

UKRAINIAN CATHOLIC UNIVERSITY

BACHELOR THESIS

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# Recognition of Outer $k$ -planar Graphs

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by Ivan SHEVCHENKO

## *Abstract*

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## Chapter 1

# Introduction

### 1.1 Contributions

### 1.2 Structure of the thesis

## Chapter 2

# Related Work

Already in the 1980s, researchers in the field of graph drawing acknowledged the importance of reducing the edge crossings for improving visualisation clarity [2]. The suspicion that a drawing with fewer edge crossings was easier to comprehend was later confirmed by several experimental studies [13]. These studies showed that minimising the crossings in graph representations significantly improves the ability of humans to interpret the structure, particularly when dealing with complex or large graphs.

The ideal form of crossing minimising drawings – planar ones, has been focused on by the research community for a long time, with the first of linear-time algorithms for recognising planar graphs presented already in 1974 [10]. However, requiring the drawing to be completely crossing free imposes severe limitations on an underlying graph. While providing a clean structure, these restrictions are often too constraining for many real-world graphs, especially large ones. This has led to a growing interest in exploring graphs close to being planar; see the survey by Didimo et al. [5]. Such graphs allow a limited number of crossings while still retaining some of the beneficial structural properties of planar graphs.

## 2.1 Difficulty of dealing with beyond-planar graphs

Most relaxations of strict planarity dramatically increase the complexity of recognising such graphs. So, the general problem of minimising edge crossings in a graph drawing was known to be computationally intractable already in 1983 when Garey and Johnson [7] demonstrated that the CROSSING NUMBER problem determining whether a given graph can be drawn with at most  $k$  crossings, is NP-hard. Their proof relies on a reduction from the OPTIMAL LINEAR ARRANGEMENT problem, which is known to be NP-hard.

Minimising the number of local crossings is also hard. Korzhik and Mohar [12] showed that testing the 1-planarity, recognising whether a graph can be drawn with at most one crossing per edge, is NP-hard. Later, Cabello and Mohar [3] showed that testing 1-planarity is still NP-hard even for near-planar graphs, that is, graphs that can be obtained from planar graphs by adding a single edge.

Given the complexity of recognising  $k$ -planarity, researchers considered exploring more restrictive classes of graphs, hoping that imposed limitations could simplify the recognition. One of the considered classes is the class of outer  $k$ -planar graphs, a subclass of  $k$ -planar graphs that admit drawings where all vertices lie on the outer face.

## 2.2 Efficient recognition of some outer $k$ -planar graph

Although the general recognition problem for outer  $k$ -planar graphs is NP-hard, efficient algorithms have been developed for specific values of  $k$ .

For  $k = 0$ , the recognition task simplifies to an outerplanarity test. Recognition can be accomplished by augmenting the graph with a new vertex connected to all original vertices and testing whether the resulting graph is planar. An alternative approach, described in [14], introduces the concept of 2-reducible graphs, which are totally disconnected or can be made totally disconnected by repeated deletion of edges adjacent to a vertex with a degree at most two. The proposed outerplanarity test is based on an algorithm for testing 2-reducibility.

In the case of  $k = 1$ , two research groups independently presented linear-time algorithms [1, 8]. Both algorithms use the SPQR decomposition of a graph for the test. Notably, the latter solution extends the graph to a maximal outer 1-planar configuration if such a drawing exists, unlike the former, which employs a bottom-up strategy which does not require any transformations of the original graph.

Considering a special case of this problem, Hong and Nagamochi [9] proposed a linear-time algorithm for recognising full outer 2-planar graphs. An outer  $k$ -planar drawing is *full* if no crossings lie on the boundary of the outer face. Later, Chaplick et al. [4] extended their result by introducing an algorithm for recognising full outer  $k$ -planar graphs for every  $k$ . Their algorithm runs in  $O(f(k) \cdot n)$  time, where  $f$  is a computable function.

