1. 设$\dfrac{\ln x}{x}$是$f(x)$的一个原函数，则$\int xf(x)\mathrm{d}x=$（　　）

A、 $\ln x-\ln(\ln x)+C$．　　　B、$\dfrac{1}{4}\left(x^2\ln^2 x-x^2\ln x+2x\right)+C$．

C、 $x\ln x-x+C$．　　　D、$\ln x-\dfrac{1}{2}\ln^2 x+C$．

解：$f(x)=\left(\dfrac{\ln x}{x}\right)'=\dfrac{1-\ln x}{x^2}$

$$\int xf(x)\mathrm{d}x=\int\left(\frac{1}{x}-\frac{\ln x}{x}\right)\mathrm{d}x=\ln x-\frac{1}{2}\ln^2 x+C$$

2. $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{4n^2-1^2}}+\dfrac{1}{\sqrt{4n^2-2^2}}+\cdots+\dfrac{1}{\sqrt{4n^2-n^2}}\right)$

解：$\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{4n^2-1^2}}+\dfrac{1}{\sqrt{4n^2-2^2}}+\cdots+\dfrac{1}{\sqrt{4n^2-n^2}}\right)$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{4-\left(\frac{i}{n}\right)^2}}=\int_0^1\frac{1}{\sqrt{4-x^2}}\mathrm{d}x$$

因为$f(x)=\dfrac{1}{\sqrt{4-x^2}}$在$[0,1]$上连续，所以在$[0,1]$上可积．现在对$[0,1]$ n等分，$[0,\frac{1}{n}]$，$[\frac{1}{n},\frac{2}{n}]$，$\cdots$，$[\frac{n-1}{n},\frac{n}{n}]$，$x_0=0$，$x_1=\frac{1}{n}$，$\cdots$，$x_n=\frac{n}{n}$

$\Delta x_i=\dfrac{1}{n}$，$\xi_i=\dfrac{i}{n}$ $(i=1,2,\cdots,n)$，则

$$\int_0^1\frac{1}{\sqrt{4-x^2}}\mathrm{d}x=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{4-\left(\frac{i}{n}\right)^2}}\cdot\frac{1}{n}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{4-\left(\frac{i}{n}\right)^2}}$$

3. $\lim\limits_{n\to\infty}\dfrac{1^5+2^5+\cdots+n^5}{n^6}=$ ( ).

A. 0. B. $\dfrac{1}{6}$. C. $\dfrac{1}{5}$.

$$\lim_{n\to\infty}\frac{1^5+2^5+\cdots+n^5}{n^6}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^5=\int_0^1 x^5\mathrm{d}x=\frac{1}{6}$$

4. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}=\int_0^1\frac{1}{1+x}\mathrm{d}x=\ln 2$$

5. $\lim\limits_{n\to\infty}\sqrt[n]{f\left(\dfrac{1}{n}\right)\cdot f\left(\dfrac{2}{n}\right)\cdots f\left(\dfrac{n}{n}\right)}$, $f(x)>0$

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)}=\lim_{n\to\infty}\left(f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

$$=e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)\right)}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdot\cdots\cdot f\left(\frac{n}{n}\right)\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln f\left(\frac{i}{n}\right)=\int_0^1\ln f(x)\mathrm{d}x$$

$$\lim_{n\to\infty}\sqrt[n]{f\left(\frac{1}{n}\right)\cdot f\left(\frac{2}{n}\right)\cdots f\left(\frac{n}{n}\right)}=e^{\int_0^1\ln f(x)\mathrm{d}x}$$

6. $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}=\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\cdots\cdot\frac{n}{n}\right)^{\frac{1}{n}}=e^{\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\cdots\cdot\frac{n}{n}\right)}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\cdots\cdot\frac{n}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{i}{n}\right)=\int_0^1\ln x\mathrm{d}x=-1$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=e^{-1}$$

7. 若 $f(x)$ 在 $[a,b]$ 上非负、连续, 且不恒为零，则 $\int_a^b f(x)\mathrm{d}x > 0$.

证明：由已知，至少存在 $x_0 \in [a,b]$，使 $f(x_0) > 0$

若 $x_0 \in (a,b)$，因为 $\lim\limits_{x \to x_0} f(x) = f(x_0) > 0$，由保号性知，存在 $\delta > 0$，

$x \in [x_0 - \delta, x_0 + \delta]$，使 $f(x) > 0$，于是由积分中值定理有

$$\int_{x_0-\delta}^{x_0+\delta} f(x)\mathrm{d}x = f(\xi)2\delta > 0，\quad \xi \in [x_0 - \delta, x_0 + \delta]$$

所以

$$\int_a^b f(x)\mathrm{d}x = \int_a^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0-\delta}^{x_0+\delta} f(x)\mathrm{d}x + \int_{x_0+\delta}^b f(x)\mathrm{d}x > 0$$

若 $x_0 = a$，即 $f(a) > 0$

因为 $\lim\limits_{x \to a^+} f(x) = f(a) > 0$，由保号性知，存在 $\delta > 0$，$x \in [a, a+\delta]$，

使 $f(x) > 0$，于是由积分中值定理有

$$\int_a^{a+\delta} f(x)\mathrm{d}x = f(\xi)\delta > 0，\quad \xi \in [a, a+\delta]$$

所以

$$\int_a^b f(x)\mathrm{d}x = \int_a^{a+\delta} f(x)\mathrm{d}x + \int_{a+\delta}^b f(x)\mathrm{d}x > 0$$

若 $x_0 = b$，即 $f(b) > 0$ 可类似证明.

8. $\lim\limits_{n \to \infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{n}{n^2+n^2}\right)$

$$\lim_{n \to \infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{n}{n^2+n^2}\right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{4}$$

9. $\lim\limits_{n\to\infty}\left(\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots+\frac{\sqrt{n^2-n^2}}{n^2}\right)=(\qquad)$

解： A、$\frac{\pi}{4}$． B、$\frac{1}{2}\ln 2$． C、0． D、1．

解：
$$\lim_{n\to\infty}\left(\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\cdots+\frac{\sqrt{n^2-n^2}}{n^2}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}=\int_0^1\sqrt{1-x^2}\,\mathrm{d}x=\frac{\pi}{4}$$

10. 求$\int\frac{\arctan e^x}{e^{2x}}\mathrm{d}x$

解：$u=\arctan e^x$ ，$v'=e^{-2x}$ ，$v=-\frac{1}{2}e^{-2x}$

$$\int\frac{\arctan e^x}{e^{2x}}\mathrm{d}x=-\frac{1}{2}e^{-2x}\arctan e^x+\frac{1}{2}\int e^{-2x}\frac{e^x}{1+e^{2x}}\mathrm{d}x$$
$$=-\frac{1}{2}e^{-2x}\arctan e^x+\frac{1}{2}\int\frac{1+e^{2x}-e^{2x}}{e^{2x}(1+e^{2x})}\mathrm{d}e^x$$
$$=-\frac{1}{2}e^{-2x}\arctan e^x+\frac{1}{2}\int\left(e^{-2x}-\frac{1}{1+e^{2x}}\right)\mathrm{d}e^x$$
$$=-\frac{1}{2}e^{-2x}\arctan e^x-\frac{1}{2}e^{-x}-\frac{1}{2}\arctan e^x+C$$