Systems and Control
Basic Control Concepts

***N.S.Nise** Third Edition*
Selections from Chapters 4 & 7

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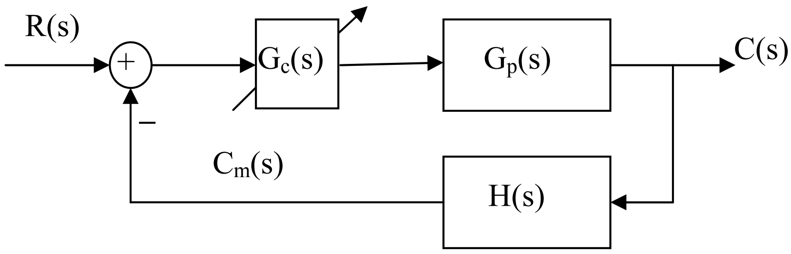
1. Closing the loop

In order to control a process we have to

1. Model the system dynamics of $G_p(s)$
2. Measure the output we want to control and try and regulate it in some way.

A general control system has the following:



Here:

- R(s) is the reference/regulated input
- $G_c(s)$ is an adjustable controller, which in a simple case may be a just variable K (called the controller gain)
- $G_p(s)$ is the plant or process that we are trying to control
- H(s) is the measurement system
- C(s) is the controlled output

We can show that:

$$G(s) = G_c(s)G_p(s)$$
$$C(s) = G(s)\big(R(s) - C_m(s)\big)$$
$$C_m(s) = H(s)C(s)$$

Combining these in terms of C(s) gives:

$$C(s) = G(s)[R(s) - H(s)C(s)]$$

resulting in the closed loop transfer function (*T(s)*):

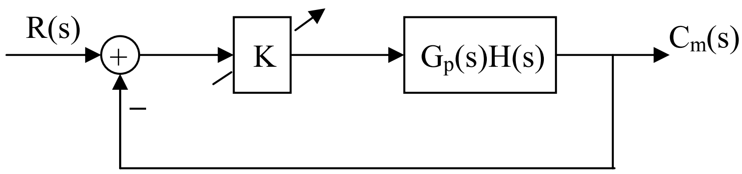
$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Nb: *G(s)H(s)* must be dimensionless

For unity feedback systems *H(s)* = 1, thus:

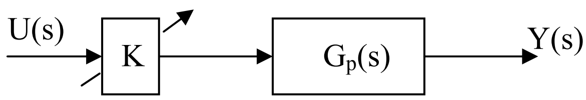
$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

In many systems the dynamics of *H(s)* are much faster than that of $G_p(s)$ so that $C_m(s)$ is a good measure of the output *C(s)*. This is true for most applications: ie *H(s)* may simply be a filter. In such cases the system may be thought of as a unity feedback system as shown below:



Why do we need closed loop control?

Open loop systems:



To determine the steady state output, we can use the final value theorem for Laplace transforms:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Ex:

$$G_p(s) = \frac{b}{s+a} \qquad \Rightarrow \qquad y(t)_{ss} = \lim_{s\to 0} s[KG_p(s)U(s)]$$

If U(s) is a unit step input (1/s) then:

$y(t)_{ss} = \dfrac{Kb}{a}$ … so that the final value of the output will grow with *K*.

A problem is that the plant dynamics may change from what we expect due to uncertainty. For instance parameter changes due to temperature (etc) may change the values of b and a. Under such circumstances, increasing *K* simply amplifies the uncertainty in the output, and will contribute to output error.

Closed loop systems:

Here the feedback is used to help remove this uncertainty in the response of the output. Consequently we can use feedback control to force the output to track a controlled input.

Ex: If we have a unity feedback system as *K* increases:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{KG_p(s)}{1+KG_p(s)} \xrightarrow[K increasing]{} \frac{KG_p(s)}{KG_p(s)} \to 1$$

Hence, changes in the plant $G_p(s)$ have much less effect.

In practice we have made the system less sensitive to plant variations by the factor (1+*G(s)*).

We can define the closed loop system sensitivity (unity feedback) as:

$$S = (1+G(s))^{-1}$$

nb: $S = (1+G(s)H(s))^{-1}$ (non unity feedback)

<u>What sort of control is necessary?</u>

There are two approaches to controlling a plant

Servo Control: Here we are interested in the performance of the system when it has to follow changes in the reference or demand signal. Thus the control system is designed based on transient and steady state performance objectives in the time and frequency domain. Applications include torque and variable speed controlled drives, vehicles, cranes etc.

Process Control: Here we are interested in the performance of the loop as a regulator. The input is generally a "set point" and the system is designed to try and avoid the effects of disturbances. Applications include petrochemical, food, steel, glass, pulp & paper and energy systems.

In both control systems the difference between the reference input *R(s)* and the output signal *C(s)* is important. This signal is called the output error signal: *E(s)*.

In a general system the *output error* signal can be defined as:

$$E(s) = R(s) - C(s) = R(s)\left(1 - \frac{C(s)}{R(s)}\right) = R(s)\big(1 - T(s)\big)$$

resulting in:

$$E(s) = R(s)\left(1 - \frac{G(s)}{1 + G(s)H(s)}\right)$$

For <u>unity feedback</u> signals:

$$E(s) = R(s)\left(\frac{1}{1 + G(s)}\right)$$

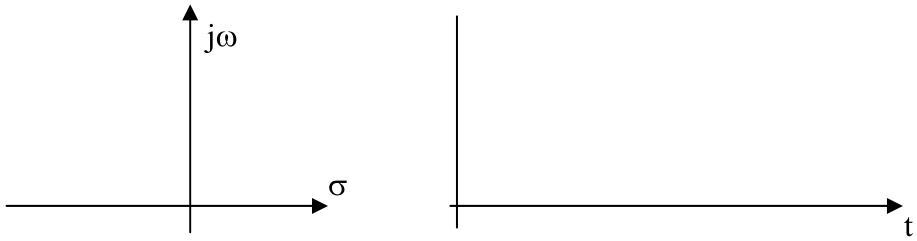
Ideally we want e(t) $\rightarrow$ 0 under both transient and steady state conditions.

2. Time Domain Performance Considerations

In a closed loop system with fixed gain the performance and stability of the system is determined by looking at the position of the **fixed** poles in the plant and determine the time domain performance

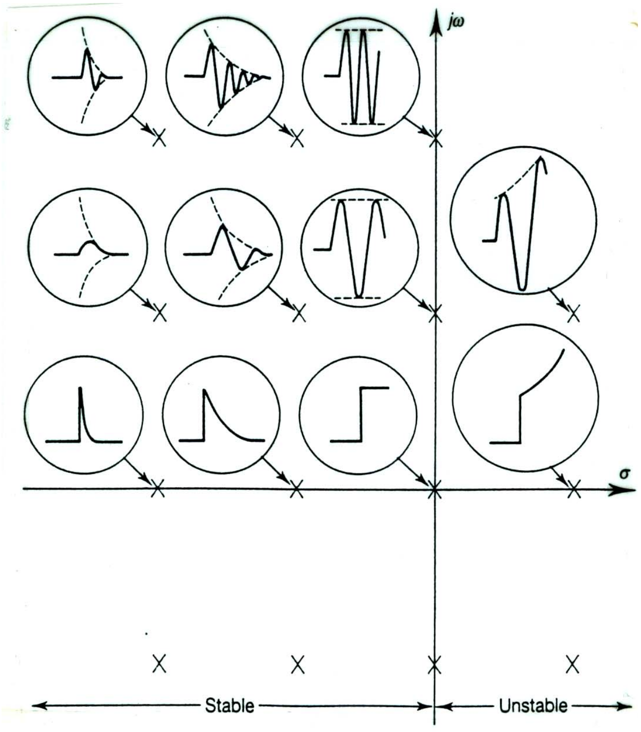
Example:

$$\frac{C(s)}{R(s)} = K\frac{1}{s+a}$$



For a step input in $r(t)$ we have: $$c(t) = \frac{K}{a}\left(1 - e^{-at}\right)$$

With variations in K we can only vary the steady state output, not the time constant of the system, and as discussed previously we are dependent on the plant dynamics and stability for good performance.



