1. $\lim\limits_{x\to\infty}\left(\sqrt[3]{1-x^6}-ax^2-b\right)=0$ 求 a， b

$$\lim_{x\to\infty}\left(\sqrt[3]{1-x^6}-ax^2-b\right)=\lim_{x\to\infty}x^2\left(\sqrt[3]{\frac{1}{x^6}-1}-a-\frac{b}{x^2}\right)=0\Rightarrow a=-1$$

$$b=\lim_{x\to\infty}\left(\sqrt[3]{1-x^6}+x^2\right)=\lim_{x\to\infty}\frac{\left(\sqrt[3]{1-x^6}+x^2\right)\left((1-x^6)^{\frac{2}{3}}-(1-x^6)^{\frac{1}{3}}x^2+x^4\right)}{\left((1-x^6)^{\frac{2}{3}}-(1-x^6)^{\frac{1}{3}}x^2+x^4\right)}$$

$$=\lim_{x\to\infty}\frac{1-x^6+x^6}{\left((1-x^6)^{\frac{2}{3}}-(1-x^6)^{\frac{1}{3}}x^2+x^4\right)}=\lim_{x\to\infty}\frac{1}{x^4\left(\left(\frac{1}{x^6}-1\right)^{\frac{2}{3}}-\left(\frac{1}{x^6}-1\right)^{\frac{1}{3}}+1\right)}=0$$

2. $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{3^n+4^n}{5^n+n}}$

解： $\dfrac{4}{5}=\sqrt[n]{\dfrac{4^n}{5^n}}\le\sqrt[n]{\dfrac{3^n+4^n}{5^n+n}}\le\sqrt[n]{\dfrac{3^n+4^n}{5^n}}=\sqrt[n]{\left(\dfrac{3}{5}\right)^n+\left(\dfrac{4}{5}\right)^n}$

$$\frac{4^n}{5^n}\le\frac{3^n+4^n}{5^n+n}$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{3^n+4^n}{5^n+n}}=\frac{4}{5}$$

3. $\lim\limits_{x\to\infty}\dfrac{2x^2+5}{x+1}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$

解：

$$\lim_{x\to\infty}\frac{2x^2+5}{x+1}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\frac{2(2x^2+5)}{(x+1)\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}$$

$$\lim_{x\to+\infty}\frac{4+\frac{10}{x^2}}{\left(1+\frac{1}{x}\right)\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)}=2，\quad\lim_{x\to-\infty}\frac{4+\frac{10}{x^2}}{\left(-1-\frac{1}{x}\right)\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)}=-2$$

4. $\lim\limits_{x\to\infty}\dfrac{e^x}{\left(1+\dfrac{1}{x}\right)^{x^2}}$

解：$\lim\limits_{x\to\infty}\dfrac{e^x}{\left(1+\dfrac{1}{x}\right)^{x^2}}=\lim\limits_{x\to\infty}\dfrac{e^x}{e^{x^2\ln\left(1+\frac{1}{x}\right)}}=\lim\limits_{x\to\infty}e^{x-x^2\ln\left(1+\frac{1}{x}\right)}$

$$=e^{\lim\limits_{x\to\infty}\left(x-x^2\ln\left(1+\frac{1}{x}\right)\right)}=e^{\frac{1}{2}}$$

令$x=\dfrac{1}{t}$，$\lim\limits_{x\to\infty}\left(x-x^2\ln\left(1+\dfrac{1}{x}\right)\right)=\lim\limits_{t\to0}\left(\dfrac{1}{t}-\dfrac{\ln(1+t)}{t^2}\right)=\lim\limits_{t\to0}\dfrac{t-\ln(1+t)}{t^2}$

$$=\lim_{t\to0}\frac{1-\frac{1}{1+t}}{2t}=\lim_{t\to0}\frac{t}{2t(1+t)}=\frac{1}{2}$$

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5. 设函数$g(x)$在点a处连续，证明函数$f(x)=(x-a)g(x)$在点a处可导，并求$f'(a)$.

解：

$$\lim_{\Delta x\to0}\frac{f(\Delta x+a)-f(a)}{\Delta x}=\lim_{\Delta x\to0}\frac{\Delta xg(\Delta x+a)-0}{\Delta x}=\lim_{\Delta x\to0}g(\Delta x+a)=g(a)$$

6. 讨论函数$f(x)=\begin{cases}x^2, x\text{为有理数}\\ -x^2, x\text{为无理数}\end{cases}$在点$x=0$处的连续性和可导性.

解：$\lim\limits_{x\to0}f(x)=\lim\limits_{x\to0}\left(\pm x^2\right)=0=f(0)$ 故在点$x=0$处连续.

$\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{\left(\pm x^2\right)}{x}=0$ 故在点$x=0$处可导.