For the general version of the problem and values of  $k > 1$ , no research had been conducted until recently when a group of researchers proposed an algorithm for the general case, which we discuss in the next section.

## 2.3 Recognizing general outer $k$ -planar graphs

For a given outer  $k$ -planar drawing of a graph, Firman et al. [6] proposed a method for constructing a triangulation with the property that each edge of the triangulation is crossed by at most  $k$  edges of the graph drawing. Since the edges of the triangulation do not necessarily belong to the original graph, they are termed *links* to distinguish them from the original graph edges. The construction is done recursively.

Initially, the algorithm selects an edge on the outer face and labels it as the active link. At each recursive step, the active link partitions the graph into two regions: a left part already triangulated and a right part not yet explored. The objective of each step is to triangulate the right portion. To achieve this, a splitting vertex is chosen within the right region, dividing it into two smaller subregions. The splitting vertex is selected so that the two new links, connecting the split vertex with the endpoints of the active link, are each intersected by at most  $k$  edges, which allows including them into the triangulation. The algorithm then recurses, treating each of these newly formed links as the active one.

Later, Kobayashi et al. [11] extended this approach to address the recognition problem for outer  $k$ -planar graphs. In contrast to the triangulation task, where the drawing is provided, the recognition problem requires determining whether a given graph admits an outer  $k$ -planar drawing. Although the core idea remains analogous, the absence of a drawing requires the exploration of all possible configurations. Here, each recursive step verifies whether the unexplored right portion of the graph can be drawn as an outer  $k$ -planar graph that is compatible with the left part.

Moreover, instead of relying on recursion, the method utilises a dynamic programming approach. This framework combines solutions of smaller subproblems retrieved

from a table to solve larger ones. To populate this table, the algorithm iterates over all possible configurations corresponding to different recursion steps. Several parameters characterise each such configuration. The first parameter is the active link – a pair of vertices that divides the graph into a left and a right region. The second parameter is a set of vertices in the right part, which is not uniquely determined as in the triangulation case. Additionally, the configuration depends on the order in which edges intersect the active link and the number of intersections on the right side for each one of them. These parameters are used to ensure that the drawing of the right part is compatible with the left part. For each configuration, the algorithm considers all possible ways to split the right region further. For each of them, the method checks whether these splits are compatible with each other and the left part of the drawing.

Using the restriction on the number of edges crossing each link, the authors demonstrated that for a fixed  $k$ , the number of possible right subgraphs grows only polynomially with respect to the size of the graph. They proceeded by arguing that the overall time complexity of the algorithm is  $2^{O(k \log k)} n^{3k+O(1)}$ , showing that the algorithm is efficient for any fixed parameter  $k$ .

## 2.4 Our contribution

A common drawback of the methods described above is the lack of practical validation. Although these algorithms have been analysed and discussed in a theoretical context, they have not been implemented or empirically tested. We address this gap by implementing the most recent recognition algorithm and introducing two alternative approaches based on Integer Linear Programming (ILP) and Satisfiability (SAT) formulations. While it is NP-hard to solve the general integer linear program or to find a satisfying truth assignment for general Boolean formulas, there are very advanced solvers for such formulations that could allow us to find exact solutions for small- to medium-sized instances within an acceptable amount of time. Then we evaluate the performance and efficiency of these methods, demonstrating their practical applicability and limitations.

## Chapter 3

# Theoretical Background



## Chapter 4

# Proposed Solution

In this chapter we will discuss the methods considered in this work to recognise the outer  $k$ -planar graphs. Besides recognition, these methods also provide the outer  $k$ -planar drawings of the given graphs if it is possible. We represent drawing as a sequence of vertices in the circular order they appear on the boundary of the outer face. Considering only straight line drawings this order uniquely determines all the edge crossings and so also the number of crossings per edge.