Relationship between pole location and transient shape

General Systems

Most complex systems have dominant features that typically can be approximated by either a first or second order system response. Thus an understanding of both first and second order system responses is very useful.

Dominant First order systems *(Nise Chapter 4.3)*

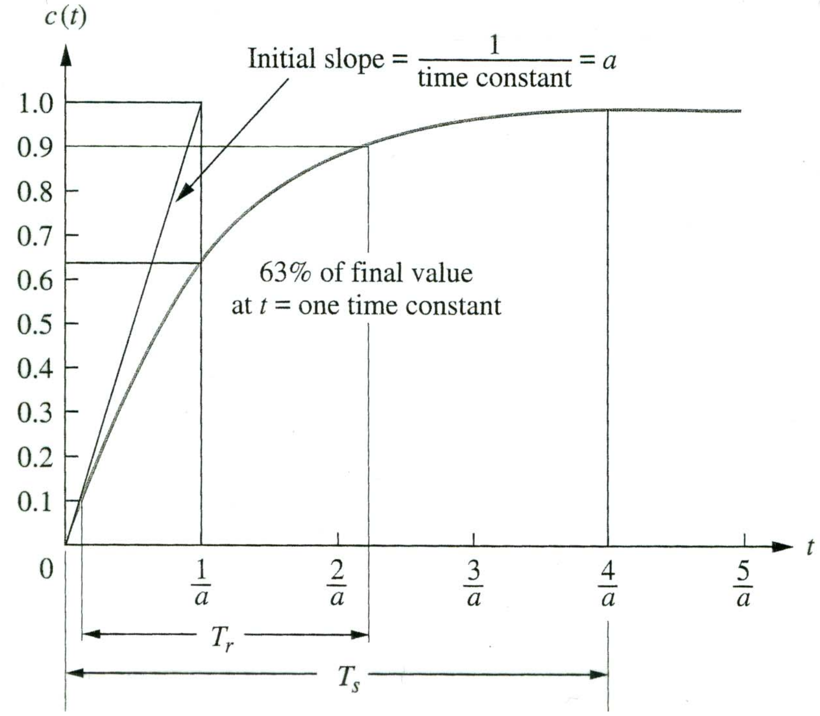
First order systems have monotonic step response. The response to a unit step input can be expressed as:

$$c(t) = 1 - e^{-t/\tau}$$

The *response time* is often redefined for first order systems as time to rise to 63% of the final value ($t_r = \tau$)

The *settling time* can be defined as the time taken for the response to reach and stay within some percentage of its final value.

A 2% settling time: $c(t) = 0.98$ and $t_s = -\tau \ln(0.02) \approx 3.9\tau$



Dominant Second order systems *(Nise Chapter 4.4)*

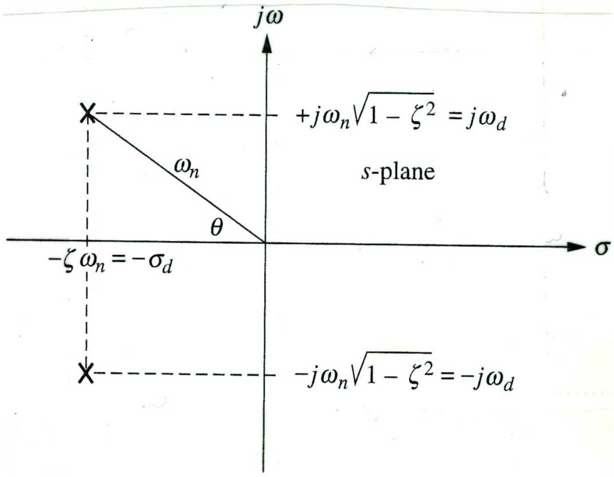
The step response of a stable second order system (ie underdamped) is defined by the pole locations in the s-plane resulting from a complex pole pair. Here *K* is assumed to be a known fixed value.

Since the following is true in the s-domain:

$$\frac{C(s)}{R(s)} = \frac{K}{(s+\sigma_d + j\omega_d)(s+\sigma_d - j\omega_d)}$$

$$= \frac{K}{s^2 + 2\sigma_d s + (\sigma_d^2 + \omega_d^2)}$$

$$= \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

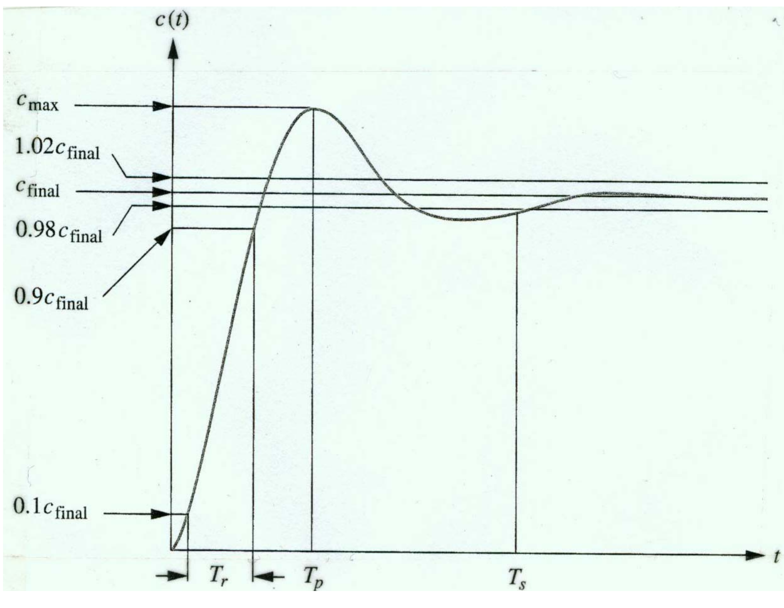


ω_n: (undamped natural frequency) ω_d: (damped natural frequency)

σ_d: (the rate of decay or the exponential damping frequency) ζ: (the damping factor)

For such a system the transient response to a step input can be written as:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$$



The *time to first peak* is half the oscillation period (ie the damped natural frequency ω_d) and the *rise time* is approximately half of this.

$$t_p = \frac{1}{2}\frac{2\pi}{\omega_d} = \frac{\pi}{\omega_d} \approx 2t_r$$

The *percentage overshoot* can be written in terms of how much the system output peaks above the steady state value:

$$\%overshoot = 100 \times \frac{y_{\max} - y_{final}}{y_{final}} = 100 \times \frac{y(t_p) - 1}{1}$$

$$= 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

rearranging for damping factor:

$$\zeta = \frac{-\ln(\%O.S./100)}{\sqrt{\pi^2 + \ln^2(\%O.S./100)}}$$

ex: if we want no more than 18% O.S. $\Rightarrow \zeta \geq 0.479$ (≈ 0.5)
if 10% O.S. is acceptable then $\Rightarrow \zeta \geq 0.591$ (≈ 0.6)

The *settling time* is mainly governed by the time for the exponential envelope to decay to within a set tolerance band. This is directly related to the exponential decay in the transient response:

$$e^{-\zeta\omega_n t} = e^{-\sigma_d t} = e^{\sigma t}$$

Similar to the first order system the 2% settling time is:

