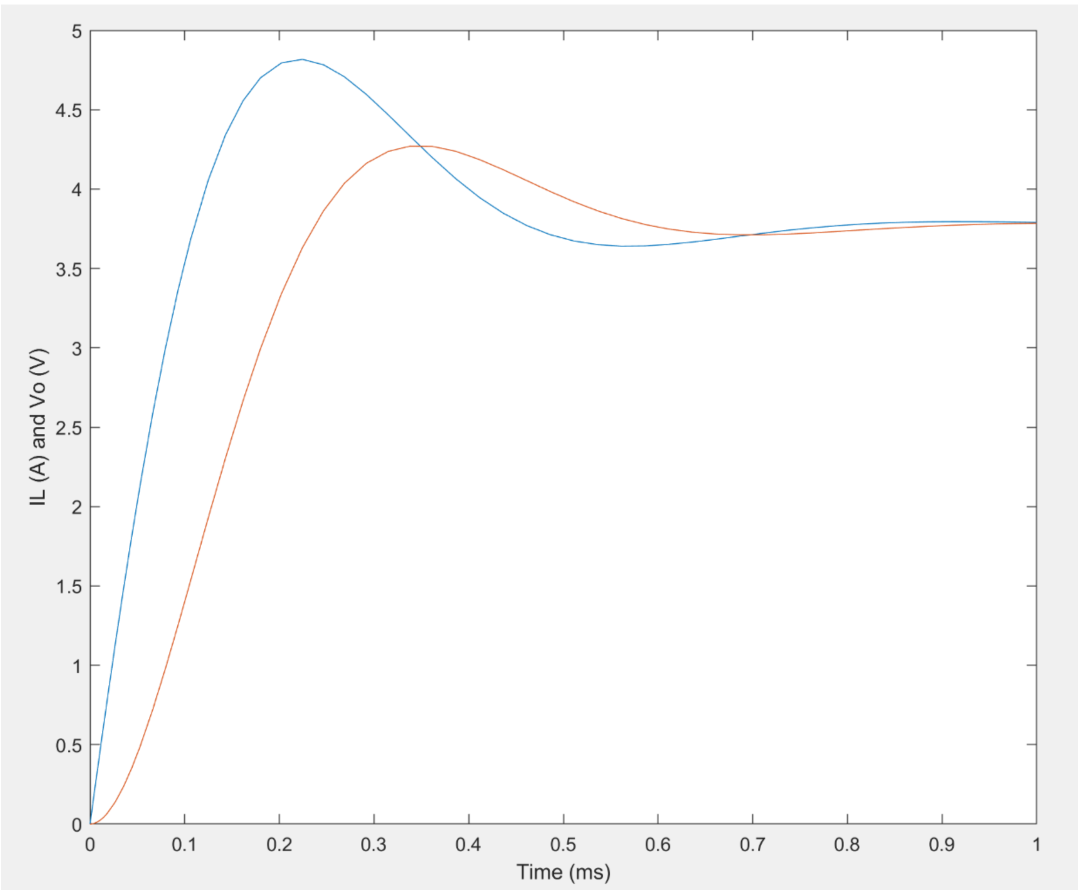
First create a function buck converter and save as buck_converter.m

```
function dx=buck_converter(t,x);
% Circuit parameters
Vin = 20;
Rds_on = 0.2;
Vd = 0.8;
Rd = 0.02;
R_L = 0.1;
R_ESR = 0.01;
L = 100e-6;
C = 100e-6;
R = 1;
f = 200e3;
D1 = 0.25;
D2 = 1-D1;
% Resistor divider constant
cd = R/(R + R_ESR);
% Average matrices
A_avg = [((-D1*Rds_on) - D2*Rd - R_L - cd*R_ESR)/L -
cd/L; cd/C -cd/(R*C)];
B_avg = [(D1*Vin-D2*Vd)/L; 0];
% Derivates of states
x1dot = A_avg(1,1)*x(1)+A_avg(1,2)*x(2)+B_avg(1,1);
x2dot = A_avg(2,1)*x(1)+A_avg(2,2)*x(2);
dx = [x1dot;x2dot];
```

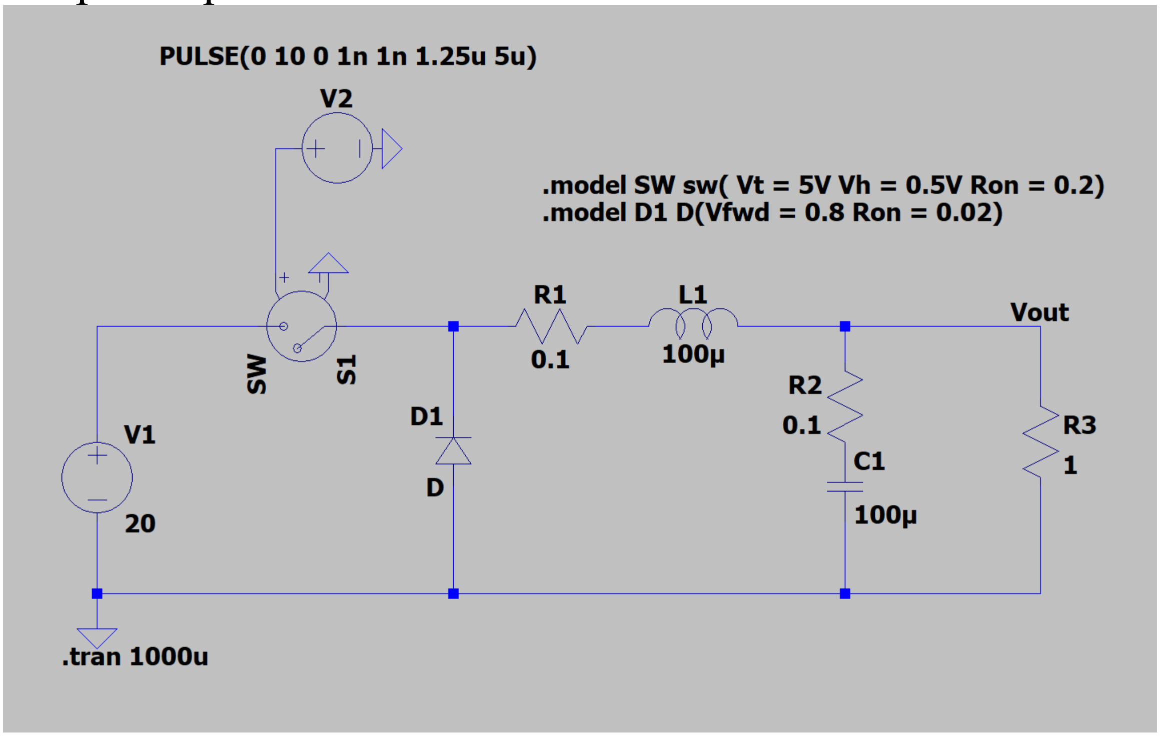
Now we can call the buck converter using the following code. It is easier to write all this in another .m file which is selected and run.

```
clear
clc

t_step = 250e-6/100;
t_start = 0.0;
t_final = 250e-6;

% Initial conditions
x0=[0.0 0.0];
%x1(1)=x0(1,1); % Save the state variables
%x2(1)=x0(1,2);
[t,x]=ode45(@buck_converter,[t_start t_final],x0);
plot(t,x)
xlim([0 250e-6])
ylim([0 5])
```

