Ukrainian Catholic University

Bachelor Thesis

Practical Aspects of Recognising Outer k-Planar Graphs

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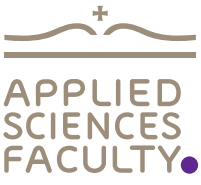
A thesis submitted in fulfillment of the requirements for the degree of Bachelor of Science

in the

Department of Computer Sciences and Information Technologies
Faculty of Applied Sciences



Lviv 2025
Saturday 26th April, 2025 22:10

UKRAINIAN CATHOLIC UNIVERSITY

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by Ivan Shevchenko

Abstract

The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too...

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor...

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Introduction

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Related Work

Already in the 1980s, researchers in the field of graph drawing acknowledged the importance of reducing the edge crossings for improving visualisation clarity [2]. The suspicion that a drawing with fewer edge crossings was easier to comprehend was later confirmed by several experimental studies [19]. These studies showed that minimising the crossings in graph representations significantly improves the ability of humans to interpret the structure, particularly when dealing with complex or large graphs.

The ideal form of crossing minimising drawings – planar ones, has been focused on by the research community for a long time, with the first of linear-time algorithms for recognising planar graphs presented already in 1974 [13]. However, requiring the drawing to be completely crossing free imposes severe limitations on an underlying graph. While providing a clean structure, these restrictions are often too constraining for many real-world graphs, especially large ones. This has led to a growing interest in exploring graphs close to being planar; see the survey by Didimo et al. [7]. Such graphs allow a limited number of crossings while still retaining some of the beneficial structural properties of planar graphs.

2.1 Difficulty of dealing with beyond-planar graphs

Most relaxations of strict planarity dramatically increase the complexity of recognising such graphs. So, the general problem of minimising edge crossings in a graph drawing was known to be computationally intractable already in 1983 when Garey and Johnson [9] demonstrated that the Crossing Number problem, where the task is to check whether a given graph can be drawn with at most k crossings, is NP-hard. Their proof relies on a reduction from the Optimal Linear Arrangement problem, which is known to be NP-hard.

Minimising the number of local crossings is also hard. Korzhik and Mohar [17] showed that testing 1-planarity, that is, recognising whether a graph can be drawn with at most one crossing per edge, is NP-hard. Later, Cabello and Mohar [4] showed that testing 1-planarity is NP-hard even for near-planar graphs, that is, graphs that can be obtained from planar graphs by adding a single edge.

Given the complexity of recognising k-planarity, researchers considered exploring more restrictive settings, hoping that imposed limitations could simplify the recognition. One of the considered is a circular setting, requiring the vertices to be placed on a circle and the edges to be drawn as straight lines. This gives rise to the circular local crossing number of a graph, which we study in this thesis.

2.2 Efficient recognition of some outer k-planar graph

Although the recognition problem for outer k-planar graphs is NP-hard if k is a part of the input, efficient algorithms have been developed for any constant value of k.

For $k = 0$, the recognition task simplifies to an outerplanarity test. Recognition can be accomplished by augmenting the graph with a new vertex connected to all original vertices and testing whether the resulting graph is planar. An alternative approach, described in [20], introduces the concept of 2-reducible graphs, which are totally disconnected or can be made totally disconnected by repeated deletion of edges adjacent to a vertex with a degree at most two. The proposed outerplanarity test is based on an algorithm for testing 2-reducibility.

In the case of $k = 1$, two research groups independently presented linear-time algorithms [1, 11]. Both algorithms use the SPQR decomposition of a graph for the test. Notably, the latter solution extends the graph to a maximal outer 1-planar configuration if such a drawing exists, unlike the former, which employs a bottom-up strategy which does not require any transformations of the original graph.

Considering a special case of this problem, Hong and Nagamochi [12] proposed a linear-time algorithm for recognising full outer 2-planar graphs. An outer k-planar drawing is *full* if no crossings lie on the boundary of the outer face. Later, Chaplick et al. [5] extended their result by introducing an algorithm for recognising full outer k-planar graphs for every k. Their algorithm runs in $O(f(k) \cdot n)$ time, where f is a computable function.

For the general version of the problem and values of $k > 1$, no research was conducted until recently when a group of researchers proposed an algorithm for the general case, which we discuss in the Section 2.3.

2.3 Recognizing general outer k-planar graphs

For a given outer k-planar drawing of a graph, Firman et al. [8] proposed a method for constructing a triangulation with the property that each edge of the triangulation is crossed by at most k edges of the graph drawing. Since the edges of the triangulation do not necessarily belong to the original graph, they are termed *links* to distinguish them from the original graph edges. The construction is done recursively.

Initially, the algorithm selects an edge on the outer face and labels it as the active link. At each recursive step, the active link partitions the graph into two regions: a left part already triangulated and a right part not yet explored. The objective of each step is to triangulate the right portion. To achieve this, a splitting vertex is chosen within the right region, dividing it into two smaller subregions. The splitting vertex is selected so that the two new links, connecting the split vertex with the endpoints of the active link, are each intersected by at most k edges, which allows including them into the triangulation. The algorithm then recurses, treating each of these newly formed links as the active one.

Later, Kobayashi et al. [16] extended this approach to address the recognition problem for outer k-planar graphs. In contrast to the triangulation task, where the drawing is provided, the recognition problem requires determining whether a given graph admits an outer k-planar drawing. Although the core idea remains analogous, the absence of a drawing requires the exploration of all possible configurations. Here, each recursive step verifies whether the unexplored right portion of the graph can be drawn as an outer k-planar graph that is compatible with the left part.

Moreover, instead of relying on recursion, the method utilises a dynamic programming approach. This framework combines solutions of smaller subproblems retrieved

from a table to solve larger ones. To populate this table, the algorithm iterates over all possible configurations corresponding to different recursion steps. Several parameters characterise each such configuration. The first parameter is the active link – a pair of vertices that divides the graph into a left and a right region. The second parameter is a set of vertices in the right part, which is not uniquely determined as in the triangulation case. Additionally, the configuration depends on the order in which edges intersect the active link and the number of intersections on the right side for each one of them. These parameters are used to ensure that the drawing of the right part is compatible with the left part. For each configuration, the algorithm considers all possible ways to split the right region further. For each of them, the method checks whether these splits are compatible with each other and the left part of the drawing.

Using the restriction on the number of edges crossing each link, the authors demonstrated that for a fixed k, the number of possible right subgraphs grows only polynomially with respect to the size of the graph. They proceeded by arguing that the overall time complexity of the algorithm is $2^{O(k \log k)} n^{3k+O(1)}$, showing that the algorithm is efficient for any fixed parameter k.

2.4 Our contribution

A common drawback of the methods described above is the lack of practical validation. Although these algorithms have been analysed and discussed in a theoretical context, they have not been implemented or empirically tested. We address this gap by implementing the most recent recognition algorithm and introducing two alternative approaches based on Integer Linear Programming (ILP) and Satisfiability (SAT) formulations. While it is NP-hard to solve the general integer linear program or to find a satisfying truth assignment for general Boolean formulas, there are very advanced solvers for such formulations that could allow us to find exact solutions for small- to medium-sized instances within an acceptable amount of time. Then, we evaluate the performance and efficiency of these methods, demonstrating their practical applicability and limitations.