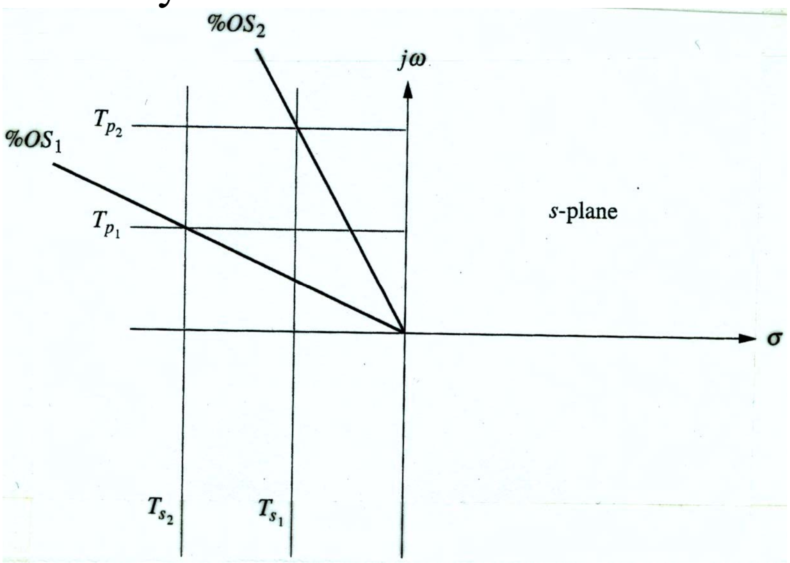
$$t_s \approx 4\tau = \frac{4}{-\sigma} = \frac{4}{\sigma_d} = \frac{4}{\zeta\omega_n}$$

3. S-plane Design Regions using Time Domain Specifications

The step performance criteria are related to the position of the poles on the s-plane.

Intuitively, we know that a best damping ratio will exist for any second order system, since lightly damped systems will overshoot (possibly many times) before settling and heavily damped systems will be sluggish.

For our second order system:



The *percentage overshoot* is determined by ζ.
Generally the allowed overshoot will be specified to be within some limits. These limits correspond locating the poles between two lines of constant damping ratio.

The *settling time* is determined by σ_d, (the real part of the poles).
A maximum desirable value may be set in which case the system poles must lie to the left of a vertical line on the s-plane.

The *time of first peak* (approx related to *rise time*) depends on ω_d.
If this time is has some minimum limit then the pole in the upper part of the s-plane must lie above a horizontal line.

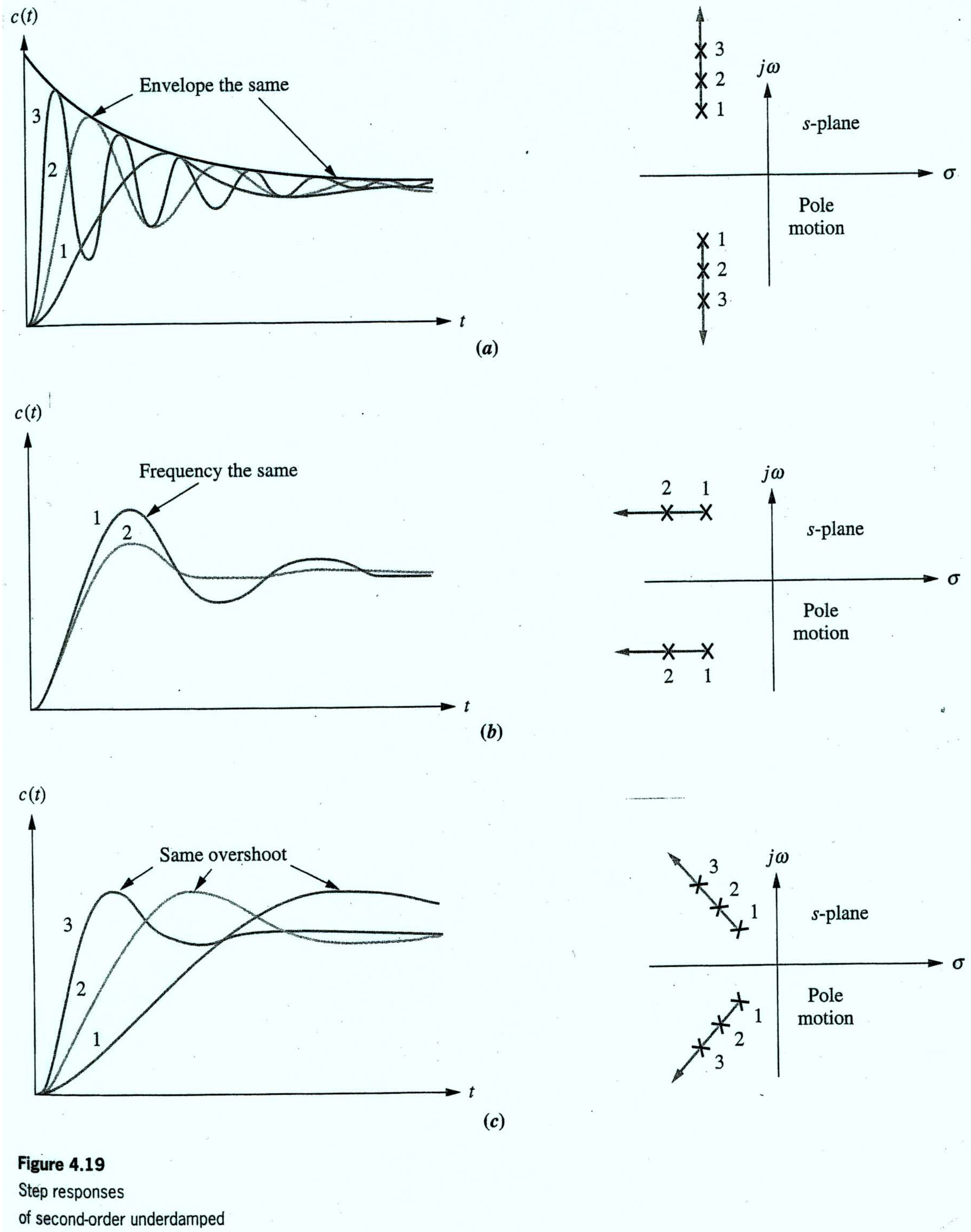


Figure 4.19
Step responses
of second-order underdamped
systems as poles move:
a. with constant real part;
b. with constant imaginary part;
c. with constant damping ratio

4. Closed Loop Systems: Controlling Stability

The poles of the closed loop system change as the gain *K* of the plant varies. This affects both the transient and steady state response in the time and frequency domains, so that the system can be made to respond much faster, but if we're not careful we could make it poorly damped or even go unstable.

Because of this the system performance and stability must be investigated with changing *K*. Usually by looking at how the poles vary with *K* in the *s*-plane.

Nyquist/Bode analysis techniques are also used to obtain an overall picture of system behaviour in the frequency domain, however we can't measure a frequency response if the system is already unstable, whereas we can determine this by looking to see if any poles exist in the right half plane (RHP) of the *s*-plane.

The poles of the closed loop system are now defined by:

$$1 + G(s)H(s) = 0$$

This is called the systems *Characteristic Equation.*
The poles of $C(s)/R(s)$ are the roots of this equation.