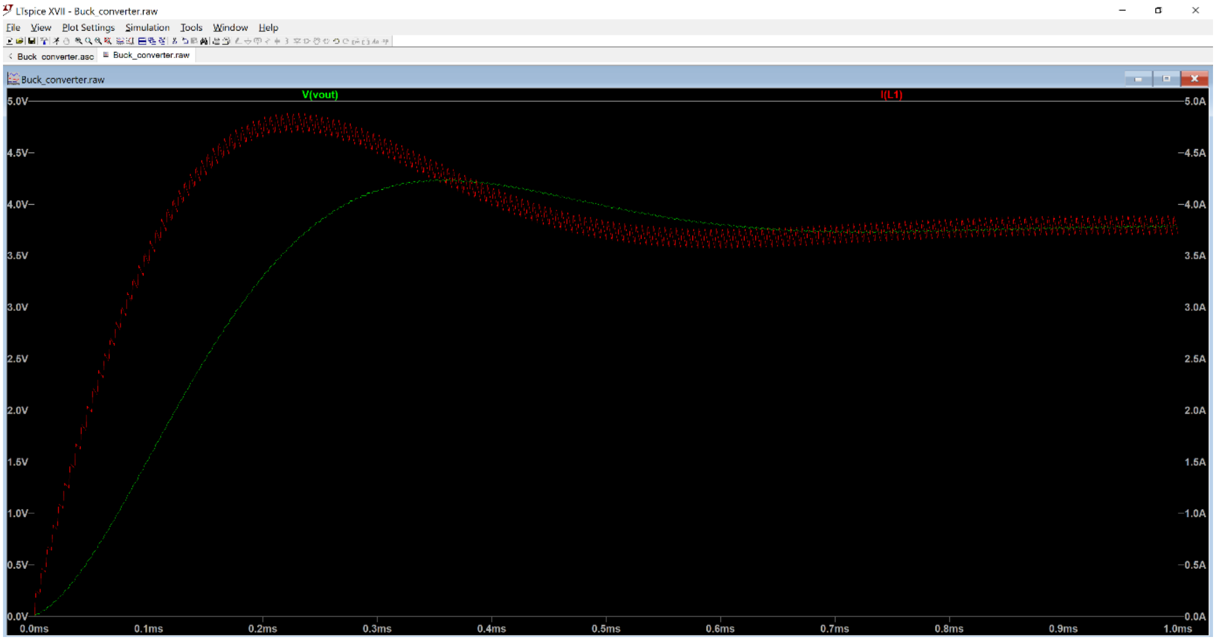
The output is as follows:



(Purely out of interest)
A Spice equivalent of this circuit in LTSPICE is as follows.

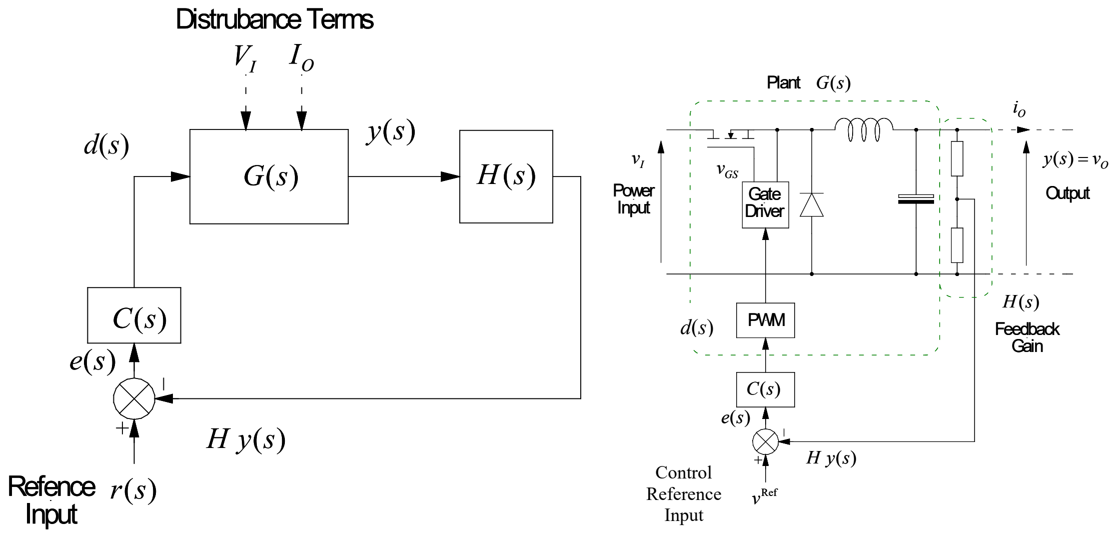


As shown the output is the same, only the switching ripple is missing.



10. *Analysis and Controller Design using State Space*

The presence of inductors and capacitors within this switched mode power supply will mean that when duty cycle d is changed there will be a transient response before the new steady state output voltage is achieved.



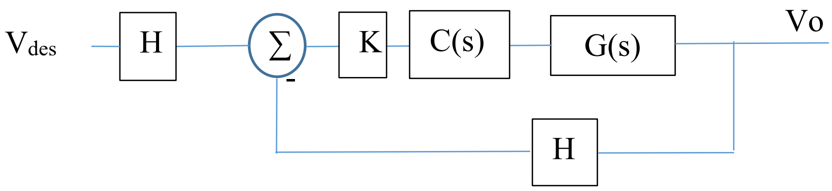
$H(s)$ is the feedback gain - normally the output is sensed via a simple potential divider.

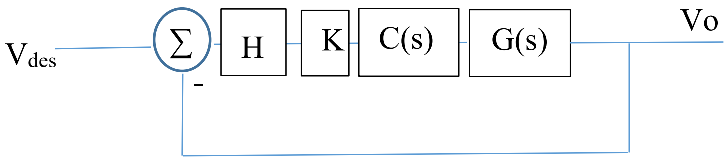
A reference is used to generate the desired output voltage.
A controller $C(s)$ (comprising either a gain or more complex compensator) may be used to adjust the response and postion of the closed loop response (poles and zeros).

