对　　　反

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例 3.1.43 求$\int e^x \sin x\mathrm{d}x$.

解

$$\int e^x \sin x\mathrm{d}x = \int \sin x\mathrm{d}e^x = e^x \sin x - \int e^x \cos x\mathrm{d}x$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x\mathrm{d}x.$$

令$\int e^x \sin x\mathrm{d}x = I$，则上式变为$I = e^x \sin x - e^x \cos x - I$，于是

$$2I = e^x \sin x - e^x \cos x$$

$$I = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$$

即$\int e^x \sin x\mathrm{d}x = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$.

$u = e^x$ ，$v' = \sin x$，　$v = -\cos x$

$\int e^x \sin x\mathrm{d}x = -e^x \cos x + \int e^x \cos x\mathrm{d}x$　　$u = e^x$ ，$v' = \cos x$，　$v = \sin x$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x\mathrm{d}x$$

$$\int e^x \sin x\mathrm{d}x = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$$

$$1.\ \int \sin^n x\mathrm{d}x = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\mathrm{d}x$$

证明： $u=\sin^{n-1}x$，$v'=\sin x$ ， $v=-\cos x$

$$\begin{aligned}\int \sin^n x\mathrm{d}x &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\mathrm{d}x\\ &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\mathrm{d}x\\ &= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\mathrm{d}x - (n-1)\int \sin^n x\mathrm{d}x\end{aligned}$$

$$\int \sin^n x\mathrm{d}x = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\mathrm{d}x$$

$$2.\ \int \frac{1}{\sin^n x}\mathrm{d}x = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2}x}\mathrm{d}x\ (n\geq 2)$$

证明： $u=\dfrac{1}{\sin^{n-1}x}$， $v'=\sin x$ ， $v=-\cos x$

$$\begin{aligned}\int \frac{1}{\sin^{n-2}x}\mathrm{d}x = \int \frac{\sin x}{\sin^{n-1}x}\mathrm{d}x &= -\frac{\cos x}{\sin^{n-1}x} - (n-1)\int \frac{\cos^2 x}{\sin^n x}\mathrm{d}x\\ &= -\frac{\cos x}{\sin^{n-1}x} - (n-1)\int \frac{1-\sin^2 x}{\sin^n x}\mathrm{d}x\\ &= -\frac{\cos x}{\sin^{n-1}x} - (n-1)\int \frac{1}{\sin^n x}\mathrm{d}x + (n-1)\int \frac{1}{\sin^{n-2}x}\mathrm{d}x\end{aligned}$$

$$\int \frac{1}{\sin^n x}\mathrm{d}x = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}\int \frac{1}{\sin^{n-2}x}\mathrm{d}x$$

3. 求 $\int \frac{1}{1+e^x}\mathrm{d}x$.

解： $\int \frac{1}{1+e^x}\mathrm{d}x = \int \frac{e^{-x}}{1+e^{-x}}\mathrm{d}x = -\int \frac{1}{1+e^{-x}}\mathrm{d}(e^{-x}+1) = -\ln(1+e^{-x}) + C$

也可以如下求解

$$\int \frac{e^x+1-e^x}{1+e^x}\mathrm{d}x = \int\left(1-\frac{e^x}{1+e^x}\right)\mathrm{d}x = x - \int \frac{1}{1+e^x}\mathrm{d}(e^x+1) = x - \ln(e^x+1) + C.$$

4. 已知 $f(x)$ 的一个原函数为 $(1+\sin x)\ln x$，求 $\int xf'(x)\mathrm{d}x$

解：$\int xf'(x)\mathrm{d}x = xf(x) - \int f(x)\mathrm{d}x = x\left((1+\sin x)\ln x\right)' - (1+\sin x)\ln x + C$

$$= x\left(\cos x\ln x + \frac{1+\sin x}{x}\right) - (1+\sin x)\ln x + C$$

5. 设 $f(\sin^2 x) = \frac{x}{\sin x}$，求 $\int \frac{\sqrt{x}}{\sqrt{1-x}} f(x)\mathrm{d}x$