Examples:

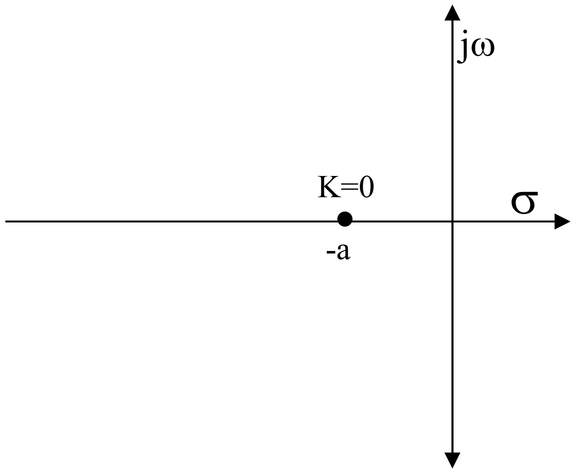
4.1 A First Order System

Using a simple plant $G_p(s)$ with a series controller $G_c(s) = K$ and unity feedback gives:

$$G(s) = G_c(s)G_p(s) = K\frac{1}{s+a}$$

$$CLTF = \frac{C(s)}{R(s)} = \frac{\frac{K}{s+a}}{1+\frac{K}{s+a}} = \frac{K}{s+(K+a)}$$

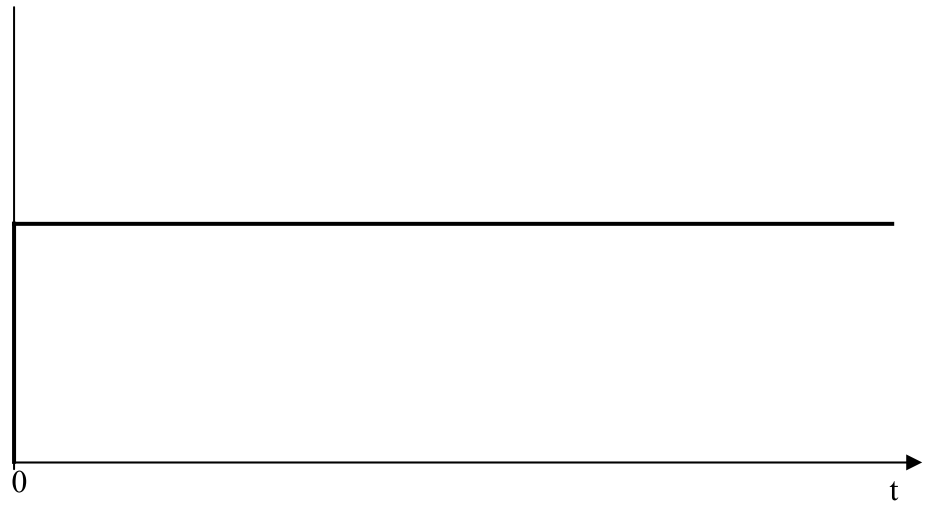
the root locus:



the time response:

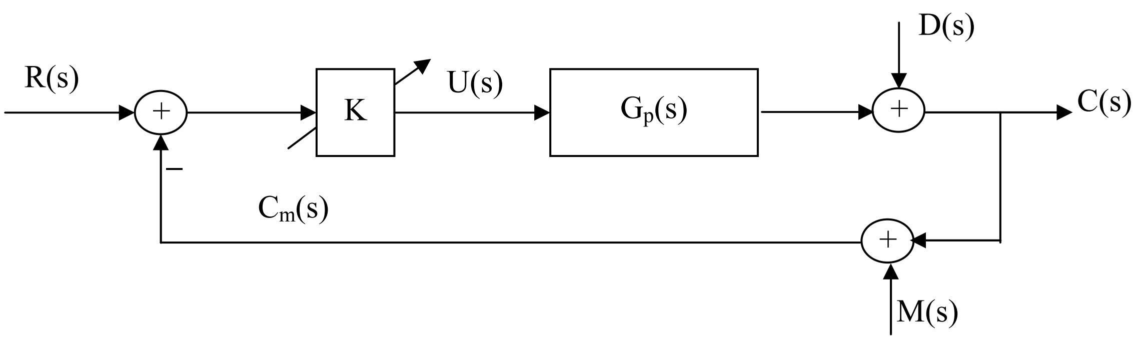
$$c(t) = \frac{K}{K+a}\left[1 - e^{-(K+a)t}\right]$$

The time response



Practical Systems

In a more practical system we have disturbances and measurement inaccuracies (noise):



$H(s) = 1$

$$C(s) = G(s)\big(R(s) - C_m(s)\big) + D(s)$$
$$C_m(s) = C(s) + M(s)$$
$$\therefore C(s) = G(s)R(s) - G(s)\big(C(s) + M(s)\big) + D(s)$$

rewrite

$$C(s) = \frac{G(s)}{1+G(s)}R(s) - \frac{G(s)}{1+G(s)}M(s) + \frac{1}{1+G(s)}D(s)$$

Recall for unity feedback the *closed loop transfer function* (CLTF) is:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{KG_p(s)}{1+KG_p(s)}$$

Thus if we want:

$C(s) = R(s)$ $\Rightarrow T(s) \to 1$ $\Rightarrow$ K must be …

to keep the effect of M small: $\Rightarrow T(s) \to 0$ $\Rightarrow$ K must be …

to keep the effect of D small: $\Rightarrow S \to 0$ $\Rightarrow$ K must be …

Futhermore, assuming the measurement error is small (i.e. *M(s)* is ignored), the *control effort* to the plant is:

$$U(s) \cong K\big(R(s) - C(s)\big) = KE(s) = K\frac{R(s)}{1+KG_p(s)}$$

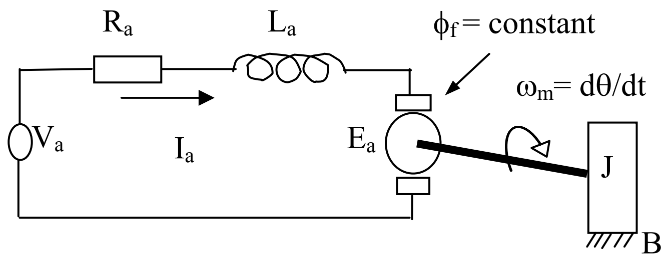
Increasing *K* increases the control effort to the plant!

In practice this may exceed the available drive energy, causing actuators to limit (saturate), thereby possibility exciting non-linear modes of behaviour.

4.2 A Second Order System

(Nise 2.8 & 8.2)

A seperately excited DC servo drive with proportional control



$$V_a = I_a R_a + L_a \frac{dI_a}{dt} + E_a$$

$$T_e = K_T I_a \quad \& \quad E_a = K_E \omega_m$$

$$T_e - T_L = J \frac{d\omega_m}{dt} + B\omega_m$$

As the time constant of the armature winding (L_a/R_a) is small, then:

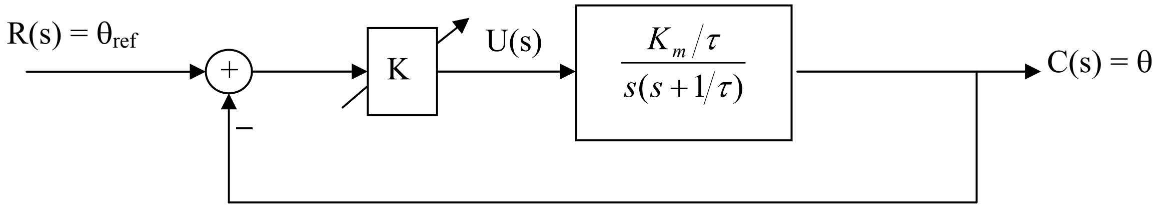
$$I_a \approx \frac{V_a - E_a}{R_a} \quad \Rightarrow \quad T_e = K_T\left(\frac{V_a}{R_a} - K_E \frac{\omega_m}{R_a}\right)$$

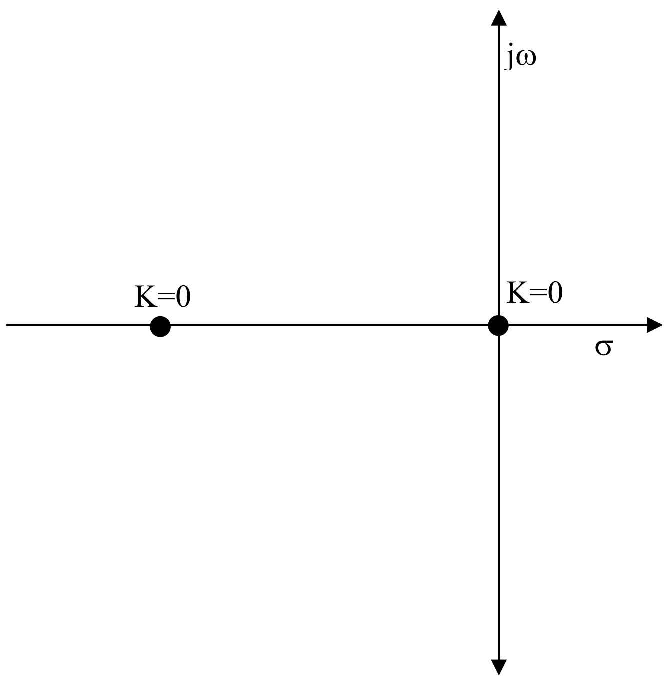
ignoring T_L

$$K_T \frac{V_a}{R_a} = J \frac{d^2\theta}{dt^2} + \left(B + \frac{K_T K_E}{R_a}\right)\frac{d\theta}{dt}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_T / R_a}{s(Js + B + \frac{K_T K_E}{R_a})} \equiv \frac{K_m}{s(\tau s + 1)} = \frac{K_m / \tau}{s(s + 1/\tau)}$$

