1. 设 $f(x)=\begin{cases}\dfrac{\sin x}{x}, x\neq 0\\ 1,\ x=0\end{cases}$，则（　　）

A、 $f'(0)=0$，$f''(0)=-\dfrac{1}{3}$.　　B、 $f'(0)=0$，$f''(0)=-\dfrac{1}{6}$.

C、 $f'(0)=1$，　$f''(0)=\dfrac{1}{3}$.　　D、 $f'(0)=1$，　$f''(0)=\dfrac{1}{6}$.

解：

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{\frac{\sin x}{x}-1}{x}=\lim_{x\to 0}\frac{\sin x-x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}=0$$

$x\neq 0$，$f'(x)=\dfrac{x\cos x-\sin x}{x^2}$

$$f''(0)=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{\frac{x\cos x-\sin x}{x^2}-0}{x}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{3x^2}=-\frac{1}{3}$$

2. 设 $f(x)=\left(x^2-1\right)\arctan\dfrac{1+2x^2}{1+x+x^2}$，则 $f'(1)=$（　　）

A、 $\dfrac{\pi}{2}$.　B、 $\dfrac{\pi}{3}$.　C、 $\dfrac{\pi}{4}$.　D、 $\dfrac{\pi}{6}$.

解： $f'(x)=2x\arctan\dfrac{1+2x^2}{1+x+x^2}+\left(x^2-1\right)\left(\arctan\dfrac{1+2x^2}{1+x+x^2}\right)'$

$$f'(1)=2\cdot\frac{\pi}{4}=\frac{\pi}{2}$$

3. 设 $\begin{cases}x=t-\ln(1+t)\\ y=t^3+t^2\end{cases}$，则（　　）

A、 $\dfrac{d^2y}{dx^2}=\dfrac{(6t+5)(1+t)}{t}$.　　B、 $\dfrac{d^2y}{dx^2}=6t+5$.

C、 $\dfrac{d^2y}{dx^2}=(6t+2)(1+t)^2$.　　D、 $\dfrac{d^2y}{dx^2}=-(6t+2)(1+t)^2$.

解：$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3t^2+2t}{1-\dfrac{1}{1+t}}=(3t+2)(1+t)=3t^2+5t+2$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=(6t+5)\cdot\frac{(1+t)}{t}=\frac{(6t+5)(1+t)}{t}$$

4. 求 $y=\dfrac{x^2}{x^2-2x-3}$ 的 n 阶导数

解：$y=\dfrac{x^2}{x^2-2x-3}=\dfrac{x^2-2x-3}{x^2-2x-3}+\dfrac{2x+3}{x^2-2x-3}=1+\dfrac{2x+3}{(x-3)(x+1)}$

$$=1+\frac{1}{4}\left(\frac{9}{x-3}-\frac{1}{x+1}\right)$$

$$y^{(n)}=\frac{1}{4}\left(9\frac{(-1)^n n!}{(x-3)^{n+1}}-\frac{(-1)^n n!}{(x+1)^{n+1}}\right)=\frac{(-1)^n n!}{4}\left(\frac{9}{(x-3)^{n+1}}-\frac{1}{(x+1)^{n+1}}\right)$$

5. 设 $f_n(x)=x^{n-1}e^{\frac{1}{x}}$，求证：$f_n^{(n)}(x)=\dfrac{(-1)^n}{x^{n+1}}e^{\frac{1}{x}}$

解：$n=1$，$\left(e^{\frac{1}{x}}\right)'=\dfrac{-1}{x^2}e^{\frac{1}{x}}$ 成立

假设 $n=k$ 成立，即 $\left(x^{k-1}e^{\frac{1}{x}}\right)^{(k)}=\dfrac{(-1)^k}{x^{k+1}}e^{\frac{1}{x}}$

现证 $n=k+1$ 成立，

$$\left(x^k e^{\frac{1}{x}}\right)^{(k+1)}=\left(x\left(x^{k-1}e^{\frac{1}{x}}\right)\right)^{(k+1)}=\left(x^{k-1}e^{\frac{1}{x}}\right)^{(k+1)}\cdot x+(k+1)\left(x^{k-1}e^{\frac{1}{x}}\right)^{(k)}$$

$$=\frac{(-1)^{k+1}(k+1)}{x^{k+1}}e^{\frac{1}{x}}+\frac{(-1)^{k+1}}{x^{k+2}}e^{\frac{1}{x}}+(k+1)\frac{(-1)^k}{x^{k+1}}e^{\frac{1}{x}}$$

$$=\frac{(-1)^{k+1}}{x^{k+2}}e^{\frac{1}{x}}$$

6. $x=2$是函数$f(x)=\arctan\frac{1}{2-x}$的（　　）

A．跳跃间断点　　　　B．无穷间断点

C．连续点　　　　　　D．可去间断点

7. $\lim\limits_{x\to 0}\frac{\tan x-\cos x+1}{\ln(1+x)+x^2}$

解：$\lim\limits_{x\to 0}\frac{\tan x-\cos x+1}{x}=\lim\limits_{x\to 0}\frac{\tan x}{x}+\lim\limits_{x\to 0}\frac{1-\cos x}{x}=1$