Chapter 3

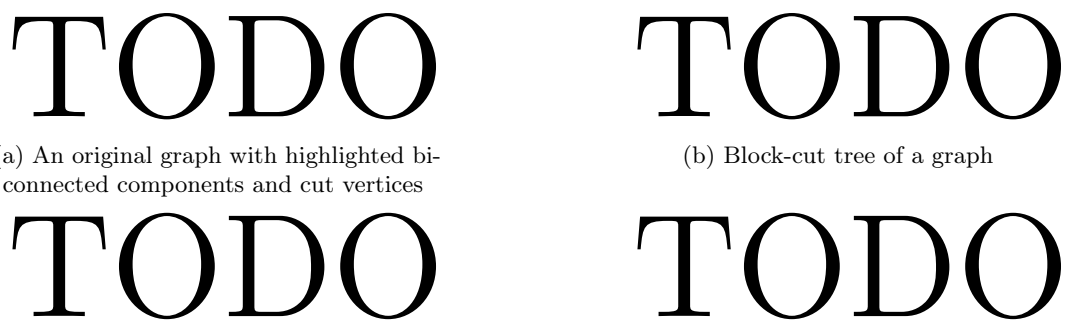
Algorithms

This chapter will discuss the methods this work considers to recognise the outer k-planar graphs. Besides recognition, these methods provide the outer k-planar drawings of the given graphs if possible. We represent the drawing as a sequence of vertices in the circular order in which they appear on the boundary of the outer face.

For operations with graphs, we use C++ Boost Graph Library [14]. As a graph class, we use adjacency_list as for all methods described below, we require both VertexList and EdgeList concepts to be able to iterate over both vertices and edges. We prefer this class to an adjacency_matrix as, according to the documentation[1], it trades memory consumption and speed of graph traversal for the speed of edge insertion and deletion, and neither of these operations is used for algorithms described below.

Explicily specify the goals of the work: implement three algorithms provide an interface.

3.1 Bicomponent decomposition



(a) An original graph with highlighted biconnected components and cut vertices

(b) Block-cut tree of a graph

(c) Outer k-planar drawing of each component

(d) Outer k-planar drawing of the original graph

Figure 3.1: An example of bicomponent decomposition

For complex problems, a decomposition into smaller subproblems often leads to a significant increase in performance. In our context of recognising outer k-planar graphs, an effective strategy to do so is to partition the graph into subgraphs in such a manner that allows us to process each part independently by the recognition algorithm. One of the plausible ways to accomplish this is to split the graph into biconnected components using block-cut decomposition, as shown in Figure 3.1a. It is worth noting that each edge of the graph belongs to a single biconnected component, referred to as a block. However, any two bicomponents may share a vertex, referred to as a cut vertex. Considering blocks and cut vertices as graph nodes, we can construct

[1] https://www.boost.org/doc/libs/1_88_0/libs/graph/doc/adjacency_matrix.html

a so-called block-cut tree, wherein a block node is connected to a cut node if and only if the corresponding biconnected component contains a corresponding cut vertex, see Figure 3.1b.

Due to the nature of bi-connectedness, after getting outer k-planar drawings of all biconnected components separately, we can combine them easily into an outer k-planar drawing of the whole graph. To be more specific, if some component does not admit an outer k-planar drawing, neither does the whole graph. Otherwise, if for all of them such a drawing exists, they can be merged by combining duplicates of each cut vertex see Figure 3.1c and Figure 3.1d. This merging process does not introduce any additional edge crossings since both components are located on the outer face of each other. Moreover, as no new faces are created during this process (due to the acyclic structure of the block-cut tree), every vertex remains on the outer face of the graph during this process. Consequently, the resulting drawing of an original graph is outer k-planar, and it exists if and only if each biconnected component of the graph admits such a drawing.

In this work, we implemented this decomposition using the method biconnected_components[2] from the Boost Graph Library [14]. This function assigns an index of the bicomponent to each edge to which it belongs. Additionally, it provides a list of cut vertices. Afterwards, we copy each block as an independent graph and create mappings to translate new *local* vertices back to their original identifiers. Finally, we construct a supergraph representing the structure of a block-cut tree wherein each node references a copied block alongside corresponding mapping or a cut vertex.

Do we really need this here? I have mentioned Boost in the introduction to the chapter

To construct a drawing of the whole graph, after performing the decomposition, we perform a depth-first search on the block-cut tree, recording the predecessor for each node upon discovery. Additionally, each time a block vertex is discovered, we use one of the methods described in Sections 3.2, 3.3 and 3.5 to check whether the component admits an outer k-planar drawing and obtain it if so. Afterwards, we merge the new drawing with the already existing one by combining the common cut vertex if such exists. To be more specific, if the considered block is the first encountered one, its drawing is directly copied into a sequence that will form the final drawing. Otherwise, the block necessarily has a predecessor. Due to the structure of a tree, it is a cut node corresponding to a vertex that is shared with some other block. Due to how we traverse the tree, that other block has already been considered and thus added to a final drawing. As a result, the corresponding cut vertex is present in both global and local drawings. Since each drawing is represented as a cyclic sequence of vertices, we can rotate the local one so that the corresponding cut vertex appears as the first one in a sequence. Finally, we insert the local drawing starting from the second element into the global one immediately after the corresponding cut vertex.

3.2 ILP-based algorithm

The problem of recognising the outer k-planar graphs is NP-hard. This justifies the use of exponential-time methods to approach our problem. By doing so, we benefit from decade-long studies conducted for these problems that resulted in the development of extremely optimised algorithms for their solving. One of them is an Integer Linear Programming problem (ILP). This problem asks to find a vector $\mathbf{x}$ that optimises the linear objective $\mathbf{c}^T\mathbf{x}$ subject to specific constraints $\mathbf{Ax} \leqslant \mathbf{b}$. Additionally, some variables in the ILP problem are restricted to integer values. Since researchers have extensively studied the problem and developed efficient solvers, we decided to use

[2] https://www.boost.org/doc/libs/1_87_0/libs/graph/doc/biconnected_components.html

their results to build an algorithm for recognising outer k-planar graphs. This section discusses the process we use to encode the recognition task as the ILP problem.

As an implementation of ILP solver, we used Gurobi Optimizer [10] under the free academic licence. We chose it due to its outstanding performance denonstraded by Luppold et al. [18].

To encode a recognition problem to an ILP, we have to represent its structure using variables and constraints. We start with a graph drawing, which is represented, as described above, as a sequence of vertices. For the ILP, we can encode it using the so-called "ordering variables", which indicate a relative order of two vertices. Specifically, for every pair of vertices u and v, we create a binary variable $a_{u,v}$ introducing Equation (3.18) as a constraint for the ILP problem. We interpret the value 1 as an indication of vertex u being located before vertex v and the value 0 as an indication of either v being located before u or u and v being the same vertex.