The transient response of $G(s) = \frac{\hat{v}_O(s)}{\hat{d}(s)}$ can be determined using state space

Note that in these systems typically both the feedback voltage and the reference are scaled by the same gain H(s)=H.

Thus the closed loop system appears as:

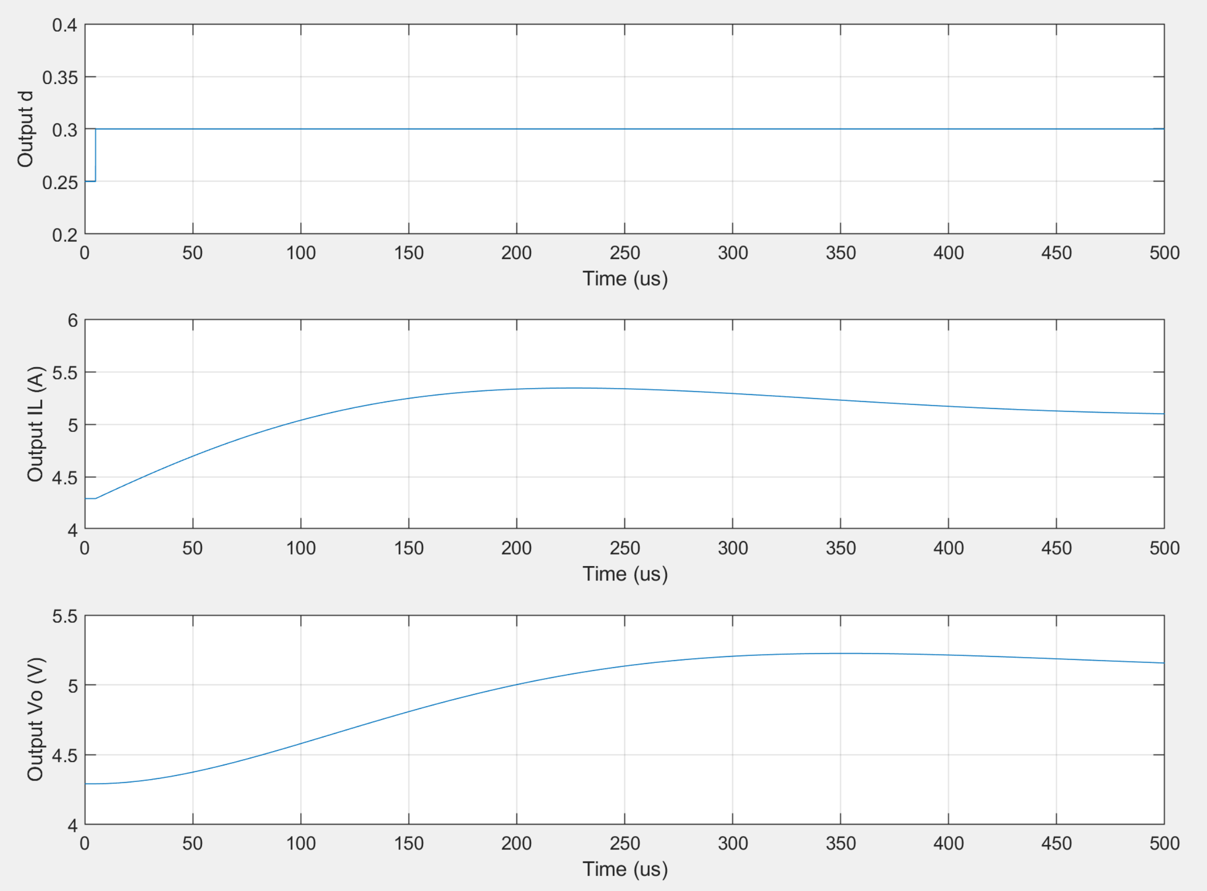




Consider a Buck converter with the circuit parameters derived earlier (the diode voltage drop is considered to be ideal V_d =0) operating at steady state with D = 0.25

```
% State space average model of a Buck converter
% d(t)= duty cycle d; x = [iL; Vc]; y=[Vo]
% circuit conditions
D1 = 0.25;
D2 = 1-D1;
Vin = 20;
% circuit losses elements, Vd is ideal so not included
Rds_on = 0.2;
Rd = 0.02;
R_L = 0.1;
R_ESR = 0.01;
% Circuit parameters and load
L = 100e-6;
C = 100e-6;
RLOAD = 1;
%f = 200e3; this helps determine T
% Resistor divider constant
a = RLOAD/(RLOAD + R_ESR);
A1 = [ (-Rds_on-R_L-a*R_ESR)/L -a/L; a/C -a/(RLOAD*C)] ;
A2 = [(-Rd-R_L-a*R_ESR)/L -a/L; a/C -a/(RLOAD*C) ];
B1 = [ 1/L; 0 ];
B2 = [ 0; 0 ];
C1 = [ a*R_ESR,a ];
C2 = [ a*R_ESR,a ];

% Create Steady State Model to get the operating point
% averaged Matricies

A = D1*A1 + D2*A2
B = D1*B1 + D2*B2
C = D1*C1 + D2*C2
```

```
U=Vin
X=-inv(A)*B*U
Y=C*X

% Ideal ANS Should be Y =D1*Vin but will be less with losses

% Small-signal model
% Average matrices
A_ss = A1 - A2 ;
B_ss = B1 - B2 ;
C_ss = C1-C2 ;
% determine x_dot and y matricies
E = A_ss*X + B_ss*U;
F = C_ss*X;
% creating xdot = Ax+ [(A1-A2)X+(B1-B2)Vin]dstep
% and    vo = y = Cx + [(C1-C2)X]dstep
% as xdot = Ax + E dstep
%         y = Cx + F dstep
buck_ss = ss(A,E,C,F) ;
% Unit step response 100 steps from 0 to 500us
T = 0:5e-6:500e-6;
[y,T,x] = step(buck_ss,T);
% scale step reponse as d/s where d=0.05 and add operating point
iL = 0.05*x(:,1)+X(1)*ones(length(T),1);
y = 0.05*y+Y*ones(length(T),1);
% Add time prior to step
T = [0; T+5e-6];
iL = [X(1); iL];
y = [Y; y];
% Plot results to same scale as Spice simulation
subplot(111);
figure(1)

% fixed plot of a duty cycle in us
subplot(3,1,1)
plot([0, 5, 5.01, 500], [0.25, 0.25, 0.3, 0.3])
axis([0, 500, 0.2, 0.4])
grid
xlabel('Time (us)')
ylabel('Output d')

% plot of iL with T in us
subplot(3,1,2)
plot(T*1e6,iL)
axis([0, 500, 4, 6])
grid
xlabel('Time (us)')
ylabel('Output IL (A)')

% plot of Vout
subplot(3,1,3)
plot(T*1e6,y)
axis([0, 500, 4, 5.5])
grid
xlabel('Time (us)')
ylabel('Output Vo (V)')
```


