

1. 函数 $z = z(x, y)$ 由方程 $x - 2019z = \varphi(y - 2020z)$ 确定, 其中 φ 为可微函数, 则 $2019 \frac{\partial z}{\partial x} + 2020 \frac{\partial z}{\partial y} = (\quad)$.

(A) 0; (B) 1; (C) 2; (D) 3.

解 (B)

$$1 - 2019 \frac{\partial z}{\partial x} = \varphi'(y - 2020z) \cdot (-2020 \frac{\partial z}{\partial x}) = \varphi' \cdot (-2020 \frac{\partial z}{\partial x})$$

$$\frac{\partial z}{\partial x} = \frac{1}{2019 - 2020\varphi'}$$

$$-2019 \frac{\partial z}{\partial y} = \varphi'(y - 2020z) \cdot (1 - 2020 \frac{\partial z}{\partial y}) = \varphi' \cdot (1 - 2020 \frac{\partial z}{\partial y})$$

$$\frac{\partial z}{\partial y} = \frac{-\varphi'}{2019 - 2020\varphi'}$$

$$2019 \frac{\partial z}{\partial x} + 2020 \frac{\partial z}{\partial y} = 1$$

2. 函数 $z = z(x, y)$ 由方程 $F\left(\frac{y}{x}, \frac{z}{x}\right) = 0$ 确定, 其中 F 有连续偏导数, 且 $F'_2 \neq 0$ 则 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = (\quad)$.

(A) $-x$; (B) $-z$; (C) x ; (D) z .

解 (D)

$$F'_1 \left(-\frac{y}{x^2} \right) + F'_2 \left(\frac{x \frac{\partial z}{\partial x} - z}{x^2} \right) = 0, \quad \frac{\partial z}{\partial x} = \frac{yF'_1 + zF'_2}{xF'_2}$$

$$F'_1 \left(\frac{1}{x} \right) + F'_2 \left(\frac{\frac{\partial z}{\partial y}}{x} \right) = 0, \quad \frac{\partial z}{\partial y} = \frac{-F'_1}{F'_2}$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

3. 设 $z = f\left(xy, \frac{1}{2}(x^2 - y^2)\right)$, 其中 f 具有二阶连续偏导数, 则

$$\frac{\partial^2 z}{\partial x \partial y} = (\quad).$$

(A) $xy(f''_{11} - f''_{22}) + f'_1 + (x^2 - y^2)f''_{12}$; (B) $xy(f''_{11} + f''_{22}) + f'_1 + (x^2 - y^2)f''_{12}$;

(C) $xy(f''_{11} + f''_{22}) + f'_1 + (x^2 + y^2)f''_{12}$; (D) $xy(f''_{11} - f''_{22}) + f'_1 + (x^2 + y^2)f''_{12}$.

解 (A)

$$\frac{\partial z}{\partial x} = yf'_1 + xf'_2$$

$$\frac{\partial^2 z}{\partial x \partial y} = f'_1 + y(xf''_{11} - yf''_{12}) + x(xf''_{21} - yf''_{22}) = xy(f''_{11} - f''_{22}) + f'_1 + (x^2 - y^2)f''_{12}$$

4. 设函数 $z = f(xy, yg(x))$, 其中 f 具有二阶连续偏导数, 函数 $g(x)$ 可导, 且

在 $x=1$ 处取得极值 $g(1)=1$, 求 $\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}}$.

解: $\frac{\partial z}{\partial x} = f'_1(xy, yg(x)) \cdot y + f'_2(xy, yg(x)) \cdot yg'(x)$.

$$\frac{\partial^2 z}{\partial x \partial y} = f'_1(xy, yg(x)) + y[f''_{11}(xy, yg(x)) \cdot x + f''_{12}(xy, yg(x)) \cdot g(x)]$$

$$+ f'_2(xy, yg(x)) \cdot g'(x) + yg'(x)[f''_{21}(xy, yg(x)) \cdot x + f''_{22}(xy, yg(x)) \cdot g(x)]$$

由于 $g(x)$ 在 $x=1$ 处取得极值, 故 $g'(1)=0$ 。将 $g(1)=1$, $g'(1)=0$ 代入上式

得

$$\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}} = f'_1(1,1) + f''_{11}(1,1) + f''_{12}(1,1) \quad .$$

5. 设有三元方程 $xy - z \ln y + e^{xz} = 1$, 则在点 $(0, 1, 1)$ 的一个邻域内, 该方程__.