To ensure that these variables encode a valid sequence, we also have to enforce the transitivity. That is, for every ordered pair of distinct vertices u and w, and every other vertex v, if $a_{u,v} = 1$ and $a_{v,w} = 1$, meaning u is located before v and v is located before w, then u must be located before w, so the following should hold $a_{u,w} = 1$. Including also the implication for the reversed order, we get:

$$a_{u,v} = 1 \land a_{v,w} = 1 \implies a_{u,w} = 1 \tag{3.1}$$

$$a_{u,v} = 0 \land a_{v,w} = 0 \implies a_{u,w} = 0 \tag{3.2}$$

If we instead consider a pair w, u and the same vertex v, the constraints would look like follows:

$$a_{w,v} = 1 \land a_{v,u} = 1 \implies a_{w,u} = 1 \tag{3.3}$$

$$a_{w,v} = 0 \land a_{v,u} = 0 \implies a_{w,u} = 0 \tag{3.4}$$

Note that for any distinct vertices x and y the equality $a_{x,y} = 1 - a_{y,x}$ always holds, thus Equations (3.1) and (3.4) alike Equations (3.2) and (3.3) are equivalent. Consequently, it is enough to ensure only the first constraint as long as we do it for every ordered pair of vertices. Considering that the variables are binary, to limit $a_{u,w}$ to 1 it is enough to impose a constraint $a_{u,w} \geqslant \epsilon$ for any $\epsilon \in (0; 1]$. In a constraint for ILP, this ϵ must be represented as a linear function of $a_{u,v}$ and $a_{v,w}$. The values of this function must lie in the half-interval $(0; 1]$ if and only if both binary variables are 1. Using an expression $a_{u,v} + a_{v,w} - 1$ for this leads to Equation (3.7) in the encoded ILP formulation below.

The next step of the algorithm is to encode the intersections. To represent them, for every unordered pair of edges, uv and st, we introduce a binary variable $c_{uv,st}$, hence Equation (3.17) representing this constraint. The endpoints of the edges can be arranged in 24 different ways, among which only eight result in an intersection, as demonstrated in Figure 3.2. To encode this, we must ensure that the value of the variable $c_{uv,st}$ equals 1 if the corresponding "ordering variables" indicate one of these eight arrangements[3]. For example, considering the arrangement in Figure 3.2b, we have to limit the value of $c_{uv,st}$ to 1 if the endpoints are arranged in the order $usvt$. This order of vertices is implied by the model if and only if each equivalence out of

[3]As the objective of the program is to minimise the number of crossings, we do not constraint $c_{uv,st}$ to 0 when uv and st do not cross, leaving this to the optimiser. Doing so, we simplify the problem by reducing the number of constraints for every pair of edges from 24 to 8.

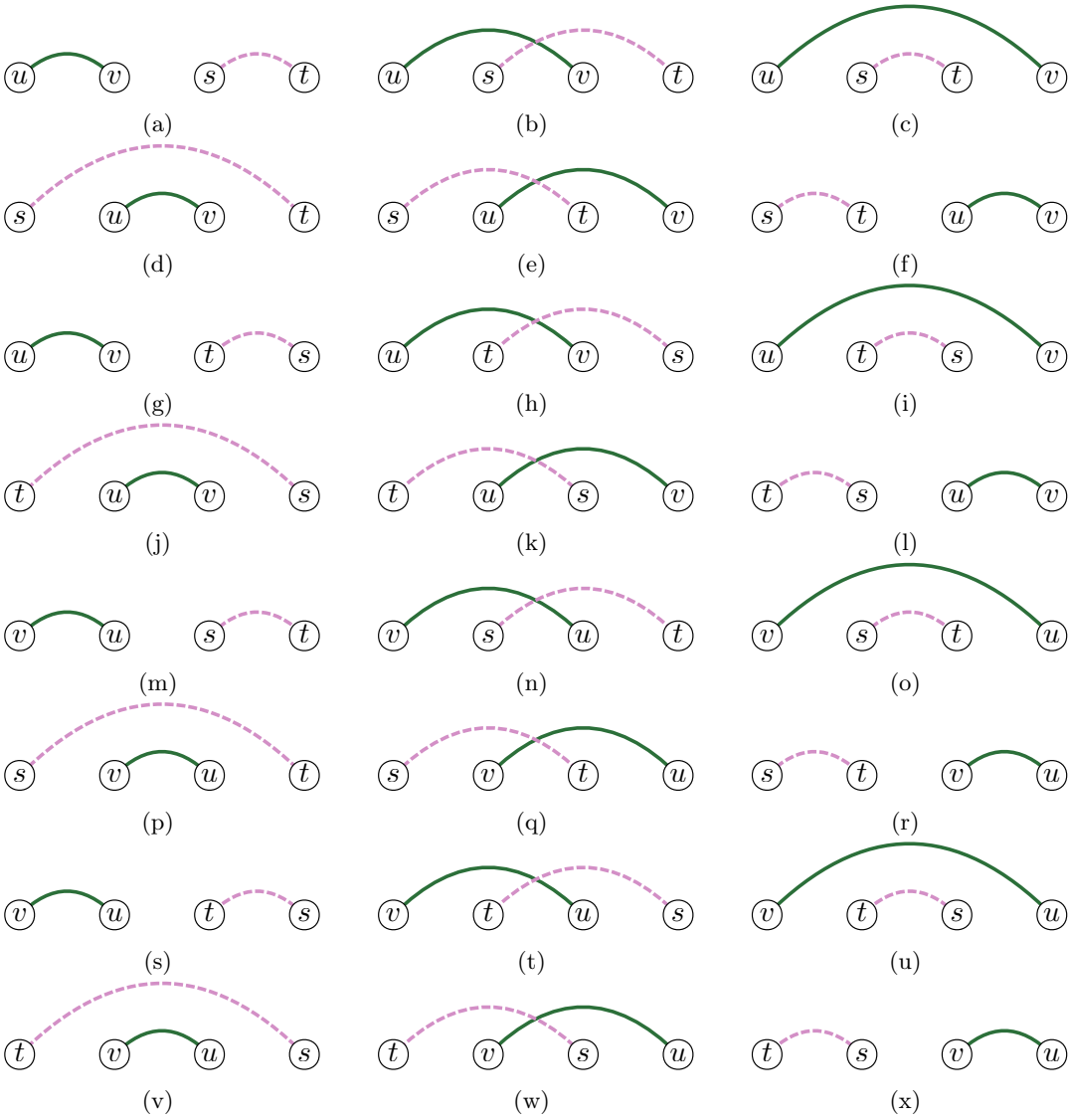


Figure 3.2: All 24 possible arrangements of the endpoints of two edges, only 8 of which result in an intersection; see the central column.

$a_{u,s} = 1$, $a_{s,v} = 1$ and $a_{v,t} = 1$ holds. Thus, we can represent this limitation as follows:

$$a_{u,s} = 1 \wedge a_{s,v} = 1 \wedge a_{v,t} = 1 \Longrightarrow c_{uv,st} = 1$$

To transform this into a constraint for an ILP formulation, we can use the same logic as for encoding Equation (3.1), getting as a result Equation (3.8). Equations (3.9) to (3.15) are constructed analogously for the other seven intersecting arrangements.