解：令 $x = \sin^2 t, t \in [0, \frac{\pi}{2})$，

$$\int \frac{\sqrt{x}}{\sqrt{1-x}} f(x)\mathrm{d}x = -\int \frac{\sin t}{\cos t} f(\sin^2 t) 2\sin t\cos t\mathrm{d}t$$

$$= \int \frac{\sin t}{\cos t}\cdot\frac{t}{\sin t} 2\sin t\cos t\mathrm{d}t$$

$$= 2\int t\sin t\mathrm{d}t = 2(-t\cos t + \int \cos t\mathrm{d}t)$$

$$= 2(-t\cos t + \sin t) + C$$

$$= 2(-\sqrt{1-x}\arcsin\sqrt{x} + \sqrt{x}) + C$$

6. 已知$\dfrac{\sin x}{x}$是$f(x)$的一个原函数，求$\int x^3 f'(x)\mathrm{d}x$

解：$\int x^3 f'(x)\mathrm{d}x = x^3 f(x) - 3\int x^2 f(x)\mathrm{d}x = x^3\left(\dfrac{\sin x}{x}\right)' - 3\int x^2\left(\dfrac{\sin x}{x}\right)'\mathrm{d}x$

$$= x^3\left(\frac{x\cos x - \sin x}{x^2}\right) - 3\int x^2\left(\frac{x\cos x - \sin x}{x^2}\right)\mathrm{d}x$$

$$= x^2\cos x - x\sin x - 3\int(x\cos x - \sin x)\mathrm{d}x$$

$$= x^2\cos x - x\sin x - 3\int x\cos x\mathrm{d}x + 3\int \sin x\mathrm{d}x$$

$$= x^2\cos x - 4x\sin x - 6\cos x + C$$

7. 已知$f(x) = \dfrac{e^x + e^{-x}}{2}$，求$\int\left[\dfrac{f'(x)}{f(x)} + \dfrac{f(x)}{f'(x)}\right]\mathrm{d}x$

解：$f'(x) = \dfrac{e^x - e^{-x}}{2}$

$$\int\left[\frac{f'(x)}{f(x)} + \frac{f(x)}{f'(x)}\right]\mathrm{d}x = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}} + \frac{e^x + e^{-x}}{e^x - e^{-x}}\right]\mathrm{d}x = \int\frac{e^x - e^{-x}}{e^x + e^{-x}}\mathrm{d}x + \int\frac{e^x + e^{-x}}{e^x - e^{-x}}\mathrm{d}x$$

$$= \int\frac{1}{e^x + e^{-x}}\mathrm{d}(e^x + e^{-x}) + \int\frac{1}{e^x - e^{-x}}\mathrm{d}(e^x - e^{-x})$$

$$= \ln(e^x + e^{-x}) + \ln\left|e^x - e^{-x}\right| + C == \ln\left|e^{2x} - e^{-2x}\right| + C$$

8. 设$f(x)$的一个原函数$F(x) = \ln^2(x + \sqrt{1 + x^2})$，求$\int xf'(x)\mathrm{d}x$

解：$\int xf'(x)\mathrm{d}x = xf(x) - \int f(x)\mathrm{d}x = xF'(x) - F(x) + C$

9. $\int\frac{x^3}{(1+x^8)^2}\mathrm{d}x=\frac{1}{4}\int\frac{1}{(1+x^8)^2}\mathrm{d}x^4=\frac{1}{4}\int\frac{1}{(1+u^2)^2}\mathrm{d}u$ （$u=x^4$）

解： $\int\frac{1}{(1+u^2)^2}\mathrm{d}u=\int\frac{1+u^2-u^2}{(1+u^2)^2}\mathrm{d}u=\int\left(\frac{1}{1+u^2}-\frac{u^2}{(1+u^2)^2}\right)\mathrm{d}u$

$$\int\frac{1}{1+u^2}\mathrm{d}u=\arctan u$$

$$\int\frac{u^2}{(1+u^2)^2}\mathrm{d}u=-\frac{1}{2}\int u\mathrm{d}\frac{1}{(1+u^2)}=-\frac{1}{2}\left(\frac{u}{1+u^2}-\int\frac{1}{1+u^2}\mathrm{d}u\right)$$ （分部积分）

$$=-\frac{1}{2}\left(\frac{u}{1+u^2}-\arctan u\right)=\frac{1}{2}\arctan u-\frac{u}{2(1+u^2)}$$

$$\int\frac{1}{(1+u^2)^2}\mathrm{d}u=\frac{1}{2}\arctan u+\frac{u}{2(1+u^2)}+C$$

$$\int\frac{x^3}{(1+x^8)^2}\mathrm{d}x=\frac{1}{4}\int\frac{1}{(1+x^8)^2}\mathrm{d}x^4=\frac{1}{4}\int\frac{1}{(1+u^2)^2}\mathrm{d}u=\frac{1}{8}\arctan x^4+\frac{x^4}{8(1+x^8)}+C$$