If we now add a proportional controller then:



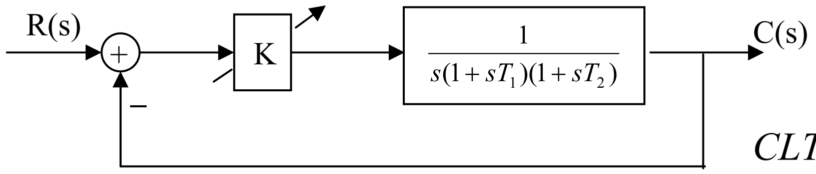


Nb: This second order system is unconditionally stable with $K \rightarrow \infty$, however damping becomes very poor. Thus K must be selected to give a desirable damped response.

4.3 Higher Order Systems

In higher order closed loop systems, the possibility of instability grows. Hence we must determine bounds on K.

eg:



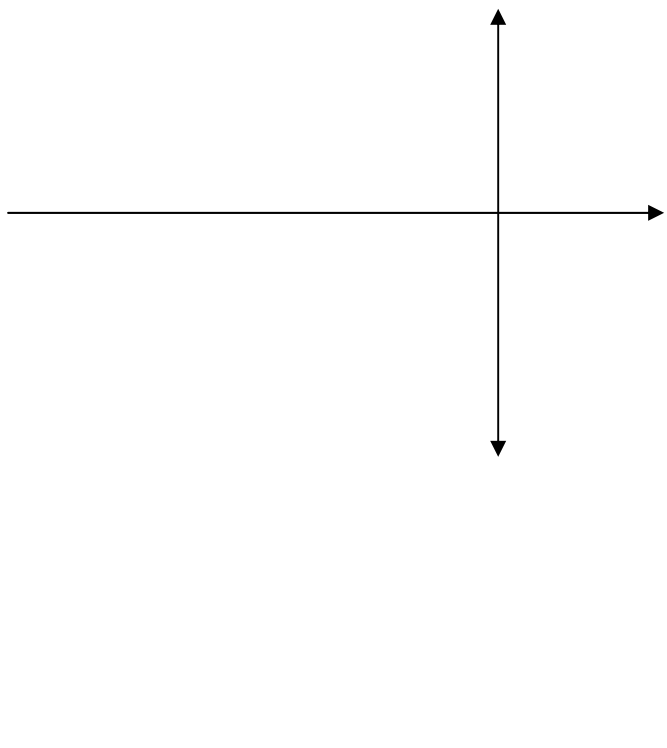
$$CLTF = \frac{K}{s(1+sT_1)(1+sT_2)+K}$$

The characteristic equation is:

$$T_1T_2s^3 + (T_1 + T_2)s^2 + s + K = 0$$

For stability, a necessary but not sufficient condition is that all the terms of the characteristic equation have the same sign and all consecutive s terms be non-zero.

With varying K the complex plane gives a better picture of the stability of the system.



5. Closed Loop Systems: Identifying & Controlling the Dominant Poles & Zeros

(Nise 4.7-4.8)

Dominant poles/zeros in CLTFs with fixed K

Given:

$$\frac{C(s)}{R(s)} = \frac{K_g(s+z_1)(s+z_2)...}{(s+p_1)(s+p_2)(s+p_3)...}$$

z_1, z_2 ... are the zeros of the CLTF,
p_1, p_2 ... are the poles of the CLTF.

The positions of p_1, p_2, p_3 ... in the complex plane determine the stability and the time response of the output c(t) to an input r(t).

eg. Suppose r(t) is a unit step function [ie R(s) =1/s]
using partial fractions we obtain:

$$C(s) = \frac{A}{s} + \frac{B}{s+p_1} + \frac{C}{s+p_2} + \frac{D}{s+p_3} + ...$$

and taking the inverse Laplace Transform results in:

$$c(t) = Au(t) + Be^{-p_1 t} + Ce^{-p_2 t} + ...$$

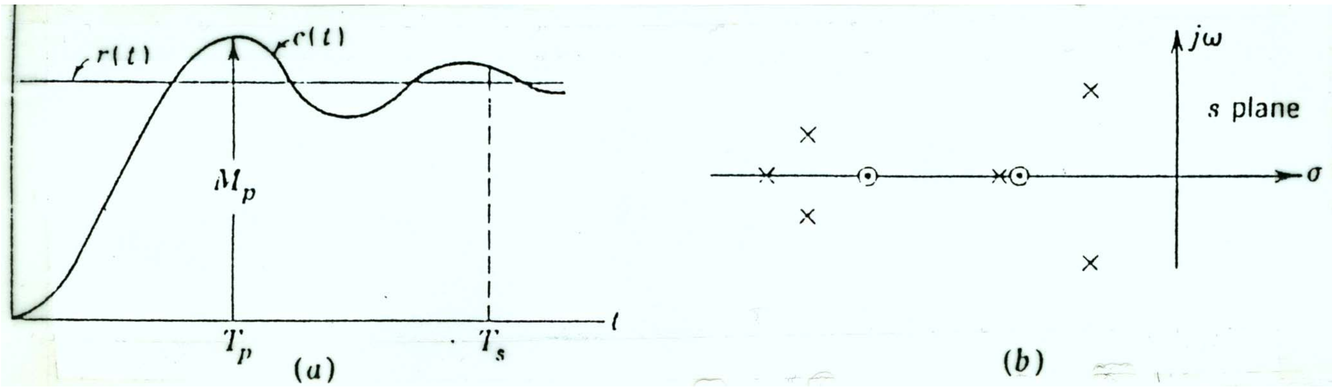
and A,B,C can be determined for a known system

The closed loop system response will be dominated by either a single pole or a pair of complex poles if the pole-zero pattern (with K fixed) has the following characteristics:

- All "other" poles and any zeros must be far to the left of the dominant pole(s), so that the transients due to these are small in amplitude and die out quickly

- Any **pole** which is not far to the left of the *dominant pole(s)* must be near a **zero** so that the magnitude of the transient term due to that pole is small (effectively cancelled out).

Ex:



As shown, peak overshoot (M_p), time to peak (t_p), settling time (t_s), damping (ζ) etc, can all be approximated from the dominant poles positions … as for a second order system.

<u>Multiple Complex Poles</u>
The distance from the imaginary axis to a pole is called the *pole attenuation* and can be used to determine which poles have a lasting effect. If a poles attenuation is 5x bigger than another pole we can neglect its *transient* effect

Normally the *radius* (from the centre point) is a better guide to a pole(s) overall dominance. Poles which are at a radius of more than three times that of the dominant pole(s) will have only a small *transient* contribution.