A. 只能确定一个具有连续偏导数的隐函数 $z = z(x, y)$

B. 可确定两个具有连续偏导数的隐函数 $y = y(x, z)$ 和 $z = z(x, y)$

C. 可确定两个具有连续偏导数的隐函数 $x = x(y, z)$ 和 $y = y(x, z)$

D. 可确定两个具有连续偏导数的隐函数 $x = x(y, z)$ 和 $z = z(x, y)$

答案: C

6. 求函数 $z = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)$ 在点 $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$ 处沿曲线 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 在这点的

内法线方向的方向导数.

$$\text{解 由 } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0 \Rightarrow \left. \frac{dy}{dx} \right|_{\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)} = -\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}} = -\frac{b}{a}$$

即该点的切向量为 $(-a, b)$, 所以该点法向量为 $\pm(b, a)$, 该点的内法线方向

$$\text{为 } l = -(b, a) = (-b, -a), \quad e_l = \left(\frac{-b}{\sqrt{a^2 + b^2}}, \frac{-a}{\sqrt{a^2 + b^2}} \right)$$

$$\left. \frac{\partial z}{\partial x} \right|_{\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)} = -\frac{2x}{a^2} \bigg|_{x=\frac{a}{\sqrt{2}}} = -\frac{\sqrt{2}}{a}, \quad \left. \frac{\partial z}{\partial y} \right|_{\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)} = -\frac{2y}{b^2} \bigg|_{y=\frac{b}{\sqrt{2}}} = -\frac{\sqrt{2}}{b}$$

$$\frac{\partial z}{\partial l} \bigg|_{\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)} = \left(-\frac{\sqrt{2}}{a} \right) \cdot \left(\frac{-b}{\sqrt{a^2 + b^2}} \right) + \left(-\frac{\sqrt{2}}{b} \right) \cdot \left(\frac{-a}{\sqrt{a^2 + b^2}} \right) = \frac{\sqrt{2}}{ab} \sqrt{a^2 + b^2}$$

7. 求曲线 $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 2y \end{cases}$ 在点 $M_0(1, 1, \sqrt{2})$ 处的切线方程与法平面方程.

解 $2x + 2y \frac{dy}{dx} + 2z \frac{dz}{dx} = 0$

$$2x + 2y \frac{dy}{dx} = 2 \frac{dy}{dx} \quad \frac{dy}{dx} = \frac{x}{1-y}$$

$$2x \frac{dx}{dy} + 2y + 2z \frac{dz}{dy} = 0 \Rightarrow \frac{dz}{dy} = -\frac{1}{\sqrt{2}}$$

$$2x \frac{dx}{dy} + 2y = 2 \Rightarrow \frac{dx}{dy} = \frac{1-y}{x} = 0$$

曲线在点 $M_0(1, 1, \sqrt{2})$ 处的切向量为 $s = \left(0, 1, -\frac{1}{\sqrt{2}}\right) = (0, \sqrt{2}, -1)$, 因此所

求的切线方程为

$$\frac{x-1}{0} = \frac{y-1}{\sqrt{2}} = \frac{z-\sqrt{2}}{-1},$$

法平面方程为

$$\sqrt{2}(y-1) - (z-\sqrt{2}) = \sqrt{2}y - z = 0$$

8. 在曲线 $x=t, y=-t^2, z=t^3$ 的所有切线中, 与平面 $x+2y+z=4$ 平行的切线 ().

(A) 只有 1 条; (B) 只有 2 条; (C) 至少有 3 条; (D) 不存在.

解 (B) 只有 2 条;

$$x'(t)=1, \quad y'(t)=-2t, \quad z'(t)=3t^2, \quad (1, 2, 1)$$

$$1-4t+3t^2=0$$

9. 设函数 $z = f(x, y)$ 在点 $(0, 1)$ 的某邻域内可微, 且

$f(x, y+1) = 1 + 2x + 3y + o(\rho)$, 其中 $\rho = \sqrt{x^2 + y^2}$, 则曲面 $z = f(x, y)$ 在点 $(0, 1, f(0, 1))$ 处的切平面方程().

(A) $2x + 3y + z = 2$;

(B) $2x + 3y - z = 2$;

(C) $2x + 3y + z = 1$;

(D) $2x + 3y - z = 1$.

解 (B)

$$f(x, y+1) = 1 + 2x + 3y + o(\rho) \Rightarrow f(0, 1) = 1$$

$$f(x, y+1) - f(0, 1) = 2x + 3y + o(\rho)$$

$$\Delta z = f(\Delta x, \Delta y + 1) - f(0, 1) = 2\Delta x + 3\Delta y + o(\rho) \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$f_x(0, 1) = 2, \quad f_y(0, 1) = 3$$

$$F(x, y, z) = f(x, y) - z$$

曲面在点 $(0, 1, f(0, 1)) = (0, 1, 1)$ 的法向量 $(2, 3, -1)$

切平面方程 $2(x-0) + 3(y-1) - (z-1) = 0 \Rightarrow 2x + 3y - z = 2$