Lastly, the algorithm has to encode each edge's crossing number and minimise the maximal value out of them. The crossing number of each edge can be easily represented as a sum of the corresponding "crossing variables":

$$cr_{e_1} \leqslant \sum_{e_2 \in E(G)} c_{e1,e2} \tag{3.5}$$

However, as the maximum is not a linear function, constructing the objective function out of the per-edge crossing numbers is not as simple. To get around this limitation, we have to introduce a new continuous variable k, which represents the crossing number

of the whole graph G. To ensure that, we have to bound k from below by the crossing number of each edge: $k \geqslant cr_{e_1} \forall e_1 \in E(G)$. Combining this with Equation (3.5), we get Equation (3.16) as a constraint for the ILP problem. As a result, minimising for k would give the desired result.

Combining everything together, we get the following formulation of the ILP problem:

$$\textbf{minimize} \quad k \tag{3.6}$$

$$\begin{aligned}
\textbf{subject to} \quad a_{u,v} &\geqslant a_{u,v} + a_{v,w} - 1, && \forall u, v, w \in V(G) && (3.7)\\
c_{uv,st} &\geqslant a_{u,s} + a_{s,v} + a_{v,t} - 2, && \forall uv, st \in E(G) && (3.8)\\
c_{uv,st} &\geqslant a_{u,t} + a_{t,v} + a_{v,s} - 2, && \forall uv, st \in E(G) && (3.9)\\
c_{uv,st} &\geqslant a_{v,s} + a_{s,u} + a_{u,t} - 2, && \forall uv, st \in E(G) && (3.10)\\
c_{uv,st} &\geqslant a_{v,t} + a_{t,u} + a_{u,s} - 2, && \forall uv, st \in E(G) && (3.11)\\
c_{uv,st} &\geqslant a_{s,u} + a_{u,t} + a_{t,v} - 2, && \forall uv, st \in E(G) && (3.12)\\
c_{uv,st} &\geqslant a_{t,u} + a_{u,s} + a_{s,v} - 2, && \forall uv, st \in E(G) && (3.13)\\
c_{uv,st} &\geqslant a_{s,v} + a_{v,t} + a_{t,u} - 2, && \forall uv, st \in E(G) && (3.14)\\
c_{uv,st} &\geqslant a_{t,v} + a_{v,s} + a_{s,u} - 2, && \forall uv, st \in E(G) && (3.15)\\
k &\geqslant \sum_{st \in E(G)} c_{uv,st}, && \forall uv \in E(G) && (3.16)\\
c_{uv,st} &\in \{0, 1\}, && \forall uv, st \in E(G) && (3.17)\\
a_{u,v} &\in \{0, 1\}, && \forall u, v \in V(G) && (3.18)
\end{aligned}$$

After encoding the problem as an ILP problem, as we described above, we run the solver. After optimisation, it returns assigned values for each variable used in the program. The variables of interest for us are "ordering variables" $a_{u,v}$ and k. The latter one indicates the minimal possible crossing number that is reported. The former one we use to reconstruct an outer k-planar drawing of the original graph. As the desired drawing is a sequence of vertices, we must order them. To do so, we can use the values of "ordering variables" to define the strict total order relation. We say that for two vertices u and v $u < v$ if and only if $a_{u,v}$ is assigned to 1. Due to the construction of these variables and transitivity constraints represented with Equation (3.7), this order satisfies all requirements of the strict total order. Thus, it can sort the vertices, resulting in an outer k-planar drawing.

3.3 SAT-based algorithm

Another example of a thoroughly studied problem with an exponential-time solver is a Boolean Satisfiability problem (SAT). As an input, the problem gets a boolean formula in a conjunctive normal form (CNF). For the output, it asks for such an assignment of logic values True and False to the variables used in the input, such that the expression evaluates to True. This section discusses our process of encoding the recognition task into the SAT problem. To do so, we represent our problem as a boolean expression, which is satisfied if and only if the graph admits an outer k-planar drawing. As an implementation of SAT solver, we used kissat [3, 15].

Explain why we have chosen kissat

To encode the drawing, we use the "ordering variables" $a_{u,v}$ we used in the ILP-based algorithm described in the Section 3.2 interpreting the value 1 as True and 0 as False. To encode the transitivity constraint, we follow the same logic, ending up with the same Equation (3.1). In terms of boolean algebra, this can be encoded as

follows:

$$a_{u,v} \wedge a_{v,w} \rightarrow a_{u,w}$$

To transform this into CNF, we expand the implication and apply De Morgan's law, getting the first set of clauses in the SAT representation:

$$\begin{aligned} a_{u,v} \wedge a_{v,w} \rightarrow a_{u,w} &\equiv \overline{(a_{u,v} \wedge a_{v,w})} \vee a_{u,w} \\ &\equiv \overline{a_{u,v}} \vee \overline{a_{v,w}} \vee a_{u,w} \end{aligned} \tag{3.19}$$

To encode the edges' intersections, we also reuse "crossing variables" $c_{uv,st}$ from the ILP-based algorithm. Similarly, we ensure that the variable is set to True if the endpoints of the corresponding edges are arranged in one of eight crossing patterns (see Figure 3.2). This leads to eight sets of clauses, each of which contains ones representing one of these arrangements for all "crossing variables". For example, for the arrangement from Figure 3.2b the constraint for the variable $c_{uv,st}$ can be encoded as follows:

$$a_{u,s} \wedge a_{s,v} \wedge a_{v,t} \rightarrow c_{uv,st}$$

To get a clause out of it, we expand the implication and apply De Morgan's law:

$$\begin{aligned} a_{u,s} \wedge a_{s,v} \wedge a_{v,t} \rightarrow c_{uv,st} &\equiv \overline{a_{u,s} \wedge a_{s,v} \wedge a_{v,t}} \vee c_{uv,st} \\ &\equiv \overline{a_{u,s}} \vee \overline{a_{s,v}} \vee \overline{a_{v,t}} \vee c_{uv,st} \end{aligned} \tag{3.20}$$

The last thing to encode is the limit of k intersections per edge. Unlike the ILP-based algorithm, we cannot add the corresponding variable. To impose this restriction, we ensure that no edges are crossed at least $k+1$ times. For that, for each edge e_0 and every set of $k+1$ edges $E = \{e_1, e_2, \ldots, e_{k+1}\}$ we add a following clause to the boolean formula:

$$\overline{c_{e_0,e_1}} \vee \overline{c_{e_0,e_2}} \vee \cdots \vee \overline{c_{e_0,e_{k+1}}} \tag{3.21}$$

which evaluates to True if and only if at least one variable is set to False. By inserting this clause for every possible set E, we ensure that no $k+1$ edges cross the same edge. Thus, if the resulting boolean expression is satisfiable by some realisation of the variables, the values of "ordering variables" from this realisation would indicate an outer k-planar drawing.