Modelling higher order systems by their dominant features

Poles with negligible contribution can be neglected providing their steady state gain contributions are retained. e.g.

$$\frac{C(s)}{R(s)} = \frac{100}{(s^2+s+1)(s^2+s+9)(s+3)}$$

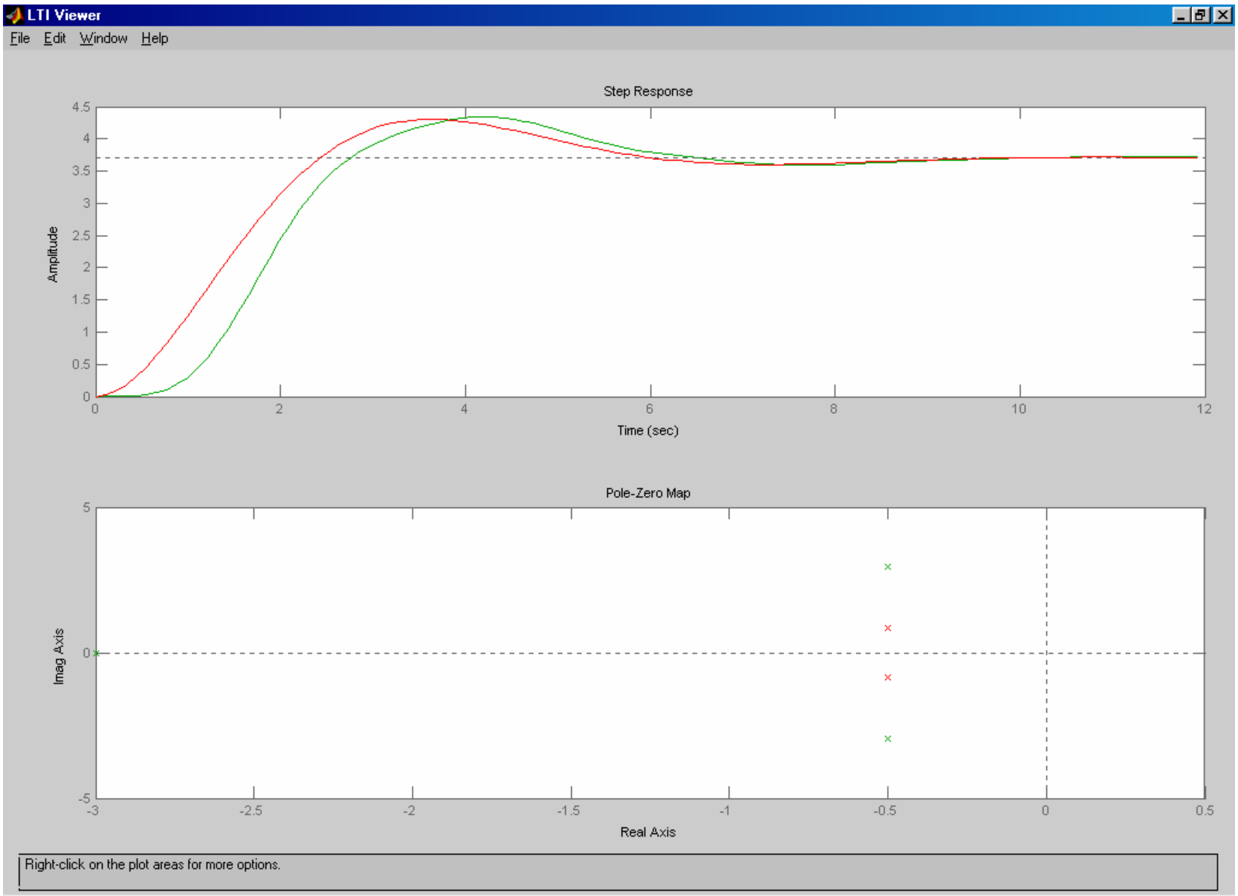
The closed loop system poles are located at:

$$s = -0.5 \pm j0.886, -0.5 \pm j2.96, \text{and} -3$$

The first complex pair is dominant, as all other poles have a radius > 3 times that of the dominant poles.

Remodel the system by approximating non-dominant poles with their steady state gain only. The simplified model is therefore:

$$\frac{C(s)}{R(s)} = \frac{100}{9\times 3\times(s^2+s+1)} \cong \frac{3.7}{(s^2+s+1)}$$

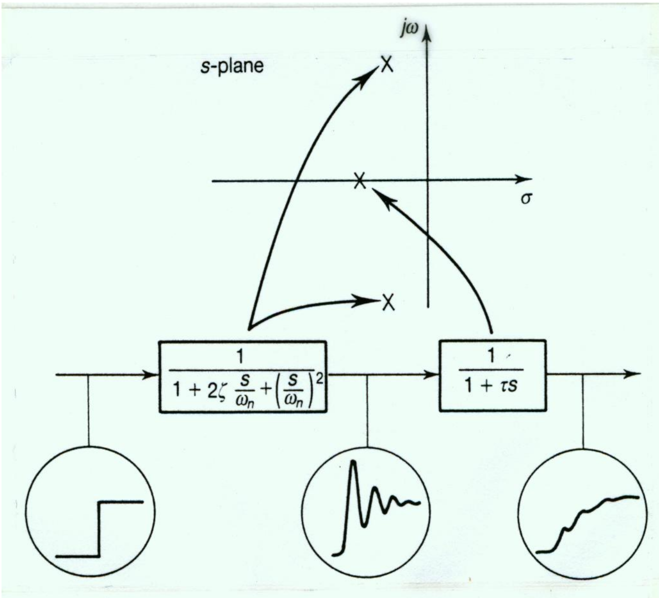


Mixed Dominant Systems

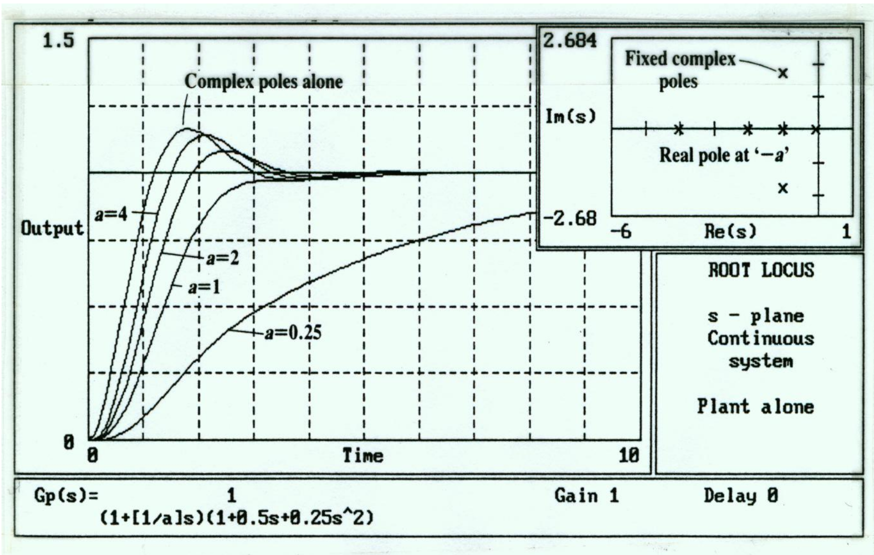
If there is not a single pole or pole pair which is dominant then the response will be a mix of the individual pole &/or zero contributions.

Pole-Pole Mix

A dominant mix of a complex pole pair with a single real pole causes the system response to be **slower** (a filtered second order response)
The extent of this filtering depends on how close the real pole is to the jω axis.