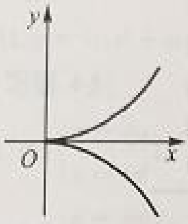
$$\lim_{x\to 0}\frac{\ln(1+x)+x^2}{x}=\lim_{x\to 0}\frac{\ln(1+x)}{x}+\lim_{x\to 0}\frac{x^2}{x}=1$$

$$\lim_{x\to 0}\frac{\tan x-\cos x+1}{\ln(1+x)+x^2}=1$$

附录　几种常见曲线

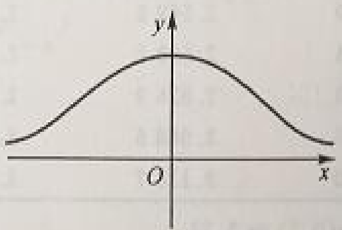
(1) 半立方抛物线

$y^2 = ax^3$



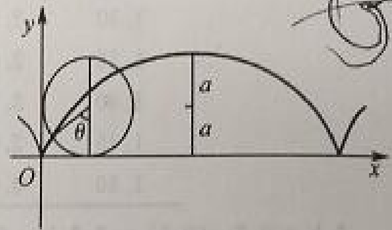
(2) 概率曲线

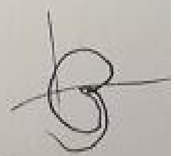
$y = \mathrm{e}^{-x^2}$



(3) 摆线

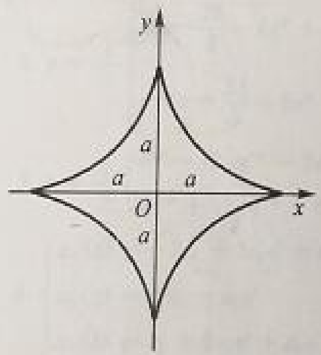
$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$





(4) 星形线(内摆线的一种)

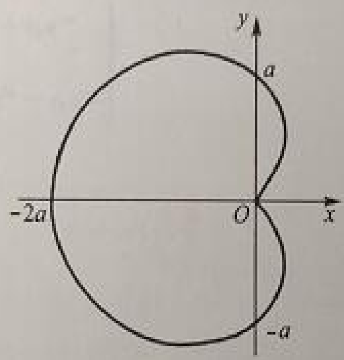
$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, \begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \end{cases}$



(5) 心形线(外摆线的一种)

$x^2 + y^2 + ax = a\sqrt{x^2 + y^2}$

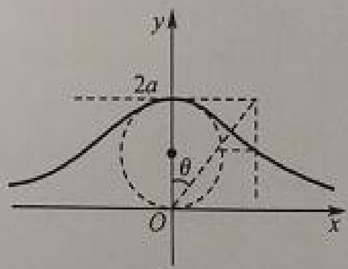
或 $r = a(1 - \cos\theta)$



(6) 箕舌线

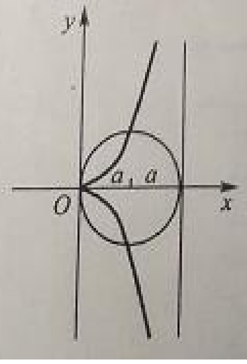
$y = \dfrac{8a^3}{x^2 + 4a^2}$

或 $\begin{cases} x = 2a\tan\theta \\ y = 2a\cos^2\theta \end{cases}$



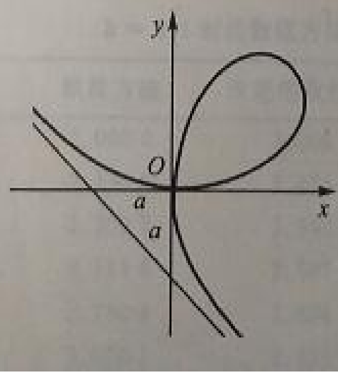
(7) 蔓叶线

$y^2(2a - x) = x^3$



(8) 笛卡儿叶形线

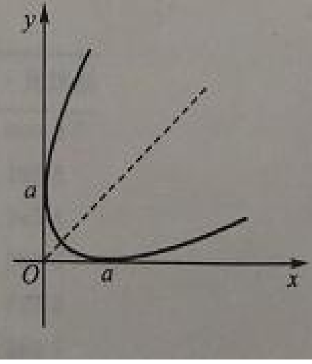
$x = \dfrac{3at}{1+t^3},\ y = \dfrac{3at^2}{1+t^3}$