Lastly we combine clauses from Equation (3.19) for all triplets of vertices u, v and w, with clauses from Equation (3.20) for all pairs of edges uv and st, and with clauses from Equation (3.21) for all edges e_0 and all sets E of $k+1$ edges. We combine the clauses using logical and operator ($\wedge$), as every single clause must be satisfied to graph to be outer k-planar. Then, we run the SAT solver using the resulting boolean expression as an input. If the solver fails to find a satisfiable instance, the algorithm halts, indicating that the graph is not outer k-planar. Otherwise, the algorithm halts returning the desired graph drawing. To reconstruct an outer k-planar drawing, similar to the ILP approach, we sort the vertices using the values of the "ordering variables" assigned by the solver to specify the order.

Unfortunately, unlike the algorithm based on ILP, this one does not solve the optimisation problem of finding the minimal possible k. The resulting algorithm solves the decision problem – whether the graph admits an outer k-planar drawing or not. To transform this into an optimisation, we incrementally check each integer,

starting from 0, until the boolean expression becomes satisfiable.

3.4 Optimisations for ILP- and SAT-based algorithms

In both ILP- and SAT-based algorithms, for encoding the intersection of two edges uv and st, we used binary variable $c_{uv,st}$. However, despite 24 possible arrangements of edges' endpoints (see Figure 3.2), we imposed only 8 constraints in each algorithm. Each of these constraints covers one of the arrangements that result in the intersection, leaving the rest 16 for the solver to optimise. This section discusses two ways we considered helping the solver optimise these variables.

The first optimisation is a simple extension of the same logic we applied for the intersecting arrangements. To ensure the correct values of "crossing variables", we can impose all 24 constraints on each one. The first eight we described in the corresponding Sections 3.2 and 3.3. The other 16 we construct in a similar way. For example, to construct a constraint for an arrangement in Figure 3.2a, we have to encode the following:

$$a_{u,v} = 1 \wedge a_{v,s} = 1 \wedge a_{s,t} = 1 \Longrightarrow c_{uv,st} = 0$$

To represent it as a linear constraint for an ILP-based algorithm, we have to limit $c_{uv,st}$ from above if all three variables are 1. We can do so with an equation $3 - a_{u,v} - a_{v,s} - a_{s,t}$, resulting in a constraint:

$$c_{uv,st} \leqslant 3 - a_{u,v} - a_{v,s} - a_{s,t}, \quad \forall uv, st \in E(G)$$

For a SAT-based algorithm, we can transform the implication in the similar way we did it for the clause from Equation (3.20):

$$\begin{aligned} a_{u,v} \wedge a_{v,s} \wedge a_{s,t} \rightarrow \overline{c_{uv,st}} &\equiv \overline{a_{u,v} \wedge a_{v,s} \wedge a_{s,t}} \vee \overline{c_{uv,st}} \\ &\equiv \overline{a_{u,v}} \vee \overline{a_{v,s}} \vee \overline{a_{s,t}} \vee \overline{c_{uv,st}} \end{aligned}$$

By repeating these constraints for all non-crossing arrangements, we get the algorithms with exact crossing variables.

Another optimisation that we considered can be implemented only for an ILP-based algorithm. The primary problem caused by this inaccuracy in ILP formulation is that the variables' influence on the objective function is not direct. The "crossing variables" constrain another variable k, which in turn affects the objective function, so it might be hard for an optimiser to estimate the effect of "crossing variables" accurately. With this optimisation, we help the solver by introducing a small direct influence on an objective function bypassing the intermediate variable k. To do so, we include an extra term in the objective function from Equation (3.6): $\frac{\sum c_{e_1,e_2}}{|E|^2}$. By using $|E|^2$ as a dominator in the fraction, we ensure that the value of the inserted term never exceeds 1 so that the optimiser would always prioritise decreasing k over this term. As a result, the optimised objective function for the ILP problem is:

$$k + \frac{\sum_{e_1,e_2 \in E(G)} c_{e_1,e_2}}{|E|^2}$$

3.5 DP algorithm

The last algorithm we considered was introduced by Kobayashi et al. [16]. Unlike previously discussed ones, this algorithm was explicitly designed to solve the recognition problem. This method uses the approach of dynamic programming, where the solution for a problem is built based on solutions of similar but smaller problems. Thus, the final drawing of the graph is built incrementally each time for a larger part of the original graph.

TODO (a) TODO (b)

Figure 3.3: An example of (a) a graph $G_{uv,R_{uv}}$ representing a configuration and (b) a corresponding triangle. *Something similar to[16, Figure 4]*

The whole process can be divided into steps. Each one of them can be parameterised by three parameters. The first is a pair of vertices u and v that split a graph into two parts, denoted as a link. The next is a set R_{uv} of vertices lying to the right of the link. And lastly the set $E_{uv} = \{e_1, e_2, \ldots, e_l\}$ of l edges crossing the uv link from the right to the left side. To represent each step separately, we introduce a graph $G_{uv,R_{uv}}$ (see Figure 3.3a) which consists of vertices $\{u, v\} \cup R_{uv}$ alongside all connecting edges from an original graph G with inserted vertices $t_1, t_2, \ldots, t_l$ connected to corresponding vertices by edges $e_1, e_2, \ldots, e_l$. We call a configuration on each step *drawable* if exists an outer k-planar drawing of a corresponding graph $G_{uv,R_{uv}}$ which cyclic order contains $(u, t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(l)}, v)$ as a consecutive subsequence for some permutation τ. On each step, the algorithm finds all possible permutations for which the configuration is *drawable* and stores them in the lookup table.

In the implementation of the algorithm, we first construct an index of all possible configurations. It allows us to simply iterate through it later without searching for the next configuration. Since, on each step, we try to draw a configuration using two smaller *drawable* ones, we group them by the size of the right part, guaranteeing that all smaller *drawable* configurations are already discovered at any point of the process. Unfortunately, it is unfeasible to consider all possible configurations due to the sheer number of them. For a graph with n vertices, there are $\frac{n(n-1)}{2}$ links with 2^{n-2} possible right sides each. As the link together with the right side uniquely determines the edges that cross the link, totally there are $n(n-1) \cdot 2^{n-3}$ possible configurations.

To significantly reduce the search space, the authors used the result of Firman et al. They showed [8] that there is a vertex w from the right side of any *drawable* configuration that splits it into two smaller *narrow* configurations for which $|E_{uw}| \leqslant k$ and $|E_{vw}| \leqslant k$. By reversing their argument, we get that $G_{uv,R_{uv}}$ admits an outer k-planar drawing if and only if we can combine it from two *narrow drawable* configurations. This allows us to limit the considered configurations only to *narrow* once, reducing their number to $2^{O(k)}m^{k+O(1)}$ [16, Lemma 15] instances.

Despite this optimisation, there is still a massive number of configurations. To minimise the memory consumption and make it feasible, we represent the right sides as binary masks stored as 64-bit integers. This decision limits the current implementation to graphs with at most 64 vertices. However, considering the complexity of

the algorithm, we believe the graphs of bigger sizes would require an unreasonable amount of resources anyway[4].