Real pole acts as a filter in oscillatory systems

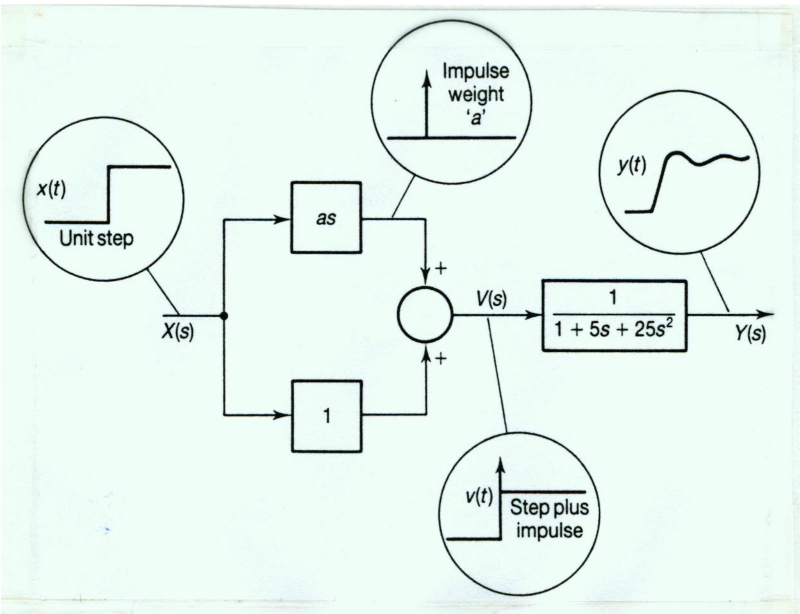


Effect of real pole location on the step response of a third-order system

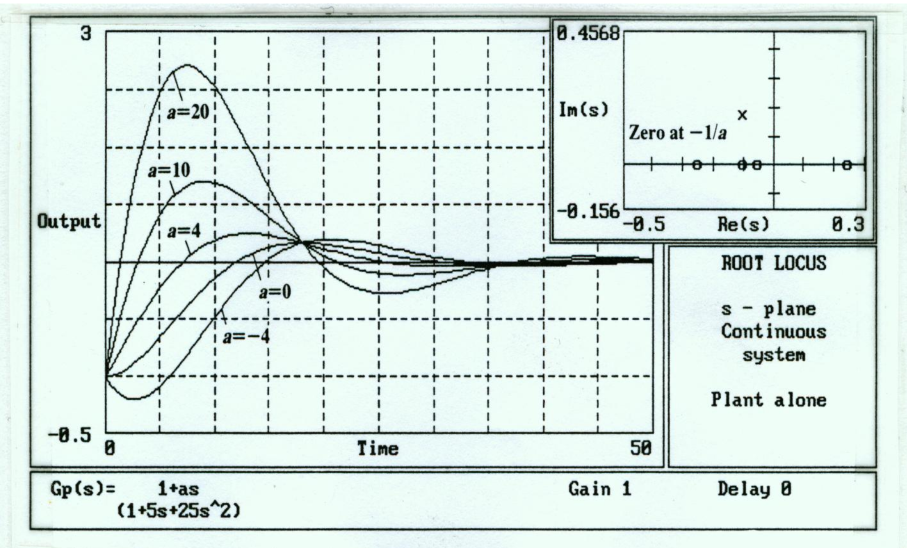
Pole-Zero Mixes

The effect of adding a dominant zero to a system (ie, a zero with a small radius) is equivalent to modifying the input function as shown below.
e.g: A step input receives an additional "kick", as the zero contributes an impulse with a weighting determined by its *attenuation*.

If the zero is in the right half plane this "kick" will be in the opposite direction to the input function forcing the initial system response to move in the opposite direction to its final steady state value.



Showing how a zero gives an extra kick to the system step response



Effect of a zero on a second-order system

Fixing K on the root locus to control the dominant system poles
CLTF poles are functions of K, therefore:

Design Objectives:

- Identify the dominant root locus "branches" that will have the largest influence on the time response.
- Try and fix the dominant closed loop pole positions such that the transient time response approximates that of a *second order system* (a pair of complex poles).
- Use the "*S-plane design region*" to fix these poles (by fixing K) somewhere that gives acceptable damping, response and settling times.

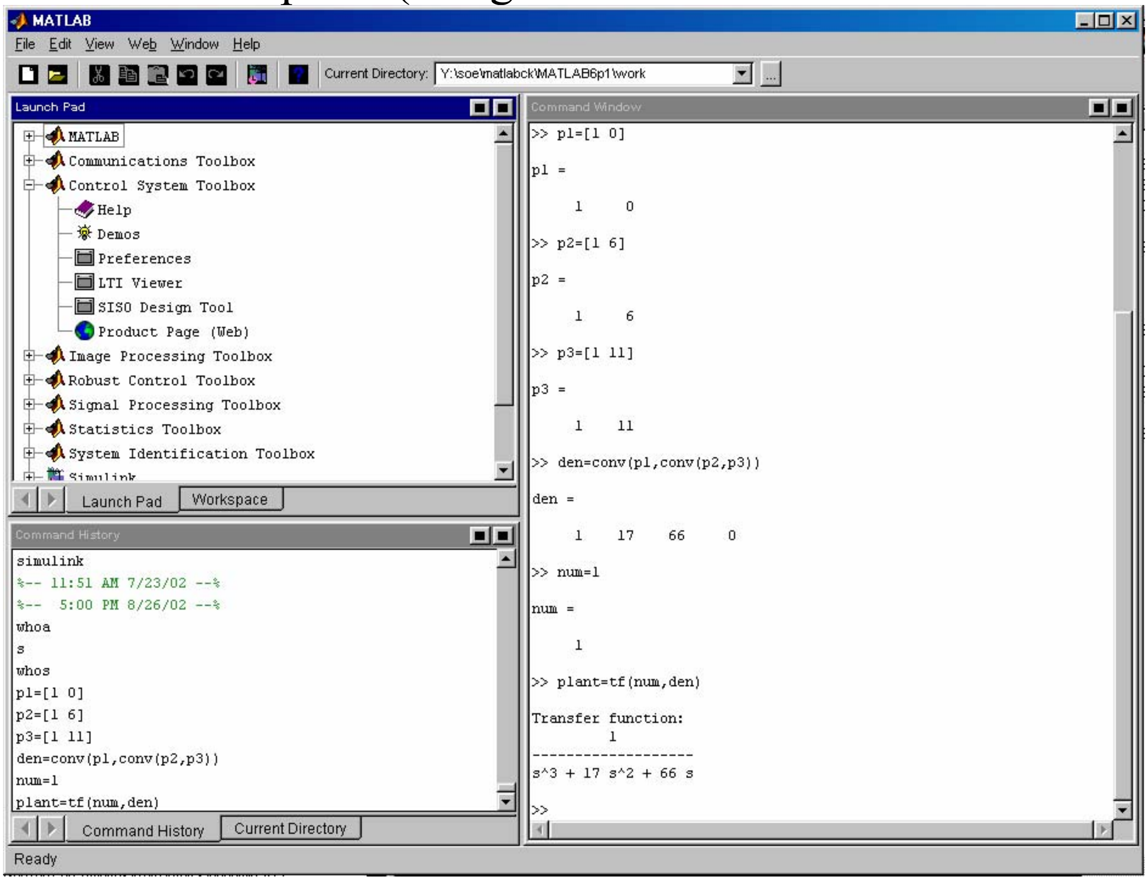
Eg: (Exam question 2000): A unity feedback control system has:

$$\frac{C(s)}{R(s)} = \frac{KG_p(s)}{1+KG_p(s)}, \qquad G_p(s) = \frac{1}{s(s+6)(s+11)}$$