Output d
Time (us)
Output IL (A)
Time (us)
Output Vo (V)
Time (us)

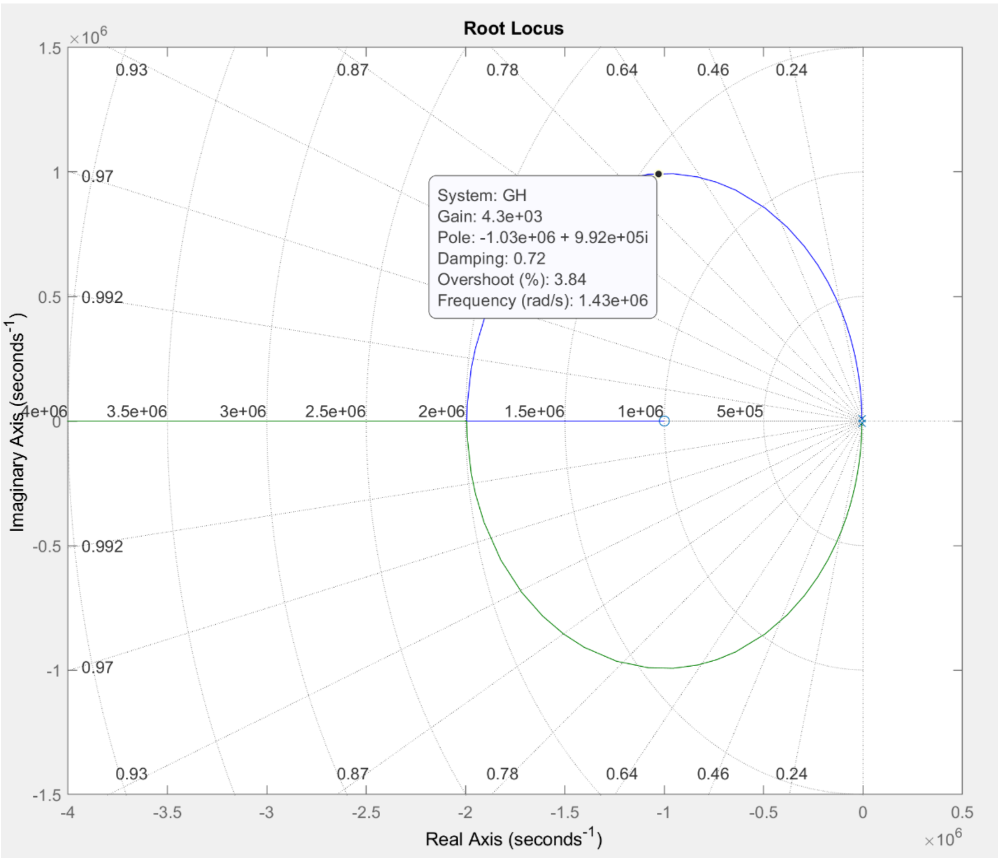
The time-domain transfer function (*AS notes module 3 pg 14*)

```
% Find the system transfer function Vo(s)/d(s) from the small
% signal matricies
% xdot = Ax + E dstep
%    y = Cx + F dstep
% determine the plant G(s)
[num den]=ss2tf(A,E,C,F);
sys=tf(num,den)
% The reference and feedback gain H
H=0.25
% Loop gain G(s)H
GH = sys*H
figure(2)
rlocus(GH)
sgrid
% K = 4300 gives a damping of 0.72
```

produces:

sys =

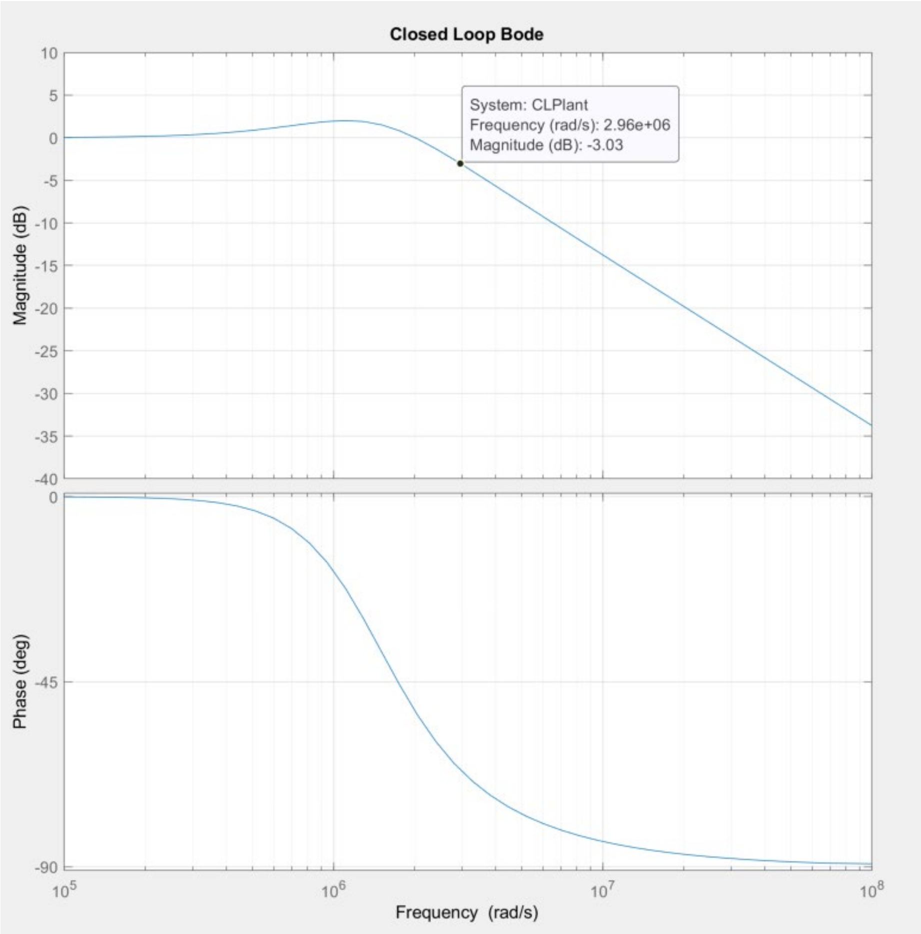
$$\frac{1904\ s + 1.904e09}{s^2 + 11650\ s + 1.153e08}$$



Poles are chosen here at 1.43x10^6 or around 200kHz

The Bode response of the closed loop system

```
K=4300;
CLPlant = feedback(K*GH,1);

figure(3)
bode(CLPlant)
grid
title('Closed Loop Bode')
```



High gain at high frequency is a problem in switched converters

- It can result in duty cycle demands that are impossible.
- The bandwidth extends to 470kHz which is impractical given the converter switches at 200kHz. Thus the large loop gain means switching frequency voltage components will appear in the feedback which will pass to the pulse-width modulator and impact operation (not seen in matlab) with subcycle oscillations.