To populate this index, we start by iterating over possible values for l[5]. For each choice of l and each link uv, we consider an augmented graph H obtained by removing u and v from the original graph G alongside all connected edges. Then, we select exactly l edges from H to cross the link uv. These edges further subdivide some connected components of H into connected subcomponents. Crucially, as these subcomponents do not contain edges that cross the link, each one of them must be located entirely on one side. Thus, finding all valid right sides for a given link means finding all valid black-white colourings of subcomponents, where white indicates belonging to the right and black to the left side. Consequently, each selected edge has to connect subcomponents of different colours, or in other words, the metagraph of H with subcomponents as vertices connected by selected edges has to be bipartite. After ensuring this holds, we construct all possible right sides for the selected link. As each connected bipartite graph can be coloured in exactly two ways, there are exactly 2^d possible right sides, where d is the number of connected components in H.

After constructing an index, we proceed to fill the lookup table. In there, for each configuration, we record all discovered sets of arrangements of E_{uv} that can appear in an outer k-planar drawing of $G_{uv,R_{uv}}$ grouped by the link uv and the right side R_{uv}. Each arrangement A_{uv} apart from a permutation τ of edges in E_{uv} also contains a map $f_{uv} : E_{uv} \to \mathbb{N}_+$ that matches each edge from E_{uv} with its number of intersections in the drawing of $G_{uv,R_{uv}}$.

I think I should use another letter for an arrangement

To fill the corresponding cell of the lookup table, we have to find all arrangements for which a specific configuration is *drawable*. We start by selecting a split vertex w that belongs to the right side R_{uv}. For each w, we iterate over all configurations with a link uw and a right side R_{uw} that is a subset of R_{uv}. Additionally, we also consider a complementary configuration with a link vw and right side $R_{vw} = R_{uv} \setminus (R_{uw} \cup \{w\})$. For each such pair of configurations, we iterate over all pairs of *drawable* arrangements A_{uw} and A_{vw} saved in the lookup table and search for all possible ways to combine them into an outer k-planar drawing of $G_{uv,R_{uv}}$.

There is only one way to glue drawings of two configurations together. However, to form a valid drawing of $G_{uv,R_{uv}}$, we also have to decide on the order of edges crossing the link uv represented by a permutation τ_{uv}. To ensure the correctness of the solution, we go through all possible ones. For each permutation, we check whether the resulting drawing is valid – each edge is crossed at most k times – and construct a mapping f_{uv} if so. To do that, we focus on an inner triangle (see Figure 3.3b on page 12) consisting of three vertices u, v and w, and all the edges crossing at least one of the links uv, uw and vw. Additionally, we include edge (u, v) if such exists. Crucially, to calculate all the intersections apart from the edges, we also need the order in which they enter the triangle. As sides of the triangle are exactly the links, this order is represented in corresponding permutations: τ_{uw} from A_{uw}, τ_{vw} from A_{vw} and τ_{uv} which is considered one by one. To represent the order in the triangle, we insert l_{uv} helper vertices between u and v, given that $l_{uv} := |E_{uv}|$. We treat them as endpoints of corresponding edges that enter the triangle by crossing the link uv. Similarly, we insert vertices along the links uw and vw. Importantly, each inserted vertex is an endpoint only for one edge, and along each link, they are ordered according to the corresponding permutations. By considering these vertices as edges' endpoints, we

[4]Potentially it is possible to develop a specified bitmask object which could handle any number of vertices by using multiple integers stored in an array.

[5]By iterating over this first, we ensure that it is easy to extend the index for $k + 1$ edges if the check for outer k-planarity is unsuccessful.

limit the view to the intersections created by the combination of two parts, ignoring those in R_{uw} and R_{vw}. So, to get the crossing number for each edge, apart from the ones we count in the triangle, we also have to add those accounted by f_{uw} and f_{vw}. If the crossing number of any edge exceeds k, we discard the permutation τ_{uv} and proceed to the next one. Otherwise, we extract the mapping f_{uv} and, together with the current permutation τ_{uv}, add it as a new arrangement to the lookup table.

If at any moment, the algorithm finds a *drawable* configuration with the link uv and right side $R_{uv} = V(G) \setminus \{u, v\}$, it halts indicating that G is outer k-planar. In this case, the graph $G_{uv,R_{uv}}$ is equivalent to the original graph G and admits an outer k-planar drawing. If, on the other hand, we reach the end of the index and do not find such a configuration, the algorithm halts, indicating that the graph is not outer k-planar.

In case of success, apart from a positive answer, we also have to return the graph drawing. There are multiple ways to accomplish this. One is to use a backtracking algorithm after finding the configuration to rediscover all smaller configurations with which the last one was built. Another approach is to store information on how each configuration was constructed alongside each arrangement. The latter trades the memory consumption required for additional information in each arrangement for the time required to perform backtracking. We used the second option in our implementation as it is much easier. Moreover, the bottleneck of the current implementation is running time and not memory. As additional information, we opted to store the drawing of the right side. To get this, we combine the drawings of two parts on each step, inserting the split vertex w between them. As a result, to get the drawing of G having the *drawable* configuration $G_{uv,R_{uv}}$, it is enough to add vertex u at the start and vertex v at the end of the drawing stored alongside the arrangement A_{uv} in the lookup table.

Similar to the SAT-based algorithm from the Section 3.3, this method only tests whether the graph admits an outer k-planar drawing or not for a fixed k, so to find the minimal possible crossing number, we have to check each value incrementally.

3.6 Interface of the implementation

Graphviz format, neato layout engine, algorithm specification, result format

Chapter 4

Experiments and Results

This chapter discusses our experiments that evaluate the performance of the methods described above.

4.1 Results

Demonstration of the implementation

4.2 Data

To evaluate the implementation, we require a set of graphs which will be provided as input to the algorithm. To acquire them, we used the online database The House of Graphs [6] that contains interesting graphs. We decided to conduct experiments on these graphs, as they are most likely to be typical targets of the algorithms. In this work, we experimented with a small subset of all those graphs.

The task of recognising outer k-planar graphs is NP-hard. Consequently, the resources required by all implemented algorithms grow exponentially with the input size. To be able to perform the experiments, we had to limit the size of the graphs. We have done this by picking only ones with at most 10 vertices. This constraint leaves plenty of graphs to experiment with while significantly limiting the requirement for resources. Also, as we designed the implementation only for connected graphs, we filtered out ones with multiple components.

This query resulted in 2007 different graphs. Among them 1326 are biconnected and 681 are not.

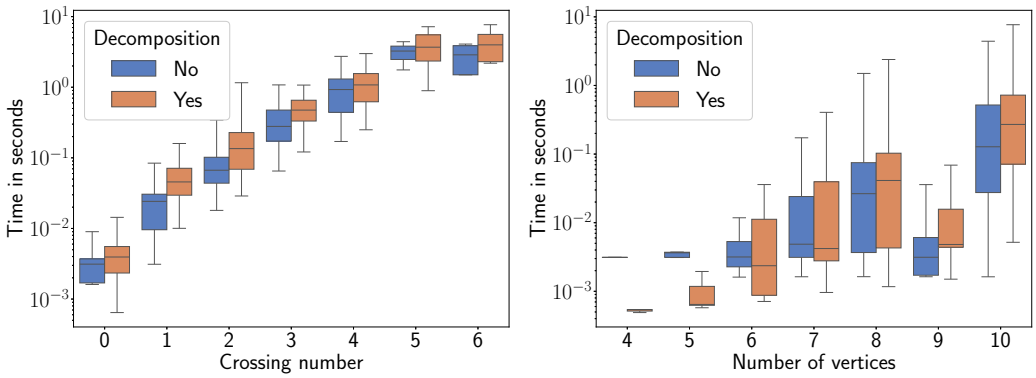
4.3 Experiment setup

All experiments described below were conducted on a virtual cloud server provided by Amazon Web Services. The hardware available was limited to a single core of an AWS Graviton4 Processor and 2GiB of random access memory. As an operating system, we used Linux.