### 4.1 ILP Formulation (unchanged)

The result of the integral linear program should be an arrangement of vertices on a line. Given the arrangement of the vertices, we can definitively identify whether two edges cross each other. Indeed, consider two edges,  $uv$  and  $st$ . Without loss of generality, we can assume that  $u$  is located before  $v$  in the arrangement,  $s$  before  $t$ , and  $u$  before  $s$ . Under these assumptions, the edges  $uv$  and  $st$  cross if and only if  $s$  is located before  $v$  and  $t$  after  $v$ .

The arrangement is represented using the so-called “ordering variables”. For every pair of vertices  $u$  and  $v$ , we create a binary variable  $a_{u,v}$  that defines the order in which they appear in the arrangement. Equality  $a_{u,v} = 1$  indicates that  $u$  is located before  $v$  and vice versa, and  $a_{u,v} = 0$  indicates that  $v$  is located before  $u$  or  $u$  and  $v$  is the same vertex.

For these variables, it is crucial to ensure transitivity. That is, if  $a_{u,v} = 1$  and  $a_{v,w} = 1$  which means  $u$  is located before  $v$  and  $v$  is located before  $w$ , then  $u$  must be located before  $w$ , so the following should hold  $a_{u,w} = 1$ . This can be ensured by the following constraint:  $a_{u,w} \geq a_{u,v} + a_{v,w} - 1$ . This constraint will restrict the value of  $a_{u,w}$  if and only if both  $a_{u,v}$  and  $a_{v,w}$  equal 1. Otherwise, the constraint will have no impact on the system at all.

Having the arrangement of the vertices, we can now deduce for each pair of edges  $uv$  and  $st$  whether they intersect or not. Naturally, there are 24 different arrangements of the vertices, as demonstrated in figure ???. Among them, there are only eight in which the edges intersect. Thus, the edge  $uv$  crosses the edge  $st$  if and only if one of the following holds:

- $a_{u,s} = 1$ ,  $a_{s,v} = 1$ , and  $a_{v,t} = 1$
- $a_{u,t} = 1$ ,  $a_{t,v} = 1$ , and  $a_{v,s} = 1$
- $a_{v,s} = 1$ ,  $a_{s,u} = 1$ , and  $a_{u,t} = 1$
- $a_{v,t} = 1$ ,  $a_{t,u} = 1$ , and  $a_{u,s} = 1$
- $a_{s,u} = 1$ ,  $a_{u,t} = 1$ , and  $a_{t,v} = 1$
- $a_{t,u} = 1$ ,  $a_{u,s} = 1$ , and  $a_{s,v} = 1$
- $a_{s,v} = 1$ ,  $a_{v,t} = 1$ , and  $a_{t,u} = 1$
- $a_{t,v} = 1$ ,  $a_{v,s} = 1$ , and  $a_{s,u} = 1$

To describe this in the linear program, we create a binary variable  $c_{uv,st}$  for every pair of edges  $uv$  and  $st$ . Equality  $c_{uv,st} = 0$  indicates that the edge  $uv$  does not cross  $st$ . To ensure the correctness of these values, we impose eight constraints on each variable, one for each case from above. For example, considering the case  $a_{u,s} = 1$ ,  $a_{s,v} = 1$ , and  $a_{v,t} = 1$ , we impose the following restriction:  $c_{uv,st} \geq a_{u,s} + a_{s,v} + a_{v,t} - 2$ , which ensures that  $c_{uv,st}$  equals 1 whenever the vertices are ordered as  $usvt$ . Making this for each case restricts  $c_{uv,st}$  to the value 1 if the vertices are arranged in one of the eight “intersecting” configurations, ensuring that  $c_{uv,st} = 0$  is possible only if  $uv$  does not cross  $st$ .

The algorithm’s objective is to minimize the maximal number of crossings per edge. This value can be written as follows:  $\max_{uv \in E(G)} \sum_{st \in E(G)} c_{uv,st}$ . Unfortunately, it cannot represent an objective function for a linear program as the max operation is not linear. To solve this, we introduce a new integer variable  $k$ . To ensure that it is equal to the objective value, we impose the following constraint on  $k$ :  $k \geq \sum_{st \in E(G)} c_{uv,st}$  for each edge  $uv \in E(G)$ .