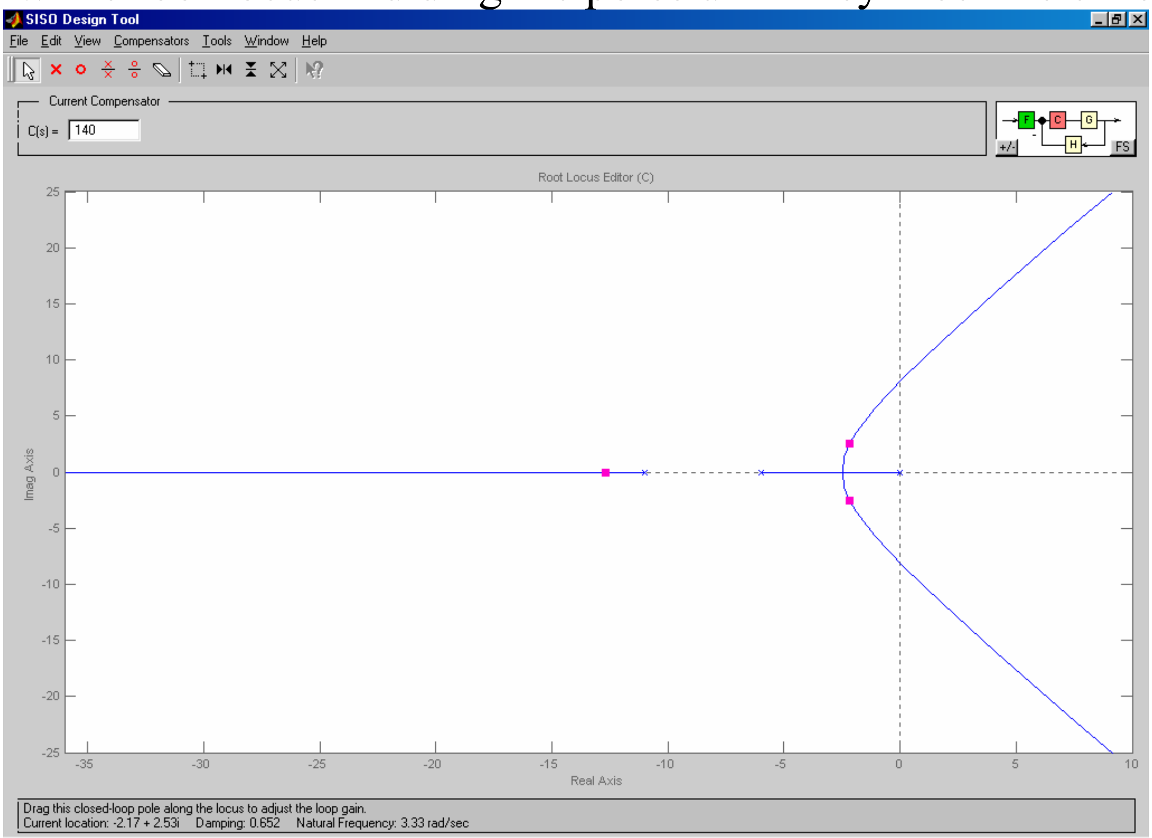
1. Draw the root-locus of its poles
2. Fix K such that dominant pole pair is at $-2.17 \pm j\,2.52$
3. Determine the position of the other pole and write down the CLTF
4. Create a reduced order model of the dominant terms in the system
5. Determine ζ and ω_n, and hence likely %O.S., t_p and t_s for a step input.

Answers:

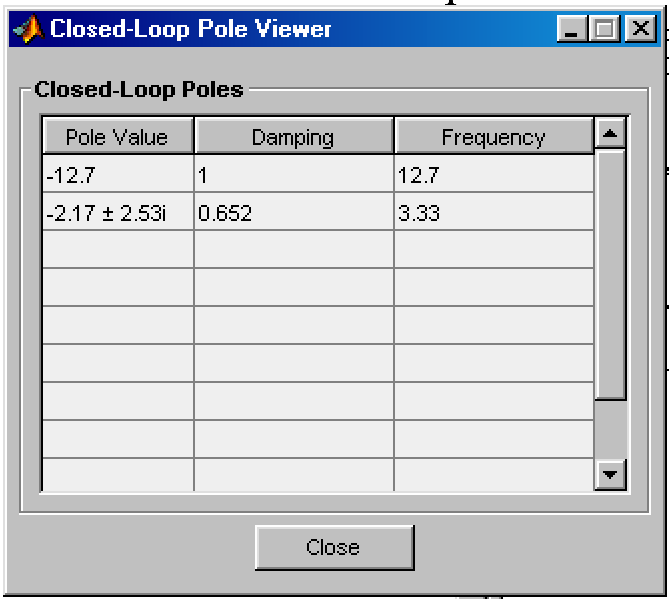
1. Root locus of its poles (using matlab SISO in the matlab toolbox)



2. draw the root locus and drag the poles until they meet the criteria



then "view" the CLTF poles:



3. The CLTF is (K taken from SISO as value for compensator):

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+6)(s+11)+K} \cong \frac{140}{(s+12.7)(s+2.17 \pm j2.53)}$$

4. The reduced order model is:

$$\frac{C(s)}{R(s)} \cong \frac{140/12.7}{s^2+4.34s+11.11} \quad \text{c.f. with} \quad \frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

5. $\sigma_d = -2.17$, $\omega_d = 2.53$, and $\zeta = 0.652$, and $\omega_n = 3.33$r/s,

thus:

$$\%O.S. = 100 \times e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cong 6.7\%$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.53} = 1.25$$

$$t_s(5\%) = \frac{-3}{\sigma_d} = \frac{-3}{-2.17} = 1.38$$

6. Closed Loop Systems: Controlling Steady State Error

(Nise Ch: 7)

Not only are we interested in the transient performance of practical feedback systems, but we also want to know how good the system response is after the transient has died away. ie the steady state response of the output to the reference input. Three standard inputs, which approximate most command inputs, are:

- unit step (1/s)
- unit ramp ($1/s^2$)
- unit acceleration. ($1/s^3$)

We will consider the steady state output error response of several common stable system types assuming unity feedback (*H(s)*=1).

Let the forward loop transfer function of the CLTF take on the general form:

$$G(s) = \frac{K(1+sT_1)(1+sT_2)\dots(1+sT_m)}{s^n(T_a s^2 + T_b s + 1)(1+sT_c)\dots(1+sT_p)}$$

where

s^n = multiple pole at the origin of the complex plane

n = system type number = no of integrations in the transfer function.

<u>For Unity Feedback</u>

$$e(t)_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \quad \text{where} \quad E(s) = \frac{R(s)}{1+G(s)}$$

thus:

$$e(t)_{ss} = \lim_{s\to 0} s\left[\frac{s^n[(T_a s^2 + T_b s + 1)(1+sT_c)\dots(1+sT_p)]R(s)}{s^n[(T_a s^2 + T_b s + 1)(1+sT_c)\dots(1+sT_p)] + K[(1+sT_1)(1+sT_2)\dots(1+sT_m)]}\right]$$

The most common (practical) system types are type: 0, 1 and 2.

<u>*Unity feedback systems: steady state error to a unit step*</u>

From our definition of $e(t)_{ss}$ then:

$$e(t)_{ss} = \lim_{s \to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim\limits_{s \to 0} G(s)}$$

Thus if we define a step (position) co-efficient as:

$$K_p = \lim_{s \to 0} G(s)$$

thus: $e(t)_{ss} = \dfrac{1}{1+K_p}$

The response of each system type to this input is summarised below:

System Type N°	K_p	$e(t)_{ss}$

Unity feedback systems: steady state error to a unit ramp

$$e(t)_{ss} = \lim_{s \to 0} \frac{s(1/s^2)}{1+G(s)} = \lim_{s \to 0} \frac{1}{sG(s)}$$

if we define a ramp (velocity) co-efficient as:

$$K_v = \lim_{s \to 0} sG(s)$$

thus: $e(t)_{ss} = \dfrac{1}{K_v}$

The system response follows as:

System Type N^o	K_v	$e(t)_{ss}$

<u>Unity feedback systems: steady state error to a unit acceleration</u>

$$e(t)_{ss} = \lim_{s\to 0} \frac{s(1/s^3)}{1+G(s)} \qquad = \lim_{s\to 0} \frac{1}{s^2 G(s)}$$

We define a parabolic (acceleration) co-efficient as:

$$K_a = \lim_{s\to 0} s^2 G(s)$$

thus: $e(t)_{ss} = \dfrac{1}{K_a}$

The system response follows is:

System Type N^o	K_a	$e(t)_{ss}$

Example of error calculation for a Type 1 system:

(a) with step input

(b) with ramp input

(c) with acceleration input