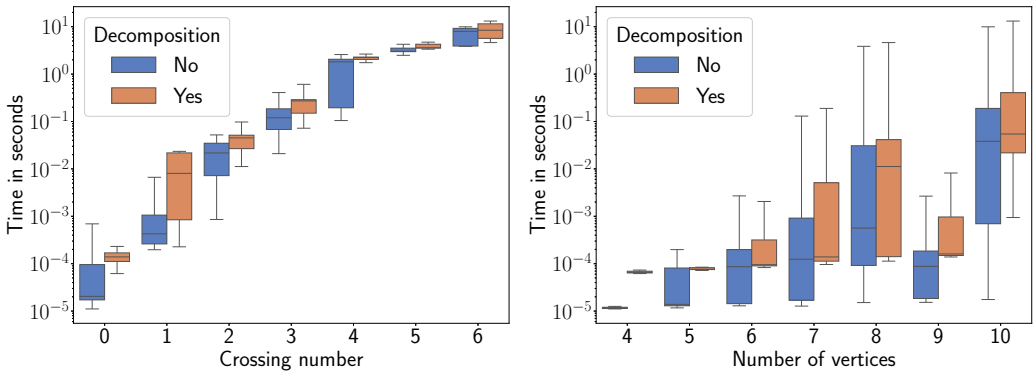
For each experiment, we chose a set of graphs, a set of methods and a set of configurations. We construct all possible triplets of these parameters. For each one of them, we find the smallest k for which the given graph is outer k-planar using the specified method in a given configuration. Apart from recording the k itself, we also track the time required by the corresponding algorithm, which does not include the time required to start up and parse the graph, as it is part of any algorithm. Unfortunately, some methods require an unreasonable amount of time to calculate the result for some graphs. To mitigate this, we limit the execution time to 10 minutes for each triplet, indicating the outcome in the results.

We save the results of each experiment in the CSV format. For each entry, we specify the Graphviz representation of the graph, the configuration, the method, and the execution results. The latter includes crossing number k, the time required to find it and whether the algorithm succeeded or not.

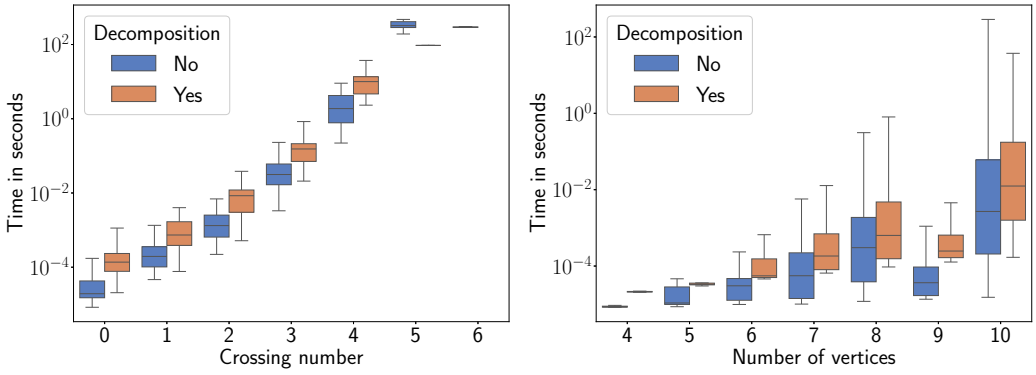
4.4 Biconnected decomposition



(a) ILP-based algorithm



(b) SAT-based algorithm



(c) DP-based algorithm

Figure 4.1: Results of the experiment, demonstrating the influence of biconnected decomposition on the running time of ILP- , SAT- and DP-based algorithms.

The first experiment we considered was comparing the performance of the algorithms with and without bicomponent decomposition. Here, we used only non-biconnected graphs from the dataset to show the difference between the two configurations. We ran this for all three methods. The results are presented in the Figure 4.1.

The results prove that using decomposition indeed boosts performance for almost all methods and graphs. The only exceptions are small graphs for the ILP, for which the cost of initialising multiple environments outweighs the cost of decoupling the problem.

As we demonstrated in this experiment, the non-biconnectivity of the graphs artificially decreases the complexity of the recognition task, as each one of them requires multiple times fewer resources compared to equally sized biconnected graphs. Thus, in the following experiments, we consider only biconnected graphs to exclude this source of noise from the results.

4.5 Comparison of the algorithms

In this experiment, we compared the performance of the algorithms. We grouped results by crossing numbers and the algorithm used for solving and displayed in Figure 4.2. Due to the complexity of the problem, some runs ran out of allocated resources. So, to display the results, we used only the measurements from runs that successfully found the minimal crossing number. As a result, starting from $k = 6$, boxes for SAT and DP depict fewer runs compared to an ILP as they required more resources for some graphs than were available.