So, the integer linear program can be described as follows:

$$\begin{array}{ll}
\text{minimize} & k \\
\text{subject to} & k \geq \sum_{st \in E(G)} c_{uv,st}, \quad \forall uv \in E(G) \\
& c_{uv,st} \geq a_{u,s} + a_{s,v} + a_{v,t} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{u,t} + a_{t,v} + a_{v,s} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{v,s} + a_{s,u} + a_{u,t} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{v,t} + a_{t,u} + a_{u,s} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{s,u} + a_{u,t} + a_{t,v} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{t,u} + a_{u,s} + a_{s,v} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{s,v} + a_{v,t} + a_{t,u} - 2, \quad \forall uv, st \in E(G) \\
& c_{uv,st} \geq a_{t,v} + a_{v,s} + a_{s,u} - 2, \quad \forall uv, st \in E(G) \\
& a_{u,w} \geq a_{u,v} + a_{v,w} - 1, \quad \forall u, v, w \in V(G) \\
& c_{uv,st} \in \{0, 1\}, \quad \forall uv, st \in E(G) \\
& a_{u,v} \in \{0, 1\}, \quad \forall u, v \in V(G)
\end{array}$$

## 4.2 SAT Formulation (unchanged)

Another approach to solving this problem is to check for a specific  $k$  whether the given graph is outer- $k$ -planar. This check can be encoded as a boolean satisfiability problem. This problem asks whether it is possible to assign logic values TRUE or FALSE so that all disjunctive clauses are satisfied. A disjunctive clause is a single literal or a disjunction of several. Literal is either a variable or a negation of a variable, with the former being the positive and the latter the negative literal.

Similarly to the ILP algorithm described in 4.1, this algorithm uses the same “ordering variables”  $a_{u,v}$  for each pair of vertices  $u$  and  $v$  that represent the arrangement of the vertices. If the boolean variable  $a_{u,v}$  is TRUE, the vertex  $u$  is located before the vertex  $v$  and vice versa otherwise.

Similarly, these variables must account for transitivity, which means that for every triple of vertices  $u$ ,  $v$ , and  $w$   $a_{u,v} \equiv \text{TRUE}$  and  $a_{v,w} \equiv \text{TRUE}$  implies  $a_{u,w} \equiv \text{TRUE}$ . This can be written as follows:  $a_{u,v} \wedge a_{v,w} \rightarrow a_{u,w}$ . Expanding the implication,

this transforms into  $\overline{a_{u,v}} \wedge \overline{a_{v,w}} \vee a_{u,w}$ . After applying De Morgan's law, we receive  $\overline{a_{u,v}} \vee \overline{a_{v,w}} \vee a_{u,w}$ , which represents a clause in the SAT problem.

The next step is to represent the crossing variables  $c_{uv,st}$  in terms of the ordering ones for each pair of edges  $uv$  and  $st$ . Similarly to the ILP algorithm, we can restrict  $c_{uv,st}$  to TRUE if  $uv$  and  $st$  cross by adding new clauses to the problem. The clauses are constructed by making the implications for each of the eight intersecting cases shown in figure ??, expanding them, and applying De Morgan's law. For example, for the case  $a_{u,s} = 1$ ,  $a_{s,v} = 1$ , and  $a_{v,t} = 1$ , we start with the logical equation as follows:  $a_{u,s} \wedge a_{s,v} \wedge a_{v,t} \rightarrow c_{uv,st}$ . Afterwards, we expand the implication:  $\overline{a_{u,s}} \wedge \overline{a_{s,v}} \wedge \overline{a_{v,t}} \vee c_{uv,st}$ . Finally, we apply De Morgan's law:  $\overline{a_{u,s}} \vee \overline{a_{s,v}} \vee \overline{a_{v,t}} \vee c_{uv,st}$  receiving one of the eight clauses for  $c_{uv,st}$ .