A compensator can be added to solve this problem and help lower the gain and bandwidth

11. *Controller Design using State Space Matricies*

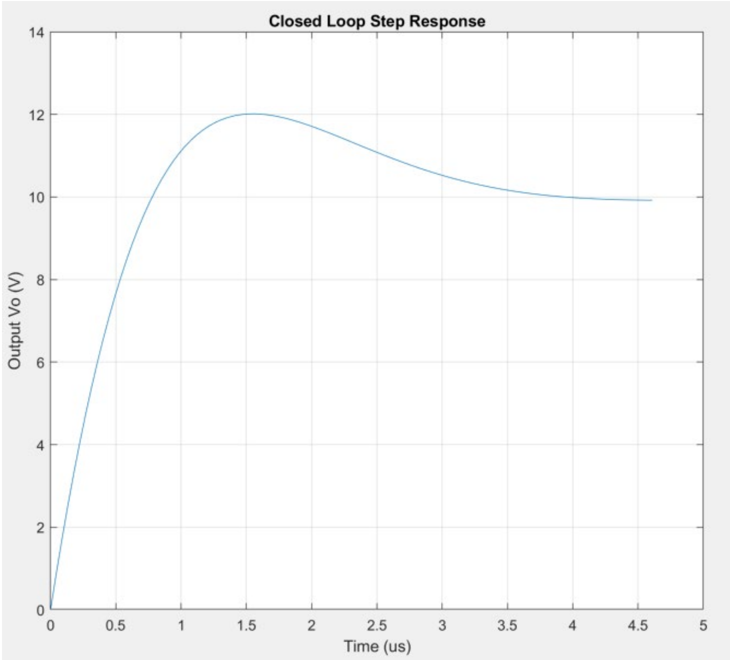
11.1 *Open loop analysis*

This code evlautes control using the state space matricies directly. It plots the openloop HG(s) rootl locus and bode plots the rootlocus and the closed loop poles, bode plot and step response.

```
% OPEN LOOP EVALUATION
% NOTE - THIS USES THE MORE PRACTICAL STATE SPACE TF OF THE
% BUCK AS THE LOSSES SHOW THE NEEDED REAL DAMPING
% openloop evaluation of system operating at D=0.25
% H samples voltage and brings it back as 1/4 into the
% regulator (assumes that Vref is impacted similarly)

H=0.25;
Hbuck_ss=H*buck_ss;
figure(4)
rlocus(Hbuck_ss);
sgrid ;

figure(5)
bode(Hbuck_ss);
grid ;
```

11.2 *Closed loop design*

```
% A FIRST SIMPLE CLOSED LOOP SYSTEM
% Using the Open loop system with damping = .72 and k=4300
% closedloop evaluation of HG(s) system operating at D=0.25

k=4300;
khbuck_closed = feedback(k*Hbuck_ss,1);
figure (6)
pzmap(khbuck_closed);
title('Closed Loop Poles and Zeros')

figure (7)
bode(khbuck_closed);
grid ;
title('Closed Loop Bode')
% Set the reference to 10V
Vref=10*H ;
figure(8)
[y,T] =step(Vref*khbuck_closed);
plot(T,y);
grid ;
xlabel('Time (us)')
ylabel('Output Vo (V)')
title('Closed Loop Step Response')
```



In practice this step response is not really practical.

11.3 *Compensator design*

Using a compensator we can reshape the root locus so that a damping factor around 0.7 can be chosen at a much lower gain resulting also in narrower bandwidth in the closed loop system.

A lead compensator is an excellent choice.
This places a zero closer to the orign and a pole further away.
The zero pulls the root locus away from the imaginary axis improving both the response at lower gain, and stability.

Overall as the poles and zeros added are the samle the overall low and high frequency behaviour remains the same.
The zero and pole should not be more than one decade apart.

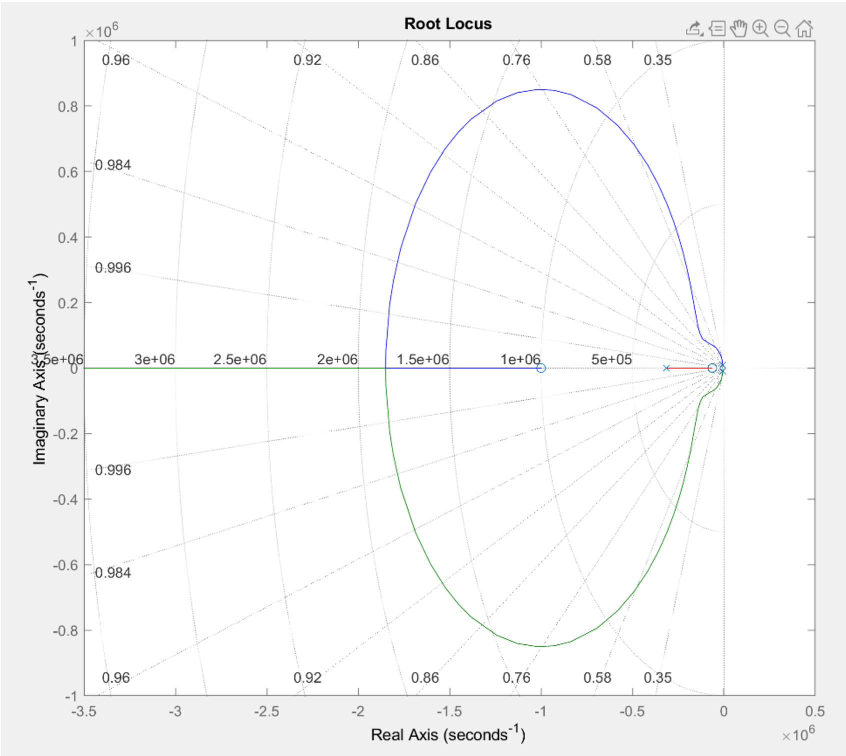
$$C(s) = \frac{1 + \tau_{Lead} s}{1 + \tau_{Lag} s}$$

Looking at the original root locus there is already a zero at 10^6 (160 kHz).