(9) 抛物线

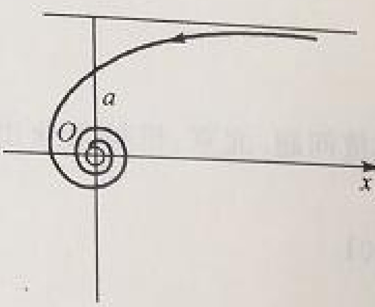
$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

或 $\begin{cases} x = a\cos^4 t \\ y = a\cos^4 t \end{cases}$



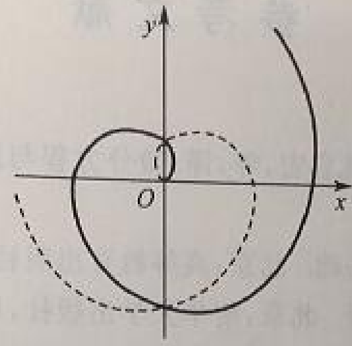
(10) 双曲螺线

$r\theta = a$



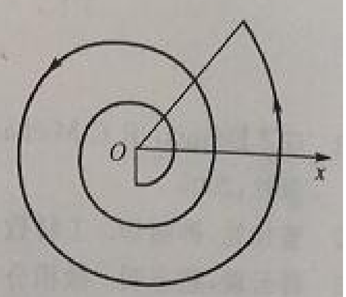
(11) 阿基米德螺线

$r = a\theta$



(12) 对数螺线

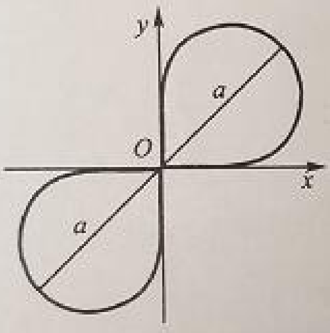
$r = e^{a\theta}$



(13) 伯努利双纽线

$(x^2 + y^2)^2 = 2a^2xy$

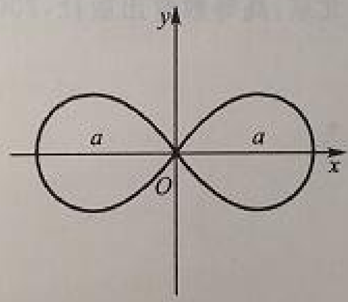
$r^2 = a^2 \sin 2\theta$



(14) 伯努利双纽线

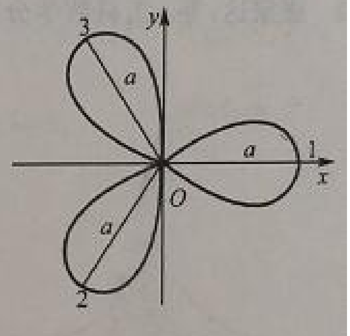
$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

$r^2 = a^2 \cos 2\theta$



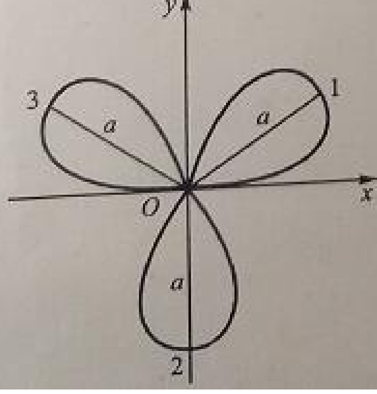
(15) 三叶玫瑰线

$r = a\cos 3\theta$



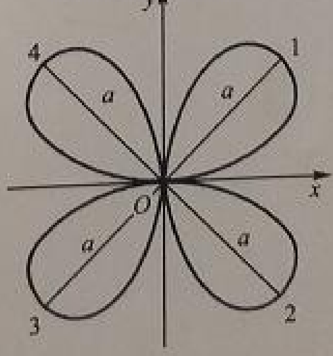
(16) 三叶玫瑰线

$r = a\sin 3\theta$



(17) 四叶玫瑰线

$r = a\sin 2\theta$



(18) 四叶玫瑰线

$r = a\cos 2\theta$