The last step in the construction of the problem is to count the number of crossings for each edge. The goal of this solver is to check whether the number of crossings can be smaller or equal to some constant  $k$  for each edge. To ensure this, we can build a set of clauses that prevent the problem from being satisfiable if the value  $k$  is too small. To do so, for every edge  $e_0$ , we consider all combinations of  $e_1, e_2, \dots, e_{k+1}$  for each of which we construct the following clause:  $\overline{c_{e_0,e_1}} \vee \overline{c_{e_0,e_2}} \vee \dots \vee \overline{c_{e_0,e_{k+1}}}$ . Doing so, we ensure that for each edge  $e_0$ , no  $k+1$  different edges intersect  $e_0$ , which effectively means that each edge has at most  $k$  crossings if all clauses are satisfied.

### 4.3 Bicomponent decomposition

Considering the complexity of the recognition of such graphs, splitting the problem into smaller independent subproblems yields a huge increase in performance.

For solving this problem particularly, such a method that can propose a way to split a graph into pieces in such a way that implementations can benefit of that is block-cut decomposition. The idea behind this decomposition is to split the graph into biconnected components (blocks). It is worth noting that each edge of the graph belongs to a single block. However, any two bicomponents can share a vertex (cut). Considering blocks and cuts as vertices of a graph we can form a so-called block-cut tree, where block vertex is connected to a cut vertex if and only if the corresponding component contains corresponding vertex.

The advantage of performing this decomposition is that now, any method for recognising outer  $k$ -planar graphs can be applied independently to each component, and results can be combined easily afterwards. Indeed, if there is a component that does not admit an outer  $k$ -planar drawing then neither does the whole graph. Furthermore, if all of them admit it, then such drawings can be glued together by combining duplicated of each cut vertex into a single one. It is worth noting that this combining does not add any edge crossings, as both components are located in the outer face of each other. Moreover, all vertices stay on the outer face of the graph during this process, as no new faces are created, as the block-cut tree does not contain loops. Thus, the resulting drawing is outer  $k$ -planar drawing of a graph, and it exists if and only if such a drawing exists for each biconnected component of this graph.

In this work we implemented this decomposition using `BICONNECTED_COMPONENTS`<sup>1</sup> function from Boost `[boost]`. This function maps each edge of a graph to its bicomponent number and provides a list of cut vertices. Afterwards we created a copy of each block as a separate graph and mappings to translate new *local* vertices in original ones. Finally, we constructed a supergraph that represents the structure of block-cut

<sup>1</sup>[https://www.boost.org/doc/libs/1\\_87\\_0/libs/graph/doc/biconnected\\_components.html](https://www.boost.org/doc/libs/1_87_0/libs/graph/doc/biconnected_components.html)

tree and references a copied block and a mapping or a cut vertex in each node of this tree.

To construct a resulting drawing of a graph we walk through the block-cut tree using depth-first search storing the predecessor for each node upon discovery. Also, each time block vertex is discovered we use one of the methods described in other sections to check whether the component is outer  $k$ -planar and get the drawing if it is. Afterwards we glue this new drawing to already existing one by combining the corresponding cut vertex as described before. If the considered block is the first one, its drawing is simply copied into a sequence that stores the final one. Otherwise, the block necessarily has a predecessor. Due to a structure of a tree it is a cut node, that corresponds to a vertex that is shared with some other block already considered and added to a final drawing. As a result, we can find a corresponding cut vertex in both global and local drawings. As drawing is a sequence of vertices in a cyclic order, we can rotate it in such a way, that this cut vertex is located as a first item in a local drawing. Finally, we insert the local drawing starting from the second element into the global one immediately after cut vertex.

Alexander, maybe we should not exclude the first element from the local drawing but include it at the end of it, so for two touching loops  $ABC$ , and  $BDE$  we would have the following order  $ABDEBC$  rather than  $ABDEC$  or  $ADEBC$  as vertex  $B$  indeed occurs twice on the boundary?

## Chapter 5

# Experiments and Results

## Chapter 6

# Conclusions

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