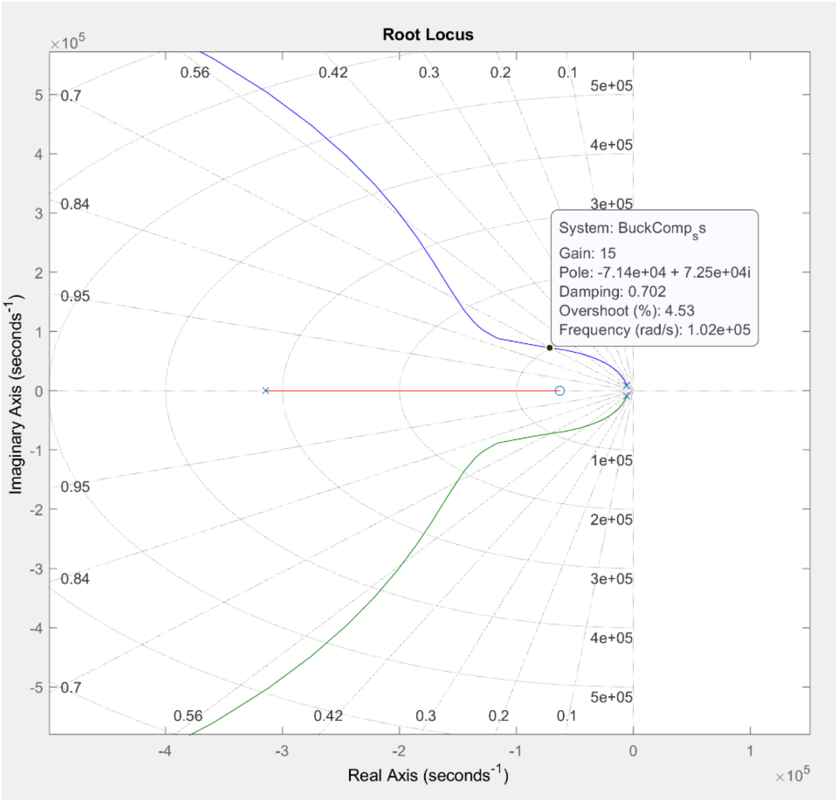
Place the compensator zero much closer.
e.g. Zero at 10kHz and the pole at 50kHz

```
% A lead controller is added in the open loop system.
% (a lead requires its zero to be closer to the origin than
% its pole,and no more than a decade between zero and pole
% often half a decade is good)
% The lead zero is placed a lot closer than the present
% CL zero to pull the RL away from the imag axis and allow
% good damping at lower frequency - narrowing the operating
% bandwidth so that we are not so responsive to switching noise
% in the feedback loop
% Try placing a pole at 50kHz and the the zero at 10kHz
Tlag=1/(2*pi*50e3)
Tlead=1/(2*pi*10e3)
Comp=tf([Tlead, 1],[Tlag,1]);
BuckComp_ss=Comp*H*buck_ss;
figure(9)
rlocus(BuckComp_ss)
sgrid
```

The new root locus is shown below



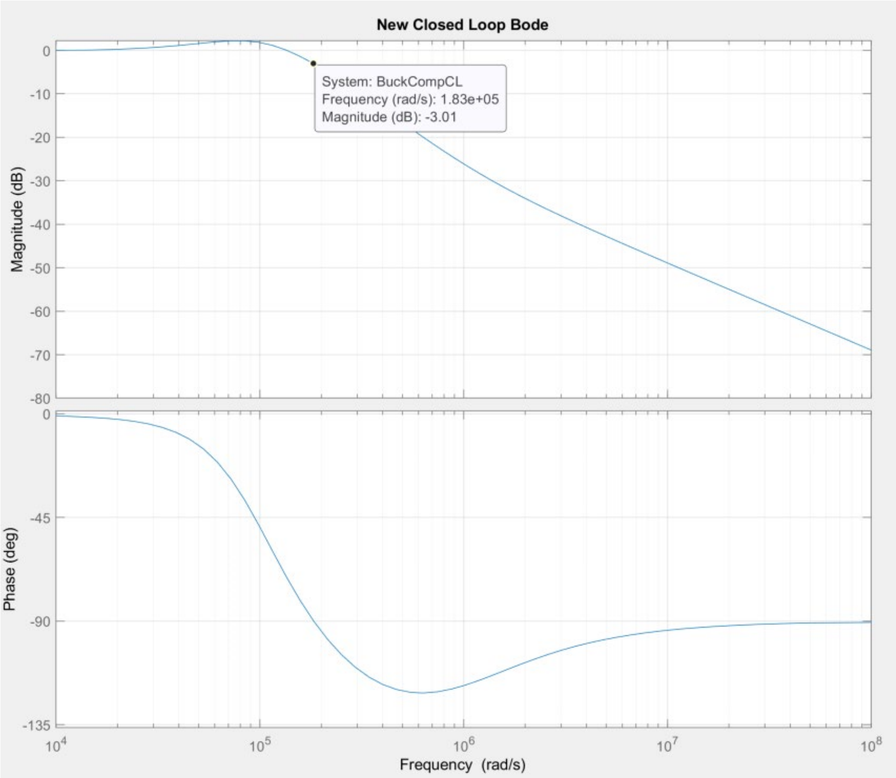
Zooming in k=15 gives a damping of 0.7 with poles selected at 16kHz