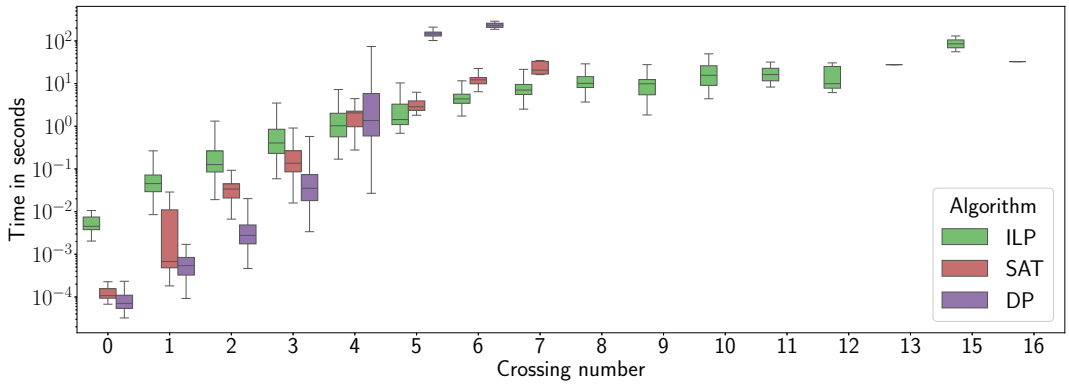
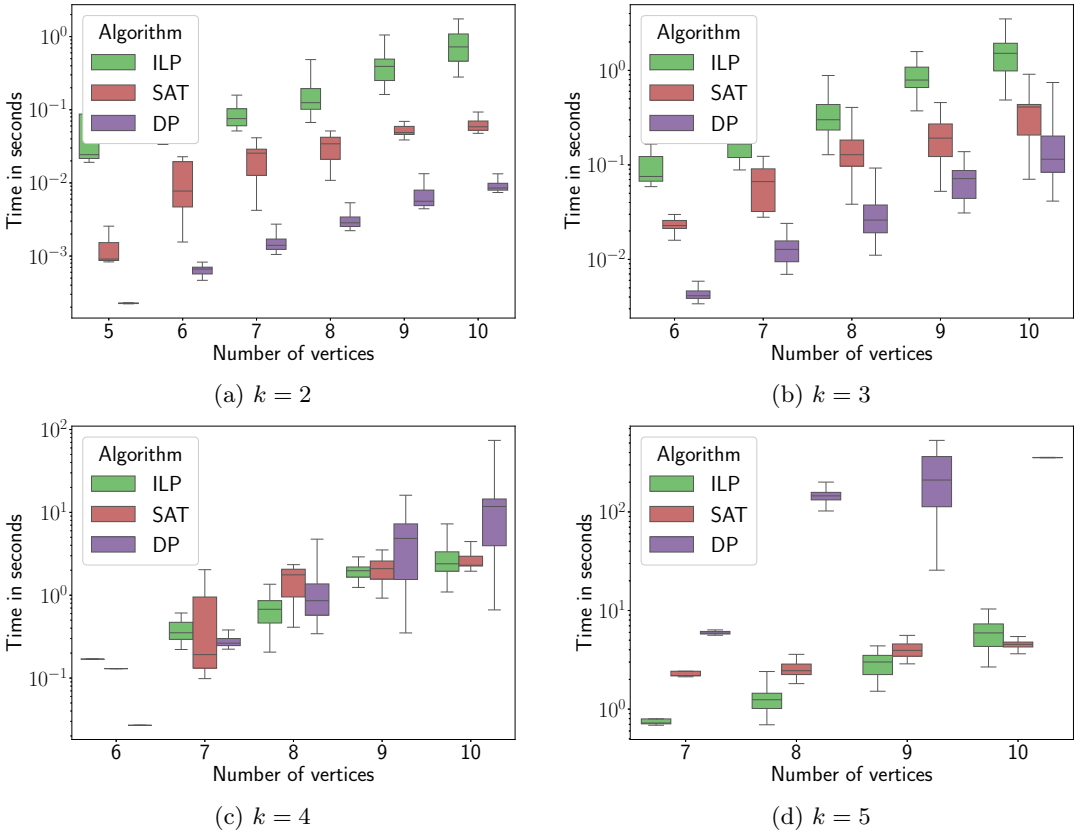


Figure 4.2: Comparison of running times of different algorithms for graphs with different crossing numbers.

SAT and DP, however, were limited primarily by different factors. Unlike ILP, encoding in SAT contains an exponential number of clauses. As a result, the solver ran out of allocated memory for 185 out of 1326 biconnected graphs. Similarly, the DP ran out of memory 16 times. However, the bigger limiting factor for this algorithm is time, as 317 runs exceeded the 10-minute time limit imposed on each one. All other runs finished successfully.

The results show that the time required by both SAT and DP grows much faster than the time required by ILP. The latter requires each instance to set up an environment for the graphs. With smaller crossing numbers, these costs outweigh the solver's speed. However, for instances with bigger k, the time required for setup is negligible.

As the algorithm's running time depends not only on crossing numbers but also on the size of the graph, we can represent the results more accurately by grouping them by both the crossing number and the number of vertices. To demonstrate this



(a) $k = 2$ (b) $k = 3$

(c) $k = 4$ (d) $k = 5$

Figure 4.3: Comparison of running times required by algorithms to recognise outer k-planar graphs for various values of k.

dependency, in Figure 4.3, we created a plot for each crossing number represented by at least 150 graphs. These plots show that, for $k \in \{2, 3\}$, DP and SAT are consistently the two fastest methods, with the former being faster than the latter. However, starting from $k = 4$ and graphs with at least 9 vertices, ILP outperforms the other two methods. Most importantly, these results agree with the ones displayed in Figure 4.2, which aggregate them for each value k.

4.6 Optimisation benchmark

The last experiment we considered is the comparison of optimisations we discussed for ILP and SAT in Section 3.4. For the ILP, we ran four configurations for each biconnected graph using none, one or both optimisations. For the SAT, we used two configurations with and without optimisation. The results are presented in Figure 4.4 and Figure 4.5 respectively.

In the first plot, we can clearly see that adding additional constraints to enforce the exact value for each crossing variable degrades the performance. For SAT, however, this change does not influence this much. We can see a slight rise in execution time, but the difference is within the margin of error. Here, the difference may be caused by the algorithm writing down the required additional constraints, not the SAT solver.

The objective optimisation for the ILP, which includes an extra term in the objective function, makes small but still an improvement. As a result, we consider the algorithm that includes this optimisation to be the most efficient ILP. Thus, we have

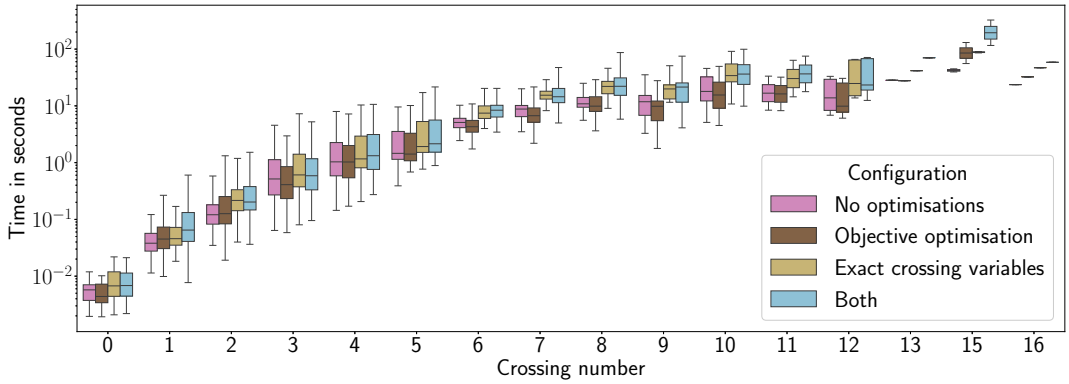


Figure 4.4: Comparison of different configurations for ILP-based algorithm.

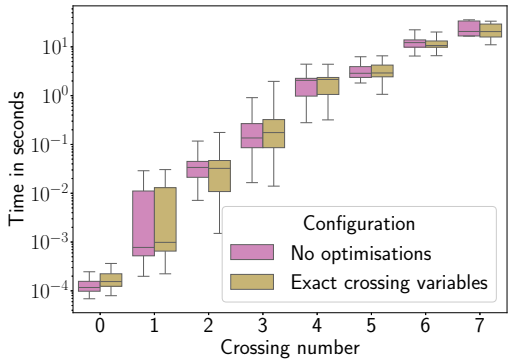


Figure 4.5: Comparison of different configurations for SAT-based algorithm.

used this configuration for all other experiments. For the SAT, we used a configuration that did not include optimisation.

Chapter 5

Conclusions

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