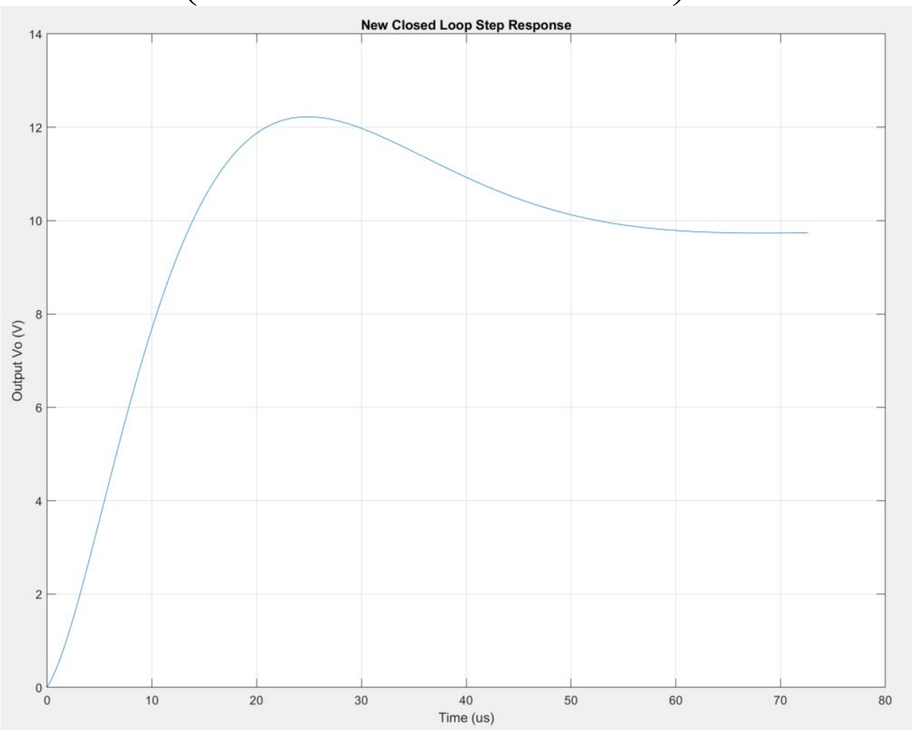
The new closed loop response can now be determined

```
% A NEW CLOSED LOOP SYSTEM
% k=15 gives damping of 0.707 from the above new open loop RL;
KC = 15
BuckCompCL=feedback(KC*BuckComp_ss,1)
figure(10)
pzmap(BuckCompCL);
title('New Closed Loop Poles and Zeros')
figure(11)
bode(BuckCompCL)
grid
title('New Closed Loop Bode')
% Set the reference to 10V
Vref=10;
figure(12)
[y,T] =step(Vref*BuckCompCL);
plot(T*1e6,y);
grid ;
xlabel('Time (us)')
ylabel('Output Vo (V)')
title('New Closed Loop Step Response')
```

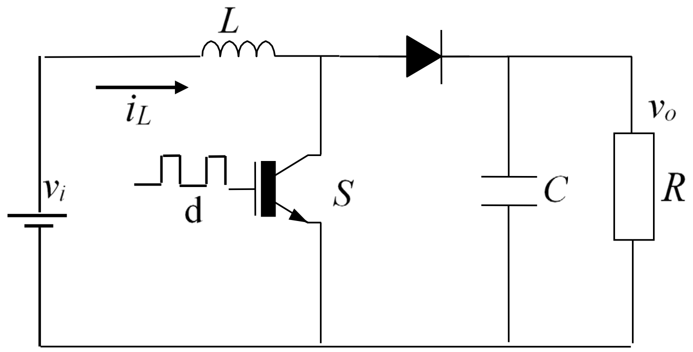
The new closed loop bode response bandwidth is 29kHz which is almost a decade below the switching frequency of 200kHz



The new step response is excellent but >10 times slower and would still have to be checked to ensure duty cycle demands are not saturated (can use simulink for that):



Example: An Ideal Boost Converter:



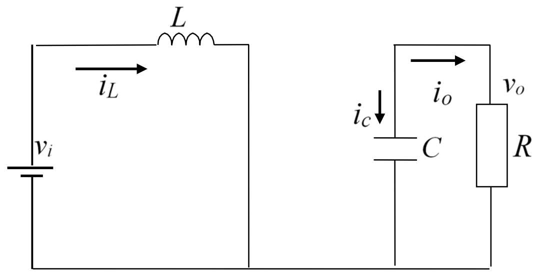
Here the output is larger than the input voltage, because the average voltage formed across the inductor adds to the input to create the output voltage

Analysising the ideal circuit using state space requires the equations to be written in terms of the state variables.

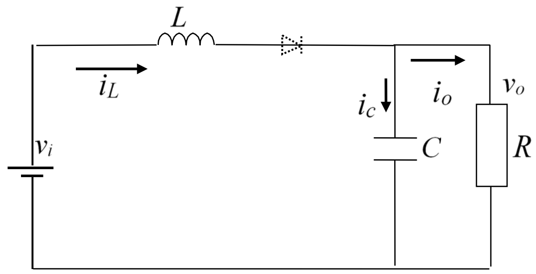
Again these are:

$$\underset{\sim}{x} = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

Again analyse in terms of each circuit operational state.



Switch On



Switch Off

For *S* on then:

$$v_i = L\frac{di_L}{dt} \Rightarrow \qquad \frac{di_L}{dt} = \frac{v_i}{L}$$

$$i_c + i_o = 0 \Rightarrow C\frac{dv_c}{dt} + \frac{v_c}{R} = 0 \qquad \Rightarrow \frac{dv_c}{dt} = -\frac{v_c}{RC}$$

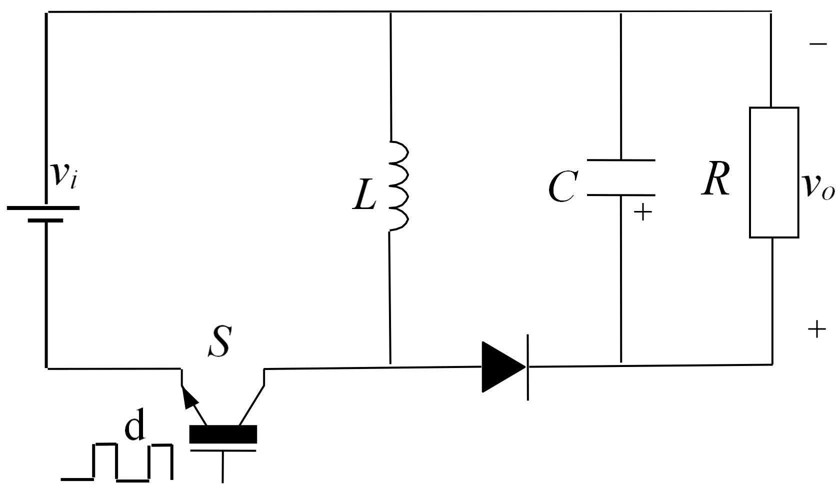
For *S* off then:

$$v_i - v_o = L\frac{di_L}{dt} \Rightarrow \qquad \frac{di_L}{dt} = \frac{v_i - v_c}{L}$$

$$i_L = i_c + i_o \Rightarrow \qquad \frac{dv_c}{dt} = \frac{i_L}{C} - \frac{v_c}{RC}$$

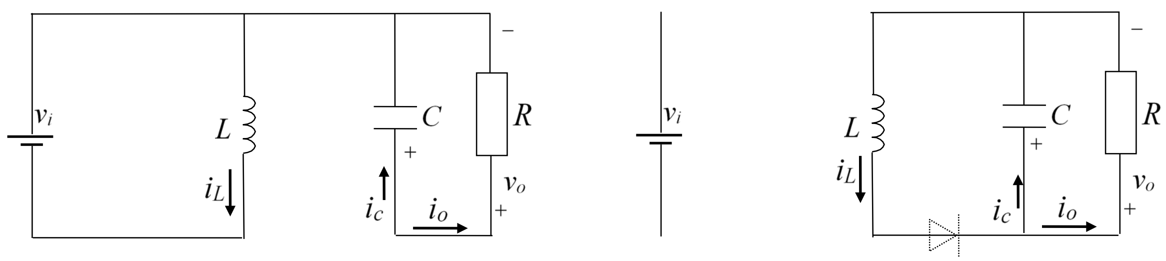
From here the same process can be followed as for the Buck Converter

Example: An Ideal Buck-Boost Converter



Again:

$$\underset{\sim}{x} = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$$



Switch On ***Switch Off***

Analysing in terms of each circuit operational state.

For S on then:

$$v_i = L\frac{di_L}{dt} \qquad \Rightarrow \qquad \frac{di_L}{dt} = \frac{v_i}{L}$$

$$i_c + i_o = 0 \qquad \Rightarrow \quad C\frac{dv_c}{dt} + \frac{v_c}{R} \qquad \Rightarrow \qquad \frac{dv_c}{dt} = -\frac{v_c}{RC}$$

For S off then:

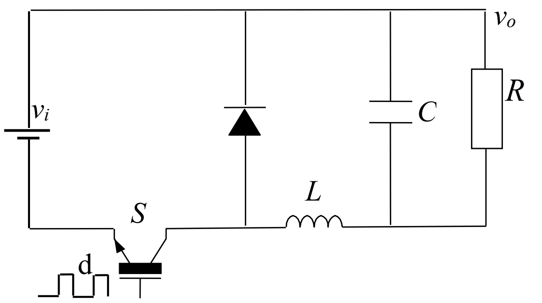
$$-v_c = L\frac{di_L}{dt} \qquad \Rightarrow \qquad \frac{di_L}{dt} = -\frac{v_o}{L}$$

$$i_L = i_c + i_o \qquad \Rightarrow \qquad \frac{dv_c}{dt} = \frac{i_L}{C} - \frac{v_c}{RC}$$

Summary of DC-DC Controller Transfer Functions:

Each converter has an inductor, a capacitor, a diode and a controllable switch, arranged differently. Normally the switch is controlled on and off to regulate the output voltage to some desired level. System second order transfer functions can be derived in terms of the steady state duty cycle (D) of the switch.

Buck:



$$H(s) = \frac{1/LC}{s^2 + \frac{1}{CR}s + \frac{1}{LC}} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

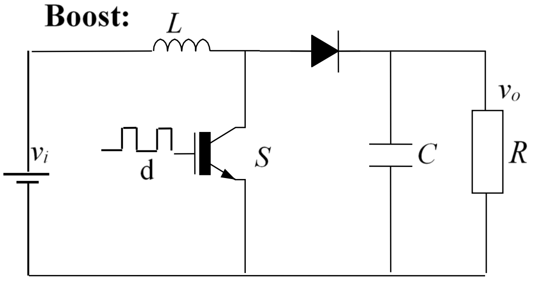
Thus $\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$

$$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = DH(s) \quad \& \quad \frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_o}{D}H(s)$$

For the following converters the derived expression includes an effective inductance L_e:

$$L_e = \frac{L}{(1-D)^2} : \quad H_e(s) = \frac{1/L_eC}{s^2 + \frac{1}{CR}s + \frac{1}{L_eC}} \quad \text{with} \quad \zeta = \frac{1}{2R}\sqrt{\frac{L_e}{C}} = \frac{1}{2(1-D)R}\sqrt{\frac{L}{C}}$$

Boost:

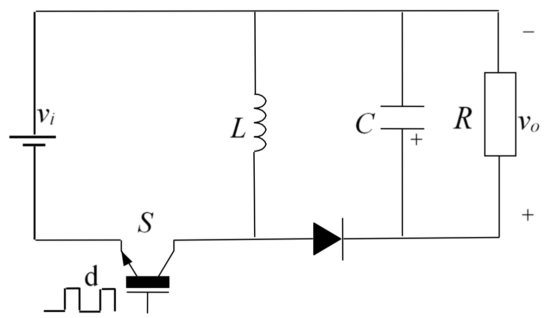


$$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = \frac{1}{(1-D)}H_e(s) \;\&$$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{(1-D)^2}\left(1 - \frac{sL_e}{R}\right)H_e(s)$$

Note: $\hat{\boldsymbol{v}}_{\boldsymbol{o}}(\boldsymbol{s})/\hat{\boldsymbol{d}}(\boldsymbol{s})$ has a RHP zero (non-minimum phase)

Buck-Boost:



$$\frac{\hat{v}_o(s)}{\hat{v}_i(s)} = \frac{D}{(1-D)}H_e(s) \;\&$$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{(1-D)^2}\left(1 - \frac{sDL_e}{R}\right)H_e(s)$$

Again $\hat{v}_o(s)/\hat{d}(s)$ has a RHP zero (non-minimum phase)

Observations:

Even the simplest transfer function of the buck converter has a poor response, because typically the poles are highly underdamped, and vary with load. In practice the damping is improved a little when practical loss terms are included as determined.

When the inductor winding resistance (r_L) and capacitor equivalent series resistance (r_{ESR}) are included then: Simplifying $a \approx 1$ and $\left(1 + \frac{r_L}{R}\right) \approx 1$

$$\frac{\hat{v}_o(s)}{\hat{d}(s)} = \frac{V_i}{LC}\left(\frac{(r_{ESR}Cs + 1)}{s^2 + \left(\frac{1}{CR} + (r_{ESR} + r_L)/L\right)s + 1/LC}\right)$$

where

$$\zeta = \frac{1/CR + (r_c + r_L)/L}{2\omega_0} \quad \& \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

The loss terms improve the damping however, *but add* extra LHP ESR-zero to the transfer function. Similar extended analyses can be applied to the boost and buck-boost converters.

RHP zeros are shown to be present in the Boost and Buck-Boost. These complicate the design of a controller since with an instantaneous increase in duty cycle the output voltage actually decreases momentarily before it finally rises. This can be explained from the circuit point of view noting that the inductor current cannot change instantaneously where as d can!!. Thus an instantaneous increase in $\hat{d}$ begins to increase the energy in the inductor but at the same time instantaneously decreases the time interval *(1-d)*T* where inductor energy is transferred to the output, thereby momentarily decreasing the output voltage.

REFERENCES

[1] S. R. Sanders, J. M. Noworolski, X. Z. Liu, and G. C. Verghese, "Generalised averaging method for power conversion circuits", *IEEE Trans. on Power Electronics*, vol.6, no.2, pp. 251-258, April 1991.

[2] N. Mohan, T. M. Underland, and W. P. Robbins, "*Power Electronics: Converters, Applications and Design*" 2nd Edn, John Wiley & Sons Inc, 1995, pp. 82-87